Willi-Hans Steeb

PROBLEMS AND SOLUTIONS IN INTRODUCTORY AND ADVANCED MATRIX CALCULUS

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Preface

The purpose of this book is to supply a collection of problems in introductory and advanced matrix problems together with their detailed solutions which will prove to be valuable to undergraduate and graduate students as well as to research workers in these fields. Each chapter contains an introduction with the essential definitions and explanations to tackle the problems in the chapter. If necessary, other concepts are explained directly with the present problems. Thus the material in the book is self-contained. The topics range in difficulty from elementary to advanced. Students can learn important principles and strategies required for problem solving. Lecturers will also find this text useful either as a supplement or text, since important concepts and techniques are developed in the problems.

A large number of problems are related to applications. Applications include wavelets, linear integral equations, Kirchhoff's laws, global positioning systems, Floquet theory, octonians, random walks, Kronecker product and images. A number of problems useful in quantum physics and graph theory are also provided. Advanced topics include groups and matrices, Lie groups and matrices and Lie algebras and matrices. Exercises for matrix-valued differential forms are also included.

The book can also be used as a text for linear and multilinear algebra or matrix theory. The material was tested in my lectures given around the world.

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the author if you want to do this course or other courses of the ISSC.

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Contents

Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbf{N}$	set of natural numbers
$\mathbf{Z}$	set of integers
$\mathbf{Q}$	set of rational numbers
$\mathbf{R}$	set of real numbers
$\mathbf{R}^+$	set of nonnegative real numbers
$\mathbf{C}$	set of complex numbers
$\mathbf{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbf{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2}, \quad x, y \in \mathbf{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector

$\Vert \cdot \Vert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
$\operatorname{tr}(A)$	trace of a square matrix A
$\operatorname{rank}(A)$	rank of matrix A
A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z .

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Chapter 1

Basic Operations

Let $\mathcal{F}$ be a field, for example the set of real numbers $\mathbf{R}$ or the set of complex numbers $\mathbf{C}$. Let m, n be two integers ≥ 1. An array A of numbers in $\mathcal{F}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$$

is called an $m \times n$ *matrix* with entry a_{ij} in the ith row and jth column. A *row vector* is a $1 \times n$ matrix. A *column vector* is an $n \times 1$ matrix. We have a *zero matrix*, in which $a_{ij} = 0$ for all i, j.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define $A + B$ to be the $m \times n$ matrix whose entry in the i-th row and j-th column is $a_{ij} + b_{ij}$. Matrix multiplication is only defined between two matrices if the number of columns of the first matrix is the same as the number of rows of the second matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the matrix product AB is an $m \times p$ matrix defined by

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

for each pair i and j, where $(AB)_{ij}$ denotes the (i, j)th entry in AB. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices with entries in some field. Then their Hadamard product is the entry-wise product of A and B, that is the $m \times n$ matrix $A \bullet B$ whose (i, j)th entry is $a_{ij} b_{ij}$.

Problem 1. Let $\mathbf{x}$ be a column vector in $\mathbf{R}^n$ and $\mathbf{x} \neq \mathbf{0}$. Let

$$A = \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$$

where ^{T} denotes the transpose, i.e. $\mathbf{x}^T$ is a row vector. Calculate A^2.

Solution 1. Obviously $\mathbf{x}\mathbf{x}^T$ is a nonzero $n \times n$ matrix and $\mathbf{x}^T\mathbf{x}$ is a nonzero real number. We find

$$\begin{aligned} A^2 &= \left(\frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}\right)^2 \\ &= \frac{(\mathbf{x}\mathbf{x}^T)(\mathbf{x}\mathbf{x}^T)}{(\mathbf{x}^T\mathbf{x})^2} \\ &= \frac{\mathbf{x}(\mathbf{x}^T\mathbf{x})\mathbf{x}^T}{(\mathbf{x}^T\mathbf{x})^2} \\ &= \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}} \\ &= A \end{aligned}$$

where we used the fact that matrix multiplication is *associative*. Thus $A^2 = A$.

Problem 2. Consider the vector in $\mathbf{R}^8$

$$\mathbf{x}^T = (20.0\ 6.0\ 4.0\ 2.0\ 10.0\ 6.0\ 8.0\ 4.0)$$

where ^{T} denotes transpose. Consider the matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$G_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad G_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$H_3 = \frac{1}{2}(1 \quad 1), \qquad G_3 = \frac{1}{2}(1 \quad -1).$$

(i) Calculate

$$H_1\mathbf{x}, \qquad G_1\mathbf{x}$$

$$H_2H_1\mathbf{x}, \quad G_2H_1\mathbf{x}, \quad H_2G_1\mathbf{x}, \quad G_2G_1\mathbf{x}$$

$$H_3H_2H_1\mathbf{x}, \quad G_3H_2H_1\mathbf{x}, \quad H_3G_2H_1\mathbf{x}, \quad G_3G_2H_1\mathbf{x},$$

$$H_3H_2G_1\mathbf{x}, \quad G_3H_2G_1\mathbf{x}, \quad H_3G_2G_1\mathbf{x}, \quad G_3G_2G_1\mathbf{x}$$

(ii) Calculate

$$H_jH_j^T, \quad G_jG_j^T, \quad H_jG_j^T$$

for $j = 1, 2, 3$.

(iii) How can we reconstruct the original vector $\mathbf{x}$ from the vector

$$(H_3H_2H_1\mathbf{x}, G_3H_2H_1\mathbf{x}, H_3G_2H_1\mathbf{x}, G_3G_2H_1\mathbf{x}, H_3H_2G_1\mathbf{x}, G_3H_2G_1\mathbf{x}, H_3G_2G_1\mathbf{x}, G_3G_2G_1\mathbf{x})$$

The problem plays a role in *wavelet theory.*

Solution 2. (i) We find

$$H_1\mathbf{x} = \begin{pmatrix} 13.0 \\ 3.0 \\ 8.0 \\ 6.0 \end{pmatrix}, \qquad G_1\mathbf{x} = \begin{pmatrix} 7.0 \\ 1.0 \\ 2.0 \\ 2.0 \end{pmatrix}.$$

Thus we have the vector

$$(13.0 \;\; 3.0 \;\; 8.0 \;\; 6.0 \;\; 7.0 \;\; 1.0 \;\; 2.0 \;\; 2.0)^T.$$

Next we find

$$H_2H_1\mathbf{x} = \begin{pmatrix} 8.0 \\ 7.0 \end{pmatrix}, \quad G_2H_1\mathbf{x} = \begin{pmatrix} 5.0 \\ 1.0 \end{pmatrix},$$

$$H_2G_1\mathbf{x} = \begin{pmatrix} 4.0 \\ 2.0 \end{pmatrix}, \quad G_2G_1\mathbf{x} = \begin{pmatrix} 3.0 \\ 0.0 \end{pmatrix}.$$

Thus we have the vector

$$(8.0 \;\; 7.0 \;\; 5.0 \;\; 1.0 \;\; 4.0 \;\; 2.0 \;\; 3.0 \;\; 0.0)^T.$$

Finally we have

$$H_3H_2H_1\mathbf{x} = 7.5, \quad G_3H_2H_1\mathbf{x} = 0.5, \quad H_3G_2H_1\mathbf{x} = 3.0, \quad G_3G_2H_1\mathbf{x} = 2.0$$

$$H_3H_2G_1\mathbf{x} = 3.0, \quad G_3H_2G_1\mathbf{x} = 1.0, \quad H_3G_2G_1\mathbf{x} = 1.5, \quad G_3G_2G_1\mathbf{x} = 1.5.$$

Thus we obtain the vector

$$(7.5 \;\; 0.5 \;\; 3.0 \;\; 2.0 \;\; 3.0 \;\; 1.0 \;\; 1.5 \;\; 1.5).$$

(ii) We find

$$H_1H_1^T = \frac{1}{2}I_4, \quad G_1G_1^T = \frac{1}{2}I_4, \quad H_1G_1^T = 0_4$$

$$H_2H_2^T = \frac{1}{2}I_2, \quad G_2G_2^T = \frac{1}{2}I_2, \quad H_2G_2^T = 0_2$$

$$H_3H_3^T = \frac{1}{2}, \quad G_3G_3^T = \frac{1}{2}, \quad H_3G_3^T = 0$$

where 0_4 is the 4×4 zero matrix and 0_2 is the 2×2 zero matrix.
(iii) Let

$$\mathbf{w} = (7.5\ 0.5\ 3.0\ 2.0\ 3.0\ 1.0\ 1.5\ 1.5)^T$$

Consider the matrices

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then

$$X_1\mathbf{w} = \begin{pmatrix} 8 \\ 7 \\ 5 \\ 1 \end{pmatrix}, \qquad Y_1\mathbf{w} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

Now let

$$X_2 = Y_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Then

$$X_2(X_1\mathbf{w}) = \begin{pmatrix} 13 \\ 3 \\ 8 \\ 6 \end{pmatrix}, \qquad Y_2(Y_1\mathbf{w}) = \begin{pmatrix} 7 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Thus the original vector $\mathbf{x}$ is reconstructed from

$$\begin{pmatrix} X_2(X_1\mathbf{w}) \\ Y_2(Y_1\mathbf{w}) \end{pmatrix}.$$

The odd entries come from $X_2(X_1\mathbf{w}) + Y_2(Y_1\mathbf{w})$ and the even ones from $X_2(X_1\mathbf{w}) - Y_2(Y_1\mathbf{w})$.

Problem 3. Consider the 8×8 *Hadamard matrix*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

(i) Do the 8 column vectors in the matrix H form a basis in $\mathbf{R}^8$? Prove or disprove.
(ii) Calculate HH^T, where ^{T} denotes transpose. Compare the results from (i) and (ii) and discuss.

Solution 3. (i) Calculating the scalar product of the column vectors (and obviously also the row vectors) we find that they are pairwise orthogonal to each other. Since they all nonzero vectors we have a basis in $\mathbf{R}^8$, which is not normalized.
(ii) We find

$$HH^T = 8I_8 .$$

Thus the matrix H is invertible and the column or row vectors must form a basis. Since the right-hand side is $8I_8$ the basis must be orthogonal.

Problem 4. Show that any 2×2 complex matrix has a unique representation of the form

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3$$

for some $a_0, a_1, a_2, a_3 \in \mathbf{C}$, where I_2 is the 2×2 identity matrix and σ_1, σ_2, σ_3 are the *Pauli spin matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution 4. Since

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix}$$

we obtain

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$. Thus

$$a_0 = \frac{\alpha + \delta}{2}, \quad a_1 = \frac{\beta + \gamma}{2i}, \quad a_2 = \frac{\beta - \gamma}{2}, \quad a_3 = \frac{\alpha - \delta}{2i}.$$

Problem 5. Let A, B be $n \times n$ matrices such that $ABAB = 0_n$. Can we conclude that $BABA = 0_n$?

Solution 5. There are no 1×1 or 2×2 counterexamples. For 2×2 matrices $ABAB = 0_n$ implies $B(ABAB)A = 0_n$. Therefore the matrix BA is *nilpotent*, i.e. $(BA)^3 = 0_n$. If a 2×2 matrix C is nilpotent, its characteristic polynomial is λ^2 and therefore $C^2 = 0_n$ by the *Cayley-Hamilton theorem*. Thus $BABA = 0_n$. For $n = 3$ we find the counterexample

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$BABA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 6. A square matrix A over $\mathbf{C}$ is called *skew-hermitian* if $A = -A^*$. Show that such a matrix is *normal*, i.e., we have $AA^* = A^*A$.

Solution 6. We have

$$AA^* = -A^*A^* = (-A^*)(-A) = A^*A.$$

Problem 7. Let A be an $n \times n$ skew-hermitian matrix over $\mathbf{C}$, i.e. $A^* = -A$. Let U be an $n \times n$ *unitary matrix*, i.e., $U^* = U^{-1}$. Show that $B := U^*AU$ is a skew-hermitian matrix.

Solution 7. We have $A^* = -A$. Thus from $B = U^*AU$ and $U^{**} = U$ it follows that

$$B^* = (U^*AU)^* = U^*A^*U = U^*(-A)U = -U^*AU = -B.$$

Problem 8. Let A, X, Y be $n \times n$ matrices. Assume that

$$XA = I_n, \qquad AY = I_n$$

where I_n is the $n \times n$ unit matrix. Show that $X = Y$.

Solution 8. We have

$$X = XI_n = X(AY) = (XA)Y = I_nY = Y.$$

Problem 9. Let A, B be $n \times n$ matrices. Assume that A is nonsingular, i.e. A^{-1} exists. Show that if $BA = 0_n$, then $B = 0_n$.

Solution 9. We have

$$B = BI_n = B(AA^{-1}) = (BA)A^{-1} = 0_n A^{-1} = 0_n .$$

Problem 10. Let A, B be $n \times n$ matrices and

$$A + B = I_n, \qquad AB = 0_n .$$

Show that $A^2 = A$ and $B^2 = B$.

Solution 10. Multiplying $A + B = I_n$ with A we obtain $A^2 + AB = A$ and therefore $A^2 = A$. Multiplying $A+B = I_n$ with B yields $AB+B^2 = B$ and therefore $B^2 = B$.

Problem 11. Consider the normalized vectors in $\mathbf{R}^2$

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

is normalized. A vector $\mathbf{x} \in \mathbf{R}^n$ is called *normalized* if $\|\mathbf{x}\| = 1$, where $\| \; \|$ denotes the Euclidean norm.

Solution 11. From the condition that the vector

$$\begin{pmatrix} \cos\theta_1 + \cos\theta_2 \\ \sin\theta_1 + \sin\theta_2 \end{pmatrix}$$

is normalized it follows that

$$(\sin\theta_1 + \sin\theta_2)^2 + (\cos\theta_1 + \cos\theta_2)^2 = 1 .$$

Thus we have

$$\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 = -\frac{1}{2} .$$

It follows that

$$\cos(\theta_1 - \theta_2) = -\frac{1}{2} .$$

Therefore, $\theta_1 - \theta_2 = 2\pi/3$ or $\theta_1 - \theta_2 = 4\pi/3$.

Problem 12. Let

$$A := \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

and $\theta \in \mathbf{R}$. Find the matrix A.

Solution 12. We find

$$A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ form a basis in $\mathbf{R}^2$. Equation (1) is called the *completeness relation.*

Problem 13. Find a 2×2 matrix A over $\mathbf{R}$ such that

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution 13. We find

$$a_{11} = a_{12} = a_{21} = \frac{1}{\sqrt{2}}, \qquad a_{22} = -\frac{1}{\sqrt{2}}.$$

Thus

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This matrix is a Hadamard matrix.

Problem 14. Consider the 2×2 matrix over the complex numbers

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j\sigma_j\right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ $(n_j \in \mathbf{R})$ is a unit vector, i.e., $n_1^2 + n_2^2 + n_3^2 = 1$. Here σ_1, σ_2, σ_3 are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 unit matrix.

(i) Describe the property of $\Pi(\mathbf{n})$, i.e., find $\Pi^*(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$, where tr denotes the trace. The trace is the sum of the diagonal elements of a square matrix.
(ii) Find the vector

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}.$$

Discuss.

Solution 14. (i) For the Pauli matrices we have $\sigma_1^* = \sigma_1$, $\sigma_2^* = \sigma_2$, $\sigma_3^* = \sigma_3$. Thus $\Pi(\mathbf{n}) = \Pi^*(\mathbf{n})$. Since

$$\mathrm{tr}\sigma_1 = \mathrm{tr}\sigma_2 = \mathrm{tr}\sigma_3 = 0$$

and the trace operation is linear, we obtain $\mathrm{tr}(\Pi(\mathbf{n})) = 1$. Since

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$$

and

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0_2, \quad \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0_2, \quad \sigma_3\sigma_1 + \sigma_1\sigma_3 = 0_2$$

the expression

$$\Pi^2(\mathbf{n}) = \frac{1}{4}\left(I_2 + \sum_{j=1}^{3} n_j\sigma_j\right)^2 = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3}\sum_{k=1}^{3} n_j n_k \sigma_j \sigma_k$$

simplifies to

$$\Pi^2(\mathbf{n}) = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3} n_j^2 I_2 .$$

Using $n_1^2 + n_2^2 + n_3^2 = 1$ we obtain $\Pi^2(\mathbf{n}) = \Pi(\mathbf{n})$.
(ii) We find

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+n_3)e^{i\phi}\cos\theta + (n_1 - in_2)\sin\theta \\ (n_1 + in_2)e^{i\phi}\cos\theta + (1-n_3)\sin\theta \end{pmatrix}.$$

Problem 15. Let

$$\mathbf{x} = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\phi, \theta \in \mathbf{R}$.
(i) Find the matrix $\rho := \mathbf{x}\mathbf{x}^*$.

(ii) Find $\mathrm{tr}\rho$.
(iii) Find ρ^2.

Solution 15. (i) Since $\mathbf{x}^* = (e^{-i\phi}\cos\theta\ ,\ \sin\theta)$ we obtain the 2×2 matrix

$$\rho = \mathbf{x}\mathbf{x}^* = \begin{pmatrix} \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

(ii) Since $\cos^2\theta + \sin^2\theta = 1$ we obtain from (i) that $\mathrm{tr}\rho = 1$.
(iii) We have

$$\rho^2 = (\mathbf{x}\mathbf{x}^*)(\mathbf{x}\mathbf{x}^*) = \mathbf{x}(\mathbf{x}^*\mathbf{x})\mathbf{x}^* = \mathbf{x}\mathbf{x}^* = \rho$$

since $\mathbf{x}^*\mathbf{x} = 1$.

Problem 16. Consider the vector space $\mathbf{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{j=1}^{p} \mathbf{x}_j \mathbf{x}_j^T$$

and find the eigenvalues and eigenvectors of this matrix.

Solution 16. The number of vectors p cannot exceed 4 since that would imply $\dim(\mathbf{R}^4) > 4$. A solution is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Thus

$$\sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalue is 4 with multiplicity 4. The eigenvectors are all $\mathbf{x} \in \mathbf{R}^4$ with $\mathbf{x} \neq \mathbf{0}$. Another solution is given by

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 17. Let

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

(i) Let X be an $m \times n$ matrix. The *column rank* of X is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of X are equal (called the *rank* of X). Find the rank of A and denote it by k.
(ii) Locate a $k \times k$ submatrix of A having rank k.
(iii) Find 3×3 permutation matrices P and Q such that in the matrix PAQ the submatrix from (ii) is in the upper left portion of A.

Solution 17. (i) The vectors in the first two columns are linearly dependent. Thus the rank of A is 2.
(ii) A 2×2 submatrix having rank 2 is

$$B = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}.$$

(iii) Let

$$P = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$PAQ = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 6 & -2 \\ 2 & -2 & 2 \end{pmatrix}.$$

Problem 18. Find 2×2 matrices A, B such that $AB = 0_n$ and $BA \neq 0_n$.

Solution 18. An example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 19. Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Then the *direct sum* of A and B, denoted by $A \oplus B$, is the $(m+p) \times (n+q)$ matrix defined by

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let A_1, A_2 be $m \times m$ matrices and B_1, B_2 be $n \times n$ matrices. Calculate

$$(A_1 \oplus B_1)(A_2 \oplus B_2).$$

Solution 19. We find

$$(A_1 \oplus B_1)(A_2 \oplus B_2) = (A_1 A_2) \oplus (B_1 B_2).$$

Problem 20. Let A be an $n \times n$ matrix over $\mathbf{R}$. Find all matrices that satisfy the equation

$$A^T A = 0_n.$$

Solution 20. From $A^T A = 0_n$ we find

$$\sum_{i=1}^{n} a_{ij} a_{ij} = 0$$

where $j = 1, 2, \ldots, n$. Thus A must be the zero matrix.

Problem 21. Let π be a permutation on $\{1, 2, \ldots, n\}$. The matrix P_π for which $p_{i*} = e_{\pi(i)*}$ is called the *permutation matrix* associated with π, where p_{i*} is the ith row of P_π and $e_{ij} = 1$ if $i = j$ and 0 otherwise. Let $\pi = (3\ 2\ 4\ 1)$. Find P_π.

Solution 21. P_π is obtained from the $n \times n$ identity matrix I_n by applying π to the rows of I_n. We find

$$P_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} p_{1*} &= e_{3*} = e_{\pi(1)*} \\ p_{2*} &= e_{2*} = e_{\pi(2)*} \\ p_{3*} &= e_{4*} = e_{\pi(3)*} \\ p_{4*} &= e_{1*} = e_{\pi(4)*}. \end{aligned}$$

Problem 22. A matrix A for which $A^p = 0_n$, where p is a positive integer, is called *nilpotent*. If p is the least positive integer for which $A^p = 0_n$ then A is said to be nilpotent of index p. Find all 2×2 matrices over the real numbers which are nilpotent with $p = 2$, i.e. $A^2 = 0_2$.

Solution 22. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = 0_2$ we obtain the four equations

$$a_{11}^2 + a_{12}a_{21} = 0, \quad a_{12}(a_{11}+a_{22}) = 0, \quad a_{21}(a_{11}+a_{22}) = 0, \quad a_{12}a_{21} + a_{22}^2 = 0.$$

Thus we have to consider the cases $a_{11} + a_{22} \neq 0$ and $a_{11} + a_{22} = 0$. If $a_{11} + a_{22} \neq 0$, then $a_{12} = a_{21} = 0$ and therefore $a_{11} = a_{22} = 0$. Thus we have the 2×2 zero matrix. If $a_{11} + a_{22} = 0$ we have $a_{11} = -a_{22}$ and $a_{11} \neq 0$, otherwise we would find the zero matrix again. Thus $a_{12}a_{21} = -a_{11}^2 = -a_{22}^2$ and for this case we find the solution

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{11}^2/a_{12} & -a_{11} \end{pmatrix}.$$

Problem 23. A square matrix is called *idempotent* if $A^2 = A$. Find all 2×2 matrices over the real numbers which are idempotent and $a_{ij} \neq 0$ for $i, j = 1, 2$.

Solution 23. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = A$ we obtain

$$a_{11}^2 + a_{12}a_{21} = a_{11}, \quad a_{12}(a_{11} + a_{22}) = a_{12},$$

$$a_{21}(a_{11} + a_{22}) = a_{21}, \quad a_{12}a_{21} + a_{22}^2 = a_{22}.$$

Since $a_{ij} \neq 0$ we obtain $a_{11} + a_{22} = 1$ and

$$a_{21} = \frac{1}{a_{12}}(a_{11} - a_{11}^2).$$

Thus the matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ (a_{11} - a_{11}^2)/a_{12} & 1 - a_{11} \end{pmatrix}$$

with a_{11} and a_{12} arbitrary and nonzero.

Problem 24. A square matrix A such that $A^2 = I_n$ is called *involutory*. Find all 2×2 matrices over the real numbers which are involutory. Assume that $a_{ij} \neq 0$ for $i, j = 1, 2$.

Solution 24. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = I_2$ we obtain

$$a_{11}^2 + a_{12}a_{21} = 1, \quad a_{12}(a_{11}+a_{22}) = 0, \quad a_{21}(a_{11}+a_{22}) = 0, \quad a_{12}a_{21}+a_{22}^2 = 1.$$

Since $a_{ij} \neq 0$ we have $a_{11} + a_{22} = 0$ and

$$a_{21} = (1 - a_{11}^2)/a_{12}.$$

Then the matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ (1 - a_{11}^2)/a_{12} & -a_{11} \end{pmatrix}.$$

Problem 25. Show that an $n \times n$ matrix A is involutary if and only if $(I_n - A)(I_n + A) = 0_n$.

Solution 25. Suppose that $(I_n - A)(I_n + A) = 0_n$. Then

$$(I_n - A)(I_n + A) = I_n - A^2 = 0_n.$$

Thus $A^2 = I_n$ and A is involutary. Suppose that A is involutary. Then $A^2 = I_n$ and

$$0_n = I_n - A^2 = (I_n - A)(I_n + A).$$

Problem 26. Let A be an $n \times n$ symmetric matrix over $\mathbf{R}$. Let P be an arbitrary $n \times n$ matrix over $\mathbf{R}$. Show that $P^T AP$ is symmetric.

Solution 26. Using that $A^T = A$ and $(P^T)^T = P$ we have

$$(P^T AP)^T = P^T A^T (P^T)^T = P^T AP.$$

Thus $P^T AP$ is symmetric.

Problem 27. Let A be an $n \times n$ skew-symmetric matrix over $\mathbf{R}$, i.e. $A^T = -A$. Let P be an arbitrary $n \times n$ matrix over $\mathbf{R}$. Show that $P^T AP$ is skew-symmetric.

Solution 27. Using $A^T = -A$ and $(P^T)^T = P$ we have

$$(P^T AP)^T = P^T A^T (P^T)^T = -P^T AP.$$

Thus $P^T AP$ is skew-symmetric.

Problem 28. Let A be an $m \times n$ matrix. The *column rank* of A is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of A are equal (called the rank of A). Find the rank of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Solution 28. The *elementary transformations* do not change the rank of a matrix. We subtract the third column from the fourth column, the second column from the third column and the first column from the second column, i.e.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 5 & 6 & 7 & 1 \\ 9 & 10 & 11 & 1 \\ 13 & 14 & 15 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 5 & 6 & 1 & 1 \\ 9 & 10 & 1 & 1 \\ 13 & 14 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 9 & 1 & 1 & 1 \\ 13 & 1 & 1 & 1 \end{pmatrix}.$$

From the last matrix we see (three columns are the same) that the rank of A is 2. It follows that two eigenvalues must be 0.

Problem 29. Let A be an invertible $n \times n$ matrix over $\mathbf{C}$ and B be an $n \times n$ matrix over $\mathbf{C}$. We define the $n \times n$ matrix

$$D := A^{-1}BA.$$

Calculate D^n, where $n = 2, 3, \ldots$.

Solution 29. Since $AA^{-1} = I_n$ we obtain

$$D^n = A^{-1}B^n A.$$

Problem 30. A *Cartan matrix* A is a square matrix whose elements a_{ij} satisfy the following conditions:

1. a_{ij} is an integer, one of $\{-3, -2, -1, 0, 2\}$
2. $a_{jj} = 2$ for all diagonal elements of A

3. $a_{ij} \leq 0$ off of the diagonal
4. $a_{ij} = 0$ iff $a_{ji} = 0$
5. There exists an invertible diagonal matrix D such that DAD^{-1} gives a symmetric and positive definite quadratic form.

Give a 2×2 non-diagonal Cartan matrix.

Solution 30. An example is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The first four conditions are obvious for the matrix A. The last condition can be seen from

$$\mathbf{x}^T A \mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2(x_1^2 - x_1 x_2 + x_2^2) \geq 0$$

for $\mathbf{x} \neq \mathbf{0}$. That the symmetric matrix A is positive definite can also be seen from the eigenvalues of A which are 3 and 1.

Problem 31. Let A, B, C, D be $n \times n$ matrices over $\mathbf{R}$. Assume that AB^T and CD^T are symmetric and $AD^T - BC^T = I_n$, where ^{T} denotes transpose. Show that

$$A^T D - C^T B = I_n.$$

Solution 31. From the assumptions we have

$$\begin{aligned} AB^T &= (AB^T)^T = BA^T \\ CD^T &= (CD^T)^T = DC^T \\ AD^T - BC^T &= I_n. \end{aligned}$$

Taking the transpose of the third equation we have

$$DA^T - CB^T = I_n.$$

These four equations can be written in the form of block matrices in the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

Thus the matrices are $(2n) \times (2n)$ marices. If X, Y are $m \times m$ matrices with $XY = I_m$, the identity matrix, then $Y = X^{-1}$ and $YX = I_m$ too. Applying this to our matrix equation with $m = 2n$ we obtain

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

Equating the lower right blocks shows that $-C^TB + A^TD = I_n$.

Problem 32. Let n be a positive integer. Let A_n be the $(2n+1)\times(2n+1)$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Give A_1 and A_2. Find the rank of A_n.

Solution 32. We have

$$A_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

We use induction on n to prove that $\text{rank}(A_n) = 2n$. The rank of A_1 is 2 since the first vector in the matrix A_1 is linear combination of the second and third vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and the second and third vectors are linearly independent. Suppose $n \geq 2$ and that $\text{rank}(A_{n-1}) = 2(n-1)$ is known. Adding multiples of the first two rows of A_n to the other rows transforms A_n to a matrix of the form

$$\begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & * \\ \mathbf{0} & -A_{n-1} \end{pmatrix}$$

in which $\mathbf{0}$ and $*$ represent blocks of size $(2n-1)\times 2$ and $2\times(2n-1)$, repectively. Thus $\text{rank}(A_n) = 2 + \text{rank}(A_{n-1}) = 2 + 2(n-1) = 2n$.

Problem 33. A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is called a *probability vector* if the components are nonnegative and their sum is 1. Is the vector

$$\mathbf{u} = (1/2,\ 0,\ 1/4,\ 1/4)$$

a probability vector? Can the vector

$$\mathbf{v} = (2,\ 3,\ 5,\ 1,\ 0)$$

be "normalized" so that we obtain a probability vector?

Solution 33. Since all the components are nonnegative and the sum of entries is 1 we find that $\mathbf{u}$ is a probability vector. All the entries in $\mathbf{v}$ are nonnegative but the sum is 11. Thus we can construct the probability vector

$$\tilde{\mathbf{v}} = \frac{1}{11}(2,\, 3,\, 5,\, 1,\, 0)\,.$$

Problem 34. An $n \times n$ matrix $P = (p_{ij})$ is called a *stochastic matrix* if each of its rows is a probability vector, i.e., if each entry of P is nonnegative and the sum of the entries in each row is 1. Let A and B be two stochastic $n \times n$ matrices. Is the matrix product AB also a stochastic matrix?

Solution 34. From matrix multiplication we have for the (ij) entry of the product

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}\,.$$

It follows that

$$\sum_{j=1}^{n} (AB)_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{n} b_{kj} = 1$$

since

$$\sum_{k=1}^{n} a_{ik} = 1, \qquad \sum_{j=1}^{n} b_{kj} = 1\,.$$

Problem 35. The *numerical range*, also known as the *field of values*, of an $n \times n$ matrix A over the complex numbers, is defined as

$$F(A) := \{\, \mathbf{z}^* A \mathbf{z} \,:\, \|\mathbf{z}\| = 1,\ \mathbf{z} \in \mathbf{C}^n \,\}\,.$$

Find the numerical range for the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Find the numerical range for the 2×2 matrix

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *Toeplitz-Hausdorff convexity theorem* tells us that the numerical range of a square matrix is a convex compact subset of the complex plane.

Solution 35. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$\mathbf{z} = \begin{pmatrix} e^{i\phi}\cos\theta \\ e^{i\chi}\sin\theta \end{pmatrix}, \qquad \phi, \chi, \theta \in \mathbf{R}.$$

Therefore $\mathbf{z}$ is an arbitrary complex number of length 1, i.e., $\|\mathbf{z}\| = 1$. Then

$$\begin{aligned} \mathbf{z}^* A\mathbf{z} &= \begin{pmatrix} e^{-i\phi}\cos\theta & e^{-i\chi}\sin\theta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e^{i\phi}\cos\theta \\ e^{i\chi}\sin\theta \end{pmatrix} \\ &= a_{11}\cos^2\theta + (a_{12}e^{i(\chi-\phi)} + a_{21}e^{i(\phi-\chi)})\sin\theta\cos\theta + a_{22}\sin^2\theta. \end{aligned}$$

Thus for the matrix B we have $\mathbf{z}^T B\mathbf{z} = \cos^2\theta$, where $\cos^2\theta \in [0,1]$ for all $\theta \in \mathbf{R}$. For the matrix C we have

$$\mathbf{z}^* C\mathbf{z} = e^{i(\phi-\chi)}\sin\theta\cos\theta + \sin^2\theta.$$

Thus the numerical range $F(C)$ is the closed elliptical disc in the complex plane with foci at $(0,0)$ and $(1,0)$, minor axis 1, and major axis $\sqrt{2}$.

Problem 36. Let A be an $n \times n$ matrix over $\mathbf{C}$. The *field of values* of A is defined as the set

$$F(A) := \{\, \mathbf{z}^* A\mathbf{z} \,:\, \mathbf{z} \in \mathbf{C}^n \,,\; \mathbf{z}^*\mathbf{z} = 1 \,\}.$$

Let $\alpha \in \mathbf{R}$ and

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$$

(i) Show that the set $F(A)$ lies on the real axis.
(ii) Show that

$$|\mathbf{z}^* A\mathbf{z}| \le \alpha + 16.$$

Solution 36. (i) Since

$$\mathbf{z}^* = (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6}, \overline{z_7}, \overline{z_8}, \overline{z_9})$$

with $\mathbf{z}^*\mathbf{z} = 1$ and applying matrix multiplication we obtain

$$\mathbf{z}^* A\mathbf{z} = \alpha + \sum_{j=1}^{8} (\overline{z_j} z_{j+1} + \overline{z_{j+1}} z_j) .$$

Let $z_j = r_j e^{i\theta_j}$. Then $\overline{z_j} = r_j e^{-i\theta_j}$. Let $j \neq k$. Then

$$\overline{z_j} z_k + z_j \overline{z_k} = 2r_j r_k \cos(\theta_j - \theta_k)$$

with $0 \leq r_j \leq 1$. Thus we have

$$\mathbf{z}^* A\mathbf{z} = \alpha + 2\sum_{j=1}^{8} r_j r_{j+1} \cos(\theta_{j+1} - \theta_j) .$$

Thus the set $F(A)$ lies on the real axis.
(ii) Since $0 \leq r_j \leq 1$ and $|\cos\theta| \leq 1$ we obtain

$$|\mathbf{z}^* A\mathbf{z}| \leq \alpha + 16 .$$

Problem 37. Let A be an $n \times n$ matrix over $\mathbf{C}$ and $F(A)$ the field of values. Let U be an $n \times n$ unitary matrix.
(i) Show that

$$F(U^* AU) = F(A) .$$

(ii) Apply the theorem to the two matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are unitarily equivalent.

Solution 37. (i) Since a unitary matrix leaves invariant the surface of the Euclidean unit ball, the complex numbers that comprise the sets $F(U^* AU)$ and $F(A)$ are the same. If $\mathbf{z} \in \mathbf{C}^n$ and $\mathbf{z}^*\mathbf{z} = 1$, we have

$$\mathbf{z}^*(U^* AU)\mathbf{z} = \mathbf{w}^* A\mathbf{w} \in F(A)$$

where $\mathbf{w} = U\mathbf{z}$, so that $\mathbf{w}^*\mathbf{w} = \mathbf{z}^* U^* U\mathbf{z} = \mathbf{z}^*\mathbf{z} = 1$. Thus $F(U^* AU) \subset F(A)$. The reverse containment is obtained similarly.
(ii) For A_1 we have

$$\mathbf{z}^* A_1\mathbf{z} = \overline{z_1} z_2 + \overline{z_2} z_1 = 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

with the constraints $0 \leq r_1, r_2 \leq 1$ and $r_1^2 + r_2^2 = 1$. For A_2 we find

$$\mathbf{z}^* A_2\mathbf{z} = \overline{z_1} z_1 - \overline{z_2} z_2 = r_1^2 - r_2^2$$

with the constraints $0 \leq r_1, r_2 \leq 1$ and $r_1^2 + r_2^2 = 1$. Both define the same set, namely the interval $[-1, 1]$.

Problem 38. Can one find a unitary matrix U such that

$$U^* \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} U = \begin{pmatrix} 0 & ce^{i\theta} \\ de^{-i\theta} & 0 \end{pmatrix}$$

where $c, d \in \mathbf{C}$ and $\theta \in \mathbf{R}$?

Solution 38. We find

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Problem 39. An $n^2 \times n$ matrix J is called a *selection matrix* such that J^T is the $n \times n^2$ matrix

$$[E_{11}\ E_{22}\ \ldots\ E_{nn}]$$

where E_{ii} is the $n \times n$ matrix of zeros except for a 1 in the (i,i)th position.
(i) Find J for $n = 2$ and calculate $J^T J$.
(ii) Calculate $J^T J$ for arbitrary n.

Solution 39. (i) We have

$$J^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $J^T J = I_2$.
(ii) For the general case we find $J^T J = I_n$.

Problem 40. Let A and B be $m \times n$ matrices. The *Hadamard product* $A \circ B$ is defined as the $m \times n$ matrix

$$A \bullet B := (a_{ij} b_{ij}).$$

(i) Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 7 & 1 \end{pmatrix}.$$

Calculate $A \bullet B$.
(ii) Let C, D be $m \times n$ matrices. Show that

$$\text{rank}(A \bullet B) \leq (\text{rank}A)(\text{rank}B)\,.$$

Solution 40. (i) Entrywise multiplication of the two matrices yields

$$A \bullet B = \begin{pmatrix} 0 & 4 \\ 7 & 0 \end{pmatrix}.$$

(ii) Any matrix of rank r can be written as a sum of r rank one matrices, each of which is an outer product of two vectors (column vector times row vector). Thus, if rank$A = r_1$ and rank$B = r_2$, we have

$$A = \sum_{i=1}^{r_1} \mathbf{x}_i \mathbf{y}_i^*, \qquad B = \sum_{j=1}^{r_2} \mathbf{u}_j \mathbf{v}_j^*$$

where $\mathbf{x}_i, \mathbf{u}_j \in \mathbf{C}^m$, $\mathbf{y}_i, \mathbf{v}_j \in \mathbf{C}^n$, $i = 1, \ldots, r_1$ and $j = 1, \ldots, r_2$. Then

$$A \bullet B = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (\mathbf{x}_i \bullet \mathbf{u}_j)(\mathbf{y}_i \bullet \mathbf{v}_j)^*\,.$$

This shows that $A \bullet B$ is a sum of at most $r_1 r_2$ rank one matrices. Thus rank$(A \bullet B) \leq r_1 r_2 = (\text{rank}A)(\text{rank}B)$.

Problem 41. Consider a symmetric matrix A over $\mathbf{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

and the orthonormal basis (so-called *Bell basis*)

$$\mathbf{x}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{y}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{y}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The Bell basis forms an orthonormal basis in $\mathbf{R}^4$. Let $\tilde{A}$ denote the matrix A in the Bell basis. What is the condition on the entries a_{ij} such that the matrix A is diagonal in the Bell basis?

Solution 41. Obviously we have

$$\widetilde{a}_{ij} = \widetilde{a}_{ji},$$

i.e., the matrix $\widetilde{A}$ is also symmetric. Straightforward calculation yields

$$\begin{aligned}
\widetilde{a}_{11} &= (\mathbf{x}^+)^T A\mathbf{x}^+ = \frac{1}{2}(a_{11} + 2a_{14} + a_{44}) \\
\widetilde{a}_{12} &= (\mathbf{x}^+)^T A\mathbf{x}^- = \frac{1}{2}(a_{11} - a_{44}) \\
\widetilde{a}_{13} &= (\mathbf{x}^+)^T A\mathbf{y}^+ = \frac{1}{2}(a_{12} + a_{13} + a_{24} + a_{34}) \\
\widetilde{a}_{14} &= (\mathbf{x}^+)^T A\mathbf{y}^- = \frac{1}{2}(a_{12} - a_{13} + a_{24} - a_{34}) \\
\widetilde{a}_{22} &= (\mathbf{x}^-)^T A\mathbf{x}^- = \frac{1}{2}(a_{11} - 2a_{14} + a_{44}) \\
\widetilde{a}_{23} &= (\mathbf{x}^-)^T A\mathbf{y}^+ = \frac{1}{2}(a_{12} + a_{13} - a_{24} - a_{34}) \\
\widetilde{a}_{24} &= (\mathbf{x}^-)^T A\mathbf{y}^- = \frac{1}{2}(a_{12} - a_{13} - a_{24} + a_{34}) \\
\widetilde{a}_{33} &= (\mathbf{y}^+)^T A\mathbf{y}^+ = \frac{1}{2}(a_{22} + 2a_{23} + a_{33}) \\
\widetilde{a}_{34} &= (\mathbf{y}^+)^T A\mathbf{y}^- = \frac{1}{2}(a_{22} - a_{33}) \\
\widetilde{a}_{44} &= (\mathbf{y}^-)^T A\mathbf{y}^- = \frac{1}{2}(a_{22} - 2a_{23} + a_{33}).
\end{aligned}$$

The condition that the matrix $\widetilde{A}$ should be diagonal leads to

$$a_{11} - a_{44} = 0, \qquad a_{22} - a_{33} = 0$$

and

$$a_{12} = a_{13} = a_{24} = a_{34} = 0$$

with the entries a_{14} and a_{23} arbitrary. Thus the matrix A has the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{22} & 0 \\ a_{14} & 0 & 0 & a_{11} \end{pmatrix}.$$

Problem 42. Let A be an $m \times n$ matrix over $\mathbf{C}$. The *Moore-Penrose pseudoinverse matrix* A^+ is the unique $n \times m$ matrix which satisfies

$$\begin{aligned}
AA^+A &= A \\
A^+AA^+ &= A^+ \\
(AA^+)^* &= AA^+ \\
(A^+A)^* &= A^+A.
\end{aligned}$$

We also have that

$$\mathbf{x} = A^+\mathbf{b} \tag{1}$$

is the shortest length least square solution to the problem

$$A\mathbf{x} = \mathbf{b}\,. \tag{2}$$

(i) Show that if $(A^*A)^{-1}$ exists, then $A^+ = (A^*A)^{-1}A^*$.
(ii) Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix}.$$

Find the Moore-Penrose matrix inverse A^+ of A.

Solution 42. (i) Suppose that $(A^*A)^{-1}$ exists we have

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^*A\mathbf{x} &= A^*\mathbf{b} \\ \mathbf{x} &= (A^*A)^{-1}A^*\mathbf{b}\,. \end{aligned}$$

Using (1) we obtain

$$A^+ = (A^*A)^{-1}A^*\,.$$

(ii) We have

$$A^*A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 26 \\ 26 & 50 \end{pmatrix}.$$

Since $\det(A^*A) \neq 0$ the inverse of A^*A exists and is given by

$$(A^*A)^{-1} = \frac{1}{12}\begin{pmatrix} 25 & -13 \\ -13 & 7 \end{pmatrix}.$$

Thus

$$\begin{aligned} A^+ = (A^*A)^{-1}A^* &= \frac{1}{12}\begin{pmatrix} 25 & -13 \\ -13 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \\ &= \frac{1}{12}\begin{pmatrix} -14 & -2 & 10 \\ 8 & 2 & -4 \end{pmatrix}. \end{aligned}$$

Problem 43. Given a signal as the column vector

$$\mathbf{x} = (3.0\ \ 0.5\ \ 2.0\ \ 7.0)^T\,.$$

The *pyramid algorithm* (for *Haar wavelets*) is as follows: The first two entries $(3.0\ 0.5)^T$ in the signal give an average of $(3.0 + 0.5)/2 = 1.75$ and a difference average of $(3.0 - 0.5)/2 = 1.25$. The second two entries $(2.0\ 7.0)$ give an average of $(2.0 + 7.0)/2 = 4.5$ and a difference average of $(2.0 - 7.0)/2 = -2.5$. Thus we end up with a vector

$$(1.75\ 1.25\ 4.5\ -2.5)^T .$$

Now we take the average of 1.75 and 4.5 providing $(1.75 + 4.5)/2 = 3.125$ and the difference average $(1.75 - 4.5)/2 = -1.375$. Thus we end up with the vector

$$\mathbf{y} = (3.125\ -1.375\ 1.25\ -2.5)^T .$$

(i) Find a 4×4 matrix A such that

$$\mathbf{x} \equiv \begin{pmatrix} 3.0 \\ 0.5 \\ 2.0 \\ 7.0 \end{pmatrix} = A\mathbf{y} \equiv A \begin{pmatrix} 3.125 \\ -1.375 \\ 1.25 \\ -2.5 \end{pmatrix} .$$

(ii) Show that the inverse of A exists. Then find the inverse matrix of A.

Solution 43. (i) Since we can write

$$\begin{pmatrix} 3.0 \\ 0.5 \\ 2.0 \\ 7.0 \end{pmatrix} = 3.125 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1.375 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 1.25 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 2.5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

we obtain the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} .$$

(ii) All the column vectors in the matrix A are nonzero and all the pairwise scalar products are equal to 0. Thus the column vectors form a basis (not normalized) in $\mathbf{R}^n$. Thus the matrix is invertible. The inverse matrix is given by

$$A^{-1} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} .$$

Problem 44. A *Hadamard matrix* is an $n \times n$ matrix H with entries in $\{-1, +1\}$ such that any two distinct rows or columns of H have inner product 0. Construct a 4×4 Hadamard matrix starting from the column vector

$$\mathbf{x}_1 = (1 \;\; 1 \;\; 1 \;\; 1)^T .$$

Solution 44. The vector

$$\mathbf{x}_2 = (1 \;\; 1 \;\; -1 \;\; -1)^T$$

is perpendicular to the vector $\mathbf{x}_1$. Next the vector

$$\mathbf{x}_3 = (-1 \;\; 1 \;\; 1 \;\; -1)^T$$

is perpendicular to $\mathbf{x}_1$ and $\mathbf{x}_2$. Finally the vector

$$\mathbf{x}_4 = (1 \;\; -1 \;\; 1 \;\; -1)$$

is perpendicular to $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Thus we obtain the 4×4 Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} .$$

Problem 45. A *binary Hadamard matrix* is an $n \times n$ matrix M (where n is even) with entries in $\{0, 1\}$ such that any two distinct rows or columns of M have *Hamming distance* $n/2$. The Hamming distance between two vectors is the number of entries at which they differ. Find a 4×4 binary Hadamard matrix.

Solution 45. Hadamard matrices (see previous problem) are in *bijection* with binary Hadamard matrices with the mapping $1 \to 1$ and $-1 \to 0$. Thus using the result from the previous problem we obtain

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Problem 46. Let $\mathbf{x}$ be a normalized column vector in $\mathbf{R}^n$, i.e. $\mathbf{x}^T\mathbf{x} = 1$. A matrix T is called a *Householder matrix* if

$$T := I_n - 2\mathbf{x}\mathbf{x}^T .$$

Calculate T^2.

Solution 46. Since the matrix product is associative we have

$$\begin{aligned}T^2 &= (I_n - 2\mathbf{x}\mathbf{x}^T)(I_n - 2\mathbf{x}\mathbf{x}^T)\\ &= I_n - 2\mathbf{x}\mathbf{x}^T - 2\mathbf{x}\mathbf{x}^T + 4\mathbf{x}(\mathbf{x}^T\mathbf{x})\mathbf{x}^T\\ &= I_n - 4\mathbf{x}\mathbf{x}^T + 4\mathbf{x}\mathbf{x}^T\\ &= I_n .\end{aligned}$$

Problem 47. An $n \times n$ matrix P is a *projection matrix* if

$$P^* = P, \qquad P^2 = P .$$

(i) Let P_1 and P_2 be projection matrices. Is $P_1 + P_2$ a projection matrix?
(ii) Let P_1 and P_2 be projection matrices. Is $P_1 P_2$ a projection matrix?
(iii) Let P be a projection matrix. Is $I_n - P$ a projection matrix? Calculate $P(I_n - P)$.
(iv) Is

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

a projection matrix?

Solution 47. (i) Obviously $(P_1 + P_2)^* = P_1^* + P_2^* = P_1 + P_2$. We have

$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_1P_2 + P_2P_1 + P_2 .$$

Thus $P_1 + P_2$ is a projection matrix only if $P_1P_2 = 0_n$, where we used that from $P_1P_2 = 0_n$ we can conclude that $P_2P_1 = 0_n$. From $P_1P_2 = 0_n$ it follows that

$$(P_1P_2)^* = P_2^*P_1^* = P_2P_1 = 0_n .$$

(ii) We have $(P_1P_2)^* = P_2^*P_1^* = P_2P_1$ and

$$(P_1P_2)^2 = P_1P_2P_1P_2 .$$

Thus we see that P_1P_2 is a projection matrix if and only if $P_1P_2 = P_2P_1$.
(iii) We have $(I_n - P)^* = I_n^* - P^* = I_n - P$ and

$$(I_n - P)^2 = I_n - P - P + P^2 = I_n - P .$$

Thus $I_n - P$ is a projection matrix. We have

$$P(I_n - P) = P - P^2 = P - P = 0_n .$$

(iv) We find $P^* = P$ and $P^2 = P$. Thus P is a projection matrix.

Problem 48. Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be vectors in $\mathbf{R}^3$. Let $\times$ denote the vector product.
(i) Show that we can find a 3×3 matrix $S(\mathbf{a})$ such that

$$\mathbf{a} \times \mathbf{b} = S(\mathbf{a})\mathbf{b}.$$

(ii) Express the *Jacobi identity*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}$$

using the matrices $S(\mathbf{a})$, $S(\mathbf{b})$ and $S(\mathbf{c})$.

Solution 48. (i) The *vector product* is defined as

$$\mathbf{a} \times \mathbf{b} := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Thus $S(\mathbf{a})$ is the skew-symmetric matrix

$$S(\mathbf{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

(ii) Using the result from (i) we obtain

$$\begin{aligned} \mathbf{a} \times (S(\mathbf{b})\mathbf{c}) + \mathbf{c} \times (S(\mathbf{a})\mathbf{b}) + \mathbf{b} \times (S(\mathbf{c})\mathbf{a}) &= \mathbf{0} \\ S(\mathbf{a})(S(\mathbf{b})\mathbf{c}) + S(\mathbf{c})(S(\mathbf{a})\mathbf{b}) + S(\mathbf{b})(S(\mathbf{c})\mathbf{a}) &= \mathbf{0} \\ (S(\mathbf{a})S(\mathbf{b}))\mathbf{c} + (S(\mathbf{c})S(\mathbf{a}))\mathbf{b} + (S(\mathbf{b})S(\mathbf{c}))\mathbf{a} &= \mathbf{0} \end{aligned}$$

where we used that matrix multiplication is associative.

Problem 49. Let s (*spin quantum number*)

$$s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \right\}.$$

Given a fixed s. The indices j, k run over $s, s-1, s-2, \ldots, -s+1, -s$. Consider the $(2s+1)$ unit vectors (standard basis)

$$\mathbf{e}_{s,s} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_{s,s-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \mathbf{e}_{s,-s} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Obviously the vectors have $(2s+1)$ components. The $(2s+1)\times(2s+1)$ matrices s_+ and s_- are defined as

$$\begin{aligned} s_+\mathbf{e}_{s,m} &:= \hbar\sqrt{(s-m)(s+m+1)}\,\mathbf{e}_{s,m+1}, \quad m = s-1, s-2, \ldots, -s \\ s_-\mathbf{e}_{s,m} &:= \hbar\sqrt{(s+m)(s-m+1)}\,\mathbf{e}_{s,m-1}, \quad m = s, s-1, \ldots, -s+1 \end{aligned}$$

and $s_+\mathbf{e}_{ss} = \mathbf{0}$, $s_-\mathbf{e}_{s-s} = \mathbf{0}$, where $\hbar$ is the Planck constant.
(i) Find the matrix representation of s_+ and s_-.
(ii) The $(2s+1)\times(2s+1)$ matrix s_z is defined as

$$s_z\mathbf{e}_{s,m} := m\hbar\mathbf{e}_{s,m}, \quad m = s, s-1, \ldots, -s\,.$$

Let

$$\mathbf{s} := (s_x, s_y, s_z)$$

where $s_+ = \frac{1}{2}(s_x + is_y)$ and $s_- = \frac{1}{2}(s_x - is_y)$. Find the $(2s+1)\times(2s+1)$ matrix

$$\mathbf{s}^2 := s_x^2 + s_y^2 + s_z^2\,.$$

(iii) Calculate the expectation values

$$\mathbf{e}_{s,s}^* s_+ \mathbf{e}_{s,s}, \quad \mathbf{e}_{s,s}^* s_- \mathbf{e}_{s,s}, \quad \mathbf{e}_{s,s}^* s_z \mathbf{e}_{s,s}\,.$$

Solution 49. (i) We have

$$\begin{aligned} (s_+)_{jk} = (s_-)_{kj} &= \hbar\sqrt{(s-k)(s+k+1)}\,\delta_{j,k+1} \\ &= \hbar\sqrt{(s+j)(s-j+1)}\,\delta_{j,k+1} \end{aligned}$$

$$\begin{aligned} (s_-)_{jk} = (s_+)_{kj} &= \hbar\sqrt{(s+k)(s-k+1)}\,\delta_{j,k-1} \\ &= \hbar\sqrt{(s-j)(s+j+1)}\,\delta_{j,k-1}\,. \end{aligned}$$

Therefore

$$s_+ = \hbar\begin{pmatrix} 0 & \sqrt{2s} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \sqrt{3(2s-2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \sqrt{2s} \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Thus $s_- = (s_+)^*$.
(ii) We have

$$\mathbf{s}^2 = \frac{1}{2}(s_+s_- + s_-s_+) + s_z^2\,.$$

Thus

$$(\mathbf{s})^2_{jk} = s(s+1)\hbar^2\delta_{jk}\,.$$

Therefore $\mathbf{s}^2$ is a diagonal matrix.
(iii) We find

$$\begin{aligned}\mathbf{e}^*_{ss}s_+\mathbf{e}_{ss} &= 0\\ \mathbf{e}^*_{ss}s_-\mathbf{e}_{ss} &= 0\\ \mathbf{e}^*_{ss}s_z\mathbf{e}_{ss} &= \hbar s\,.\end{aligned}$$

Problem 50. The *Fibonacci numbers* are defined by the recurrence relation (linear difference equation of second order with constant coefficients)

$$s_{n+2} = s_{n+1} + s_n$$

where $n = 0, 1, \ldots$ and $s_0 = 0$, $s_1 = 1$. Write this recurrence relation in matrix form. Find s_6, s_5, and s_4.

Solution 50. We have $s_2 = 1$. We can write

$$\begin{pmatrix} s_{n+1} & s_n \\ s_n & s_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n, \qquad n = 1, 2, \ldots\quad .$$

Thus

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^5 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}.$$

It follows that $s_6 = 8$, $s_5 = 5$ and $s_4 = 3$.

Problem 51. (i) Find four unit (column) vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ in $\mathbf{R}^3$ such that

$$\mathbf{x}_j^T\mathbf{x}_k = \frac{4}{3}\delta_{jk} - \frac{1}{3} = \begin{cases} 1 & \text{for } j = k \\ -1/3 & \text{for } j \neq k\,. \end{cases}$$

Give a geometric interpretation.
(ii) Calculate

$$\sum_{j=1}^{4} \mathbf{x}_j\,.$$

(iii) Calculate

$$\sum_{j=1}^{4} \mathbf{x}_j\mathbf{x}_j^T\,.$$

Solution 51. (i) The four vectors consist of the vectors pointing from the center of a cube to nonadjacent corners. Alternatively, one may picture these vectors as the normal vectors for the faces of the tetrahedron that is defined by the other four corners of the cube. Thus an example is

$$\mathbf{x}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

$$\mathbf{x}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{x}_4 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

(ii) We obviously find

$$\sum_{j=1}^{4} \mathbf{x}_j = \mathbf{0}$$

which states that the four vectors are linearly dependent.
(iii) We obtain

$$\sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T = \frac{4}{3} I_3$$

where I_3 is the 3×3 identity matrix.

Problem 52. Assume that

$$A = A_1 + iA_2$$

is a nonsingular $n \times n$ matrix, where A_1 and A_2 are real $n \times n$ matrices. Assume that A_1 is also nonsingular. Find the inverse of A using the inverse of A_1.

Solution 52. We have the identity

$$(A_1 + iA_2)(I_n - iA_1^{-1}A_2) \equiv A_1 + A_2 A_1^{-1} A_2 .$$

Thus we find the inverse

$$A^{-1} = (A_1 + A_2 A_1^{-1} A_2)^{-1} - iA_1^{-1} A_2 (A_1 + A_2 A_1^{-1} A_2)^{-1} .$$

Problem 53. The 8×8 matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

plays a role in the *discrete wavelet transform*.
(i) Show that the matrix is invertible without calculating the determinant.
(ii) Find the inverse.

Solution 53. (i) All the column vectors in the matrix H are nonzero. All the pairwise scalar products of the column vectors are 0. Thus the matrix has maximum rank, i.e. rank$H$ = 8 and the column vectors form a basis (not normalized) in $\mathbf{R}^n$.
(ii) The inverse matrix is given by

$$H^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\ 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

Problem 54. Let A and B be $n \times n$ matrices over $\mathbf{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2B = B^2A$. Is $A^2 + B^2$ invertible?

Solution 54. We have

$$(A^2 + B^2)(A - B) = A^3 - B^3 - A^2B + B^2A = 0_n .$$

Since $A \neq B$, we can conclude that $A^2 + B^2$ is not invertible.

Problem 55. Let A be a positive definite $n \times n$ matrix over $\mathbf{R}$. Let $\mathbf{x} \in \mathbf{R}$. Show that $A + \mathbf{x}\mathbf{x}^T$ is also positive definite.

Solution 55. For all vector $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y} \neq \mathbf{0}$, we have $\mathbf{y}^T A\mathbf{y} > 0$. We have

$$\mathbf{y}^T(\mathbf{x}\mathbf{x}^T)\mathbf{y} = (\mathbf{y}^T\mathbf{x})(\mathbf{x}^T\mathbf{y}) = \left(\sum_{j=1}^n y_j x_j\right)\left(\sum_{j=1}^n x_j y_j\right) = \left(\sum_{j=1}^n x_j y_j\right)^2 \geq 0$$

and therefore we have $\mathbf{y}^T(A + \mathbf{x}\mathbf{x}^T)\mathbf{y} > 0$ for all $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y} \neq \mathbf{0}$.

Problem 56. Let A, B be $n \times n$ matrices over $\mathbf{C}$. The matrix A is called *similar* to the matrix B if there is an $n \times n$ invertible matrix S such that

$$A = S^{-1}BS .$$

If A is similar to B, then B is also similar to A, since $B = SAS^{-1}$.
(i) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Are the matrices similar?
(ii) Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Are the matrices similar?

Solution 56. (i) From $A = S^{-1}BS$ we obtain $SA = BS$. Let

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Then from $SA = BS$ we obtain

$$\begin{pmatrix} s_{11} + 2s_{12} & s_{12} \\ s_{21} + 2s_{22} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

It follows that $s_{12} = 0$ and $s_{22} = 0$. Thus S is not invertible and therefore A and B are not similar.
(ii) The matrices C and D are similar. We find

$$s_{11} = s_{21}, \qquad s_{12} = -s_{22}.$$

Since S must be invertible, all four matrix elements are nonzero. For example, we can select

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Chapter 2

Linear Equations

Let A be an $m \times n$ matrix over a field $\mathcal{F}$. Let $b_1, \dots, b_m$ be elements of the field $\mathcal{F}$. The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is called a system of linear equations. We also write $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ and $\mathbf{b}$ are considered as column vectors. The system is said to be homogeneous if all the numbers $b_1, \dots, b_m$ are equal to 0. The number n is called the number of unknowns, and m is called the number of equations. The system of homogeneous equations also admits the trivial solution $x_1 = x_2 = \cdots = x_n = 0$.

A system of homogeneous equations of m linear equations in n unknowns with $n > m$ admits a non-trivial solution. An underdetermined linear system is either inconsistent or has infinitely many solutions.

An important special case is $m = n$. Then for the system of linear equations $A\mathbf{x} = \mathbf{b}$ we investigate the cases A^{-1} exists and A^{-1} does not exist. If A^{-1} exists we can write the solution as $\mathbf{x} = A^{-1}\mathbf{b}$.

If $m > n$, then we have an overdetermined system and it can happen that no solution exists. One solves these problems in the least-square sense.

Problem 1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Find the solutions of the system of linear equations

$$A\mathbf{x} = \mathbf{b}.$$

Solution 1. Since A is invertible we have the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

From the equations

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 5 \end{aligned}$$

we obtain by addition of the two equations $3x_1 = 6$ and thus $x_1 = 2$. It follows that $x_2 = -1$.

Problem 2. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ \alpha \end{pmatrix}$$

where $\alpha \in \mathbf{R}$. What is the condition on α so that there is a solution of the equation $A\mathbf{x} = \mathbf{b}$?

Solution 2. From $A\mathbf{x} = \mathbf{b}$ we obtain

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= \alpha. \end{aligned}$$

Multiplying the first equation by 2 and then subtracting from the second equation yields $6 = \alpha$. Thus if $\alpha \neq 6$ there is no solution. If $\alpha = 6$ the line $x_1 + x_2 = 3$ is the solution.

Problem 3. (i) Find all solutions of the system of linear equations

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \theta \in \mathbf{R}.$$

(ii) What type of equation is this?

Solution 3. (i) We obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ -\sin(\theta/2) \end{pmatrix}$$

where we used the identities

$$\sin\theta \equiv 2\sin(\theta/2)\cos(\theta/2), \qquad \cos\theta \equiv 2\cos^2(\theta/2) - 1 .$$

(ii) This is an eigenvalue equation with eigenvalue 1.

Problem 4. Let $A \in \mathbf{R}^{n\times n}$ and $\mathbf{x}, \mathbf{b} \in \mathbf{R}^n$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Show that it can be written as $\mathbf{x} = T\mathbf{x}$, i.e., find $T\mathbf{x}$.

Solution 4. Let $C = I_n - A$. Then we can write

$$\mathbf{x} = C\mathbf{x} + \mathbf{b} .$$

Thus $\mathbf{x} = T\mathbf{x}$ with $T\mathbf{x} := C\mathbf{x} + \mathbf{b}$.

Problem 5. If the system of linear equations $A\mathbf{x} = \mathbf{b}$ admits no solution we call the equations inconsistent. If there is a solution, the equations are called consistent. Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in n unknowns and suppose that the rank of A is m. Show that in this case $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution 5. Since $[A|\mathbf{b}]$ is an $m \times (n+1)$ matrix we have $m \geq \text{rank}[A|\mathbf{b}]$. We have $\text{rank}[A|\mathbf{b}] \geq \text{rank}A$ and by assumption $\text{rank}A = m$. Thus

$$m \geq \text{rank}[A|\mathbf{b}] \geq \text{rank}A = m .$$

Hence $\text{rank}[A|\mathbf{b}] = m$ and therefore the system of equations $A\mathbf{x} = \mathbf{b}$ are consistent.

Problem 6. Show that the *curve fitting problem*

j	0	1	2	3	4
t_j	−1.0	−0.5	0.0	0.5	1.0
y_j	1.0	0.5	0.0	0.5	2.0

by a quadratic polynomial of the form

$$p(t) = a_2 t^2 + a_1 t + a_0$$

leads to an overdetermined linear system.

Solution 6. From the interpolation conditions $p(t_j) = y_j$ with $j = 0, 1, \ldots, 4$ we obtain the overdetermined linear system

$$\begin{pmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} .$$

Problem 7. Consider the overdetermined linear system $A\mathbf{x} = \mathbf{b}$. Find an $\hat{\mathbf{x}}$ such that

$$\|A\hat{\mathbf{x}} - \mathbf{b}\|_2 = \min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2 \equiv \min_{\mathbf{x}} \|\mathbf{r}(\mathbf{x})\|_2$$

with the *residual vector* $\mathbf{r}(\mathbf{x}) := \mathbf{b} - A\mathbf{x}$ and $\|\,.\,\|_2$ denotes the Euclidean norm.

Solution 7. From

$$\|\mathbf{r}(\mathbf{x})\|_2^2 = \mathbf{r}^T\mathbf{r} = (\mathbf{b} - A\mathbf{x})^T(\mathbf{b} - A\mathbf{x}) = \mathbf{b}^T\mathbf{b} - 2\mathbf{x}^T A^T\mathbf{b} + \mathbf{x}^T A^T A\mathbf{x}$$

where we used that $\mathbf{x}^T A^T \mathbf{b} = \mathbf{b}^T A\mathbf{x}$, and the necessary condition

$$\nabla\|\mathbf{r}(\mathbf{x})\|_2^2|_{\mathbf{x}=\hat{\mathbf{x}}} = \mathbf{0}$$

we obtain

$$A^T A\hat{\mathbf{x}} - A^T\mathbf{b} = \mathbf{0} .$$

This system is called *normal equations.* We can also write this system as

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) \equiv A^T\mathbf{r}(\hat{\mathbf{x}}) = \mathbf{0} .$$

This justifies the name normal equations.

Problem 8. Consider the overdetermined linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 444 \\ 458 \\ 478 \\ 493 \\ 506 \\ 516 \\ 523 \\ 531 \\ 543 \\ 571 \end{pmatrix} .$$

Solve this linear system in the least squares sense (see previous problem) by the normal equations method.

Solution 8. From the normal equations

$$A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$$

we obtain

$$\begin{pmatrix} 10 & 55 \\ 55 & 385 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 5063 \\ 28898 \end{pmatrix}$$

with $\det(A^T A) \neq 0$. The solution is approximately

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 436.2 \\ 12.7455 \end{pmatrix} .$$

Problem 9. An underdetermined linear system is either inconsistent or has infinitely many solutions. Consider the underdetermined linear system

$$H\mathbf{x} = \mathbf{y}$$

where H is an $n \times m$ matrix with $m > n$ and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} .$$

We assume that $H\mathbf{x} = \mathbf{y}$ has infinitely many solutions. Let P be the $n \times m$ matrix

$$P = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} .$$

We define $\hat{\mathbf{x}} := P\mathbf{x}$. Find

$$\min_{\mathbf{x}} \|P\mathbf{x} - \mathbf{y}\|_2^2$$

subject to the constraint $\|H\mathbf{x} - \mathbf{y}\|_2^2 = 0$. We assume that $(\lambda H^T H + P^T P)^{-1}$ exists for all $\lambda > 0$. Apply the *Lagrange multiplier method.*

Solution 9. We have

$$V(\mathbf{x}) = \|P\mathbf{x} - \mathbf{y}\|_2^2 + \lambda \|H\mathbf{x} - \mathbf{y}\|_2^2$$

where λ is the Lagrange multiplier. Thus $V(\mathbf{x}) \to \min$ if λ is sufficiently large. The derivative of $V(\mathbf{x})$ with respect to the unknown $\mathbf{x}$ is

$$\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) = 2\lambda H^T(H\mathbf{x} - \mathbf{y}) + 2P^T(P\mathbf{x} - \mathbf{y}).$$

Thus

$$(\lambda H^T H + P^T P)\mathbf{x} = (\lambda H^T + P^T)\mathbf{y}.$$

It follows that

$$\hat{\mathbf{x}} = \left(\lambda H^T H + P^T P\right)^{-1} (\lambda H + P)^T \mathbf{y}.$$

Problem 10. Show that solving the system of nonlinear equations with the unknowns x_1, x_2, x_3, x_4

$$\begin{aligned}
(x_1 - 1)^2 + (x_2 - 2)^2 + x_3^2 &= a^2(x_4 - b_1)^2 \\
(x_1 - 2)^2 + x_2^2 + (x_3 - 2)^2 &= a^2(x_4 - b_2)^2 \\
(x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 &= a^2(x_4 - b_3)^2 \\
(x_1 - 2)^2 + (x_2 - 1)^2 + x_3^2 &= a^2(x_4 - b_4)^2
\end{aligned}$$

leads to a linear underdetermined system. Solve this system with respect to x_1, x_2 and x_3.

Solution 10. Expanding all the squares and rearranging that the linear terms are on the left-hand side, yields

$$\begin{aligned}
2x_1 + 4x_2 - 2a^2 b_1 x_4 &= 5 - a^2 b_1^2 + x_1^2 + x_2^2 + x_3^2 - a^2 x_4^2 \\
4x_1 + 4x_3 - 2a^2 b_2 x_4 &= 8 - a^2 b_2^2 + x_1^2 + x_2^2 + x_3^2 - a^2 x_4^2 \\
2x_1 + 2x_2 + 2x_3 - 2a^2 b_3 x_4 &= 3 - a^2 b_3^2 + x_1^2 + x_2^2 + x_3^2 - a^2 x_4^2 \\
4x_1 + 2x_2 - 2a^2 b_4 x_4 &= 5 - a^2 b_4^2 + x_1^2 + x_2^2 + x_3^2 - a^2 x_4^2 .
\end{aligned}$$

The quadratic terms in all the equations are the same. Thus by subtracting the first equation from each of the other three, we obtain an underdetermined system of three linear equations

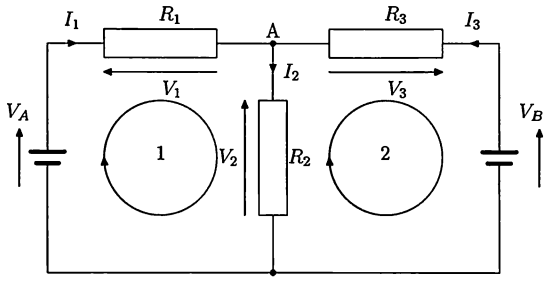
$$\begin{aligned}
2x_1 - 4x_2 + 4x_3 - 2a^2(b_2 - b_1)x_4 &= 3 + a^2(b_1^2 - b_2^2) \\
-2x_2 + 2x_3 - 2a^2(b_3 - b_1)x_4 &= -2 + a^2(b_1^2 - b_3^2) \\
2x_1 - 2x_2 - 2a^2(b_4 - b_1)x_4 &= a^2(b_1^2 - b_4^2) .
\end{aligned}$$

Solving these equations we obtain

$$x_1 = a^2(b_1 + b_2 - 2b_3)x_4 + \frac{a^2}{2}(-b_1^2 - b_2^2 + 2b_3^2) + \frac{7}{2}$$
$$x_2 = a^2(2b_1 + b_2 - 2b_3 - b_4)x_4 + \frac{a^2}{2}(-2b_1^2 - b_2^2 + b_3^2 + b_4^2) + \frac{7}{2}$$
$$x_3 = a^2(b_1 + b_2 - b_3 - b_4)x_4 + \frac{a^2}{2}(-b_1^2 - b_2^2 + b_3^2 + b_4^2) + \frac{5}{2}.$$

Inserting these solutions into one of the nonlinear equations provides a quadratic equation for x_4. Such a system of equations plays a role in the *Global Positioning System* (GPS), where x_4 plays the role of time.

Problem 11. *Kirchhoff's current law* states that the algebraic sum of all the currents flowing into a junction is 0. *Kirchhoff's voltage law* states that the algebraic sum of all the voltages around a closed circuit is 0. Use Kirchhoff's laws and *Ohm's law* ($V = RI$) to setting up the system of linear equations for the circuit depicted in the figure.



Given the voltages V_A, V_B and the resistors R_1, R_2, R_3. Find I_1, I_2, I_3.

Solution 11. From Kirchhoff's voltage law we find for loop 1 and loop 2 that

$$V_1 + V_2 = V_A$$
$$V_2 + V_3 = V_B.$$

From Kirchhoff's current law we obtain for node A

$$I_1 - I_2 + I_3 = 0.$$

Thus the linear system can be written in matrix form

$$\begin{pmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & R_3 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_A \\ V_B \\ 0 \end{pmatrix}.$$

The determinant of the matrix on the left-hand side is given by

$$R_1R_2 + R_2R_3 + R_1R_3 .$$

Since $R_1, R_2, R_3 > 0$ the inverse matrix exists. The solution is given by

$$I_1 = \frac{(R_2+R_3)V_A - R_2V_B}{R_1R_2 + R_1R_3 + R_2R_3}$$
$$I_2 = \frac{R_3V_A + R_1V_B}{R_1R_2 + R_2R_3 + R_1R_3}$$
$$I_3 = \frac{-R_2V_A + (R_1+R_2)V_B}{R_1R_2 + R_2R_3 + R_1R_3} .$$

Problem 12. Let A be an $m \times n$ matrix over $\mathbf{R}$. We define

$$N_A := \{ \mathbf{x} \in \mathbf{R}^n \; : \; A\mathbf{x} = \mathbf{0} \} .$$

N_A is called the *kernel* of A and

$$\nu(A) := \dim(N_A)$$

is called the *nullity* of A. If N_A only contains the zero vector, then $\nu(A) = 0$.
(i) Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix} .$$

Find N_A and $\nu(A)$.
(ii) Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -9 \end{pmatrix} .$$

Find N_A and $\nu(A)$.

Solution 12. (i) From

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we find the system of linear equations

$$x_1 + 2x_2 - x_3 = 0$$
$$2x_1 - x_2 + 3x_3 = 0 .$$

Eliminating x_3 yields

$$x_1 = -x_2 .$$

It follows that $x_3 = -x_1$. Thus N_A is spanned by the vector

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore $\nu(A) = 1$.
(ii) From

$$\begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we find the system of linear equations

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 - 2x_2 + 6x_3 &= 0 \\ -6x_1 + 3x_2 - 9x_3 &= 0. \end{aligned}$$

The three equation are the same. Thus from $2x_1 - x_2 + 3x_3 = 0$ we find that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

is a basis for N_A and $\nu(A) = 2$.

Problem 13. Let V be a vector space over a field $\mathcal{F}$. Let W be a subspace of V. We define an *equivalence relation* $\sim$ on V by stating that $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The *quotient space* V/W is the set of equivalence classes $[v]$ where $v_1 - v_2 \in W$. Thus we can say that v_1 is equivalent to v_2 modulo W if $v_1 = v_2 + w$ for some $w \in W$. Let

$$V = \mathbf{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbf{R} \right\}$$

and the subspace

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} : x_1 \in \mathbf{R} \right\}.$$

(i) Is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 1 \end{pmatrix} ?$$

(ii) Give the quotient space V/W.

Solution 13. (i) We have

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \in W$$

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \in W$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \not\sim \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin W .$$

(ii) Thus the elements of the quotient space consists of straight lines parallel to the x_1 axis.

Problem 14. Let $x_1, x_2, x_3 \in \mathbf{Z}$. Find all solutions of the system of linear equations

$$7x_1 + 5x_2 - 5x_3 = 8$$
$$17x_1 + 10x_2 - 15x_3 = -42 .$$

Find all positive solutions.

Solution 14. Eliminating x_2 yields $3x_1 - 5x_3 = -58$ or

$$3x_1 \equiv -58 \pmod 5 .$$

The solution is $x_1 \equiv 4 \pmod 5$ or

$$x_1 = 4 + 5s, \qquad s \in \mathbf{Z} .$$

Thus using $3x_1 - 5x_3 = -58$ we obtain

$$x_3 = 14 + 3s$$

and using $7x_1 + 5x_2 - 5x_3 = 8$ we find

$$x_2 = 10 - 4s .$$

For x_2 positive we have $s \le 2$, for x_1 positive we have $0 \le s$ and x_3 remains positive. Thus the solution set for positive x_1, x_2, x_3 is

$$(4,\ 10,\ 14), \quad (9,\ 6,\ 17), \quad (14,\ 2,\ 20) .$$

We can write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 14 \end{pmatrix} + s \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$$

where $s = 0, 1, 2$.

Problem 15. Consider the inhomogeneous linear integral equation

$$\int_0^1 (\alpha_1(x)\beta_1(y) + \alpha_2(x)\beta_2(y))\varphi(y)dy + f(x) = \varphi(x) \qquad (1)$$

for the unknown function φ, $f(x) = x$ and

$$\alpha_1(x) = x, \quad \alpha_2(x) = \sqrt{x}, \quad \beta_1(y) = y, \quad \beta_2(y) = \sqrt{y}\,.$$

Thus α_1 and α_2 are continuous in $[0,1]$ and likewise for β_1 and β_2. We define

$$B_1 := \int_0^1 \beta_1(y)\varphi(y)dy, \qquad B_2 := \int_0^1 \beta_2(y)\varphi(y)dy$$

and

$$a_{\mu\nu} := \int_0^1 \beta_\mu(y)\alpha_\nu(y)dy, \qquad b_\mu := \int_0^1 \beta_\mu(y)f(y)dy$$

where $\mu, \nu = 1, 2$. Show that the integral equation can be cast into a system of linear equations for B_1 and B_2. Solve this system of linear equations and thus find a solution of the integral equation.

Solution 15. Using B_1 and B_2 equation (1) can be written as

$$\varphi(x) = \alpha_1(x)B_1 + \alpha_2(x)B_2 + f(x) \tag{2}$$

or

$$\varphi(y) = \alpha_1(y)B_1 + \alpha_2(y)B_2 + f(y)\,. \tag{3}$$

We insert (2) into the left-hand side and (3) into the right-hand side of

$$\varphi(x) = \alpha_1(x)\int_0^1 \beta_1(y)\varphi(y)dy + \alpha_2(x)\int_0^1 \beta_2(y)\varphi(y)dy + f(x)\,.$$

Using $a_{\mu\nu}$ and b_μ we find

$$\begin{aligned}\alpha_1(x)B_1 + \alpha_2(x)B_2 = {} & \alpha_1(x)B_1a_{11} + \alpha_1(x)B_2a_{12} + \alpha_1(x)b_1\\ & +\alpha_2(x)B_1a_{21} + \alpha_2(x)B_2a_{22} + \alpha_2(x)b_2\,.\end{aligned}$$

Now α_1 and α_2 are linearly independent. Comparing the coefficients of α_1 and α_2 we obtain the system of linear equations for B_1 and B_2

$$\begin{aligned}B_1 &= a_{11}B_1 + a_{12}B_2 + b_1\\ B_2 &= a_{21}B_1 + a_{22}B_2 + b_2\end{aligned}$$

or in matrix form

$$\begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Since

$$a_{11} = \int_0^1 y^2 dy = \frac{1}{3}, \qquad a_{12} = \int_0^1 y^{3/2}dy = \frac{2}{5}$$

$$a_{21} = \int_0^1 y^{3/2} dy = \frac{2}{5}, \qquad a_{22} = \int_0^1 y dy = \frac{1}{2}.$$

$$b_1 = \int_0^1 y^2 dy = \frac{1}{3}, \qquad b_2 = \int_0^1 y^{2/3} dy = \frac{2}{5}.$$

Therefore

$$\begin{pmatrix} 2/3 & -2/5 \\ -2/5 & 1/2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/5 \end{pmatrix}$$

with the unique solution

$$B_1 = \frac{49}{26}, \qquad B_2 = \frac{30}{13}.$$

Since

$$\varphi(x) = \alpha_1(x)B_1 + \alpha_2(x)B_2 + x$$

we obtain the solution of the integral equation

$$\varphi(x) = \frac{75}{26}x + \frac{30}{13}\sqrt{x}.$$

Chapter 3

Determinants and Traces

A function on $n \times n$ matrices $\det : \mathbf{C}^{n\times n} \to \mathbf{C}$ is called a determinant function if and only if it satisfies the following conditions:
1) det is linear in each row if the other rows of the matrix are held fixed.
2) If the $n \times n$ matrix A has two identical rows then $\det A = 0$.
3) If I_n is the $n \times n$ identity matrix, then $\det I_n = 1$.

Let A, B be $n \times n$ matrices and $c \in \mathbf{C}$. Then we have

$$\det(AB) = \det A \det B, \qquad \det(cA) = c^n \det A .$$

The determinant of A is the product of the eigenvalues of A

$$\det A = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n .$$

Let A be an $n \times n$ matrix. Then the *trace* is defined as

$$\mathrm{tr} A := \sum_{j=1}^{n} a_{jj} .$$

The trace is independent of the underlying basis. The trace is the sum of the eigenvalues of A, i.e.

$$\mathrm{tr} A = \sum_{j=1}^{n} \lambda_j .$$

The trace and determinant of a square matrix A are related by the identity

$$\det \exp(A) \equiv \exp(\mathrm{tr} A) .$$

Problem 1. Let A be a 2×2 matrix over $\mathbf{C}$. Using the *Cayley-Hamilton theorem* show that

$$(\mathrm{tr}A)^2 = \mathrm{tr}(A^2) + 2\det(A). \tag{1}$$

Cayley-Hamilton theorem. Let A be an $n \times n$ matrix over a field (in our case $\mathbf{C}$) with characteristic polynomial $p_A(\lambda) = \det(\lambda I_n - A)$, where I_n is the $n \times n$ identity matrix. Then $p_A(A)$ is the $n \times n$ zero matrix.

Solution 1. Thus for $n = 2$ we have

$$A^2 - A\mathrm{tr}A + I_2 \det(A) = 0.$$

Taking the trace of the left and right hand side and $\mathrm{tr}I_2 = 2$ we obtain (1).

Problem 2. Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Can we find an invertible 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix?

Solution 2. The answer is no. Let

$$\tilde{A} = Q^{-1}AQ \tag{1}$$

with

$$\tilde{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $a, b \in \mathbf{C}$. Taking the trace of (1) we find

$$a + b = \mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}A = 0.$$

Taking the determinant of (1) we obtain

$$ab = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \det A = 0.$$

Thus from $a + b = 0$ and $ab = 0$ we find $a = b = 0$. However from (1) it also follows that

$$Q\tilde{A}Q^{-1} = A.$$

Therefore A is the zero matrix which contradicts the assumption for the matrix A. Thus no invertible Q can be found. This means A is not diagonalizable.

Problem 3. Let A be a 2×2 matrix over $\mathbf{R}$. Assume that $\mathrm{tr}A = 0$ and $\mathrm{tr}A^2 = 0$. Can we conclude that A is the 2×2 zero matrix?

Solution 3. No we cannot conclude that A is the 2×2 zero matrix. For example,

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfies $\mathrm{tr}A = 0$ and $\mathrm{tr}A^2 = 0$. What happens if we also assume that A is symmetric over $\mathbf{R}$?

Problem 4. Consider the $(n-1) \times (n-1)$ matrix

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ 1 & 1 & 1 & 6 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}.$$

Let D_n be the determinant of this matrix. Is the sequence $\{ D_n/n! \}$ bounded?

Solution 4. If we expand the last row we obtain the recursion

$$D_n = nD_{n-1} + (n-1)!, \qquad n = 3, 4, \dots$$

with the initial condition $D_2 = 3$. We divide by $n!$ to obtain

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{1}{n}.$$

Thus

$$D_n = n! \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

Therefore the sequence $\{ D_n/n! \}$ is the nth partial sum of the *harmonic series*, which is unbounded as $n \to \infty$.

Problem 5. For an integer $n \geq 3$, let $\theta := 2\pi/n$. Find the determinant of the $n \times n$ matrix $A + I_n$, where I_n is the $n \times n$ identity matrix and the matrix $A = (a_{jk})$ has the entries $a_{jk} = \cos(j\theta + k\theta)$ for all $j, k = 1, 2, \dots, n$.

Solution 5. The determinant of a square matrix is the product of its eigenvalues. We compute the determinant by calculating the eigenvalues of $I_n + A$. The eigenvalues of $I_n + A$ are obtained by adding 1 to each of the eigenvalues of A. Thus we only have to calculate the eigenvalues of A and then add 1. We show that the eigenvalues of A are $n/2, -n/2, 0, \dots, 0$, where 0 occurs with multiplicity $n-2$. We define column vectors $\mathbf{v}^{(m)}$,

$0 \le m \le n-1$, componentwise by $v_k^{(m)} = e^{ikm\theta}$, where $\theta = 2\pi/n$. We form a matrix from the column vectors $\mathbf{v}^{(m)}$. Its determinant is a Vandermonde product and hence is nonzero. Thus the vectors $\mathbf{v}^{(m)}$ form a basis in $\mathbf{C}^n$. Since $\cos z \equiv (e^{iz} + e^{-iz})/2$ for any $z \in \mathbf{C}$ we obtain

$$\begin{aligned}(A\mathbf{v}^{(m)})_j &= \sum_{k=1}^{n} \cos(j\theta + k\theta) e^{ikm\theta} \\ &= \frac{e^{ij\theta}}{2} \sum_{k=1}^{n} e^{ik(m+1)\theta} + \frac{e^{-ij\theta}}{2} \sum_{k=1}^{n} e^{ik(m-1)\theta}\end{aligned}$$

where $j = 1, 2, \ldots, n$. Since

$$\sum_{k=1}^{n} e^{ik\ell\theta} = 0$$

for integer ℓ unless $n|\ell$, we conclude that $A\mathbf{v}^{(m)} = \mathbf{0}$ for $m = 0$ and for $2 \le m \le n-1$. In addition, we find that

$$(A\mathbf{v}^{(1)})_j = \frac{n}{2} e^{-ij\theta} = \frac{n}{2} (\mathbf{v}^{(n-1)})_j, \qquad (A\mathbf{v}^{(n-1)})_j = \frac{n}{2} e^{ij\theta} = \frac{n}{2} (\mathbf{v}^{(1)})_j .$$

Thus

$$A(\mathbf{v}^{(1)} \pm \mathbf{v}^{(n-1)}) = \pm \frac{n}{2} (\mathbf{v}^{(1)} \pm \mathbf{v}^{(n-1)}) .$$

Consequently

$$\{ \mathbf{v}^{(0)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n-2)}, \mathbf{v}^{(1)} + \mathbf{v}^{(n-1)}, \mathbf{v}^{(1)} - \mathbf{v}^{(n-1)} \}$$

is a basis for $\mathbf{C}^n$ of eigenvectors of A with the claimed eigenvalues. Since the determinant of $I_n + A$ is the product of $(1 + \lambda)$ over all eigenvalues λ of A, we obtain

$$\det(I_n + A) = (1 + n/2)(1 - n/2) = 1 - n^2/4 .$$

Problem 6. Let α, β, γ, δ be real numbers.
(i) Is the matrix

$$U = e^{i\alpha} \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} \begin{pmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

unitary?
(ii) What the determinant of U?

Solution 6. (i) Each of the three matrices on the right-hand side is unitary and $e^{i\alpha}$ is unitary. The product of two unitary matrices is again a unitary matrix. Thus U is unitary.

(ii) The determinat of each of the three matrices on the right-hand side is 1. Thus $\det(U) = e^{2i\alpha}$.

Problem 7. Let A and B be two $n \times n$ matrices over $\mathbf{C}$. If there exists a non-singular $n \times n$ matrix X such that

$$A = XBX^{-1}$$

then A and B are said to be *similar matrices.* Show that the spectra (eigenvalues) of two similar matrices are equal.

Solution 7. We have

$$\begin{aligned}\det(A - \lambda I_n) &= \det(XBX^{-1} - X\lambda I_n X^{-1}) \\ &= \det(X(B - \lambda I_n)X^{-1}) \\ &= \det(X)\det(B - \lambda I_n)\det(X^{-1}) \\ &= \det(B - \lambda I_n).\end{aligned}$$

Problem 8. Let A be an $n \times n$ matrix. Assume that the inverse matrix of A exists. The inverse matrix can be calculated as follows (*Csanky's algorithm*). Let

$$p(x) := \det(xI_n - A) \tag{1}$$

where I_n is the $n \times n$ unit matrix. The roots are, by definition, the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A. We write

$$p(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \tag{2}$$

where

$$c_n = (-1)^n \det A\,.$$

Since A is nonsingular we have $c_n \neq 0$ and vice versa. The *Cayley-Hamilton theorem* states that

$$p(A) = A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_n = 0_n\,. \tag{3}$$

Multiplying this equation with A^{-1} we obtain

$$A^{-1} = \frac{1}{-c_n}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I_n)\,. \tag{4}$$

If we have the coefficients c_j we can calculate the inverse matrix A. Let

$$s_k := \sum_{j=1}^{n} \lambda_j^k\,.$$

Then the s_j and c_j satisfy the following $n \times n$ lower triangular system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ s_1 & 2 & 0 & \dots & 0 \\ s_2 & s_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & \dots & s_1 & n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} -s_1 \\ -s_2 \\ -s_3 \\ \vdots \\ -s_n \end{pmatrix} .$$

Since

$$\operatorname{tr}(A^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = s_k$$

we find s_k for $k = 1, 2, \ldots, n$. Thus we can solve the linear equation for c_j. Finally, using (4) we obtain the inverse matrix of A. Apply Csanky's algorithm to the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Solution 8. Since

$$A^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

we find

$$\operatorname{tr} A = 2 = s_1, \quad \operatorname{tr} A^2 = 2 = s_2, \quad \operatorname{tr} A^3 = 5 = s_3 .$$

We obtain the system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -5 \end{pmatrix}$$

with the solution

$$c_1 = -2, \quad c_2 = 1, \quad c_3 = -1 .$$

Since $c_3 = -1$ the inverse exists and $\det A = 1$. The inverse matrix of A is given by

$$A^{-1} = \frac{1}{-c_3}(A^2 + c_1 A + c_2 I_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} .$$

Problem 9. Let U be the $n \times n$ unitary matrix

$$U := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and V be the $n \times n$ unitary diagonal matrix ($\zeta \in \mathbf{C}$)

$$V := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0 & \dots & 0 \\ 0 & 0 & \zeta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix}$$

where $\zeta^n = 1$. Then the set of matrices

$$\{ U^j V^k \ : \ j, k = 0, 1, 2, \dots, n-1 \}$$

provide a basis in the Hilbert space for all $n \times n$ matrices with the *scalar product*

$$\langle A, B \rangle := \frac{1}{n} \operatorname{tr}(AB^*)$$

for $n \times n$ matrices A and B. Write down the basis for $n = 2$.

Solution 9. For $n = 2$ we have the combinations

$$(jk) \in \{ (00), (01), (10), (11) \}.$$

This yields the orthonormal basis in the vector space of 2×2 matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Problem 10. Let A and B be $n \times n$ matrices over $\mathbf{C}$. Show that the matrices AB and BA have the same set of eigenvalues.

Solution 10. Consider first the case that A is invertible. Then we have

$$AB = A(BA)A^{-1}.$$

Thus AB and BA are similar and therefore have the same set of eigenvalues. If A is singular we apply the *continuity argument*: If A is singular, consider $A + \epsilon I_n$. We choose $\delta > 0$ such that $A + \epsilon I_n$ is invertible for all ϵ, $0 < \epsilon < \delta$.

Thus $(A+\epsilon I_n)B$ and $B(A+\epsilon I_n)$ have the same set of eigenvalues for every $\epsilon \in (0,\delta)$. We equate their characteristic polynomials to obtain

$$\det(\lambda I_n - (A+\epsilon I_n)B) = \det(\lambda I_n - B(A+\epsilon I_n)), \quad 0<\epsilon<\delta\,.$$

Since both sides are continuous (even analytic) functions of ϵ we find by letting $\epsilon \to 0^+$ that

$$\det(\lambda I_n - AB) = \det(\lambda I_n - BA)\,.$$

Problem 11. An $n \times n$ *circulant matrix* C is given by

$$C := \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

For example, the matrix

$$P := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is a circulant matrix. It is also called the $n \times n$ *primary permutation matrix*.
(i) Let C and P be the matrices given above. Let

$$f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}\,.$$

Show that $C = f(P)$.
(ii) Show that C is a *normal matrix*, that is,

$$C^*C = CC^*\,.$$

(iii) Show that the eigenvalues of C are $f(\omega^k)$, $k = 0,1,\dots,n-1$, where ω is the nth primitive root of unity.
(iv) Show that

$$\det(C) = f(\omega^0)f(\omega^1)\dots f(\omega^{n-1})\,.$$

(v) Show that F^*CF is a diagonal matrix, where F is the unitary matrix with (j,k)-entry equal to

$$\frac{1}{\sqrt{n}}\omega^{(j-1)(k-1)}, \quad j,k = 1,\dots,n\,.$$

Solution 11. (i) Direct calculation of

$$f(P) = c_0 I_n + c_1 P + c_2 P^2 + \cdots + c_{n-1} P^{n-1}$$

yields the matrix C, where I_n is the $n \times n$ unit matrix. Notice that P^2, P^3, ..., P^{n-1} are permutation matrices.
(ii) We have $PP^* = P^*P$. If two $n \times n$ matrices A and B commute, then $g(A)$ and $h(B)$ commute, where g and h are polynomials. Thus C is a normal matrix.
(iii) The characteristic polynomial of P is

$$\det(\lambda I_n - P) = \lambda^n - 1 = \prod_{k=0}^{n-1} (\lambda - \omega^k) .$$

Thus the eigenvalues of P and P^j are, respectively, ω^k and ω^{jk}, where $k = 0, 1, \ldots, n-1$. It follows that the eigenvalues of $C = f(P)$ are $f(\omega^k)$, $k = 0, 1, \ldots, n-1$.
(iv) Using the result from (iii) we find

$$\det(C) = \prod_{k=0}^{n-1} f(\omega^k) .$$

(v) For each $k = 0, 1, \ldots, n-1$, let

$$\mathbf{x}_k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})^T$$

where ^{T} denotes the transpose. If follows that

$$P\mathbf{x}_k = (\omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}, 1)^T = \omega^k \mathbf{x}_k$$

and

$$C\mathbf{x}_k = f(P)\mathbf{x}_k = f(\omega^k)\mathbf{x}_k .$$

Thus the vectors $\mathbf{x}_k$ are the eigenvectors of P and C corresponding to the respective eigenvalues ω^k and $f(\omega^k)$, $k = 0, 1, \ldots, n-1$. Since

$$\langle \mathbf{x}_j, \mathbf{x}_k \rangle \equiv \mathbf{x}_j^* \mathbf{x}_k = \sum_{\ell=0}^{n-1} \overline{\omega^{k\ell}} \omega^{j\ell} = \sum_{\ell=0}^{n-1} \omega^{(j-k)\ell} = \begin{cases} 0 & j \neq k \\ n & j = k \end{cases}$$

we find that

$$\left\{ \frac{1}{\sqrt{n}} \mathbf{x}_0, \ \frac{1}{\sqrt{n}} \mathbf{x}_1, \ \ldots, \ \frac{1}{\sqrt{n}} \mathbf{x}_{n-1} \right\}$$

is an orthonormal basis in the Hilbert space $\mathbf{C}^n$. Thus we obtain the unitary matrix

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

such that

$$F^*CF = \mathrm{diag}(f(\omega^0), f(\omega^1), \ldots, f(\omega^{n-1})).$$

The matrix F is unitary and is called the *Fourier matrix.*

Problem 12. An $n \times n$ matrix A is called *reducible* if there is a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square matrices of order at least 1. An $n \times n$ matrix A is called *irreducible* if it is not reducible. Show that the $n \times n$ primary permutation matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

is irreducible.

Solution 12. Suppose the matrix A is reducible. Let

$$P^T AP = J_1 \oplus J_2 \oplus \ldots \oplus J_k, \quad k \geq 2$$

where P is some permutation matrix and the J_j are irreducible matrices of order $< n$. Here $\oplus$ denotes the direct sum. The rank of $A - I_n$ is $n-1$ since $\det(A - I_n) = 0$ and the submatrix of size $n-1$ by deleting the last row and the last column from $A - I_n$ is nonsingular. It follows that

$$\mathrm{rank}(P^T AP - I_n) = \mathrm{rank}(P^T (A - I_n)P) = n - 1.$$

By using the above decomposition, we obtain

$$\mathrm{rank}(P^T AP - I_n) = \sum_{j=1}^{k} \mathrm{rank}(J_j - I_n) \leq (n-k) < (n-1).$$

This is a contradiction. Thus A is irreducible.

Problem 13. We define a linear *bijection*, h, between $\mathbf{R}^4$ and $\mathbf{H}(2)$, the set of complex 2×2 hermitian matrices, by

$$(t, x, y, z) \to \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}.$$

We denote the matrix on the right-hand side by H.
(i) Show that the matrix can be written as a linear combination of the Pauli spin matrices and the identity matrix I_2.
(ii) Find the inverse map.
(iii) Calculate the determinant of 2×2 hermitian matrix H. Discuss.

Solution 13. (i) We have

$$H = tI_2 + x\sigma_z + y\sigma_x + z\sigma_y .$$

(ii) Consider

$$\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} .$$

Compare the entries of the 2×2 matrix we obtain

$$t = \frac{a+b}{2}, \quad x = \frac{a-b}{2}, \quad y = \frac{c+c^*}{2}, \quad z = \frac{c^*-c}{2i} .$$

(iii) We obtain

$$\det H = t^2 - x^2 - y^2 - z^2 .$$

This is the *Lorentz metric.* Let U be a unitary 2×2 matrix. Then $\det(UHU^*) = \det(H)$.

Problem 14. Let A be an $n \times n$ invertible matrix over $\mathbf{C}$. Assume that A can be written as

$$A = B + iB$$

where B has only real coefficients. Show that B^{-1} exists and

$$A^{-1} = \frac{1}{2}(B^{-1} - iB^{-1}) .$$

Solution 14. Since

$$A = (1+i)B$$

and $\det A \neq 0$ we have $(1+i)^n \det B \neq 0$. Thus $\det B \neq 0$ and B^{-1} exists. We have

$$(1+i)B\frac{1}{2}(1-i)B^{-1} = I_n .$$

Problem 15. Let A be an invertible matrix. Assume that $A = A^{-1}$. What are the possible values for $\det A$?

Solution 15. Since

$$1 = \det I_n = \det(AA^{-1}) = \det A \det A^{-1}$$

and by assumption $\det A = \det A^{-1}$ we have

$$1 = (\det A)^2 .$$

Thus $\det A$ is either $+1$ or -1.

Problem 16. Let A be a skew-symmetric matrix over $\mathbf{R}$, i.e. $A^T = -A$ and of order $2n-1$. Show that $\det(A) = 0$.

Solution 16. From $A^T = -A$ we obtain

$$\det(A^T) = \det(-A) = (-1)^{2n-1}\det(A) = -\det(A) .$$

Since

$$\det(A) = \det(A^T)$$

we obtain $\det(A) = -\det(A)$ and therefore $\det(A) = 0$.

Problem 17. Show that if A is hermitian, i.e. $A^* = A$ then $\det(A)$ is a real number.

Solution 17. Since A is hermitian we have $A^* = A$ or $\overline{A} = A^T$. Furthermore $\det(A) = \det(A^T)$. Thus

$$\det(\overline{A}) = \det(A^T) = \det A .$$

Now if $\det(A) = x + iy$ then $\det(\overline{A}) = x - iy$ with $x, y \in \mathbf{R}$. Thus $x + iy = x - iy$ and therefore $y = 0$. Thus $\det(A)$ is a real number.

Problem 18. Let A, B, and C be $n \times n$ matrices. Calculate

$$\det \begin{pmatrix} A & 0_n \\ C & B \end{pmatrix} .$$

where 0_n is the $n \times n$ zero matrix.

Solution 18. Obviously we find

$$\det \begin{pmatrix} A & 0_n \\ C & B \end{pmatrix} = \det(A)\det(B) .$$

Problem 19. Let A, B are 2×2 matrices over $\mathbf{R}$. Let $H := A + iB$. Express $\det H$ as a sum of determinants.

Solution 19. Since

$$H = \begin{pmatrix} a_{11} + ib_{11} & a_{12} + ib_{12} \\ a_{21} + ib_{21} & b_{22} + ib_{22} \end{pmatrix}$$

we find

$$\begin{aligned} \det H &= a_{11}a_{22} - a_{12}a_{21} - b_{11}b_{22} + b_{12}b_{21}i(a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21}) \\ &= \det A - \det B + i \det \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix} + i \det \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix} . \end{aligned}$$

Problem 20. Let A, B are 2×2 matrices over $\mathbf{R}$. Let $H := A + iB$. Assume that H is hermitian. Show that

$$\det H = \det A - \det B .$$

Solution 20. Since H is hermitian, i.e. $H^* = H$ with $H^* = A^T - iB^T$ we have $A = A^T$ and $B = -B^T$. Thus $a_{12} = a_{21}$, $b_{11} = b_{22} = 0$ and $b_{12} = -b_{21}$. It follows that

$$\begin{aligned} \det \begin{pmatrix} a_{11} + ib_{11} & a_{12} + ib_{12} \\ a_{21} + ib_{21} & a_{22} + ib_{22} \end{pmatrix} &= \det A - \det B \\ &\quad + i(a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21}) \\ &= \det A - \det B . \end{aligned}$$

Problem 21. Let A, B, C, D be $n \times n$ matrices. Assume that $DC = CD$, i.e. C and D commute and $\det D \neq 0$. Consider the $(2n) \times (2n)$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Show that

$$\det M = \det(AD - BC) . \tag{1}$$

We know that

$$\det \begin{pmatrix} U & 0_n \\ X & Y \end{pmatrix} = \det U \det Y \tag{2}$$

and

$$\det \begin{pmatrix} U & V \\ 0_n & Y \end{pmatrix} = \det U \det Y \tag{3}$$

where U, V, X, Y are $n \times n$ matrices and 0_n is the $n \times n$ zero matrix.

Solution 21. We have the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D & 0_n \\ -C & I_n \end{pmatrix} = \begin{pmatrix} AD-BC & B \\ CD-DC & D \end{pmatrix} = \begin{pmatrix} AD-BC & B \\ 0_n & D \end{pmatrix} \tag{4}$$

where we used that $CD = DC$. Applying the determinant to the right and left-hand side of (4) and using (2),(3) and $\det D \neq 0$ we obtain identity (1).

Problem 22. Let A, B be $n \times n$ matrices. We have the identity

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} \equiv \det(A+B)\det(A-B).$$

Use this identity to calculate the determinant of the left-hand side using the right-hand side, where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 \\ 4 & 6 \end{pmatrix}.$$

Solution 22. We have

$$A+B = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}, \qquad A-B = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}.$$

Therefore $\det(A+B) = 1$ and $\det(A-B) = 5$. Finally,

$$\det(A+B)\det(A-B) = 5.$$

Problem 23. Let A, B, C, D be $n \times n$ matrices. Assume that D is invertible. Consider the $(2n) \times (2n)$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Show that

$$\det M = \det(AD - BD^{-1}CD). \tag{1}$$

Solution 23. Using the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_n & 0_n \\ -D^{-1}C & I_n \end{pmatrix} \equiv \begin{pmatrix} A-BD^{-1}C & B \\ 0_n & D \end{pmatrix}$$

equation (1) follows.

Problem 24. Let A, B be $n \times n$ positive definite (and therefore hermitian) matrices. Show that

$$\operatorname{tr}(AB) > 0.$$

Solution 24. Let U be a unitary matrix such that

$$U^*AU = D = \operatorname{diag}(d_1, d_2, \ldots, d_n).$$

Obviously, d_1, d_2, ..., d_n are the eigenvalues of A. Then $\operatorname{tr}A = \operatorname{tr}D$ and

$$\operatorname{tr}(AB) = \operatorname{tr}(U^*AUU^*BU) = \operatorname{tr}(DC)$$

where $C = U^*BU$. Now C is positive definite and therefore its diagonal entries c_{ii} are real and positive and $\operatorname{tr}C = \operatorname{tr}B$. The diagonal entries of DC are $d_i c_{ii}$ and therefore

$$\operatorname{tr}(DC) = \sum_{i=1}^{n} d_i c_{ii} > 0.$$

Problem 25. Let $P_0(x) = 1$, $P_1(x) = \alpha_1 - x$ and

$$P_k(x) = (\alpha_k - x)P_{k-1}(x) - \beta_{k-1}P_{k-2}(x), \qquad k = 2, 3, \ldots$$

where β_j, $j = 1, 2, \ldots$ are positive numbers. Find a $k \times k$ matrix A_k such that

$$P_k(x) = \det(A_k).$$

Solution 25. The matrix is

$$A_k = \begin{pmatrix} \alpha_1 - x & \beta_1 & 0 & \ldots & 0 \\ 1 & \alpha_2 - x & \beta_2 & \ldots & 0 \\ 0 & 1 & \alpha_3 - x & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \alpha_k - x \end{pmatrix}.$$

Problem 26. Let

$$A = \begin{pmatrix} \dfrac{1}{x_1 + y_1} & \dfrac{1}{x_1 + y_2} \\ \dfrac{1}{x_2 + y_1} & \dfrac{1}{x_2 + y_2} \end{pmatrix}$$