Recursion Tree Method: Example 1

$$T(n) = \begin{cases} 1 & n = 1 \\ 2\,T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

Solve the following recurrence using the Recurrence Tree Method.

Assumption: We assume that n is exact power of 2.

$$x^{\log_y n} \Longrightarrow n^{\log_y x}$$

$$x^0 + x^1 + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x-1} \qquad for\ x \neq 1$$

$$x^0 + x^1 + x^2 + \cdots = \frac{1}{1-x} \qquad for\ |x| < 1$$

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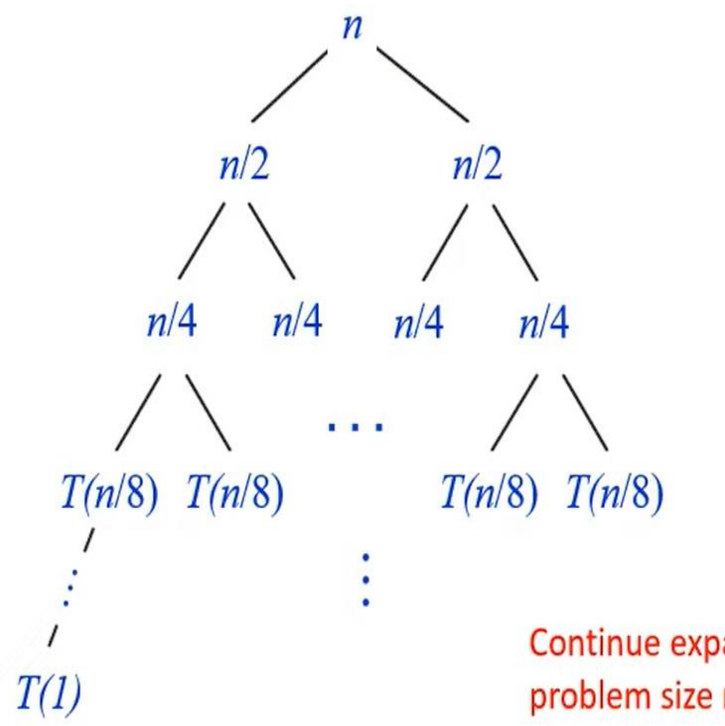
$$T(n) = 2\,T\left(\frac{n}{2}\right) + n$$

$$T(\frac{n}{2}) = 2.T(\frac{n}{2^2}) + \frac{n}{2}$$

$$T(\frac{n}{2^2}) = 2.T(\frac{n}{2^3}) + \frac{n}{2^2}$$

$$T(\frac{n}{2^{k-1}}) = 2.T(\frac{n}{2^k}) + \frac{n}{2^{k-1}}$$

$$T(\frac{n}{2^k}) = T(1)$$



Continue expanding until the problem size reduces to 1.

Total Cost = Cost of Leaf Nodes + Cost of Internal Nodes

Total Cost = (cost of leaf node x total leaf nodes) + (sum of costs at each level of internal nodes)

Total Cost = $L_c + I_c$

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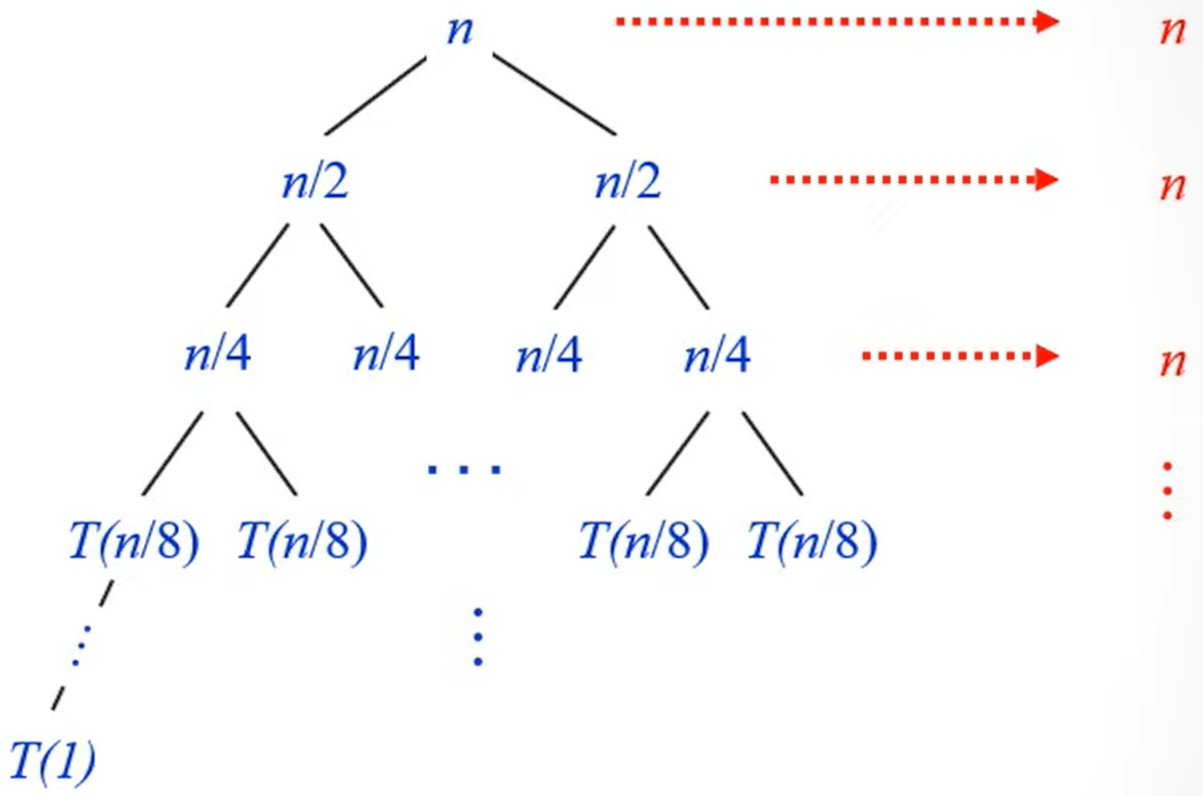
$$T(\frac{n}{2^{k-1}}) = 2.T(\frac{n}{2^k}) + \frac{n}{2^{k-1}}$$

$$T(\frac{n}{2^k}) = T(1)$$

$$n = 2^k \implies k = \lg n$$

$$L_c = 2^k \implies 2^{\lg n} \implies n^{\lg 2} \implies n$$



$$I_c = k \cdot n$$

$$I_c = n \lg n$$

Total Cost $= L_c + I_c \implies n + n \lg n$

Hence: $T(n) \in O(n \lg n)$

$T(1) = 4$ $\quad$ $T(n) = 2T(n/2) + 4n$

Recursive Call	#nodes	TREE	Row Sum
$T(n)$, $i=0$	1	$4n$	$4n$
$T\left(\frac{n}{2}\right)$, $i=1$	2	$\frac{4n}{2} + \frac{4n}{2} =$	$4n$
$T\left(\frac{n}{2^2}\right)$, $i=2$	2^2	$\frac{4n}{2^2} + \frac{4n}{2^2} + \frac{4n}{2^2} + \frac{4n}{2^2} =$	$4n$
$T\left(\frac{n}{2^3}\right)$, $i=3$	2^3	$\frac{4n}{2^3}$ $\frac{4n}{2^3}$ $\frac{4n}{2^3}$ $\dots$	$4n$
$T\left(\frac{n}{2^i}\right)$, i	2^i	$\frac{4n}{2^i}$ $\dots\dots$	$+\ 4n$

$1 = \frac{n}{2^i}$ $\quad$ $\log_2 n = i$

$$\sum_{i=0}^{\log_2 n} 4n = 4n \sum_{i=0}^{\log_2 n} 1$$

$$\underbrace{4n + 4n + \dots + 4n}_{\log_2 n + 1} = 4n(\underbrace{1 + 1 + \dots + 1}_{\log_2 n + 1})$$

$$4n(\log_2 n + 1)$$

$$4n \log_2 n + 4n$$

Recursion Tree Method: Example 2

$$T(n) = \begin{cases} 1 & n = 1 \\ 2\,T\left(\frac{n}{2}\right) + n^2 & n > 1 \end{cases}$$

Solve the following recurrence using the Recurrence Tree Method.

Assumption: We assume that n is exact power of 2.

$$T(n) = 2\,T\left(\frac{n}{2}\right) + n^2$$

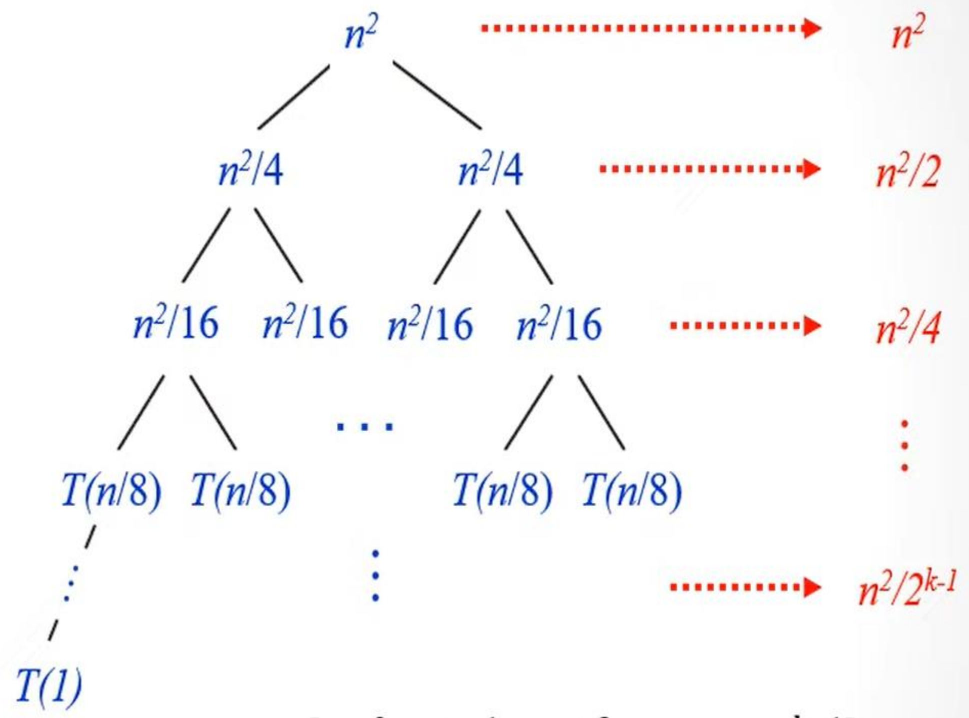
$$T\left(\frac{n}{2}\right) = 2.T\left(\frac{n}{2^2}\right) + \frac{n^2}{2^2}$$

$$T\left(\frac{n}{2^2}\right) = 2.T\left(\frac{n}{2^3}\right) + \frac{n^2}{4^2}$$

$$T\left(\frac{n}{2^k}\right) = T(1)$$

$$n = 2^k \implies k = \lg n$$

$$L_c = 2^k \implies 2^{\lg n} \implies n^{\lg 2} \implies n$$



$$I_c = n^2 \cdot \left[\left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^{k-1}\right]$$

$$I_c = n^2 \cdot \left[\frac{1}{1 - {}^1/_2}\right] \implies 2n^2$$

Total Cost $= L_c + I_c \implies n + 2n^2$

Hence: $T(n) \in O(n^2)$

Recursion Tree Method: Example 3

$$T(n) = \begin{cases} 1 & n = 1 \\ 3\,T\left(\frac{n}{4}\right) + n^2 & n > 1 \end{cases}$$

Solve the following recurrence using the Recurrence Tree Method.

Assumption: We assume that n is exact power of 4.

$$T(n) = 3\,T\left(\frac{n}{4}\right) + n^2$$

$$T\left(\frac{n}{4}\right) = 3.T\left(\frac{n}{4^2}\right) + \frac{n^2}{4^2}$$

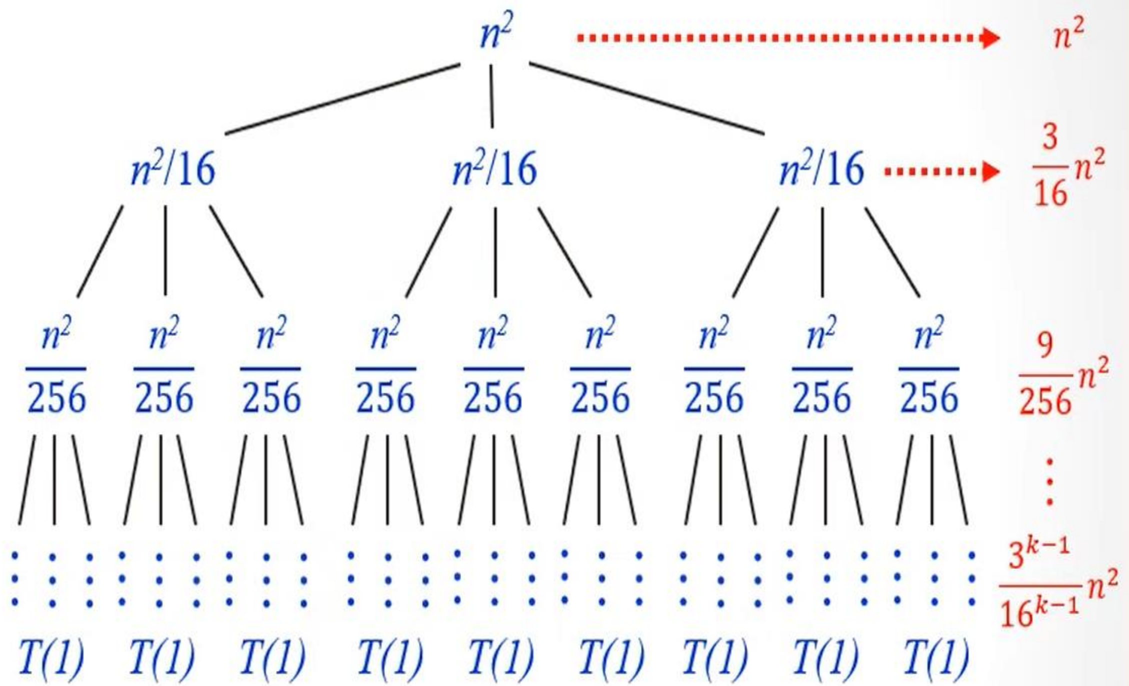
$$T\left(\frac{n}{4^2}\right) = 3.T\left(\frac{n}{4^3}\right) + \frac{n^2}{16^2}$$

$$T\left(\frac{n}{4^k}\right) = T(1)$$

$$n = 4^k$$

$$k = \log_4 n$$

$$L_c = 3^k \Rightarrow 3^{\log_4 n} \Rightarrow n^{\log_4 3}$$



$$I_c = n^2 \cdot \left[\left(\frac{3}{16}\right)^0 + \left(\frac{3}{16}\right)^1 + \left(\frac{3}{16}\right)^2 + \cdots + \left(\frac{3}{16}\right)^{k-1}\right]$$

$$I_c = n^2 \cdot \left[\frac{1}{1 - {}^3/_{16}}\right] \Rightarrow \frac{16}{13}n^2$$

Total Cost $= L_c + I_c \Rightarrow n^{\log_4 3} + \frac{16}{13}n^2$

Hence: $T(n) \in O(n^2)$

Recursion Tree Method: Example 4

$$T(n) = \begin{cases} 1 & n = 1 \\ T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2 & n > 1 \end{cases}$$

Solve the following recurrence using the Recurrence Tree Method.

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2$$

$$T\left(\frac{n}{4}\right) = T\left(\frac{n}{16}\right) + T\left(\frac{n}{8}\right) + \frac{n^2}{16}$$

$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{8}\right) + T\left(\frac{n}{4}\right) + \frac{n^2}{4}$$

$$T\left(\frac{n}{16}\right) = T\left(\frac{n}{64}\right) + T\left(\frac{n}{32}\right) + \frac{n^2}{256}$$

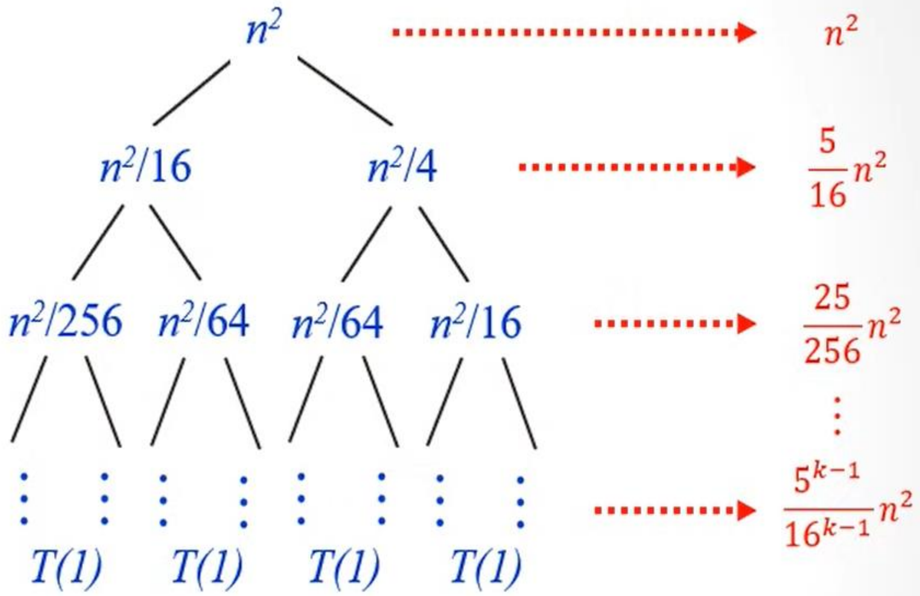
$$T\left(\frac{n}{8}\right) = T\left(\frac{n}{32}\right) + T\left(\frac{n}{16}\right) + \frac{n^2}{64}$$

$$T\left(\frac{n}{4}\right) = T\left(\frac{n}{16}\right) + T\left(\frac{n}{8}\right) + \frac{n^2}{16}$$

$$T\left(\frac{n}{2^k}\right) = T(1)$$

$$n = 2^k \implies k = \lg n$$

$$L_c = 2^k \implies 2^{\lg n} \implies n^{\lg 2} \implies n$$



$$I_c = n^2 \cdot \left[\left(\frac{5}{16}\right)^0 + \left(\frac{5}{16}\right)^1 + \left(\frac{5}{16}\right)^2 + \cdots + \left(\frac{5}{16}\right)^{k-1}\right]$$

$$I_c = n^2 \cdot \left[\frac{1}{1 - {}^5/_{16}}\right] \implies \frac{16}{11}n^2$$

Total Cost $= L_c + I_c \implies n + \frac{16}{11}n^2$ *Hence:* $T(n) \in O(n^2)$

Recursion Tree Method: Example 5

$$T(n) = \begin{cases} 1 & n = 1 \\ T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n & n > 1 \end{cases}$$

Solve the following recurrence using the Recurrence Tree Method.

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

$$T\left(\frac{n}{3}\right) = T\left(\frac{n}{9}\right) + T\left(\frac{2n}{9}\right) + \frac{n}{3}$$

$$T\left(\frac{2n}{3}\right) = T\left(\frac{2n}{9}\right) + T\left(\frac{4n}{9}\right) + \frac{2n}{3}$$

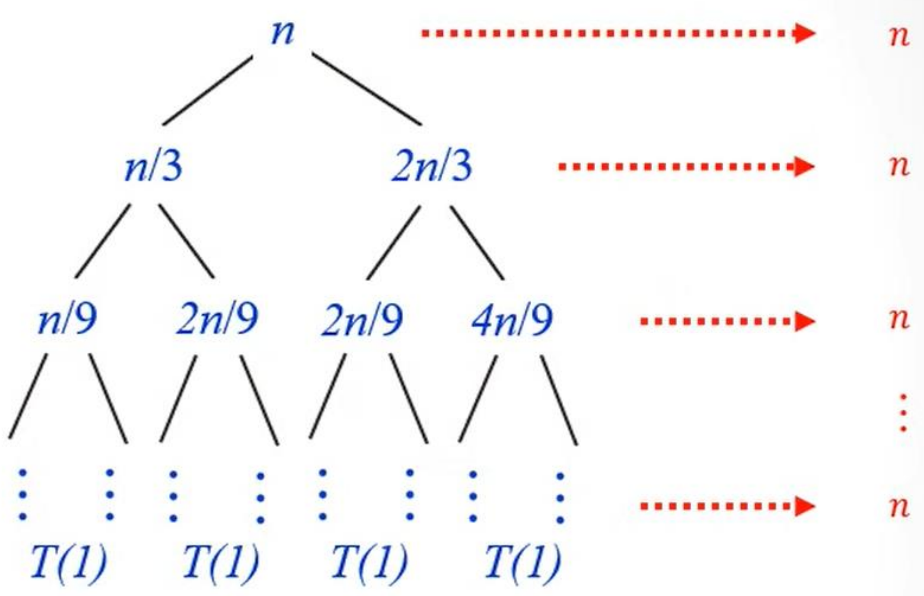
$$T\left(\frac{n}{9}\right) = T\left(\frac{n}{27}\right) + T\left(\frac{2n}{27}\right) + \frac{n}{9}$$

$$T\left(\frac{2n}{9}\right) = T\left(\frac{2n}{27}\right) + T\left(\frac{4n}{27}\right) + \frac{2n}{9}$$

$$T\left(\frac{4n}{9}\right) = T\left(\frac{4n}{27}\right) + T\left(\frac{8n}{27}\right) + \frac{4n}{9}$$

$$T\left(\frac{2^k}{3^k}n\right) = T(1) \quad \Longrightarrow n = \frac{3^k}{2^k} \quad \Longrightarrow k = \log_{3/2} n$$

$$L_c = 2^k \quad \Longrightarrow 2^{\log_{3/2} n} \Longrightarrow n^{\log_{3/2} 2}$$



$$I_c = n \cdot k$$

$$I_c = n \log_{3/2} n$$

Total Cost $= L_c + I_c \quad \Longrightarrow n^{\log_{3/2} 2} + n \log_{3/2} n \qquad T(n) \in O(n \lg n)??$