Jiangxi University of Science and Technology

DIGITAL DESIGN

Lecture 6:
Boolean algebra and DE Morgan's Theorems

Introduction

- Developed by English Mathematician *George Boole* in between 1815 - 1864.
- It is described as *an algebra of logic or an algebra of two values i.e True or False*.
- The term *logic* means a statement having binary decisions i.e *True/Yes or False/No*.

Application of Boolean algebra

- It is used to perform the logical operations in digital computer.
- In digital computer True represent by ‘1’ (high volt) and False represent by ‘0’ (low volt)
- Logical operations are performed by logical operators. The fundamental logical operators are:
  1. AND (conjunction)
  2. OR (disjunction)
  3. NOT (negation/complement)

The “WHY” slide

- Boolean Algebra
  - When we learned numbers like 1, 2, 3, we also then learned how to add, multiply, etc. with them. Boolean Algebra covers operations that we can do with 0’s and 1’s. Computers do these operations ALL THE TIME and they are basic building blocks of computation inside your computer program.
- Axioms, laws, theorems
  - We need to know some rules about how those 0’s and 1’s can be operated on together. There are similar axioms to decimal number algebra, and there are some laws and theorems that are good for you to use to simplify your operation.

How does Boolean Algebra fit into the big picture?

- It is part of the Combinational Logic topics (memoryless)
  - Different from the Sequential logic topics (can store information)

- Learning Axioms and theorems of Boolean algebra
  - Allows you to design logic functions
  - Allows you to know how to combine different logic gates
  - Allows you to simplify or optimize on the complex operations

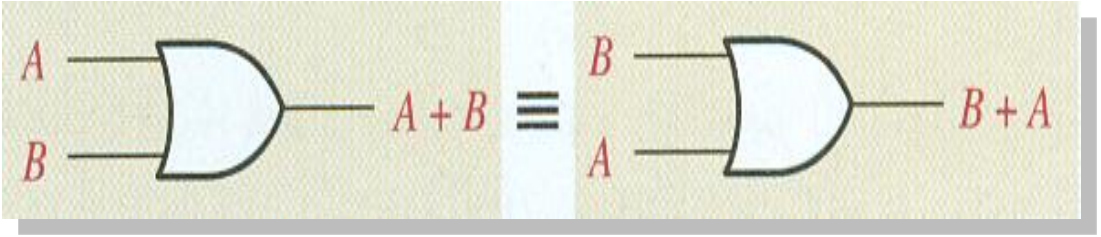
Boolean algebra

- A Boolean algebra comprises…
  - A set of elements B
  - Binary operators {+ , •} –Boolean sum and product
  - A unary operation { ' } (or { }) example: A’ or A
- …and the following axioms
  - **1. The set B contains at least two elements {a b} with a ≠ b**
  - **2. Closure: a+b is in B a•b is in B**
  - **3. Commutative: a+b = b+a a•b = b•a**
  - **4. Associative: a+(b+c) = (a+b)+c a•(b•c) = (a•b)•c**
  - **5. Identity: a+0 = a a•1 = a**
  - **6. Distributive: a+(b•c)=(a+b)•(a+c) a•(b+c)=(a•b)+(a•c)**
  - **7. Complementarity: a+a' = 1 a•a' = 0**

Laws of Boolean Algebra

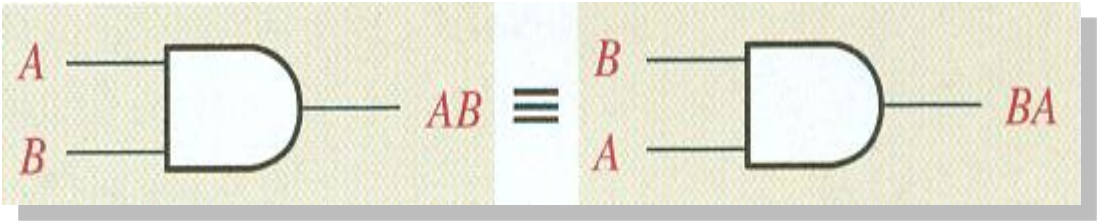
- Commutative Law of Addition:

  A + B = B + A



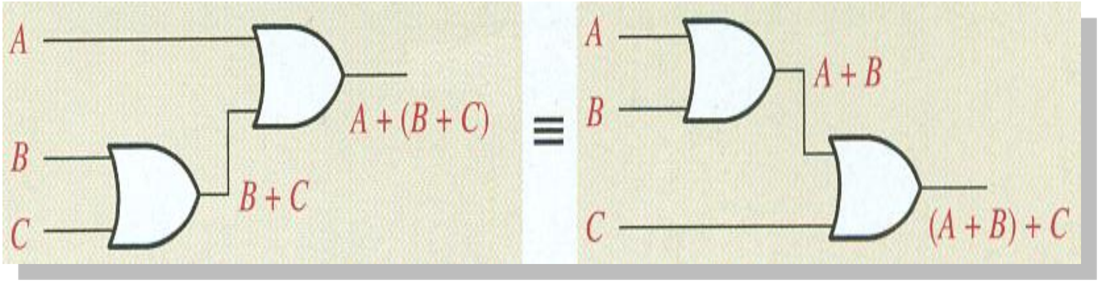
- Commutative Law of Multiplication:

  A * B = B * A



- Associative Law of Addition:

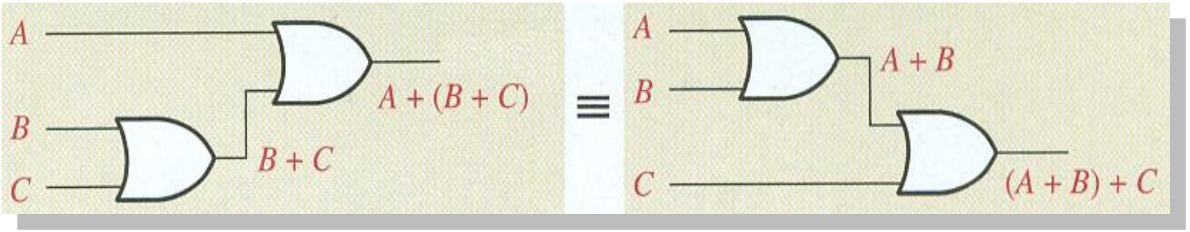
  A + (B + C) = (A + B) + C



Laws of Boolean Algebra

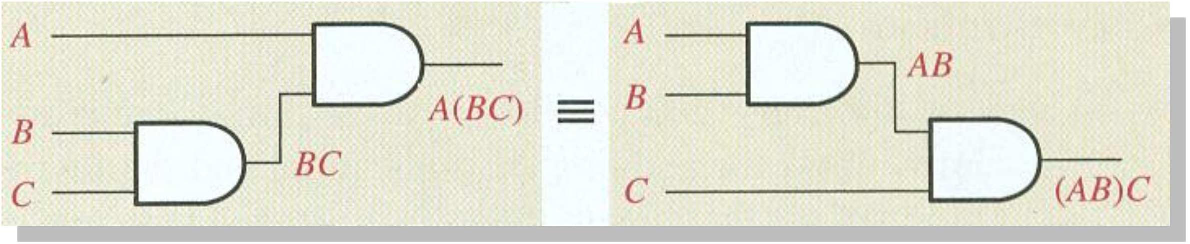
- Associative Law of Addition:

  A + (B + C) = (A + B) + C



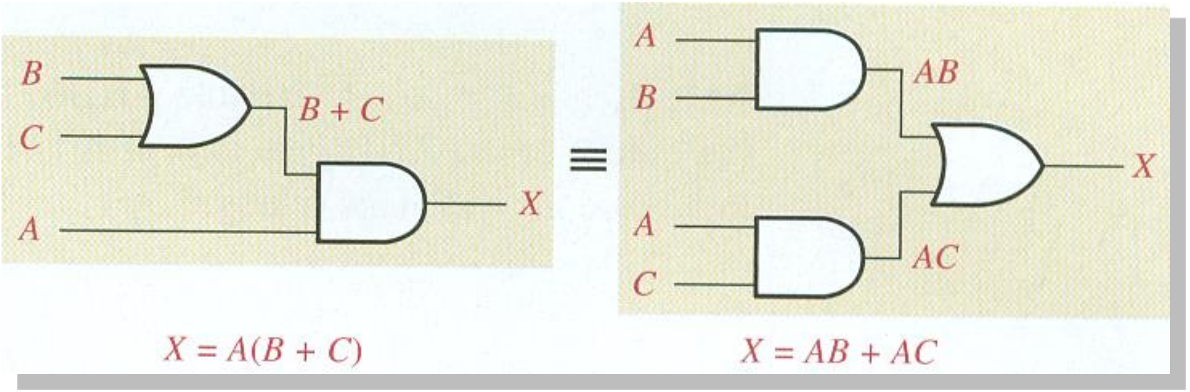
- Associative Law of Multiplication:

  A * (B * C) = (A * B) * C



- Distributive Law:

  A(B + C) = AB + AC

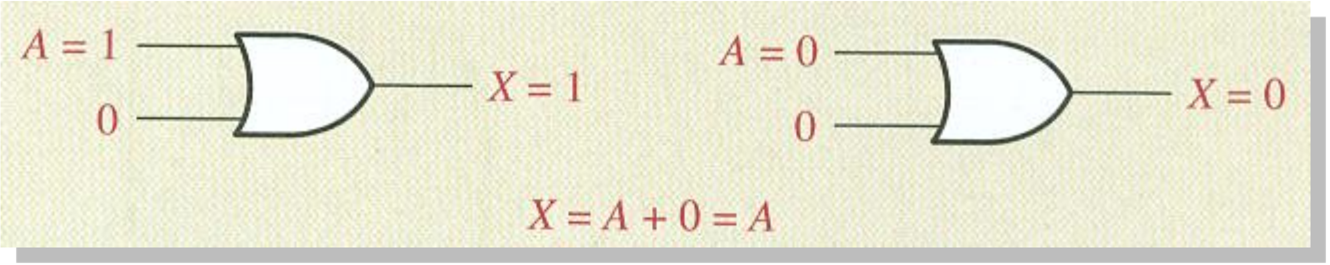


Rules of Boolean Algebra

1. $A + 0 = A$
2. $A + 1 = 1$
3. $A \cdot 0 = 0$
4. $A \cdot 1 = A$
5. $A + A = A$
6. $A + \overline{A} = 1$
7. $A \cdot A = A$
8. $A \cdot \overline{A} = 0$
9. $\overline{\overline{A}} = A$
10. $A + AB = A$
11. $A + \overline{A}B = A + B$
12. $(A + B)(A + C) = A + BC$

Rules of Boolean Algebra

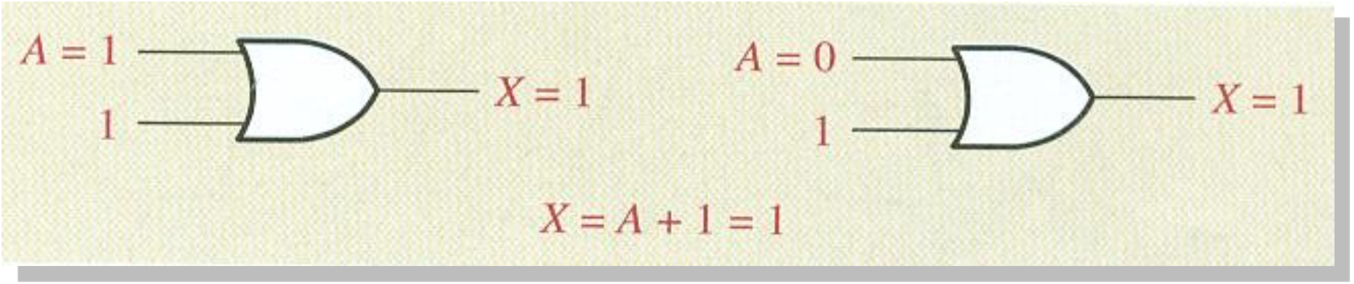
Rule 1



A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

OR Truth Table

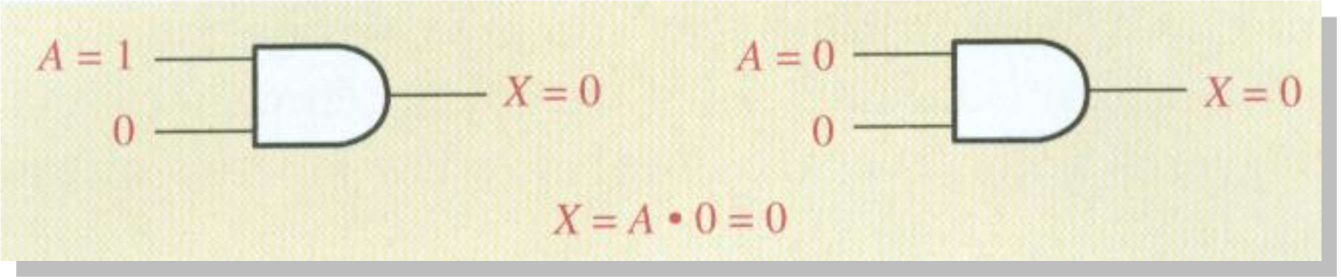
Rule 2



A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

OR Truth Table

Rule 3

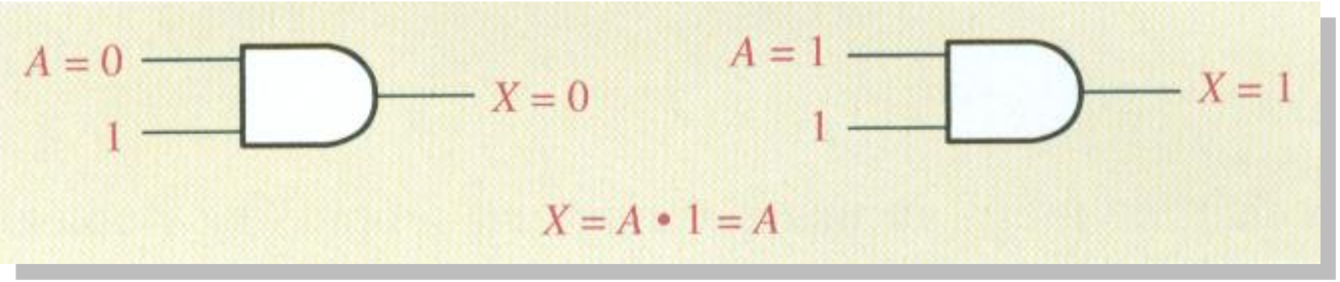


A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

Rules of Boolean Algebra

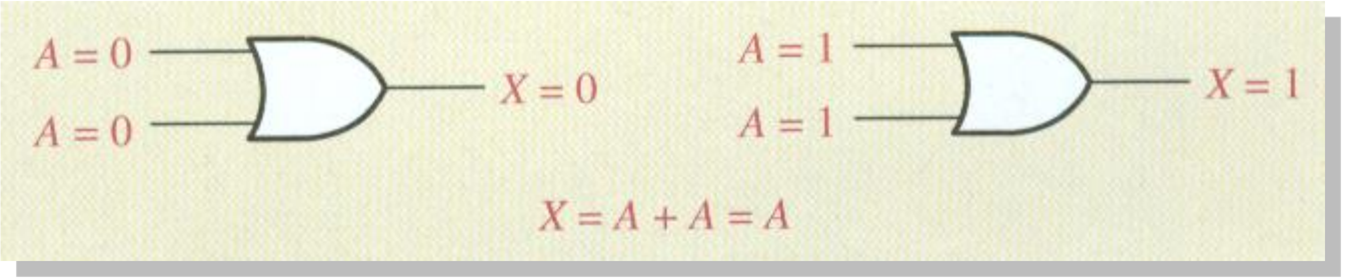
- Rule 4



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

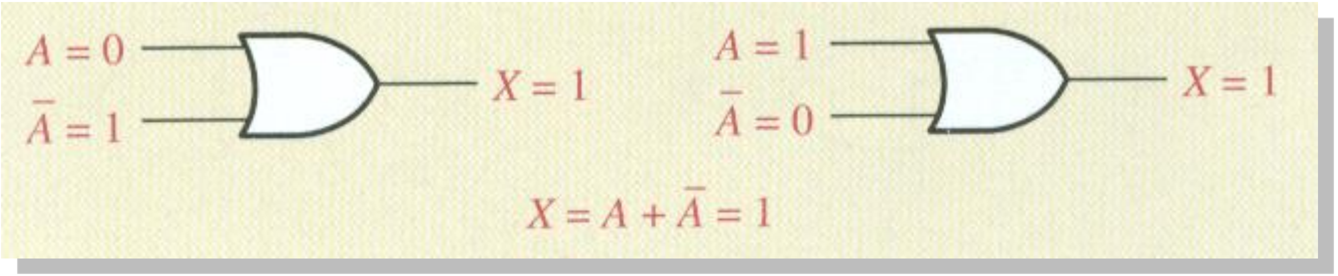
- Rule 5



A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

OR Truth Table

- Rule 6

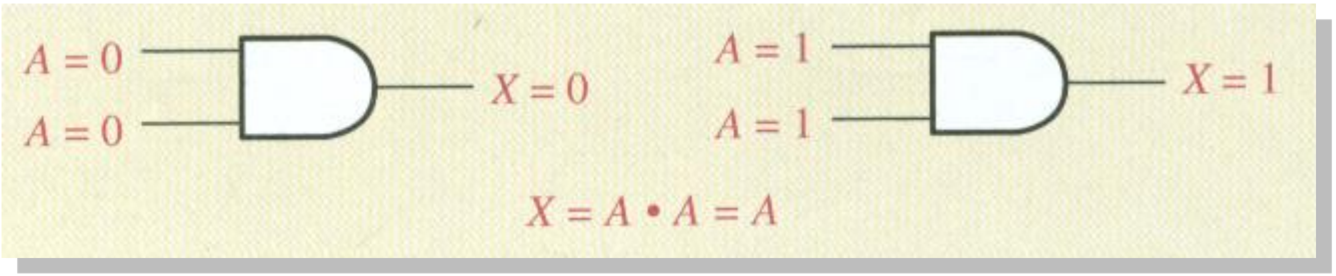


A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

OR Truth Table

Rules of Boolean Algebra

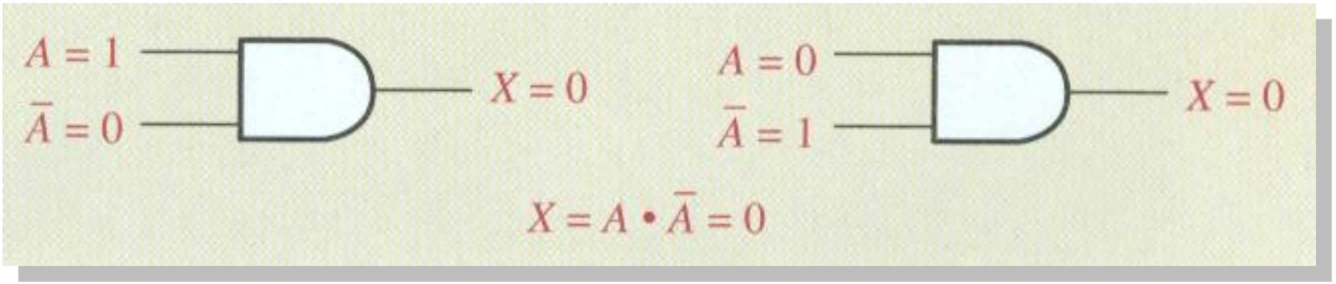
- Rule 7



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

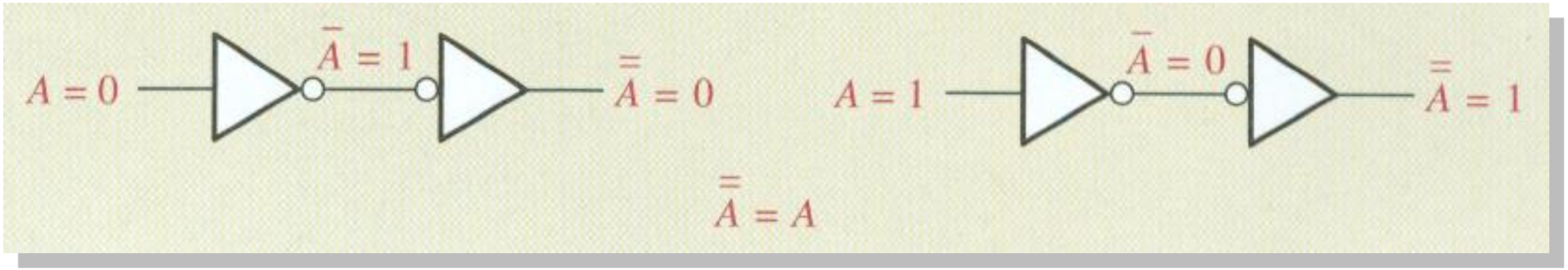
- Rule 8



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

- Rule 9

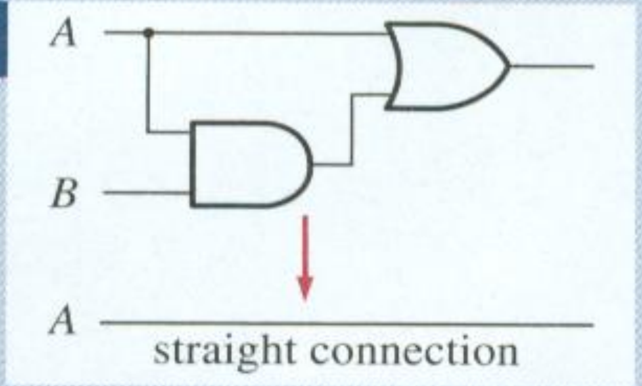


Rules of Boolean Algebra

- Rule 10: $A + AB = A$

A	B	AB	$A + AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

equal



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

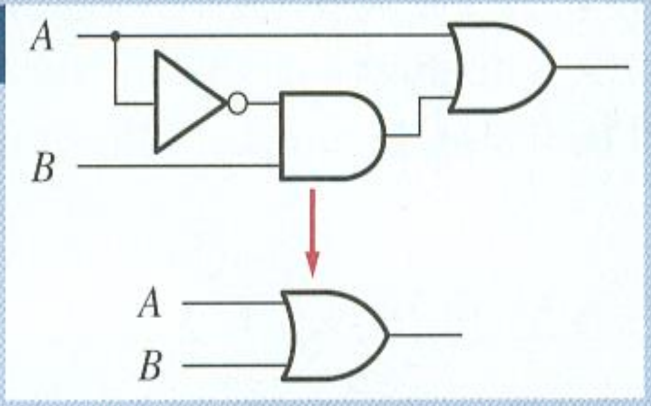
OR Truth Table

Rules of Boolean Algebra

- Rule 11: $A + \overline{A}B = A + B$

A	B	$\overline{A}B$	$A + \overline{A}B$	$A + B$
0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

equal



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

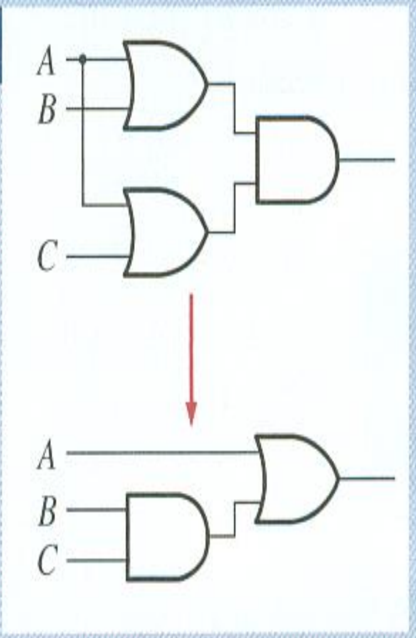
OR Truth Table

Rules of Boolean Algebra

- Rule 12: $(A + B)(A + C) = A + BC$

A	B	C	$A + B$	$A + C$	$(A + B)(A + C)$	BC	$A + BC$
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

equal



A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

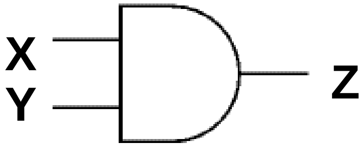
OR Truth Table

Digital (binary) logic is a Boolean algebra

- Substitute
  - {0, 1} for B
  - AND for • Boolean Product. In CSE 321 this was $\wedge$
  - OR for + Boolean Sum. In CSE 321 this was $\vee$
  - NOT for ‘ Complement. In CSE 321 this was $\neg$
- All the axioms hold for binary logic
- Definitions
  - Boolean function
    - Maps inputs from the set {0,1} to the set {0,1}
  - Boolean expression
    - An algebraic statement of Boolean variables and operators

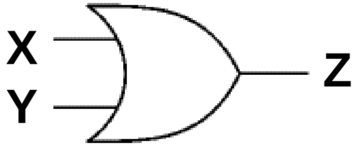
Logic Gates (AND, OR, Not) & Truth Table

- AND $X \cdot Y$ XY



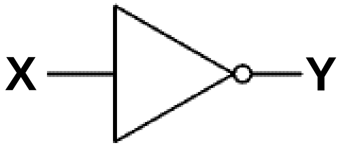
X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

- OR $X+Y$



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

- NOT $\overline{X}$ X'



X	Y
0	1
1	0

Logic functions and Boolean algebra

- *Any* logic function that is expressible as a truth table can be written in Boolean algebra using +, •, and '

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

Z=X•Y

X	Y	X'	Z
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	0

Z=X'•Y

X	Y	X'	Y'	X • Y	X' •Y'	Z
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

Z=(X•Y)+(X' •Y')

Some notation

- Priorities: $\overline{A} \bullet B + C = ((\overline{A}) \bullet B) + C$
- Variables and their complements are sometimes called literals

Two key concepts

- Duality (a meta-theorem— *a theorem about theorems*)
  - All Boolean expressions have logical duals
  - Any theorem that can be proved is also proved for its dual
  - Replace: $\cdot$ with +, + with $\cdot$, 0 with 1, and 1 with 0
  - Leave the variables unchanged

- de Morgan's Theorem
  - Procedure for complementing Boolean functions
  - Replace: $\cdot$ with +, + with $\cdot$, 0 with 1, and 1 with 0
  - Replace all variables with their complements

Useful laws and theorems

Identity:	$X + 0 = X$	Dual:	$X \cdot 1 = X$
Null:	$X + 1 = 1$	Dual:	$X \cdot 0 = 0$
Idempotent:	$X + X = X$	Dual:	$X \cdot X = X$
Involution:	$(X')' = X$		
Complementarity:	$X + X' = 1$	Dual:	$X \cdot X' = 0$
Commutative:	$X + Y = Y + X$	Dual:	$X \cdot Y = Y \cdot X$
Associative:	$(X+Y)+Z=X+(Y+Z)$	Dual:	$(X \cdot Y) \cdot Z=X \cdot (Y \cdot Z)$
Distributive:	$X \cdot (Y+Z)=(X \cdot Y)+(X \cdot Z)$	Dual:	$X+(Y \cdot Z)=(X+Y) \cdot (X+Z)$
Uniting:	$X \cdot Y+X \cdot Y'=X$	Dual:	$(X+Y) \cdot (X+Y')=X$

Useful laws and theorems (con't)

Absorption: $X+X\cdot Y=X$ Dual: $X\cdot(X+Y)=X$

Absorption (#2): $(X+Y')\cdot Y=X\cdot Y$ Dual: $(X\cdot Y')+Y=X+Y$

de Morgan's: $(X+Y+\ldots)'=X'\cdot Y'\cdot\ldots$ Dual: $(X\cdot Y\cdot\ldots)'=X'+Y'+\ldots$

Duality: $(X+Y+\ldots)^D=X\cdot Y\cdot\ldots$ Dual: $(X\cdot Y\cdot\ldots)^D=X+Y+\ldots$

Multiplying & factoring: $(X+Y)\cdot(X'+Z)=X\cdot Z+X'\cdot Y$

Dual: $X\cdot Y+X'\cdot Z=(X+Z)\cdot(X'+Y)$

Consensus: $(X\cdot Y)+(Y\cdot Z)+(X'\cdot Z)= X\cdot Y+X'\cdot Z$

Dual: $(X+Y)\cdot(Y+Z)\cdot(X'+Z)=(X+Y)\cdot(X'+Z)$

Proving theorems

- Example 1: Prove the uniting theorem-- $X \cdot Y + X \cdot Y' = X$

Distributive	$X \cdot Y + X \cdot Y' = X \cdot (Y + Y')$
Complementarity	$= X \cdot (1)$
Identity	$= X$

- Example 2: Prove the absorption theorem-- $X + X \cdot Y = X$

Identity	$X + X \cdot Y = (X \cdot 1) + (X \cdot Y)$
Distributive	$= X \cdot (1 + Y)$
Null	$= X \cdot (1)$
Identity	$= X$

Proving theorems

Example 3:

Prove the consensus theorem-- $(XY)+(YZ)+(X'Z)= XY+X'Z$

Complementarity $XY+YZ+X'Z = XY+(X+X')YZ + X'Z$

Distributive $= XYZ+XY+X'YZ+X'Z$

Use absorption {AB+A=A} with A=XY and B=Z

$= XY+X'YZ+X'Z$

Rearrange terms $= XY+X'ZY+X'Z$

Use absorption {AB+A=A} with A=X'Z and B=Y

$XY+YZ+X'Z = XY+X'Z$

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DeMorgan’s Theorems

Augustus DeMorgan

My name is Augustus DeMorgan. I'm an Englishman born in India in 1806. I was instrumental in the advancement of mathematics and am best known for the logic theorems that bear my name.

P.S. George Boolean gets WAY too much credit. He has more theorems, but mine are WAY Cooler! Take a look at them, OMG, they are the bomb.

DeMorgan's Theorems

DeMorgan's Theorems are two additional simplification techniques that can be used to simplify Boolean expressions. Again, the simpler the Boolean expression, the simpler the resulting logic.

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

- Theorem 1

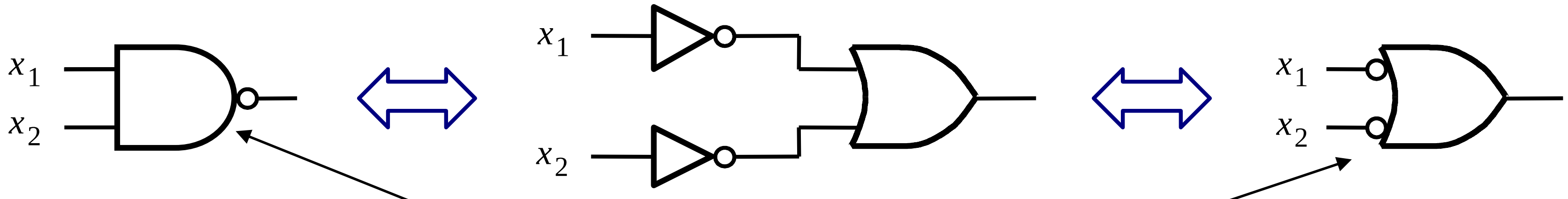
$$\overline{XY} = \overline{X} + \overline{Y}$$

- Theorem 2

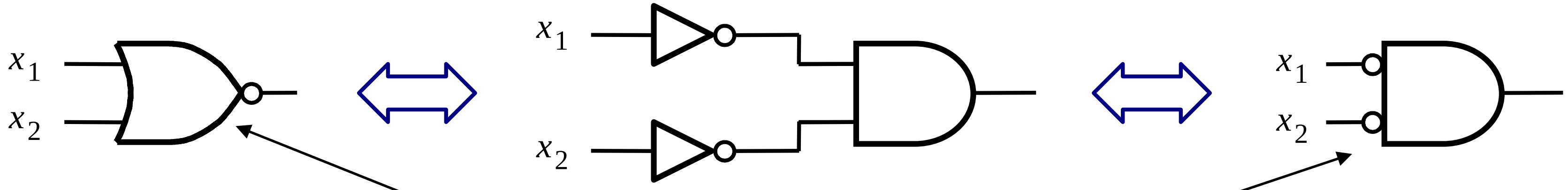
$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

Remember: "Break the bar, change the sign"

DeMorgan's Theorems



(a) $\overline{x_1 x_2} = \bar{x}_1 + \bar{x}_2$



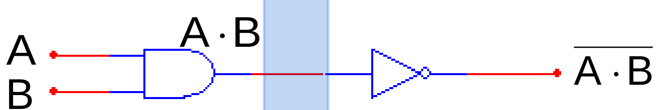
(b) $\overline{x_1 + x_2} = \bar{x}_1 \bar{x}_2$

DeMorgan's Theorem #1

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

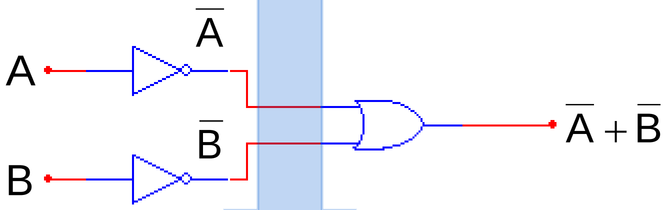
Proof

$$\overline{A \cdot B}$$



A	B	$A \cdot B$	$\overline{A \cdot B}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

$$\overline{A} + \overline{B}$$



A	B	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

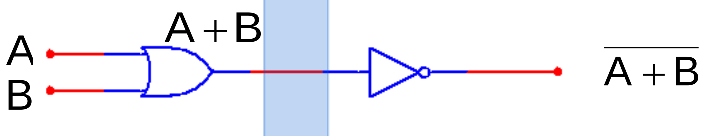
The truth-tables are equal; therefore, the Boolean equations must be equal.

DeMorgan's Theorem #2

$$\overline{A+B} = \overline{A} \cdot \overline{B}$$

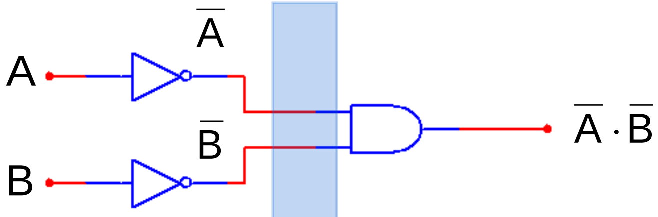
Proof

$\overline{A+B}$



A	B	$A+B$	$\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

$\overline{A} \cdot \overline{B}$



A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cdot \overline{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

The truth-tables are equal; therefore,
the Boolean equations must be equal.

Summary

Boolean & DeMorgan's Theorems

1) $X \cdot 0 = 0$
2) $X \cdot 1 = X$
3) $X \cdot X = X$
4) $X \cdot \overline{X} = 0$
5) $X + 0 = X$
6) $X + 1 = 1$
7) $X + X = X$
8) $X + \overline{X} = 1$
9) $\overline{\overline{X}} = X$

10A) $X \cdot Y = Y \cdot X$
10B) $X + Y = Y + X$

Commutative Law

11A) $X(YZ) = (XY)Z$
11B) $X + (Y + Z) = (X + Y) + Z$

Associative Law

12A) $X(Y + Z) = XY + XZ$
12B) $(X + Y)(W + Z) = XW + XZ + YW + YZ$

Distributive Law

13A) $X + \overline{X}Y = X + Y$
13B) $\overline{X} + XY = \overline{X} + Y$
13C) $X + \overline{X}\,\overline{Y} = X + \overline{Y}$
13D) $\overline{X} + X\overline{Y} = \overline{X} + \overline{Y}$

Consensus Theorem

14A) $\overline{X\,Y} = \overline{X} + \overline{Y}$
14B) $\overline{X + Y} = \overline{X}\ \overline{Y}$

DeMorgan's

DeMorgan Shortcut

BREAK THE LINE, CHANGE THE SIGN

Break the *LINE* over the two variables,
and change the *SIGN* directly under the line.

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

For Theorem #14A, break the line, and change the AND function to an OR function. Be sure to keep the lines over the variables.

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

For Theorem #14B, break the line, and change the OR function to an AND function. Be sure to keep the lines over the variables.

de Morgan's Theorem

- Use de Morgan's Theorem to find complements
- Example: F=(A+B)•(A'+C), so F'=(A'•B')+(A•C')

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A	B	C	F'
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

One more example of logic simplification

- Example:

$$Z = A'BC + AB'C' + AB'C + ABC' + ABC$$

$= A'BC + AB'(C' + C) + AB(C' + C)$	distributive
$= A'BC + AB' + AB$	complementary
$= A'BC + A(B' + B)$	distributive
$= A'BC + A$	complementary
$= BC + A$	absorption #2 Duality

$(X \cdot Y')+Y=X+Y$ *with* $X=BC$ *and* $Y=A$

Reference

http://www.ee.surrey.ac.uk/Projects/CAL/digital-logic/gatesfunc/