Jiangxi University of Science and Technology

DIGITAL DESIGN

Lecture 7:SOP and POS Standard Forms of Boolean Expressions

Summary

- Realizing truth tables
- SOP /POS
- Minterm/ Maxterm

Realizing truth tables

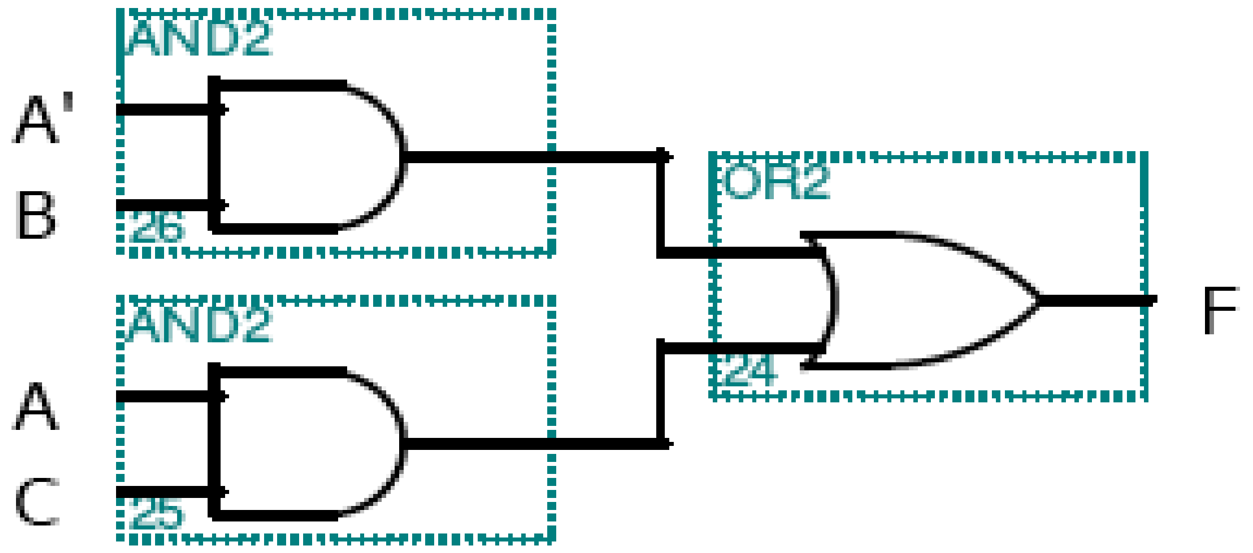
- Given a truth table

1. Write the Boolean expression
2. Minimize the Boolean expression
3. Draw as gates

Example

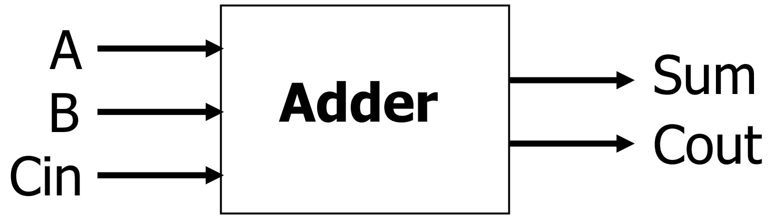
$$F = A'BC' + A'BC + AB'C + ABC$$
$$= A'B(C'+C) + AC(B'+B)$$
$$= A'B + AC$$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



Example: Binary full adder

- 1-bit binary adder
  - Inputs: A, B, Carry-in
  - Outputs: Sum, Carry-out



A	B	Cin	Cout	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Sum = A'B'Cin + A'BCin' + AB'Cin' + ABCin

Cout = A'BCin + AB'Cin + ABCin' + ABCin

Both Sum and Cout can be minimized.

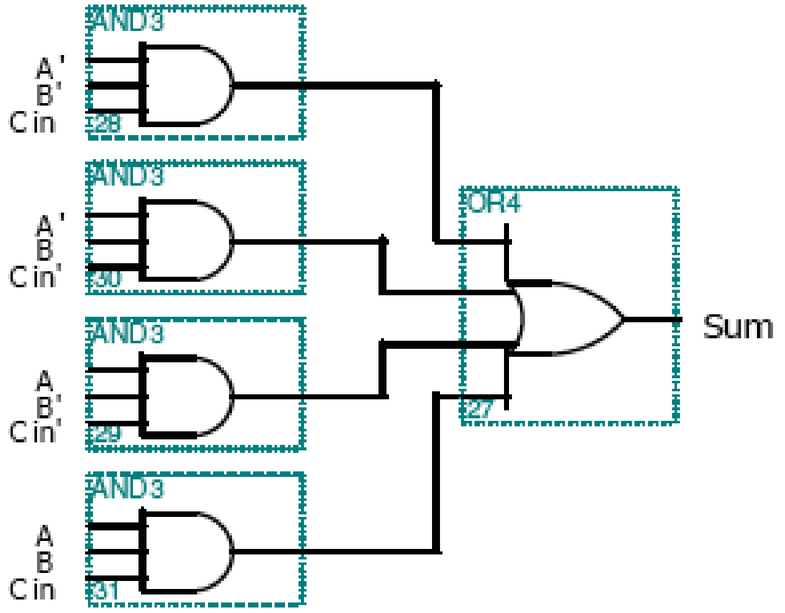
Full adder: Sum

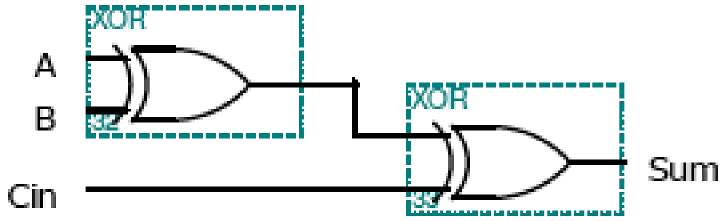
Before Boolean minimization

Sum = A'B'Cin + A'BCin' + AB'Cin' + ABCin

After Boolean minimization

Sum = (A⊕B) ⊕ Cin





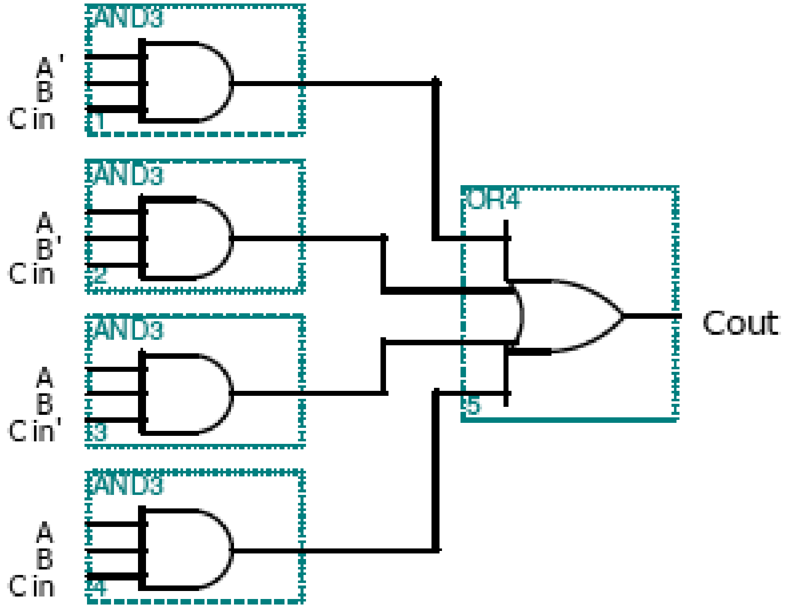
Full adder: Carry-out

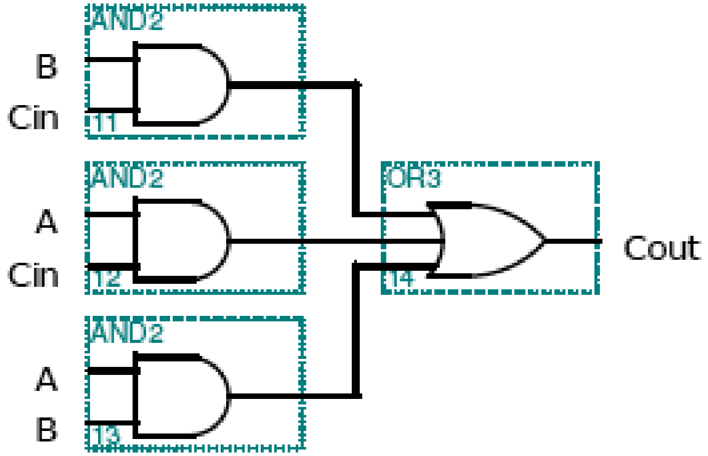
Before Boolean minimization

Cout = A'BCin + AB'Cin
+ ABCin' + ABCin

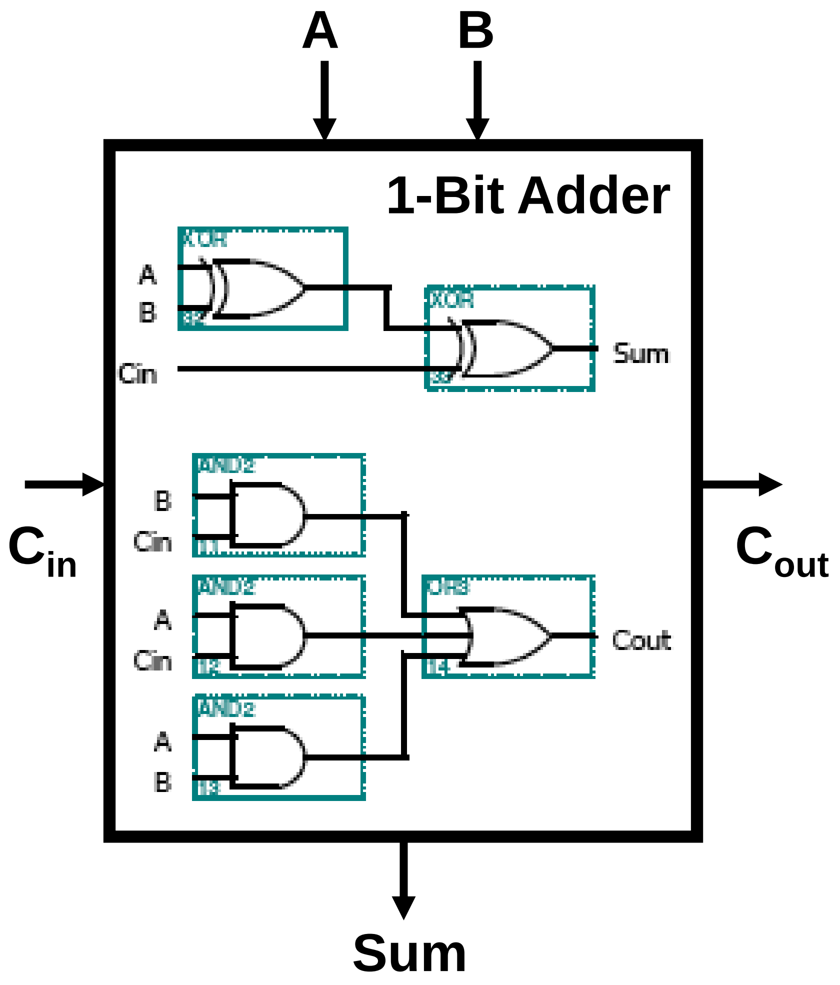
After Boolean minimization

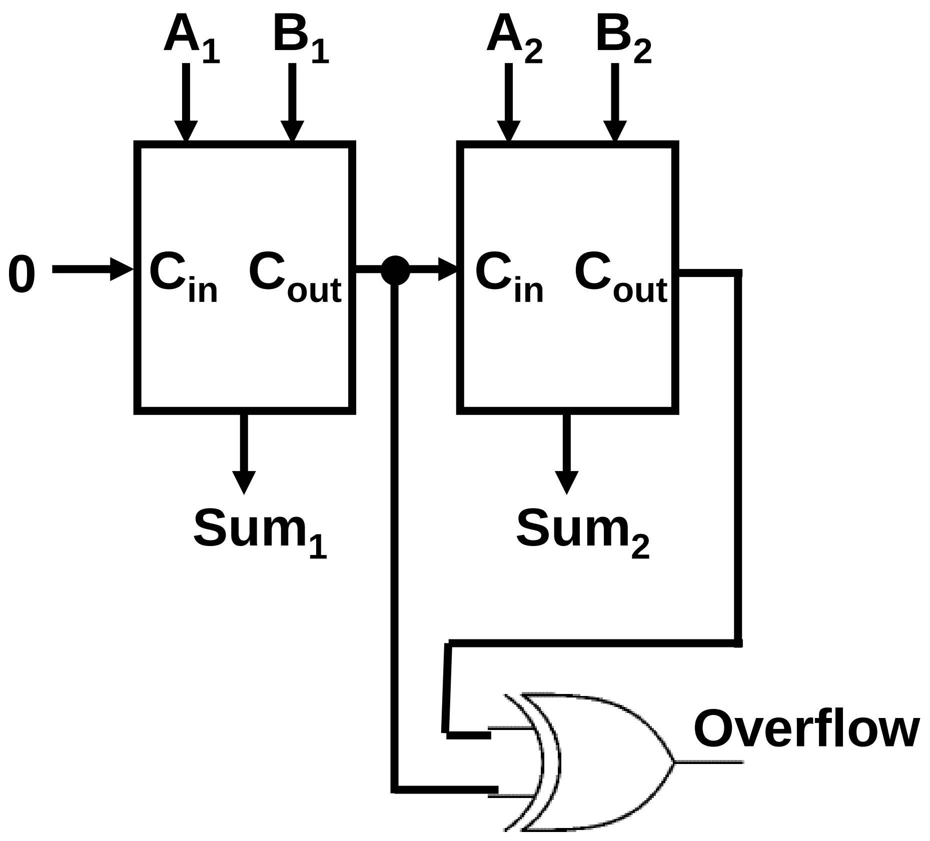
Cout = BCin + ACin + AB





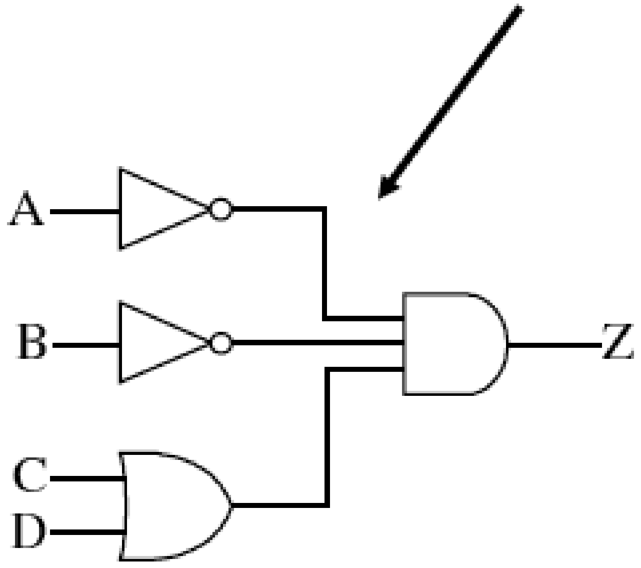
Preview: 2-bit ripple-carry adder

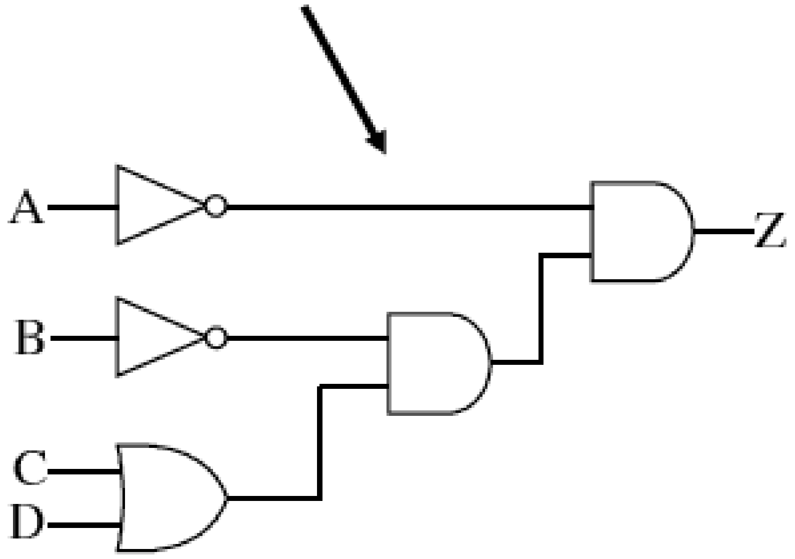




Many possible mappings

- Many ways to map expressions to gates
- Example: $Z = \overline{A} \bullet \overline{B} \bullet (C+D) = \overline{A} \bullet (\overline{B} \bullet (C+D))$





What is the optimal realization?

- We use the axioms and theorems of Boolean algebra to "optimize" our designs
- Design goals vary
  - Reduce the number of gates?
  - Reduce the number of gate inputs?
  - Reduce the number of cascaded levels of gates?
- How do we explore the tradeoffs?
  - Logic minimization: Reduce number of gates and complexity
  - Logic optimization: Maximize speed and/or minimize power
  - CAD tools

Canonical forms

- Canonical forms
  - Standard forms for Boolean expressions
  - Derived from truth table
  - Generally not the simplest forms (can be minimized)
- Two canonical forms
  - Sum-of-products (minterms)
  - Product-of-sums (maxterms)

Representation of Boolean expression

Boolean expression can be represented by either

(i)Sum of Product(SOP) form or

(ii)Product of Sum (POS form)

e.g.

AB+AC → SOP
(A+B)(A+C) → POS

In above examples both are in SOP and POS respectively but they are not in Standard SOP and POS.

Canonical form of Boolean Expression (Standard form)

- In standard SOP and POS each term of Boolean expression must contain all the literals (with and without bar) that has been used in Boolean expression.
- If the above condition is satisfied by the Boolean expression, that expression is called Canonical form of Boolean expression.

- In Boolean expression **AB+AC** the literal C is mission in the 1st term **AB** and B is mission in 2nd term **AC**. That is why AB+AC is not a Canonical SOP.

SOP and POS

Sum of Products (SOP)

A boolean expression consisting purely of Minterms (product terms) is said to be in canonical sum of products form.

Product of Sums (POS)

A boolean expression consisting purely of Maxterms (sum terms) is said to be in canonical product of sums form.

Example SOP

lets say, we have a Boolean function F defined on two variables A and B. So, A and B are the inputs for F and lets say, output of F is true i.e., F = 1 when any one of the input is true or 1. Now we draw the truth table for F.

A	B	F	Minterm
0	0	0	A'B'
0	1	1	A'B
1	0	1	AB'
1	1	1	AB

Now we will create a column for the minterm using the variables A and B. If input is 0 we take the complement of the variable and if input is 1 we take the variable as is.

To get the desired canonical SOP expression we will add the **minterms** (product terms) for which the **output is 1**.

F = A’B + AB’ + AB

Canonical form of Boolean Expression (Standard form) Contd..

Convert (A+B)(A+C) in Canonical SOP (Standard SOP)

Sol. (A+B).(A+C)

(A+B)+(C.C’) . (A+C)+(B.B’)

(A+B+C).(A+B+C’).(A+B+C)(A+B’+C)

(A+B+C).(A+B+C’)(A+B’+C)

Distributive law

Remove duplicates

Example POS

Example
Lets say, we have a boolean function F defined on two variables A and B. So, A and B are the inputs for F and lets say, output of F is true i.e., F = 1 when only one of the input is true or 1.now we draw the truth table for F

A	B	F
0	0	0
0	1	1
1	0	1
1	1	0

Now we will create a column for the maxterm using the variables A and B. If input is 1 we take the complement of the variable and if input is 0 we take the variable as is.

A	B	F	Maxterm
0	0	0	$A + B$
0	1	1	$A + B'$
1	0	1	$A' + B$
1	1	0	$A' + B'$

To get the desired canonical POS expression we will multiply the maxterms (sum terms) for which the output is 0.

$F = (A+B) . (A'+B')$

Canonical form of Boolean Expression (Standard form) Contd..

Minterm and Maxterm

- Individual term of Canonical Sum of Products (SOP) is called Minterm. In otherwords minterm is a product of all the literals (with or without bar) within the Boolean expression.

- Individual term of Canonical Products of Sum (POS) is called Maxterm. In otherwords maxterm is a sum of all the literals (with or without bar) within the Boolean expression.

Minterms & Maxterms for 2 variables (Derivation of Boolean function from Truth Table)

x	y	Index	Minterm	Maxterm
0	0	0	$m_0 = x' y'$	$M_0 = x + y$
0	1	1	$m_1 = x' y$	$M_1 = x + y'$
1	0	2	$m_2 = x y'$	$M_2 = x' + y$
1	1	3	$m_3 = x y$	$M_3 = x' + y'$

The minterm m_i should evaluate to 1 for each combination of x and y.
The maxterm is the complement of the minterm

Minterms & Maxterms for 3 variables

x	y	z	Index	Minterm	Maxterm
0	0	0	0	$m_0 = \bar{x}\,\bar{y}\,\bar{z}$	$M_0 = x + y + z$
0	0	1	1	$m_1 = \bar{x}\,\bar{y}\,z$	$M_1 = x + y + \bar{z}$
0	1	0	2	$m_2 = \bar{x}\,y\,\bar{z}$	$M_2 = x + \bar{y} + z$
0	1	1	3	$m_3 = \bar{x}\,y\,z$	$M_3 = x + \bar{y} + \bar{z}$
1	0	0	4	$m_4 = x\,\bar{y}\,\bar{z}$	$M_4 = \bar{x} + y + z$
1	0	1	5	$m_5 = x\,\bar{y}\,z$	$M_5 = \bar{x} + y + \bar{z}$
1	1	0	6	$m_6 = x\,y\,\bar{z}$	$M_6 = \bar{x} + \bar{y} + z$
1	1	1	7	$m_7 = x\,y\,z$	$M_7 = \bar{x} + \bar{y} + \bar{z}$

Maxterm M_i is the complement of minterm m_i

$M_i = \overline{m_i}$ and $m_i = \overline{M_i}$

Solved Problem

Prob. Find the minterm designation of XY’Z’

Sol. Subsitute 1’s for non barred and 0’s for barred letters

Binary equivalent = 100

Decimal equivalent = 4

Thus XY’Z’=m_4

Purpose of the Index

- Minterms and Maxterms are designated with an index
- The index number corresponds to a binary pattern
- The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - ‘1’ means the variable is “Not Complemented” and
  - ‘0’ means the variable is “Complemented”.
- For Maxterms:
  - ‘0’ means the variable is “Not Complemented” and
  - ‘1’ means the variable is “Complemented”.

Solved Problem

Write SOP form of a Boolean Function F, Which is represented by the following truth table.

Sum of minterms of entries that evaluate to '**1**'

x	y	z	F	Minterm
0	0	0	0	
0	0	1	**1**	$m_1 = x' y' z$
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	**1**	$m_6 = x\, y\, z'$
1	1	1	**1**	$m_7 = x\, y\, z$

Focus on the '**1**' entries

$$F = m_1 + m_6 + m_7 = \sum(1, 6, 7) = \bar{x}\,\bar{y}\,z + x\,y\,\bar{z} + x\,y\,z$$

Exercise:Home work

1. Write POS form of a Boolean Function F, Which is represented by the following truth table

x	y	z	F
0	0	0	1
0	0	1	**1**
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	**0**
1	1	1	**1**

2. Write equivalent canonical Sum of Product expression for the following Product of Sum Expression: F(X,Y,Z)=Π(1,3,6,7)

– – –

Summery Minimization of Boolean Expression

- Canonical SOP (Sum of Minterms) and POS (Product of Maxterm) is the derivation/expansion of Boolean Expression.

- Canonical forms are not usually minimal.

- Minimization of Boolean expression is needed to simplify the Boolean expression and thus reduce the circuitry complexity as it uses less number of gates to produce same output that can by taken by long canonical expression.

Minimization of Boolean Expression (Contd...)

➢ **Two method can by applied to reduce the Boolean expression –**

i) Algebraic
ii) Using Karnaugh Map (K-Map).

➢ Algebraic Method

- The different Boolean rules and theorems are used to simplify the Boolean expression in this method.

Minimization of Boolean Expression (Contd...)

Solved Problem

Minimize the following Boolean Expression:

1. a'bc + ab'c' + ab'c + abc' +abc

2. AB'CD' + AB'CD + ABCD' + ABCD

Minimization of Boolean Expression (Contd...)

Exercise

A. Minimize the following Boolean Expression:

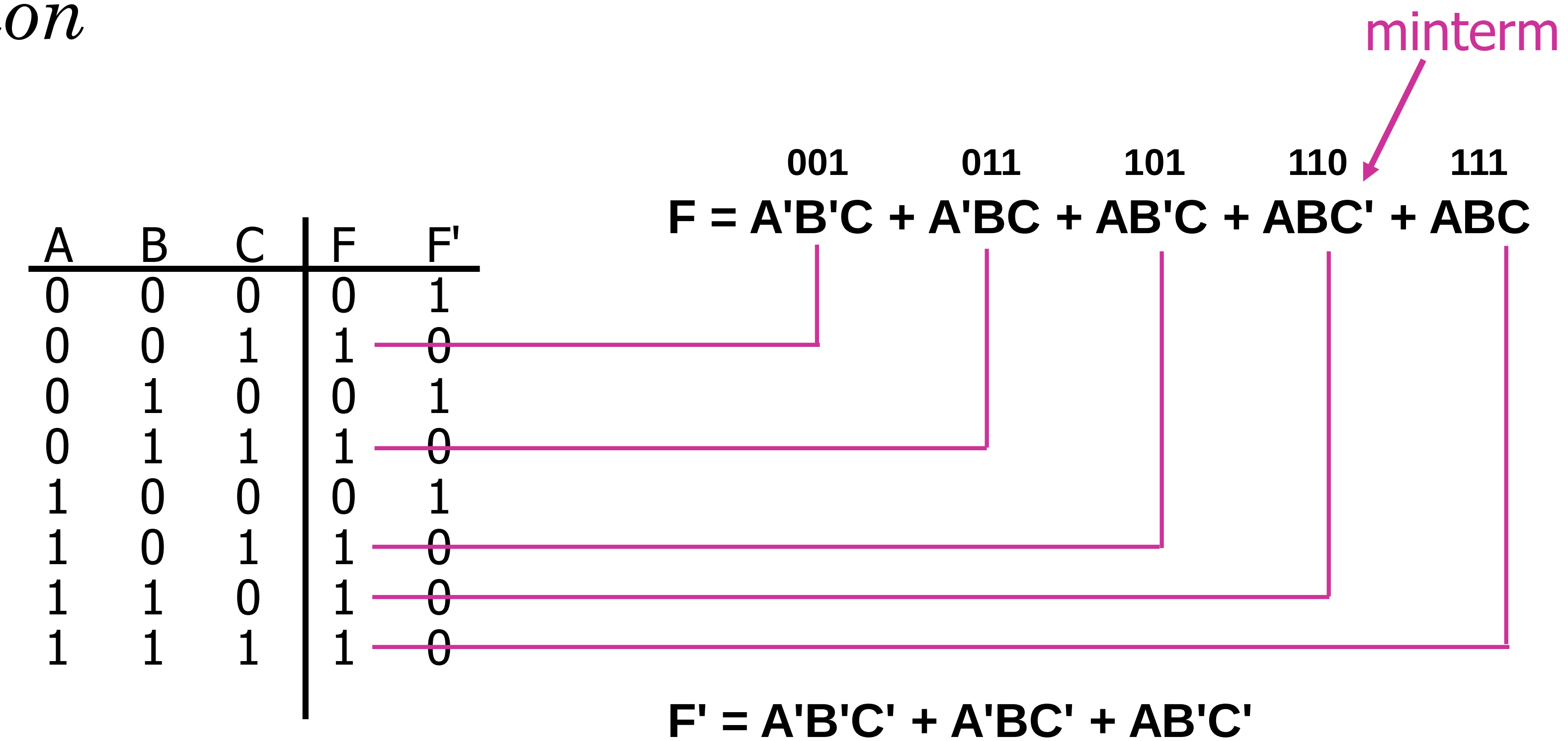
1. **X’Y’Z’ + X’YZ’ + XY’Z’ + XYZ’**
2. **a(b + b’c + b’c’)**

B. Prove algebraically that

1. **(x+y+z)(x’+y+z)=y+z**
2. **A+A’B’=A+B’**

Sum-of-products (SOP)

- Also called *disjunctive normal form* (DNF) or *minterm expansion*



A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F = A'B'C + A'BC + AB'C + ABC' + ABC

F' = A'B'C' + A'BC' + AB'C'

Minterms

- Variables appear exactly once in each minterm in true or inverted form (but not both)

A	B	C	minterms	
0	0	0	A'B'C'	m0
0	0	1	A'B'C	m1
0	1	0	A'BC'	m2
0	1	1	A'BC	m3
1	0	0	AB'C'	m4
1	0	1	AB'C	m5
1	1	0	ABC'	m6
1	1	1	ABC	m7

short-hand notation

F in canonical form:

$$\begin{aligned} F(A,B,C) &= \Sigma m(1,3,5,6,7) \\ &= m1 + m3 + m5 + m6 + m7 \\ &= A'B'C + A'BC + AB'C + ABC' + ABC \end{aligned}$$

Product-of-sums (POS)

- Also called *conjunctive normal form* (CNF) or *maxterm expansion*

A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

$$F = \overset{000}{(A + B + C)}\ \overset{010}{(A + B' + C)}\ \overset{100}{(A' + B + C)}$$

maxterm

$$F' = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C)(A'+B'+C')$$

Maxterms

- Variables appear exactly once in each maxterm in true or inverted form (but not both)

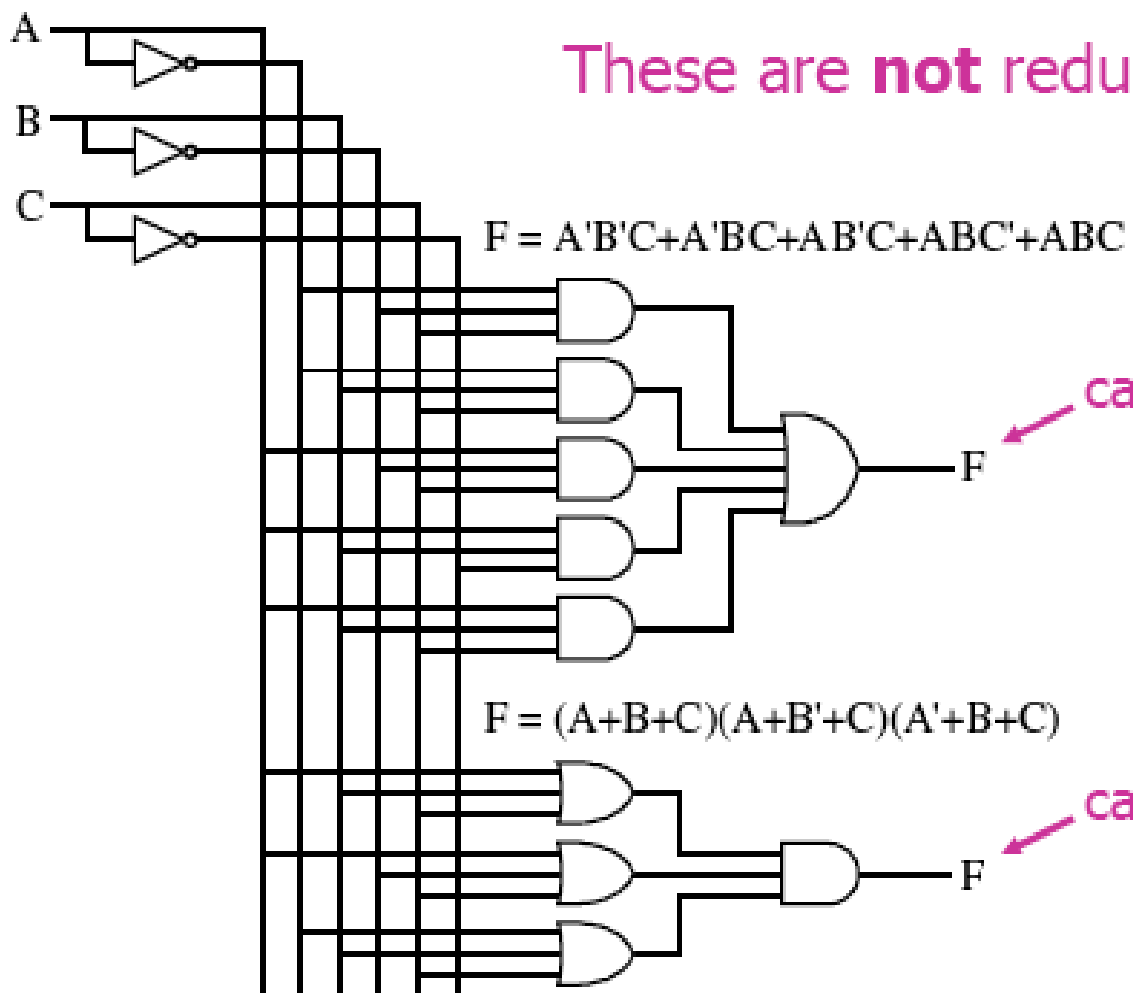
A	B	C	maxterms	
0	0	0	A+B+C	M0
0	0	1	A+B+C'	M1
0	1	0	A+B'+C	M2
0	1	1	A+B'+C'	M3
1	0	0	A'+B+C	M4
1	0	1	A'+B+C'	M5
1	1	0	A'+B'+C	M6
1	1	1	A'+B'+C'	M7

short-hand notation

F in canonical form:

$$\begin{aligned} F(A,B,C) &= \Pi M(0,2,4) \\ &= M0 \bullet M2 \bullet M4 \\ &= (A+B+C)(A+B'+C)(A'+B+C) \end{aligned}$$

Example: F = AB+C



These are **not** reduced forms for F

canonical sum-of-products

canonical product-of-sums

From SOP to POS and back

- Minterm to maxterm
  - Use maxterms that aren't in minterm expansion
  - $F(A,B,C) = \sum m(1,3,5,6,7) = \prod M(0,2,4)$
- Maxterm to minterm
  - Use minterms that aren't in maxterm expansion
  - $F(A,B,C) = \prod M(0,2,4) = \sum m(1,3,5,6,7)$

From SOP to POS and back

- Minterm of F to minterm of F'
  - Use minterms that don't appear
  - $F(A,B,C) = \sum m(1,3,5,6,7)$ $F' = \sum m(0,2,4)$
- Maxterm of F to maxterm of F'
  - Use maxterms that don't appear
  - $F(A,B,C) = \prod M(0,2,4)$ $F' = \prod M(1,3,5,6,7)$

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Converting Product of Sums (POS) to shorthand notation

From the previous example we have F = (A+B) . (A'+B')

Now, lets say we want to express the POS using shorthand notation. we have F = (A+B) . (A'+B')

$A+B = (00)_2 = M_0$
$A'+B' = (11)_2 = M_3$

Now we express F using shorthand notation.

$F = M_0 \, . \, M_3$

This can also be written as $F = \prod(0, 3)$

We saw the conversion of POS to shorthand notation. Lets check the conversion of shorthand notation to POS.

Converting shorthand notation to Product of Sums (POS)

Lets say, we have a boolean function F defined on two variables A and B so, A and B are the inputs for F and lets say, the maxterm are expressed as shorthand notation given below.

$F = \prod(1, 2, 3)$

Our task is to get the POS.

F has two input variables A and B and output of F = 0 for M_1, M_2 and M_3 i.e., 2nd, 3rd and 4th combination.

we have, $F = \prod(1, 2, 3)$
$= M_1 . M_2 . M_3$
$= 01 . 10 . 11$

To convert from shorthand notation to POS we follow the given rules. If the variable is 0 then it is taken as is and if the variable is 1 then we take its complement.

we have, $F = \prod(1, 2, 3) = (A+B') . (A'+B) . (A'+B')$

And we have the required POS.

SOP, POS, and DeMorgan's

- Sum-of-products
  - F' = A'B'C' + A'BC' + AB'C'
- Apply DeMorgan's to get POS
  - (F')' = (A'B'C' + A'BC' + AB'C')'
  - F = (A+B+C)(A+B'+C)(A'+B+C)

- Product-of-sums
  - F' = (A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C')
- Apply DeMorgan's to get SOP
  - (F')' = ((A+B+C')(A+B'+C')(A'+B+C')(A'+B'+C'))'
  - F = A'B'C + A'BC + AB'C + ABC

Standard Forms of Boolean Expressions

- The sum-of-product (SOP) form
  - Example: X = AB + CD + EF

- The product of sum (POS) form
  - Example: X = (A + B)(C + D)(E + F)

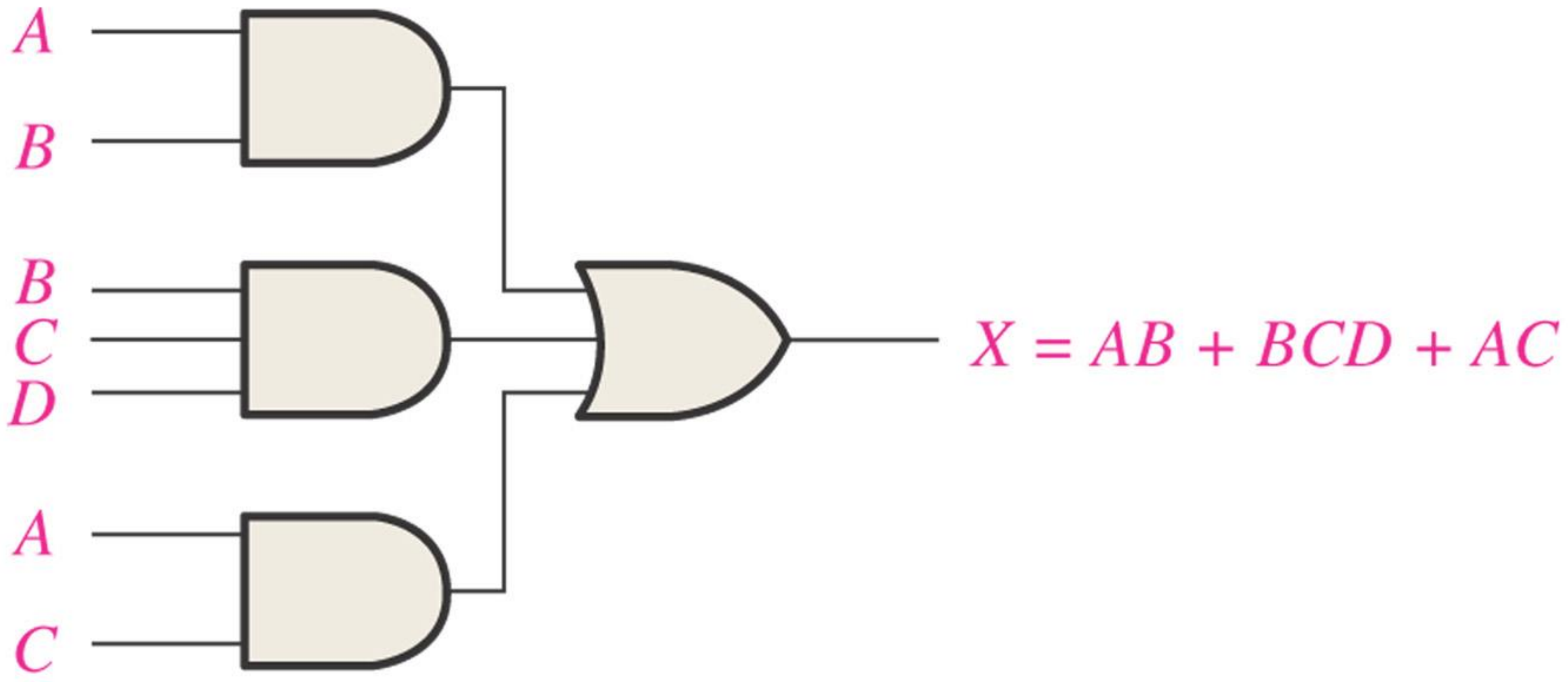


Figure 4–18
Implementation of the SOP expression AB + BCD + AC.

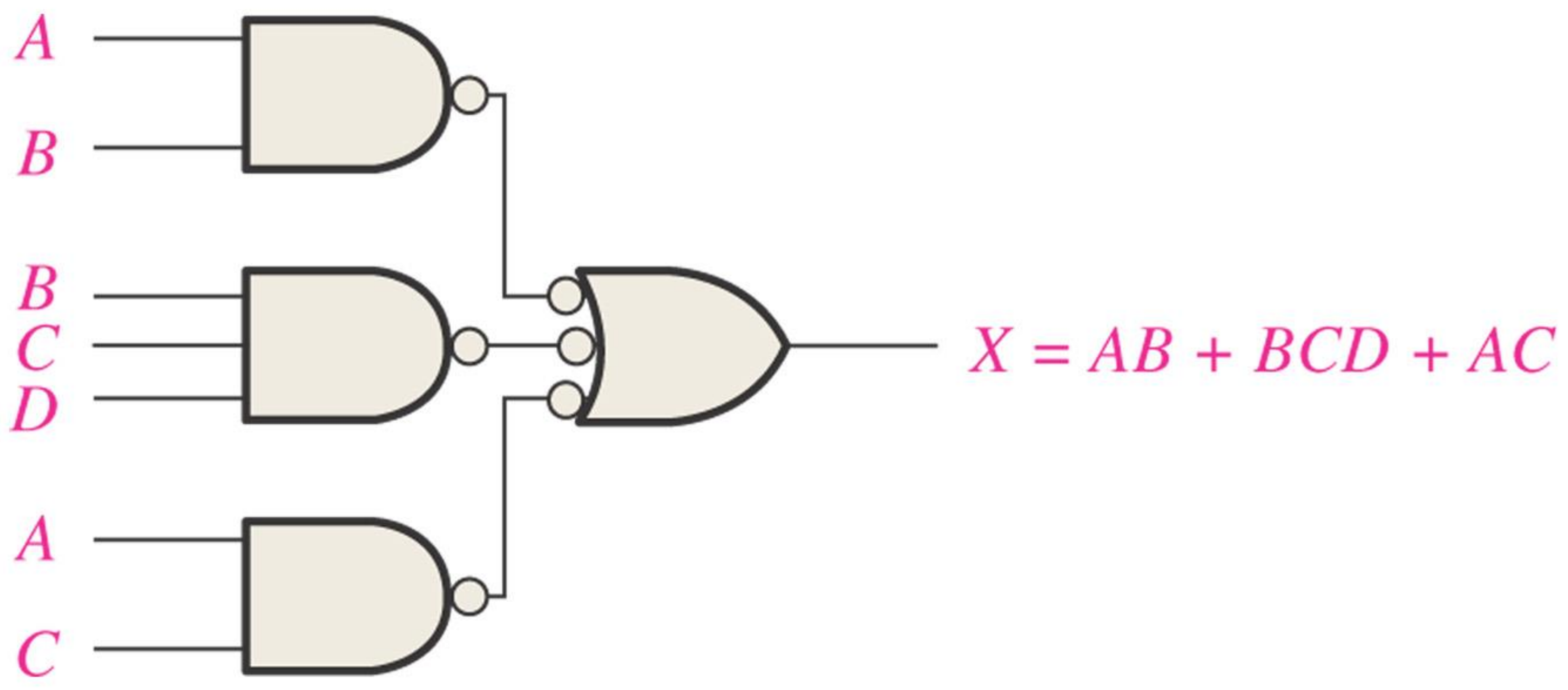


Figure 4–19
This NAND/NAND implementation is equivalent to the AND/OR in Figure 4–18.

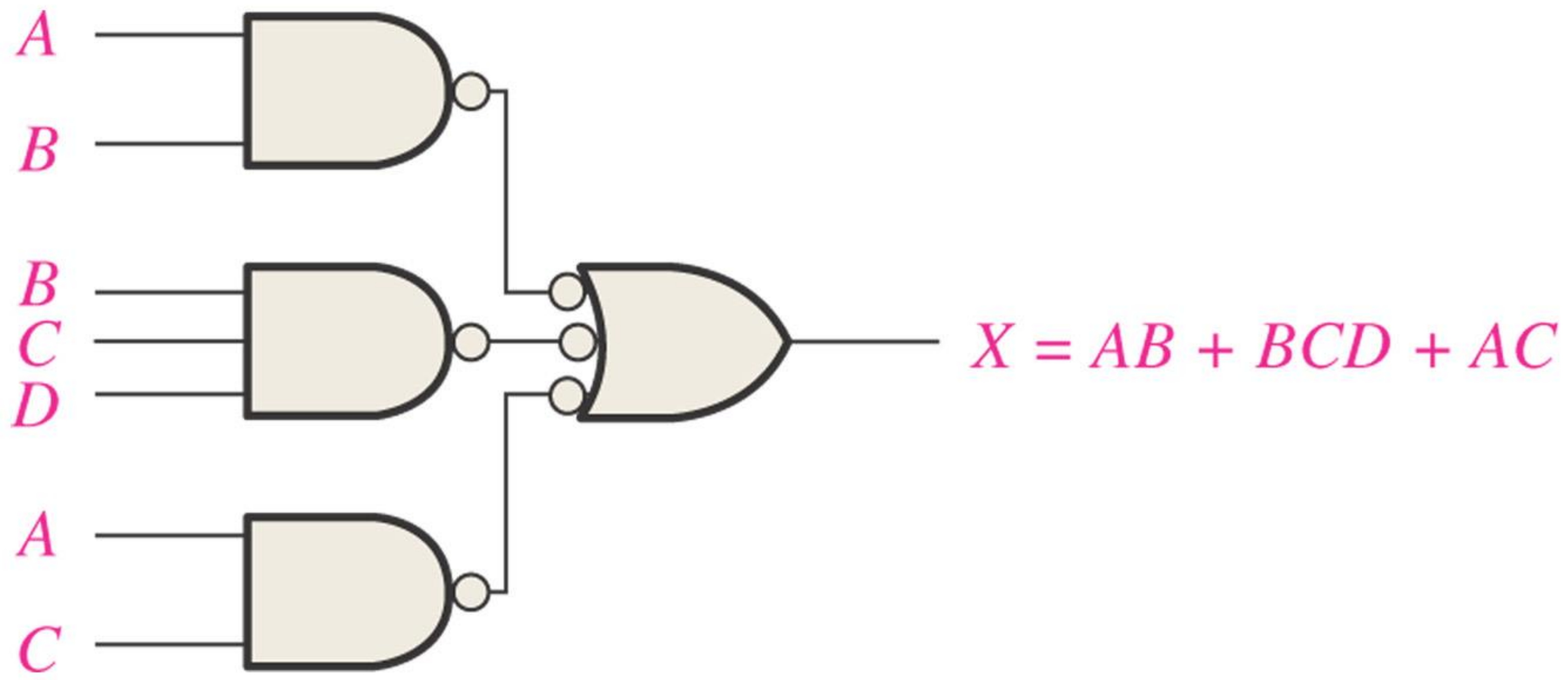


Figure 4–19
This NAND/NAND implementation is equivalent to the AND/OR in Figure 4–18.

DeMorgan’s: Example #1

Example

Simplify the following Boolean expression and note the Boolean or DeMorgan’s theorem used at each step. Put the answer in SOP form.

$$F_1 = \overline{(\overline{X \cdot \overline{Y}}) \cdot (\overline{Y} + Z)}$$

DeMorgan's: Example #1

Example

Simplify the following Boolean expression and note the Boolean or DeMorgan's theorem used at each step. Put the answer in SOP form.

$$F_1 = \overline{\overline{(X \cdot \overline{\overline{Y}})} \cdot (\overline{Y} + Z)}$$

Solution

$$F_1 = \overline{\overline{(X \cdot \overline{\overline{Y}})} \cdot (\overline{Y} + Z)}$$

$$F_1 = \overline{\overline{(X \cdot \overline{\overline{Y}})}} + \overline{(\overline{Y} + Z)}$$; Theorem #14A

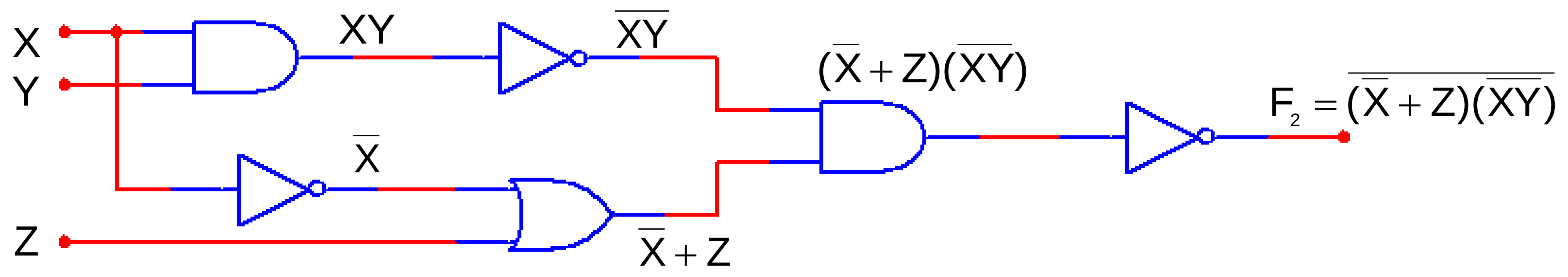
$$F_1 = (X \cdot \overline{Y}) + (\overline{\overline{Y}} \cdot \overline{Z})$$; Theorem #9 & #14B

$$F_1 = (X \cdot \overline{Y}) + (Y \cdot \overline{Z})$$; Theorem #9

$$\boxed{F_1 = X\overline{Y} + Y\overline{Z}}$$; Rewritten without AND symbols and parentheses

DeMorgan's: Example #2

So, where would such an odd Boolean expression come from? Take a look at the VERY poorly designed logic circuit shown below. If you were to analyze this circuit to determine the output function F_2, you would obtain the results shown.



X
Y
Z
XY
$\overline{XY}$
$\overline{X}$
$\overline{X}+Z$
$(\overline{X}+Z)(\overline{XY})$
$F_2 = \overline{(\overline{X}+Z)(\overline{XY})}$

Example

Simplify the output function F_2. Be sure to note the Boolean or DeMorgan's theorem used at each step. Put the answer in SOP form.

DeMorgan’s: Example #2

Solution

$$F_2 = \overline{(\overline{X} + Z)(\overline{XY})}$$

$$F_2 = (\overline{\overline{X} + Z}) + (\overline{\overline{XY}})$$; Theorem #14A

$$F_2 = (\overline{\overline{X} + Z}) + (XY)$$; Theorem #9

$$F_2 = (\overline{\overline{X}}\,\overline{Z}) + (XY)$$; Theorem #14B

$$F_2 = (X\,\overline{Z}) + (XY)$$; Theorem #9

$$F_2 = X\,\overline{Z} + X\,Y$$; Rewritten without AND symbols

Reference

http://www.ee.surrey.ac.uk/Projects/CAL/digital-logic/gatesfunc/