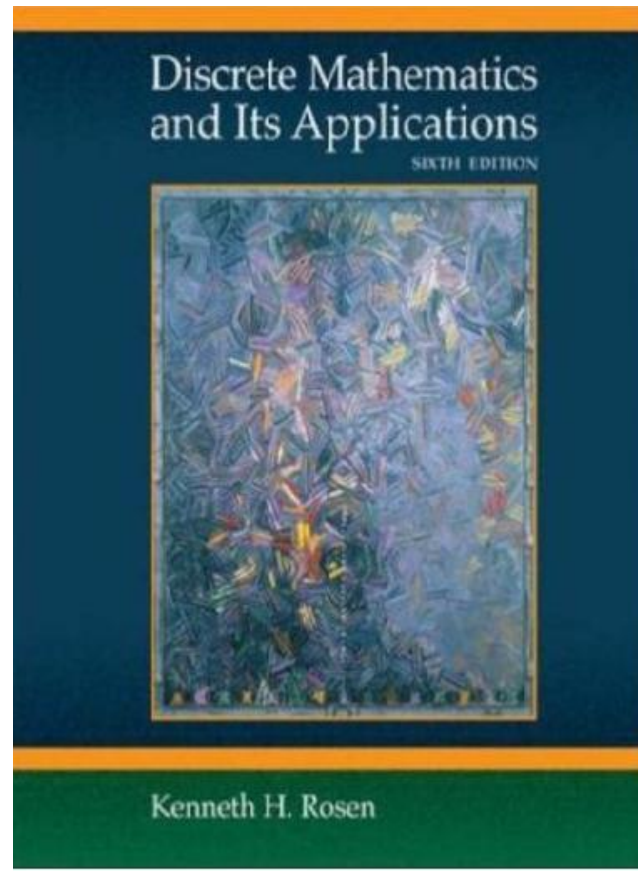
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications

Lecture 4:

Applications of Propositional Logic



Acknowledgement

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Short review: universal quantification

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Short review: EXAMPLE

- Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true

Short review: existential quantification

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

Short review: example

- Let$Q(x)$ denote the statement "$x = x + 1$.

"What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false.

Quantifiers with restricted domain

What do these statements mean (domain real)

- $\forall x<0\ (x2>0)$ same as $\forall x\ (x<0 \rightarrow x2>0)$

"the square of a negative real number is positive"

- $\forall y\neq 0\ (y^3 \neq 0)$ same as $\forall y(y\neq 0 \rightarrow y^3 \neq 0)$

"the cube of every nonzero real number is nonzero"

- $\exists z >0\ (z2=2)$ same as $\exists z(z>0 \wedge z2=2)$

"there is a positive square root of 2"

Binding variables

- $\exists x\ (x+y=1)$

The variable x is bounded by the existential quantification $\exists x$ and the variable y is **free**

- $\exists x\ (P(x) \wedge Q(x)) \vee \forall x\ R(x)$

All variables are bounded

The **scope** of the **first** quantifier $\exists x$ is the expression $P(x) \wedge Q(x)$

second quantifier $\forall x$ is the expression $R(x)$

existential quantifier binds the variable x in $P(x) \wedge Q(x)$

Universal quantifier binds the variable x in $R(x)$

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Logical equivalence involving quantifiers

Statements involving predicates and quantifier are logically equivalent iff they have the same truth value

Show that $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- suppose that $\forall x\,(P(x) \wedge Q(x))$ is true,
  this means that if a is in the domain then $P(a) \wedge Q(a)$ is true

Hence, both $P(a)$ and $Q(a)$ are true for every element in the domain

So $\forall x\, P(x)$ and $\forall x\, Q(x)$ are both true.

This means that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true.

Logical equivalence involving quantifiers

Show that $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- suppose that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true, it follows that both $\forall x\, P(x)$ and $\forall x\, Q(x)$ are true hence,
- if a is in the domain then P(a) is true and Q(a) is true

This means that $\forall x\, (P(x) \wedge Q(x))$ is true.

Now we can conclude that

$$\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

Logical equivalence involving quantifiers

$$\forall x\ (P(x) \wedge Q(x)) \equiv \forall x\ P(x) \wedge \forall x\ Q(x)$$

- The universal quantifier can be distributed over the a conjunction ($\wedge$).
- The universal quantifier can be distributed over the a disjunction ($\vee$).
- The existential quantifier can not be distributed over the a conjunction ($\wedge$) and disjunction ($\vee$).

Negating quantified expression

“Every student in your class has taken a course in calculus” $\forall x\, P(x)$
Where, $P(x)$ is “x has taken a course in calculus”
and the domain consists of the student in your class

Negation

“It is not the case that every student in your class has taken a course in calculus”

or

“There is a student in your class who has not taken a course in calculus” $\exists x \neg P(x)$

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$

Negating quantified expression

"There is a student in this class who has taken a course in calculus" $\exists x\ Q(x)$

Where, $Q(x)$ is "x has taken a course in calculus"

and the domain consists of the student in your class

Negation

"It is not the case that there is a student in this class who has taken a course in calculus"

or

"Every student in this class has not taken a course in calculus" $\forall x\ \neg Q(x)$

"Not all students in this class have taken a course in calculus" is not used.

$$\neg\exists x\ Q(x) \equiv \forall x\ \neg Q(x)$$

Negating quantified expression

Show that (homework)

- $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$
- $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

The rules for negations for quantifiers are called De Morgan's laws for quantifiers.

Negating quantified expression

De Morgan's laws for quantifiers

$\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$

The negation is true when for every x, P(x) is false

The negation is false when there is an x for which P(x) is true

$\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$

The negation is true when there is an x for which P(x) is false

The negation is false when for every x, P(x) is true

Negating quantified expression

De Morgan's laws for quantifiers

When the domain of a predicate $Q(x)$
consists of n elements $x_1, x_2, x_3, x_4, \ldots\ldots, x_n$

$\forall x\, Q(x)$ is the same as the conjunction $Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$
$\neg\forall x\, Q(x)$ is the same as the disjunction $\neg Q(x_1) \vee \neg Q(x_2) \vee \ldots \vee \neg Q(x_n)$

$\exists x\, Q(x)$ is the same as the disjunction $Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$
$\neg \exists x\, Q(x)$ is the same as the conjunction $\neg Q(x_1) \wedge \neg Q(x_2) \wedge \ldots \wedge \neg Q(x_n)$

Negating quantified expression

Examples: what are the negation of

- $\forall x\ (x^2>x)$
- $\exists x\ (x^2=x)$

- $\forall x\ (x^2>x)$

$\neg\forall x\ (x^2>x) \equiv \exists x\ \neg(x^2>x) \equiv \exists x\ (x^2\le x)$

- $\exists x\ (x^2=x)$

$\neg\exists x\ (x^2=x) \equiv \forall x\ \neg(x^2=x) \equiv \forall x\ (x^2\ne x)$

Negating quantified expression

Show that

$$\neg\forall x\,[P(x) \rightarrow Q(x)] \equiv \exists x\,[P(x) \wedge \neg Q(x)]$$

$$\begin{aligned}\neg\forall x\,[P(x) \rightarrow Q(x)] &\equiv \exists x\,\neg[P(x) \rightarrow Q(x)] \\ &\equiv \exists x\,\neg[\neg P(x) \vee Q(x)] \\ &\equiv \exists x\,[\neg\,\neg P(x) \wedge \neg Q(x)] \\ &\equiv \exists x\,[P(x) \wedge \neg Q(x)]\end{aligned}$$

Translating from English into Logical Expressions

Express the statement “Every student in the class has studied calculus” using predicates and quantifiers.

- Rewrite the statement to identify the appreciate quantifiers to use

“For every student in the class,that student has studied calculus”

- Introduce the variable x

“For every student x in the class,x has studied calculus”

- introduce C(x) “x has studied calculus”

$$\forall x\ C(x)$$

The domain for x consists of the students in the class

Translating from English into Logical Expressions

If we change the domain to consists of people

"For every person x, if person x is a student in the class,then x has studied calculus"

C(x) "x has studied calculus"
S(x) "person x is a student in the class"

$\forall x\ (S(x) \rightarrow C(x))$

Note $\forall x\ (S(x) \wedge C(x))$ is wrong
All people are students in this class and have studied calculus

Translating from English into Logical Expressions

Express the statements "some students in the class has visited Cairo" , "every student in the class has visited either Aswan or Cairo " using predicates and quantifiers.

- "some students in the class has visited Cairo"

some students ~ there is a student

C(x) "x has visited Cairo"

$$\exists x\ C(x)$$

- "every student in the class has visited either Aswan or Cairo "

A(x) "x has visited Aswan "

$$\forall x\ (C(x) \vee A(x))$$

Translating from English into Logical Expressions

System specifications

Express the statement

using predicates and quantifiers

S(m,y) “Mail message m is larger than y megabyte”

Where m has the domain of all mail message

y is a positive real number

C(m) “Mail message m will be compressed”

$\forall m\ (S(m,1) \rightarrow C(m))$

Translating from English into Logical Expressions

System specifications

Express the statement "If a user is active, at least one network link will be available" using predicates and quantifiers.

$A(u)$ "user u is active"

Where u has the domain of all users

$S(n,x)$ "network link n is in state x"

Where n has the domain of all network links

x has the domain of all possible states

for a network link

$$\exists u\, A(u) \rightarrow \exists n\, S(n, \text{available})$$

Chapter 1 Exercises

Pages (46-50)

1-4

5(a,d)

8(b,c)

10

12(b,d,g)

17

20(e)

21-22

24, 29, 31 ,36

40-42

43, 48