Common Mistakes in Mathematical Induction

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1 No Basis Step

2 Wrong Inductive Step

Examples 1 **Prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.**

Proof. **Basis Step:** Want to show: $2^{2 \cdot 1} - 1$ is divisible by 3.

Note that $2^{2 \cdot 1} - 1 = 2^2 - 1 = 3 = 3 \cdot 1$, therefore $2^{2 \cdot 1} - 1$ is divisible by 3. Therefore $P(1)$ is true.

Inductive Step: Suppose for $k \geq 1$, $P(k)$ is true, i.e., $2^{2k} - 1$ is divisible by 3, i.e.,

$$2^{2k} - 1 = 3 \cdot m$$

for some $m \in \mathbb{Z}$.

We want to show: $P(k+1)$ is true, i.e., $2^{2(k+1)} - 1$ is divisible by 3.

We know that $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1)$, so we have $2^{2(k+1)} - 1 = (2^2)^{k+1} - 1 = 4^{k+1} - 1 = (4-1)(4^k + 4^{k-1} + \cdots + 4^2 + 4 + 1) = 3(4^k + 4^{k-1} + \cdots + 4^2 + 4 + 1)$.

Since $k \in \mathbb{Z}$, the number in the last bracket is an integer, so $2^{2(k+1)} - 1$ is divisible by 3.

Therefore $P(k+1)$ is true. Therefore the statement is true by mathematical induction. $\square$

Remark In the inductive step, we assume that $P(k)$ is true, so when deriving $P(k+1)$, we must use the result of $P(k)$.

Correct Proof

Proof. **Basis Step:** Want to show: $2^{2 \cdot 1} - 1$ is divisible by 3.

Note that $2^{2 \cdot 1} - 1 = 2^2 - 1 = 3 = 3 \cdot 1$, therefore $2^{2 \cdot 1} - 1$ is divisible by 3. Therefore $P(1)$ is true.

Inductive Step: Suppose for $k \geq 1$, $P(k)$ is true, i.e., $2^{2k} - 1$ is divisible by 3, i.e.,

$$2^{2k} - 1 = 3 \cdot m$$

for some $m \in \mathbb{Z}$.

We want to show: $P(k+1)$ is true, i.e., $2^{2(k+1)}-1$ is divisible by 3.

Note that

$$2^{2(k+1)}-1 = 2^{2k+2}-1 = 4\cdot 2^{2k}-1 = 4(2^{2k}-1)+3 = 4\cdot 3m+3 = 3(4m+1)$$

is divisible by 3.

Therefore $P(k+1)$ is true. Therefore the statement is true by mathematical induction. □

Examples 2 **Determine whether for all odd integers n, 7^n+1 is divisible by 8.**

Proof. **Basis Step:** When n=1, $7^1+1=8$ is divisible by 8, therefore $P(1)$ is true.

Inductive Step: Suppose for $k\geq 1$, $P(k)$ is true, i.e., 7^k+1 is divisible by 8, i.e.,

$$7^k+1=8m$$

for some $m\in\mathbb{Z}$.

Consider $P(k+1)$.

Note that

$$7^{k+1}+1 = 7\cdot 7^k+1 = 77^k+1-6 = 7\cdot 8m-6,$$

which is not divisible by 8. Therefore the statement is false.

Therefore the statement is not true by mathematical induction. □

Remark Notice that in the question, n must be an odd number, so we cannot use $P(k+1)$ directly, but $P(k+2)$ instead.

Correct Proof

Proof. **Basis Step:** When $n=1$, $7^1+1=8$ is divisible by 8, therefore $P(1)$ is true.

Inductive Step: Suppose for odd integer $k\geq 1$, $P(k)$ is true, i.e., 7^k+1 is divisible by 8, i.e.,

$$7^k+1=8m$$

for some $m\in\mathbb{Z}$.

Consider $P(k+2)$.

Note that

$$7^{k+2}+1 = 49\cdot 7^k+1 = 49(7^k+1)-48 = 49\cdot 8m-48 = 8(49m-6)$$

is divisible by 8.

Therefore $P(k+1)$ is true. Therefore the statement is true by mathematical induction. □

Examples 3 **Prove that** $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1$.

Proof. **Basis Step:** When $n = 1$, $LHS = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$, therefore $P(1)$ is true.

Inductive Step: Suppose for odd integer $k \geq 1$, $P(k)$ is true, i.e.,

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k+1} > 1.$$

Consider $P(k+2)$.
We want to show: $P(k+1)$ is true.
Note that

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{3k} + \frac{1}{3(k+1)+1} > 1 + \frac{1}{3(k+1)+1}$$

Therefore $P(k+1)$ is true. Therefore the statement is true by mathematical induction. □

Remark We must pay attention to the difference between $P(k)$ and $P(k+1)$, namely the denominators are consecutive integers.

Correct Proof

Proof. **Basis Step:** When $n = 1$, $LHS = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$, therefore $P(1)$ is true.

Inductive Step: Suppose for odd integer $k \geq 1$, $P(k)$ is true, i.e.,

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k+1} > 1.$$

Consider $P(k+2)$.
We want to show: $P(k+1)$ is true.
Note that

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{3k} + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3(k+1)+1}$$

$$= (\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k} + \frac{1}{3k+1}) + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3(k+1)+1} - \frac{1}{k+1}$$

$$> 1 + \frac{1}{3k+2} + \frac{1}{3(k+1)+1} - \frac{2}{3(k+1)} = 1 + \frac{6k+6}{(3k+2)(3k+4)} - \frac{6k+6}{(3(k+1))^2} > 1$$

Therefore $P(k+1)$ is true. Therefore the statement is true by mathematical induction. □