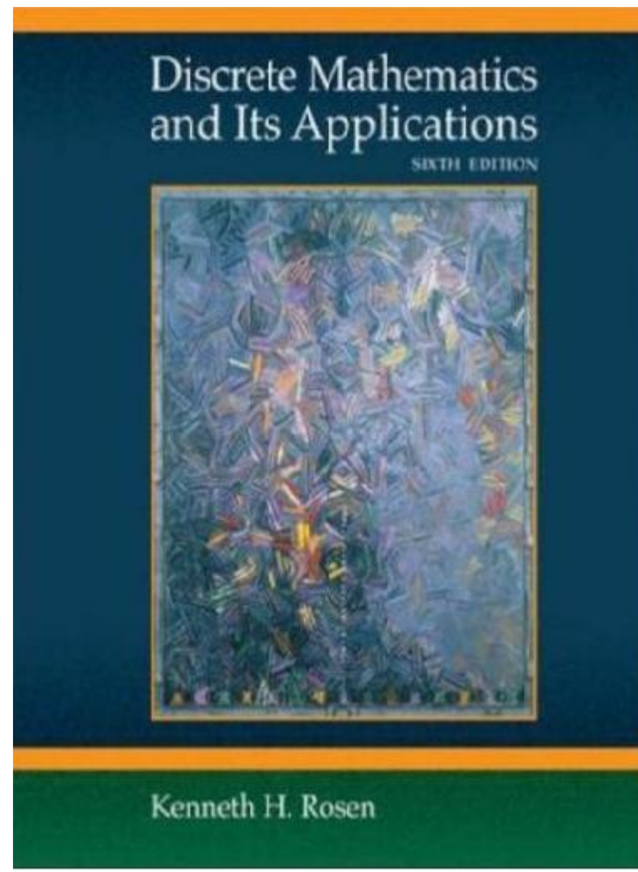
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications

Logic Module (Part 1)

Section 1.5 Rules of Inference



Acknowledgement

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1.5 Propositional logic:
Rules of Inference or Methods of Proof
How to produce additional wffs (sentences) from other ones?

What steps can we perform to show that a conclusion follows logically from a set of hypotheses?

Example

Modus Ponens

$$P$$

$$P \to Q$$

$$\therefore\ Q$$

The hypotheses (premises) are written in a column and the conclusions below the bar
The symbol $\therefore$ denotes "therefore".
Given the hypotheses, the conclusion follows.

The basis for this rule of inference is the **tautology** $(P \land (P \to Q)) \to Q$

[aside: check tautology with truth table to make sure]

In words: when P and $P \to Q$ are True, then Q must be True also. (meaning of second implication)

Section 1.5 Rules of Inference

Propositional logic:

Rules of Inference or Methods of Proof

- Example: Modus Ponens

If you study the CS 230322 material $\rightarrow$ You will pass

You study the CS23022 material

$\therefore$ you will pass

- Nothing "deep", but again remember the formal reason is that

$((P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

Propositional logic: Rules of Inference

See Table 1, p. 66, Rosen.

Rule of Inference	Tautology (Deduction Theorem)	Name
P $\therefore P \vee Q$	$P \rightarrow (P \vee Q)$	**Addition**
$P \wedge Q$ $\therefore P$	$(P \wedge Q) \rightarrow P$	**Simplification**
P Q $\therefore P \wedge Q$	$[(P) \wedge (Q)] \rightarrow (P \wedge Q)$	**Conjunction**
P $P \rightarrow Q$ $\therefore Q$	$[(P) \wedge (P \rightarrow Q)] \rightarrow P$	**Modus Ponens**
$\neg Q$ $P \rightarrow Q$ $\therefore \neg P$	$[(\neg Q) \wedge (P \rightarrow Q)] \rightarrow \neg P$	**Modus Tollens**
$P \rightarrow Q$ $Q \rightarrow R$ $\therefore P \rightarrow R$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	**Hypothetical Syllogism (“chaining”)**
$P \vee Q$ $\neg P$ $\therefore Q$	$[(P \vee Q) \wedge (\neg P)] \rightarrow Q$	**Disjunctive syllogism**
$P \vee Q$ $\neg P \vee R$ $\therefore Q \vee R$	$[(P \vee Q) \wedge (\neg P \vee R)] \rightarrow (Q \vee R)$	**Resolution**

Section 1.5 Rules of Inference
Valid Arguments

- An argument is a sequence of propositions. The final proposition is called the conclusion of the argument while the other proposition are called the premises or hypotheses of the argument.
- An argument is valid whenever the truth of all its premises implies the truth of its conclusion.
- How to show that **q** logically follows from the hypotheses

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n)?$$

Show that $(p1 \wedge p2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology

One can use the rules of inference to show the validity of an argument.

Section 1.5 Rules of Inference
Proof Tree

- Proofs can also be based on partial orders

– we can represent them using a tree structure:
  - Each node in the proof tree is labeled by a wff, corresponding to a wff in the original set of hypotheses or be inferable from its parents in the tree using one of the rules of inference;
  - The labeled tree is a proof of the label of the root node.

Example:

Given the set of wffs:

$P, R, P \rightarrow Q$

Give a proof of $Q \wedge R$

Section 1.5 Rules of Inference
Tree Proof

P, $P \to Q$, Q, R, $Q \wedge R$

$$\dfrac{\dfrac{P \qquad P \to Q}{Q}\,\text{MP} \qquad R}{Q \wedge R}\,\text{Conj.}$$

What rules of inference did we use?

Section 1.5 Rules of Inference
Length of Proofs

- Why bother with inference rules? We could always use a truth table
- to check the validity of a conclusion from a set of premises.

But, *resulting proof can be much shorter than truth table method*.

Consider premises:

p_1, $p\_1 \to p\_2$, $p\_2 \to p\_3$, …, $p\_(n-1) \to p\_n$

To prove conclusion: p_n

Inference rules: n-1 MP steps Truth table: 2^n

Key open question: Is there always a short proof for any valid conclusion? Probably not. The NP vs. co-NP question.
(The closely related: P vs. NP question carries a $1M prize.)

1.3-1.4 BEYOND PROPOSITIONAL LOGIC: PREDICATES AND QUANTIFIERS

Predicates

- Propositional logic assumes the world contains facts that are true or false.
- But let's consider a statement containing a variable:

- $x > 3$ since we don't know the value of x we cannot say whether the expression is true or false

Predicate, i.e. a property of x

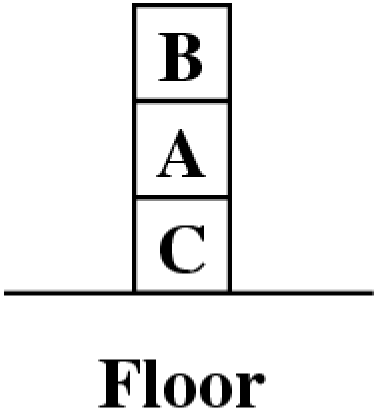
- $x > 3$ which corresponds to "x is greater than 3"

- “x is greater than 3” can be represented as P(x),
  where P denotes “greater than 3”

- In general a statement involving n
  variables $x_1, x_2, \ldots x_n$ can be denoted by **$P(x_1, x_2, \ldots x_n)$**

- P is called a predicate or the propositional function P at the n-tuple $(x_1, x_2, \ldots x_n)$.

When all the variables in a predicate are assigned values → Proposition, with a certain truth value.



Predicate: On(x,y)

Propositions:

ON(A,B) is False (in figure)

ON(B,A) is True

Clear(B) is True

Variables and Quantification

- How would we say that every block in the world has a property – say “clear”?

  We would have to say:

- Clear(A); Clear(B); … for all the blocks… (it may be long or worse we may have an infinite number of blocks…)

What we need is: Quantifiers

$\forall$ Universal quantifier **$\forall x$ P(x)**

- P(x) is true for all the values x in the universe of discourse

$\exists$ Existential quantifier **$\exists x$ P(x)**

- there exists an element x in the universe of discourse such that P(x) is true

Universal quantification

- Everyone at Kent State is smart:

$$\forall x \ \ At(x,Kent\ State) \rightarrow Smart(x)$$

- Implicitly equivalent to the conjunction of instantiations of Predicate “At“

$$\wedge \quad At(Mary,Kent\ State) \rightarrow Smart(Mary)$$
$$\wedge \quad At(Richard,Kent\ State) \rightarrow Smart(Richard)$$
$$\wedge \quad At(John,Kent\ State) \rightarrow Smart(John)$$
$$\wedge \ \ldots$$

A common mistake to avoid

- Typically, $\rightarrow$ is the main connective with $\forall$
- Common mistake: Using $\wedge$ as the main connective with $\forall$:

$\forall x$ At(x,Kent State) $\wedge$ Smart(x) means:

"Everyone is at Kent State and everyone is smart."

Existential quantification

- Someone at Kent State is smart:
- $\exists x$ (At(x,Kent State) $\wedge$ Smart(x))

$\exists x\ P(x)$ “ There exists an element x in the universe of discourse such that P(x) is true”

- Equivalent to the disjunction of instantiations of *P*

$$
\begin{aligned}
&(\text{At(John,Kent State)} \wedge \text{Smart(John)})\\
\vee\ &(\text{At(Mary,Kent State)} \wedge \text{Smart(Mary)})\\
\vee\ &(\text{At(Richard,Kent State)} \wedge \text{Smart(Richard)})\\
\vee\ &\ldots
\end{aligned}
$$

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\rightarrow$ as the main connective with $\exists$:
  - $\exists x$ At(x,Harvard) $\rightarrow$ Smart(x)
- when is this true?
- " **Is true if there is either (anyone who is not at Harvard) or (there is anyone who is smart)**

Above is equivalent to

$\exists x$ [$\neg$ At(x,Harvard) $\vee$ Smart(x)]

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Quantified formulas

If α is a wff and x is a variable symbol, then both $\forall x \ \alpha$ and $\exists x \ \alpha$ are wffs.

x is the variable quantified over

α is said to be within the scope of the quantifier

if all the variables in α are quantified over in α, we say that we have a *closed wff* or *closed sentence*.

Examples:

$$\forall x \, [P(x) \rightarrow R(x)]$$

$$\exists x \, [P(x) \rightarrow (\exists y \, [R(x,y) \rightarrow S(x))]]$$

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$

- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$
- $\forall x\ \exists y$ Loves(x,y)
  - "Everyone in the world is loves at least one person"
- $\exists y\ \forall x$ Loves(x,y)
  - "There is a person who is loved by everyone in the world"

- Quantifier duality: each can be expressed using the other
- $\forall x$ Likes(x,IceCream) $\neg\exists x\ \neg$Likes(x,IceCream)
- $\exists x$ Likes(x,Broccoli) $\neg\forall x\ \neg$Likes(x,Broccoli)

Love Affairs Loves(x,y) x loves y

Everybody loves Jerry

$\forall x \; Loves(x, Jerry)$

Everybody loves somebody

$\forall x \, \exists y \; Loves(x, y)$

There is somebody whom somebody loves

$\exists y \, \exists x \; Loves(x, y)$

Nobody loves everybody

$$\neg \exists x \, \forall y \; Loves(x, y) \equiv \forall x \, \exists y \, \neg Loves(x, y)$$

There is somebody whom Lydia doesn't love

$\exists y \; \neg Loves(Lydia, y)$

Note: flipping quantifiers when $\neg$ moves in.

Love Affairs continued…

There is somebody whom no one loves

$$\exists y\ \forall x\ \neg Loves\ (x, y)$$

There is exactly one person whom everybody loves (**uniqueness**)

$$\exists y\ (\forall x\ Loves(x,y) \wedge \forall z((\forall w\ Loves\ (w\ ,z) \rightarrow z=y))$$

There are exactly two people whom Lynn Loves

$$\exists x\ \exists y\ ((x\neq y) \wedge Loves(Lynn,x)\ Loves(Lynn,y) \wedge$$
$$\forall z(\ Loves\ (Lynn\ ,z)\rightarrow (z=x \vee z=y)))$$

Everybody loves himself or herself

$$\forall x\ Loves(x,x)$$

There is someone who loves no one besides herself or himself

$$\exists x\ \forall y\ \ Loves(x,y) \leftrightarrow (x=y)$$ (note biconditional – why?)

- Let Q(x,y) denote "x+y =0"; consider the domain of discourse the real numbers

- What is the truth value of

a) $\exists y\ \forall x\ \ Q(x,y)$? **False**

b) $\forall x\ \exists y\ \ Q(x,y)$? **True** **(additive inverse)**

Statement	When True	When False
$\forall x \, \forall y \, P(x,y)$ $\forall y \, \forall x \, P(x,y)$	P(x,y) is true for every pair	There is a pair for which P(x.y) is false
$\forall x \, \exists y \, P(x,y)$	For every x there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y.
$\exists x \, \forall y \, P(x,y)$	There is an x such that P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false
$\exists x \, \exists \, y \, P(x,y)$ $\exists y \, \exists \, x \, P(x,y)$	There is a pair x, y for which P(x,y) is true	P(x,y) is false for every pair x,y.

Negation

Negation	Equivalent Statement	When is the negation True	When is False
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	For every x, P(x) is false	There is an x for which P(x) is true.
$\neg\, \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which P(x) is false.	For every x, P(x) is true.

- The kinship domain:
- Brothers are siblings

  $\forall x,y \quad Brother(x,y) \rightarrow Sibling(x,y)$

- One's mother is one's female parent

  $\forall m,c \quad Mother(c) = m \leftrightarrow (Female(m) \wedge Parent(m,c))$ *[uses function]*

- "Sibling" is symmetric

  $\forall x,y \quad Sibling(x,y) \leftrightarrow Sibling(y,x)$

Rules of Inference for Quantified Statements

$\dfrac{(\forall x)\, P(x)}{\therefore P(c)}$	Universal Instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore (\forall x)\, P(x)}$	Universal Generalization
$\dfrac{\exists(x)\, P(x)}{\therefore P(c) \text{ for some element } c}$	Existential Instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists(x)\, P(x)}$	Existential Generalization

Example:

Let CS23022(x) denote: x is taking the CS23022 class

Let CS(x) denote: x is taking a course in CS

Consider the premises $\forall x\ (CS23022(x) \rightarrow CS(x))$

CS23022(Ron)

We can conclude CS(Ron)

Arguments

- Argument (formal):

- <u>Step</u> <u>Reason</u>
- 1 $\forall x\,(CS23022(x) \rightarrow CS(x))$ premise
- 2 $CS23022(Ron) \rightarrow CS(Ron)$ Universal Instantiation
- 3 CS23022(Ron) Premise
- 4 CS(Ron) Modus Ponens (2 and 3)

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Example

- Show that the premises:
- 1- A student in this class has not read the textbook;
- 2- Everyone in this class passed the first homework

- Imply
- Someone who has passed the first homework has not read the textbook

Example

- Solution:
  - Let $C(x)$ denote that x is in this class;
  - $T(x)$ denote that x has read the textbook;
  - $P(x)$ denote that x has passed the first homework

- Premises:
  - $\exists x\,(C(x) \land \neg T(x))$
  - $\forall x\,(C(x) \rightarrow P(x))$
  - Conclusion: we want to show $\exists x\,(P(x) \land \neg T(x))$

Step		Reason
1	$\exists x\,(Cx \wedge \neg T(x))$	Premise
2	$C(a) \wedge \neg T(a)$	Existential Instantiation from 1
3	$C(a)$	Simplification 2
4	$\forall x\,(C(x) \rightarrow P(x))$	Premise
5	$C(a) \rightarrow P(a)$	Universal Instantiation from 4
6	$P(a)$	Modus ponens from 3 and 5
7	$\neg T(a)$	Simplification from 2
8	$P(a) \wedge \neg T(a)$	Conjunction from 6 and 7
9	$\exists x\, P(x) \wedge \neg T(x)$	Existential generalization from 8

Next: methods for proving theorems.

Possible Classroom Examples

1. What is the negation of "There is no pollution in New Jersey".
2. p denote "The election is decided" and q denote "The votes have been counted. Express $\neg p \rightarrow \neg q$ as an English Sentence
3. For hiking on the trail, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in this area. (this is the question discussed in class earlier)
4. Determine the truth value of 1 + 1 = 3 if and only if monkeys can fly.
5. Determine if the exclusive or is intended:
   - You can pay using dollars or euros.
6. To take discrete mathematics, you must have taken a course in calculus or a course in computer science
7. Use a truth table to verify the first De Morgan law
   - $\neg(p \wedge q) \equiv \neg p \vee \neg q$

The END