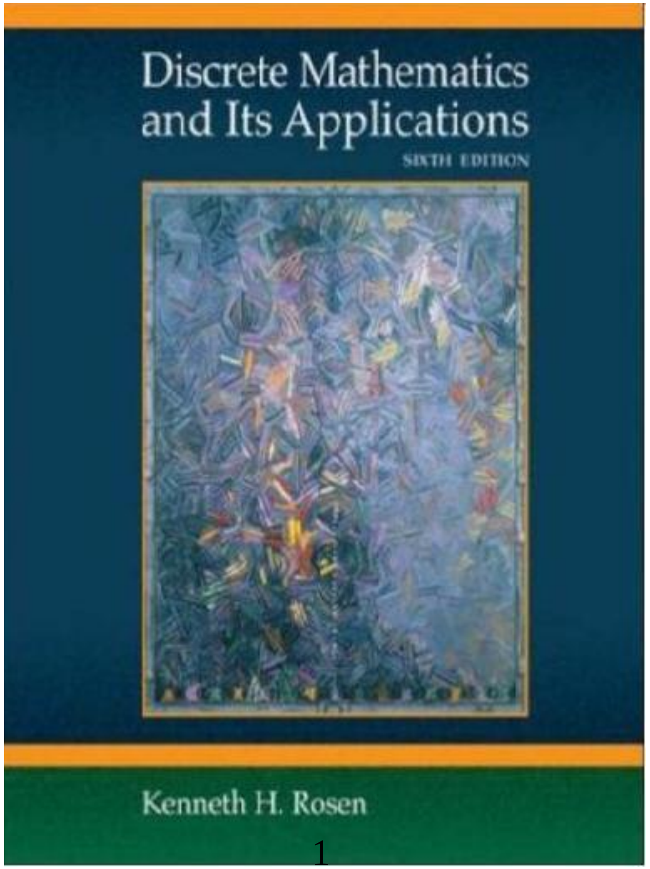
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications

Lecture017:

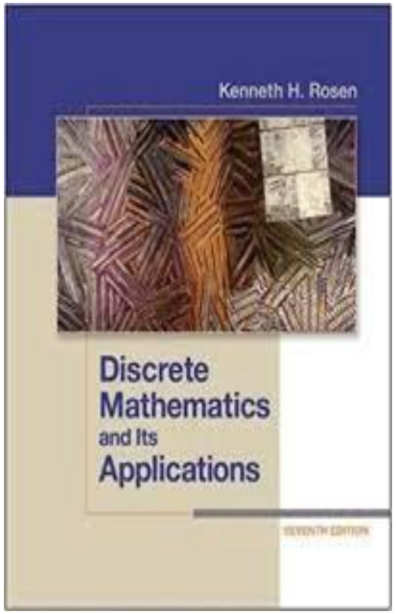
Chapter 5: Counting



Acknowledgement

Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University ,

and Dr Johnnie Baker and **Discrete Mathematics and Its Applications** **(Seventh Edition)** **Kenneth H. Rosen**



5.1 The Basics of Counting

Basic Counting Principles

Examples

A password on a computer system consists of six, seven or eight characters. Each of these characters must be a digit or a letter. Each password must contain at least one digit.

How many such passwords are there?

Examples

Each computer on the internet is assigned an IP address. If each IP address is a string of 32 bits. How many different IP addresses are available?

The Product Rule

– applies when a procedure is made up of separate tasks.

THE PRODUCT RULE

Suppose that a procedure can be broken down into a sequence of two tasks.

If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are n_1n_2 ways to do the procedure.

5.1 The Basics of Counting

- Example
- A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
- Solution: 12 ways to assign an office to Sanchez, and 11 ways to assign an office to Patel. By the product rule, there are 12 * 11 = 132 ways.

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: 32 * 24 = 768 ports

How many different bit strings of length seven are there?

Solution: Each of the seven bit can be chosen in two ways, 0 or 1.

$2*2*2*2*2*2*2 = 2^7 = 128$ different bit strings of length 7.

5.1 The Basics of Counting (Example)

–The Telephone Numbering Plan.

- The format of telephone numbers in North America is specified by a numbering plan. Let *X* denote a digit with values 0 – 9, *N* denote a digit with values 2 – 9, and *Y* denote a digit either 0 or 1. The old plan used in the 1960s has the format *NYX-NNX-XXXX*. The new plan under use now has the format *NXX-NXX-XXXX*. How many different telephone numbers are possible under the old plan and the new plan?

Solution:

Old plan: (8*2*10) * (8*8*10) *(10*10*10*10) = 1,024,000,000
New plan: (8*10*10)*(8*10*10)*(10*10*10*10) = 6,400,000,000

What is the value of k after the following code has been executed?

```
k:= 0
for i1:=1 to n1
    for i2 : = 1 to n2
                …
            for im :=1 to nm
                k := k + 1
```

Solution: $n_1 * n_2 * \ldots * n_m$

5.1 The Basics of Counting

THE SUM RULE:

If a task can be done either in one of n_1 ways or in one of n_2 ways , where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

- Examples:
- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

  Solution: 23 + 15 + 19 = 57

- What is the value of k after the following code has been executed?

  Solution: $n_1 + n_2 + \ldots + n_m$

k:= 0
for i_1:=1 to n_1 $\quad k := k + 1$
for i_2 : = 1 to n_2 $\quad k := k + 1$
…
for i_m :=1 to n_m $\quad k := k + 1$

5.1 The Basics of Counting

More Complex Counting Problems

- Examples

In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the five string of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

Solution:

Let V_1 be the number of these that are one character long, and V_2 be the number of these that are two characters long.

Then $V = V_1 + V_2$.

$V_1 = 26$ because a variable name must begin with a letter.

$V_2 = 26 * 36 – 5 = 931$.

$V = 26 + 931 = 957$

5.3 Permutations and Combinations

Permutations

- Many counting problems can be solved by arranging distinct elements where the order of these elements matters (permutation).

- Many counting problems can be solved by arranging distinct elements where the order of these elements does not matter(combination).

Examples of permutation:

- Examples of permutation:

  – In how many ways can we select three students from a group of five students to stand in line for a picture?

Solution:

The order matters. There are 5*4*3 = 60 ways.

- A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

- The number of r-permutation of a set with n elements is denoted by $P(n,r)$. We can find $P(n,r)$ using the product rule.

Example:

Let $S = \{1,2,3\}$. The ordered arrangement 3,1,2 is a permutation of S. The ordered arrangement 3,2 is a 2-permutation of S.

Let S = {a,b,c}. The 2-permutation of S are the ordered arrangements *a*,*b*; *a*,*c*; *b*,*a*;*b*,*c*; *c*,*a*; and *c*,*b*. $P(3,2) = 3 * 2 = 6$

5.3 Permutations and Combinations

THEOREM 1

If n is a positive integer and r is an integer with $1 <= r <= n$,

then there are $P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$

r-permutations of a set with n distinct elements.

- $P(n,0) = 1$ whenever n is a nonnegative integer because there is exactly one way to order zero elements, i.e., there is exactly one list with no elements in it, namely the empty list.

Example:

How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who have entered a contest?

Solution: $P(100, 3) = 100*99*98 = 970,200$

– How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Solution: Because *ABC* must occur as a block, we can find the answer by finding the number of permutations of six objects, namely the block *ABC*, and the individual letters *D*,*E*,*F*,*G*, and *H*. There are $6! = 720$ permutations.

5.3 Permutations and Combinations

Combinations

- *Order does not matter!*

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by $C(n,r)$ or $\binom{n}{r}$

Example:

Let S be the set $\{1,2,3,4\}$. The $\{1,3,4\}$ is a 3-combination from S.
We see that $C(4,2) = 6$, because the 2-combinations of $\{a,b,c,d\}$ are the six subsets $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$ and $\{c,d\}$

THEOREM 2

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 <= r <= n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

5.3 Permutations and Combinations (Example)

- How many poker hands of five cards can be dealt from a standard deck of 52 cards?

- How many ways are there to select 47 cards from a standard deck of 52 cards?

Solution:

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$C(52,47) = \frac{52!}{47!5!}$$

江西理工大学

JIANGXI UNIVERSITY OF SCIENCE AND TECHNOLOGY

5.6 Generating Permutations and Combinations

Generating Permutations

- Sometimes permutations or combinations need to be generated, not just counted. E.g. given a set of positive integers, and find a subset that has 100 as their sum, if such a subset exists.

- One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their elements.

- Any set with n elements can be placed in one-to-one correspondence with the set $\{1,2,3,\ldots,n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements.

E.g. {*a*,*b*,*c*} ->{1,2,3} Permutation:

123->*abc*, 132 ->*acb*, 213 -> *bac*

- Permutation $a_1a_2\ldots a_n$ precedes the permutation $b_1b_2\ldots b_n$, if for some k, with $1 <= k <= n$, $a_1 = b_1$, $a_2 = b_2$, …, $a_{k-1} = b_{k-1}$, and $a_k < b_k$.
  E.g. 13245 precedes 13254 ->*acbde* precedes *acbed*

5.6 Generating Permutations and Combinations

Example:

– **Generate the permutations of the integers 1,2,3 in lexicographic order.**

Solution:

Begin with 123.

123, 132, 213, 231, 312, 321

What is the next permutation in lexicographic order after 362541?

Solution:

362541 – 364521 – 364125. No number between 362541 and 364125

Algorithm:

1. From the last digit forward, find the first a_j so that $a_j < a_{j+1}$

2. To the right of a_j, find the smallest number a_k that is greater than a_j

3. Swap a_j and a_k

4. Place all the numbers after *jth* position in order.

– What's the next permutation of 234165?

5.6 Generating Permutations and Combinations

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

Procedure next Permutation($a_1a_2\ldots a_n$: permutation of $\{1,2,\ldots,n\}$ not equal to n n-1 … 2 1)

$j := n - 1$
while $a_j > a_{j+1}$
    $j := j - 1$
$\{j$ is the largest subscript with $a_j < a_{j+1}\}$

Step 1

$k := n$
while $a_j > a_k$
    $k := k - 1$
$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$
Interchange a_j and a_k

Step 2

$r := n$
$s := j + 1$

Step 3

while $r > s$
begin
    interchange a_r and a_s
    $r := r - 1$
    $s := s + 1$
end

step 4

{this puts the tail end of the permutation after the *jth* position in increasing order}

5.6 Generating Permutations and Combinations

Generating Combinations

- A combination is just a subset, thus we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.
  E.g. bit string 110100 represents subset $\{a,b,d\}$ of the set $\{a,b,c,d,e,f\}$

To find all the subsets, start with the bit string 000..00, with n zeros.

Then successively find the next expansion until the bit string 111..11 is obtained. At each stage the next expansion is found by locating the first position from the right that is not a 1, then changing all the 1s to the right of the position to 0s and making this first 0 (from the right) a 1.

Example:

Find the next bit string after 10 0010 0111

Solution: 10 0010 1000

Finding: add 1 to the bit string

5.6 Generating Permutations and Combinations

ALGORITHM 2 Generating the Next Larger Bit String.

Procedure nextPermutation($b_{n-1}b_{n-2}\ldots b_1b_0$: bit string not equal to 11…11)

i :=0

while $b_i = 1$

begin

  $b_i := 0$

  $i := i + 1$

end

$b_i := 1$

5.6 Generating Permutations and Combinations

Example:

From the set {1,2,3,4,5}:

Find the next larger combination of {1,2,3,4}.

Solution: {1,2,3,5}

Find the next larger combination of {1,3,5}.

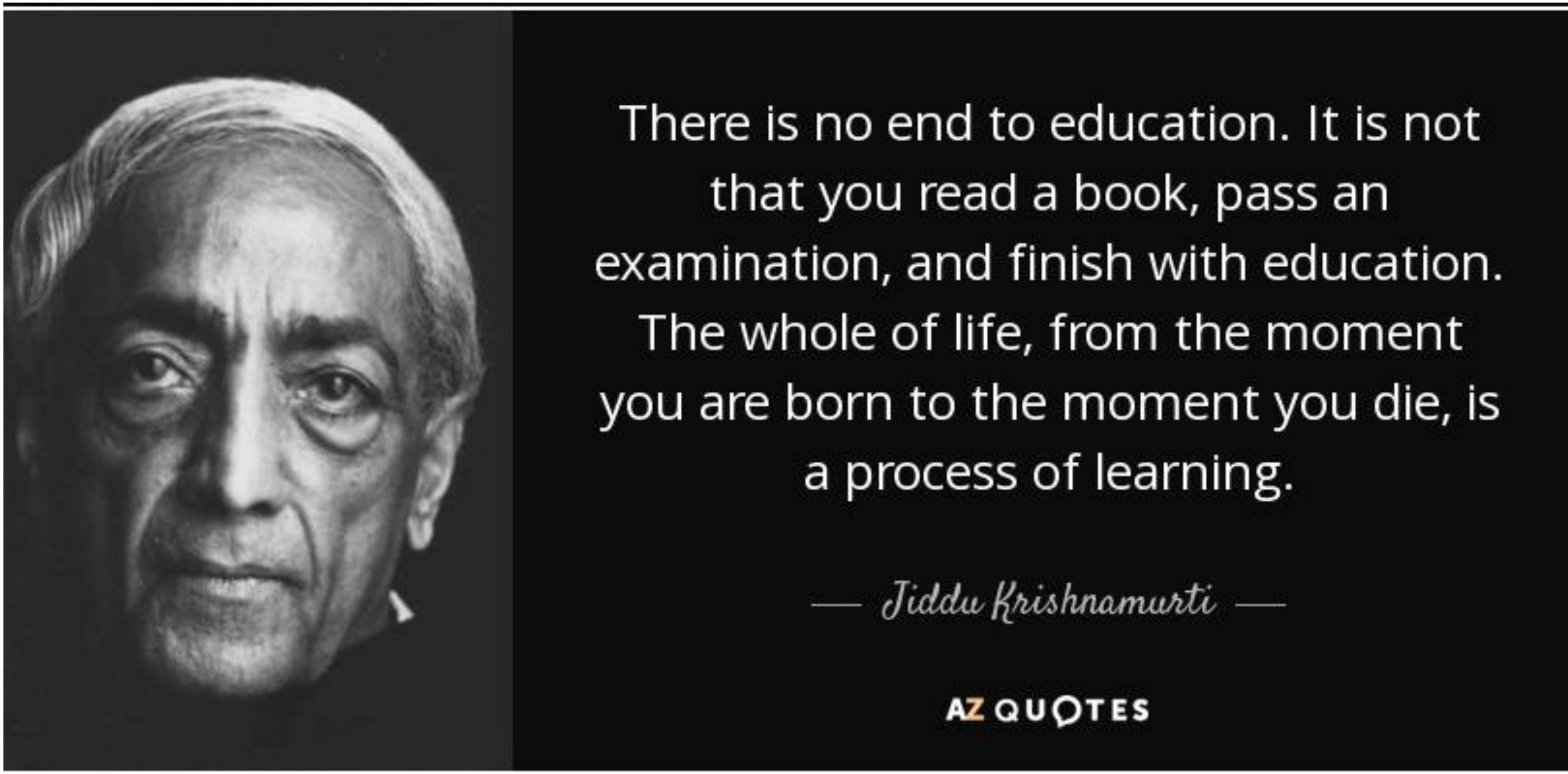
Solution: {1,4,5}

Find the next larger combination of {1,4,5}.

Solution: {2,3,4}

Algorithm:

1. Sort the combination.
2. From right to left, find the first position i so that a_i can be increased.
3. Increase a_i by 1.
4. For the numbers to the right of a_i(if any), set to increased order starting from $a_{i.}$


There is no end to education. It is not that you read a book, pass an examination, and finish with education. The whole of life, from the moment you are born to the moment you die, is a process of learning.
— Jiddu Krishnamurti —
AZ QUOTES