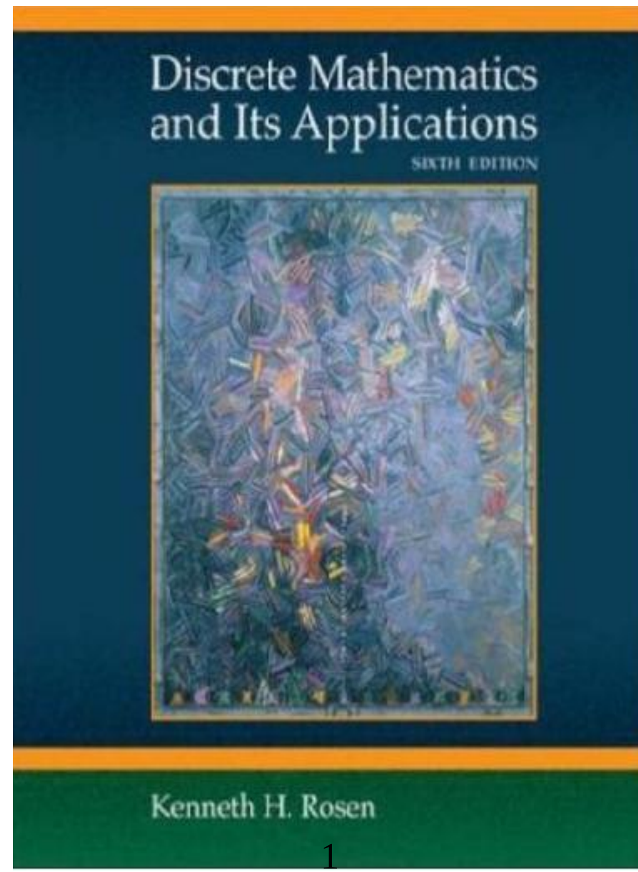
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications



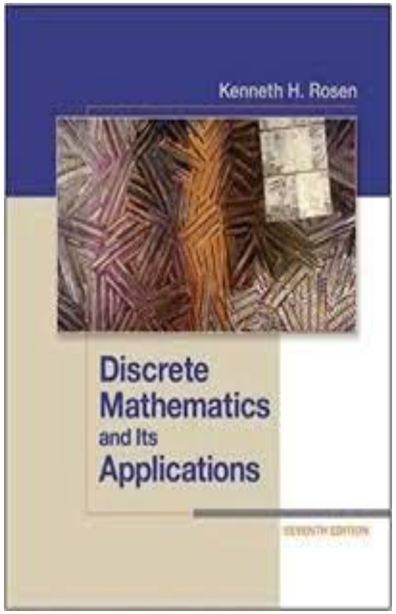
Lecture015:

Strong Induction vs. Induction

Acknowledgement

Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University ,

and Dr Johnnie Baker and **Discrete Mathematics and Its Applications** **(Seventh Edition)** **Kenneth H. Rosen**



Strong Induction vs. Induction

- Sometimes strong induction is easier to use.
- It can be shown that strong induction and induction are equivalent:

- any proof by induction is also a proof by strong induction (why?)

- any proof by strong induction can be converted into a proof by induction

- Strong induction also referred to as complete induction; in this context induction is referred to as incomplete induction.

Strong Induction

- Show that if n is an integer greater than 1, then *n* can be written as the product of primes.
- 1 - Hypothesis P(n) - *n* can be written as the product of primes.
- 2 – Base case – P(2) 2 can be written a 2 (the product of itself)
- 3 – Inductive Hypothesis - P(j) is true for $\forall\ 2 \leq j \leq k$, j integer.
- 4 – Inductive step?
-

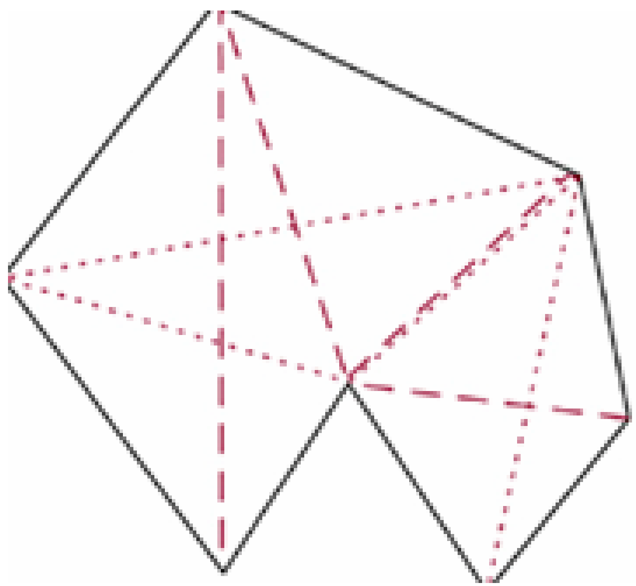
a) k+1 is prime – in this case it's the product of itself;

b) k+1 is a composite number and it can be written as the product of two positive integers *a* and *b*, with $2 \leq a \leq b \leq k+1$. By the inductive hypothesis, a and b can be written as the product of primes, and so does k+1

QED

Strong Induction: Polygon Triangulation

- Theorem: A simple polygon with n sides, where n is an integer with n≥3, can be triangulated into (n-2) triangles.



n=7
5 triangles
(2 different triangulations)

How would we prove it?

Strong Induction: Polygon Triangulation

Hypothesis:

T(n) – every polygon with n sides can be triangulated in n-2 triangles

Basis Step: T(3), a polygon with three sides is a triangle;

Inductive Hypothesis:

T(j), i.e, all triangles with j sides can be triangulated in j-2 triangles, is true for all integers $3 \le j \le k$.

Inductive Step

– assuming inductive hypothesis, show T(k+1), i.e., every single polygon of k+1 sides can be triangulated in k+1-2 = k-1 triangles

Inductive Step

– assuming T(j), i.e, all triangles with j sides can be triangulated in j-2 triangles, is true for all integers 3≤j ≤k, show T(k+1), i.e., every single polygon of k+1 sides can be triangulated in k+1-2 = k-1 triangles.

- First, we split the polygon with (k+1) sides into two polygons:

  Q with *s* sides and R with *t* sides.

#sides of P = k+1 = #sides of Q + #sides of R – 2 = s + t – 2

(we counted the new diagonal twice)

Also 3≤s ≤k and 3≤t ≤k both Q and R have at least one fewer side than P, and therefore by IH we can triangulate Q into s-2 and R into t-2 triangles respectively, and these triangulations with

s-2+t-2 = s+t-4= (k+1)-2 triangles constitute a valid triangulation for P.

QED

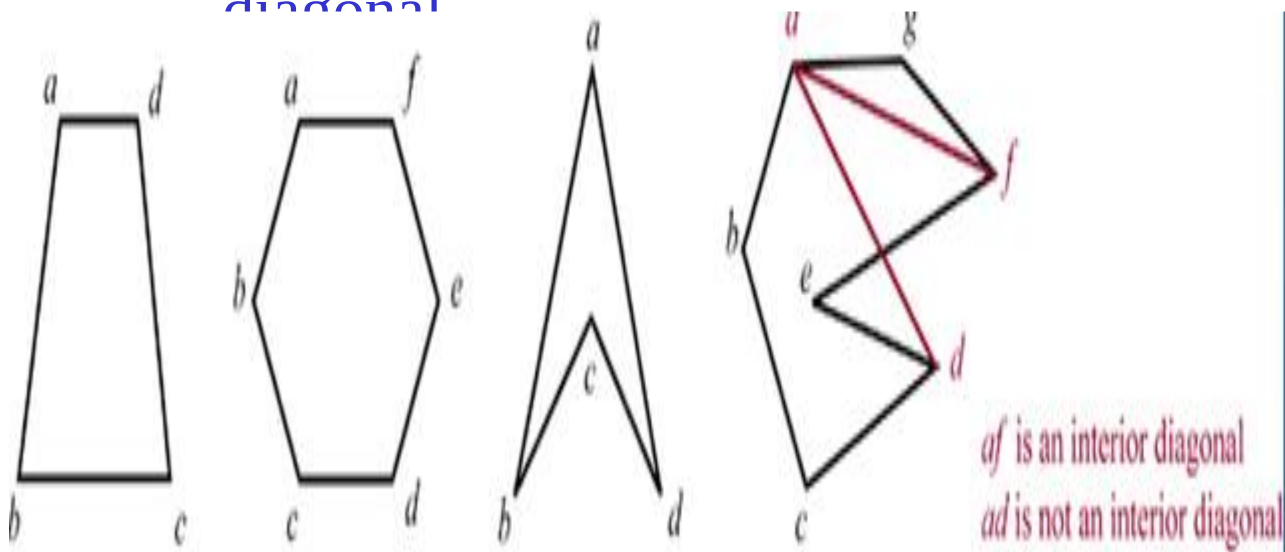
Strong Induction: Polygon Triangulation

Subtlety:

we assumed the following lemma (not so easy to prove! see Rosen):

Every simple polygon

(i.e., one in which no non-consecutive sides intersect) has an interior diagonal



(a) (b) (c) (d)

Winning Strategy:Strong Induction

Example:

Consider the game where there are 2 piles of *n* matches.

Players take turns removing any number of matches they want from one of the two piles. The player who removes the last match wins the game.

Show that the second player can always guarantee a win.

Think about this for a moment: what strategy could the second player use?

Hint: it’s the “annoying” strategy. ☺

Hyp.: P(n) The second player always has a winning strategy for two piles of n matches.

Basic step:

P(1) when there are 2 piles with 1 match each the second player always wins.

Inductive Hypothesis: $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$

Inductive Step: $(P(1) \wedge P(2) \wedge \ldots \wedge P(k) \rightarrow P(k+1))$

Assume player 2 wins when there are 2 piles of k matches.

Can player 2 win when there are 2 piles of k+1 matches?

Suppose that the first player takes r matches ($1 \le r \le k$), leaving k+1-r matches in the pile. By removing the same number of matches from the other pile, the second player creates the situation where both piles have the same number of matches ($\le k$), which we know, by the inductive hypothesis, there is a winning strategy for player two.

Note this proof actually also provides the winning strategy for the 2nd player. (constructive)

QED

Postage: Induction

- **Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.**

Hypothesis: Every amount of postage of 12 cents or more can be formed using just 4-cent or 5-cent stamps.

Base case: P(12) postage of 12 cents can be formed using just 4-cent or 5-cent stamps, 12=3(4).

Inductive Hypothesis: P(k) postage of k cents can be formed using just 4-cent or 5-cent stamps

Inductive step: $P(k) \rightarrow P(k+1)$, given P(k).

Let's assume P(k), $k \geq 12$. There are two cases:

a) at least one 4-cent stamp was used to form postage of k cents --- in that case with the extra cent we replace this stamp with a 5-cent stamp.

b) no extra 4-cent was used to form postage of k cents --- in that case we only used 5 cent stamps; given that k>=12, it has to be at least 15, in which case we need at least three 5-cent stamps. We can replace three 5 cent stamps with four 4-cent stamps to form postage of k+1 cents.

QED

Postage: Strong Induction

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

- Hypothesis: Every amount of postage of 12 cents or more can be formed using just 4-cent or 5-cent stamps
- Base case: P(12) 12=3(4); P(13) 13=2(4)+1(5); P(14) 14=1(4)+2(5) P(15) 15=3(5), so $\forall 12 \le n \le 15$, $P(n)$.
- Inductive Hypothesis: P(j) postage, $\forall$j, $12 \le j \le k$, $k \ge 15$ cents can be formed using just 4-cent or 5-cent stamps

- Inductive step: Assuming $\forall$j $12 \le j \le k$ $P(j)$, $k \ge 15$, we want to show P(k+1).
- Note $12 \le k-3 \le k$, so $P(k-3)$, so add a 4-cent stamp to get postage for k+1.

So, shortens/simplifies standard induction proof. QED

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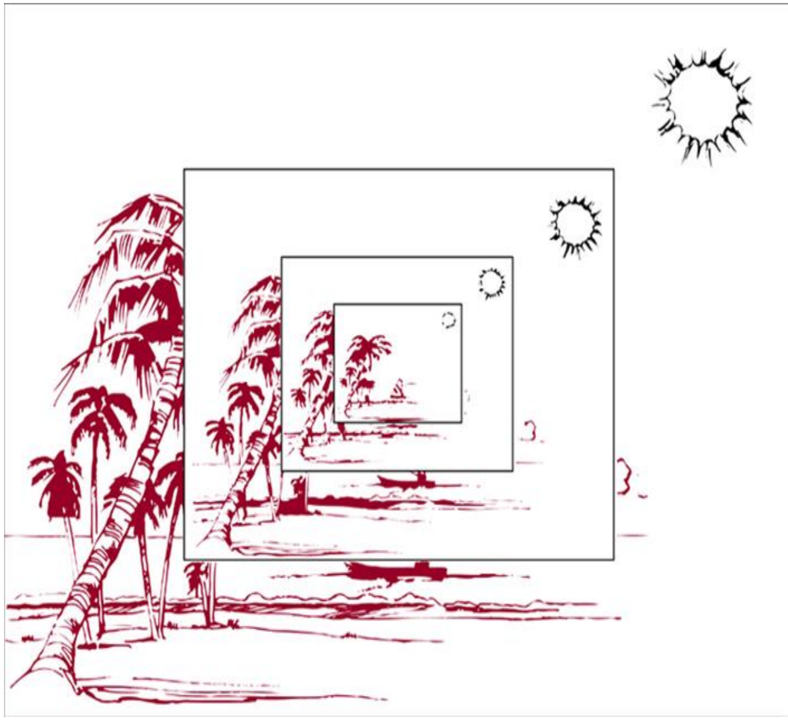
Recursive Definitions and Structural Induction

Recursive or Inductive Definitions

- **Sometimes it is difficult to define an object explicitly. However, it may be easy to define the object in terms of itself. This process is called recursion.**

- Recursion is useful to define sequences, functions, sets, and algorithms.
- When a sequence is defined recursively, by specifying how terms are formed from previous terms, we can use induction to prove results about the sequence.



Recursive or Inductive Function Definition

- Basis Step:

Specify the value of the function for the base case.

- Recursive Step:

Give a rule for finding the value of a function from its values at smaller integers greater than the base case.

Inductive Definitions

- We completely understand the function f(n) = n!   right?

    - $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n,\ n \geq 1$

But equivalently, we could define it like this:

Recursive Case

Base Case

$$n! = \begin{cases} n \cdot (n-1)!, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$

Inductive (Recursive) Definition

Inductive Definitions

- The 2nd most common example:
- Fibonacci Numbers

Note why you need two base cases.

$$f(n)=\begin{cases}0 & \text{if} \quad n=0 \\ 1 & \text{if} \quad n=1 \\ f(n-1)+f(n-2) & \text{if} \quad n>1\end{cases}$$

Base Cases

Recursive Case

Is there a non-recursive definition for the Fibonacci Numbers?

$$f(n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(Prove by induction.)

All linear recursions have a closed form.

Recursively Defined Sets: Inductive Definitions

- Examples so far have been inductively defined functions.
- Sets can be defined inductively, too.

Give an inductive definition of $T = \{x: x \text{ is a positive integer divisible by } 3\}$

$3 \in S$ Base Case

$x, y \in S \rightarrow x + y \in S$ Recursive Case

Exclusion Rule: No other numbers are in S.

Exclusion rule:
The set contains nothing other than
Those elements specified in the basic
Step or generated by the recursive step.

How can we prove it's correct?

We want to show that the definition of S:

rule 1 - $3 \in S$

rule 2 - $x,y \in S \rightarrow x + y \in S$

Contains the same elements as the set: $T=\{x: x$ is a positive integer divisible by $3\}$

To prove S = T, show

$T \subseteq S$

$S \subseteq T$

Perhaps the "trickiest" aspect of this exercise is realizing that there *is* something to prove! ☺

First, we prove $T \subseteq S$.

T = {x: x is a positive integer, multiple of 3}

If $x \in T$, then $x = 3k$ for some integer k. We show by induction on $|k|$ that $3k \in S$.

Hypothesis: P(n) – 3 n belongs to S, for all positive integers n.

Base Case P(1) = 3 $\in$ S since 3 $\in$ S by rule 1.

Inductive Hypothesis: $3k \in S$

Inductive Step: Assume 3k, $\in$ S, show that 3(k+1), $\in$ S.

Inductive Step:

$3k \in S$ by inductive hypothesis.

$3 \in S$ by rule 1.

$3k + 3 = 3(k+1) \in S$ by rule 2.

Next, we show that $S \subseteq T$.

That is, if an number x is described by S, then it is a positive multiple of 3.

Observe that the exclusion rule, all numbers in S are created by a finite number of applications of rules 1 and 2. We use the number of rule applications as our induction counter.

For example:

$3 \in S$ by 1 application of rule 1.

$9 \in S$ by 3 applications (rule 1 once and rule 2 twice).

Base Case (k=1): If $x \in S$ by 1 rule application, then it must be rule 1 and $x = 3$, which is clearly a multiple of 3.

Inductive Hypothesis: Assume any number described by k or fewer applications of the rules in S is a multiple of 3

Inductive Step: Prove that any number described by (k+1) applications of the rules is also a multiple of 3, assuming IH.

Suppose the (k+1)st rule is applied (rule 2), and it results in value

$x = a + b$. Then a and b are multiples of 3 by inductive hypothesis, and thus x is a multiple of 3.

Aside --- Message here: in a proof, follow a well-defined sequence of steps. This avoids subtle misstakes.

QED

Structural Induction

- Basic Step: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.

- Recursive step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

Validity of Structural Induction follows Mathematical Induction(for the nonnegative integers)

- P(n) the claim is true for all elements of the set that are generated by *n* or fewer applications of the rules in the recursive step of the recursive definition.
- So, we will do induction on the number of rules applications.
- We show that P(n) is true whenever n is a nonnegative integer.

Basis case - we show that P(0) is true (i.e., it's true for the elements specified in the basis step of recursive definition).

From recursive step, if we assume P(k), it follows that P(k+1) is true.
Therefore when we complete a structural induction proof we have shown that P(0) is true, and that $P(k) \rightarrow P(k+1)$.

So, by mathematical induction P(n) follows for all nonnegative numbers.

Well-Formed Formulas

T is a wff

F is a wff

p is a wff for any propositional variable p

If p is a wff, then $(\neg p)$ is a wff

If p and q are wffs, then $(p \vee q)$ is a wff

If p and q are wffs, then $(p \wedge q)$ is a wff

Basic Cases

Recursive Step

For example, a statement like $((\neg r) \vee (p \wedge r))$ can be proven to be a wff by arguing that $(\neg r)$ and $(p \wedge r)$ are wffs by induction and then applying rule 5.

Note: we have three recursive/construction rules to create new elements.

Structural induction --- illustrative example

Show that every well-formed formula for compound propositions, contains an equal number of left and right parentheses.

Basic Step --- **True since each formula T, F, and p contains no parentheses;**

Recursive Step:

Assume p and q are well formed formulas with an equal number of left and right parentheses (lp = rp; lq=rq)

We need to show that $(\neg p)$, $(p \vee q)$, and $(p \wedge q)$ contain an equal number of parentheses. Follows directly be considering each rule: Each rule adds a left and a right parenthesis.

The key aspect of structural induction proofs is to show that the base case satisfies the property that we want to prove *and the recursive steps/rules maintain it!*

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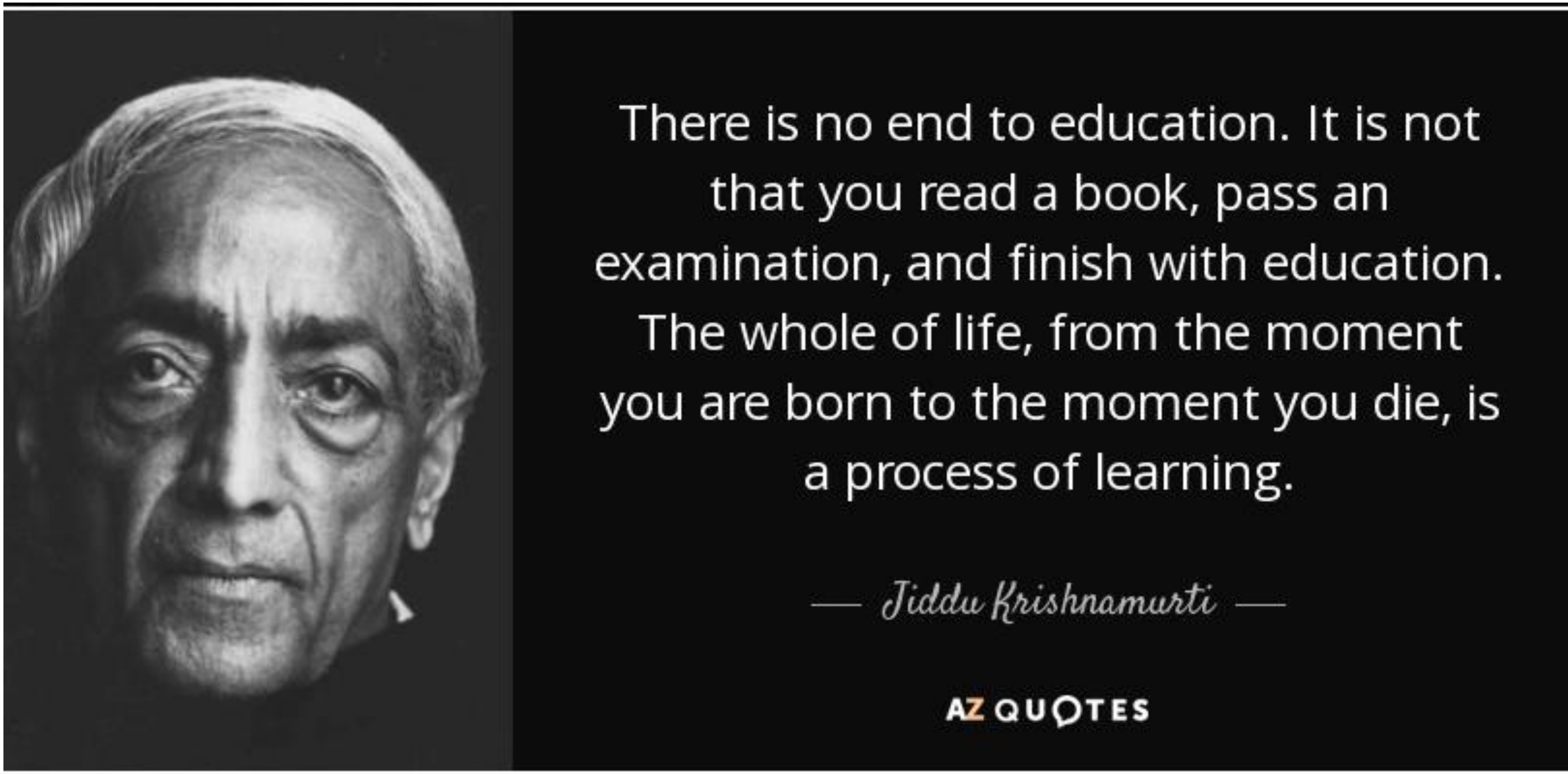
Can reformulate into induction by doing induction on the # of rule applications.



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