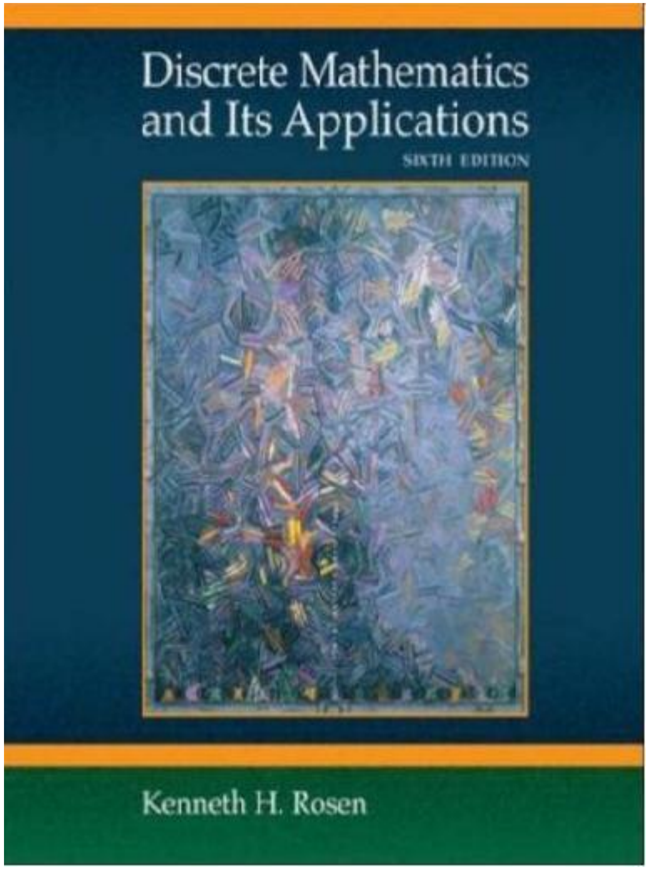
Jiangxi University of Science and Technology

# Discrete Mathematics
# and
# Its Applications

Chapter 2

Sequences, Sums, and Matrices



# Acknowledgement


- Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University and Dr Johnnie Baker

# Summation

- The symbol ∑ (Greek letter sigma) is used to denote summation.

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$$

*i* is the **index of the summation**, and the choice of letter *i* is arbitrary;

the index of the summation runs through all integers, with its **lower limit** 1 and ending **upper limit** k.

- The limit:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

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# Summation

- The laws for arithmetic apply to summations

$$\sum_{i=1}^{k}(ca_i + b_i) = c\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$$

Use associativity to separate the b terms from the a terms.

Use distributivity to factor the c's.

# Summations you should know…

- What is S = 1 + 2 + 3 + … + n?

(little) Gauss in 4th grade. ☺

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| S | = | 1 | + | 2 | + | … | + | n | Write the sum. |
| S | = | n | + | n-1 | + | … | + | 1 | Write it again. |
| 2s | = | n+1 | + | n+1 | + | … | + | n+1 | Add together. |

You get n copies of (n+1). But we've over added by a factor of 2.
So just divide by 2.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Why whole number?

Sum of first n odds.

- What is S = 1 + 3 + 5 + … + (2n - 1)?

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

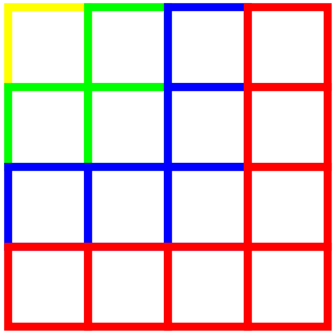
$$= 2\left(\frac{n(n+1)}{2}\right) - n$$

$$= n^2$$

Sum of first n odds.

- What is S = 1 + 3 + 5 + … + (2n - 1)?

$$= n^2$$



Geometric Series

- What is $S = 1 + r + r^2 + \ldots + r^n$

$$\sum_{k=0}^{n} r^k = 1 + r + \ldots + r^n$$

Multiply by r

$$r\sum_{k=0}^{n} r^k = r + r^2 + \ldots + r^{n+1}$$

Subtract the summations

$$\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

factor

$$(1-r)\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

divide

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{(1-r)}$$

DONE!

- What about:

$$\sum_{k=0}^{\infty} r^k = 1 + r + \ldots + r^n + \ldots$$

If r ≥ 1 this blows up.

If r < 1 we can say something.

$$\sum_{k=0}^{\infty} r^k = \lim_{n \to \infty} \sum_{k=0}^{n} r^k$$

$$= \lim_{n \to \infty} \frac{1 - r^{n+1}}{(1 - r)} \qquad = \frac{1}{(1 - r)}$$

Try r = ½.

# Useful Summations

| Sum | Closed Form |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r - 1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$ | $\frac{1}{(1-x)^2}$ |

# Infinite Cardinality

- How can we extend the notion of cardinality to infinite sets?

- Definition: Two sets **A and B have the same cardinality** if and only if there exists a bijection (or a one-to-one correspondence) between them, A ~ B.

We split infinite sets into two groups:

1. Sets with the **same cardinality as the set of natural numbers**
2. Sets with **different cardinality as the set of natural numbers**

# Infinite Cardinality

- Definition: A set is **countable** if it is **finite** or has the same **cardinality as the set of positive integers.**
- Definition: A set is un**countable** if it is **not countable.**
- Definition: The cardinality of an infinite set S that is countable is denotes by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet).
- We write $|S| = \aleph_0$ and say that S has cardinality "aleph null"

**Note: Georg Cantor defined the notion of cardinality and was the first to realize that infinite sets can have different cardinalities. $\aleph_0$ is the cardinality of the natural numbers; the next larger cardinality is aleph-one $\aleph_1$, then, $\aleph_2$ and so on.**

# Infinite Cardinality: Odd Positive Integers

Example: The set of odd positive integers is a countable set.

Let's define the function f, from $Z^+$ to the set of odd positive numbers,

$$f(n) = 2n - 1$$

We have to show that f is both one-to-one and onto.

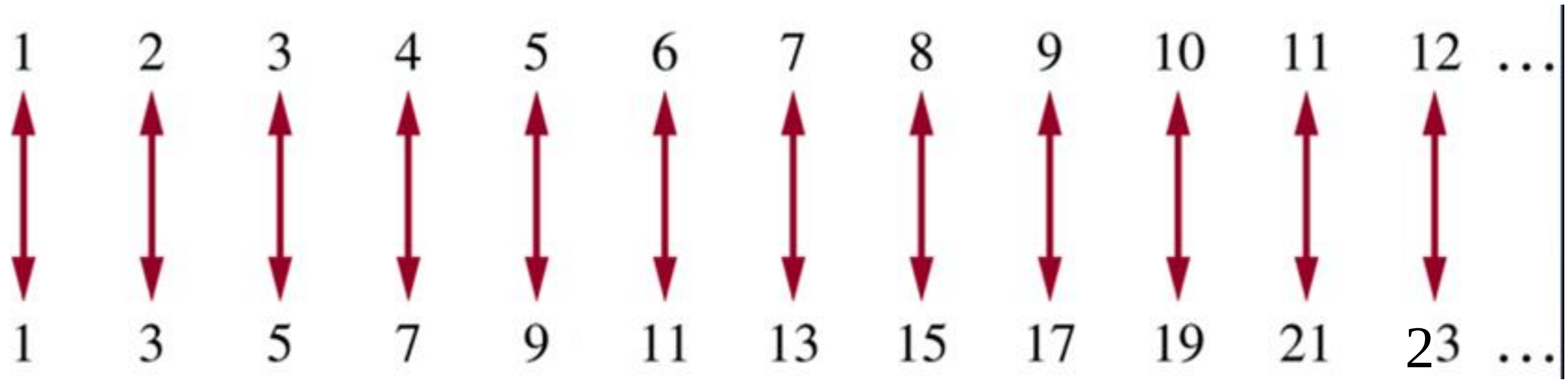
one-to-one

Suppose $f(n) = f(m) \rightarrow 2n-1 = 2m-1 \rightarrow n=m$

onto

Suppose that t is an odd positive integer. Then t is 1 less than an even integer 2k, where k is a natural number. hence $t=2k-1= f(k)$.

# Infinite Cardinality: Odd Positive Integers



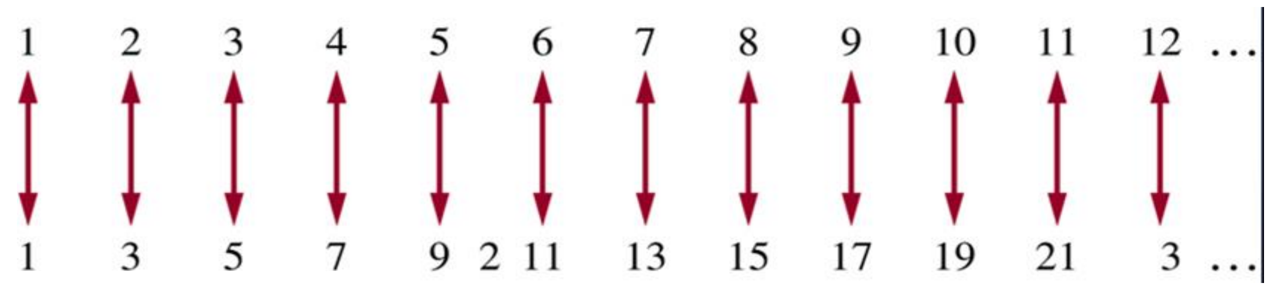
# Infinite Cardinality:Integers

Example: **The set of integers is a countable set.**

**Lets consider the sequence of all integers, starting with 0: 0,1,-1,2,-2,….**

**We can define this sequence as a function:**

$$f(n) = \begin{cases} n/2 & n \in N, even \\ \dfrac{-(n-1)}{2} & n \in N, odd \end{cases}$$




0 1 -1 2 2

Show at home that it’s one-to-one and onto

# Infinite Cardinality: Rational Numbers

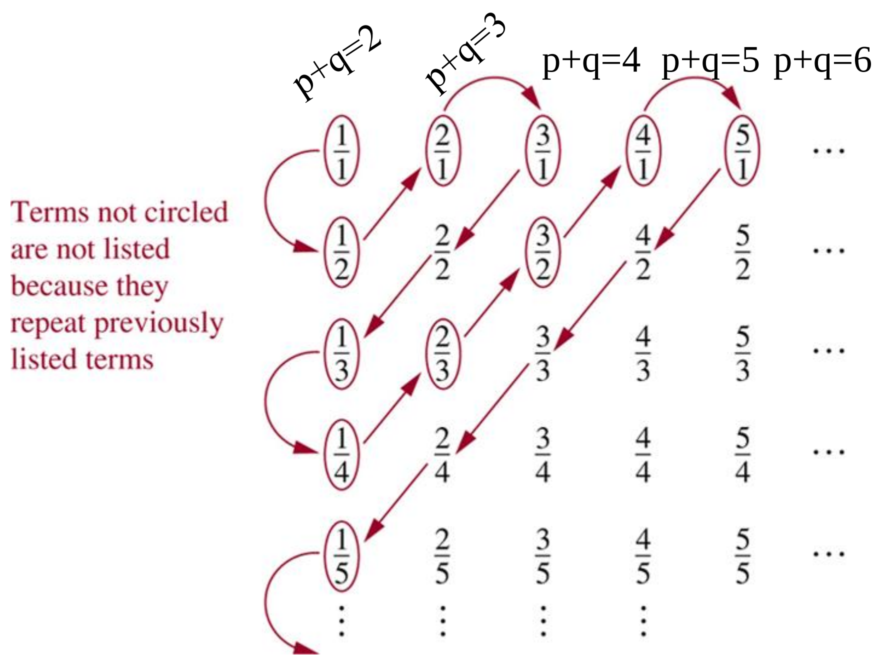
- Example: The set of **positive rational numbers** is a **countable** set.
- Hmm…

# Infinite Cardinality: Rational Numbers

Example: The set of **positive rational numbers** is a **countable** set

Key aspect to list the rational numbers as a sequence – every positive number is the quotient p/q of two positive integers.

Visualization of the proof.




Since all positive rational numbers are listed once, the set of positive rational numbers is countable.

# Uncountable Sets: Cantor's diagonal argument

The set of all **infinite sequences of zeros and ones** is **uncountable.**

Consider a sequence,

$$a_1, a_2, \cdots, a_n, n \to \infty, a_i = 0 \; or \; a_i = 1$$

For example:

$$s_1 = (0, 0, 0, 0, 0, 0, 0, \ldots)$$
$$s_2 = (1, 1, 1, 1, 1, 1, 1, \ldots)$$
$$s_3 = (0, 1, 0, 1, 0, 1, 0, \ldots)$$
$$s_4 = (1, 0, 1, 0, 1, 0, 1, \ldots)$$
$$s_5 = (1, 1, 0, 1, 0, 1, 1, \ldots)$$
$$s_6 = (0, 0, 1, 1, 0, 1, 1, \ldots)$$
$$s_7 = (1, 0, 0, 0, 1, 0, 0, \ldots)$$

So in general we have:

$$s_n = (s_{n,1}, s_{n,2}, s_{n,3}, s_{n,4}, \ldots)$$

i.e., $s_{n,m}$ is the $m^{th}$ element of the $n^{th}$ sequence on the list.

# Uncountable Sets: Cantor's diagonal argument

- It is possible to build a sequence, say $s_0$, in such a way that its first element is different from the first element of the first sequence in the list, its second element is different from the second element of the second sequence in the list, and, in general, its *n*th element is different from the $n^{th}$ element of the $n^{th}$ sequence in the list. In other words, $s_{0,m}$ will be 0 if $s_{m,m}$ is 1, and $s_{0,m}$ will be 1 if $s_{m,m}$ is 0.

# Uncountable Sets: Cantor's diagonal argument

$s_1 = (\underline{\mathbf{0}}, 0, 0, 0, 0, 0, 0, \ldots)$
$s_2 = (1, \underline{\mathbf{1}}, 1, 1, 1, 1, 1, \ldots)$
$s_3 = (0, 1, \underline{\mathbf{0}}, 1, 0, 1, 0, \ldots)$
$s_4 = (1, 0, 1, \underline{\mathbf{0}}, 1, 0, 1, \ldots)$
$s_5 = (1, 1, 0, 1, \underline{\mathbf{0}}, 1, 1, \ldots)$
$s_6 = (0, 0, 1, 1, 0, \underline{\mathbf{1}}, 1, \ldots)$
$s_7 = (1, 0, 0, 0, 1, 0, \underline{\mathbf{0}}, \ldots)$
...
$s_0 = (\underline{\mathbf{1}}, \underline{\mathbf{0}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{0}}, \underline{\mathbf{1}}, \ldots)$

Note: the diagonal elements are highlighted, showing why this is called the **diagonal argument**

- The sequence $s_0$ is distinct from all the sequences in the list. Why?
- Let's say that $s_0$ is identical to the 100th sequence, therefore, $s_{0,100}=s_{100,100}$.
- In general, if it appeared as the *n*th sequence on the list, we would have $s_{0,n} = s_{n,n}$,
- which, due to the construction of $s_0$, is impossible.

# Uncountable Sets: Cantor's diagonal argument

From this it follows that the set $T$, consisting of all infinite sequences of zeros and ones, cannot be put into a list $s_1$, $s_2$, $s_3$, ...

Otherwise, it would be possible by the above process to construct a sequence $s_0$ which would both be in $T$ (because it is a sequence of 0's and 1's which is by the definition of $T$ in $T$) and at the same time not in $T$ (because we can deliberately construct it not to be in the list). $T$, containing all such

sequences, must contain $s_0$, which is just such a sequence. But since $s_0$ does not appear anywhere on the list, $T$ cannot contain $s_0$.

Therefore $T$ cannot be placed in one-to-one correspondence with the natural numbers. In other words, the set of infinite binary strings is **uncountable**.

## Real Numbers

Example;

The set of real numbers is an uncountable set.

Let's assume that the set of real numbers is countable.

Therefore any subset of it is also countable, in particular the interval [0,1].

How many real numbers are in interval [0, 1]?

# Real Numbers

- How many real numbers are in interval [0, 1]?

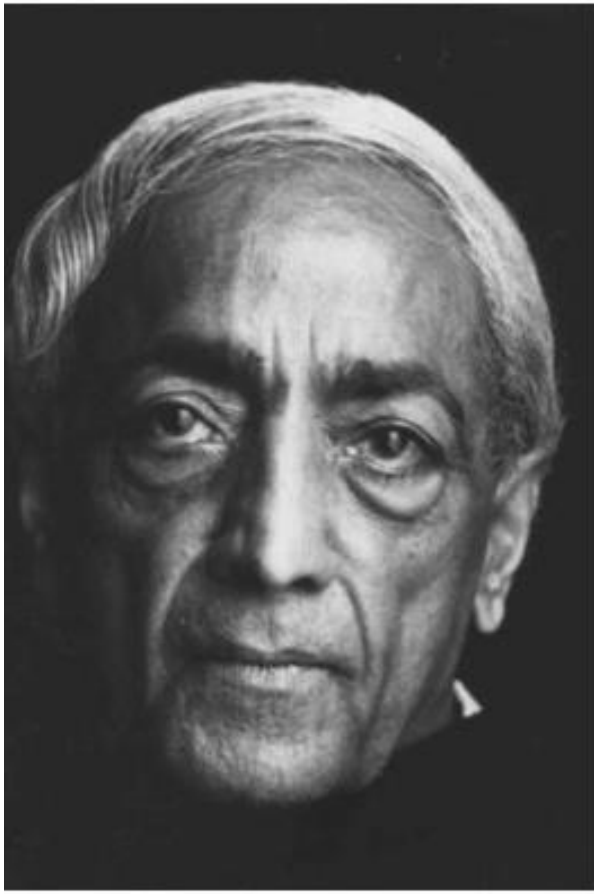
0.4 3 2 9 0 1 3 2 9 8 4 2 0 3 9 …
0.8 2 5 9 9 1 3 2 7 2 5 8 9 2 5 …
0.9 2 5 3 9 1 5 9 7 4 5 0 6 2 1 …
⋮

"Countably many! There's part of the list!"

"Are you sure they're all there?"

Counterexample:
Use diagonalization
to create a new number
that differs in the ith
position of the
ith number
by 1.

0.5 3 6 …
So we say the reals are "uncountable."



There is no end to education. It is not that you read a book, pass an examination, and finish with education. The whole of life, from the moment you are born to the moment you die, is a process of learning.

— Jiddu Krishnamurti —

AZ QUOTES