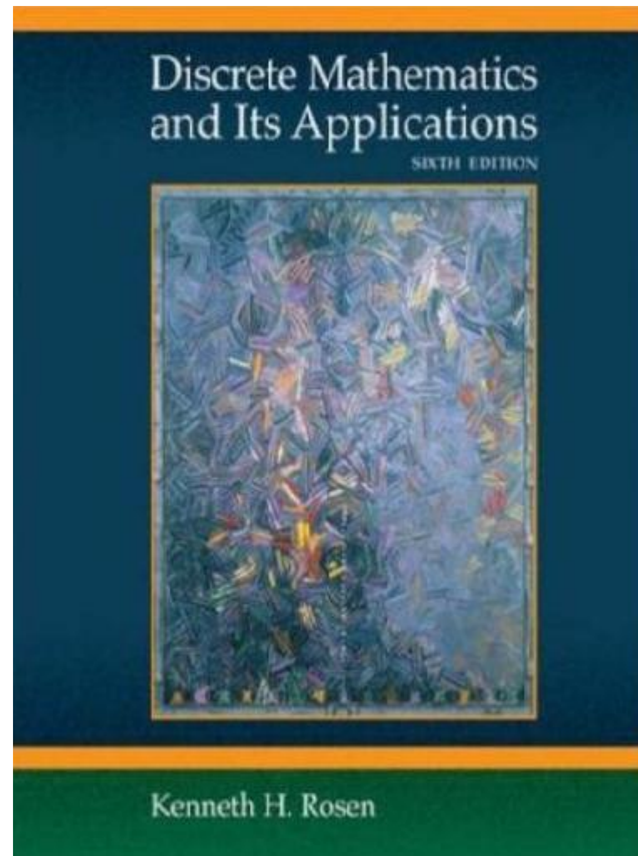
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications



Chapter 2

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Acknowledgement

- Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University and Dr Johnnie Baker

Section Summary

Definition of sets

Describing Sets

- Roster Method
- Set-Builder Notation

Some Important Sets in Mathematics

Empty Set and Universal Set

Subsets and Set Equality

Cardinality of Sets

Tuples

Cartesian Product

Section 2.1:Sets Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- **Set theory is an important branch of mathematics.**
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room

- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

Each distinct object is either a member or not; listing more than once does not change the set.

$$\boldsymbol{S = \{a,b,c,d\} = \{a,b,c,b,c,d\}}$$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

Set of all positive integers less than 100:

$S = \{1,2,3,\ldots\ldots..,99\}$

Set of all integers less than 0:

$S = \{\ldots., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z⁺ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R⁺ = set of *positive real numbers*

C = set of *complex numbers.*

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

$$S = \{x \mid P(x)\}$$

- Example: $\boldsymbol{S = \{x \mid \mathrm{Prime}(x)\}}$

Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

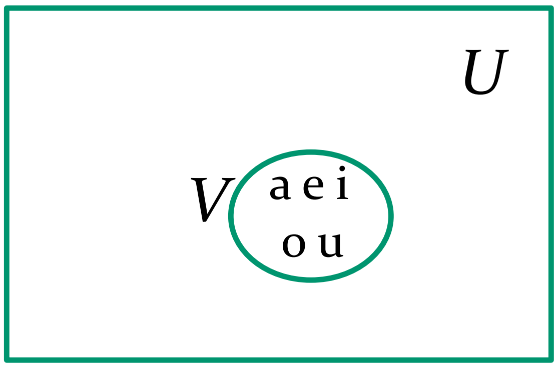
- $[a,b] = \{x \mid a \le x \le b\}$ $\quad [a,b) = \{x \mid a \le x < b\}$
- $(a,b] = \{x \mid a < x \le b\}$ $\quad (a,b) = \{x \mid a < x < b\}$
  - *closed interval* [a,b]
  - *open interval* (a,b)

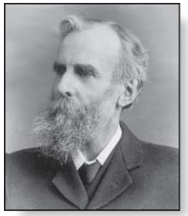
Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.

- **The empty set is the set with no elements. Symbolized ∅, but{} also used.**

Venn Diagram





John Venn (1834-1923)
Cambridge, UK

Russell’s Paradox

- Let S be the set of all sets which are not members of themselves. A paradox results from trying to answer the question “Is S a member of itself?”

Related Paradox:

Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question “Does Henry shave himself?”

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b,c\}\}$

  $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Equality

Definition:

Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

- **We write *A* = *B* if *A* and *B* are equal sets.**

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition:

The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff $\forall x(x \in A \leftrightarrow x \in B)$
- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A$$

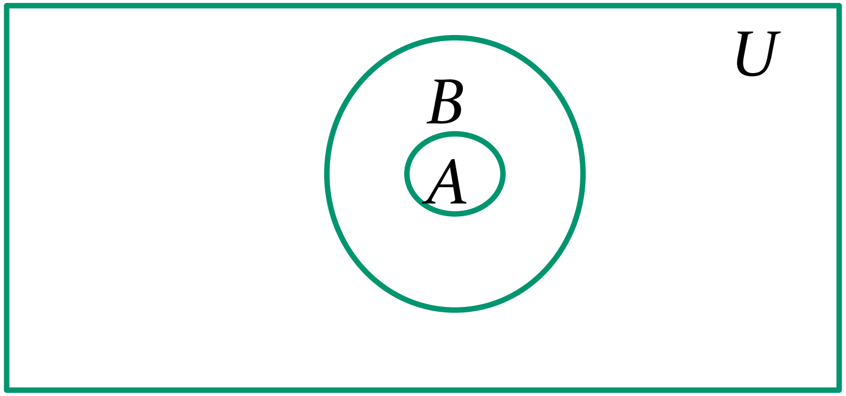
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by |*A*|, is the number of (distinct) elements of *A*.

Examples:

1. |ø| = 0
2. Let S be the letters of the English alphabet. Then |*S*| = 26
3. |{1,2,3}| = 3
4. |{ø}| = 1
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If *A* = {a,b} then

$$\mathcal{P}(A) = \{\varnothing, \{a\},\{b\},\{a,b\}\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.
The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition:

A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. ***(Relations will be covered in depth in Chapter 9.)***

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example:
What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution:
$A \times B \times C =$
{(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

Example:
The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1,1}

Section Summary Section 2.2 :Set Operations

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*.

  This is discussed in Chapter 12.

- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

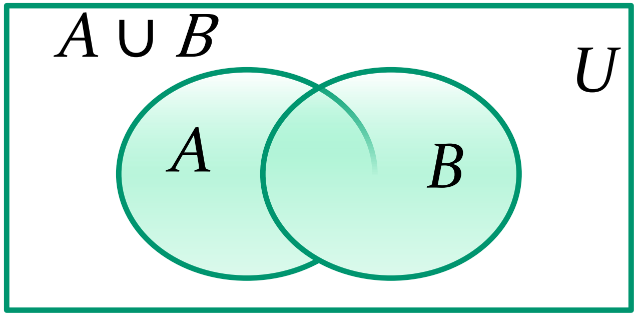
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x | x \in A \wedge x \in B\}$$

- Note if the intersection is empty, then A and B are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

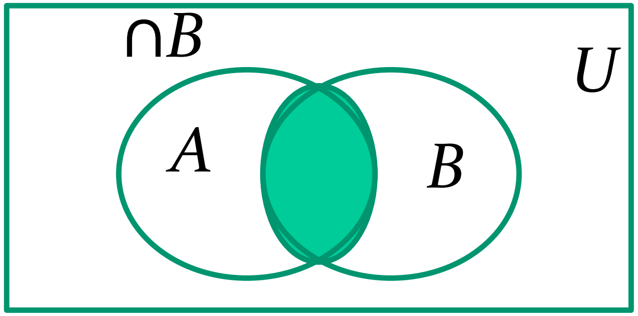
  Solution: $\{3\}$

- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

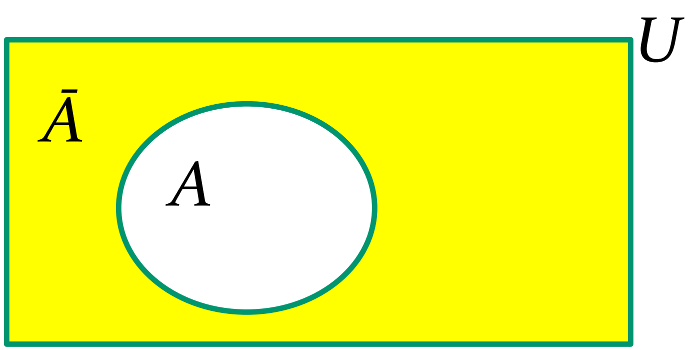
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement



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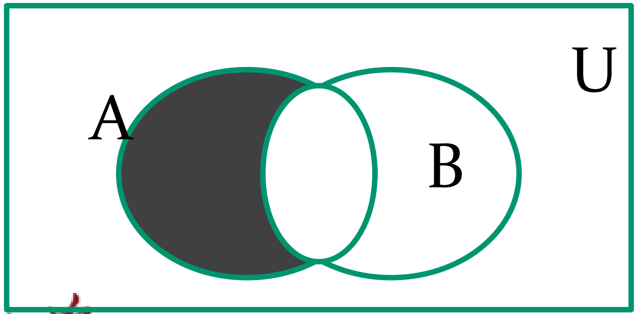
Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

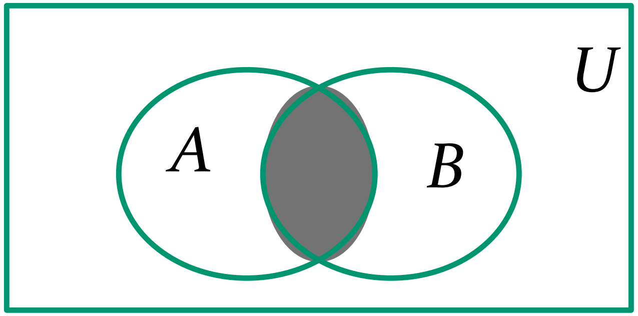


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$$|A \cup B| = |A| + |B| + |A \cap B|$$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

Example:
Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: {1,2,3,4,5,6,7,8}

2. $A \cap B$

   Solution: {4,5}

3. $\bar{A}$

   Solution: {0,6,7,8,9,10}

4. $\bar{B}$

   Solution: {0,1,2,3,9,10}

5. $A - B$

   Solution: {1,2,3}

6. $B - A$

   Solution: {6,7,8}

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

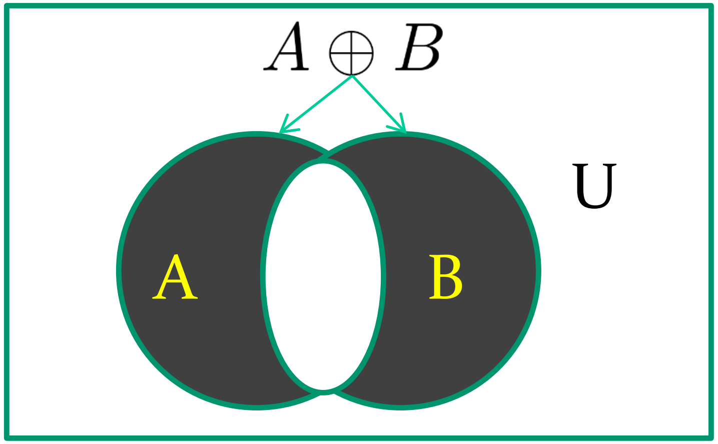
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

– **Solution**: {1,2,3,6,7,8}



Venn Diagram

Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

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Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:

1. **Prove that each set (side of the identity) is a subset of the other.**
2. **Use set builder notation and propositional logic.**
3. **Membership Tables: Verify that elements in the same**

- combination of sets always either belong or do not belong to the same side of the identity.
- Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{lcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.
  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

- For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

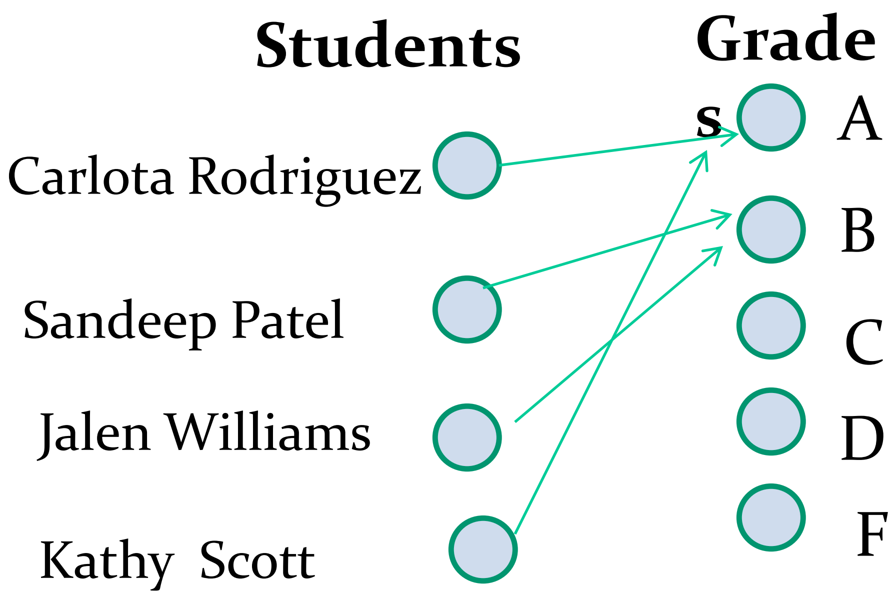
Section Summary

- Definition of a Function.
  - Domain, Cdomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation).
- This subset is restricted to be a relation where no two elements of the relation have the same first element.

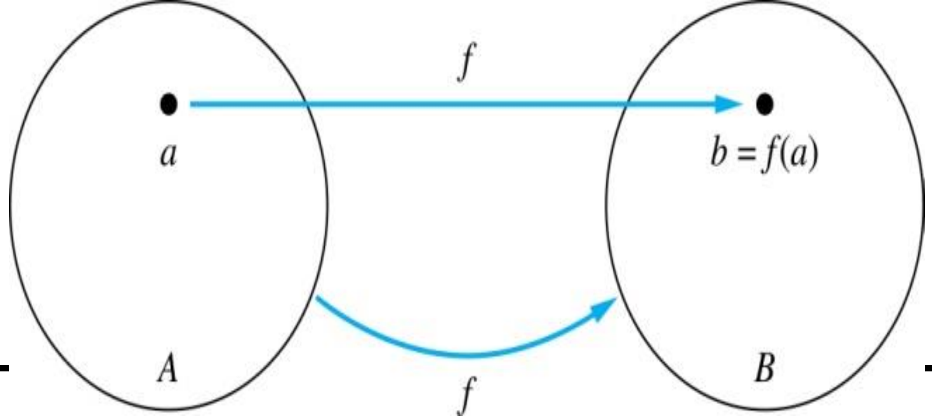
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions



Given a function f: $A \to B$**:**

- We say *f maps A* to *B or f* is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.

- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.

- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.

- **Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.**

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.

  - A formula.

    $f(x) = x + 1$

  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also inChapter 5).

Questions

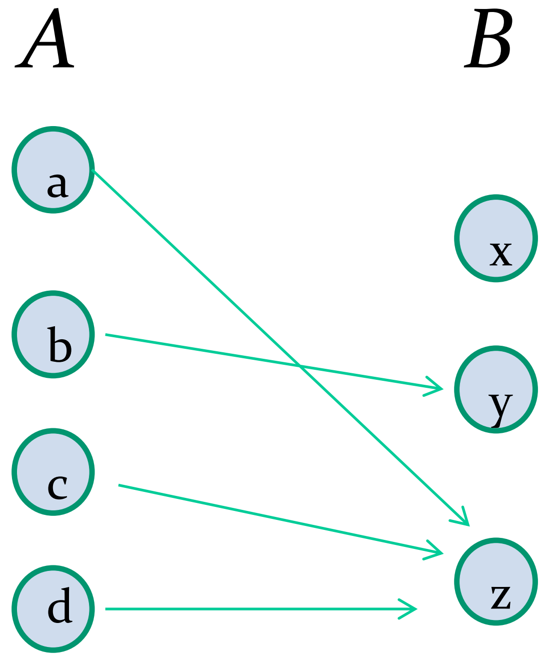
f(a) = ? z

The image of d is ? z

The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b



f(A) = ?

The preimage(s) of z is (are) ? {a,c,d}

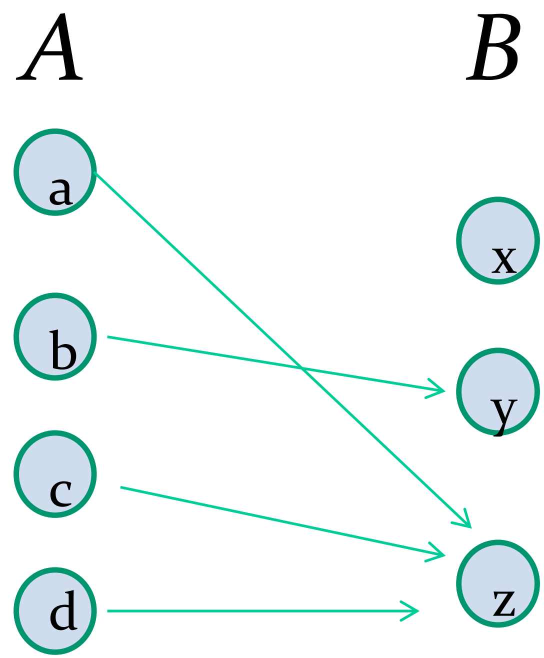
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

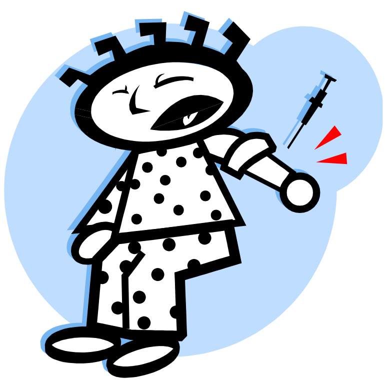
f {a,b,c,} is ? {y,z}

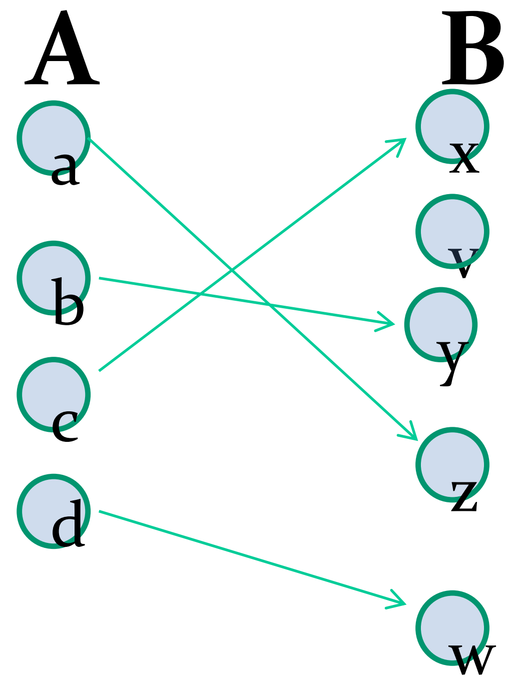
f {c,d} is ? {z}



Injections

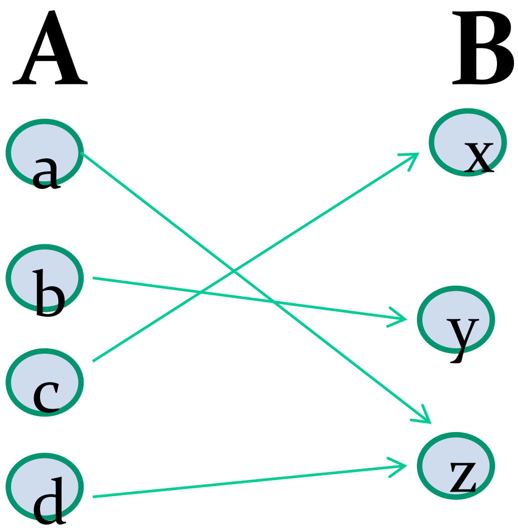
Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.





Surjections

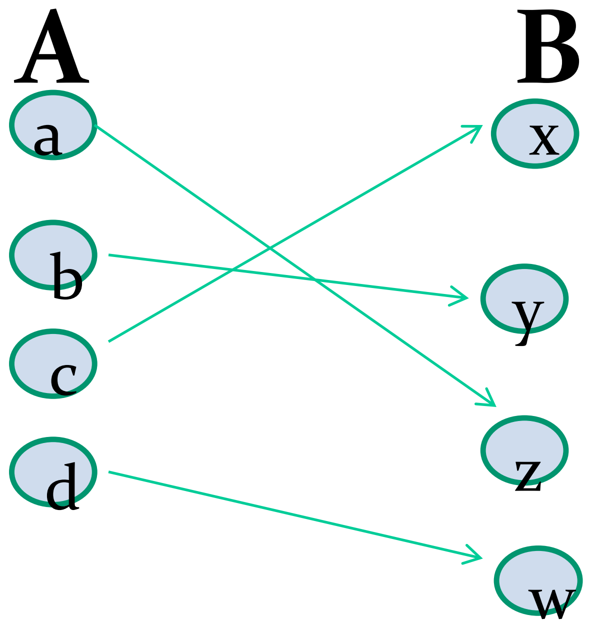
Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.



Bijections

Definition:

A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

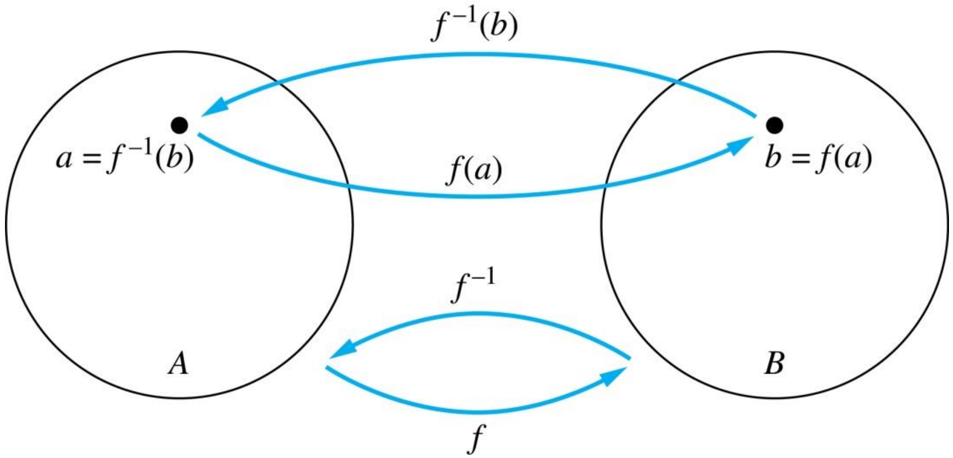
Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

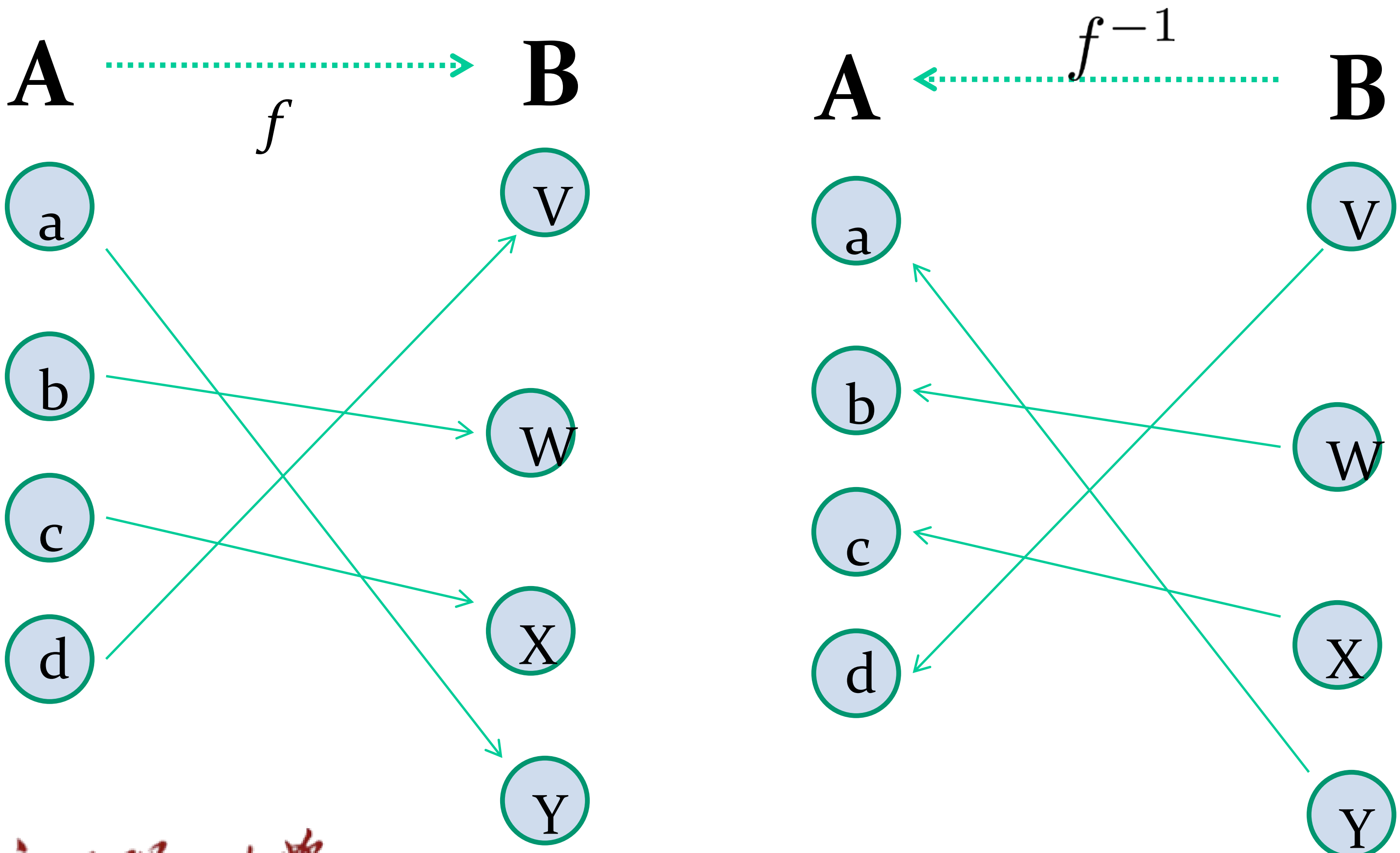
Inverse Functions

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

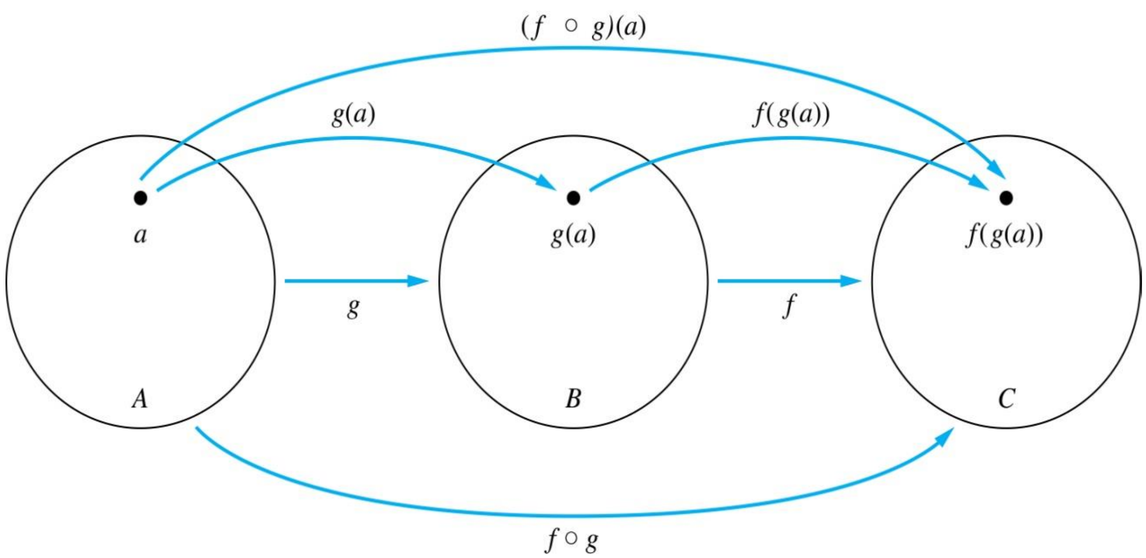
Questions

Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$.
Is f invertible, and if so, what is its inverse?

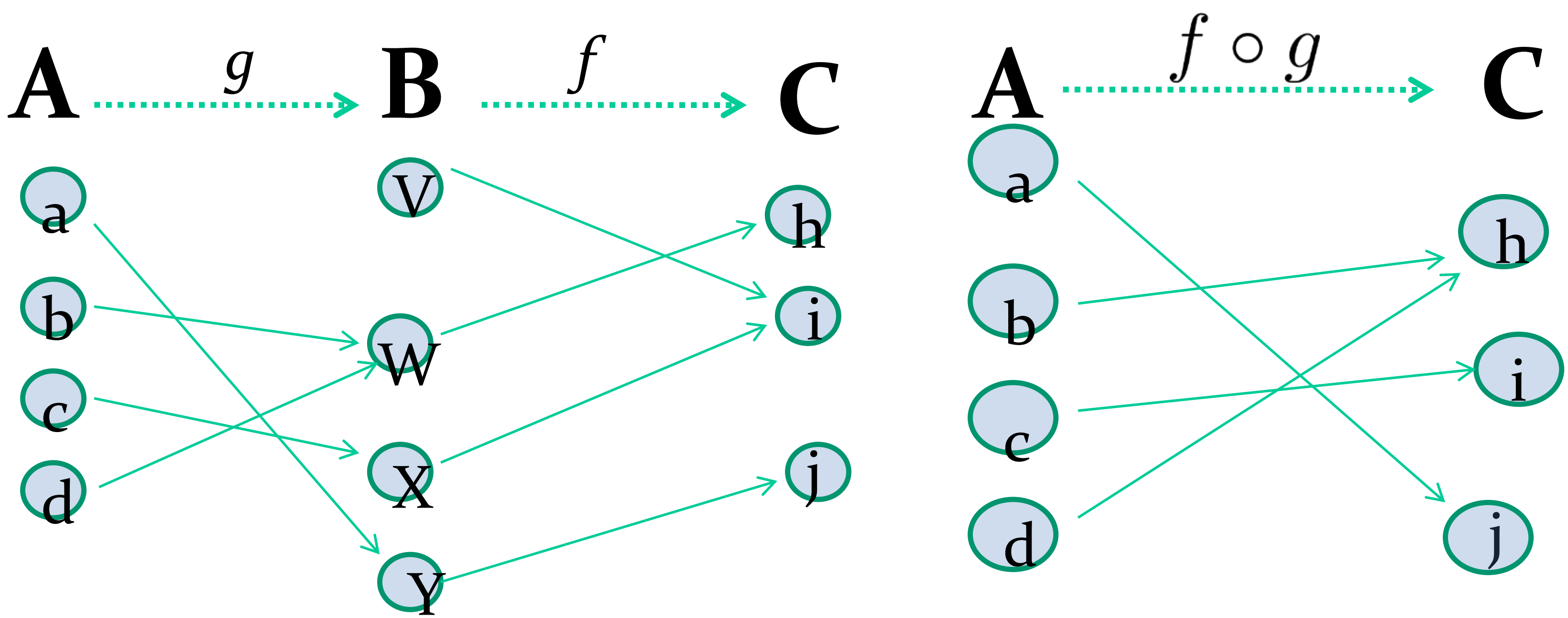
Solution:
The function f is not invertible because it is not one-to-one .

Composition

- **Definition**: Let *f*: $B \to C$, *g*: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $$f \circ g(x) = f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f ?

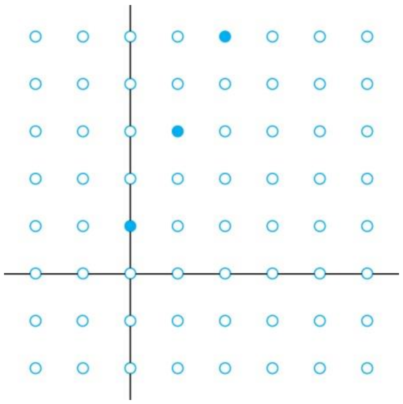
Solution:

$$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

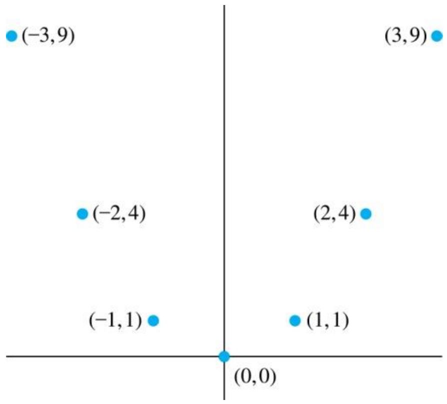
$$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

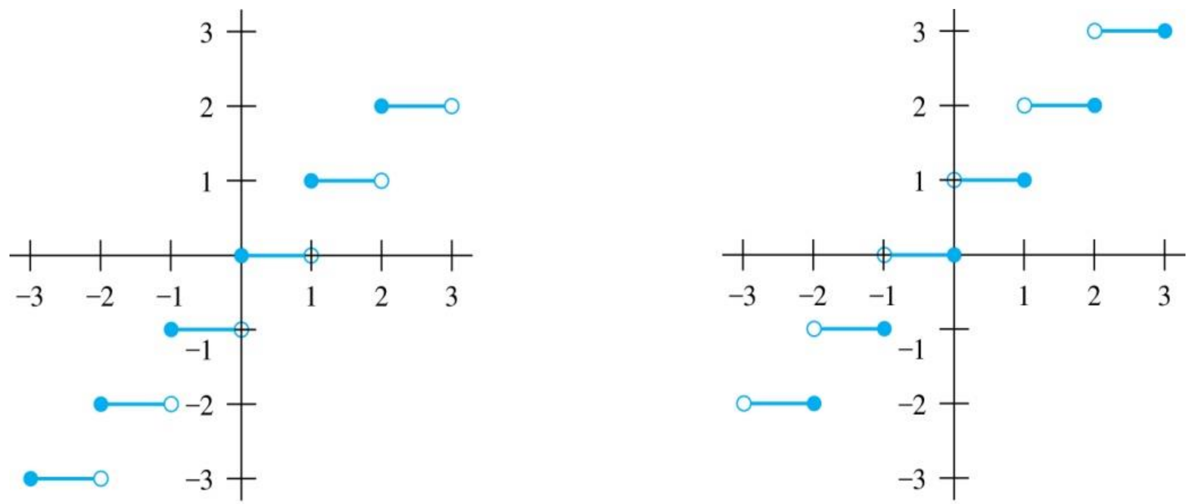
$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



(a) $y = [x]$ (b) $y = [x]$

Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

Factorial Function

Definition: f: $\mathbf{N} \rightarrow \mathbf{Z}^+$ **,** denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n,$ $\quad f(0) = 0! = 1$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Partial Functions (*optional*)

Definition: A *partial function* f from a set A to a set B is an assignment to each element a in a subset of A**,** called the *domain of definition* of f, of a unique element b in B.

- The sets A and B are called the *domain* and *codomain* of f, respectively.
- We day that f is *undefined* for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a *total function*.

Example: f: $\mathbf{N} \rightarrow \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

Some important functions

Absolute value:

Domain R; Co-Domain = $\{0\} \cup R^+$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|-3| = 3$; $|3| = 3$

Floor function (or greatest integer function):

Domain = R; Co-Domain = Z

$\lfloor x \rfloor$ = largest integer not greater than x

Ex: $\lfloor 3.2 \rfloor = 3$; $\lfloor -2.5 \rfloor = -3$

Some important functions

- **Ceiling function:**

  Domain = R;

  Co-Domain = Z

  $\lceil x \rceil$ = smallest integer greater than x

  Ex: $\lceil 3.2 \rceil = 4$; $\lceil -2.5 \rceil = -2$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
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(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Some important functions

Factorial function: Domain = Range = N **Error on range**

n! = n (n-1)(n-2) …, 3 x 2 x 1

Ex: 5! = 5 x 4 x 3 x 2 x 1 = 120

Note: 0! = 1 by convention.

Some important functions

Mod (or remainder):

- Domain = $N \times N^+ = \{(m,n) | m \in N, n \in N+ \}$
  Co-domain Range = N

  $m \textit{ mod } n = m - \lfloor m/n \rfloor n$

Ex: $8 \text{ mod } 3 = 8 - \lfloor 8/3 \rfloor 3 = 2$
$57 \text{ mod } 12 = 9;$

Note: This function computes the remainder when m is divided by n.
The name of this function is an abbreviation of m modulo n, where modulus means with respect to a modulus (size) of n, which is defined to be the remainder when m is divided by n. Note also that this function is an example in which the domain of the function is a 2-tuple.

Some important functions: Exponential Function

Exponential function:

- Domain = $R^+ \times R$ = $\{(a,x) | a \in R+, x \in R\}$
  Co-domain Range = R^+
  $f(x) = a^x$

Note: *a* is a **positive** constant; x varies.

Ex: $f(n) = a^n = a \times a \ldots, \times a$ (n times)

How do we define f(x) if x is not a positive integer?

Some important functions: Exponential function

Exponential function:

- How do we define f(x) if x is not a positive integer?

Important properties of exponential functions:

(1) $a^{(x+y)} = a^x a^y$; (2) $a^1 = a$ (3) $a^0 = 1$

See:

$$a^2 = a^{1+1} = a^1 a^1 = a \times a;$$

$$a^3 = a^{2+1} = a^2 a^1 = a \times a \times a;$$

$$\ldots$$

$$a^n = a \times \cdots \times a \quad (n\ times)$$

We get:

$$a = a^1 = a^{1+0} = a \times a^0 \quad therefore \quad a^0 = 1$$

$$1 = a^0 = a^{b+(-b)} = a^b \times a^{-b} \quad therefore \quad a^{-b} = 1/a^b$$

$$a = a^1 = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2 therefore \quad a^{\frac{1}{2}} = \sqrt{a}$$

By similar arguments:

$$a^{\frac{1}{k}} = \sqrt[k]{a}$$

$$a^{mx} = a^x \times \cdots a^x \ (m \quad times) = (a^x)^m, \quad therefore \quad a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$$

Note: This determines a^x for all x rational. x is irrational by continuity (we'll skip "details").

Some important functions: Logarithm Function

Logarithm base a:

Domain = $R^+ \times R$ = $\{(a,x) \mid a \in R+, a>1, x \in R\}$
Co-domain Range = R
$y = \log_a(x) \Leftrightarrow a^y = x$

Ex: $\log_2(8) = 3$; $\log_2(16) = 3$; $3 < \log_2(15) < 4$.

Key properties of the log function (they follow from those for exponential):

1. $\log_a(1) = 0$ (because $a^0 = 1$)
2. $\log_a(a) = 1$ (because $a^1 = a$)
3. $\log_a(xy) = \log_a(x) + \log_a(x)$ (similar arguments)
4. $\log_a(x^r) = r \log_a(x)$
5. $\log_a(1/x) = -\log_a(x)$ (note $1/x = x^{-1}$)
6. $\log_b(x) = \log_a(x) / \log_a(b)$

Logarithm Functions

Examples:

$\log_2(1/4) = -\log_2(4) = -2.$

$\log_2(-4)$ undefined

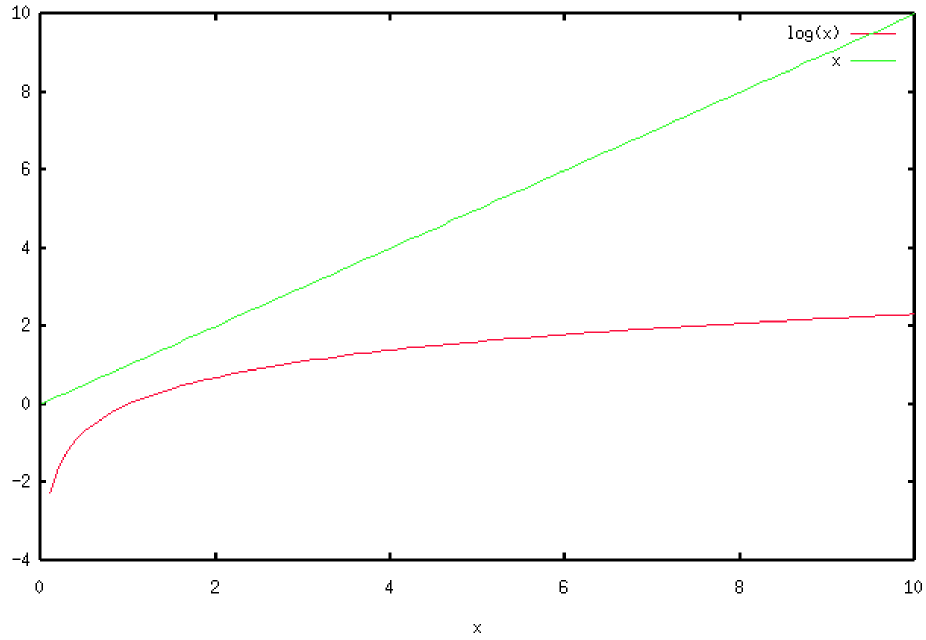
$\log_2(2^{10}\,3^5) = \log_2(2^{10}) + \log_2(3^5) = 10\log_2(2) + 5\log_2(3) =$

$= 10 + 5\log_2(3)$

Limit Properties of Log Function

$$\lim_{x \to \infty} \log(x) = \infty$$

$$\lim_{x \to \infty} \frac{\log(x)}{x} = 0$$



As x gets large, *log(x)* grows without bound.

But *x* grows **MUCH** faster than *log*(*x*)…more soon on growth rates.

Some important functions:Polynomials

- **Polynomial function:**
  Domain = usually R
  Co-domain Range = usually R

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

n, a nonnegative integer is the degree of the polynomial;
$a_n \neq 0$ (so that the term $a_n x^n$ actually appears)

$(a_n, a_{n-1}, \dots, a_1, a_0)$ are the coefficients of the polynomial.

Ex:

$y = P_1(x) = a_1 x^1 + a_0$ linear function

$y = P_2(x) = a_2 x^2 + a_1 x^1 + a_0$ quadratic polynomial or function

•Exponentials grow MUCH faster than polynomials:

$$\lim_{x\to\infty} \frac{a_0 + \cdots + a_k x^k}{b^x} = 0 \ if \ \ b > 1$$

We’ll talk more about growth rates in the next module….

Sequences

- Definition:

  A sequence $\{a_i\}$ is a function $f: A \subseteq N \cup \{0\} \to S$, where we write a_i to indicate $f(i)$. We call a_i term I of the sequence.

- Examples:
- Sequence $\{a_i\}$, where $a_i = i$ is just $a_0 = 0, a_1 = 1, a_2 = 2, \ldots$
- Sequence $\{a_i\}$, where $a_i = i^2$ is just $a_0 = 0, a_1 = 1, a_2 = 4, \ldots$

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science.
(always check whether sequence starts at a_0 or a_1)
These finite sequences are also called strings. The length of a string is the number of terms in the string. The empty string, denoted by λ, is the string that has no terms.

Geometric and Arithmetic Progressions

•Definition: A **geometric progression** is a sequence of the form

$$a, ar, ar^2, ar^3, \cdots, ar^n, \cdots$$

The **initial term** ***a*** and the common **ratio** ***r*** are real numbers

Definition: An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, a+3d, \cdots, a+nd, \cdots$$

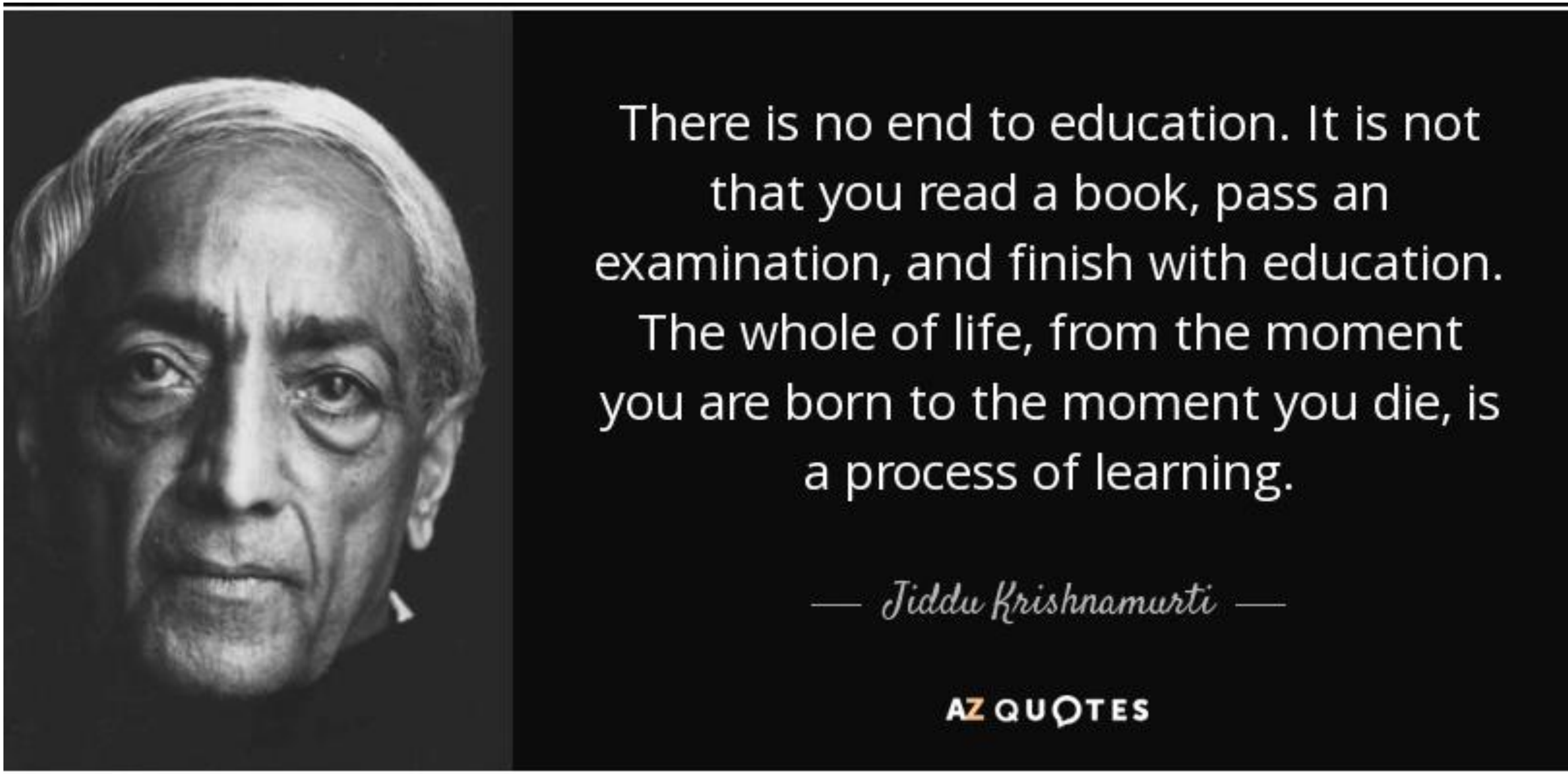
The **initial term** ***a*** and the common **difference** ***d*** are real numbers

Note: An arithmetic progression is a discrete analogue of the linear function f(x) = dx + a

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, … .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, … .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, … .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, … .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, … .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, … .

Notice differences in growth rate.


There is no end to education. It is not that you read a book, pass an examination, and finish with education. The whole of life, from the moment you are born to the moment you die, is a process of learning.
— Jiddu Krishnamurti —
AZ QUOTES