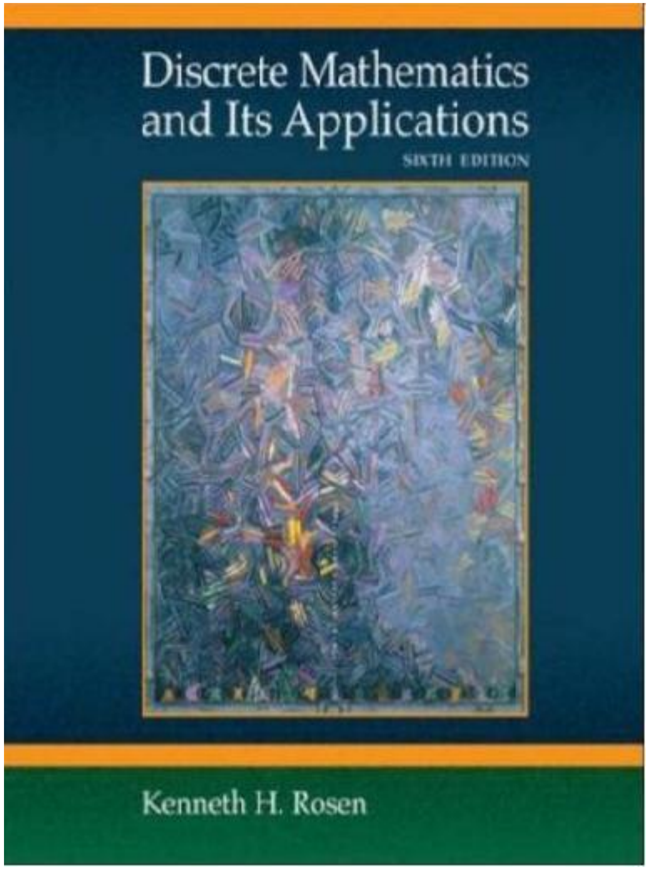
Jiangxi University of Science and Technology

# Discrete Mathematics and Its Applications

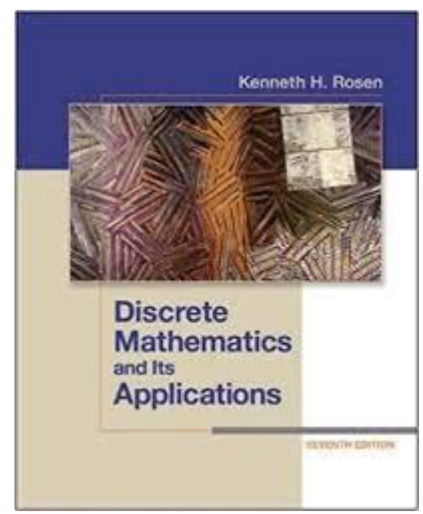



## Lecture012: CHP2:

## Sequences and Summations

# Acknowledgement

- Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University and Dr Johnnie Baker and **Discrete Mathematics and Its Applications** **(Seventh Edition)** **Kenneth H. Rosen**

# Summation

- The symbol ∑ (Greek letter sigma) is used to denote summation.

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$$

*i* is the **index of the summation**, and the choice of letter *i* is arbitrary;
the index of the summation runs through all integers, with its **lower limit** 1 and ending **upper limit** k.

- The limit:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

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# Summation

- The laws for arithmetic apply to summations

$$\sum_{i=1}^{k}(ca_i + b_i) = c\sum_{i=1}^{k}a_i + \sum_{i=1}^{k}b_i$$

Use associativity to separate the b terms from the a terms.

Use distributivity to factor the c's.

# Sequences and Summations (2.4)

- ***Definition***: A ***sequence*** is a function from a subset of the natural numbers (usually of the form {0, 1, 2, . . . } to a set S.

  Note: the sets

  $$\{0, 1, 2, 3, \ldots, k\} \text{ and } \{1, 2, 3, 4, \ldots, k\}$$

  are called *initial segments* of N.

  Notation: if f is a function from {0, 1, 2, . . .} to S we usually denote f(i) by $a_i$ and we write

  $$\{a_0, a_1, a_2, \ldots\} = \{a_i\}_{i=0}^{k} = \{a_i\}_0^k$$

  where k is the upper limit (usually ∞).

# Sequences and Summations (2.4) (cont.)

Examples:

Using zero-origin indexing, if f(i) = 1/(i + 1).

then the Sequence

$$f = \{1, 1/'2, 1/3, 1/4, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$$

Using one-origin indexing the sequence f becomes

$$\{1/2, 1/3, \ldots\} = \{a_1, a_2, a_3, \ldots\}$$

# Sequences and Summations (2.4) (cont.)

- Summation Notation $\{a_i\}_0^k$

Given a sequence we can add together a subset of the sequence by using the summation and function notation

$$a_{g(m)} + a_{g(m+1)} + \ldots + a_{g(n)} = \sum_{j=m}^{n} a_{g(j)}$$

or more generally

$$\sum_{j \in S} a_j$$

# Sequences and Summations (2.4) (cont.)

Examples:

$$r^0 + r^1 + r^2 + \ldots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + = \sum_1^\infty \frac{1}{i}$$

$$a_{2m} + a_{2(m+1)} + \ldots + a_{2(n)} = \sum_{j=m}^{n} a_{2j}$$

$$\text{if} \quad S = \{2,5,7,10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Similarity for the *product* notation:

$$\prod_{j=m}^{n} a_j = a_m a_{m+1} \ldots a_n$$

# Sequences and Summations (cont.)

**Definition:** A *geometric progression* is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots.$$

Your book has a proof that

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \quad \text{if } r \neq 1$$

(you can figure out what it is if r = 1).

You should also be able to determine the sum

- if the index starts at k vs. 0
- if the index ends at something other than n (e.g., n-1, n+1, etc.).

# Sequences and Summations (2.4) (cont.)

- Cardinality

  **Definition:** :

The cardinality of a set A is equal to the cardinality of a set B, denoted $|A| = |B|$,

if there exists a bijection from A to B.

# Sequences and Summations (2.4)

**Definition:**

- If a set has the same cardinality as a subset of the natural numbers N, then the set is called *countable*.

- If |A| = |N|, the set A is *countably infinite*.
- The (transfinite) cardinal number of the set N is *aleph null* = $\aleph_0$.
- If a set is not countable we say it is *uncountable*.

# Sequences and Summations (2.4)

- **Examples:**

  The following sets are uncountable (we show later)

  The real numbers in [0, 1]

    - P(N), the power set of N

  – **Note:** With infinite sets proper subsets can have the same cardinality. This cannot happen with finite sets.

  – Countability carries with it the implication that there is a *listing* of the elements of the set.

# Sequences and Summations (2.4)

– **Definition:** | A | ≤ | B | if there is an injection from A to B.
  Note: as you would hope,
– **Theorem:** If | A | ≤ | B | and | B | ≤ | A | then | A | = | B |.

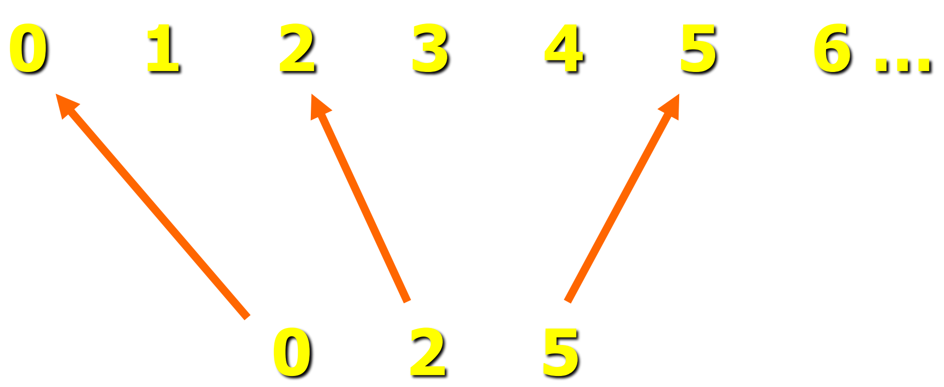
  This implies
  - if there is an injection from A to B
  - if there is an injection from B to A then
  - there must be a bijection from A to B

This is <u>difficult</u> to prove but is an example of demonstrating existence without construction.
It is often easier to build the injections and then conclude the bijection exists.

# Sequences and Summations (2.4)

– **Example:**
**Theorem:** If A is a subset of B then | A | ≤ | B |.
Proof: the function f(x) = x is an injection from A to B.

– **Example:** $\{0, 2, 5\}| \leq \aleph_0$
The injection f: {0, 2, 5} → N defined by f(x) = x is shown below:



# Sequences and Summations (2.4)

- Some Countably Infinite Sets
  - The set of even integers E ( 0 is considered even) is countably infinite. Note that E is a proper subset of N,

    Proof: Let f(x) = 2x. Then f is a bijection from N to E

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 \ldots \end{array}$$

  - $Z^+$, the set of positive integers is countably infinite.

# Sequences and Summations (2.4)

– **The set of positive rational numbers $Q^+$ is countably infinite.**
  Proof: $Z^+$ is a subset of $Q^+$ so $|Z^+| = \aleph_0 \leq |Q^+|$.
– Now we have to show that $|Q^+| \leq \aleph_0$.
– To do this we show that the positive rational numbers with repetitions, $Q_R$, is countably infinite.

Then, since $Q^+$ is a subset of $Q_R$, it follows that
$|Q^+| \leq \aleph_0$ and hence $|Q^+| = \aleph_0$.

# Sequences and Summations (2.4)

| y \ x | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1/1 | 2/1 | 3/1 | 4/1 | 5/1 | 6/1 | 7/1 |
| 2 | 1/2 | 2/2 | 3/2 | 4/2 | 5/2 | 6/2 | 7/2 |
| 3 | 1/3 | 2/3 | 3/3 | 4/3 | 5/3 | 6/3 | 7/3 |
| 4 | 1/4 | 2/4 | 3/4 | 4/4 | 5/4 | 6/4 | 7/4 |
| 5 | | | | | | | |

# Sequences and Summations (2.4)

– The position on the path (listing) indicates the image of the bijective function f from N to $Q_R$:

**$f(0) = 1/1$, $f(1) = 1/2$, $f(2) = 2/1$, $f(3) = 3/1$, and so forth.**

Every rational number appears on the list at least once, some many times (repetitions).

**Hence, $|N| = |QR| = \aleph 0$.** **Q. E. D**

– The set of all rational numbers Q, positive and negative, is countably infinite.

# Sequences and Summations (2.4)

– The set of (finite length) strings S over a finite alphabet A is countably infinite.

  To show this we assume that

  - A is nonvoid
  - There is an “alphabetical” ordering of the symbols in A

  Proof: List the strings in lexicographic order:

  - all the strings of zero length,
  - then all the strings of length 1 in alphabetical order,
  - then all the strings of length 2 in alphabetical order,

  etc

  This implies a bijection from N to the list of strings and hence it is a countably infinite set.

# Sequences and Summations (2.4)

– For **example**:

Let A = {a, b, c}.

Then the lexicographic ordering of A is

{$\lambda$ , a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, aba, ....} = {f(0), f(1), f(2), f(3), f(4), . . . .}

# Sequences and Summations (2.4)

- The set of all C programs is *countable*.
- Proof: Let S be the set of legitimate characters which can appear in a C program.
  - A C compiler will determine if an input program is a syntactically correct C program (the program doesn't have to do anything useful).
  - Use the lexicographic ordering of S and feed the strings into the compiler.
    - If the compiler says YES, this is a syntactically correct C program, we add the program to the list.
    - Else we move on to the next string.
- In this way we construct a list or an implied bijection from N to the set of C programs.
- Hence, the set of C programs is countable. Q. E. D.

# Sequences and Summations (2.4)

- Cantor Diagonalization
  An important technique used to construct an object which is not a member of a countable set of objects with (possibly) infinite descriptions

**Theorem**: The set of real numbers between 0 and 1 is uncountable.

Proof: We assume that it is countable and derive a contradiction.

If it is countable we can list them

(i.e., there is a bijection from a subset of N to the set).

We show that no matter what list you produce we can construct a real number between 0 and 1 which is not in the list.

Hence, there cannot exist a list and therefore the set is not countable

# Sequences and Summations (2.4)

It's actually much bigger than countable. It is said to have the *cardinality of the continuum,* c.

Represent each real number in the list using *its decimal expansion.*

e.g., 1/3 = .3333333........

1/2 = .5000000........

= .4999999........

If there is more than one expansion for a number, it doesn't matter as long as our construction takes this into account.

# Sequences and Summations (2.4)

– THE LIST....

$$r_1 = .d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}. \ . \ . \ . \ .$$
$$r_2 = .d_{21}d_{22}d_{23}d_{24}d_{25}d_{26} \ . \ . \ . \ .$$
$$r_3 = .d_{31}d_{32}d_{33}d_{34}d_{35}d_{36} \ . \ . \ . \ .$$
...

Now construct the number $x = .x_1x_2x_3x_4x_5x_6x_7. \ . \ . \ .$

$$x_i = 3 \text{ if } d_{ii} \neq 3$$
$$x_i = 4 \text{ if } d_{ii} = 3$$

(Note: choosing 0 and 9 is not a good idea because of the non uniqueness of decimal expansions.)

Then x is not equal to any number in the list.

Hence, no such list can exist and hence the interval (0,1) is uncountable.
Q. E. D.

# Sequences and Summations (2.4)

– An extra goody:

**Definition:** a number x between 0 and 1 is *computable* if there is a C program which when given the input i, will produce the ith digit in the decimal expansion of x.

**Example:**
The number 1/3 is computable.

The C program which always outputs the digit 3,regardless if the input, computes the number.

# Sequences and Summations (2.4)

**Theorem:** There is exists a number x between 0 and 1 which is *not computable*.

There *does not exist* a C program (or a program in any other language) which will compute it!

Why? Because there are more numbers between 0 and 1 than there are C programs to compute them.

(in fact there are c such numbers!)

Our second example of the *nonexistence* of programs to compute things!

# Useful Summations

| Sum | Closed Form |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r - 1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$ | $\frac{1}{(1-x)^2}$ |

# Infinite Cardinality

- How can we extend the notion of cardinality to infinite sets?

- Definition: Two sets **A and B have the same cardinality** if and only if there exists a bijection (or a one-to-one correspondence) between them, A ~ B.

We split infinite sets into two groups:

1. Sets with the **same cardinality as the set of natural numbers**
2. Sets with **different cardinality as the set of natural numbers**

# Infinite Cardinality

- Definition: A set is **countable** if it is **finite** or has the same **cardinality as the set of positive integers.**
- Definition: A set is un**countable** if it is **not countable.**
- Definition: The cardinality of an infinite set S that is countable is denotes by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet).
- We write $|S| = \aleph_0$ and say that S has cardinality "aleph null"

**Note: Georg Cantor defined the notion of cardinality and was the first to realize that infinite sets can have different cardinalities. $\aleph_0$ is the cardinality of the natural numbers; the next larger cardinality is aleph-one $\aleph_1$, then, $\aleph_2$ and so on.**

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# Infinite Cardinality: Odd Positive Integers

Example: The set of odd positive integers is a countable set.

Let's define the function f, from $Z^+$ to the set of odd positive numbers,

$$f(n) = 2n - 1$$

We have to show that f is both one-to-one and onto.

one-to-one

Suppose $f(n)= f(m) \rightarrow 2n-1 = 2m-1 \rightarrow n=m$

onto

Suppose that t is an odd positive integer. Then t is 1 less than an even integer 2k, where k is a natural number. hence $t=2k-1= f(k)$.

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# Infinite Cardinality: Odd Positive Integers

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | … |
| ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | |
| 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | … |

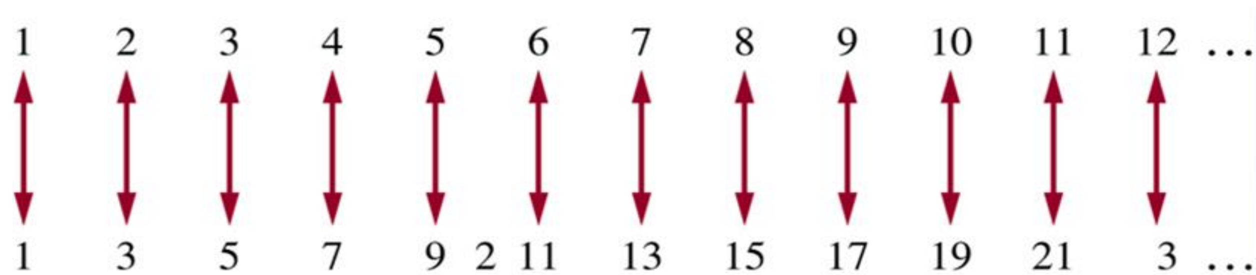
# Infinite Cardinality:Integers

Example: **The set of integers is a countable set.**

**Lets consider the sequence of all integers, starting with 0: 0,1,-1,2,-2,….**

**We can define this sequence as a function:**

$$f(n) = \begin{cases} n/2 & n \in N, even \\ \dfrac{-(n-1)}{2} & n \in N, odd \end{cases}$$




0 1 -1 2 2

Show at home that it's one-to-one and onto

# Infinite Cardinality: Rational Numbers

- Example: The set of **positive rational numbers** is a **countable** set.
- Hmm…

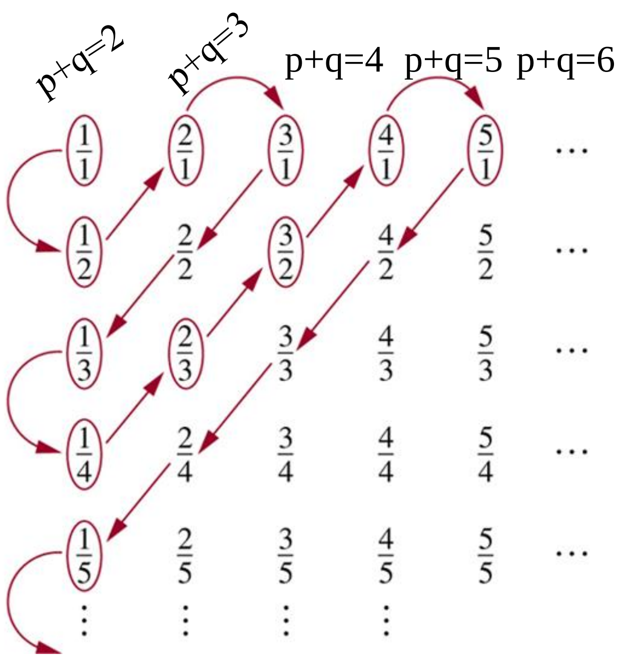
# Infinite Cardinality: Rational Numbers

Example: The set of **positive rational numbers** is a **countable** set

Key aspect to list the rational numbers as a sequence – every positive number is the quotient p/q of two positive integers.

Visualization of the proof.

Terms not circled are not listed because they repeat previously listed terms




Since all positive rational numbers are listed once, the set of positive rational numbers is countable.

# Uncountable Sets: Cantor's diagonal argument

The set of all **infinite sequences of zeros and ones** is **uncountable.**

Consider a sequence,

$$a_1, a_2, \cdots, a_n, n \to \infty, a_i = 0 \ or \ a_i = 1$$

For example:

$$s_1 = (0, 0, 0, 0, 0, 0, 0, \ldots)$$
$$s_2 = (1, 1, 1, 1, 1, 1, 1, \ldots)$$
$$s_3 = (0, 1, 0, 1, 0, 1, 0, \ldots)$$
$$s_4 = (1, 0, 1, 0, 1, 0, 1, \ldots)$$
$$s_5 = (1, 1, 0, 1, 0, 1, 1, \ldots)$$
$$s_6 = (0, 0, 1, 1, 0, 1, 1, \ldots)$$
$$s_7 = (1, 0, 0, 0, 1, 0, 0, \ldots)$$

So in general we have:

$$s_n = (s_{n,1}, s_{n,2}, s_{n,3}, s_{n,4}, \ldots)$$

i.e., $s_{n,m}$ is the $m^{th}$ element of the $n^{th}$ sequence on the list.

# Uncountable Sets: Cantor's diagonal argument

- It is possible to build a sequence, say $s_0$, in such a way that its first element is different from the first element of the first sequence in the list, its second element is different from the second element of the second sequence in the list, and, in general, its *n*th element is different from the $n^{th}$ element of the $n^{th}$ sequence in the list. In other words, $s_{0,m}$ will be 0 if $s_{m,m}$ is 1, and $s_{0,m}$ will be 1 if $s_{m,m}$ is 0.

# Uncountable Sets: Cantor's diagonal argument

$s_1 = (\mathbf{\underline{0}}, 0, 0, 0, 0, 0, 0, \ldots)$
$s_2 = (1, \mathbf{\underline{1}}, 1, 1, 1, 1, 1, \ldots)$
$s_3 = (0, 1, \mathbf{\underline{0}}, 1, 0, 1, 0, \ldots)$
$s_4 = (1, 0, 1, \mathbf{\underline{0}}, 1, 0, 1, \ldots)$
$s_5 = (1, 1, 0, 1, \mathbf{\underline{0}}, 1, 1, \ldots)$
$s_6 = (0, 0, 1, 1, 0, \mathbf{\underline{1}}, 1, \ldots)$
$s_7 = (1, 0, 0, 0, 1, 0, \mathbf{\underline{0}}, \ldots)$
...
$s_0 = (\mathbf{\underline{1}}, \mathbf{\underline{0}}, \mathbf{\underline{1}}, \mathbf{\underline{1}}, \mathbf{\underline{1}}, \mathbf{\underline{0}}, \mathbf{\underline{1}}, \ldots)$

Note: the diagonal elements are highlighted, showing why this is called the **diagonal argument**

- The sequence $s_0$ is distinct from all the sequences in the list. Why?
- Let's say that $s_0$ is identical to the 100th sequence, therefore, $s_{0,100}=s_{100,100}$.
- In general, if it appeared as the *n*th sequence on the list, we would have $s_{0,n} = s_{n,n}$,
- which, due to the construction of $s_0$, is impossible.

# Uncountable Sets: Cantor's diagonal argument

From this it follows that the set *T*, consisting of all infinite sequences of zeros and ones, cannot be put into a list $s_1$, $s_2$, $s_3$, ...

Otherwise, it would be possible by the above process to construct a sequence $s_0$ which would both be in *T* (because it is a sequence of 0's and 1's which is by the definition of *T* in *T*) and at the same time not in *T* (because we can deliberately construct it not to be in the list). *T*, containing all such

sequences, must contain $s_0$, which is just such a sequence. But since $s_0$ does not appear anywhere on the list, *T* cannot contain $s_0$.

Therefore *T* cannot be placed in one-to-one correspondence with the natural numbers. In other words, the set of infinite binary strings is **uncountable**.

# Real Numbers

Example;

The set of real numbers is an uncountable set.

Let’s assume that the set of real numbers is countable.

Therefore any subset of it is also countable, in particular the interval [0,1].

How many real numbers are in interval [0, 1]?

# Real Numbers

- How many real numbers are in interval [0, 1]?

0.4 3 2 9 0 1 3 2 9 8 4 2 0 3 9 …
0.8 2 5 9 9 1 3 2 7 2 5 8 9 2 5 …
0.9 2 5 3 9 1 5 9 7 4 5 0 6 2 1

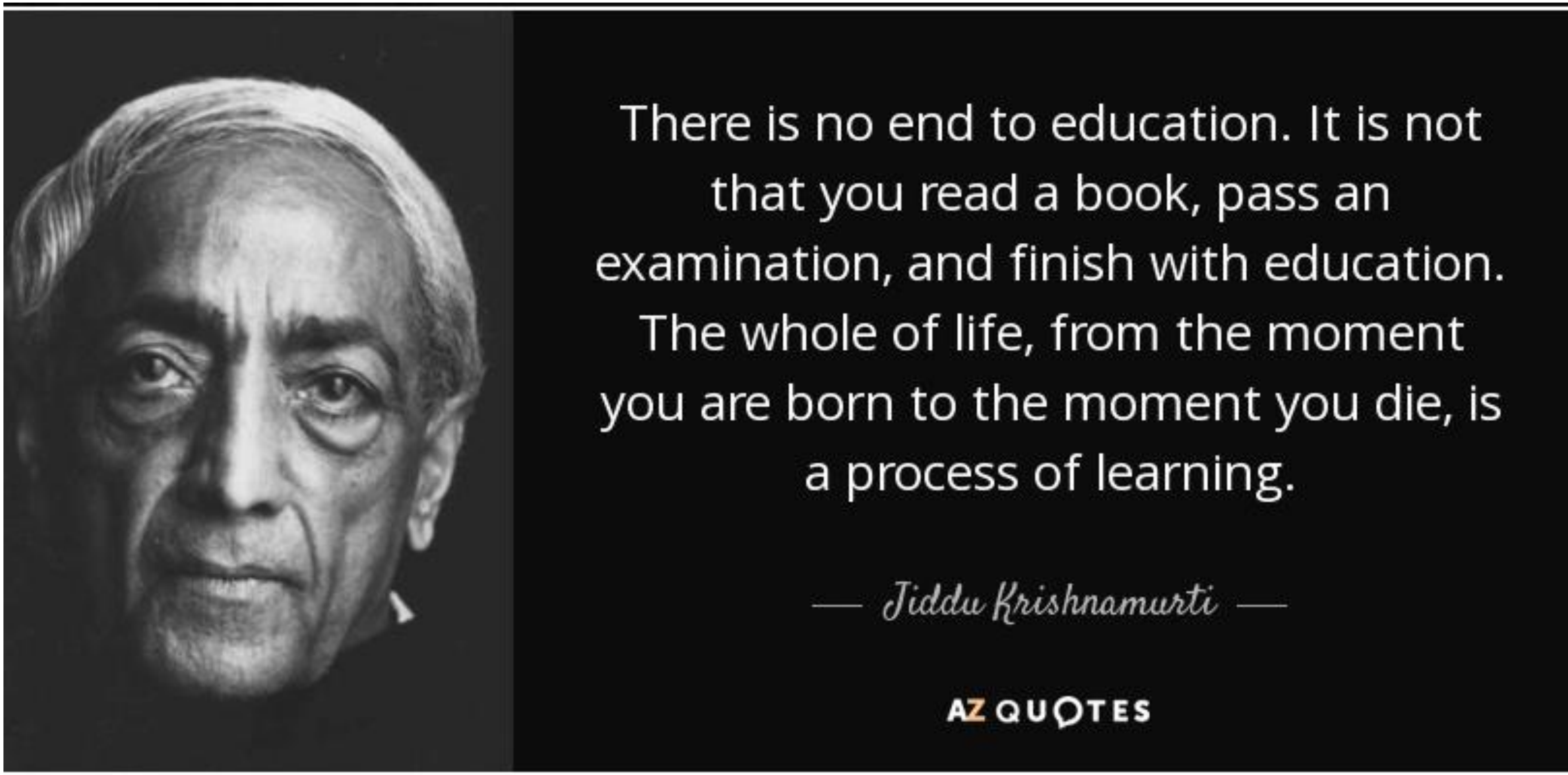
⋮

"Countably many! There's part of the list!"

"Are you sure they're all there?"

Counterexample:
Use diagonalization
to create a new number
that differs in the ith
position of the
ith number
by 1.

0.5 3 6 …
So we say the reals are "uncountable."


There is no end to education. It is not that you read a book, pass an examination, and finish with education. The whole of life, from the moment you are born to the moment you die, is a process of learning.
— Jiddu Krishnamurti —
AZ QUOTES