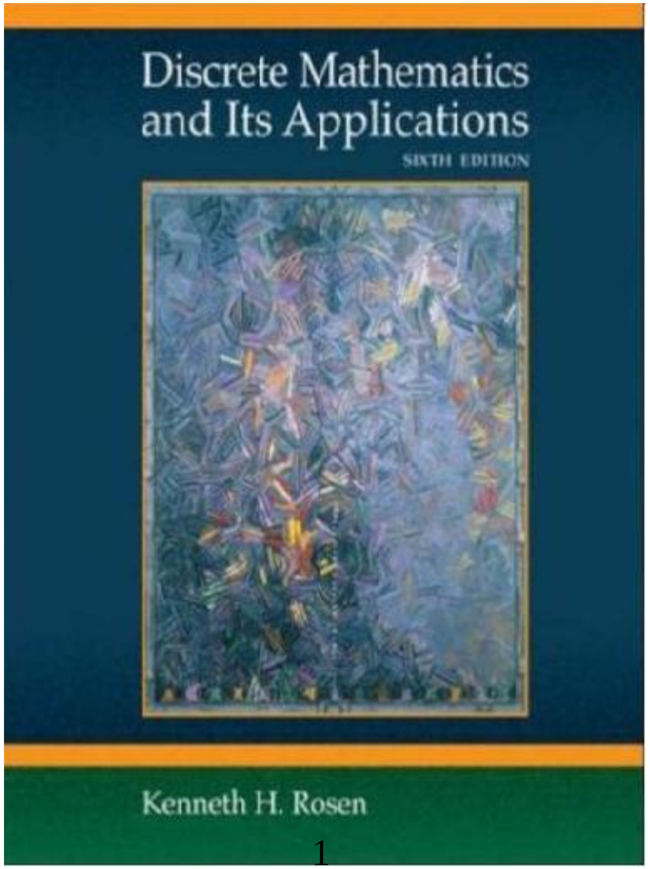
Jiangxi University of Science and Technology

Discrete Mathematics and Its Applications



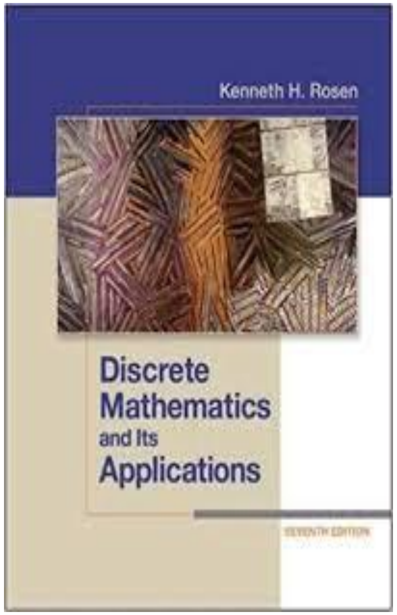
Lecture013: CHP3:

The Fundamentals: Algorithms, the Integers, and Matrices

Acknowledgement

Most of these slides are adapted from ones created by Professor Bart Selman at Cornell University ,

and Dr Johnnie Baker and **Discrete Mathematics and Its Applications** **(Seventh Edition)** **Kenneth H. Rosen**



What’s is Induction About?

- Many statements assert that a property is an universal true – i.e., all the elements of the universe exhibit that property;

- Examples:

1. For every positive integer n: $n! \leq n^n$

2. For every set with n elements, the cardinality of its power set is 2^n.

- **Induction is one of the most important techniques for proving statements about universal properties.**

Mathematical induction,

is a technique for proving results or establishing statements for natural numbers. This part illustrates the method through a variety of examples.

Definition

- **Mathematical Induction** is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
- The technique involves two steps to prove a statement, as stated below −
  - **Step 1(Base step)** − It proves that a statement is true for the initial value.
  - **Step 2(Inductive step)** − It proves that if the statement is true for the nthiteration (or number n), then it is also true for $(n+1)$th iteration (or number $n+1$).

We know that:

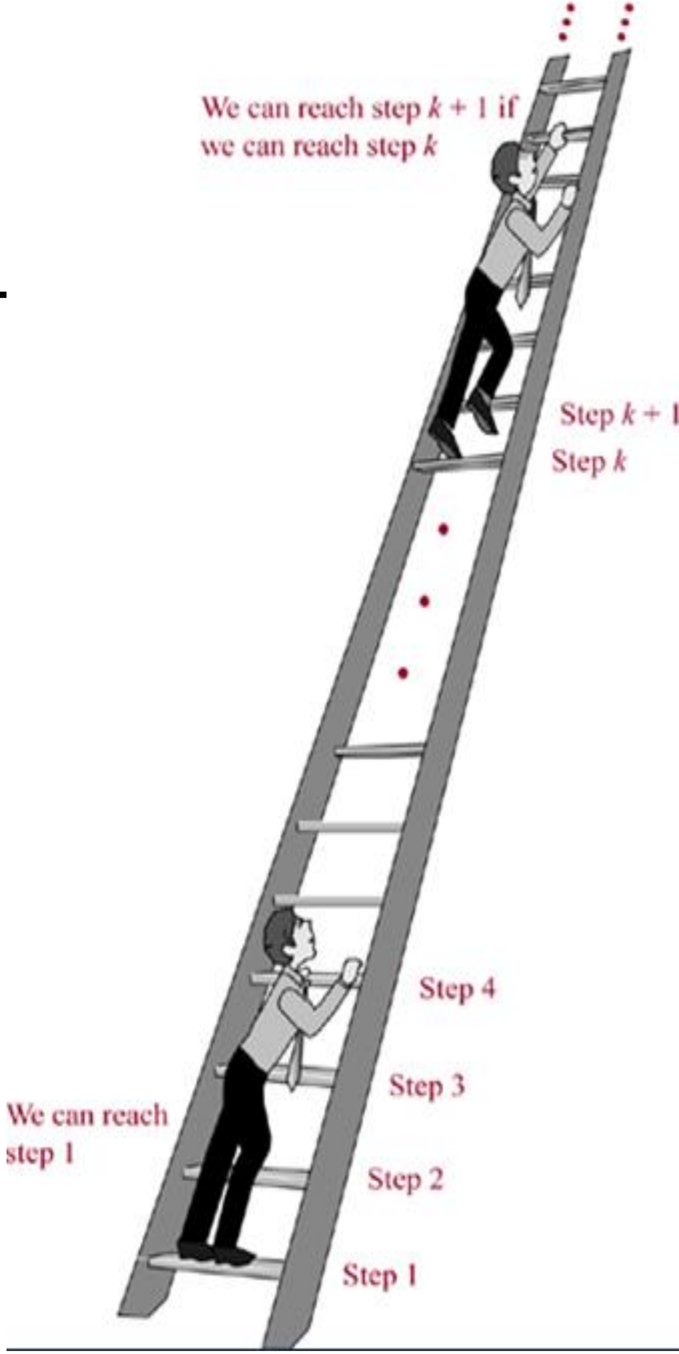
We can reach the first rung of this ladder;
If we can reach a particular rung of the ladder,
then we can reach the next rung of the ladder.

Can we reach every step of this infinite ladder?

Yes, using Mathematical Induction which is
a rule of inference that tells us:

$P(1)$
$\forall k\ (P(k) \rightarrow P(k+1))$

$\therefore\ \forall n\ (P(n))$



Principle of Mathematical Induction

- Hypothesis: $P(n)$ is true for all integers $n \geq b$
- To prove that $P(n)$ is true for all integers $n \geq b$ (*), where $P(n)$ is a propositional function, follow the steps:
- Basic Step or Base Case: Verify that $P(b)$ is true;
- Inductive Hypothesis: assume $P(k)$ is true for some $k \geq b$;
- Inductive Step: Show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers $k \geq b$.
- ***This can be done by showing that under the inductive hypothesis that $P(k)$ is true, $P(k+1)$ must also be true.***

(*) quite often $b=1$, but b can be any integer number.

How to Do It

- **Step 1** − Consider an initial value for which the statement is true. It is to be shown that the statement is true for n = initial value.

- **Step 2** − Assume the statement is true for any value of $n = k$. Then prove the statement is true for $n = k+1$.
- We actually break $n = k+1$ into two parts, one part is $n = k$ (which is already proved) and try to prove the other part.

Writing a Proof by Induction

1. State the hypothesis very clearly:

   P(n) is true for all integers $n \geq b$ – state the property P in English

2. Identify the the base case

   P(b) holds because …

3. Inductive Hypothesis

   Assume P(k)

4. Inductive Step - Assuming the inductive hypothesis P(k), prove that P(k+1) holds; i.e.,

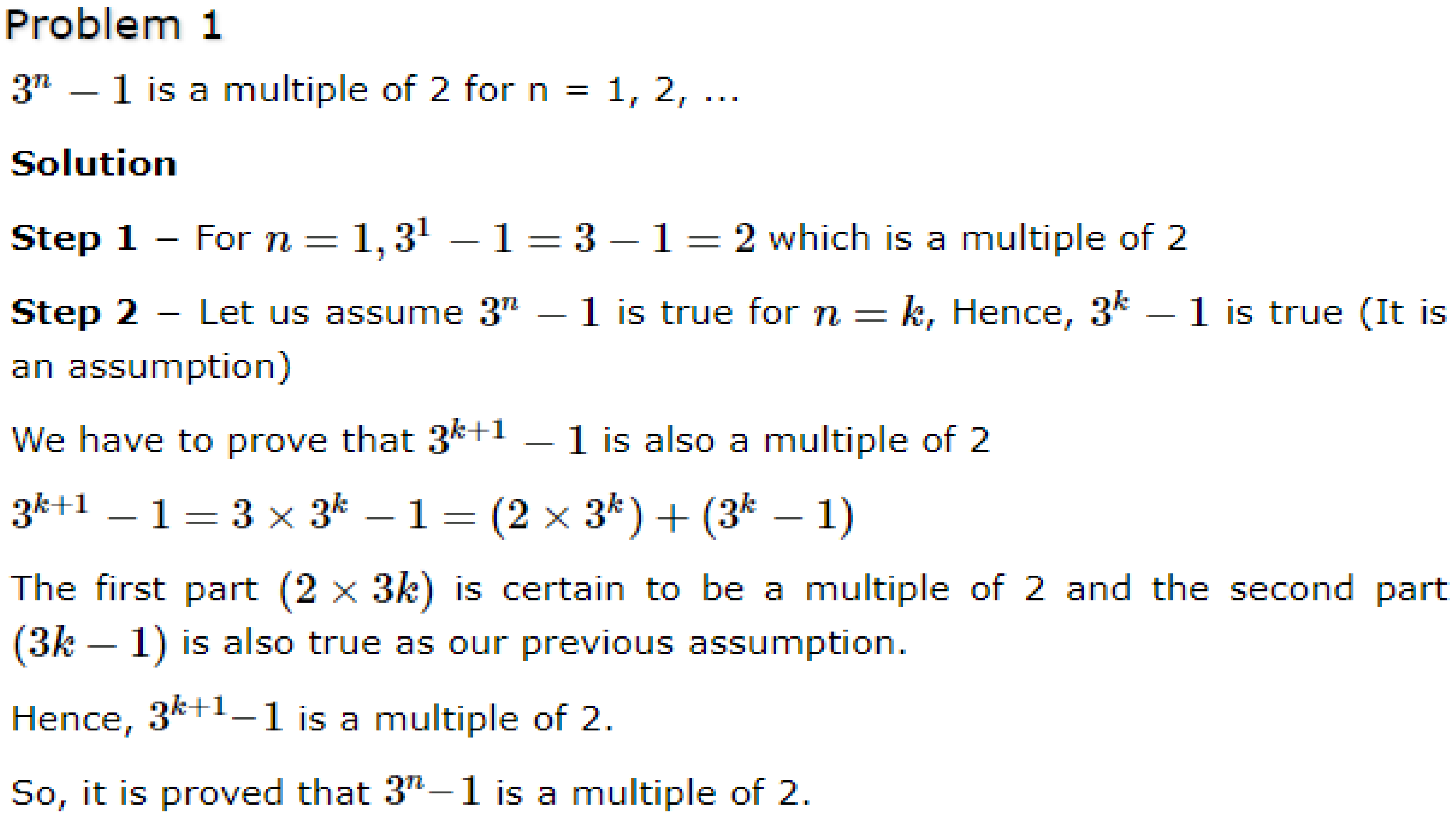
   $P(k) \rightarrow P(k+1)$

Conclusion

By induction we have shown that P(k) holds for all $k \geq b$

(b is what was used for the base case).

Problem 1



Problem 1

$3^n - 1$ is a multiple of 2 for n = 1, 2, …

Solution

Step 1 – For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2

Step 2 – Let us assume $3^n - 1$ is true for $n = k$, Hence, $3^k - 1$ is true (It is an assumption)

We have to prove that $3^{k+1} - 1$ is also a multiple of 2

$$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3k - 1)$ is also true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

Problem 2

$1+3+5+\ldots+(2n-1)=n^2$ for $n=1,2,\ldots$

Solution

Step 1 – For $n=1, 1=1^2$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n=k$.

Hence, $1+3+5+\cdots+(2k-1)=k^2$ is true (It is an assumption)

We have to prove that $1+3+5+\ldots+(2(k+1)-1)=(k+1)^2$ also holds

$$\begin{aligned}
&1+3+5+\cdots+(2(k+1)-1)\\
&=1+3+5+\cdots+(2k+2-1)\\
&=1+3+5+\cdots+(2k+1)\\
&=1+3+5+\cdots+(2k-1)+(2k+1)\\
&=k^2+(2k+1)\\
&=(k+1)^2
\end{aligned}$$

So, $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ hold which satisfies the step 2.

Hence, $1+3+5+\cdots+(2n-1)=n^2$ is proved.

Mathematical Induction

Prove a base case (n=1)

- Use induction to prove that the sum of the first n odd integers is n^2. What's the hypothesis?

Prove $P(k) \rightarrow P(k+1)$

1 – Hypothesis: P(n) – sum of first n odd integers = n^2.

2 - Base case (n=1): the sum of the first 1 odd integer is 1^2.

Since $1 = 1^2$ ☺

3 - Assume P(k): the sum of the first k odd ints is k^2.

$1 + 3 + \dots + (2k - 1) = k^2$

Inductive hypothesis

4 – Inductive Step: show that $\forall(k)\, P(k) \rightarrow P(k+1)$, assuming P(k).

How?

By inductive hypothesis

$$P(k+1) = \underbrace{1 + 3 + \dots + (2k-1)}_{p(k)} + (2k+1) = k^2 + (2k + 1)$$

$$= (k+1)^2$$

QED

Mathematical Induction

Prove a base case (n=?)

Prove P(k)→P(k+1)

- Use induction to prove that the **$1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$** for all non-negative integers n.

1 — Hypothesis?

$P(n) = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

for all non-negative integers n.

2 - Base case?

$n = 0 \quad 1^0 = 2^1 - 1.$

← not n=1! The base case can be negative, zero,or positive

3 – Inductive Hypothesis

Assume P(k) $= 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

Inductive hypothesis

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 – For $n = 1, (ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n = k$, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

Mathematical Induction

4 – Inductive Step: show that ∀(k) P(k) → P(k+1), assuming P(k). How?

$$P(k+1) = \underbrace{1 + 2 + 2^2 + \ldots + 2^k}_{p(k)} + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1}$$

By inductive hypothesis

$$= 2\ 2^{k+1} - 1$$

$$P(k+1) = 2^{k+2} - 1$$

$$= 2^{(k+1)+1} - 1$$

QED

Mathematical Induction

- Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1, \quad \forall$ positive integers

1 – Hypothesis $P(n) = 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1, \forall$ positive integers

2 - Base case $(n=1)$: $1 \cdot 1! = (1+1)! - 1$?

$1 \cdot 1! = 1$ and $2! - 1 = 1$

3 - Assume $P(k)$: $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$

Inductive hypothesis

4 – Inductive Step - show that $\forall (k)\ P(k) \rightarrow P(k+1)$, assuming $P(k)$.

I.e, prove that $1 \cdot 1! + \ldots + k \cdot k! + (k+1)(k+1)! = (k+2)! – 1$, assuming $P(k)$

$$\boxed{1 \cdot 1! + \ldots + k \cdot k! +}\ (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (1 + (k+1))(k+1)! - 1$$

$$= (k+2)(k+1)! - 1$$

QED

$$= (k+2)! - 1$$

Mathematical Induction

Prove that a set with n elements has 2^n subsets.

1-Hypothesis: set with n elements has 2n subsets

2- Base case (n=0): S=ø, P(S) = {ø} and |P(S)| = 1 = 2^0

3- Inductive Hypothesis - P(k): given |S| = k, |P(S)| = 2^k

Inductive hypothesis

4- Inductive Step: ∀(k) P(k) → P(k+1), assuming P(k). i.e,

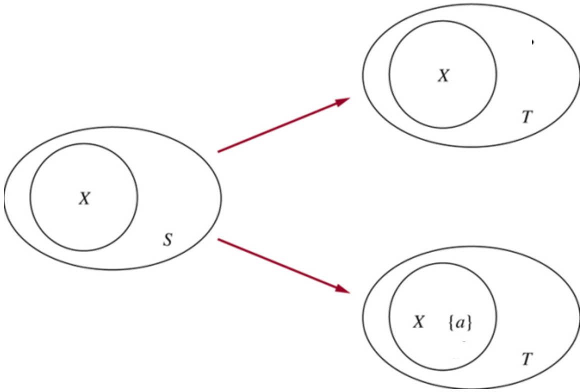
Prove that if |T| = k+1, then |P(T)| = $2^{k+1,}$ given that P(k)=2^k

Inductive Step: Prove that if $|T| = k+1$, then $|P(T)| = 2^{k+1}$ assuming $P(k)$ is true.

$T = S \cup \{a\}$ for some $S \subset T$ with $|S| = k$, and $a \in T$

How to obtain the subsets of T?

For each subset X of S there are exactly two subsets of T, namely X and $X \cup \{a\}$



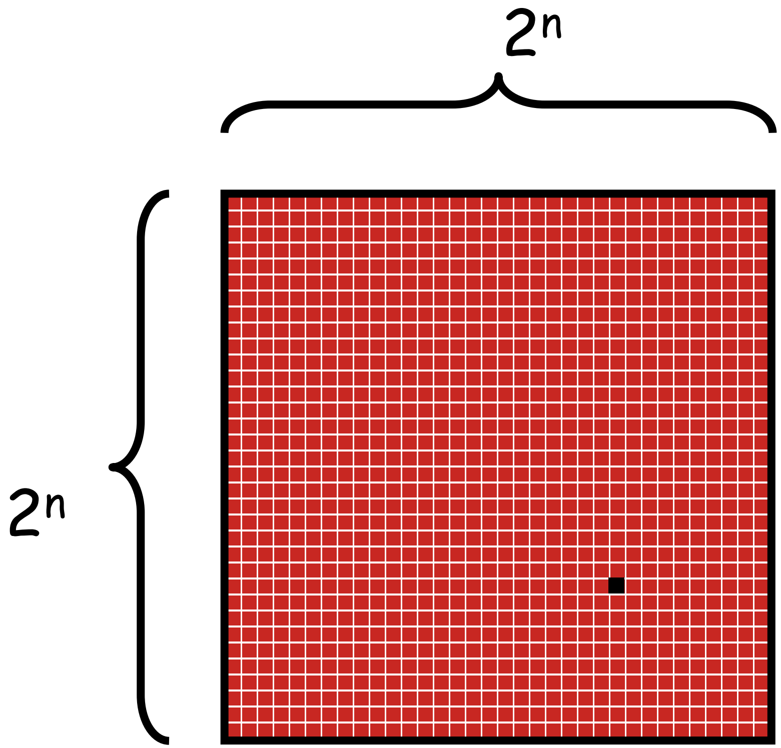
Generating subsets of a set T with k+1 elements
from a set S with K elements

Because there are 2^k subsets of S (inductive hypothesis), there are 2×2^k subsets of T.

QED

Deficient Tiling

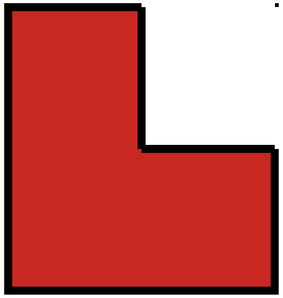
- A 2^n x 2^n sized grid is *deficient* if all but one cell is tiled.



Mathematical Induction - a cool example

Hypothesis:

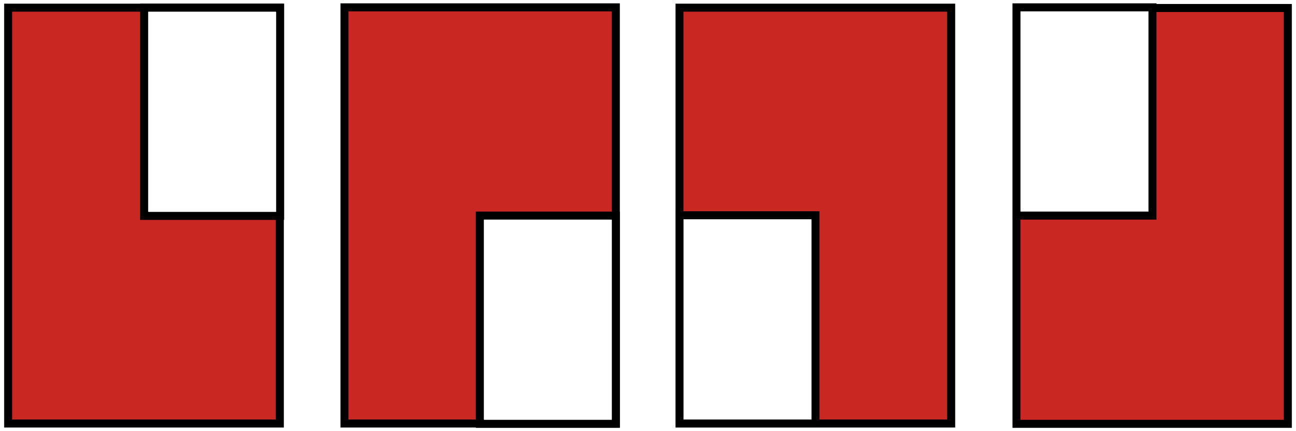
P(n) - We want to show that all 2^n x 2^n sized deficient grids can be tiled with right triominoes, which are pieces that cover three squares at a time, like this:



Mathematical Induction - a cool example

- Base Case:

P(1) - Is it true for 2^1 x 2^1 grids?

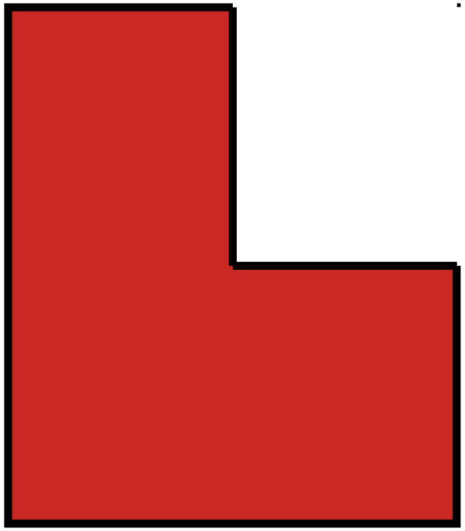


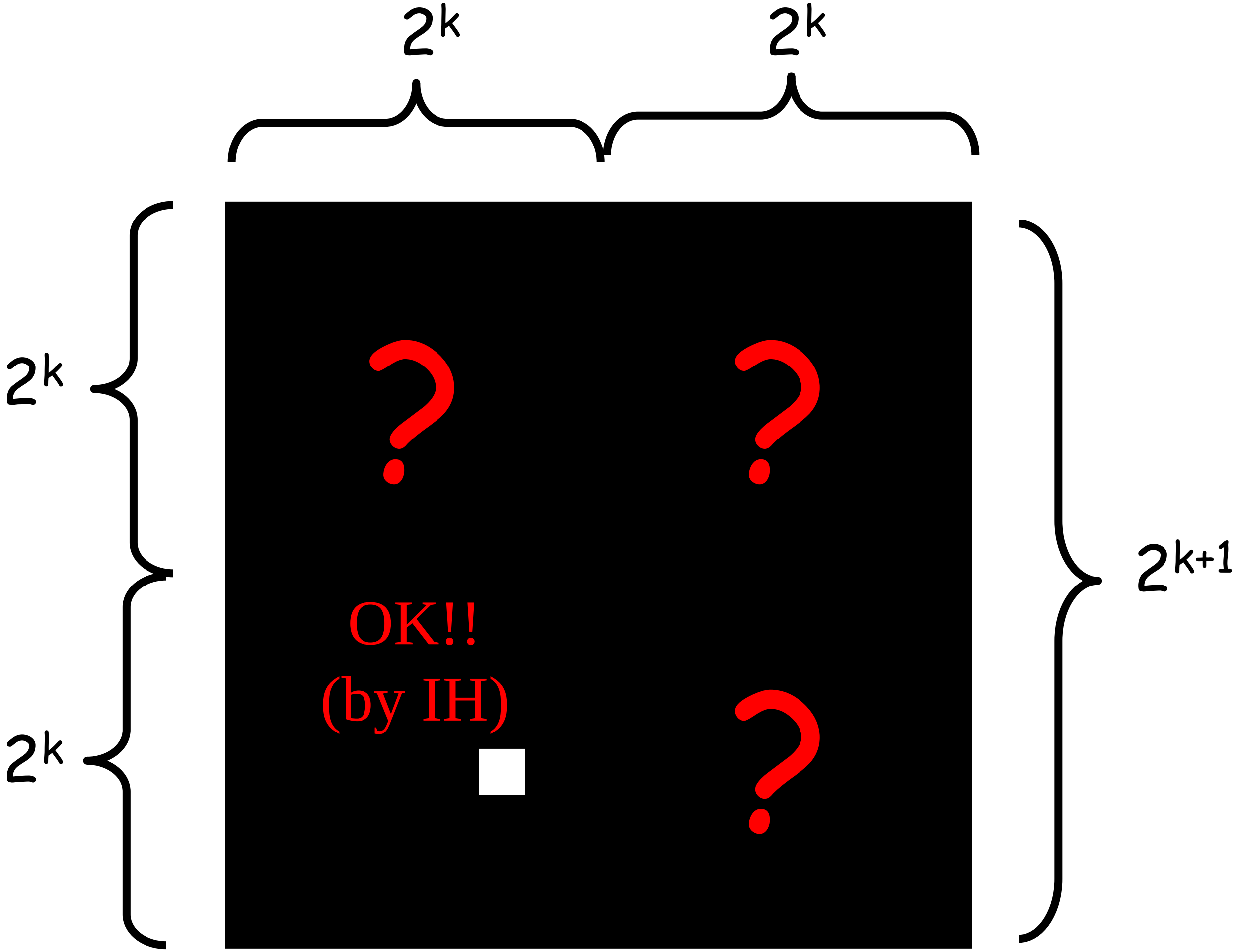
YES ☺

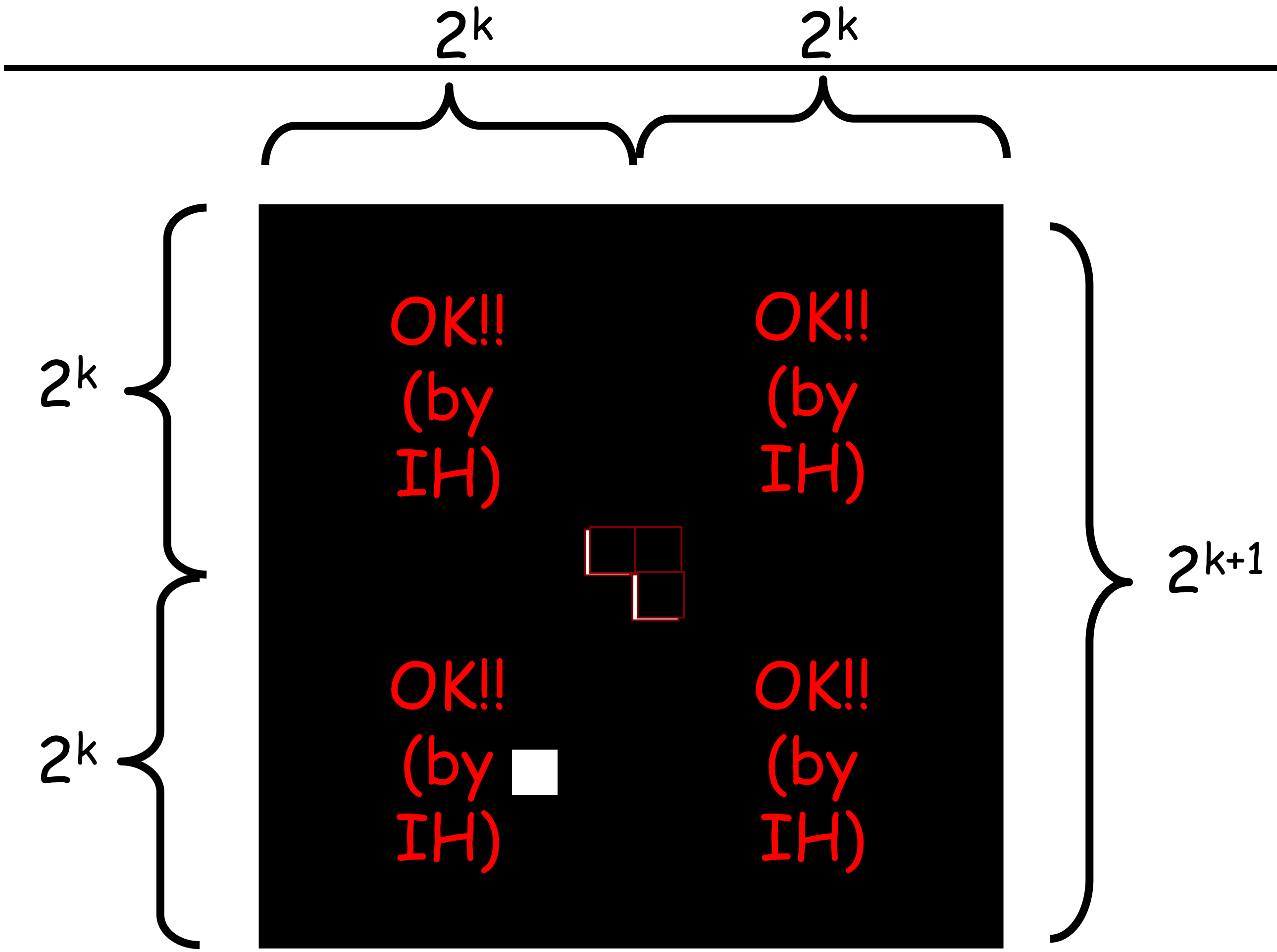
Mathematical Induction - a cool example

- Inductive Hypothesis:
- We can tile a 2^k x 2^k deficient board using our designer tiles.

- Inductive Step:
- Use this to prove that we can tile a 2^{k+1} x 2^{k+1} deficient board using our designer tiles.




2^k
2^k
2^k
2^k
2^{k+1}
?
?
OK!!
(by IH)
?



2^k 2^k

2^k

2^k

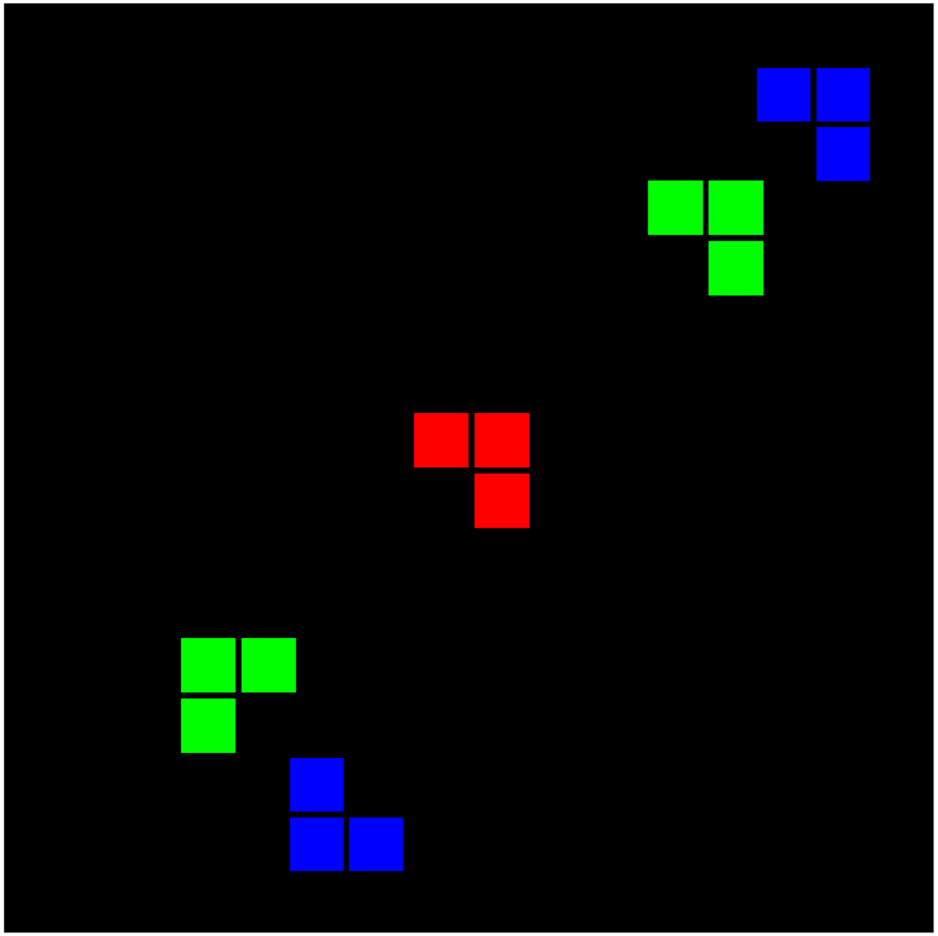
2^{k+1}

OK!! (by IH)

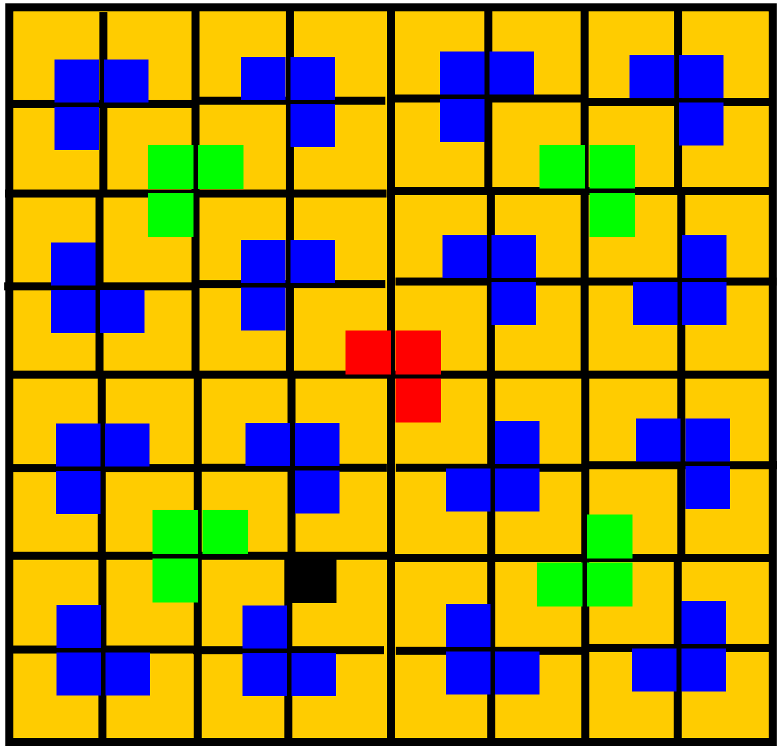
OK!! (by IH)

OK!! (by IH)

OK!! (by IH)

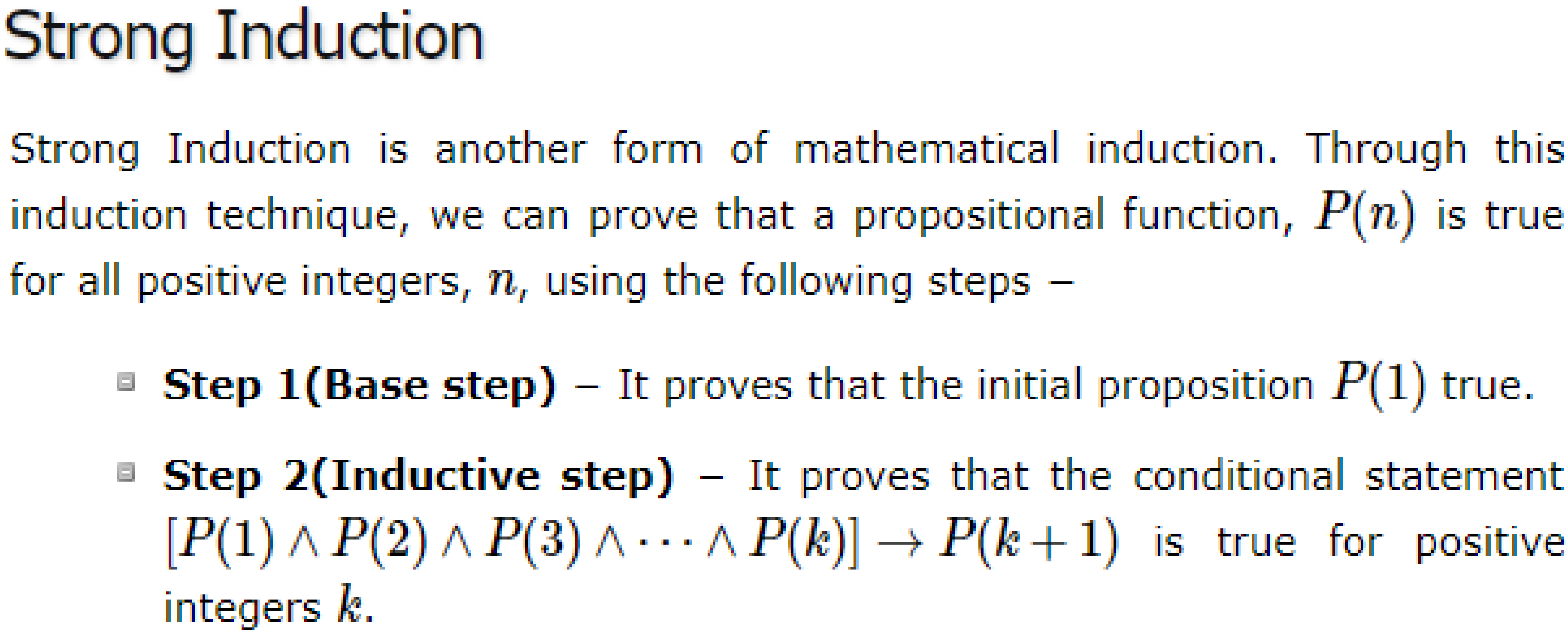


So, we can tile a 2^k x 2^k deficient board using our designer tiles.



What does this mean for 2^{2k} mod 3? **= 1** (also do direct proof by induction)

Strong Induction



Strong Induction

Strong Induction is another form of mathematical induction. Through this induction technique, we can prove that a propositional function, $P(n)$ is true for all positive integers, n, using the following steps –

- **Step 1(Base step)** – It proves that the initial proposition $P(1)$ true.
- **Step 2(Inductive step)** – It proves that the conditional statement $[P(1) \land P(2) \land P(3) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true for positive integers k.

Mathematical Induction - why does it work?

- Definition:
- A set S is “well-ordered” if every non-empty subset of S has a least element.
- Given (we take as an axiom): the set of natural numbers (N) is well-ordered.
- Is the set of integers (Z) well ordered?

No.
$\{ x \in Z : x < 0 \}$ has no least element.

Mathematical Induction - why does it work?

- Is the set of non-negative reals (R) well ordered?

No.
{ $x \in R : x > 1$ } has no least element.

Mathematical Induction - why does it work?

- Proof of Mathematical Induction:
- We prove that
  - $(P(0) \wedge (\forall k\ P(k) \rightarrow P(k+1))) \rightarrow (\forall n\ P(n))$

Proof by contradiction.

Assume

$P(0)$

$\forall k\ P(k) \rightarrow P(k+1)$

$\neg \forall n\ P(n)$

$\exists n\ \neg P(n)$

Mathematical Induction - why does it work?

Assume ____

$P(0)$

$\forall n\, P(n) \rightarrow P(n+1)$

$\neg\forall n\, P(n)$

$\exists n\, \neg P(n)$

Let $S = \{\, n : \neg P(n) \,\}$

Since N is well ordered, S has a least element. Call it k.

What do we know?

$P(k)$ is false because it’s in S.

$k \neq 0$ because $P(0)$ is true.

$P(k-1)$ is true because $P(k)$ is the least element in S.

But by (2), $P(k-1) \rightarrow P(k)$. Contradicts $P(k-1)$ true, $P(k)$ false.

Done.

Strong Induction

1. State the hypothesis very clearly:

- P(n) is true for all integers $n \geq b$ – state the property P is English

2. Identify the the base case

- P(b) holds because …

3. Inductive Hypothesis

- $(P(b) \wedge P(b+1) \wedge \ldots \wedge P(k))$
- 4 . Inductive Step - Assuming P(k) is true for all positive integers not exceeding k (inductive hypothesis), prove that P(k+1) holds; i.e.,

  $(P(b) \wedge P(b+1) \wedge \ldots \wedge P(k) \rightarrow P(k+1))$

Strong Mathematical Induction

- If
- $P(0)$ and
- $\forall n \geq 0\ (P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$
- Then
- $\forall n \geq 0\ P(n)$

In our proofs, to show P(k+1), our inductive hypothesis assures that ALL of P(b), P(b+1), … P(k) are true, so we can use ANY of them to make the inference.

Strong Induction vs. Induction

- Sometimes strong induction is easier to use.
- It can be shown that strong induction and induction are equivalent:

- any proof by induction is also a proof by strong induction (why?)

- any proof by strong induction can be converted into a proof by induction

- Strong induction also referred to as complete induction; in this context induction is referred to as incomplete induction.

Strong Induction

- Show that if n is an integer greater than 1, then *n* can be written as the product of primes.

- 1 - Hypothesis P(n) - *n* can be written as the product of primes.

- 2 – Base case – P(2) 2 can be written a 2 (the product of itself)

- 3 – Inductive Hypothesis - P(j) is true for ∀ $2 \leq j \leq k$, j integer.

- 4 – Inductive step?

-

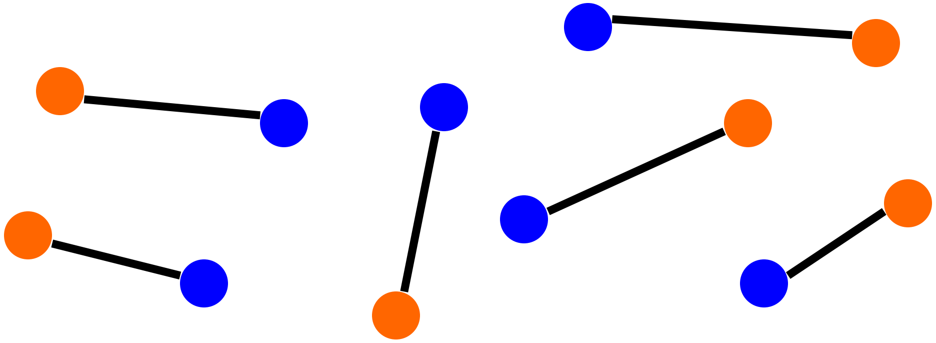
a) k+1 is prime – in this case it's the product of itself;

b) k+1 is a composite number and it can be written as the product of two positive integers *a* and *b*, with $2 \leq a \leq b \leq k+1$. By the inductive hypothesis, a and b can be written as the product of primes, and so does k+1

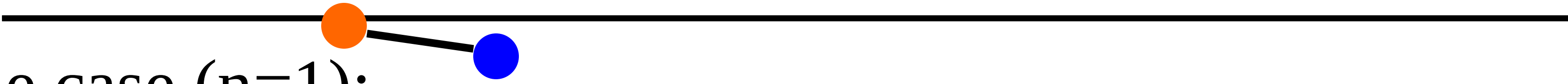
QED

Strong Mathematical Induction

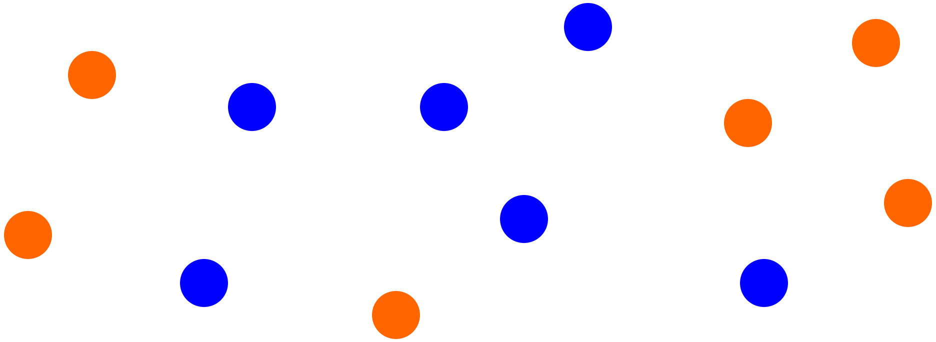
- An example.
- Given *n* blue points and *n* orange points in a plane with no 3 collinear, prove there is a way to match them, blue to orange, so that none of the segments between the pairs intersect.



Strong Mathematical Induction

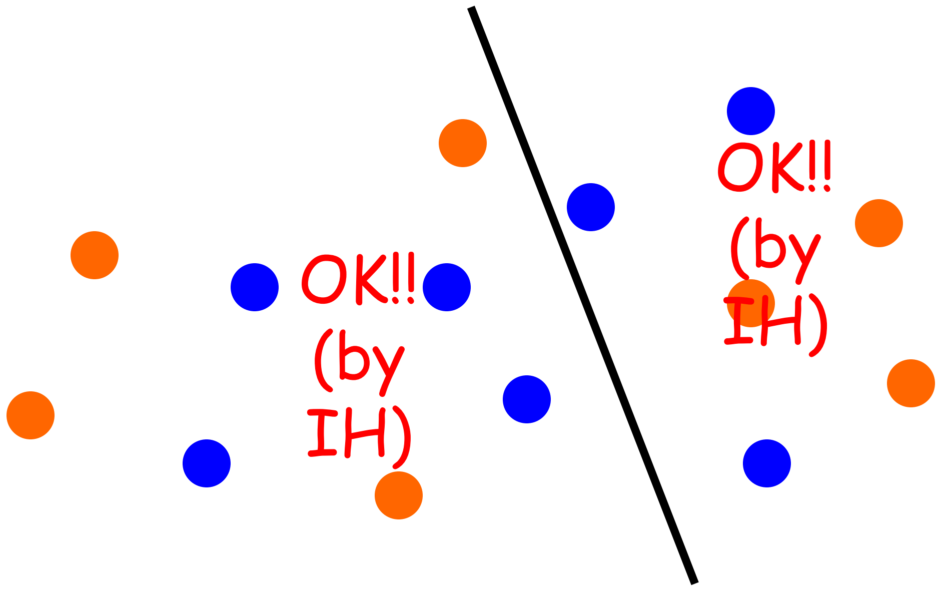


- Base case (n=1):
- Assume any matching problem of size less than (k+1) can be solved.
- Show that we can match (k+1) pairs.



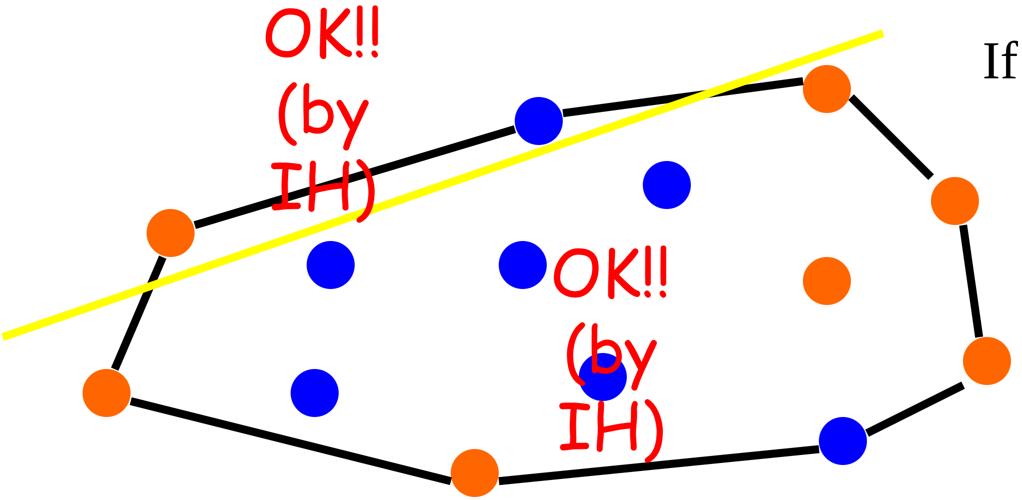
Strong Mathematical Induction

- Show that we can match (k+1) pairs.
- Suppose there is a line partitioning the group into a smaller one of j blues and j oranges, and another smaller one of (k+1)-j blues and (k+1)-j oranges.



Strong Mathematical Induction

- ***But, how do we know such a line always exists?***
- Consider the convex hull of the points:

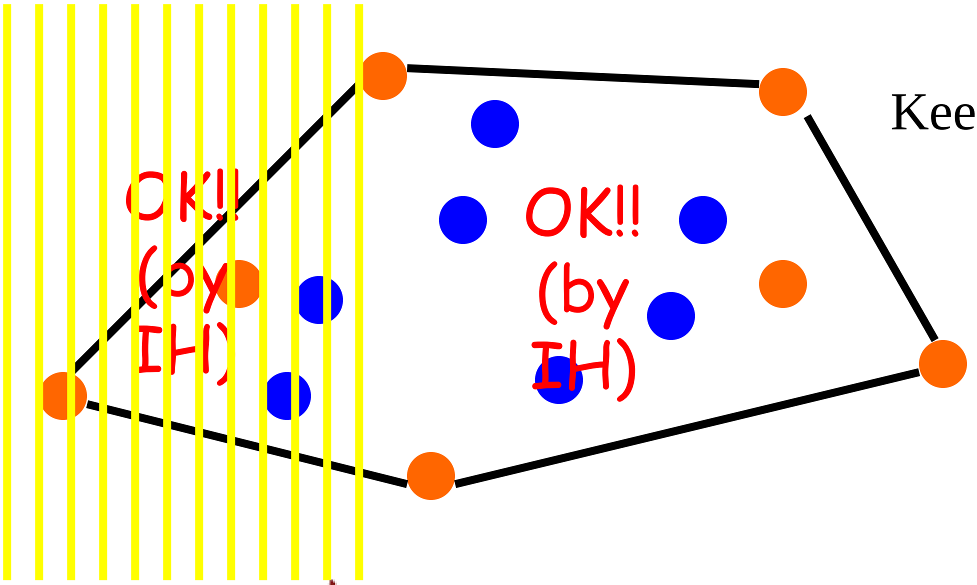


If there is an alternating pair of colors on the hull, we're done!

Strong Mathematical Induction

- If there is no alternating pair, all points on hull are the same color. ☹

Notice that any sweep of the hull hits an orange point first and also last. We sweep on some slope not given by a pair of points.



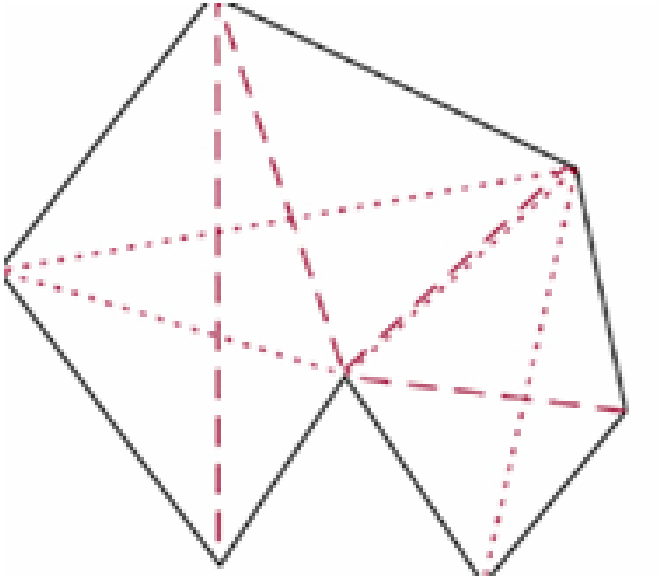
Keep score of # of each color seen. Orange gets the early lead, and then comes from behind to tie at the end.

There must be a tie along the way

Jiangxi University of Science and Technology

Strong Induction: Polygon Triangulation

- Theorem: A simple polygon with n sides, where n is an integer with n≥3, can be triangulated into (n-2) triangles.



n=7
5 triangles
(2 different triangulations)

How would we prove it?

Hypothesis:

T(n) – every polygon with n sides can be triangulated in n-2 triangles

Basis Step: T(3), a polygon with three sides is a triangle;

Inductive Hypothesis: T(j), i.e, all triangles with j sides can be triangulated in j-2 triangles, is true for all integers $3\leq j \leq k$.

Inductive Step – assuming inductive hypothesis, show T(k+1), i.e., every single polygon of k+1 sides can be triangulated in k+1-2 = k-1 triangles

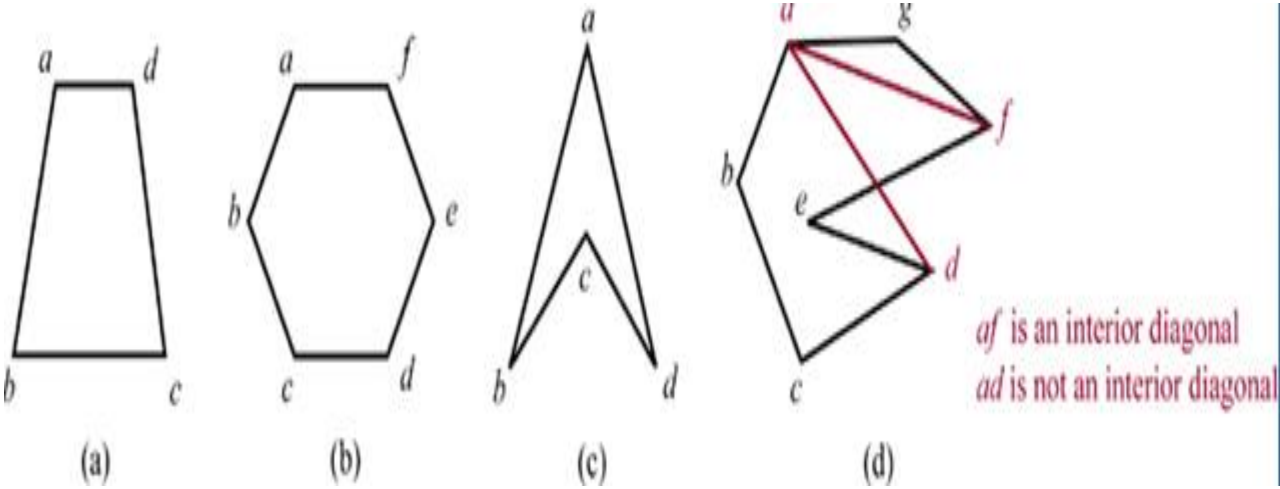
- Inductive Step – assuming T(j), i.e, all triangles with j sides can be triangulated in j-2 triangles, is true for all integers 3≤j ≤k, show T(k+1), i.e., every single polygon of k+1 sides can be triangulated in k+1-2 = k-1 triangles.

- First, we split the polygon with (k+1) sides into two polygons:
- Q with *s* sides and R with *t* sides.
- #sides of P = k+1 = #sides of Q + #sides of R – 2 = s + t - 2 (we counted the new diagonal twice)
- Also 3≤s ≤k and 3≤t ≤k both Q and R have at least one fewer side than P, and therefore by IH we can triangulate Q into s-2 and R into t-2 triangles respectively, and these triangulations with
- s-2+t-2 = s+t-4= (k+1)-2 triangles constitute a valid triangulation for P.

QED

- Subtlety: we assumed the following lemma (not so easy to prove! see Rosen):

- Every simple polygon (i.e., one in which no non-consecutive sides intersect) has an interior diagonal.



(a) (b) (c) (d)

af is an interior diagonal
ad is not an interior diagonal

Winning Strategy: Strong Induction

Example:

Consider the game where there are 2 piles of *n* matches.
Players take turns removing any number of matches they want from one of the two piles. The player who removes the last match wins the game.
Show that the second player can always guarantee a win.

Think about this for a moment: what strategy could the the second player use?

Hint: it’s the “annoying” strategy. ☺

Hyp.: P(n) The second player always has a winning strategy for two piles of n matches.

Basic step: P(1) when there are 2 piles with 1 match each the second player always wins.

Inductive Hypothesis: $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$

Inductive Step: $(P(1) \wedge P(2) \wedge \ldots \wedge P(k) \rightarrow P(k+1))$

Assume player 2 wins when there are 2 piles of k matches.

Can player 2 win when there are 2 piles of k+1 matches?

Suppose that the first player takes r matches ($1 \leq r \leq k$), leaving k+1-r matches in the pile. By removing the same number of matches from the other pile, the second player creates the situation where both piles have the same number of matches ($\leq k$), which we know, by the inductive hypothesis, there is a winning strategy for player two.

Note this proof actually also provides the winning strategy for the 2nd player. (constructive)

QED

Postage: Induction

- **Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.**
- Hypothesis: Every amount of postage of 12 cents or more can be formed using just 4-cent or 5-cent stamps.
- Base case: P(12) postage of 12 cents can be formed using just 4-cent or 5-cent stamps, 12=3(4).
- Inductive Hypothesis: P(k) postage of k cents can be formed using just 4-cent or 5-cent stamps

- Inductive step: P(k)→ P(k+1), given P(k).
- Let's assume P(k), $k \geq 12$. There are two cases:
- a) at least one 4-cent stamp was used to form postage of k cents --- in that case with the extra cent we replace this stamp with a 5-cent stamp.
- b) no extra 4-cent was used to form postage of k cents --- in that case we only used 5 cent stamps; given that k>=12, it has to be at least 15, in which case we need at least three 5-cent stamps. We can replace three 5 cent stamps with four 4-cent stamps to form postage of k+1 cents.

QED

Postage: Strong Induction

- **Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.**
- Hypothesis: Every amount of postage of 12 cents or more can be formed using just 4-cent or 5-cent stamps
- Base case: P(12) 12=3(4); P(13) 13=2(4)+1(5); P(14) 14=1(4)+2(5) P(15) 15=3(5), so $\forall 12 \leq n \leq 15, P(n)$.
- Inductive Hypothesis: P(j) postage, ∀j, $12 \leq j \leq k$, $k \geq 15$ cents can be formed using just 4-cent or 5-cent stamps

- Inductive step: Assuming ∀j $12 \leq j \leq k$ $P(j)$, $k \geq 15$, we want to show P(k+1).
- Note $12 \leq k-3 \leq k$, so $P(k-3)$, so add a 4-cent stamp to get postage for $k+1$.

So, shortens/simplifies standard induction proof.

QED

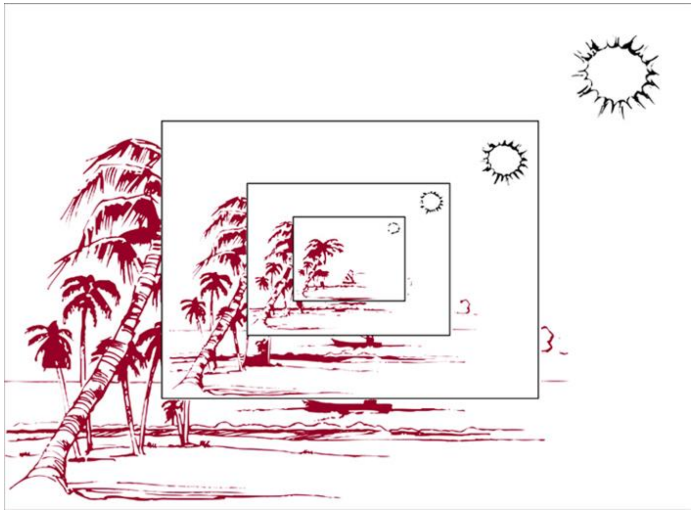
Recursive Definitions and Structural Induction

Recursive or Inductive Definitions

- Sometimes it is difficult to define an object explicitly. However, it may
- be easy to define the object in terms of itself. This process is called
- recursion.

- Recursion is useful to define sequences,
- functions, sets, and algorithms.

- When a sequence is defined recursively,
- by specifying how terms are formed from
- previous terms, we can use induction
- to prove results about the sequence.



Recursive or Inductive Function Definition

- Basis Step: Specify the value of the function for the base case.
- Recursive Step: Give a rule for finding the value of a function from its values at smaller integers greater than the base case.

Inductive Definitions

- We completely understand the function f(n) = n!   right?

  - n! = 1 · 2 · 3 · … · (n-1) · n, n ≥ 1

But equivalently, we could define it like this:

Recursive Case

Base Case

$$n! = \begin{cases} n \cdot (n-1)!, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$

Inductive (Recursive) Definition

Inductive Definitions

- The 2nd most common example:
- Fibonacci Numbers

Note why you need two base cases.

$$f(n)=\begin{cases}0 & \text{if} \quad n=0\\ 1 & \text{if} \quad n=1\\ f(n-1)+f(n-2) & \text{if} \quad n>1\end{cases}$$

Base Cases

Recursive Case

Is there a non-recursive definition for the Fibonacci Numbers?

$$f(n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(Prove by induction.)

All linear recursions have a closed form.

Recursively Defined Sets: Inductive Definitions

- Examples so far have been inductively defined functions.
- Sets can be defined inductively, too.

Give an inductive definition of T = {x: x is a positive integer divisible by 3}

$3 \in S$

Base Case

$x,y \in S \rightarrow x + y \in S$

Recursive Case

Exclusion Rule: No other numbers are in S.

Exclusion rule:
The set contains nothing other than
Those elements specified in the basic
Step or generated by the recursive step.

How can we prove it's correct?

We want to show that the definition of S:

rule 1 - $3 \in S$

rule 2 - $x, y \in S \rightarrow x + y \in S$

Contains the same elements as the set: $T=\{x: x \text{ is a positive integer divisible by } 3\}$

To prove S = T, show

$T \subseteq S$

$S \subseteq T$

Perhaps the “trickiest” aspect of this exercise is realizing that there *is* something to prove! ☺

First, we prove $T \subseteq S$.

$T = \{x$: x is a positive integer, multiple of $3\}$

If $x \in T$, then $x = 3k$ for some integer k. We show by induction on $|k|$ that $3k \in S$.

Hypothesis: $P(n)$ – 3 n belongs to S, for all positive integers n.

Base Case $P(1) = 3 \in S$ since $3 \in S$ by rule 1.

Inductive Hypothesis: $3k \in S$

Inductive Step: Assume $3k, \in S$, show that $3(k+1), \in S$.

Inductive Step:

$3k \in S$ by inductive hypothesis.

$3 \in S$ by rule 1.

$3k + 3 = 3(k+1) \in S$ by rule 2.

Next, we show that $S \subseteq T$.

That is, if an number x is described by S, then it is a positive multiple of 3.

Observe that the exclusion rule, all numbers in S are created by a finite number of applications of rules 1 and 2. We use the number of rule applications as our induction counter.

For example:

$3 \in S$ by 1 application of rule 1.

$9 \in S$ by 3 applications (rule 1 once and rule 2 twice).

- Base Case (k=1): If x $\in$ S by 1 rule application, then it must be rule 1 and x = 3, which is clearly a multiple of 3.

Inductive Hypothesis: Assume any number described by k or fewer applications of the rules in S is a multiple of 3

Inductive Step: Prove that any number described by (k+1) applications of the rules is also a multiple of 3, assuming IH.

Suppose the (k+1)st rule is applied (rule 2), and it results in value

x = a + b. Then a and b are multiples of 3 by inductive hypothesis, and thus x is a multiple of 3.

Aside --- Message here: in a proof, follow a well-defined sequence of steps. This avoids subtle misstakes.

QED

Structural Induction

- Basic Step: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.
- Recursive step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

Validity of Structural Induction follows Mathematical Induction

(for the nonnegative integers)

- P(n) the claim is true for all elements of the set that are generated by *n* or fewer applications of the rules in the recursive step of the recursive definition.

- So, we will do induction on the number of rules applications.

- We show that P(n) is true whenever n is a nonnegative integer.
- Basis case - we show that P(0) is true (i.e., it’s true for the elements specified in the basis step of recursive definition).
- From recursive step, if we assume P(k), it follows that P(k+1) is true.
- Therefore when we complete a structural induction proof we have shown that P(0) is true, and that P(k) →P(k+1).
- So, by mathematical induction P(n) follows for all nonnegative numbers.

Well-Formed Formulas

T is a wff

F is a wff

p is a wff for any propositional variable p

If p is a wff, then $(\neg p)$ is a wff

If p and q are wffs, then $(p \vee q)$ is a wff

If p and q are wffs, then $(p \wedge q)$ is a wff

Basic Cases

Recursive Step

For example, a statement like $((\neg r) \vee (p \wedge r))$ can be proven to be a wff by arguing that $(\neg r)$ and $(p \wedge r)$ are wffs by induction and then applying rule 5.

Note: we have three recursive/construction rules to create new elements.

Structural induction --- illustrative example

- Show that every well-formed formula for compound propositions, contains an equal number of left and right parentheses.

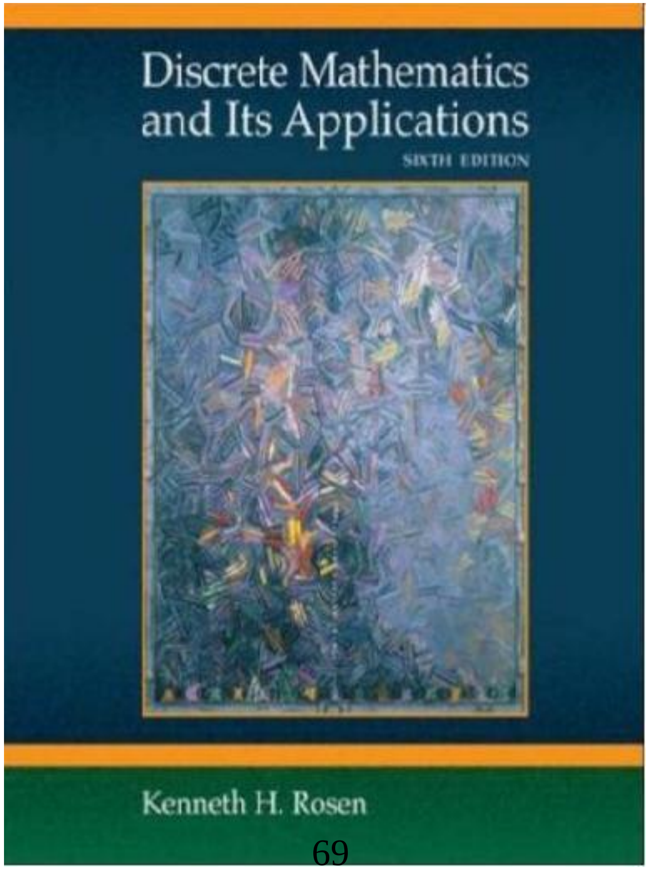
- Basic Step --- True since each formula T, F, and p contains no parentheses;
- Recursive Step:
- Assume p and q are well formed formulas with an equal number of left and right parentheses (lp = rp; lq=rq)
- We need to show that $(\neg p)$, $(p \lor q)$, and $(p \land q)$ contain an equal number of parentheses. Follows directly be considering each rule: Each rule adds a left and a right parenthesis.

The key aspect of structural induction proofs is to show that the base case satisfies the property that we want to prove *and the recursive steps/rules maintain it!*

Can reformulate into induction by doing induction on the # of rule applications.

Jiangxi University of Science and Technology

Recursive Algorithms



- A recursive algorithm is an algorithm that solves the problem by reducing it to an instance of the same problem with smaller input.

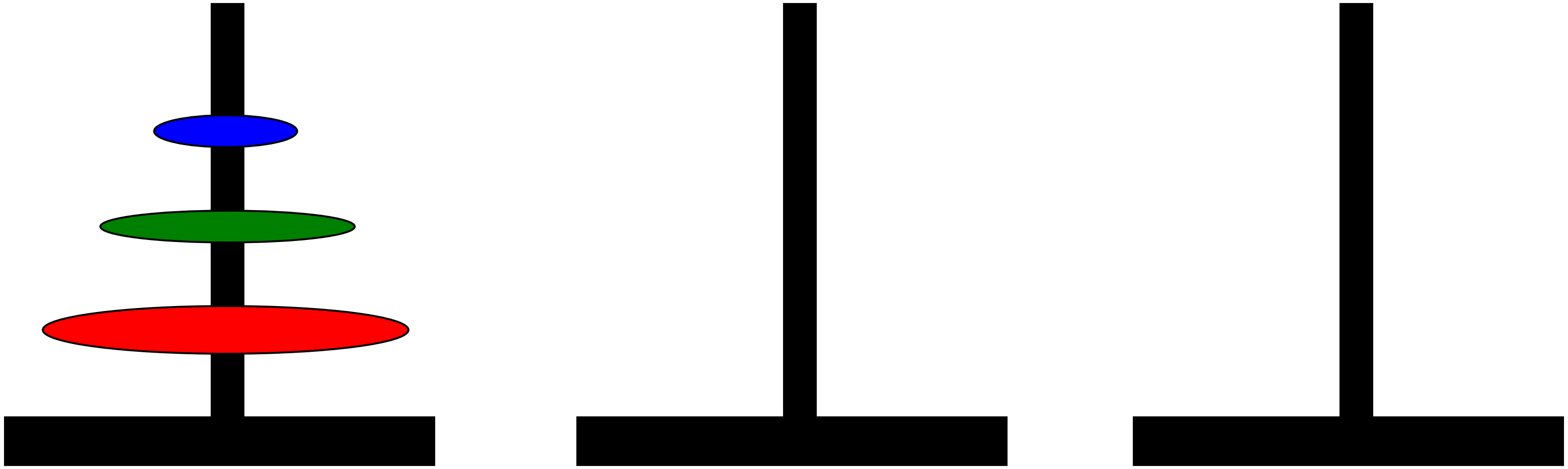
- Recursive Linear Search

```
Procedure search( i, j, x: i, j, x integers,   $1\leq i \leq n$, $1\leq j \leq n$)
if $a_i = x$ then
        location := i
else if i=j then
        location := 0
else
        search(i+1,j,x)
```

- Recursive Binary Search

```
Procedure binary search (i, j, x: i, j, x integers,   1≤i ≤n, 1≤j ≤n)
m := ⌊(i+j)/2⌋
if x = $a_m$ then
        location := m
else if (x < $a_m$ and i < m) then
        binary search( i, m-1,x)
else if (x > $a_m$ and j >  m) then
        binary search(m+1,j,x)
else location :=0
```

Towers of Hanoi (N=3)



Towers of Hanoi

- There are three pegs.
- 64 gold disks, with decreasing sizes, placed on the first peg.
- You need to move all of the disks from the first peg to the second peg.
- Larger disks cannot be placed on top of smaller disks.
- The third peg can be used to temporarily hold disks.

Tower of Hanoi

- The disks must be moved within one week. Assume one disk can be moved in 1 second. Is this possible?
- To create an algorithm to solve this problem, it is convenient to generalize the problem to the “N-disk” problem, where in our case N = 64.
-

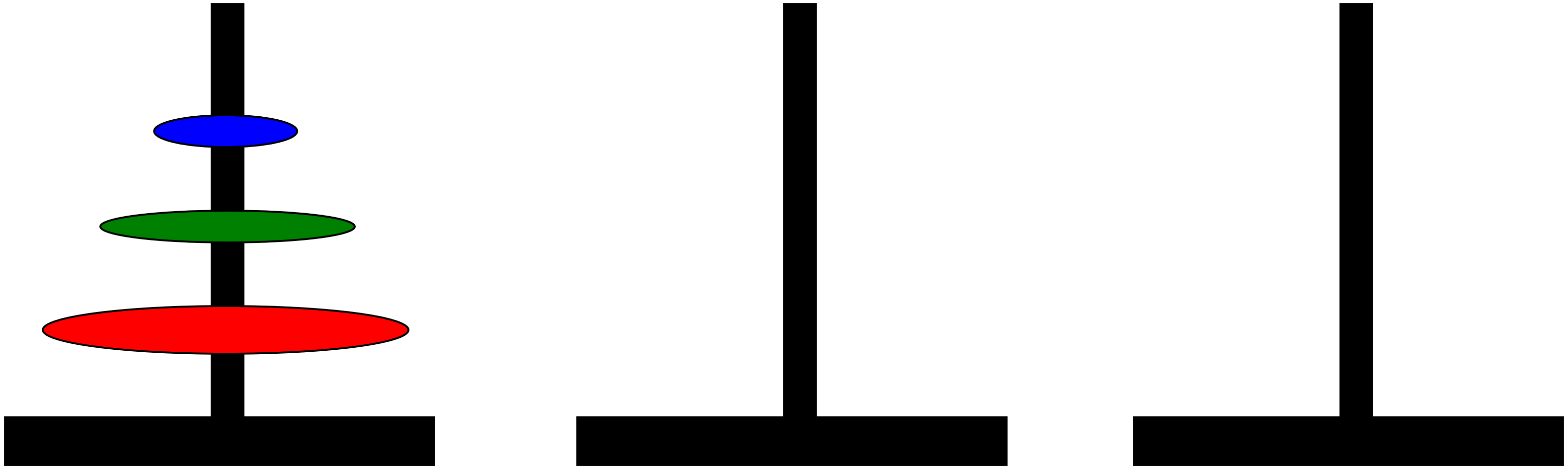
Tower of Hanoi

- How to solve it?
- Think recursively!!!!
- Suppose you could solve the problem for n-1 disks, i.e., you can move (n-1) disks from one tower to another, without ever having a large disk on top of a smaller disk. How would you do it for n?

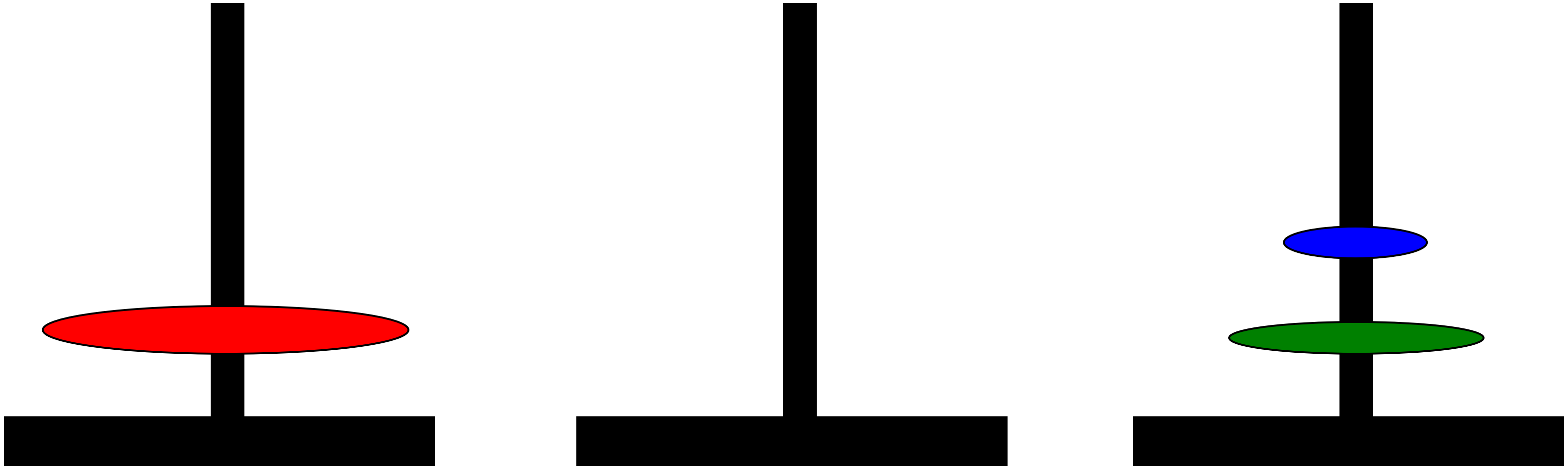
- Solution:

1. Move top (n-1) disks from tower 1 to tower 3 (you can do this by assumption – just pretend the largest ring is not there at all).
2. Move largest ring from tower 1 to tower 2.
3. Move top (n-1) rings from tower 3 to tower 2 (again, you can do this by assumption).

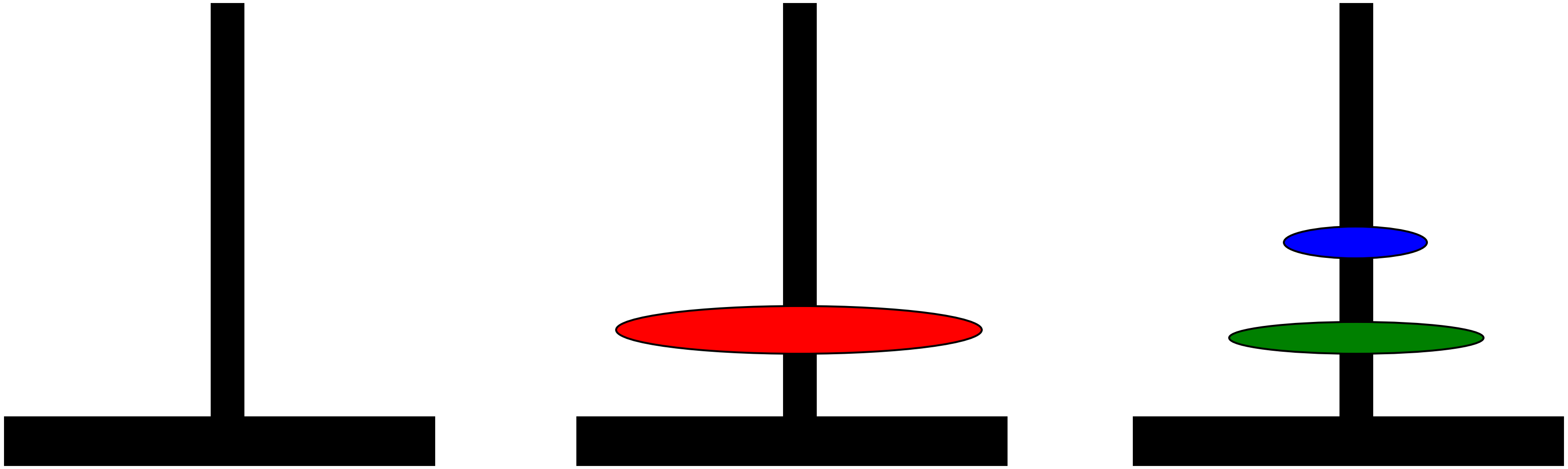
Recursive Solution



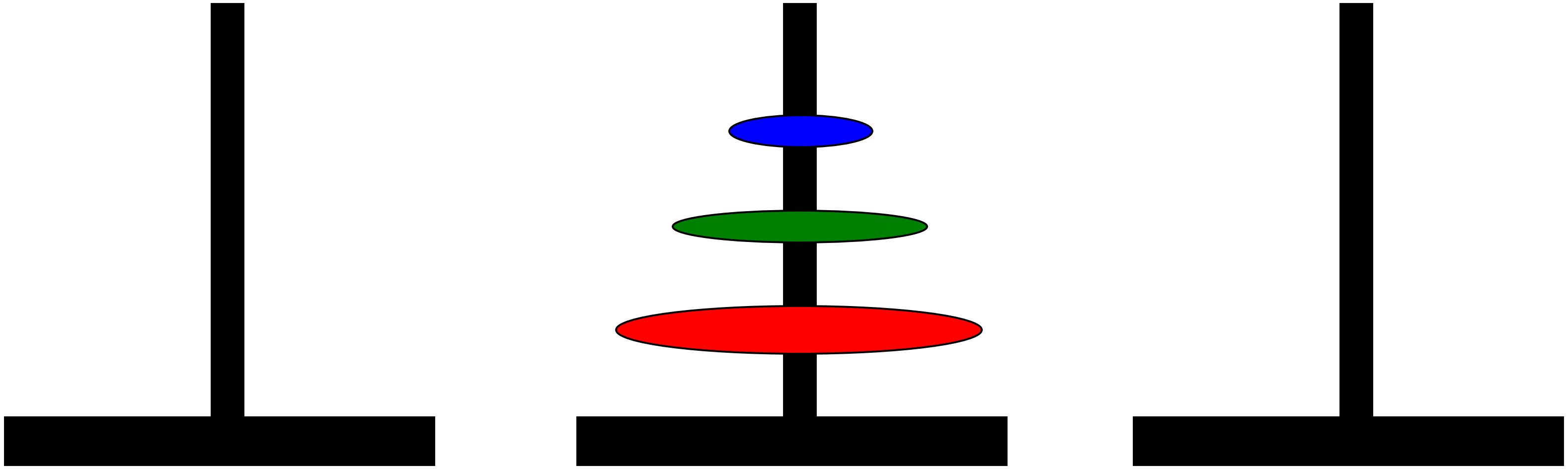
Recursive Solution



Recursive Solution



Recursive Solution

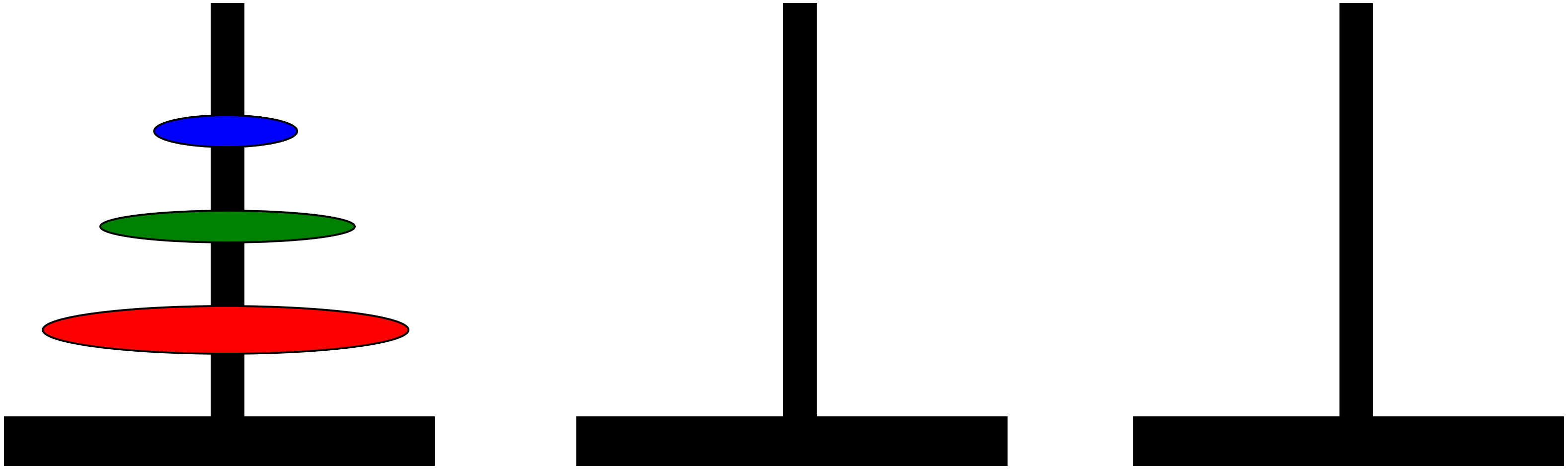


Towers of Hanoi

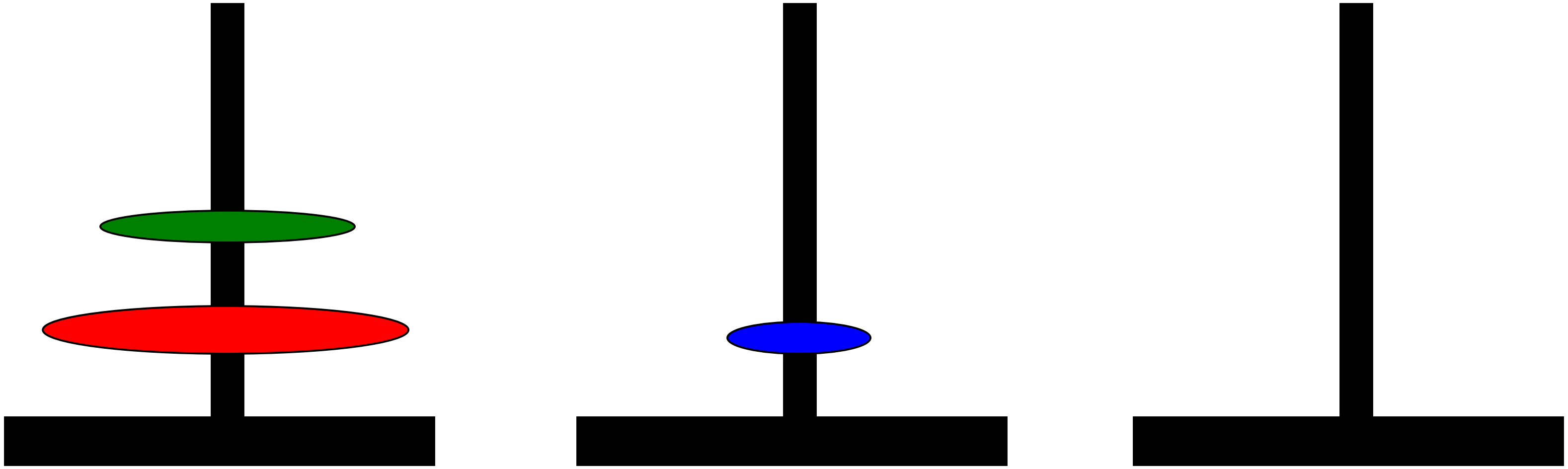
```
• Procedure TowerHanoi (n, a, b, c: n, x, y, z integers,
  1≤a≤3, 1≤b≤3, 1≤c≤3 )
• if n= 1 then
•       move(a,b)
• else
• begin
•       TowerHanoi(n-1, a, c, b)
•       move(a,b);
•     TowerHanoi (n-1,c,b,a);
```

{*TowerHanoi* is the procedure to move *n* disks from tower *a* to tower *b* using tower *c* as an intermediate tower; *move* is the procedure to move a disk from tower a to tower b)

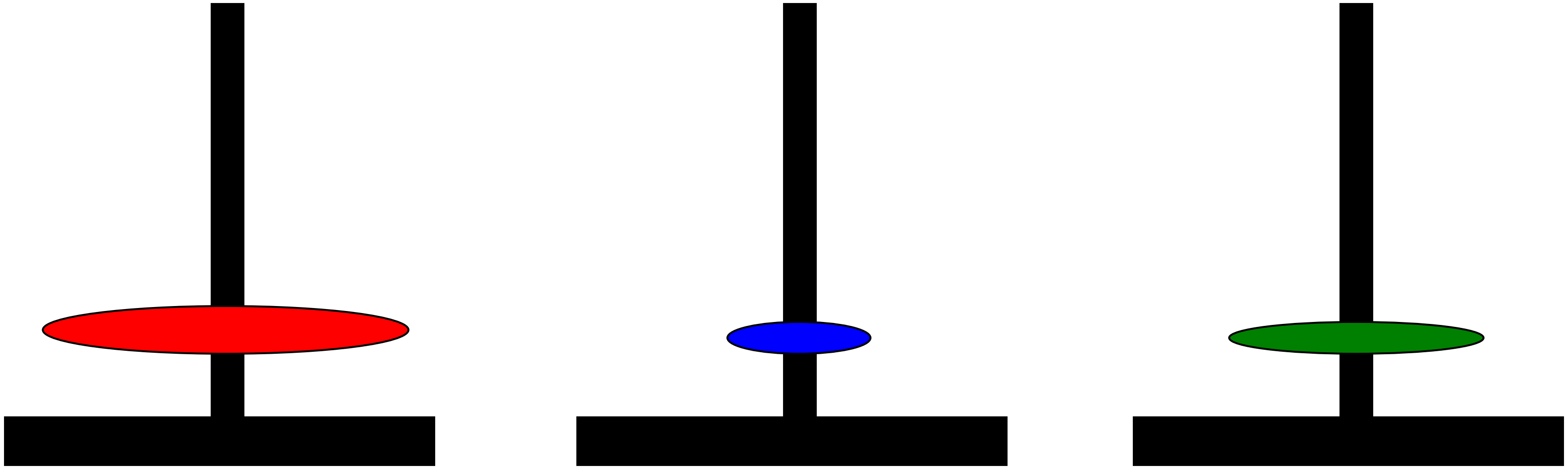
Tower of Hanoi



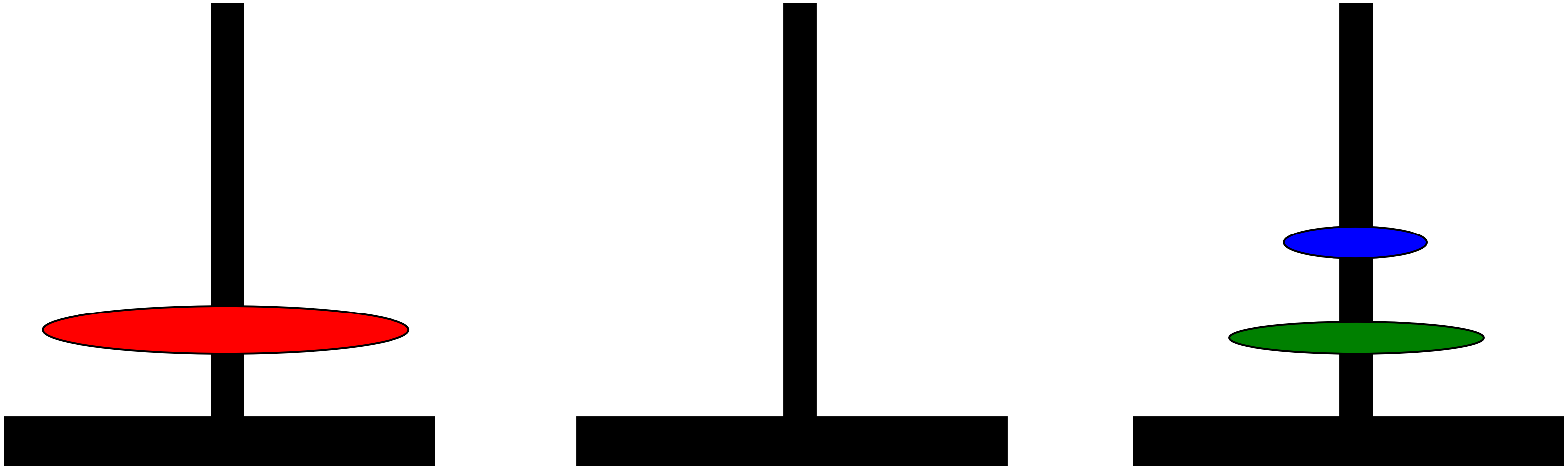
Tower of Hanoi



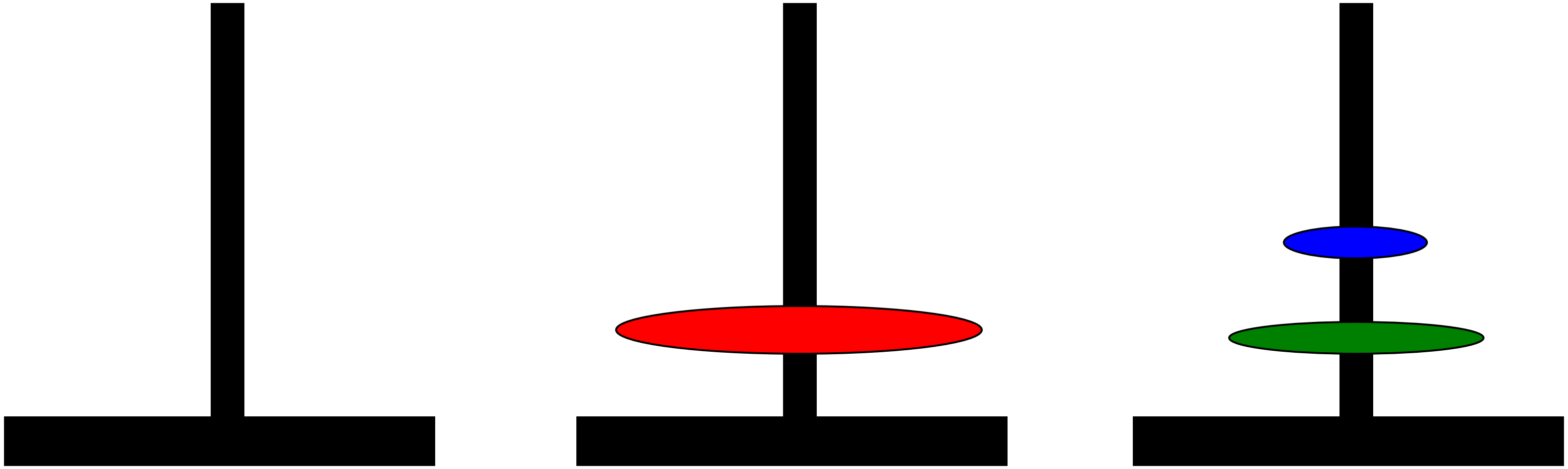
Tower of Hanoi



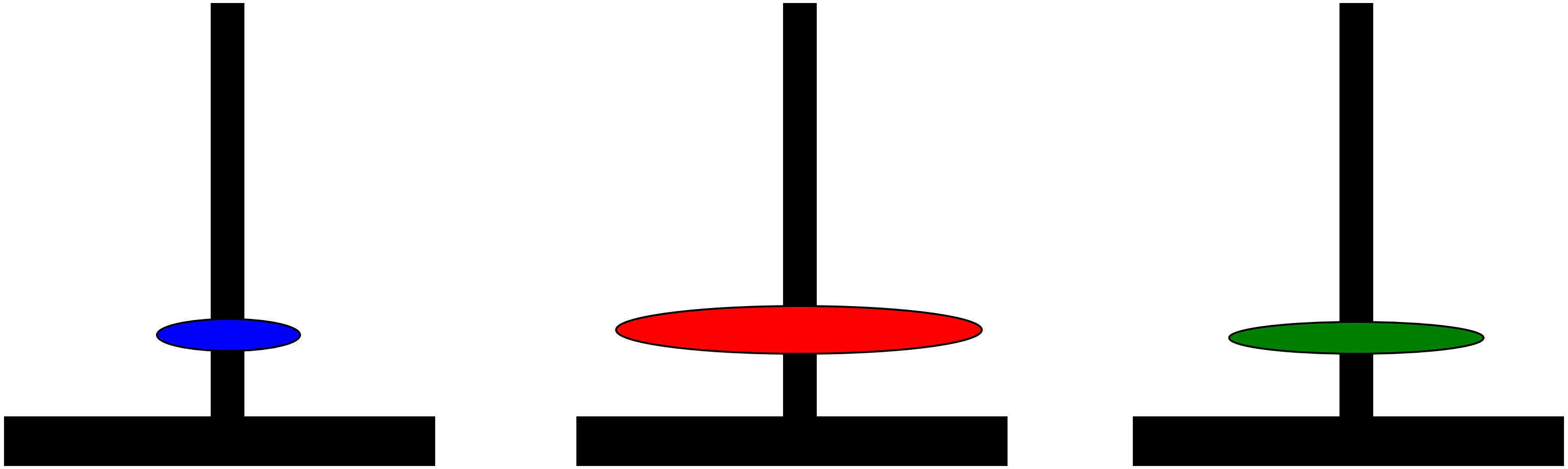
Tower of Hanoi



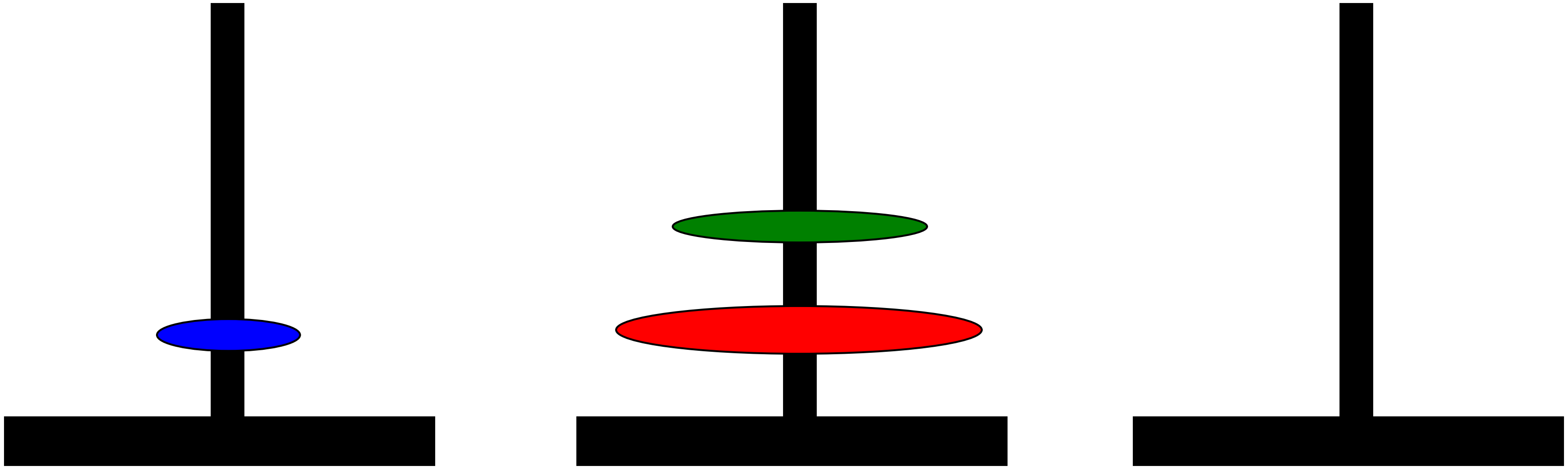
Tower of Hanoi



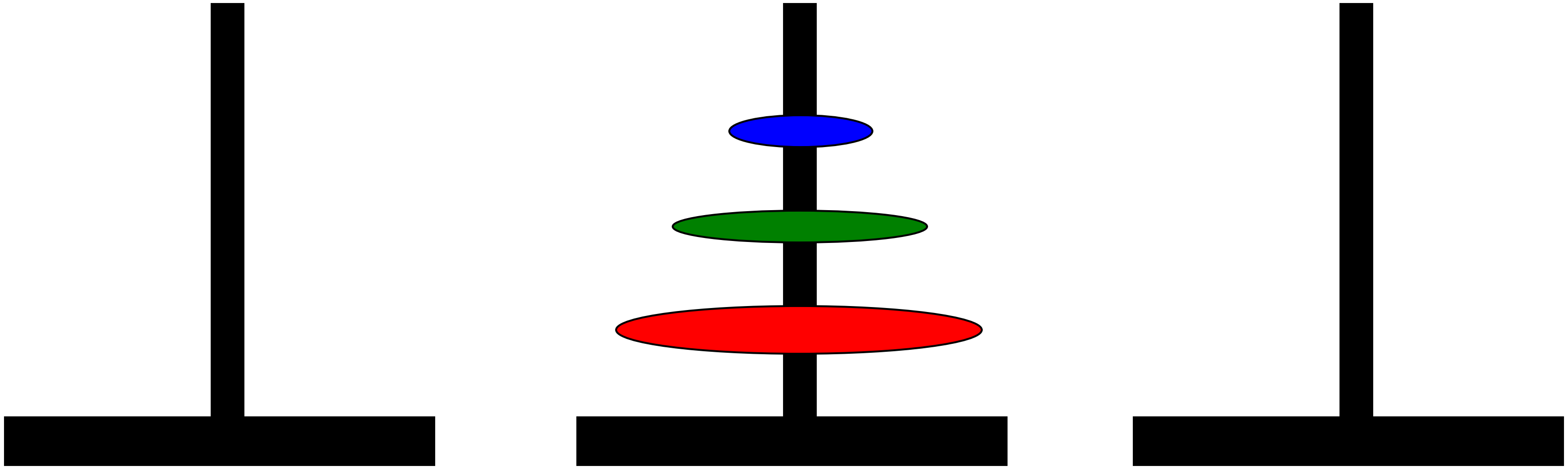
Tower of Hanoi



Tower of Hanoi



Tower of Hanoi



Analysis of Towers of Hanoi

- Hypothesis --- it takes 2^n -1 moves to perform TowerHanoi(n,a,b,c) for all positive n.

- Proof:
- Basis: P(1) – we can do it using move(a,b) i.e., $2^1 - 1 = 1$
- Inductive Hypothesis: P(n) - it takes 2^n -1 moves to perform TowerHanoi(n,a,b,c)
- Inductive Step: In order to perform TowerHanoi(n+1,a,b,c)
- we do: TowerHanoi(n,a,c,b), move(a,c), and TowerHanoi(n,c,b,a);
- Assuming the IH this all takes $2^n - 1 + 1 + 2^n - 1 = 2 \times 2^n - 1 = 2^{(n+1)} - 1$

N = 64 **Note: (2^64) - 1 = 1.84467441 × 10^{19}**

QED

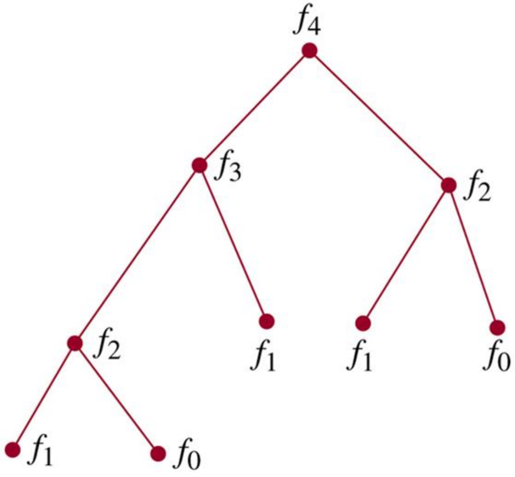
Recursion and Iteration

- A recursive definition expresses the value of a function at a positive integer in terms of the values of the function at smaller integers.
- But, instead of successively reducing the computation to the evaluation of the function at smaller integers, we can start by considering the base cases and successively apply the recursive definition to find values of the function at successive larger integers.

Recursive Fibonacci

```
procedure fibonacci (n: nonnegative integer)
if n = 0 then fibonacci(0) :=0
else if n =1 then fibonacci(1) := 1
else fibonacci(n): = fibonacci(n-1) + fibonacci(n-2)
```

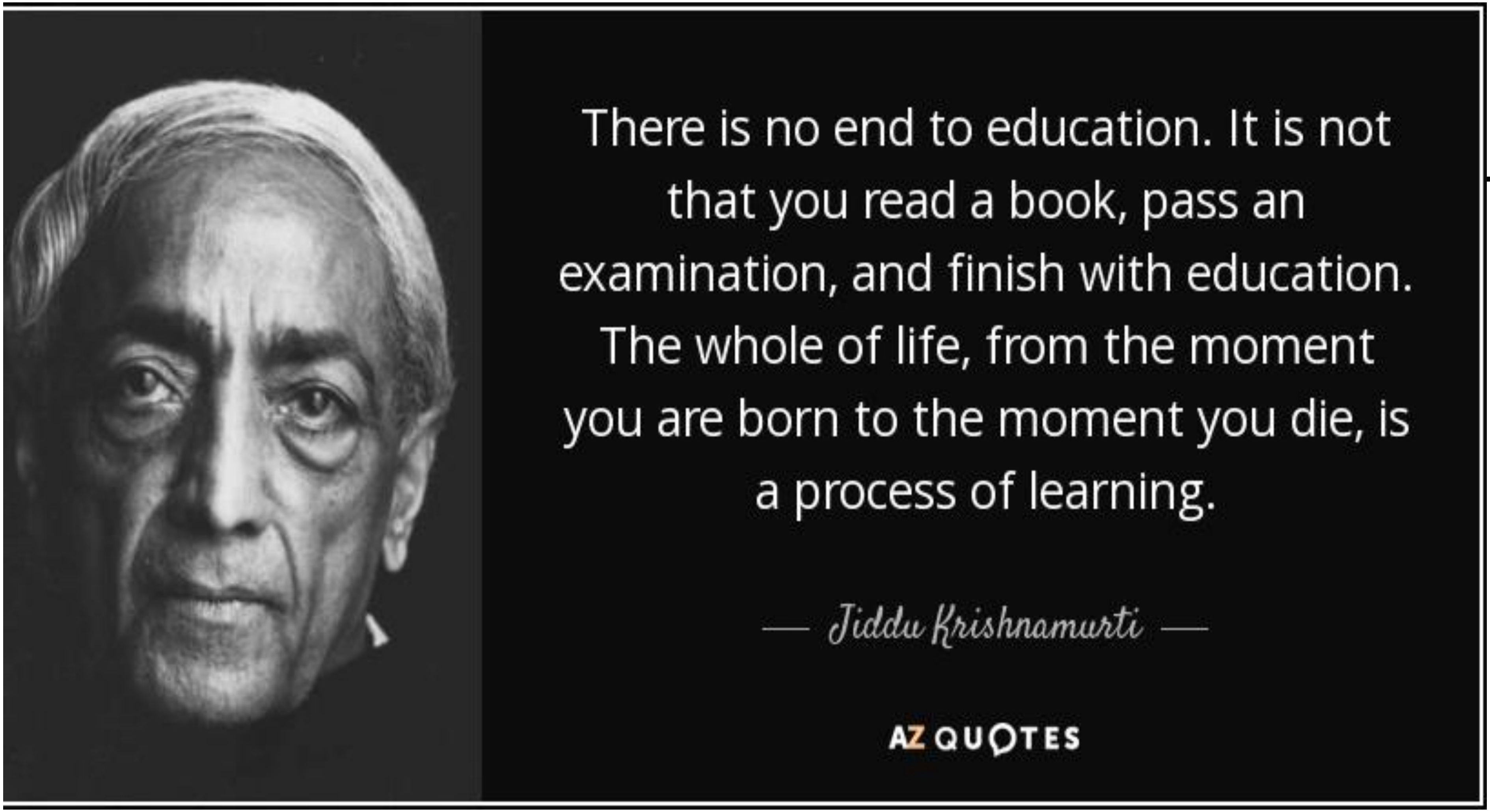
What's the "problem" with this algorithm?



Iterative Fibonacci

```
procedure iterativefibonacci(n: nonnegative integer)
if n=0 then y:= 0
else
begin
        x := 0
        y := 1
        for i := 1 to (n-1)
        begin
                z := x + y
                x := y
                y := z
        end
end
{y is the nth Fibonacci number}
```

$$f(n) = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ f(n-1) + f(n-2) & \text{if} \quad n > 1 \end{cases}$$



There is no end to education. It is not that you read a book, pass an examination, and finish with education. The whole of life, from the moment you are born to the moment you die, is a process of learning.

— Jiddu Krishnamurti —

AZ QUOTES

Reference

https://www.tutorialspoint.com/computer_logical_organization/binary_arithmetic.htm