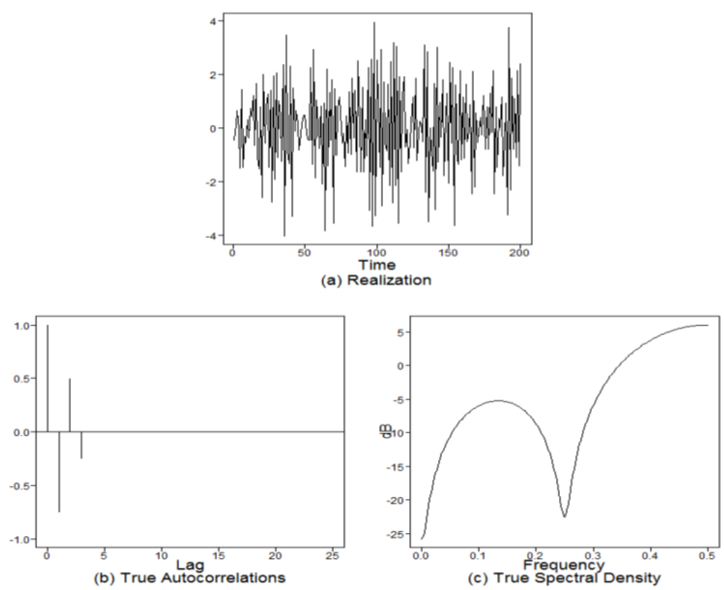
DS 6373: Time Series: Unit 5 HW Solutions

Below are the homework (HW) problems for this Unit. You do not need to submit the solutions rather double check your solutions to the solutions posted. Solutions will be posted to the Wall a few days after the release of the HW. This is intended to let the student think about the problem and attempt it without the temptation to first look at the solution. Please write any questions to the Wall or in an email to myself and/or bring them up during office hours or even in the next Live Session. Remember that the concepts covered below are fundamental to the course and are fair game for the midterm and final.

Have a blast!

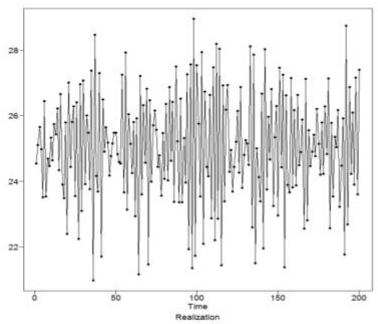
Problems from Chapter 3 of the Textbook:
3.1

Problem 3.1
(a,b,c) Using the code
```
x=plotts.true.wge(n=200,theta=c(.95,-.9,.855))
```
We obtained the plots



To obtain and plot a realization with mean 25 I used thecommands
```
x25=mean(x$data)
plotts.wge(x25)
```

and obtained the following is the data with $\mu = 25$. The other two plots do not change with the mean change.



The realization shows high frequency behavior with oscillation back and forth across the mean of 25. The autocorrelations show only three non-zero autocottelations. The fact that ρ_1 is fairly large negative and ρ_2 is positive, causes the up-and-down oscillatory behavior. The spactreal density shows power at f=0.5 which

we have already noted along with some power at around f=0.15 which is not detectable using the other two plots.

(d) $\mu = 25$ and $\sigma^2 = \sigma_a^2(1+0.95^2+0.9^2+.855^2) =$ 3.44 which can be found in `x$acv[1]` after running the above code.

(e) using the base r code `mean(x25)` we get $\bar{x} = 25.013$ and var(x25) gives $\hat{\sigma}^2 = 2.979$

3.3 (c)

This is a moving average model which is always stationary since it is a linear combination of white noise terms which are individually stationary. All moving average models are stationary, they may not be invertible, but they are stationary.

3.6

Problem 3.6 AR factors are

Factor	Roots	Abs Recip	System Freq
1 - 1.000B + 0.900B^2	.5556 +/- .8958i	0.9487	0.1616
1 - 0.800B	1.2500	0.8000	0.0000
1 + 0.800B	-1.2500	0.8000	0.5000

While the factors for the MA part are

Factor	Roots	Abs Recip	System Freq
1 - 1.600B + 0.900B^2	.8889 +/- .5666i	0.9487	0.0903
1 - 0.800B	1.2500	0.8000	0.0000

(a) The factors 1-0.8B cancel. Thus, the resulting model is the ARMA(3,2)
(b) The resulting ARMA(3,2) model is

$$(1-B+.9B^2)(1+.8B)X_t = (1-1.6B+.9B^2)a_t$$

or

$$(1-.2B+.1B^2+.72B^3)X_t = (1-1.6B+.9B^2)a_t$$

(c) Based on the factor tables above, it can be seen that the model is both stationary and invertible (all roots are outside the unit cicle).

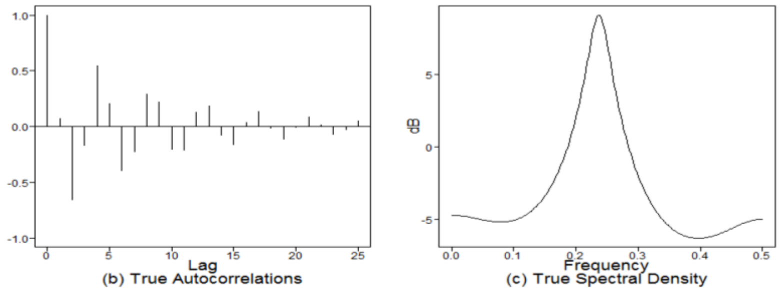
3.7

Problem 3.7

(a) Using `factor.wge`, the factor table for model (a) is

```
Factor                   Roots               Abs Recip    System Freq
1-0.1455B+0.8048B^2      0.0904+-1.1110i     0.8971       0.2371
1+0.5693B               -1.7565              0.5693       0.5000
1-0.5238B                1.9091              0.5238       0.0000
```

Since the process is AR, it is invertible and since all the roots are outside the unit circle it is stationary. Using `plotts.true.wge` the autocorrelations and spectral density are shown below.



(b) True Autocorrelations (c) True Spectral Density

The factor table shows strong frequency behavior at about f=.24 with weaker behavior at f=0 and f=.5. There is some indication in the autocorrelations of a period of 4 $(\approx 1/.24)$ with very little or no indication of frequency behavior at f=0 or f=0.5. The spectral density shows the peak at about f=.24 and very small peaks at f=0 and f=.5.

(b) Using `factor.wge`, the factor table for the AR component of model (b) is

Factor	Roots	Abs Recip	System Freq
1-0.8000B	1.2500	0.8000	0.0000
1-0.5000B	2.0000	0.5000	0.0000

while for the MA component we trivially have

Factor	Roots	Abs Recip	System Freq
1-1.9000B	0.5263	1.9000	0.0000

This process is stationary (both roots of the AR characteristic equation are outside the unit circle), but it is not invertible since the root of the MA characteristic equation (.5263) is inside the unit circle.
nSince the process is AR, it is invertible and since all the roots are outside the unit circle it is stationary.

(c) The factor table for the AR component is trivially

Factor	Roots	Abs Recip	System Freq
1-1.9000B	0.5263	1.9000	0.0000

while for the MA part it is

Factor	Roots	Abs Recip	System Freq
1-0.8000B	1.2500	0.8000	0.0000
1-0.5000B	2.0000	0.5000	0.0000

Thus, this model is not stationary and is invertible.

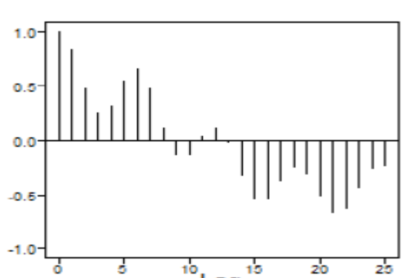
(d) Using `factor.wge`, the factor table for the AR part of model (d) is

Factor	Roots	Abs Recip	System Freq
1-1.9484B+0.9695B^2	1.0049+-0.1475i	0.9846	0.0232
1-1.0016B+0.9490B^2	0.5278+-0.8805i	0.9742	0.1641

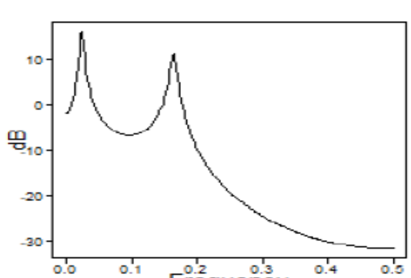
The factor table for the MA part is trivially

Factor	Roots	Abs Recip	System Freq
1-0.9000B	1.1111	0.9000	0.0000

Since all roots of the AR and MA characteristic equations are outside the unit circle, this model is both stationary and invertible. Using `plotts.true.wge` the autocorrelations and spectral density are shown below.



(b) True Autocorrelations



(c) True Spectral Density

The factor table for the AR part shows strong frequency behavior at about *f*=.023 and at *f*=.16. Both periods show up in the autocorrelations (as periods of about 1/.023=43.5 and 1/.16 = 6.25). There are peaks in the spectral density at about *f*=.023 and at *f*=.16 and a dip at f=.5 associated with the positive real root in the MA part.

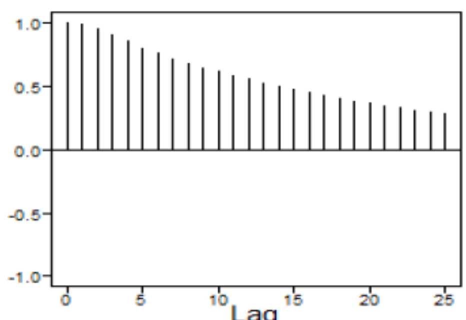
(e) Using `factor.wge`, the factor table for the AR part of model (e) is

Factor	Roots	Abs Recip	System Freq
1+0.9487B	-1.0541	0.9487	0.5000
1-0.9487B	1.0541	0.9487	0.0000
1-1.0000B+0.4100B^2	1.2195+-0.9756i	0.6403	0.1074

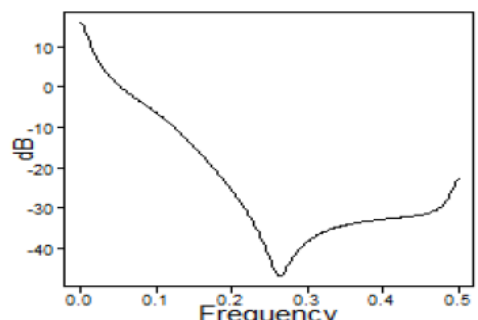
the factor table for the MA part is

Factor	Roots	Abs Recip	System Freq
1+0.1443B+0.8765B^2	-0.0823+-1.0650i	0.9362	0.2623
1+0.8557B	-1.1687	0.8557	0.5000

Since all roots of the AR and MA characteristic equations are outside the unit circle, this model is both stationary and invertible. Using `plotts.true.wge` the autocorrelations and spectral density are shown below.



(b) True Autocorrelations



(c) True Spectral Density

Only the the slow exponential-type damping behavior associated with the factor $(1-.9487B)$ is visible in the autocorrelations. The spectral density shows peaks at $f=0$ and $f=.5$ associated with factors $(1-.9487B)$ and $(1+.9487B)$, respectively. The relatively weak AR component associated with AR frequency of *f*=.1074 (the root is far removed from the unit circle) is only visible as a mild inflection point in the spectral density. The MA frequency of *f*=.26 is shown as a strong dip in the spectrum at about *f*=.26. The expected dip at *f*=.5 associated with $(1+0.8557B)$ is not seen in the spectral density because of the corresponding factor $(1+.9487B)$ in the AR part. The presence of these two factors cause near cancellation, but the fact that the AR factor is slightly “stronger”results in a peak instead of a dip at *f*=.5. If the MA factor $(1+0.8557B)$ were not in the model, the peak associated with the AR factor $(1+0.8557B)$ would be much higher.

3.8

Problem 3.8

(a) Using `factor.wge(phi=c(.5,5))` (even though the model is MA), the resulting factor table is

Factor	Roots	Abs Recip	System Freq
1-2.5000B	0.4000	2.5000	0.0000
1+2.0000B	-0.5000	2.0000	0.5000

That is, the model factors as $X_t = (1-2.5B)(1+2B)a_t$. Since the characteristic equation for the MA part has roots inside the unit circle, the process is not innvertivle. Using the `tswge` command `plotts.true.wge(theta=c(.5,5))` we see that $\rho_0 = 1, \rho_1 = 0.07619, \rho_2 = -.1905$ and $\rho_k = 0,$ for $k \geq 3$. Consider the model $Y_t = (1-.4B)(1+.5B)a_t$. Clearly, this model is invertible, and multiplying this out or using `mult.wge(fac1=.4,fac2=-.5)` we obtain the model $Y_t = (1+.1B-.2B^2)a_t$. Using `plotts.true.wge(theta=c(-.1,.2))` or using the formulas in Section 3.1.2, we can see that this model is invertible and has the same autocorrelations as the original model.

(b) (a) Using `factor.wge(phi=c(2,-1.5))` (even though the model is MA), the resulting factor table is

Factor	Roots	Abs Recip	System Freq
1-2.0000B+1.5000B^2	0.6667+-0.4714i	1.2247	0.0980

This is a model with a pair of complex conjugate roots which are inside the unit circle. Using the tswge command `plotts.true.wge(theta=c(2,-1.5))` we see that $\rho_0 = 1, \rho_1 = -0.6897, \rho_2 = 0.2069$ and $\rho_k = 0, k \geq 3$. The invertible model we want has $0.6667 \pm 0.4714i$ as the reciprocals of the roots. That is $r_1^{-1} = 0.6667 + 0.4714i$ and $r_2^{-1} = 0.6667 - 0.4714i$ In equations (3.657) and (3.58) (which were based on AR models but provide a realizationship between the roots and the parameters of an AR(2) or MA(2) model) we get that for an MA(2), $\theta_1 = r_1^{-1} + r_2^{-1} = 2(.666667) =$ 1.33333 and $\theta_2 = r_1^{-1} r_2^{-1} = .666667^2 + .4714^2 = .66667$. Thus, the invertible model we want is $Y_t = a_t + 1.33333a_{t-1} + 0.666667a_{t-2}$ Using the `tswge` command `plotts.true.wge(theta=c(-1.33333,-.666667))` we obtain the same autocorrelations as above.

3.11

Problem 3.11

(a) The factor tables of the AR and MA parts of the model can be obtained using the commands

```
factor.wge(phi=c(1,.49,-.9,.369))
factor.wge(phi=c(-1,-1,-.75))
```

Factor Table for AR Part

Factor	Roots	Abs Recip	System Freq
1+0.9487B	-1.0541	0.9487	0.5000
1-0.9487B	1.0541	0.9487	0.0000
1-1.0000B+0.4100B^2	1.2195+-0.9756i	0.6403	0.1074

Factor Table for MA Part

Factor	Roots	Abs Recip	System Freq
1+0.1443B+0.8765B^2	-0.0823+-1.0650i	0.9362	0.2623
1+0.8557B	-1.1687	0.8557	0.5000

These factor tables show that all the roots of the characteristic equations for the AR and MA parts are outside the unit circle and consequently the process is stationary and invertible.

(b) The ψ -weights can be found by a number of methods. However, we will use the `tswge` function `psi.weights.wge` and use the specific command

```
psi.weights.wge(phi=c(1,.49,-.9,.369),theta=c(-1,-1,-.75),lag.max=10)
```

And obtain the results

```
2.000000 3.490000 4.320000 4.599100 4.312900 3.966269 3.534480 3.293410 3.047123
2.943415
```

which are $\psi_i, i = 1,\ldots,10$ respectively.

3.13: Just for (i)

Problem 3.13

Model i:

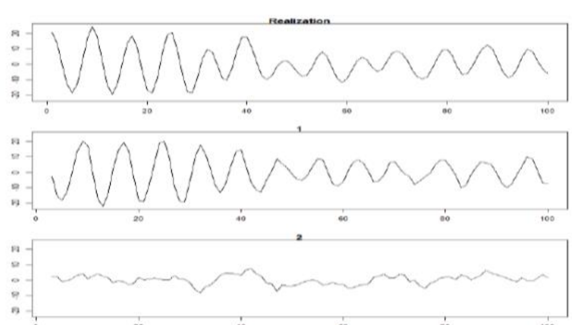
The command `factor.wge(phi=c(2.2,-2.1,.8))` produces the factor table

Factor	Roots	Abs Recip	System Freq
1-1.3685B+0.9621B^2	0.7112+-0.7305i	0.9809	0.1271
1-0.8315B	1.2026	0.8315	0.0000

Since there is a pair of complex roots and a real root, we will examine 2 additive components. Also, we will fit an AR(3) model to the data. The commands

```
xi=gen.arma.wge(n=100,phi=c(2.2,-2.1,.8),sn=1)
factor.comp.wge(xi,p=3,ncomp=2)
```

produce the following plots:



Clearly, the dominant feature is the cyclic behavior associated with frequency f=.127. The first order behavior associated with the factor 1-0.8315B plays a minimal role in the appearance of the realization.