Example Solve the system $Ax=b$ with Apr 11/14

$$A=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$ using LU factorization

Step 1: LU-factorization

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -2 & 3 & -1 \end{bmatrix} \begin{matrix} \\ R_2 = R_2 - R_1 \\ R_3 = R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -2 & 3 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 = R_3 - 3R_2 \end{matrix}$$

So $L=\begin{bmatrix} 1 & & \\ 1 & 1 & \\ -2 & 3 & 1 \end{bmatrix}$ $\quad U=\begin{bmatrix} 1 & 1 & -1 \\ & 1 & -1 \\ & & 2 \end{bmatrix}$

Step 2 We have $LUx=b$, so let $y=Ux$. Then

$$Ly=b \Rightarrow \begin{bmatrix} 1 & & \\ 1 & 1 & \\ -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= 2 - y_1 = 1 \\ y_3 &= 3 - 3y_2 + 2y_1 = 2 \end{aligned}$$

So, $y=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Step 3 Solve $Ux=y$

$$\begin{bmatrix} 1 & 1 & -1 \\ & 1 & -1 \\ & & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 1 + x_3 - x_2 = 0 \\ x_2 &= 1 + x_3 = 2 \\ x_3 &= 1 \end{aligned} \quad \therefore \; x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

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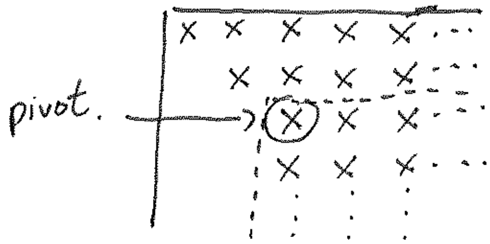
Pivoting

- Sometimes it is necessary (or desirable) to swap rows when doing GE.
- When we eliminate zeros in the k^{th} column, we refer to a_{kk} as the *pivot* element



Two situations:

i) If the pivot is zero, then it is necessary (mathematically) to swap rows, e.g.

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 1 \\ \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{matrix} \\ R_2 = R_2 - \frac{1}{2}R_1 \\ R_3 = R_3 - \frac{1}{4}R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{5}{2} \end{bmatrix} \begin{matrix} \\ R_2 = R_3 \\ R_3 = R_2 \end{matrix}$$

(ii) If the pivot is small, then it is a good idea (numerically) to swap rows to avoid large round-off error.

eg Suppose we have the system $\begin{bmatrix} 3.1\times10^{-4} & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, only 4 digits of accuracy.

Then, we do one step of GE:

- $\frac{1}{3.1e^{-4}} = 3266$
- $1 - 3266 \cdot 1 = -3255$
- $7 - 3266 \cdot 3 = -9671$

$$\begin{bmatrix} 3.1\times10^{-4} & 1 \\ 0 & -3225 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -9671 \end{bmatrix} \quad R_2 = R_2 - 3266R_1$$

And solve: $x_2 = \frac{-9671}{-3225} = 2.999$

$$x_1 = \frac{3 - 2.999}{3.1e^{-4}} = 3.266$$

This is a poor solution since $x_1 + x_2 = 6.225 \neq 7$

Problem: Cancellation error $(3 - 2.999)$ was amplified when we divided by the small pivot element when computing x_1.

If we swap rows:

$$\begin{bmatrix} 1 & 1 \\ 3.1e^{-4} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

- $\frac{3.1e^{-4}}{1} = 3.1e^{-4}$
- $1 - 3.1e^{-4} = 0.9997$
- $3 - 3.1e^{-4} \cdot 7 = 2.998$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0.9997 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.998 \end{bmatrix}$$

$$\left.\begin{aligned} \text{So, } x_2 &= \frac{2.998}{0.9997} = 2.999 \\ x_1 &= \frac{7 - 2.999}{1} = 4.001 \end{aligned}\right\} \text{ satisfies both equations to 4 digit accuracy.}$$

So, the first method is <u>unstable</u>, and the second is <u>stable</u>.

To ensure that the calculation is stable, implementations of GE use a <u>pivoting</u> strategy to ensure that the pivot is "large" before eliminating the other elements in the column:

* <u>partial pivoting</u> - swap rows so that the column element with largest absolute value is the pivot (p. 270 of text)

* <u>scaled partial pivoting</u> - swap rows so that the column element with largest absolute value <u>relative to elements in its row</u>. This can avoid issues if the matrix has elements that vary widely in magnitude

* <u>complete pivoting</u> - swap rows & columns to get the largest possible pivot (most stable, but also expensive).

As a result of pivoting, LU factorization often includes a <u>permutation matrix</u>, P, which keeps a record of row swaps. This matrix is just I with the corresponding rows swapped.

e.g multiplying A by the matrix shown at right would swap rows 2 & 4.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We then get a factorization $LU = PA$, or

$P^T LU = A$ (it can be shown that $P^{-1} = P^T$)

E.g. Compute the LU factorization of A with partial pivoting

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Swap rows 1 & 3:

$$\begin{bmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Eliminate 1st column

$$\begin{bmatrix} 7 & 8 & 0 \\ 4/7 & \frac{3}{7} & 6 \\ 1/7 & \frac{6}{7} & 3 \end{bmatrix} \quad \begin{aligned} R_2 &= R_2 - \tfrac{4}{7} R_1 \\ R_3 &= R_3 - \tfrac{1}{7} R_1 \end{aligned}$$

Swap rows 2 & 3, eliminate 2nd column

$$\begin{bmatrix} 7 & 8 & 0 \\ 1/7 & \frac{6}{7} & 3 \\ 4/7 & \frac{1}{2} & \frac{9}{2} \end{bmatrix} \quad R_3 = R_3 - \tfrac{1}{2} R_2 \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ 1/7 & 1 & \\ 4/7 & \frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 7 & 8 & 0 \\ & 6/7 & 3 \\ & & \frac{9}{2} \end{bmatrix}$$