Drake Bridgewater

Due Wednesday, April 23 by the end of class.

1. **[10 points]** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -6 & 7 & -9 \\ 1 & -5 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

(a) Calculate $\|A\|_\infty$, $\|A^{-1}\|_\infty$, and the condition number in the infinity norm, $\kappa_\infty(A)$. You can use Matlab to find A^{-1}.

$\|A\|_\infty = max_{1\leq j\leq n} \sum_{i=1} \mid A_{ij} \mid \quad \Rightarrow$ max of the rows $\Rightarrow max(23, 6, 6, 6) = 23$

$$A^{-1} = \begin{bmatrix} 125 & -124 & 130 & -225 \\ 25 & -25 & 26 & -45 \\ 5 & -5 & 5 & -9 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

$\|A^{-1}\|_\infty = max_{1\leq j\leq n} \sum_{i=1} \mid A^{-1}_{ij} \mid \quad \Rightarrow$ max of the rows $\Rightarrow max(604, 121, 24, 5) = 604$

$\kappa_\infty(A) = \|A^{-1}\|_\infty \; \|A\|_\infty$

$\Rightarrow \; \|A^{-1}\|_\infty = 604$

$\|A\|_\infty = 23$

$\Rightarrow \; 23 \times 604 = 13892$

(b) Let $b = [-7, -4, -4, -4]^T$. Then, the exact solution to the system $Ax = b$ is $x = [1, 1, 1, 1]^T$. Suppose the system is perturbed as follows:

$$\tilde{b} = b + 0.01 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Solve the system $A\tilde{x} = \tilde{b}$ in Matlab, and state the result. How much bigger is the relative error, $\|x - \tilde{x}\|_\infty / \|x\|_\infty$, compared to the relative difference in the residual, $\|b - \tilde{b}\|_\infty / \|b\|_\infty$? Verify that the magnification of the error is within the bound given by the condition number.

$A\tilde{x} = \tilde{b}$ $\Rightarrow \begin{bmatrix} 0.9496 \\ 0.9979 \\ 1.0076 \\ 1.0095 \end{bmatrix}$	$\|x - \tilde{x}\|_\infty / \|x\|_\infty = .0095$	$\|b - \tilde{b}\|_\infty / \|b\|_\infty = 0.0014$

(c) Compute the condition number of A in the 1-norm and repeat part (b) using this norm.

$$\kappa_1(A) = \|A^{-1}\|_1 \; \|A\|_1$$

$$\Rightarrow \; \|A^{-1}\|_1 = 281$$

$$\|A\|_1 = 14 \Rightarrow \; 14 \times 281 = 3934$$

$A\tilde{x} = \tilde{b}$ $\Rightarrow \begin{bmatrix} 0.9496 \\ 0.9979 \\ 1.0076 \\ 1.0095 \end{bmatrix}$	$\|x - \tilde{x}\|_1 / \|x\|_1 = .0096$	$\|b - \tilde{b}\|_1 / \|b\|_1 = .0014$

Note: You can use the file `a3q1.m` to ensure that you are working with the correct A and b values.

2. **[4 points]** Let A be a strictly diagonally dominant square matrix (i.e. the magnitude of the diagonal entry in each row is larger than the sum of the magnitudes of all the other entries in that row). Show that the Jacobi iteration always converges for matrices of this type.

   Hint: It is sufficient to show that the error goes to zero in the infinity norm.

3. **[6 points]** Consider the system $Ax = b$, with

$$A = \begin{bmatrix} 9 & -9 & 9 \\ -9 & 10 & -10 \\ 9 & -10 & 14 \end{bmatrix}, \qquad b = \begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

The exact solution to the system is $x = [1, 1, 1]^T$.

Note: Please use the file `a3q3.m` for this problem.

(a) Is the matrix A strictly diagonally dominant? What (if anything) can you conclude from this about whether the Jacobi and Gauss-Seidel algorithms will converge?

No, A is not strictly diagonally dominate and because it is not strictly diagonally dominant we cannot conclude anything about the Jacobi and Gauss-Seidel algorithms.

$|a_{11}|$ is not $\geq |a_{12}| + |a_{13}|$ since $|9|$ is not $\geq |-9| + |9|$
$|a_{22}|$ is not $\geq |a_{21}| + |a_{23}|$ since $|10|$ is not $\geq |-9| + |-10|$
$|a_{33}|$ is not $\geq |a_{31}| + |a_{32}|$ since $|14|$ is not $\geq |-10| + |9|$

(b) Compute the infinity norms of the matrices M_j and M_g used for the Jacobi and Gauss-Seidel iterations, respectively. What (if anything) can you conclude from these norms about whether the Jacobi and Gauss-Seidel algorithms will converge?

$\|M_j\|_\infty = max_{1 \leq j \leq n} \sum_{i=1} \mid M_{ij} \mid \;\; \Rightarrow$ max sum of the rows $\Rightarrow max(9, 10, 14) = 14$
$\|M_g\|_\infty = max_{1 \leq j \leq n} \sum_{i=1} \mid M_{ij} \mid \;\; \Rightarrow$ max sum of the rows $\Rightarrow max(9, 19, 33) = 33$

(c) Use Matlab to find the eigenvalues of M_j and M_g. What can you conclude from the eigenvalues about whether the Jacobi and Gauss-Seidel algorithms will converge?

$$M_g = \begin{bmatrix} 14 \\ 10 \\ 9 \end{bmatrix} \qquad M_j = \begin{bmatrix} 9 \\ 10 \\ 14 \end{bmatrix}$$

(d) Run the Jacobi and Gauss-Seidel iterations using the provided code. Do the results agree with your prediction from part (d)? If the method does converge, how many iterations does it take for the solution to be accurate to all the digits shown?

You do not need to submit any code with this assignment – just a clear writeup of your results.