1. **[5 points]** Consider the following system of equations:

$$\begin{bmatrix} 3 & -4\alpha \\ 3\alpha & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

where α is some real-valued parameter. For what value(s) of α does the system have:

(a) No solutions?

(b) Infinitely many solutions?

(c) A unique solution?

2. **[8 points]** Consider the system $Ax = b$, with

$$A = \begin{bmatrix} 2 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 5 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$\left|\begin{array}{ccc|c} 2 & 2 & 2 & \text{-1} \\ \text{-1} & 0 & 1 & 3 \\ 3 & 5 & 6 & 0 \end{array}\right| \quad \rightarrow \quad \begin{array}{l} R_2 \Leftrightarrow R_3 \\ R_2 = R_2 - 2.5R_1 \\ R_3 = R_3 - 0.5R_2 \end{array} \quad \left|\begin{array}{ccc|c} 2 & 2 & 2 & \text{-1} \\ \text{-2} & 0 & 1 & \frac{5}{2} \\ 0 & 0 & .5 & \frac{7}{4} \end{array}\right|$$

$$\rightarrow \quad R_2 = R_2 + 2R_1 \quad \left|\begin{array}{ccc|c} 2 & 2 & 2 & \text{-1} \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 0 & .5 & \frac{7}{4} \end{array}\right|$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left|\begin{array}{ccc|c} 2 & 2 & 2 & \text{-1} \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 0 & .5 & \frac{7}{4} \end{array}\right|$$

(a) Compute the LU factorization of A by hand, and use it to solve the system as shown in class.

(b) Compute the determinant of A using the usual method (see here if you do not recall how to do this).

(c) Explain how the determinant of any $n \times n$ matrix A can be easily computed from the LU factorization of A, using only $n - 1$ multiplications. Use your results from parts (a) and (b) to verify that this works for the matrix A of this problem.
Hint: you may use the following result from linear algebra: $\det(AB) = \det(A) \cdot \det(B)$.

3. **[7 points]** For certain special types of matrix, the LU factorization has a simpler form, which can be computed more efficiently. A *positive definite* matrix is a symmetric, $n \times n$ matrix A such that the product $x^T Ax$ is positive for any non-zero vector $x \in \mathbb{R}^n$. It can be shown that the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 8 & 6 \\ 1 & 6 & 19 \end{bmatrix}$$

is positive definite.

(a) Compute the LU factorization of A by hand.

(b) You may notice that every **row** of U is a scalar multiple of the corresponding **column** in L. Thus, show that you can write the LU factorization of A as

$$A = LDL^T,$$

where L is lower triangular and D is a diagonal matrix. This can be done for any symmetric matrix (even if it is not positive definite).

(c) As a consequence of the matrix being positive definite, all the elements in D should be positive. Show that you can therefore write the LU factorization of A as

$$A = LL^T$$

where L is a lower triangular matrix (it is not exactly the same lower triangular matrix as you get from the LU factorization). This is known as the Cholesky factorization, and it can be computed for any positive definite matrix.