Due Wednesday, April 9 by the end of class.

1. **[6 points]** For each of the following formulas, (i) Identify for which values of x (e.g. x large, x close to zero) we may have issues with cancellation error, and (ii) derive a better way of evaluating the expression, and explain how it avoids cancellation error.

   (a) $\log(x+1) - \log(x)$

   (b) $\sqrt{1 + \frac{1}{x}} - 1$

   (c) $\dfrac{x - \sin(x)}{x^3}$

2. **[4 points]** Recall that an *eigenvalue* of an $n \times n$ matrix A is any value λ such that $Av = \lambda v$ for some *eigenvector* $v \in \mathbb{R}^n$. An $n \times n$ matrix may have up to n distinct eigenvalues. Eigenvalues can be computed in Matlab using the `eig` command – type `help eig` at the command prompt for more information.

   (a) Compute the eigenvalues of the following two matrices in Matlab:

   $$A = \begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 10^6 \\ 10^{-6} & 1 \end{bmatrix}$$

   How large is the change in the result compared to the change from A to A^*?

   Note: You can enter matrices in Matlab as follows: `A = [1 1e6; 0 1];`, etc.

   (b) Repeat part (a) for the following two matrices:

   $$B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad B^* = \begin{bmatrix} 1 & 2 \\ 10^{-6} & 3 \end{bmatrix}$$

   (You may want to enter `format long` on the Matlab command line to display results to full precision). What can you conclude about the conditioning of problem (b) vs. problem (a)?

3. **[10 points]** In this problem we consider computing the following integral:

$$I_n = \int_0^1 x^n e^x \, dx,$$

for increasing values of the integer n. The program `a1q3.mat` provides a starting point for some parts of the question. The demo code `unstable_recursion.m` from class may also be helpful.

(a) Plot the integrand $f(x) = x^n e^x$ in Matlab for n=1, 2, 5, 10, and 20. What can you say about the value of the integral I_n as n increases? What is the limit as $n \to \infty$?

(b) Prove the following recursive identity using integration by parts:

$$\begin{aligned} I_0 &= e - 1 \\ I_{n+1} &= e - nI_n \end{aligned}$$

(c) Implement the recursive formula from part (b) in Matlab, and use it to compute I_n up to $n = 25$. A stub is provided in `a1q3.mat`. Do the values appear correct? Explain why or why not.

(d) Give an explanation for the results of part (c). What function describes how the error grows as n increases? (It is not quite the same as the example seen in class).

(e) Use the results from part (a) and (b) to derive a backwards recursion:

$$\begin{aligned} I_{50} &\approx \lim_{n \to \infty} I_n \text{ (from part (a))} \\ I_n &= \text{(some function of } I_{n+1}) \end{aligned}$$

Again, a stub is provided in `a1q3.m`. Run the backwards recursion and compare the results with those of part (c). Does this method produce better results? Discuss.

(f) Explain the results of part (e) versus part (c), in terms of what happens to the numerical error at each step.

Please submit the following for Question 3:

- Written responses to parts (a)-(f)
- A clearly labeled plot for part (a)
- Code used for parts (c) and (e)
- Output of code for part (c). You do not need to submit the output for part (e).