

# DISCO: Internal Evaluation of Density-Based Clustering

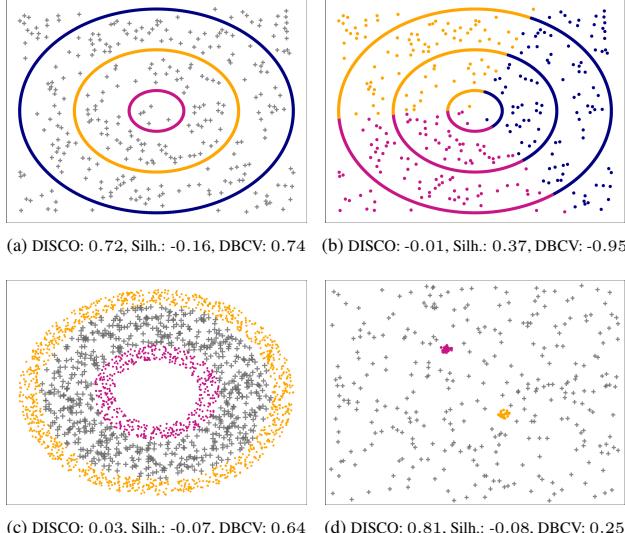
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## Abstract

In density-based clustering, clusters are areas of high object density separated by lower object density areas. This notion supports arbitrarily shaped clusters and automatic detection of noise points that do not belong to any cluster. However, it is challenging to adequately evaluate the quality of density-based clustering results. Even though some existing cluster validity indices (CVIs) target arbitrarily shaped clusters, none of them captures the quality of the labeled noise. In this paper, we propose DISCO, a **D**ensity-based **I**nternal **S**core for **C**lustering **O**utcomes, which is the first CVI that also evaluates the quality of noise labels. DISCO reliably evaluates density-based clusters of arbitrary shape by assessing compactness and separation. It also introduces a direct assessment of noise labels for any given clustering. Our experiments show that DISCO evaluates density-based clusterings more consistently than its competitors. It is additionally the first method to evaluate the *complete* labeling of density-based clustering methods, including noise labels.

## 1. Introduction

Density-based clustering is a fundamental clustering concept known from methods like DBSCAN (Ester et al., 1996) or HDBSCAN (Campello et al., 2013). However, evaluating the quality of density-based clusterings still faces open challenges, especially regarding noise. Density-based clusters consist of points in dense areas that are separated from



*Figure 1.* Clusters indicated by color, noise with + symbol. Top: The density-based CVIs DISCO and DBCV rate the ground truth density-based ring clustering in (a) higher than the centroid-based  $k$ -Means clustering cutting across rings in (b), while the Silhouette Coefficient prefers cuts (b) over dense rings (a). Bottom: (c) DISCO correctly provides a poor rating of the “arbitrary” noise assignment in the middle of the donut cluster, which is not penalized by DBCV. In (d) DISCO provides a high score to the good identification of clusters and noise, which is missed by other CVIs.

others by sparse areas. Points that do not lie in a cluster are labeled as noise. Thus, unlike centroid-based clustering, density-based clusters may have arbitrary shapes, and not all points need to be assigned to a cluster. For example, in Figure 1a, each circle corresponds to one density-based cluster, separated from noise and other clusters by sparse areas.

Internal cluster validity indices (CVIs) provide a quality score of a clustering without known ground truth (Zaki et al., 2020). Typically, they balance the *compactness* of clusters and their *separation*, e.g., in Davies-Bouldin (Davies & Bouldin, 1979), Dunn index (Dunn, 1974), or Silhouette Coefficient (Rousseeuw, 1987). They support the comparison of different clusterings, the selection of hyperparameters, and an overall assessment of cluster validity.

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Inherently, most CVIs assume centroid-based clusters and are, thus, not meaningful in our density-based setting: for example, the Silhouette Coefficient of the ground truth clustering in Figure 1a incorrectly indicates a poor clustering. In contrast, cutting the points like a pie chart and thus breaking the rings as in Figure 1b indicates that this unintuitive  $k$ -Means clustering would be preferred. Moreover, a crucial feature of density-based clustering is its ability to identify and label noise and avoid possible cluster degradation by “forced” noise assignment to clusters. However, this is not captured by existing methods, either. Consider the “arbitrary” assignment of clusters in the donut shape in Figure 1c versus the good detection of noise and clusters in Figure 1d, which receive similar Silhouette Coefficient ratings. Other centroid-based CVIs behave similarly, making them ill-suited for evaluating density-based clusterings.

While recent density-based CVIs such as DBCV (Moulavi et al., 2014) or DCSI (Gauss et al., 2024) aim to capture density-connectivity, they fail to rate noise labelings. Instead, they effectively penalize the existence of noise, even for correctly identified noise present in the ground truth clustering (cf. the low DBCV score in Figure 1d). Generally, CVIs, including DBCV, filter out noise or scale by the amount of noise points when computing their scores. This may lead to inadequately low scores for a correct clustering with noise (Figure 1d) or overly good scores in Figure 1c where the donut cluster is split in two by an (artificially labeled) ring of noise.

To overcome these limitations, we introduce DISCO, a **D**ensity-based **I**nternal **E**valuation **S**core for **C**lustering **O**utcomes. We define the score pointwise to apply different strategies for cluster and noise points, incorporating notions of density-connectivity and explicitly handling noise. In contrast to existing CVIs, DISCO is well-defined for the complete labeling (including noise points) and captures density-connectivity.

Our suggested internal cluster evaluation measure for density-based clusterings DISCO has the following properties:

- It is the first CVI to evaluate the quality of noise labels, an essential feature of density-based clustering.
- It adopts the principle of compactness and separation, as well as that of density-connectivity.
- It is simple and intuitive, and its pointwise scores allow assessing clustering quality globally and locally, enhancing interpretability.

## 2. Related Work

Internal CVIs evaluate clustering quality without the need for ground truth labels by comparing *compactness* of clusters and *separation* between clusters (Zaki et al., 2020). Compactness refers to how closely the points of the individ-

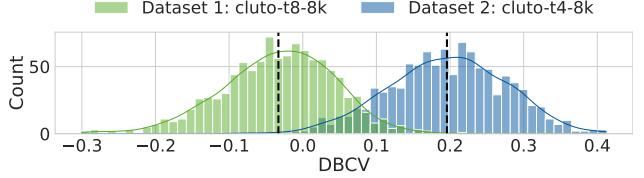


Figure 2. DBCV results reported over 1,000 runs where the order of the data points is shuffled. The black dashed line denotes the mean. Noise points are excluded in this experiment.

ual clusters are grouped together, while separation assesses how well-separated the clusters are from each other. This idea is used in CVIs like the Silhouette score (Rousseeuw, 1987), Davies-Bouldin (Davies & Bouldin, 1979), or Dunn index (Dunn, 1974). While those classical methods work well for centroid-based clusterings, an essential aspect for the evaluation of density-based clusterings, noise-labeled points, is not considered, and all clusters are assumed to be ball-shaped, which is rarely the case for real-world data.

Compactness and separation can be evaluated either at the cluster or point level. Among the clusterwise CVIs, CDbw (Halkidi & Vazirgiannis, 2008) and CVNN (Liu et al., 2013) extend centroid-based CVIs with multiple representation points to handle more complex shapes. As a downside, their scores depend on the number and choice of these representation points. CVDD (Hu & Zhong, 2019) uses local density when computing the distance between clusters, allowing it to assess cluster separation without being misled by outliers. For CVDD and CVNN, the resulting scores are not bounded, making it difficult to assess how good a clustering really is, especially as, in practice, the output spans several orders of magnitude.

In contrast, DBCV (Moulavi et al., 2014) and DCSI (Gauss et al., 2024) score compactness and separation as the longest edge within and the minimum distance between clusterwise minimal spanning trees (MSTs) under the pairwise mutual reachability distance. Both DBCV and DCSI do not consider all points to avoid outliers: DBCV builds one MST on all points of each cluster and then removes all leaves, while DCSI builds the MSTs only on core-points. As MSTs are not unique, scores computed by DBCV are not deterministic as shown in Figure 2 and discussed in detail in Appendix C.

LCCV (Cheng et al., 2018) and VIASCCKDE (Senol, 2022) aggregate pointwise scores to capture connectedness and separation. LCCV builds on points with local maximum density, while VIASCCKDE employs Kernel Density Estimation to assign higher weights to scores from points in regions of higher density.

Another promising step in the direction of density-based clustering evaluation is the work by Schlake & Beecks (2024). They suggest to use the density-connectivity dis-

tance (dc-dist) (Beer et al., 2023) that captures the essence of density-connectivity-based clustering algorithms like DBSCAN with various classic internal evaluation measures.

While all these methods use some notion of density for evaluating clusterings, they share a significant drawback – noise is either not handled at all or, for DBCV, LCCV, and CVNN, only implicitly handled. DBCV effectively assigns a score of 0 to every noise point, LCCV includes noise points in its graph-based distance, and CVNN implicitly involves noise when selecting its representation points. All of these approaches tend to penalize the existence of noise rather than to evaluate the *quality* of noise-labeled points, i.e., whether it was adequate to mark a given point as noise. We summarize key properties of these methods in Table 1.

Table 1. Features of internal evaluation measures.

Method	arbitrary shapes	evaluates noise	bounded	deterministic	$\uparrow/\downarrow$
Silhouette	✗	✗	✓	✓	↑
S_Dbw	✗	✗	✗	(✓)	↓
DBCV	✓	◆	✓	✗	↑
DCSI	✓	✗	✓	✓	↑
LCCV	✓	◆	✓	(✓)	↑
VIASCKDE	✓	✗	✓	✓	↑
CVDD	✓	✗	✗	✓	↑
CDbw	✓	✗	✗	✓	↑
CVNN	✓	◆	✗	✓	↓
DISCO (ours)	✓	✓	✓	✓	↑

✓ yes ✗ no ◆ no, but affected by noise

### 3. DISCO: A New Internal Evaluation Method

We present some necessary preliminaries (Section 3.1), followed by the definition of our new internal CVI, DISCO (Sections 3.2-3.5), and the discussion (Section 3.6).

#### 3.1. Preliminaries

DISCO evaluates a given density-based clustering  $\mathcal{C}$  on a dataset  $X \in \mathbb{R}^{n \times d}$  with  $n$   $d$ -dimensional points. A clustering  $\mathcal{C}$  is a set of clusters  $C_i$ :  $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$  with  $C_i \cap C_j = \emptyset$  for all  $i \neq j$ . An advantage of density-based clustering methods is that not every point needs to be assigned to a cluster. There may be noise points  $N = X \setminus \bigcup_i C_i$ . For  $x \in X$ , we use shorthand  $\hat{C}_x$  when referring to cluster  $C_i$  such that  $x \in C_i$ .

Internal evaluation metrics assess compactness and separation to evaluate given clusterings. Since we focus on density-based clusterings, we base our concepts on notions introduced in density-based clustering approaches like DBSCAN, or more recently HDBSCAN (Ester et al., 1996; Campello et al., 2013). Density-based clusters are defined using core points and density-connectivity.

**Core property.** *Core points* are points with more than  $\mu$  neighbors within  $\varepsilon$  distance (their neighborhood), which makes these areas dense. The *core-distance*  $\kappa(x)$  captures the density of the area around a point  $x$  as the Euclidean distance to its  $\mu$ -th nearest neighbor  $x_\mu$ :

$$\kappa(x) = d_{eucl}(x, x_\mu) \quad (1)$$

A lower core-distance thus implies a higher object-density around  $x$ .

**Density-Connectivity.** *Density-based clusters* can be obtained as the connected components in the  $\varepsilon$ -range graph over core points which are connected when their distance is at most  $\varepsilon$ . Points connected by a path in the graph are called *density-connected*.

To assess density-connectivity in a clustering, DISCO uses the *density-connectivity distance* (dc-dist)  $d_{dc}(x, y)$  (Beer et al., 2023). It is the minimax path (i.e., the path with the smallest maximum step size) distance between two points  $x, y \in X$  in the graph given by all pairwise mutual reachability distances  $d_m(x, y) = \max(\kappa(x), \kappa(y), d_{eucl}(x, y))$  (Ankerst et al., 1999):

$$d_{dc}(x, y) = \max_{e \in p(x, y)} |e| \quad \text{if } x \neq y, \text{ else } 0 \quad (2)$$

where  $|e|$  is the weight of any edge  $e$  (given by  $d_m$ ) on the path  $p(x, y)$  that connects points  $x$  and  $y$  in the minimum spanning tree (MST) over this graph.

Note that, in contrast to Euclidean distance,  $d_{dc}$  not only depends on the feature values of points  $x$  and  $y$ , but rather on how they are connected in the dataset  $X$ : the minimax path may meander through the dataset to reach the target using only small steps, effectively focusing on dense regions.

#### 3.2. Pointwise Definition

To allow the assessment of individual cluster assignments and support interpretability, we define DISCO pointwise, giving a score  $\rho(x)$  to each point  $x$ . The score for the entire dataset  $X$  is then the average over all points' scores:

$$\text{DISCO: } \rho(X) = \operatorname{avg}_{x \in X} \rho(x). \quad (3)$$

Importantly, DISCO treats cluster points and noise points differently, as we detail in the following subsections:

$$\rho(x) = \begin{cases} \rho_{cluster}(x) & \text{if } x \in C_i \text{ for any } i \in [1, \dots, k] \\ \rho_{noise}(x) & \text{if } x \in N \end{cases} \quad (4)$$

where  $\rho_{cluster}$  and  $\rho_{noise}$  are the pointwise DISCO scores for cluster and noise points which we later define in Equations (5) and (8).

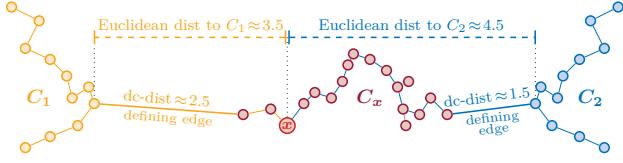


Figure 3. Regarding the dc-dist,  $C_2$  is closer to  $x$  than  $C_1$ .

### 3.3. Cluster Points: $\rho_{\text{cluster}}(x)$

When  $x \in X$  is assigned to a cluster  $\hat{C}_x$ , we compute  $\rho_{\text{cluster}}(x)$  by comparing average distances within the cluster (compactness) with those to the closest other cluster (separation). Importantly, these assessments employ the dc-dist  $d_{dc}$  to account for density-based clustering notions using  $d_{dc}(x, C_i) = \text{avg}_{y \in C_i} d_{dc}(x, y)$ :

$$\rho_{\text{cluster}}(x) = \min_{C_i \neq \hat{C}_x} \frac{\widetilde{d}_{dc}(x, C_i) - \widetilde{d}_{dc}(x, \hat{C}_x)}{\max(\widetilde{d}_{dc}(x, C_i), \widetilde{d}_{dc}(x, \hat{C}_x))} \quad (5)$$

In Equation (5) we compare the average distance from  $x$  to points in its own cluster  $\hat{C}_x$  and the “closest” other cluster. Here, the shape and density of  $\hat{C}_x$  and the “gap” to the next cluster are much more important than, e.g., the Euclidean distance to the closest point of each cluster. We illustrate this difference in Figure 3. There, the point  $x$  in the red cluster is closer (in terms of dc-dist) to  $C_2$  than to  $C_1$  because the gap between the  $\hat{C}_x$  and  $C_2$  is smaller than between  $\hat{C}_x$  and  $C_1$ . Since these gaps are the longest edges one needs to pass to reach  $C_1$  or  $C_2$  from  $x$ , the length of those edges defines the respective dc-dist. This notion of “closest” contrasts with the Euclidean distance under which  $C_1$  would be the closest cluster from  $x$  and not  $C_2$ .

### 3.4. Noise Points: $\rho_{\text{noise}}(x_n)$

One of the key advantages of density-based clustering methods is their ability to detect and label noise. Thus, in order to properly evaluate the quality of a density-based clustering, any internal evaluation method must be able to quantify the quality of the noise labels in addition to the quality of the cluster labels. Note that neither ignoring noise points in the score nor interpreting it as a single cluster gives adequate or interpretable scores (cf. also Figure 1 and Appendix D): Ignoring would assign overly positive scores to excessive noise labeling while interpreting it as a cluster would punish the poor cohesion of correctly labeled noise. Thus, DISCO actively evaluates the quality of given noise labels. To do so, we follow the notion of noise points in density-based clustering (Ester et al., 1996; Campello et al., 2013). There, noise points are points that are neither core points nor density-connected to any cluster. Thus, a noise point is in a low-density area (otherwise, the point would be a core point and would start its own cluster) and also far away from

any existing cluster (otherwise, it would be part of such a nearby cluster). We capture both properties in our score.

**(1) Noise points are not core points.** Their core-distances (see Equation (1)) are larger than some  $\varepsilon$ . If a noise point  $x_n$  is in a low-density area, measured by comparing its core-distance to the maximum core-distance of a point within a cluster, then it can “rightfully” be considered noise. We capture this by

$$\rho_{\text{sparse}}(x_n) = \min_{C_i \in \mathcal{C}} \frac{\kappa(x_n) - \kappa(C_i)}{\max(\kappa(x_n), \kappa(C_i))}, \quad (6)$$

where the core-distance threshold of a cluster  $C$  is the maximum core-distance of any point in  $C$ :  $\kappa(C) = \max_{x \in C} \kappa(x)$ . It corresponds to the smallest  $\varepsilon$  such that the entire cluster remains density-connected.

By choosing the minimum over all clusters in Equation (6) instead of just comparing to the core-distance of the closest cluster, we account for clusters of varying density. We illustrate this in Figure 4 where (a) gives an example of a dense region and (b) is prototypical noise as indicated by a core-distance larger than any core-distance appearing in any cluster. This global interpretation of sparsity ensures that a group of noise points with the same density as a cluster (somewhere else) is not rated as well-labeled noise.

**(2) Noise points are not density-connected to any cluster.** We assess this by comparing the dc-dist between the noise point and each cluster with the maximum core-distance in that cluster. If the dc-dist between the point and the cluster is smaller or equal to the maximum core-distance of the cluster, then it is density-connected to said cluster and should thus be part of it. Formally, for a noise point  $x_n$  we compute a score for not being density-connected as

$$\rho_{\text{far}}(x_n) = \min_{C_i \in \mathcal{C}} \frac{\min_{y \in C_i} d_{dc}(x_n, y) - \kappa(C_i)}{\max(\min_{y \in C_i} d_{dc}(x_n, y), \kappa(C_i))}. \quad (7)$$

In Figure 4,  $\rho_{\text{far}}$  of noise point (d) is determined by Cluster 1 and not Cluster 2, even though both have the same distance because Cluster 1 has a larger core-distance.

As noise should be neither in a dense region nor density-connected to an existing cluster, it is scored as the minimum of these two:

$$\rho_{\text{noise}}(x_n) = \min(\rho_{\text{sparse}}(x_n), \rho_{\text{far}}(x_n)). \quad (8)$$

Taking the minimum ensures that not only (c) but also (a) and (d) in Figure 4 get low DISCO scores.

### 3.5. Edge Cases

We handle edge cases as follows: For the extreme case of clusterings with only noise points and no clusters, we define  $\rho_{\text{noise}}(x_n) = -1$  as they have no clustering value.

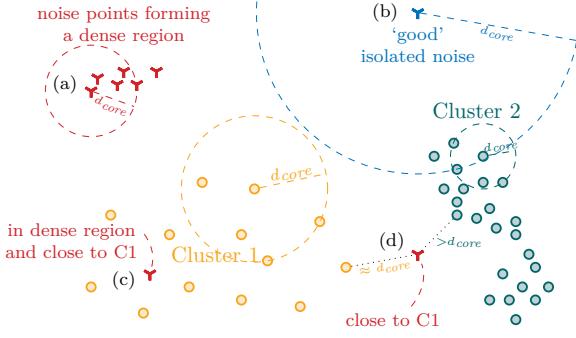


Figure 4. Assessing the quality of noise: red noise labels have low DISCO scores, and blue is prototypical noise.

Singleton clusters consist of only one point, contradicting the idea of grouping together similar points. Thus, for all points  $x$  in singleton clusters, we set  $\rho_{cluster}(x) = 0$ . Similarly, if the clustering consists of one cluster and no noise points, we let  $\rho_{cluster}(x) = 0$  for all  $x \in X$ .

If there are, in addition to the only cluster  $C_1$ , also noise points, we evaluate  $C_1$  w.r.t. the closest noise points instead of the (not existing) closest cluster:

$$\rho_{cluster}(x) = \frac{\min_{x_n \in N} d_{dc}(x, x_n) - \widetilde{d}_{dc}(x, \hat{C}_x)}{\max(\min_{x_n \in N} d_{dc}(x, x_n), \widetilde{d}_{dc}(x, \hat{C}_x))} \quad (9)$$

Note that no other CVI is defined for clusterings with less than two clusters, even though density-based methods like DBSCAN or HDBSCAN, synchronization-based clustering methods (Böhm et al., 2010), or deep clustering methods like SHADE (Beer et al., 2024) may return such clusterings.

A commonly overlooked edge case occurs when datasets have more than  $\mu$  duplicate points, making their core-distances 0. This can lead to zero denominators in, e.g., Eq. 6. However, as these points are always core points and, thus, “bad” noise, we simply set the fraction/score to 0.

### 3.6. Discussion

DISCO effectively assesses compactness and separation in the density-connectivity sense following the structure of well-established CVIs. Further benefits are its simplicity and reliability. It naturally integrates noise evaluation as scores are computed at the point level, allowing the assessment of each point’s individual label. DISCO is widely applicable: being based on density-connectivity, it is suitable not only for numeric data but also for any data type given the pairwise similarities. It covers all edge cases that might be produced by various cluster algorithms (e.g., clusterings with only one cluster or datasets with duplicate points). Figure 5 showcases several of those benefits on a 2d toy dataset with two different labelings: the density-based ground truth

clustering (left) and a  $k$ -Means clustering (right). For further examples, see Appendix E.10. The colors indicate the pointwise DISCO scores, which are high (blue) for most points on the left. Only one cluster in the middle right has mediocre (light brownish) DISCO scores, as it is the least dense cluster and relatively close to the next cluster. In contrast, the  $k$ -Means clustering on the right yields only low (orange) or mediocre (light brownish) DISCO scores for almost all points as density-separated clusters are merged and density-connected clusters are split apart. More details regarding DISCO’s properties in Appendix A.

## 4. Experiments

We highlight the difference between centroid-based and density-based evaluation in Section 4.1. We show that none of the existing CVIs evaluates the *quality* of noise labels in Section 4.2. In Section 4.3, we investigate the CVIs performances for typical use cases. In Sections 4.4 and 4.5, we perform systematic ablation studies showing DISCO’s desirable behavior for various properties in the data.

**Details.** We use the official Python implementation for Silhouette Coefficient, DBCV, and VIASKDE (VIAS.). Our implementation of all other competitors and our code for DISCO are available at <https://anonymous.4open.science/r/DISCO-8E44/>. All further details are in Appendix B and additional experiments in Appendix E.

### 4.1. Density- vs. Centroid-based Cluster Notion

Density-based CVIs should provide better scores for correct, density-based clusterings than for unintuitive, centroid-based clusterings (that might be more compact). Thus, in Table 2, we regard two different labelings of the density-based toy datasets 3-spiral and complex9: first, the density-based ground truth labels, and second, the  $k$ -means clusterings (averaged over 10 runs). We compute the CVIs described in Section 2 and mark them in green if they indeed yield better scores for the density-based clustering than for the centroid-based clustering. Most of the CVIs discussed in Section 2 for the density-based notion indeed prefer (i.e., evaluate it as better) the density-based clustering. However, CVNN does not, VIASKDE evaluates both labelings as similarly

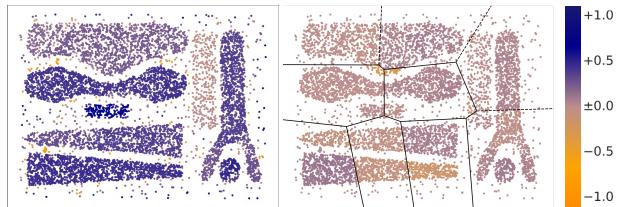


Figure 5. Pointwise DISCO scores for cluto\_t8\_8k. Left: ground truth labels. Right:  $k$ -Means labels, lines indicate cluster borders.

**Table 2.** CVIs for different notions of clustering on two density-based datasets. A density-based CVI should evaluate the DBSCAN clustering that corresponds to the ground truth (left) as better than the  $k$ -Means clusterings (right). The color indicates if the CVIs align with these expectations or not. ↓ denotes that lower is better.

CVI	3-spiral		complex9	
	DBSCAN	$k$ -Means	DBSCAN	$k$ -Means
<b>DISCO</b>	0.59	0.00	0.36	0.05
<b>Silhouette</b>	0.0	0.36	-0.01	0.40
<b>S_Dbw</b> ↓	2.79	1.90	0.59	0.49
<b>CVNN</b> ↓	5.49	3.63	5.11	4.80
<b>DBCV</b>	0.55	-0.96	0.17	-0.84
<b>DCSI</b>	0.93	0.01	0.95	0.70
<b>LCCV</b>	0.66	0.03	0.55	0.16
<b>VIASCKDE</b>	0.31	0.26	0.63	0.61
<b>CVDD</b>	189.35	0.47	689.4	20.28
<b>CDbw</b>	0.01	0.01	0.61	0.21

good, and CDbw only makes a difference for complex9, but not for 3-spiral. As expected, Silhouette and S\_Dbw prefer the  $k$ -Means clustering (orange in Table 2). As we focus on density-based clustering evaluation, we exclude S\_Dbw and CVNN in some diagrams for clarity.

## 4.2. Evaluating Noise Labels Is Important

To the best of our knowledge, no internal CVI handles noise labels *explicitly*; instead, the noise is ignored or treated as a cluster.<sup>1</sup> Thus, they might yield undesirable and unintuitive results, as shown in Table 3: Here, we regard variations of a dataset containing four clusters. The first version consists of only these clusters. The second version has additional uniformly distributed noise points outside of the clusters (red) that are correctly labeled as noise. The third version also has additional points compared to the first (in red), which are labeled as noise, but those lie *within* the clusters. I.e., the labeling of these noise points is of bad quality as they should be assigned to the clusters as they are density-connected to the clusters. The fourth version has the same data points as the second version, but the labeling does not identify the noise points and instead assigns them to the closest cluster. Thus, the quality of noise and cluster labels given for version 1 is good. For version 2, it is also good, but for versions 3 and 4, it is bad. This should be reflected in the CVIs: if so, we highlight the cell in green. However, as our competitors do not evaluate noise (see Section 2), none of them shows the desired behavior for *all* cases. For more details, see Appendix E.9.

**Table 3.** CVIs for different qualities of noise labeling. The color indicates if the CVIs align with the expectations or not. \* denotes that the CVI does not handle noise; the implementation treats noise-labeled points as a cluster per default. + indicates noise filtering. ↓ denotes that lower is better.

CVI	good	good	bad	bad
Noise Points	✗	✓	✗	✓
Noise Labels	✗	✓	✓	✗
Quality of Labels	good	good	bad	bad
<b>DISCO</b>	0.99	0.98	0.23	0.25
<b>Silhouette*</b>	0.95	0.30	0.00	0.24
<b>S_Dbw</b> + ↓	0.05	0.05	0.05	0.80
<b>CVNN</b> ↓	7.62	7.62	7.41	559.27
<b>DBCV</b>	1.00	0.50	0.50	-0.83
<b>DCSI</b> +	1.00	1.00	1.00	1.00
<b>LCCV</b>	0.62	0.51	0.27	0.43
<b>VIASCKDE*</b>	0.20	0.07	0.00	0.11
<b>CVDD*</b>	7981.55	69.81	7442.46	0.01
<b>CDbw</b> +	1.52	1.52	1.52	0.00

## 4.3. Practical Use Cases of Internal CVIs

### 4.3.1. DETERMINING BEST PARAMETER SETTINGS

A key application of CVIs is to determine “good” parameter settings for clustering methods that result in a high-quality clustering. Ideally, the highest internal quality score across different parameter settings corresponds to the best clustering regarding the ground truth. Thus, in Figure 6, we compare the scores of internal CVIs for DBSCAN clusterings across a range of  $\varepsilon$ -values (leading to  $k \in [2; 20]$  clusters) on the Synth\_high dataset that has  $k = 10$  density-connected well-separated ground truth clusters. Optimally, the circles indicating the highest score should fit the highest ARI values at  $k = 10$  (red bar). However, only DISCO and DBCV have the desired peak at ten clusters. Thus, if used to find the “best” parameter setting, other CVIs are misleading here, while DISCO and DBCV correctly guide users to the setting aligned with the ground truth.

### 4.3.2. CONSENSUS OF INTERNAL AND EXTERNAL CVIS

Ideally, internal CVIs should yield similar scores to external CVIs based on ground truth. In Table 4, we study this correspondence between internal CVIs and the (external) ARI values across several datasets and clusterings. For this, we generate clusterings by diverse standard clustering algorithms for each dataset and give the Pearson Correlation Coefficient (PCC) between the respective CVIs and the ARI values for those clusterings. Appendix E.3 shows the detailed results used to compute the PCC between DISCO and ARI scores. For the ARI calculation, points labeled as noise are treated as singleton clusters, as discussed in Appendix D.2.

<sup>1</sup>Table 3 indicates the default noise treatment method.

Table 4. Pearson Correlation Coefficient (PCC) between the CVI scores on the ground truth (GT), DBSCAN, HDBSCAN, Density Peak Clustering (DPC), Spectral Clustering (SC), Agglomerative Ward Clustering (Ward), Mean Shift Clustering,  $k$ -Means Clustering, and two random labelings and their corresponding ARI values. In gray: percentage of successfully evaluated clusterings.

Dataset	DISCO	DBCV	DCSI	LCCV	VIAS.	CVDD	CDbw	CVNN	Silh.	S_Dbw	
3-spiral aggregation	<b>90.16</b> 100%	<b>99.73</b> 80%	64.01 100%	86.94 90%	45.96 80%	79.54 80%	26.08 90%	<b>74.21</b> 90%	-13.79 90%	47.59 90%	
chainlink	76.73 100%	79.22 98%	87.67 100%	<b>91.98</b> 100%	<b>93.96</b> 98%	33.94 98%	<b>70.40</b> 100%	<b>-71.73</b> 98%	<b>88.07</b> 100%	<b>-82.12</b> 100%	
cluto-t4-8k	<b>92.51</b> 100%	90.14 90%	79.74 100%	83.67 90%	51.11 90%	<b>93.86</b> 90%	-13.16 90%	-28.63 100%	23.70 90%	30.94 90%	
cluto-t7-10k	41.70 100%	<b>52.10</b> 90%	39.80 100%	<b>72.01</b> 90%	<b>52.10</b> 90%	21.33 90%	-35.11 80%	-32.19 100%	41.83 90%	-51.63 90%	
cluto-t8-8k	41.49 100%	<b>62.91</b> 100%	34.12 100%	22.14 100%	41.12 100%	-1.10 100%	-18.18 90%	-33.40 100%	6.39 100%	-53.77 100%	
complex8	65.97 100%	49.68 100%	0.49 100%	46.30 100%	<b>66.75</b> 100%	-14.87 100%	43.31 90%	-49.42 100%	1.32 100%	-64.26 100%	
complex9	<b>90.49</b> 100%	67.41 100%	-16.02 100%	<b>86.27</b> 100%	70.36 100%	50.83 100%	48.86 100%	-47.49 100%	18.89 100%	-66.76 100%	
compound	52.31 100%	54.84 100%	<b>71.60</b> 100%	<b>70.87</b> 100%	56.91 100%	24.76 100%	65.39 100%	-37.89 100%	-4.92 100%	-58.72 100%	
dartboard1	83.98 100%	<b>86.83</b> 81%	81.86 100%	<b>92.96</b> 100%	79.46 81%	52.09 81%	59.37 100%	<b>-82.44</b> 81%	<b>64.85</b> 100%	-66.89 100%	
diamond9	89.28 100%	<b>93.61</b> 90%	68.03 100%	87.49 90%	55.13 90%	<b>99.95</b> 90%	-26.88 90%	-19.76 100%	-12.93 90%	-11.42 90%	
smile1	<b>96.26</b> 100%	62.88 100%	49.70 100%	83.52 100%	73.41 100%	63.36 100%	18.45 100%	-48.64 100%	<b>89.10</b> 100%	-90.32 100%	
Synth_low	<b>91.22</b> 100%	77.47 98%	83.79 100%	<b>90.56</b> 100%	79.61 98%	83.31 98%	56.18 100%	-46.49 98%	72.16 100%	-82.94 100%	
Synth_high	<b>90.78</b> 100%	<b>99.67</b> 60%	54.80 100%	71.05 80%	90.49 60%	4.86 60%	13.89 70%	-38.66 80%	77.33 80%	-63.93 80%	
hitru2	58.65 100%	<b>99.42</b> 60%	86.80 100%	72.72 70%	<b>95.21</b> 60%	12.84 60%	<b>58.92</b> 80%	-49.09 70%	26.58 90%	-50.48 90%	
Pendigits	36.88 100%	-34.51 90%	7.30 100%	52.29 90%	41.03 90%	<b>59.09</b> 90%	-27.74 80%	-27.59 100%	<b>64.05</b> 90%	-21.50 90%	
COIL20	44.16 100%	-1.09 90%	28.39 100%	<b>74.82</b> 90%	63.79 70%	52.73 90%	3.88 90%	-30.24 100%	<b>75.37</b> 90%	-42.78 90%	
cmu_faces	<b>93.89</b> 100%	88.01 80%	62.75 100%	<b>92.13</b> 80%	-	0%	65.17 80%	-6.43 80%	-28.75 100%	82.30 80%	-64.01 80%
Optdigits	11.05 100%	<b>100.00</b> 20%	49.57 100%	74.26 80%	-	0%	<b>80.75</b> 70%	15.04 100%	-53.60 70%	66.49 100%	8.48 100%
				<b>88.67</b> 90%	<b>88.42</b> 60%	56.05 80%	20.94 90%	-44.72 90%	82.53 90%	4.99 90%	

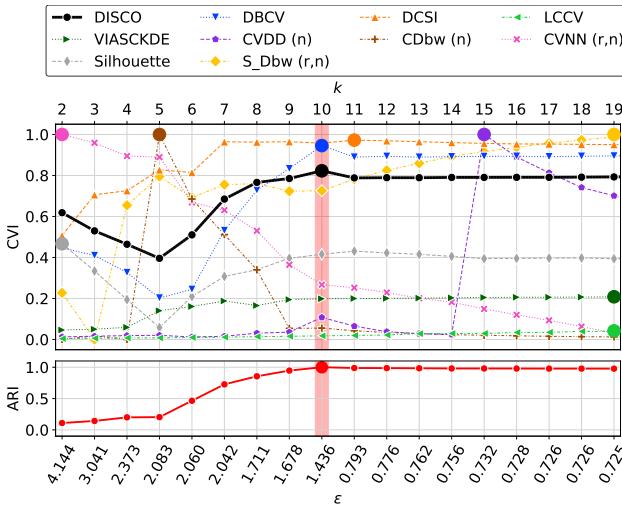


Figure 6. CVIs for DBSCAN clusterings with varying  $\epsilon$  values and resulting number of clusters  $k$  on the noisy Synth\_high dataset ( $k = 10, d = 100$ ). The corresponding ARI scores are below.

Our competitors cannot be computed for all clusterings as they are not defined for certain edge cases, see Section 3.5. Thus, Table 4 also shows (in gray) the percentage of clusterings that could be evaluated by each CVI. DISCO is the only CVI inherently designed to handle all edge cases.<sup>2</sup> Such edge cases often occur when, e.g., DBSCAN’s parameter  $\epsilon$  is set too high (only one cluster) or too low (no cluster).

A reliable CVI should always return *some* result and, ideally, have a high PCC to the ARI. Table 4 shows that DISCO and LCCV meet those criteria best. DBCV and DCSI come

<sup>2</sup>DCSI always returns a score as we implemented a feature in our code that catches exceptions by returning 0 when only one or no cluster is detected (and all other points are labeled as noise).

close; however, they return scores that contradict the ARI on some datasets.

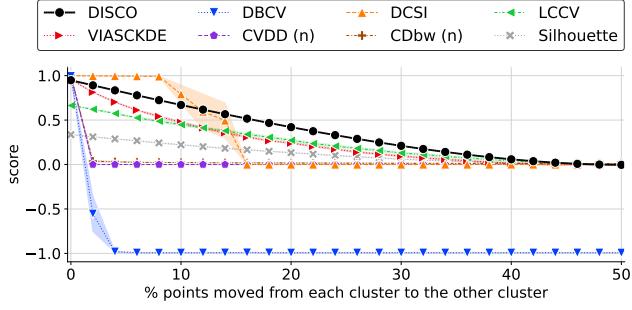
#### 4.4. Ablation of Clustering Score $\rho_{cluster}$

**Influence of mislabeled cluster points.** A good CVI should be robust against small changes in the clustering, and points with a similar role in the dataset should have a similar influence on the score. In Figure 7a,<sup>3</sup> we increase the percentage of wrongly assigned points for the two-moons dataset. While most CVIs, including DISCO, show the intended consistent decrease of quality, DCSI and DBCV show questionable behavior. DBCV drops to the worst-case evaluation of -1 as soon as only 2 of 50 points per cluster are wrongly assigned. DCSI gives a perfect score for less than 10% wrongly assigned points and the worst-case score for more than 14% wrongly assigned points, leaving only a very small range with distinguishable results.

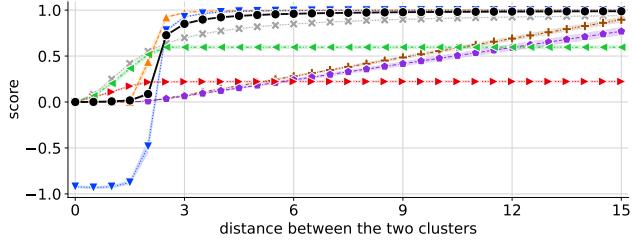
**Influence of separation.** Figure 7b shows the CVIs for increasingly distant clusters, exposing interesting behaviors for CDbw and CVDD: they display a consistent, linear increase, where it is not recognizable at which distance the switch from density-connected to density-separated clusters happens. In contrast, DISCO, DBCV, and DCSI increase sharply as soon as the clusters are clearly separated.

**Influence of fuzzy cluster borders.** To regard the influence of blending and fuzzy clusters, we increase the fuzziness (jitter) of the two moons dataset in Figure 7c. Most CVIs, including DISCO, behave as expected, starting with high values that evenly decrease. However, CVDD drops

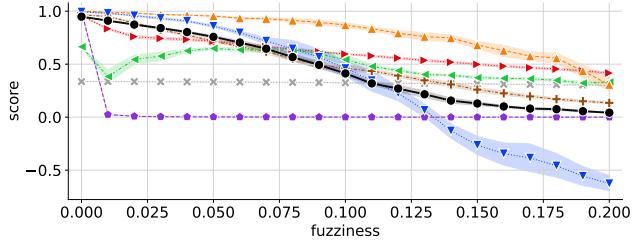
<sup>3</sup>For clarity, we linearly normalize CVDD and CDbw to [0, 1] in all plots, marked with (n). CVIs with reversed orientation are additionally subtracted from their largest value, marked with (r).



(a) Influence of mislabeled points: increasing percentage of random points assigned to the wrong cluster in the two moons dataset.



(b) Influence of separation: Increasing distance between cluster centers for uniform, spherical clusters of radius 2.



(c) Influence of fuzzy cluster borders: Increasing fuzziness of two moons (in percent of “jitter”).

Figure 7. Ablation of Clustering Score  $\rho_{cluster}$

quite rapidly for very low amounts of jitter, where the clusters are still well separated. LCCV shows an unexpected drop for 2% of jitter, yielding higher scores for less and more jitter. Being purely centroid-based, the Silhouette Coefficient stays constant throughout all settings.

#### 4.5. Ablation of Noise Score $\rho_{noise}$

As our competitors do not explicitly handle noise, we only present DISCO’s behavior.

**Hyperparameter Robustness.** In Figure 8, we analyze the effect of  $\mu$  on DISCO. We increase the amount of uniform additive noise on a synthetic dataset with 10 density-based clusters, generated with DENSIRE (Jahn et al., 2024). DISCO is stable across a large range of values for  $\mu$ . The general upward trend is due to the additional correctly labeled noise, which is rewarded by DISCO. For  $\mu = 1$ , each point has a core-distance of zero. Thus, all added noise points would ideally form singleton clusters, which leads

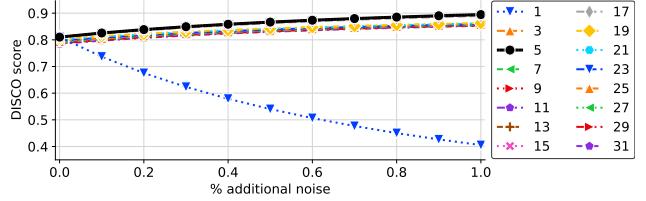


Figure 8. DISCO for varying  $\mu$ : x-axis shows noise in percent added to synthetic dataset with 10 density-based clusters.

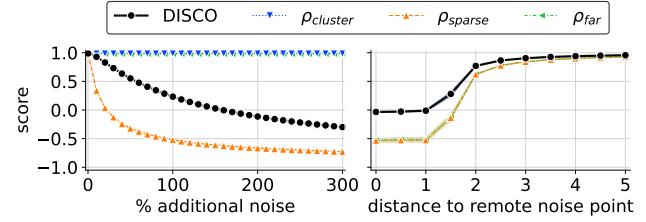


Figure 9. Left: One cluster and a distant group of points with increasing size and density, labeled as noise.

Right: A single noise point with increasing distance to the cluster.

to lower DISCO scores for higher noise percentages. Since DISCO is highly stable w.r.t.  $\mu$ , we use the same default value  $\mu = 5$  for all experiments. This aligns with heuristics for choosing  $minPts$  in the literature (Ester et al., 1996; Schubert et al., 2017).

#### Distance between noise and the closest cluster: $\rho_{far}$

Noise points should be far from any cluster, a property measured by  $\rho_{far}$ . We evaluate this part of the noise score in Figure 9 (right) on a dataset with a uniform, spherical cluster with radius  $r=2$  and one noise point at increasing distance from the cluster’s center. When the noise point is in the middle of the cluster, DISCO yields the desired outcomes around 0.  $\rho_{noise}$  and accordingly DISCO increases sharply as soon as the noise point is not density-connected to the cluster anymore, i.e., at a distance from the center larger than 2.

**Sparseness of noise:**  $\rho_{sparse}$ . True noise points lie in sparse areas as measured by  $\rho_{sparse}$ . In Figure 9 (left), we regard a dataset with a uniform, spherical cluster of points and noise points that lie far apart. We add further noise close to the first noise point by placing them uniformly within a small radius, which increases the density in this area. Increasing the density of noise points quickly deteriorates  $\rho_{sparse}$  when the noise points start forming a cluster. This lowers the overall DISCO score, as expected.

## 5. Conclusion

We introduced DISCO, a density-based internal CVI for the evaluation of arbitrarily shaped clusterings that includes

evaluating the quality of noise labels. We provide extensive experiments showcasing the ability of DISCO to properly evaluate a large variety of clusterings. We believe DISCO enhances the fair evaluation and reproducibility of research in density-based clustering and noise detection.

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<sup>4</sup><https://projekte.ffg.at/projekt/4814676>

<sup>5</sup><https://aicentre.dk>

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## A. Analysis and Discussion of DISCO

In this section, we discuss some of the properties of DISCO in more detail.

### A.1. DISCO is deterministic

We include this analysis since one competitor (DBCV, see Appendix C) is not deterministic and returns different scores for the exact same data, depending only on the order of points.

DISCO returns deterministic values as all its steps are deterministic and independent of the ordering of points: it is based on dc-dist, which gives deterministic values for each pair of points as it is a well-defined distance measure (Beer et al., 2023). Also the core-distances (see Equation (1)) used in Equation (6) are deterministic: even if several points  $x_{\mu_1}, x_{\mu_2}, \dots$  have the same distance  $d_{eucl}(x, x_{\mu_1}) = d_{eucl}(x, x_{\mu_2})$  to a point  $x$  (and thus the k-NN might not be deterministic, if they are the  $\mu$ -th nearest and  $\mu + 1$ -st nearest neighbor) the distance to those points is deterministic. All other operations are simple global agglomerative functions ( $\min$ ,  $\max$ ,  $\text{avg}$ ) or mathematical operations that are also deterministic.

### A.2. DISCO is bounded

Following Equation (3), the DISCO score of a clustering is the average of the scores for each point in the dataset. For both cluster points (Equations (5) and (9)) and noise points (Equations (6) and (7)), the values are bounded between  $-1$  and  $1$ , as we actively normalize them following the example of the Silhouette Coefficient. Thus, the final DISCO score is *bounded* as all components in the average in Equation (3) are bounded.

### A.3. Complexity

We calculate DISCO in three main steps. The first one is to compute pairwise dc-distances for the dataset. In order to do so, one first computes element-wise core-distances in  $\mathcal{O}(n)$  per datapoint so  $\mathcal{O}(n^2)$  overall. Note that we assume the  $\mu$  parameter for the core distance to be constant. Otherwise, we need to include a linear dependence on  $\mu$ . One then computes the pairwise “mutual reachability distance”, which takes  $\mathcal{O}(n^2)$  as it is a pairwise operation. On these pairwise distances, one then builds a minimum spanning tree using Kruskal’s algorithm, which is linear in the number of edges and thus also in  $\mathcal{O}(n^2)$ . Naively extracting the dc-dist from the tree would take cubic time as for every pair of nodes, one would need to traverse the tree, using up to  $n$  steps. Instead, for every node  $x$ , we extract the distances to its neighbors by interpreting  $x$  as the root and then performing a traversal of the tree in linear time. At every node  $u$ , we store (and output) the dc-dist to  $x$  by computing the max between the dc-dist between  $u$ ’s parent and  $x$  and the length of the upwards-facing edge. This way, we compute one dc-dist per (constant-time) step, leading to  $\mathcal{O}(n^2)$  overall processing time for this step.

For the pointwise score, we again need up to  $\mathcal{O}(n^2)$  time, as every point may need to look at the distance to every other point during that computation (as this involves checking distances to oneself and all nodes in the closest cluster). The final aggregation of pointwise scores happens in  $\mathcal{O}(n)$ , leading to an overall complexity of  $\mathcal{O}(n^2)$ .

All computations can be performed in linear space by not storing the complete graph but instead computing the pairwise distances for the MST computation on the fly and using Prim instead of Kruskal.

Regarding runtime, DISCO is in the same order of magnitude as most of its density-based competitors. While there are much slower CVIs such as LCCV, CVDD, and CDbw, centroid-based algorithms are significantly faster.

## B. Experiment Details

The following section includes more detailed information about experiment settings, implementations, methods, and datasets used in our experiments.

### B.1. Experiment Settings

All experiments were performed with Python 3.12 on a Linux workstation with 2x Intel 6326 with 16 cores each and multithreading, as well as 512GB RAM. We use the `sklearn` clustering implementations for our experiments with clustering algorithms.

## B.2. Datasets

In Table 5, we provide an overview of the datasets included in our experiments. We categorized the datasets into density-based benchmark data and real world. With more detailed categorization denoting the dimensionality of data. Within those categories, the datasets are sorted by descending DISCO scores. For real world data, we used famous benchmarks

Table 5. Dataset properties. Number of samples ( $n$ ), dimensions ( $d$ ), number of ground truth clusters ( $k$ ), number of noise points (#noise), DISCO score for the ground truth labels (DISCO) including noise, and the source.

	Dataset	$n$	$d$	$k$	#noise	DISCO	Source
Density-based Benchmark Data	smile1	1,000	2	4	0	0.90	(Barton & Bruna, 2015)
	dartboard1	1,000	2	4	0	0.87	(Barton & Bruna, 2015)
	chainlink	1,000	3	2	0	0.84	(Barton & Bruna, 2015)
	3-spiral	312	2	3	0	0.59	(Barton & Bruna, 2015)
	complex8	2,551	2	8	0	0.39	(Barton & Bruna, 2015)
	complex9	3,031	2	9	0	0.36	(Barton & Bruna, 2015)
	compound	399	2	6	0	0.35	(Barton & Bruna, 2015)
	aggregation	788	2	7	0	0.31	(Barton & Bruna, 2015)
	cluto-t8-8k	8,000	2	8	323	0.30	(Barton & Bruna, 2015)
	cluto-t7-10k	10,000	2	9	792	0.29	(Barton & Bruna, 2015)
Real World	cluto-t4-8k	8,000	2	6	764	0.24	(Barton & Bruna, 2015)
	diamond9	3,000	2	9	0	0.22	(Barton & Bruna, 2015)
	Synth_high	5,000	100	10	500	0.82	(Jahn et al., 2024)
	Synth_low	5,000	100	10	500	0.76	(Jahn et al., 2024)
	htru2	17,898	8	2	0	0.41	(Markelle Kelly, 2023)
Real World	COIL20	1,440	16,384	20	0	0.30	(Nene et al., 1996)
	Pendigits	10,992	16	10	0	0.11	(Markelle Kelly, 2023)
	cmu_faces	624	960	20	0	0.07	(Markelle Kelly, 2023)
	Optdigits	5,620	64	10	0	0.06	(Markelle Kelly, 2023)

for tabular and image classification, mostly from the UCI Machine Learning Repository. The datasets included from the density-based Tomas Barton benchmarks (first block in density-based benchmark data) are visualized in Figure 20. The synthetic data (Synth\_high, Synth\_low) is provided by the data generator DENSIRED (Jahn et al., 2024) that we also used for some of the systematic experiments in Section 4: Such datasets have ten density-connected clusters of different densities that are generated based on random walks in high-dimensional space. For the generated data, we include noise points that are uniformly distributed and positioned outside of the clusters, s.t. they are guaranteed to be density-separated from the clusters. The COIL20 benchmark data contains continuously changing images of 20 different objects taken from different angles.

All datasets are z-standardized for the experiments such that for tabular data, every feature has mean 0 and std deviation 1, while on image data, this step has been performed globally instead of per feature. To account for DBCV’s non-deterministic behavior, we performed 10 shuffled runs for the relatively low-dimensional datasets and reported the results with standard deviation. However, due to runtimes of multiple hours per dataset and algorithm, we decided to perform only a single run on datasets with at least 700 dimensions (i.e., cmu\_faces, COIL20).

## B.3. Other Cluster Validation Indices

Table 6 provides implementation details for the methods. We employ the default hyperparameters provided by the authors. The author implementations are linked whenever available and used when implemented in Python. In all other cases, we re-implemented the method in Python and used this implementation in our experiments; we marked those with ✓ in the last column of the table.

The table further includes information on the direction of the CVI: when marked with ↑, higher is better, and otherwise, it is marked with ↓. We distinguish between bounded methods (↑ and ↓) and unbounded ones (↑ and ↓). Both scores where lower values are better are in the range  $[0, \infty]$ , i.e., they have a lower bound.

Table 6. Implementation details of included internal CVIs.

Method	Hyperparameter (default)	Official implementation	Implemented ourselves
↑ Silhouette (Rousseeuw, 1987)	✗	(sklearn)	✗
↓ S_Dbw (Halkidi & Vazirgiannis, 2001)	✗	-	✓
↑ DBCV (Moulavi et al., 2014)	distance ( <b>squared euclidean</b> )	Matlab, Python	✗
↑ DCSI (Gauss et al., 2024)	minPts (5)	R	✓
↑ LCCV (Cheng et al., 2018)	✗	Matlab	✓
↑ VIASCKDE (Şenol, 2022)	bandwidth, kernel ( <b>0.05, gaussian</b> )	Python	✗
↑ CVDD (Hu & Zhong, 2019)	number of neighborhoods (7)	Matlab	✓
↑ CDbw (Halkidi & Vazirgiannis, 2008)	number of representative points (10)	-	✓
↓ CVNN (Liu et al., 2013)	number of nearest neighbors (10)	-	✓
↑ DISCO (ours)	$\mu$ (5)	Python (github)	✓

#### B.4. Normalization

To account for unbounded scores, we perform a normalization, bounding it to scores between [0, 1]. To do this, we divide by the maximum value assigned by the CVI for this specific experiment and dataset; unbounded scores are marked with an n in the table. Scores that indicate better clusterings with lower values are previously normalized to be in the same orientation. Reversed scores are calculated as the maximum value for the experiment – actual value and marked with (r).

### C. Discussion of DBCV

In this section, we regard DBCV, its non-deterministic behavior, and existing variants.

#### C.1. Determinism of DBCV

Some competitors in our experiments, namely LCCV and S\_Dbw, returned results with a small standard deviation because of their implementation or in very specific edge cases. However, the most well-known measure, DBCV, is even conceptually not deterministic as it builds on top of MSTs, disregarding points at the leaf level of the clusterwise MST. Even though Euclidean MSTs are in practice for most datasets unique, this is not the case when they are based on the mutual reachability distance, as is the case for DBCV. Pairwise mutual reachability distances occur per definition more often in a dataset, increasing the number of different MSTs. This effect becomes more prominent when the processing order of the data objects changes, as we show in our experiment in Figure 2, or different algorithms are used to create the MST.

To the best of our knowledge, state-of-the-art works using DBCV are not aware of this non-determinism, as it is also not discussed in (Moulavi et al., 2014), yielding potentially misleading results in multiple works in the field of density-based clustering. While the effect could be diminished, e.g., by taking the mean of several runs, users would need to know about the problem, and it is not part of the standard implementations.

#### C.2. DBCV Variations

We use DBCV on squared Euclidean distances as suggested by the authors (Moulavi et al., 2014). Note that the most common implementation of DBCV that comes with the **HDBSCAN** implementation by default uses Euclidean distances. In our experiments, DBCV provided much better results on squared Euclidean distances compared to Euclidean distances.

Varying implementations for the construction of the MST also lead to different results.

## D. Noise

In this section, we regard different notions and ways to handle noise points.

### D.1. Notions of Noise

In the literature, there are various notions of *noise*, as elaborated in [Hohma et al. \(2022\)](#). For example, noise can refer to jitter, i.e., some deviation of a specific point from some optimal position. E.g., the noise parameter of the famous two-moons generator of sklearn defines the spread of the moons. This type of noise usually goes unmentioned as it is ubiquitous as it resembles standard (unbiased) measuring errors that occur in real life.

Especially in high-dimensional data, noise can also refer to redundant or irrelevant features. This kind of noise can be removed through feature engineering before clustering. Subspace clustering algorithms find clusters using only a subspace of the features for each cluster (or the full dataset). While we do not regard this specific type of clusters in this paper, DISCO could be expanded to handle subspace clusters in future work, e.g., by combining it with ideas from [Kazempour et al. \(2020\)](#).

In this paper, we focus on the most common type of noise: additional noise points in the data that do not belong to any cluster. The points usually stem from a different source or distribution than the points within clusters. These can have external causes or come from measuring errors. Note that this kind of noise is closely related to the concept of *outliers*. However, outliers are typically few and warrant further investigation, e.g., for fraud detection. Noise points, in contrast, are expected and occur in larger quantities. Furthermore, we regard uniformly distributed noise points, in contrast to, e.g., fraudulent noise points forming chains that connect clusters as described in [Held et al. \(2019\)](#).

Note that some measures, e.g., CVNN, explicitly evaluate the impact of noise in the data on their measure; however, this refers to the structure of the dataset, and they do not regard noise as part of the labeling. In contrast, most of their experiments are based on clusterings returned by k-Means, which do not contain noise labeling.

### D.2. Biases When Not Properly Evaluating Noise

There are some common ways to (not properly) work with noise labels: one can ignore them by filtering, one can interpret them as an additional cluster, one can interpret all noise points as singleton clusters, and one can perform a global penalty as suggested by DBCV. In the following, we highlight possible downsides to all of those approaches.

**Filtering noise.** Especially when filtering noise, this opens up many options to manipulate scores by focusing on, e.g., two very dense but far apart clusters and labeling everything else as noise. This would (in most cases) lead to very high scores when the clusters are indeed well-separated. In general, filtering noise would incentivize methods to label more points as noise than necessary, possibly leading to missed clusters (as the CVI states that the one with a lot of noise is better). Note that if the noise points are filtered, the quality of their labeling can not be evaluated.

**Noise as an additional cluster.** Often, users just apply their CVI to the output of their clustering algorithm. Most implementations label noise as -1. Thus, when given to a CVI that does not explicitly handle noise, all noise points are interpreted as belonging to one cluster. The main disadvantage is that the set of noise does not form a cluster: The “noise” cluster tends to have a much lower density than others and lacks cohesion (especially for uniform noise) – in general, a (true) “noise” cluster tends to have highly different properties from a true cluster. This lack of cohesion (and possibly also low separation) of one of the clusters can lead to low overall scores, even if the noise was perfectly separated from the clusters. This would bias an algorithm from labeling noise at all or using it sparingly in cluster-like regions of the dataset. Since the CVI in this setting can not distinguish between a cluster and noise, it would report identical (high) scores when reporting one of the clusters as noise and the noise as one of the clusters. This could lead to surprising results and grave differences when comparing with other methods that handle noise in a different way.

**Interpreting each noise point as singleton cluster.** This approach is often used for external cluster evaluation and is probably the most recommendable. It allows using CVIs like ARI or NMI without any adaptations. More importantly, this solution refrains from treating noise points together as a cluster, avoiding the problems explained in the previous paragraph. However, it can lead to a biased evaluation if the applied CVI itself already has a bias depending on the number of clusters. Furthermore, this solution cannot be applied to most internal CVIs, as they are usually not defined on labelings containing singleton clusters.

Table 7. Internal CVI results on several benchmark datasets for the ground truth labeling. Colored column-wise.

Dataset	DISCO ( $\uparrow$ )	DBCV ( $\uparrow$ )	DCSI ( $\uparrow$ )	LCCV ( $\uparrow$ )	VIAS. ( $\uparrow$ )	CVDD ( $\uparrow$ )	CDbw ( $\uparrow$ )	CVNN ( $\downarrow$ )	Silh. ( $\uparrow$ )	S_Dbw ( $\downarrow$ )	
Density-based Benchmark Data	smile1	0.90	0.96 $\pm$ 0.01	0.99	0.83	0.76	2523.39	1.52	3.64	0.50	0.51
	dartboard1	0.87	0.95	1.00	0.68	0.48	44912.94	0.00	6.76	-0.04	1.68 $\pm$ 0.07
	chainlink	0.84	0.92 $\pm$ 0.01	0.96	0.70	0.39	1275.27	0.01	4.24	0.15	2.23
	3-spiral	0.59	0.55 $\pm$ 0.01	0.93	0.66	0.30	189.35	0.01	5.49	0.00	2.79
	complex8	0.39	-0.10 $\pm$ 0.09	0.85	0.60 $\pm$ 0.01	0.64	323.64	0.32	5.68	0.09	0.74
	complex9	0.36	0.17 $\pm$ 0.07	0.95	0.55	0.63	689.40	0.61	5.11	-0.01	0.59
	compound	0.35	0.32 $\pm$ 0.10	0.85	0.60 $\pm$ 0.02	0.30	48.60	0.32	4.16	0.14	0.53
	aggregation	0.31	0.49 $\pm$ 0.08	0.97	0.66 $\pm$ 0.01	0.54	541.68	0.42	3.21	0.47	0.37
	cluto-t8-8k	0.47	-0.05 $\pm$ 0.10	0.92	0.62	0.73	1034.46	0.16	5.59	0.09	0.73
	cluto-t7-10k	0.44	-0.17 $\pm$ 0.03	0.98	0.28	0.74	1164.77	0.66	5.08	-0.01	0.54
Synthetic	cluto-t4-8k	0.62	0.23 $\pm$ 0.08	0.94	0.75	0.71	2775.02	0.08	4.65	0.23	0.84
	diamond9	0.22	-0.16 $\pm$ 0.07	0.91	0.58	0.62	685.19	0.07	4.05	0.55	0.41
Real World	Synth_high	0.80	0.95	0.96	0.02	0.24	828.11	1.09	16.55	0.49	0.30
	Synth_low	0.73	0.90	0.96	0.05	0.15	765.67	0.04	8.82	0.58	0.11
Real World	COIL20	0.30	0.36	0.74	0.72	-	0.22	0.00	1847.42	0.16	0.63
	htru2	0.41	-0.99	0.00	0.62	0.04	1.46	0.00	7.97	0.55	1.08
	Pendigits	0.11	-0.77	0.36	0.33	0.00	2.34	0.00	36.58	0.17	0.68
	cmu_faces	0.07	-	0.59	0.23	-	0.32	0.00	442.29	0.17	0.66
	Optdigits	0.06	-0.38	0.50	0.52	0.01	17.28	0.00	63.05	0.17	0.75

**Adding a penalty factor.** In addition to the score itself, DBCV suggests a postprocessing step to make all CVIs noise-aware: after computing the score, we multiply it by  $\frac{|X| - |N|}{|X|}$  where  $X$  is the dataset and  $N$  the set of noise points. This effectively assumes a pointwise value of 0 for each noise point, assuming that all noise points are inherently “bad” for the quality of an overall clustering. This incentivizes to label only a few points as noise. We argue that some noise labels (i.e. those far away from clusters and in sparse regions, as described in Section 3) are indeed positive and should thus not draw down the CVI’s score. We thus include noise evaluation in DISCO, which can easily be included in all CVIs that compute pointwise scores to allow for proper noise evaluation.

## E. Additional Experiments

In this section, we include additional experiments highlighting differences between density-based and centroid-based CVIs. We furthermore investigate important properties of internal evaluation measures as, e.g., defined in Kazempour et al. (2020): the influence of separability, different densities, amount of noise, number of clusters, dimensionality, and edge case datasets. We adhere to good standards regarding our evaluation, e.g., avoiding an overoptimistic assessment as described in Ullmann et al. (2023).

### E.1. Evaluation of Benchmark Data

To compare DISCO to other methods, we use density-connected and real world benchmark datasets (cf. Appendix B.2).

To give a more complete picture, we also include the centroid-based CVIs like S\_Dbw and the Silhouette Coefficient (Silh.). Table 7 shows the CVIs for the ground truth without noise, as only DISCO evaluates the quality of noise labels.

For several competitors, computations on large or high-dimensional datasets failed by running into the 12h time or 512GB RAM limit (marked with –). Note that some CVIs are incalculable for specific edge cases that appear in some benchmark datasets (marked with –).

The table also highlights problems of unbounded scores: CVDD, CDbw, CVNN, and S\_Dbw yield different value ranges for different datasets that are hard to compare across different datasets. It is important to note that higher values do not always indicate that the CVI works better. For the diamond9 dataset, illustrated in Figure 20l, DISCO and DBCV show low scores, especially in comparison to DCSI. However, when investigating the visualization of the dataset, it becomes apparent that the diamonds are connected, and thus, a low score is actually desired while DCSI fails to recognize these density-connectivity bridges.

On the density-connected benchmarks, DISCO and most other density-based CVIs (and also CVNN) give good scores, while Silhouette and CDbw do not. DBCV results vary. It rejects multiple of the ground truth clusterings, especially in cases where the separation between clusters is not as clear. While DCSI and DBCV show similar trends as DISCO for most datasets, they deviate for the highly unbalanced 2-class dataset htru2: it consists of a large and compact cluster and a smaller and less dense one. DISCO’s pointwise metric emphasizes the quality of the larger cluster. Overall, DISCO gives high scores to datasets with density-connected clusters and low ones to such without (e.g. cmu\_faces).

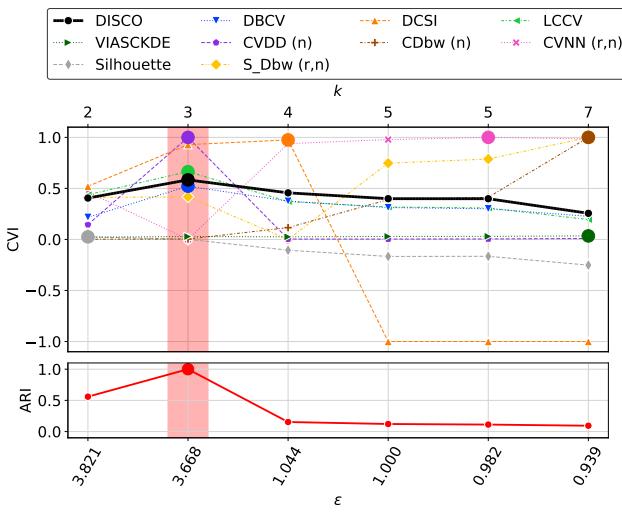
Table 7 shows the results for all included CVIs and benchmark datasets. Results for DBCV and two-dimensional datasets include the standard deviation over multiple runs of the processing order of points. We do not depict it for the other internal CVIs as they produced constant results, i.e., a standard deviation of zero. The only exceptions to this are LCCV and S\_Dbw; LCCV’s deviations are due to numerical instabilities when calculating the score. The definition of S\_Dbw seems not to capture the edge case that every cluster center and every middle point on the line between cluster centers is placed at the same coordinates, as it is the case in dartboard1, refer Figure 20g.

## E.2. Determining Best Parameter Settings

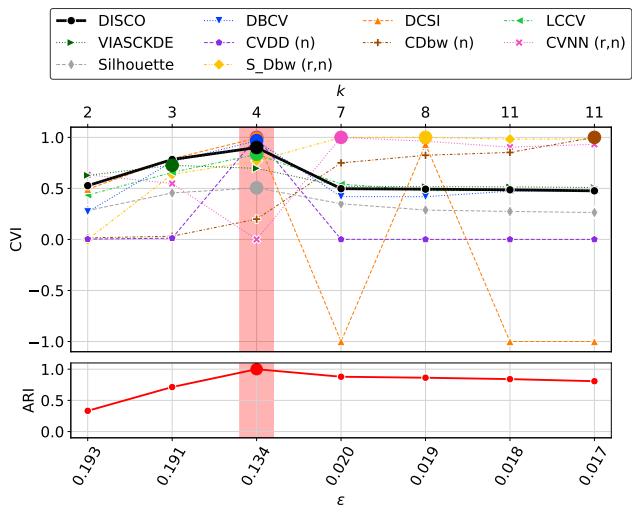
Additionally to Section 4.3.1, we illustrate the results for the use case experiment to find “good” parameter settings for clustering methods for three more datasets: 3-spiral (Figure 10a), smile1 (Figure 10b) and COIL20 (Figure 11). We compare the scores of internal CVIs for DBSCAN clusterings across a range of  $\varepsilon$ -values to find the best DBSCAN clustering. The red bar marks the clustering that fits the ground truth best (according to ARI). DISCO effectively evaluates the parameter settings yielding the optimal ARI as best for the datasets 3-spiral and smile1. On COIL20, it yields a setting that creates the ground truth number of  $k = 20$  clusters with an ARI close to the optimal ARI. This might be because of a better density-separation between classes or noise that is not labeled in the data.

## E.3. Evaluation of Clustering Algorithms

One goal of internal CVIs is to assess the quality of a clustering (without knowing the ground truth), which is similar to an external validation based on the ground truth. To evaluate this, we regard benchmark datasets that contain density-based clusters (excluding points labeled as noise) and various clusterings of them. We use the well-known density-based clustering methods DBSCAN, HDBSCAN, and Density-Peaks Clustering (DPC), where DPC is not based on density-*connectivity* but on finding local peaks of density. We also regard Spectral Clustering (SC), Agglomerative Ward Clustering (Ward), MeanShift, and  $k$ -Means to assess the quality of clusterings following a different cluster notion. Table 8 shows the DISCO and ARI scores, where the ARI here handles each noise point as an individual cluster, and the Pearson Correlation Coefficient (PCC) between both scores on the respective dataset. Positive PCC values imply the desired strong correlation, i.e., DISCO



(a) 3-spiral dataset (see Figure 20h)



(b) smile1 dataset (see Figure 20e)

Figure 10. CVIs for DBSCAN clusterings with varying  $\varepsilon$  values and the resulting number of clusters  $k$ . The corresponding ARI scores are below.

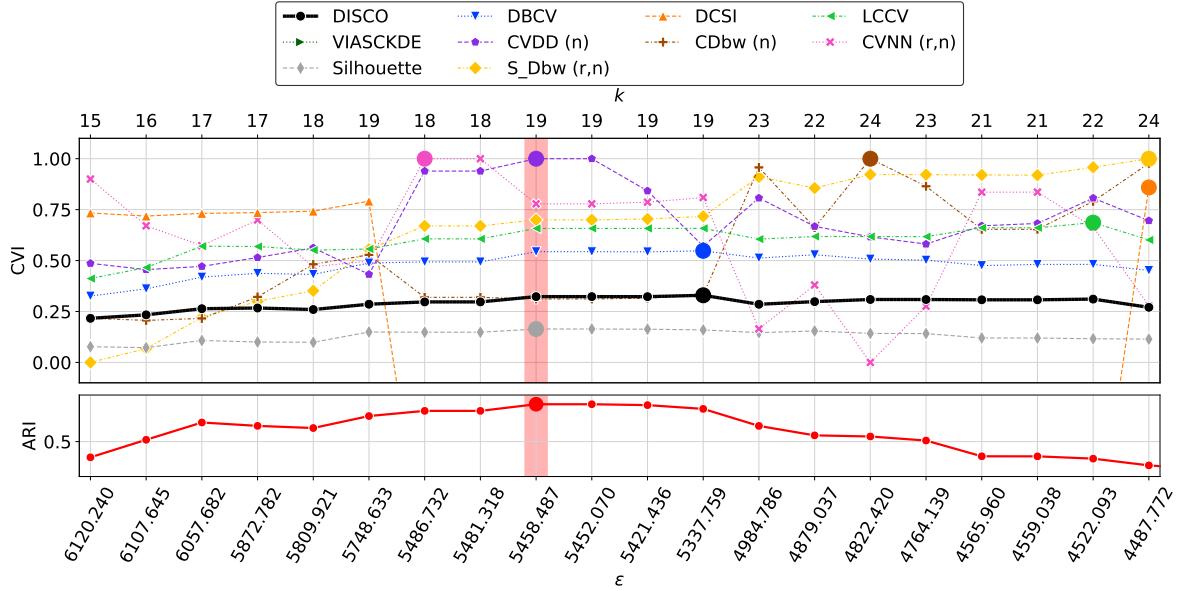


Figure 11. CVIs for DBSCAN clustering of the COIL20 dataset with varying  $\varepsilon$  values and the resulting number of clusters  $k$ . The corresponding ARI scores are below.

scores give an assessment of the quality that is similar to ARI for most datasets. For the complex9 dataset, neither score correlates as strongly because some of the clusters are very close, leading to a low separability of ground truth clusters. For the aggregation dataset, the PCC is even negative. This is because the ground truth labels do not correspond to the actual density-connected components; see Figure 20k: chains of points that are too dense to be considered noise connect different clusters, and some other clusters are contiguous. The overall PCC between ARI and DISCO is 0.69, implying a successful quality assessment by DISCO for density-based clusterings.

#### E.4. Noise Evaluation

In this experiment, we expand upon the motivational example in Figure 1 by showcasing further scenarios. In addition to the values for DISCO, DBCV, and DCSI, we also report the internal scores  $\rho_{\text{cluster}}$ ,  $\rho_{\text{sparse}}$ , and  $\rho_{\text{far}}$  to give more details on how DISCO evaluates cluster and noise points. The experiments in Figure 12 demonstrate how noise affects our DISCO score. We employ eight different scenarios of noise-labeled points within a dartboard like dataset. DCSI does not change across the different settings, except for (g) where it does not cope well with all cluster points being on the same position, since the definition does not include any evaluation for noise labeled points. When comparing (a) and (b), we observe that DISCO and DBCV both decrease. While DBCV only detects that (b) contains more noise than (a), DISCO additionally checks the quality and considers the noise “low-quality” as the right halves of the circles are density-connected to the clusters on the left. This can be seen by the low value for  $\rho_{\text{far}}$ . Furthermore, the right halves of the circles are as dense as the left halves. Marking them all as noise leads to a low value for  $\rho_{\text{sparse}}$ , as this metric captures how well noise lays within a non-dense region. While when comparing (b) and (c) the total DISCO score increases, the individual parts of DISCO change in different directions. By moving the noise to the center, the distance between the rings increases, leading to a higher  $\rho_{\text{cluster}}$ . On the other hand, all noise points are on the same position, forming a very dense region, which results in an extremely low  $\rho_{\text{sparse}}$ . This indicates that the noise points should rather form a cluster than be labeled noise.

Similar effects can be observed when comparing (e) and (f). Figure 12 (g) shows a clustering consisting of four clusters, concentrated at the same location (the four colored points in each of the quadrants) and therefore very dense and 23% of noise-labeled points placed in the space. DISCO assesses this as a perfect clustering. This can be explained by looking at the different components:  $\rho_{\text{cluster}}$  is high since the clusters included are of high density and well-separated. The two concepts denoting the quality of the noise ( $\rho_{\text{sparse}}$  and  $\rho_{\text{far}}$ ) are also high since the noise is well placed in comparison to the least dense cluster in the dataset.

Table 8. DISCO (top), ARI (bottom) for different clustering algorithms. Colored based global on min. and max. values. The last column shows the Pearson correlation values between the DISCO and ARI values. Table 4 shows the Pearson correlation values for all competitors and averaged over 10 runs.

Dataset	<i>GT</i>	<i>DBSCAN</i>	<i>HDBSCAN</i>	<i>DPC</i>	<i>SC</i>	<i>Ward</i>	<i>MeanShift</i>	<i>k-Means</i>	<i>Random1</i>	<i>Random2</i>	<i>PCC</i>
DISCO scores	58.79	58.79	59.62	58.79	-0.20	0.27	0.00	-0.14	-2.36	-52.89	
	30.57	29.80	66.97	50.04	28.02	21.47	50.80	20.88	-5.65	-54.09	
	83.50	83.50	83.50	-1.28	47.98	0.60	0.00	25.14	0.04	-54.81	
	23.57	24.21	4.77	-25.80	2.27	8.44	0.00	6.00	-0.88	-9.10	
	29.01	31.68	17.20	-26.10	-0.26	1.54	10.38	9.42	-1.18	-6.24	
	29.52	28.94	19.24	-28.98	1.73	2.12	14.26	2.32	-1.48	-8.79	
	39.00	39.19	36.48	-9.87	4.04	6.99	13.53	4.71	-1.91	-15.69	
	35.78	35.78	22.27	-20.06	0.09	0.20	27.16	2.59	-4.83	-32.25	
	34.76	32.10	44.69	59.16	16.46	10.25	58.25	17.23	-4.67	-53.31	
	87.43	87.43	87.43	47.91	-0.68	-5.27	0.00	-0.34	-3.17	-49.82	
	21.77	20.86	17.09	-8.13	20.89	21.72	4.18	21.77	-0.84	-5.43	
	90.01	90.01	90.01	52.56	43.15	36.85	35.89	33.55	-3.65	-51.64	
	75.65	72.92	75.65	0.00	0.00	51.94	48.79	51.14	-6.75	-32.55	
	82.27	78.81	82.27	0.00	74.48	46.70	82.24	25.18	-7.98	-50.57	
	40.93	61.48	59.13	0.00	46.60	46.32	53.35	46.76	0.00	-8.61	
	11.07	16.90	6.20	-11.07	12.45	9.71	0.00	11.52	-0.63	-3.45	
	29.56	28.48	30.23	0.54	0.00	24.24	0.00	18.24	-2.43	-10.50	
	6.75	8.67	13.48	-15.02	4.59	6.53	39.23	5.09	-2.22	-8.49	
	5.86	6.24	5.89	-12.79	5.55	4.87	0.00	5.80	-0.67	-2.24	
ARI values	100.00	100.00	98.56	100.00	-0.43	-0.38	0.00	-0.59	0.47	0.14	90.99
	100.00	97.47	80.89	79.33	82.88	70.18	62.85	73.53	-0.17	0.07	77.13
	100.00	100.00	100.00	20.09	50.36	-0.10	0.00	22.03	0.03	-0.01	92.79
	96.63	1.19	57.25	19.20	38.22	47.30	0.00	46.42	0.02	0.00	42.85
	97.67	0.82	60.62	19.51	31.21	29.59	15.67	39.74	0.01	0.01	42.87
	99.39	93.22	53.26	34.16	44.51	45.43	21.28	45.93	-0.01	0.00	68.41
	100.00	93.94	87.05	31.11	43.76	51.19	23.02	49.96	0.00	-0.01	90.19
	100.00	99.95	30.10	19.81	35.32	32.75	18.26	37.76	-0.05	0.00	74.71
	100.00	61.26	81.11	74.02	56.12	50.11	72.23	61.15	-0.20	0.11	83.75
	100.00	100.00	100.00	0.91	-0.29	7.98	0.00	-0.30	0.12	-0.06	88.08
	100.00	87.49	77.51	15.11	96.57	99.63	14.77	100.00	-0.01	0.01	96.33
	100.00	100.00	100.00	33.27	63.60	56.89	33.05	54.62	0.23	0.02	91.29
	95.42	94.23	95.42	0.00	0.00	62.56	14.27	65.73	0.02	-0.01	90.78
	95.42	94.12	95.42	0.00	0.04	71.76	4.01	43.99	0.02	0.04	58.65
	100.00	5.65	13.69	0.00	53.98	59.40	-1.31	74.58	0.00	0.00	36.88
	100.00	0.28	63.10	22.19	43.60	61.03	0.00	59.46	0.00	0.00	44.16
	100.00	65.39	77.37	35.99	0.00	68.55	0.00	55.12	0.12	0.13	93.89
	100.00	25.65	46.36	0.59	0.00	61.63	0.00	50.01	-0.29	0.18	11.05
	100.00	60.62	57.38	0.00	76.87	69.71	0.00	67.28	0.02	0.04	78.18

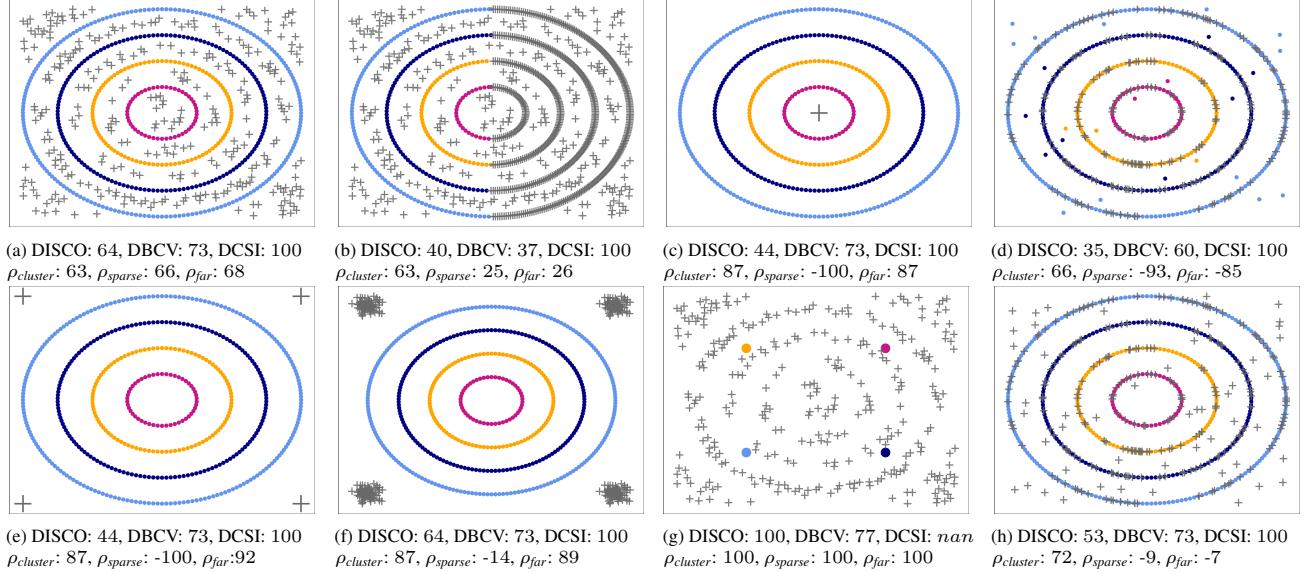


Figure 12. Values in percent. Clusterings implied by color, noise marked as +.

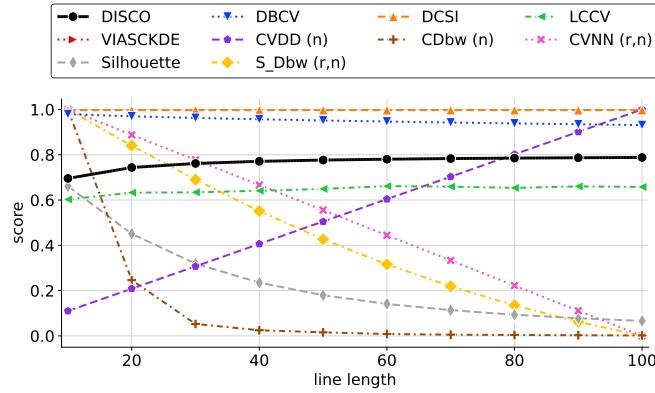


Figure 13. CVIs for lines datasets. The x-axis denotes the length of the line-shaped clusters.

## E.5. Centroid-based CVIs

The datasets used for this experiment are parallel line-shaped clusters that increase in length. The idea is that while short lines could still be interpreted as centroid-based clusters, very long lines are undoubtedly density-based, and we should get a smooth transition between the two cases. Figure 13 illustrates the results for all CVIs included in the scope of our experiments. As expected, the centroid-preferring CVIs, Silhouette, and S\_Dbw decrease the longer the lines get. LCCV, DCSI, DBCV, and DISCO produce constant scores indicating a well-separated and compact clustering. This highlights that those four methods are indeed density-based. VIASKDE does not return any score, as the KDE weight is the same for each datapoint in the individual clusters, resulting in an undefined edge case. CVDD increases strongly and prefers longer clusters. CDbw decreases rapidly, as it is a difficult problem for a method based on representatives.

## E.6. Influence of Difference in Density between Clusters

In the experiment depicted in Figure 14 we increase the density of one moon in the two-moons dataset. All CVIs, including DISCO, show similar good results except for CVDD, which drops for higher density ratios as the score depends on the number of data points (and is thus only comparable within datasets of the same size). LCCV and Silhouette are unaffected by the density ratio but give rather low scores. DISCO shows a slight increase as denser clusters have higher compactness, which increases the overall score.

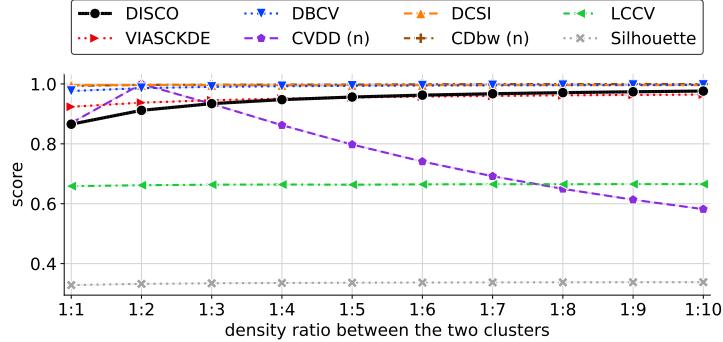


Figure 14. Influence of difference in density: Increasing the number of points in one of the two moons

### E.7. Number of Clusters

In this experiment, we randomly assigned one out of  $k$  labels – with the same probability for every label – to every data point in the aggregation dataset, which consists of 788 points in total. Figure 15 shows the mean and confidence interval of the DISCO score over 10 runs. As expected, DISCO returned decreasing scores for higher  $k$ . At approx.  $k = 150$ , the scores increased slightly again. That can be explained by the formation of singleton clusters, which we all score with 0, moving the average closer to zero, i.e., upwards.

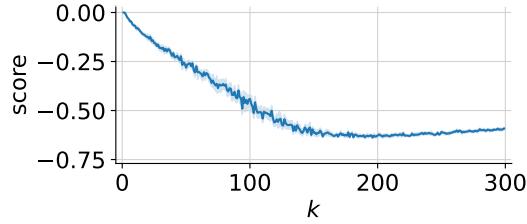


Figure 15. DISCO score of the aggregation dataset with randomly assigning one of  $k$  (x-axis) labels to a data point.

### E.8. Influence of Dimensionality

Figure 16 shows the influence of increasing the dimensionality in the dataset. For this, we add un-informative dimensions to the two-moons dataset while keeping all relative distances, i.e., all coordinates, but the first two dimensions are set to 0. While almost all CVIs are robust against this change, CDbw quickly converges to 0 for increasing dimensionality, significantly limiting its applicability. We will see the full extent of this problem for real-world and benchmark data also in Table 7.

### E.9. Notions of Noise

This section extends Table 3 from the experiments section. Figures 17a and 17b illustrate the behavior of different CVI's when adding noise points, regarding different noise labeling approaches in Figure 17a “good” noise is added, i.e., noise points that are isolated and far away from the clusters, in Figure 17b “bad” noise is added. This is done by adding noise-labeled points into the existing clusters. Every method that does not explain how noise-labeled points are handled (see Table 1) assesses noise-labeled points as their own cluster as a default functionality in our implementation. Except for this are DCSI, CDbw, and S\_Dbw, where we followed official implementations, where noise-labeled points were filtered first. Therefore, we cannot observe any change in the scores when adding noise-labeled points.

Note that DBCV, Silhouette, LCCV, and VIASCKDE treat good noise and bad noise the same, i.e., they reduce the overall score without distinguishing between the two types of noise. CVDD decreases when good noise is added because the “noise cluster” lacks density. For the case of bad noise, the curve of the score is not as steep since the added noise, which is considered a cluster here, is placed in four different quite dense regions. CVNN is not affected by additional good labeled noise, however in case of bad additional noise we observe an increase in the score. This behavior can be attributed to the

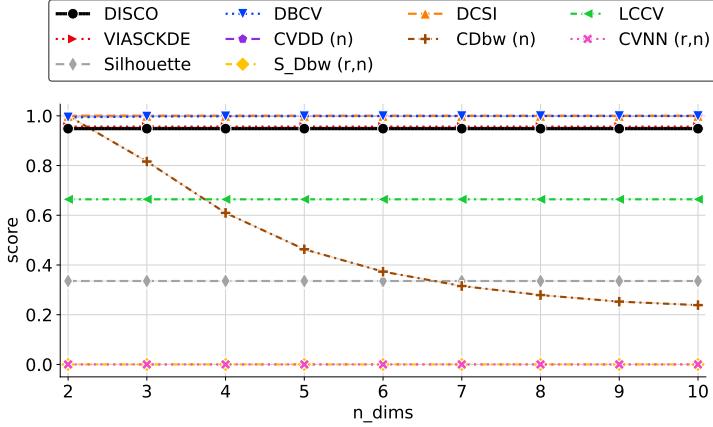


Figure 16. Influence of increasing dimensionality while keeping all relative distances in the two-moons dataset.

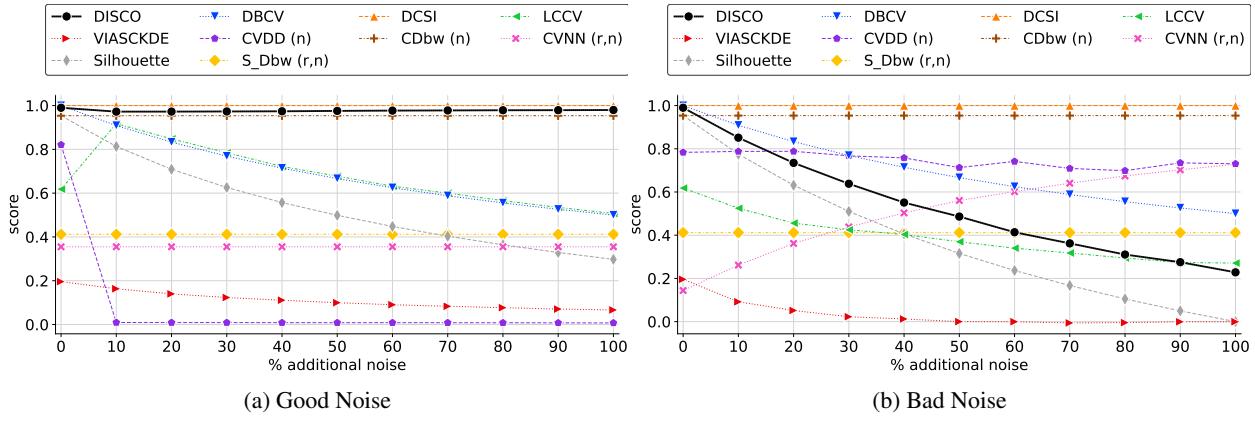


Figure 17. Good noise vs Bad noise

way the CVNN implementation treats noise, i.e., as a cluster, as well as the noise generation method for this scenario, which is adding points within the given clusters. As CVNN measures compactness based on the average pairwise distances within the clusters, it decreases with each additional noise point added, rendering the disconnected cluster more dense in compact regions. Consequently, the overall score, which is the sum of compactness and separation, that does not change, decreases. In Figure 17b, we have reversed the score for better visibility, which explains why this decline is reflected as an increase in the score.

Unlike the other metrics, DISCO evaluates the quality of noise labels. Given that the clustering is already assumed to be very good in the beginning, additional good noise does not improve the score. It results in a slight decrease by 0.01, the reason for this is that DISCO averages over all points and a perfect score of 1 is very difficult to achieve.

#### E.10. Pointwise Scores

As previously discussed in Section 3.2, DISCO is defined on a point-level basis, i.e., it assesses each point individually. Figure 18 illustrates this for the two benchmark datasets, cluto-t7-10k (top) and zelnik4 (bottom), highlighting the differences between well-labeled noise points that do not form dense regions and are far away from clusters (bottom left), and those that clearly show cohesion (top left).

The assessment of cluster points is clearly illustrated in Figure 18 (top). It is evident that cluster points only receive high values when the corresponding clusters are distinctly enclosed within the defined boundaries of the cluster. In the case of  $k$ -Means clustering, this is particularly noticeable in the second cluster at the bottom right. The high values can be attributed to the fact that the arbitrary shape in this area is fully encompassed by the cluster boundaries rather than being divided by them like other shapes in this clustering.

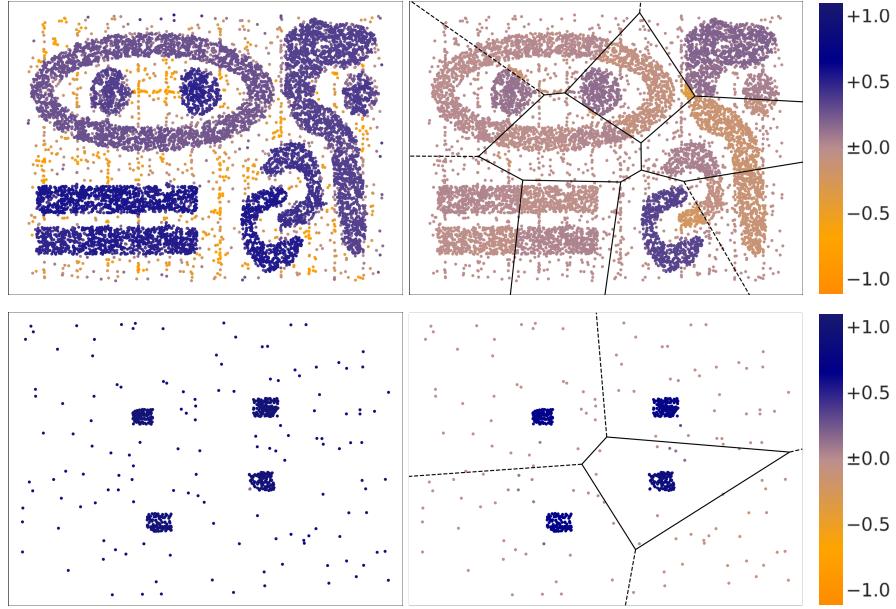


Figure 18. Pointwise DISCO scores. Left: ground truth labels. Right:  $k$ -Means labels, lines indicate cluster borders. Top: dataset cluto-t7-10k. Bottom: dataset zelnik4.

Figure 18 (bottom) shows high pointwise DISCO scores for the ground truth labels on the left, where the points in the sparse area are labeled as noise points. On the right, we see that the DISCO scores for the  $k$ -Means clustering are low (orange) for the noise points as they are not detected as noise but assigned to their closest clusters – even though they are not density-connected to the respective cluster.

### E.11. Noise on Degenerated Data

An example that is typically likely to break density-connectivity-based methods is a degenerating dataset where the MST of the mutual reachability distance is a degenerated tree, i.e., a line of maximum height where every node has only one child in the tree. That is the case for a dataset like this: Points  $p_i$  at  $x = 2^i$  and  $y = 0$ . A good noise labeling is given by labeling the sparsest points as noise, as they are relatively far apart from the rest of the points. Thus, DISCO yields high values in Figure 19 left. In contrast, when we label the densest points as noise, we get a very counterintuitive labeling that results in the lowest possible DISCO scores; see Figure 19 right.

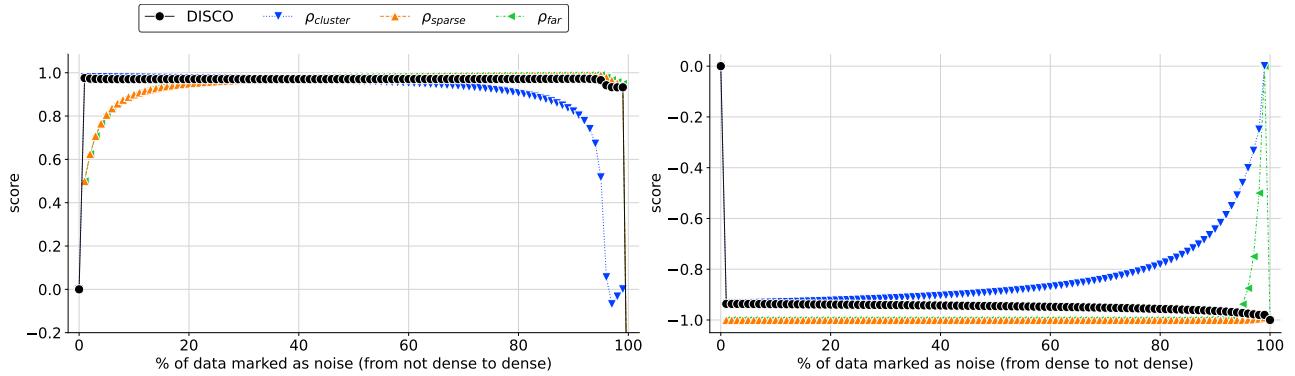


Figure 19. Points  $p_i$  at  $x = 2^i$  and  $y = 0$ . Left: Labeling points in the sparsest area as noise and assigning the rest of the data as one cluster yields very high DISCO values. Right: Labeling points in the densest area as noise and the rest as clusters yield low DISCO values.

## F. Tomas Barton Benchmark Datasets

Figure 20 shows clusterings produced by clustering algorithms, including density-based algorithms like DBSCAN, HDBSCAN, and DPC, as well as centroid-based methods, for example,  $k$ -Center or  $k$ -Means, for most of the Tomas Barton benchmark datasets.



Figure 20. Tomas Barton Benchmark datasets