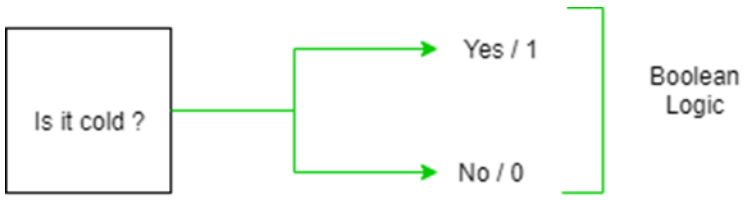
Fuzzy Logic

❖ Introduction

- It is a way to represent uncertainty.
- Uncertainty associated with vagueness, with imprecision and /or with a lack of information regarding a particular element of the problem.
- Fuzziness is found in our thinking, in our decisions, in the way we process information & particularly in our language.
- Statements can be unclear or subject to different interpretations.

  e.g. Phrases like "see you later", "little more", "Don't feel very well" are the fuzzy expressions



- **Boolean logic uses sharp distinctions. It forces us to draw lines between members of a class and non-members.**
- **Fuzzy Set expresses the degree to which an element belongs to a set.**

❖ Fuzzy Logic applications

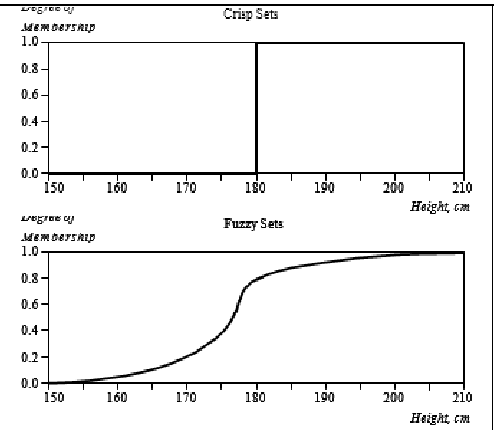
- pattern recognition
- data mining & information retrieval
- automata theory, game theory
- automatic train control
- tunnel digging machinery
- home appliances: washing machines, air conditioners

Crisp vs. Fuzzy Sets

- **The classical example in fuzzy sets is tall men. The elements of the fuzzy set tall men" are all men, but their degrees of membership depend on their height.**

- **For instance, in crisp set, we may say, Tom is tall because his height is 181 cm. If we drew a line at 180 cm, we would find that David, who is 179 cm, is small. Is David really a small man?**

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00



- **The x-axis represents the universe of discourse – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.**

- **The y-axis represents the membership value of the fuzzy set. In our case, the fuzzy set of "tall men" maps height values into corresponding membership values.**

Crisp Set

- **Let *X* be the universe of discourse and its elements be denoted as *x*. In the classical set theory, crisp set *A* of *X* is defined as function $f_A(x)$ called the characteristic function of *A***

$$f_A(x) : X \text{ in } \{0, 1\}, \text{ where } f_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

This set maps universe *X* to a set of two elements. For any element *x* of universe *X*, characteristic function $f_A(x)$ is equal to 1 if *x* is an element of set *A*, and is equal to 0 if *x* is not an element of *A*.

Fuzzy Set

In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the membership function of set A

$\mu_A(x)$: X in [0, 1], where $\mu_A(x) = 1$ if x is totally in A;

$\mu_A(x) = 0$ if x is not in A;

$0 < \mu_A(x) < 1$ if x is partly in A.

Fuzzy set A in X is defined by a set of ordered pairs

$$A = \{ (x, \mu_A(x)) \mid x \in X \}$$

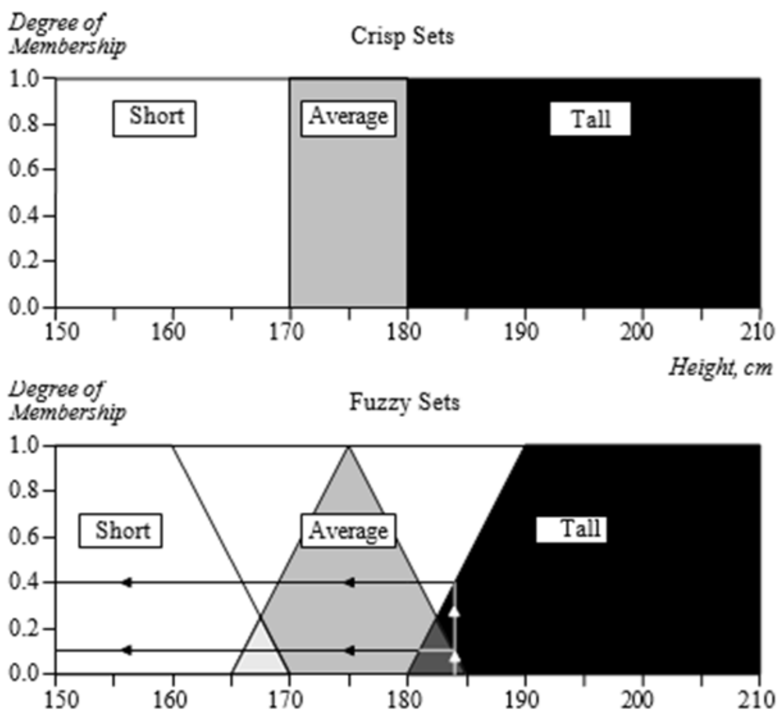
- **This definition of set allows a continuum of possible choices. For any element x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A. This degree, a value between 0 and 1, represents the degree of membership, also called membership value, of element x in set A.**

Fuzzy Set Representation

- **In our "tall men" example, we can define fuzzy sets of *tall*, *short* and *average* men.**

- **The universe of discourse for three defined fuzzy sets consist of all possible values of the men's heights.**

Height	Short	Average	Tall
150	1	0	0
155	1	0	0
160	1	0	0
165	0.45	0	0
170	0	0.45	0
175	0	1	0
180	0	0.45	0
185	0	0	0.45
190	0	0	1
195	0	0	1
200	0	0	1
205	0	0	1
210	0	0	1



- **For example, a man who is 184 cm tall is a member of the *average* men set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall* men set with a degree of 0.4**

❖ <u>Linguistic Variables</u>

- **A linguistic variable is a fuzzy variable. For example, the statement "John is tall" implies that the linguistic variable John takes the linguistic value tall.**
- **In fuzzy expert systems, linguistic variables are used in fuzzy rules.**

For example:

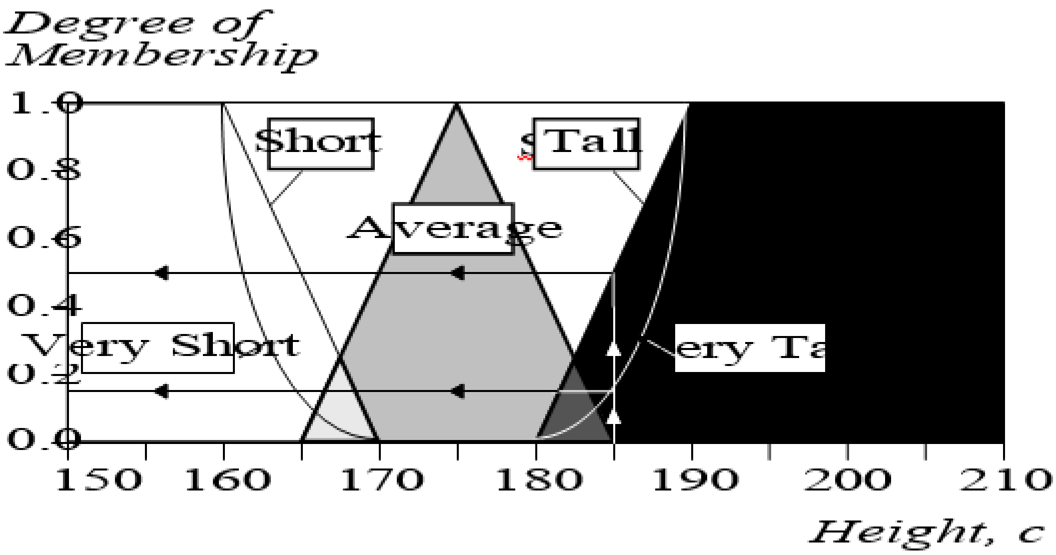
IF wind is strong THEN sailing is good

IF project_duration is long THEN completion_risk is high

IF speed is slow THEN stopping_distance is short

❖ Linguistic Variables and Hedges

- **The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.**
- **A linguistic variable carries with it the concept of fuzzy set qualifiers, called hedges.**
- **Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as very, somewhat, quite, more or less and slightly.**



❖ Fuzzy Sets & Fuzzy Rules

Complement

- **<u>Crisp Sets</u>: Who does not belong to the set?**
- **<u>Fuzzy Sets</u>: How much do elements not belong to the set?**
- **The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.**
- **If A is the fuzzy set, its complement $\neg A$ can be found as follows:**

$$\mu_{\neg A}(x) = 1 - \mu_A(x).$$

Containment

- **Crisp Sets: Which sets belong to which other sets?**
- **Fuzzy Sets: How much sets belong to other sets?**
- **Similar to a Chinese box, a set can contain other sets. The smaller set is called the subset. For example, the set of tall men contains all tall men; very tall men is a subset of tall men. However, the tall men set is just a subset of the set of men.**
- **In crisp sets, all elements of a subset entirely belong to a larger set.**
- **In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in the subset than in the larger set.**

Intersection

- **Crisp Sets: Which element belongs to both sets?**
- **Fuzzy Sets: How much of the element is in both sets?**
- **In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of tall men and the set of fat men is the area where these sets overlap.**
- **In fuzzy sets, an element may partly belong to both sets with different memberships.**
- **A fuzzy intersection is the lower membership in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:**

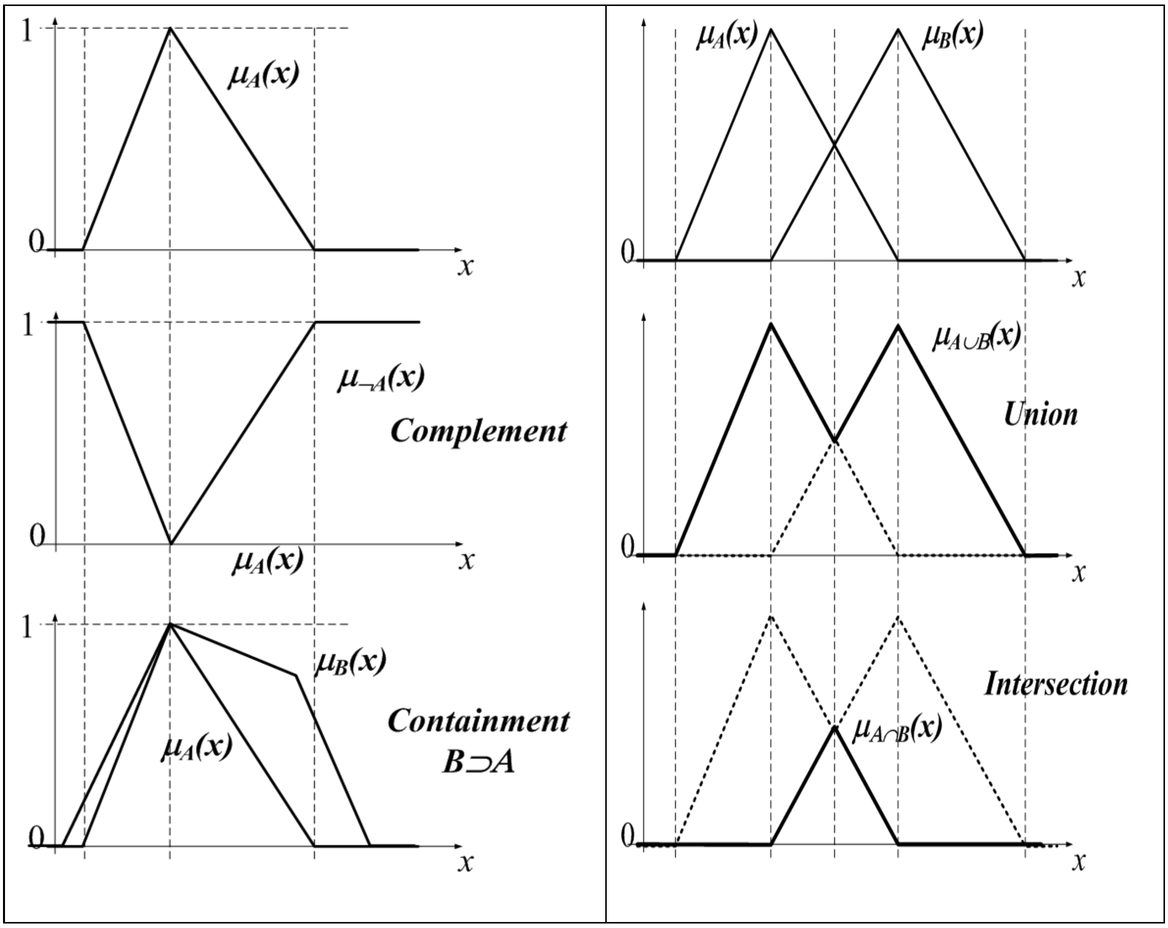
$$\mu_{A\cap B}(x) = \min\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x), \text{ where } x \in X.$$

Union

- **Crisp Sets: Which element belongs to either set?**
- **Fuzzy Sets: How much of the element is in either set?**
- **The union of two crisp sets consists of every element that falls into either set. For example, the union of tall men and fat men contains all men who are tall OR fat.**
- **In fuzzy sets, the union is the reverse of the intersection. That is, the union is the largest membership value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:**

$$\mu_{A\cup B}(x) = \max\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x), \text{ where } x \in X.$$

Operations of Fuzzy Sets



❖ Properties of Fuzzy Sets

Equality

- **Fuzzy set *A* is considered equal to a fuzzy set *B*, IF AND ONLY IF:**

  $\mu_A(x) = \mu_B(x), \forall x \in X$

- **Example: *A* = 0.3/1 + 0.5/2 + 1/3, *B* = 0.3/1 + 0.5/2 + 1/3,**

  therefore *A* = *B*.

Inclusion

- **Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:**

  $\mu_A(x) \leq \mu_B(x), \forall x \in X$

- Example: Consider $X = \{1, 2, 3\}$ and sets A

  and B $A = 0.3/1 + 0.5/2 + 1/3$;

  $B = 0.5/1 + 0.55/2 + 1/3$

  then A is a subset of B, or $A \subseteq B$

Cardinality

- Cardinality of a crisp (non-fuzzy) set Z is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

  $card_A = \mu_A(x_1) + \mu_A(x_2) + \dots \mu_A(x_n) = \Sigma\mu_A(x_i)$, for i=1..n

- Example: Consider $X = \{1, 2, 3\}$ and sets A

  and B $A = 0.3/1 + 0.5/2 + 1/3$;

  $B = 0.5/1 + 0.55/2 + 1/3$

  $card_A = 1.8$

  $card_B = 2.05$

Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:
  $\mu_A(x) = 0, \forall x \in X$

- Example: Consider $X = \{1, 2, 3\}$ and fuzzy set

  $A = 0/1 + 0/2 + 0/3$,

  then A is *empty.*

Alpha-Cut

- An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:

  $A_\alpha = \{\mu_A(x) \geq \alpha, \forall x \in X\}$.

- Example: Consider $X = \{1, 2, 3\}$ and set $A = 0.3/1 + 0.5/2$

  $+ 1/3$ then: $A_{0.5} = \{2, 3\}$, $A_{0.1} = \{1, 2, 3\}$, $A_1 = \{3\}$.

Fuzzy Set Normality

- **A fuzzy subset of X is called normal if there exists at least one element $x \in X$ such that $\mu_A(x) = 1$.**
- **A fuzzy subset that is not normal is called subnormal.**
- **All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.**
- **The height of a fuzzy set A is the largest membership grade of an element in**

  A $height(A) = \max_x(\mu_A(x))$.
- **Fuzzy set is called normal if and only if:**

  $height(A) = 1$.

Fuzzy Sets Core and Support

- **Assume A is a fuzzy set over universe of discourse X.**
- **The support of A is the crisp subset of X consisting of all elements with membership grade:**

  $$supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$$
- **The core of A is the crisp subset of X consisting of all elements with membership grade:**

  $$core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$$

❖ Fuzzy Sets Examples

- **Consider two fuzzy subsets of the set X, X = {a, b, c, d, e }**

  referred to as A and B

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e} and B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

- **Support:**

  $supp(A)$ = {a, b, c, d }

  $supp(B)$ = {a, b, c, d, e }

- **Core:**

  $core(A)$ = {a}

  $core(B)$ = {}

- **Cardinality:**

  $card(A)$ = 1+0.3+0.2+0.8+0 = 2.3

  $card(B)$ = 0.6+0.9+0.1+0.3+0.2 = 2.1

- **Complement:**

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  $\neg A$ = {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

- **Union:**

  $A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

- **Intersection:**

  $A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

- **α-cut:**

  $A_{0.2}$ = {a, b, c, d}

  $A_{0.3}$ = {a, b, d}

  $A_{0.8}$ = {a, d}

  A_{1} = {a}