Chapter 10
Sampling

Learning Objectives

- To represent a continuous time signal by a sequence of equally spaced samples.
- To establish the sampling theorem for exact reconstruction of the original signal.
- To reconstruct the original signal by means of low-pass filters.
- To choose the correct sampling rate to avoid aliasing.
- To study the important applications of sampling.

10.1 Introduction

Due to the dramatic development of digital technology in the recent past, continuous time signals are converted into discrete time signals, which are processed by discrete time systems and again converted back to continuous time signals which are applied to continuous time systems. A continuous time signal can be completely represented by and recovered from its values called samples at points equally spaced in time. This process is called sampling. The concept of sampling uses a discrete time system to implement continuous time systems and process continuous time signals. The information present in the sampled continuous time signal is retained in the discrete time signal also. While a sampled continuous time signal is represented by a sequence of impulses, a discrete time signal is represented by a sequence of numbers which carry the sample information as that of the sampled sequence.

In the discussion to follow, the concept of sampling and the process of reconstructing a continuous time signal from its samples are developed. The necessary condition under which a CT signal can be exactly reconstructed from its samples is established through the **Sampling Theorem**. Finally, the consequences that arise when the sampling theorem is not satisfied are discussed.

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10.2 The Sampling Process

Figure 10.1a shows the block diagram representation of a continuous signal $x(t)$ being multiplied by a periodic impulse train $\delta_T(t)$ to get the sampled output $g(t)$. The device used for this is called a **sampler**. The sampler is also represented by a switch which opens and closes with periodicity T_s. This is shown in Fig. 10.1b. The continuous time signal $x(t)$ is shown in Fig. 10.1c. The periodic impulse train $\delta_T(t)$ is shown in Fig. 10.1d. The product of $x(t)$ and $\delta_T(t)$ which is the sampled signal $g(t)$ is shown in Fig. 10.1e. Now we develop the sampling theorem as discussed below.

10.3 The Sampling Theorem

The signals $x(t)$, $\delta_T(t)$ and $g(t)$ shown in Fig. 10.1 are connected by the following equation:

$$g(t) = x(t)\delta_T(t) \tag{10.1}$$

where

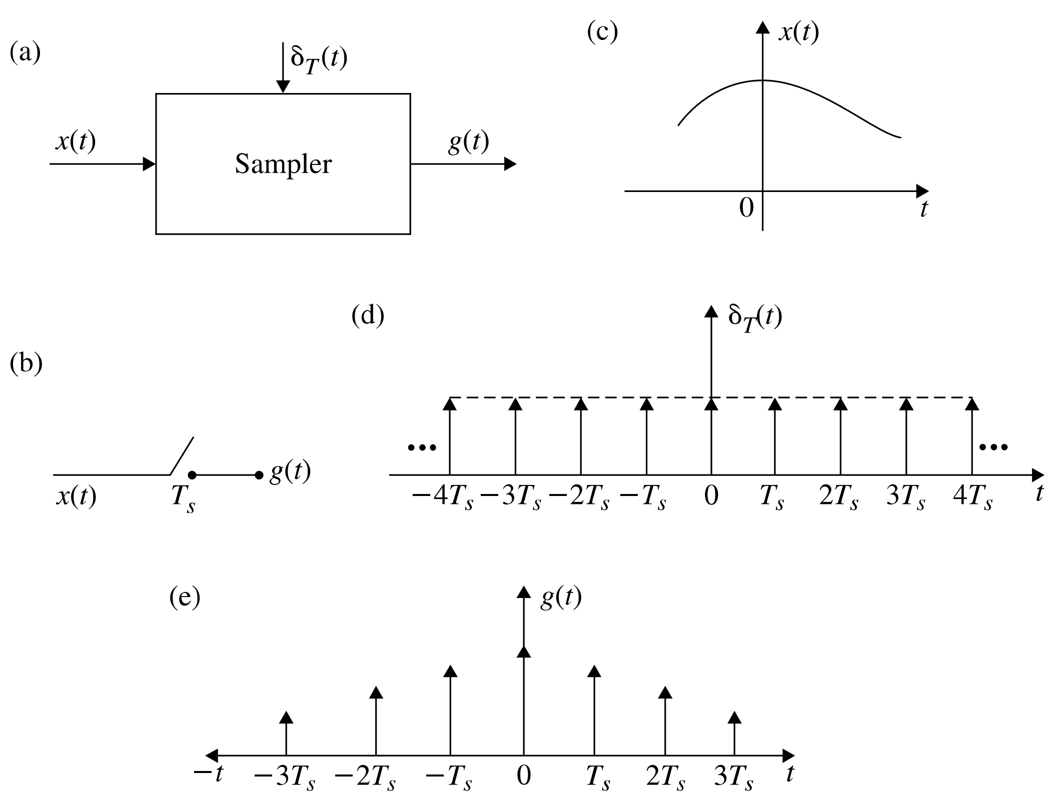

Fig. 10.1 The sampling process representation by schematic diagram and signals

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{10.2}$$

$\delta_{T_s}(t)$ is called a sampling function, T_s the sampling period and $\omega_s = \frac{2\pi}{T_s}$ the fundamental sampling frequency. This is also called radian frequency and is related to cyclic frequency as

$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

where $f_s = \frac{1}{T_s}$ is the cyclic frequency. The sampled signal $g(t)$ shown in Fig. 10.1e consists of impulses spaced every T_s seconds, which is the sampling interval. $\delta_T(t)$ can be expressed as a trigonometric Fourier series as

$$\delta_T(t) = \frac{1}{T_s}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \cdots] \tag{10.3}$$

$$\begin{aligned} g(t) &= x(t)\delta_T(t) \\ &= \frac{1}{T_s}[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + \cdots] \end{aligned} \tag{10.4}$$

From the knowledge of Fourier transform of continuous time signals, the following equations are written:

$$\begin{aligned} x(t) &\xleftrightarrow{\text{FT}} X(\omega) \\ 2x(t)\cos\omega_s t &\xleftrightarrow{\text{FT}} X(\omega - \omega_s) + X(\omega + \omega_s) \\ 2x(t)\cos 2\omega_s t &\xleftrightarrow{\text{FT}} X(\omega - 2\omega_s) + X(\omega + 2\omega_s) \\ g(t) &\xleftrightarrow{\text{FT}} G(\omega) \end{aligned} \tag{10.5}$$

Substituting Eq. (10.5) in Eq. (10.4), we get

$$\begin{aligned} G(\omega) = \frac{1}{T_s}[&X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) \\ &+ X(\omega - 2\omega_s) + X(\omega - 2\omega_s) + \cdots] \end{aligned} \tag{10.6}$$

$$\boxed{G(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)} \tag{10.7}$$

In Eq. (10.6), the first term in the bracket is $X(\omega)$. The second term is $X(\omega - \omega_s) + X(\omega + \omega_s)$. This represents the spectrum of $X(j\omega)$ shifted to ω_s and $-\omega_s$. Similarly, the third term $X(\omega - 2\omega_s) + X((\omega - 2\omega_s))$ which represents the

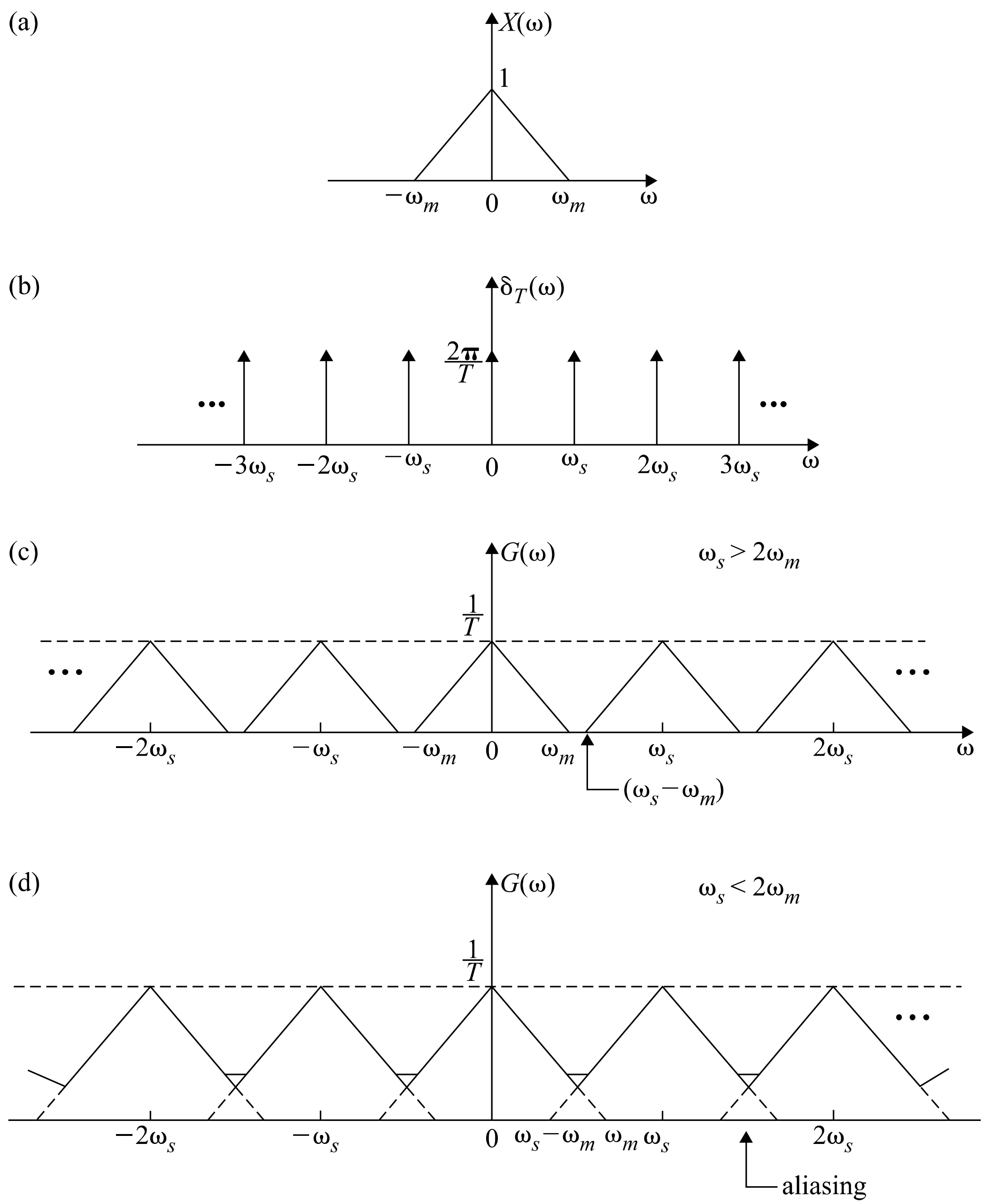

Fig. 10.2 Continuous signal, impulse train sampled signal spectrum

spectrum $X(\omega)$ shifted by $2\omega_s$ and $-2\omega_s$ and so on. The frequency spectrum $X(\omega)$ and $\delta(\omega)$ are represented in Fig. 10.2a and b, respectively. In Fig. 10.2a, ω_m is the maximum frequency content of the continuous time signal. Figure 10.2c represents the frequency spectrum $G(\omega)$ of the continuous sampled signal for $\omega_s > 2\omega_m$. Spectrum of the sampled signal $G(\omega)$ for $\omega_s < 2\omega_m$ is shown in Fig. 10.2d.

To reconstruct the continuous time signal $x(t)$ from sampled signal $g(t)$, we should be able to recover $X(\omega)$ from $G(\omega)$. This recovery is possible of there is no overlap between successive $G(\omega)$. From Fig. 10.2c, this is possible if $\omega_s > 2\omega_m$ or

$$\boxed{f_s > 2f_m} \tag{10.8}$$

The minimum sampling rate $f_s = 2f_m$ is called the **Nyquist rate** of $x(t)$, and the corresponding time interval $T_s = \frac{1}{f_s} = \frac{1}{2f_m}$ is called **Nyquist interval** of $x(t)$. Samples of a signal taken at its Nyquist rate are the **Nyquist samples**.

From Eq. (10.8), the Shannon sampling theorem or simply sampling theorem is stated as follows.

A band-limited signal of finite energy which has no frequency component higher than f_m can be completely described and recovered back if the sampling frequency is twice the highest frequency of the given signal.

The proof of the theorem is given in Eq. (10.7) and Fig. 10.2c. If $f_s < 2f_m$, Eq. (10.7) is represented in Fig. 10.2d, and here overlapping between successive samples occurs and therefore it is not possible to recover $x(t)$ from the frequency spectrum $G(\omega)$ when passed through low-pass filter.

10.4 Signal Recovery

If the condition $\omega_s > 2\omega_m$ is satisfied, $x(t)$ can be recovered exactly from $g(t)$ using an ideal low-pass filter whose characteristic is shown in Fig. 10.3d with a gain T_s and cutoff frequency greater than ω_m and less than $(\omega_s - \omega_m)$.

10.5 Aliasing

Consider the frequency spectrum of sampled signal $g(t)$ which has been obtained by sampling $x(t)$ with a sampling frequency $f_s < 2f_m$. The frequency spectrum of $G(\omega)$ is shown in Fig. 10.2d. When $f_s < 2f_m$, the signal is said to be under-sampled. The spectra located at $G(\omega_m)$, $G(\omega_m - \omega_s)$, $G(\omega_m - 2\omega_s)$, *etc.*, overlap on each other. When the high frequency interferes and appears as low frequency, then the phenomenon is called aliasing.

The effects of aliasing are as follows:

1. Distortion in signal recovery is generated when the high and low frequencies interfere with each other.
2. The data is lost and it cannot be recovered.

Different methods are available to avoid aliasing:

1. To increase the sampling rate f_s so that $f_s > 2f_m$.
2. To put anti-aliasing filter before the signal $x(t)$ is sampled.

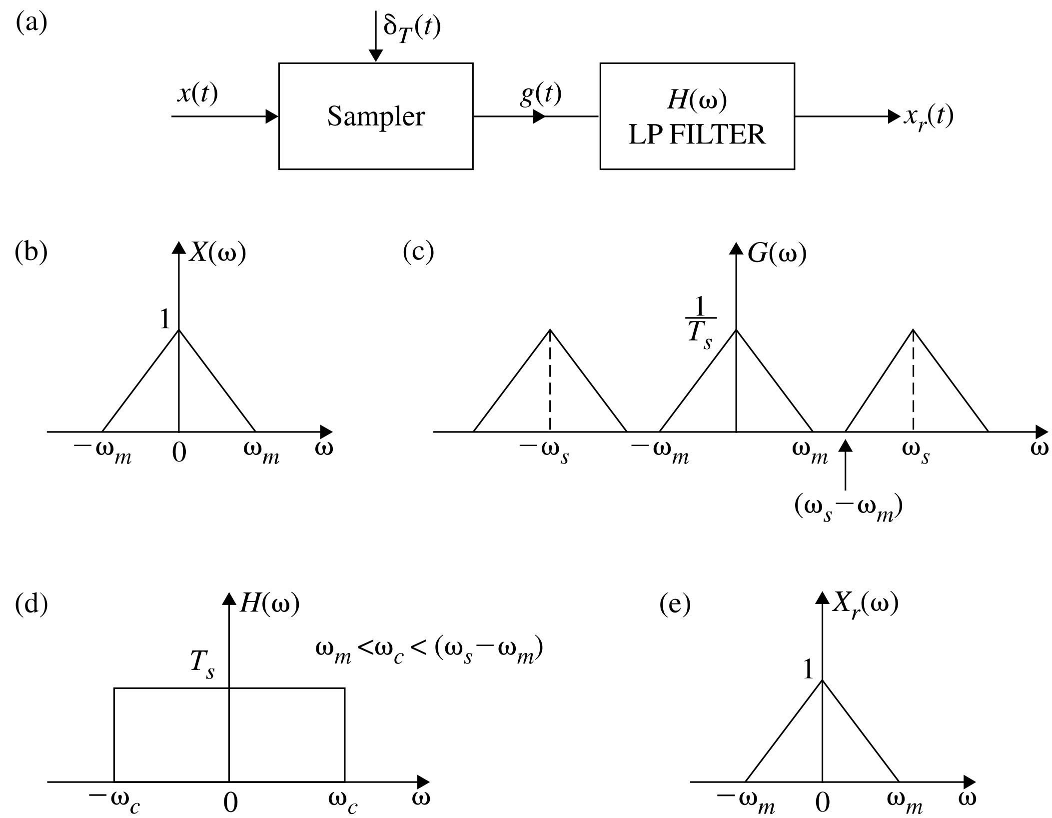

Fig. 10.3 Signal sampling and recovery. **a** Continuous signal for sampling and reconstruction; **b** Frequency spectrum of $x(t)$; **c** Frequency spectrum of sampled signal; (d) Characteristic of a low-pass filter; **e** Spectrum of recovered continuous signal

10.5.1 Sampling Rate ω_s Higher than $2\omega_m$

If the sampling rate $\omega_s > 2\omega_m$, the frequency spectrum of the sampled continuous signal is as shown in Fig. 10.2a and there is no overlapping between the samples and the original signal can be reconstructed without aliasing.

10.5.2 Anti-aliasing Filter

The anti-aliasing filter $H_{aa}(\omega)$ put before the sampler is shown in Fig. 10.4. $x(t)$ is the continuous signal and is passed through the anti-aliasing filter $H_{aa}(\omega)$ which gives the output $\bar{x}(t)$. The signal $\bar{x}(t)$ is passed through the sampler and recovered as $x_r(t)$. The continuous signal $x(t)$ is passed through an anti-aliasing filter whose cutoff frequency is $f_s/2$. All the frequency components of $x(t)$ beyond $f_s/2$ are eliminated before sampling of $x(t)$ is started. The anti-aliasing filter essentially band-limits the signal to $f_s/2$. By this, the components of $x(t)$ beyond $f_s/2$ are lost. However, these suppressed

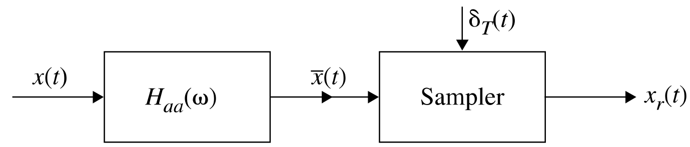

Fig. 10.4 Anti-aliasing filter before the sampler

components cannot corrupt the components of $x(t)$ whose frequency is less than $f_s/2$. Thus, the spectrum below $f_s/2$ remains intact and completely recovered. The noise produced by the aliasing is very much reduced when anti-aliasing filter is used. It also suppresses the entire noise spectrum beyond the frequency $f_s/2$.

10.6 Sampling with Zero-Order Hold

Band-limited signals which are sampled are narrow with large amplitude pulses. These impulses are difficult to generate and transmit. However, a more convenient method is to generate the sampled signal and send it through a zero-order hold. The input-output of a zero-order hold is shown in Fig. 10.5. The Zero-Order Hold (ZOH) samples $x(t)$ at a given time and holds those values until the next instant.

The input of the ZOH is shown in Fig. 10.5b and the output is shown in Fig. 10.5c. The output of the hold circuit is passed through a low-pass filter to recover the continuous time signal. The cascade-connected ZOH with a reconstruction filter is shown in Fig. 10.6.

The transfer function of ZOH is obtained from a unit step function with a time T shifted step being subtracted from that. Thus,

$$\begin{aligned} H_0(s) &= \frac{1}{s}[1 - e^{-sT}] \qquad (10.9) \\ &= \frac{e^{-sT/2}}{s}[e^{sT/2} - e^{-sT/2}] \end{aligned}$$

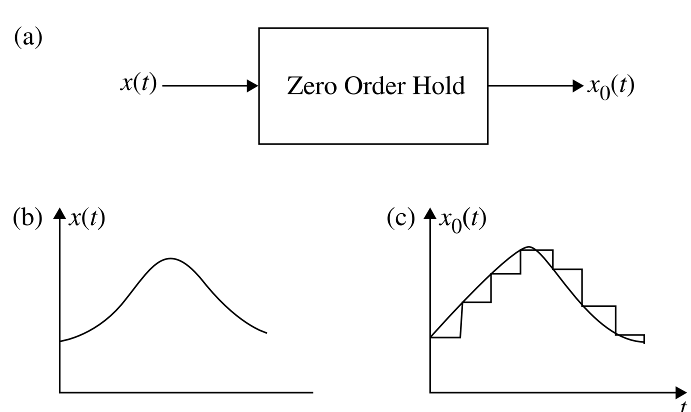

Fig. 10.5 Input-output of zero-order hold

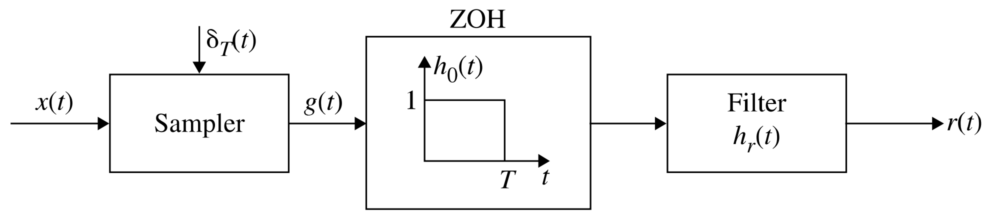

Fig. 10.6 Cascade-connected ZOH and reconstruction filter

The frequency response of the ZOH is therefore

$$\boxed{H_0(\omega) = e^{-j\omega T/2}\frac{\sin \omega T/2}{\omega}} \tag{10.10}$$

If we consider the frequency response of the ideal filter as $H(\omega)$, then

$$H_0(\omega)H_r(\omega) = H(\omega)$$

$$\boxed{H_r(\omega) = \frac{H(\omega)\omega e^{j\omega T/2}}{\sin\left(\frac{\omega T}{2}\right)}} \tag{10.11}$$

10.7 Application of Sampling Theorem

1. The sampling theorem is used in the analysis, processing and transmission of signals.
2. Processing the continuous time signal is equivalent to processing a discrete sequence of numbers, which ultimately leads to the area of digital filtering.
3. In the field of communication, the transmission of continuous time signals reduces to the transmission of a sequence of numbers. While doing so, the amplitude (PAM) of the sample, the width of the sample (PWM) or the position of the sample (PPM) can be varied and transmitted. At the receiver end, the pulse modulated signal is reconstructed and $x(t)$ is received. This process permits simultaneous transmission of several signals on a time-sharing basis which is called Time Division Multiplexing (TDM). By this, we can multiplex several signals in the same channel.
4. The transmission of digital signals is more rugged than that of analog signals because digital signals can withstand channel noise and distortion much better.

■ Example 10.1

Find the Nyquist rate and Nyquist interval for the following signals:

1. $x(t) = \sin 200\pi t$
2. $x(t) = 2 + 3\cos 100\pi t + 2\sin 200\pi t$
3. $x(t) = \dfrac{\sin 100\pi t}{\pi t}$
4. $x(t) = (\sin 200\pi t)^2$
5. $x(t) = \cos 200\pi t \cos 100\pi t$
6. $x(t) = \text{sinc}\, 2000\pi t$

Solution:

1. $\boldsymbol{x(t) = \sin 200\pi t}$

   Let $\omega_m = 200\pi$

$$f_m = \frac{\omega_m}{2\pi} = \frac{200\pi}{2\pi} = 100 \text{ Hz}$$
$$\text{Nyquist rate } f_s = 2f_m = 200 \text{ Hz}$$
$$\text{Nyquist width } W = \frac{1}{f_s} = \frac{1}{200} = 5 \text{ m.s.}$$

2. $\boldsymbol{x(t) = 2 + 3\cos 100\pi t + 2\sin 200\pi t}$

   Let $x_1(t) = \cos 100\pi t$ and $x_2(t) = \sin 200\pi t$

$$f_{m1} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$
$$f_{m2} = \frac{200\pi}{2\pi} = 100 \text{ Hz}$$
$$f_{m2} > f_{m1}$$
$$\text{Nyquist rate } f_s = 2f_{m2} = 200 \text{ Hz}$$
$$\text{Nyquist width } W = \frac{1}{f_s} = \frac{1000}{200} = 5 \text{ m.s.}$$

3. $\boldsymbol{x(t) = \dfrac{\sin 100\pi t}{\pi t}}$

   Let $\omega_m = 100\pi$

$$f_m = \frac{\omega_m}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$
$$\text{Nyquist rate } f_s = 2f_m = 2 \times 50 = 100 \text{ Hz}$$
$$\text{Nyquist width } W = \frac{1}{f_s} = \frac{1000}{100} = 10 \text{ m.s.}$$

4. $\boldsymbol{x(t) = (\sin 200\pi t)^2}$

   Let

$$\begin{aligned} x(t) &= \sin^2 200\pi t \\ &= \frac{1}{2}[1 - \cos 400\pi t] \\ \omega_m &= 400\pi \\ f_m &= \frac{400\pi}{2\pi} = 200 \text{ Hz} \\ \text{Nyquist rate } f_s &= 2f_m = 400 \text{ Hz} \\ \text{Nyquist width } W &= \frac{1}{f_s} = \frac{1000}{400} = 2.5 \text{ m.s.} \end{aligned}$$

5. $\boldsymbol{x(t) = \cos 200\pi t \cos 100\pi t}$

   Let

$$\begin{aligned} x(t) &= \frac{1}{2}[\cos(200 + 100)\pi t + \cos(200 - 100)\pi t] \\ &= \frac{1}{2}[\cos 300\pi t + \cos 100\pi t] \\ x_1(t) &= \cos 300\pi t \\ f_{m1} &= \frac{300\pi}{2\pi} = 150 \text{ Hz} \\ x_2(t) &= \cos 100\pi t \\ f_{m2} &= \frac{100\pi}{2\pi} = 50 \text{ Hz} \\ f_{m1} &> f_{m2} \\ \text{Nyquist rate } f_s &= 2f_{m1} = 2 \times 150 = 300 \text{ Hz} \\ \text{Nyquist width } W &= \frac{1}{f_s} = \frac{1000}{300} = \frac{10}{3} \text{ m.s.} \end{aligned}$$

6. $\boldsymbol{x(t) = \mathrm{sinc}\, 2000\pi t}$

   Let

$$\begin{aligned} \omega_m &= 2000\pi \\ f_m &= \frac{\omega_m}{2\pi} = \frac{2000\pi}{2\pi} = 1000 \text{ Hz} \\ \text{Nyquist rate } f_s &= 2f_m = 2000 \text{ Hz} \\ \text{Nyquist width } W &= \frac{1}{2000} = 0.5 \text{ m.s.} \end{aligned}$$

■ Example 10.2

Consider the signal

$$x(t) = \cos 2000\pi t + 10 \sin 10{,}000\pi t + 20 \cos 5000\pi t$$

Determine the (1) Nyquist rate for this signal and (2) If the sampling rate is 5000 samples per sec., then what is the discrete time signal obtained after sampling?

(*Anna University, May, 2007*)

Solution:

1.

$$\begin{aligned} x_1(t) &= \cos 2000\pi t \\ f_{m1} &= \frac{2000\pi}{2\pi} = 1000 \text{ Hz} \\ x_2(t) &= 10 \sin 10{,}000\pi t \\ f_{m2} &= \frac{10{,}000\pi}{2\pi} = 5000 \text{ Hz} \\ x_3(t) &= 20 \cos 5000\pi t \\ f_{m3} &= \frac{5000\pi}{2\pi} = 2500 \text{ Hz} \\ f_{m2} &> f_{m3} > f_{m1} \end{aligned}$$

Hence, the Nyquist rate

$$f_s = 2f_{m2} = 10{,}000 \text{ Hz}$$

2. The required sampling rate is 5000 samples per second. The maximum frequency of the given signal is 5000 Hz. The sampling rate is not equal to twice the maximum frequency content of the given signal and therefore aliasing occurs. The given signal cannot be recovered. $f_s = 5000$ Hz, $2f_m = 10{,}000$ and $f_s < 2f_m$.

10.8 Sampling of Band-Pass Signals

In the previous sections, we discussed the sampling theorem for low-pass signals. If the signal is band pass, the sampling theorem is stated as follows.

The band-pass signal $x(t)$ whose maximum bandwidth is $2W$ can be completely represented and recovered from the sample if it is sampled at a minimum rate of twice the bandwidth.

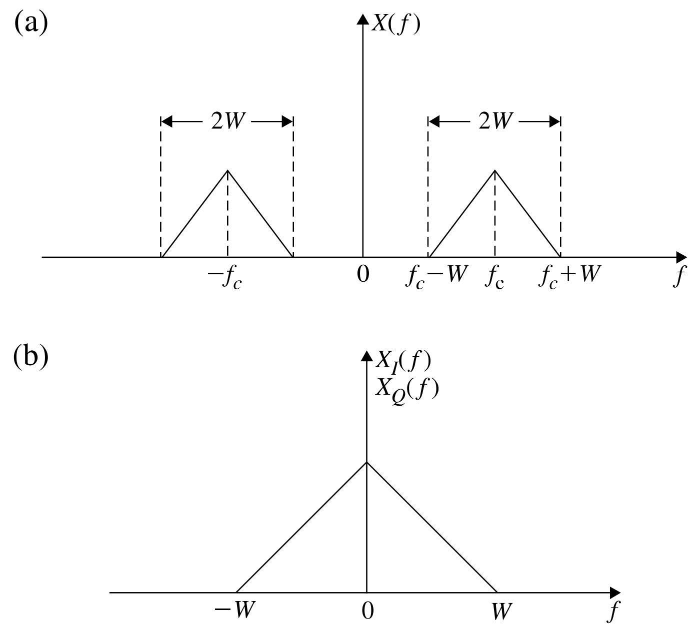

Fig. 10.7 Sampling of band-pass signal

Let $x(t)$ be the band-pass signal whose bandwidth is $2W$ as shown in Fig. 10.7a. The signal can be represented as impulse and quadrature components $x_I(t)$ and $x_Q(t)$. Thus, $x(t)$ which is centered around f_s can be represented as

$$x(t) = x_I(t)\cos 2\pi f_c t - x_Q \sin 2\pi f_c t \tag{10.12}$$

These two components are low pass in nature. Their spectrum is shown in Fig. 10.7b. The in phase and quadrature components are sampled at $f_s = 4W$ rate. The sampled $x_I(nT_s)$ and $x_Q(nT_s)$ are passed through their respective reconstruction filters and $x_I(t)$ and $x_Q(t)$ are reconstructed. The original $x(t)$ is obtained by multiplying $x_I(t)$ by $\cos 2\pi f_c t$ and $x_Q(t)$ by $\sin 2\pi f_c t$. $x(t)$ is obtained using the following relationship:

$$x(t) = x_I(t)\cos 2\pi f_c t + x_Q \sin 2\pi f_e t.$$

Summary

1. The sampling theorem states that a continuous time band-limited signal $x(t)$ can be sampled and reconstructed iff the sampling frequency is greater than twice the maximum frequency of the given signal.
2. If the sampling theorem is not satisfied while sampling $x(t)$, aliasing occurs and it is not possible to reconstruct the original signal. Further, the sampling process creates noise.

3. While the continuous time signal is sampled, if the high frequency of the spectrum interferes with the low frequency spectrum then the phenomenon is called aliasing.
4. The effect of aliasing is the distortion in the recovered signal and also loss of data.
5. To avoid sampling, sampling rate is increased such that the sampling rate (Nyquist rate) $f_s > 2f_m$. Aliasing is also avoided by putting an anti-aliasing filter just before the signal $x(t)$ is sampled.
6. Some of the applications of sampling include signal analysis processing and transmission.
7. The band-pass signal $x(t)$ whose maximum bandwidth is $2W$ can be completely represented and recovered if it is sampled at a minimum rate of twice the bandwidth.

Exercise

I. Short Answer Type Questions

1. **What is sampling theorem?**
   If a continuous time signal $x(t)$ is to be sampled and recovered, then the sampling frequency should be greater than twice the maximum frequency content of the signal. This is called sampling theorem.
2. **What is Nyquist rate and Nyquist interval?**
   The sampling frequency which is greater than twice the maximum frequency content of the signal to be sampled is called the Nyquist rate. The reciprocal of the Nyquist rate is called Nyquist width.
3. **What is aliasing?**
   When the continuous time signal is sampled and if it does not satisfy the sampling theorem, then the high-frequency spectrum of the sampled signal interferes and appears as low-frequency spectrum. This phenomenon is called aliasing.
4. **What are the effects of aliasing?**
   The effects of aliasing are as follows:
   1. Distortion in signal recovery is generated.
   2. The data is lost and the reconstruction of original signal becomes impossible.
5. **What are the methods available to avoid aliasing?**
   Aliasing can be avoided or minimized by increasing the sampling rate which satisfies the sampling theorem. Aliasing can also be minimized by putting anti-aliasing filters just before the signal is sampled.

6. **What is zero-order hold? What is its T.F.?**
   Band-limited signals which are to be sampled are passed through the zero-order hold. The ZOH samples the continuous time signal $x(t)$ at a given time and holds that value until the next instant. The T.F. of a ZOH is

$$H(s) = \frac{1}{s}\left[1 - e^{-sT}\right]$$

7. **What are the applications of the sampling theorem?**
   The sampling theorem is used in the analysis, processing and transmission of signals.
8. **State the sampling theorem as applied to band-pass signals.**
   The band-pass signal $x(t)$ whose maximum bandwidth is $2W$ can be completely represented and recovered from the sample if it is sampled at a minimum rate of twice the bandwidth. This is the sampling theorem as applied to band-pass signal.
9. **Find the Nyquist rate and Nyquist width for the signal given below.**

   1. $\boldsymbol{x(t) = 10\cos 2000\pi t + \sin 3000\pi t + 5\cos 1500\pi t}$
   2. $\boldsymbol{x(t) = (10\cos 100\pi t)^2}$
   3. $\boldsymbol{x(t) = (10\text{sinc } 2000\pi t)^2}$
   4. $\boldsymbol{x(t) = \cos 1000\pi t \cos 2000\pi t}$

   1. Nyquist rate $f_s = 3000$ Hz.
      Nyquist width $W = \frac{1}{3}$ m.s.
   2. Nyquist rate $f_s = 200$ Hz.
      Nyquist width $W = 5$ m.s.
   3. Nyquist rate $f_s = 4000$ Hz.
      Nyquist width $W = 0.25$ m.s.
   4. Nyquist rate $f_s = 3000$ Hz.
      Nyquist width $W = \frac{1}{3}$ m.s.

Appendix
Mathematical Formulae

A.1 Summation Formulae

1. $$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

2. $$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{(1-\alpha)} \qquad |\alpha| < 1$$

3. $$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{(1-\alpha)} \qquad |\alpha| < 1$$

4. $$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \qquad |\alpha| < 1$$

5. $$\sum_{n=0}^{\infty} n^2\alpha^n = \frac{\alpha^2+\alpha}{(1-\alpha)^3} \qquad |\alpha| < 1$$

6. $$\sum_{k=m}^{n} a^k = \frac{a^{n+1}-a^m}{a-1} \qquad a \neq 1$$

7. $$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

8. $$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

9. $$\sum_{k=n_2}^{k=n_1} (1)^k = n_2 - n_1 + 1$$

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A.2 Euler's Formula

1. $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$
2. $\cos\theta = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$
3. $\sin\theta = \frac{1}{2j}[e^{j\theta} - e^{-j\theta}]$

A.3 Power Series Expansion

1. $e^{\alpha} = \sum_{k=0}^{\alpha} \frac{\alpha^k}{\angle k} = 1 + \alpha + \frac{\alpha^2}{\angle 2} + \frac{\alpha^3}{\angle 3} + \cdots$
2. $(1+\alpha)^n = 1 + n\alpha + \frac{n(n-1)}{\angle 2}\alpha^2 + \cdots + \left(\frac{n}{k}\right)\alpha^k + \cdots$
3. $n(1+\alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \cdots + \frac{(-1)^{k+1}}{k}\alpha^k + \cdots$

A.4 Trigonometric Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
3. $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
4. $\sin 2\theta = 2\sin\theta\cos\theta$
5. $\cos 2\theta = 1 - 2\cos^2\theta$
6. $\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$
7. $\sin(a \mp b) = \sin a\cos b \pm \cos a\sin b$
8. $\sin a\sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
9. $\cos a\cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$
10. $\sin a\cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
11. $\cos a + \cos b = 2\cos\frac{(a+b)}{2}\cos\frac{(a-b)}{2}$