<u>Motivating Example:</u> Claim: I can predict the flip of a coin

Data: $X = 9, n = 10$: 90% correct.

Let θ be the probability gets it right

Model: $X \sim Bin(n, \theta)$

parameter of interest

Idea: estimate θ, $\hat{\theta} = x/n = 9/10$ and construct a confidence interval

$$\hat{\theta} \pm 1.96 \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} \approx (0.71, 1)$$

If we formalize the test:

$$H_0: \theta = 0.5 \quad H_A: \theta > 0.5$$

Assume the null is true, what is the prob of observing a $\hat{\theta}$ as or more extreme.

$$\text{p-value} = \mathbb{P}_0(X \geq 9) = \binom{10}{9}(1/2)^9(1/2)^{10-1} + \binom{10}{10}(1/2)^{10}(1/2)^{10-10}$$

$$= 11 \; \frac{1}{2^{10}} \approx 0.01$$

That is it is very unlikely that $X = 9$ given $\theta = 1/2$.

<u>Rmks:</u> (a) Our conclusions were based on a <u>model</u> – an abstraction or reductive approach that is meant to aid in prediction / or explain a phenomenon

(b) The model is probabilistic by nature – the outcomes are uncertain. Probability is the quantification of uncertainty

(c) The model distinguishes between observables & parameters.

$$\mathbb{P}(\text{DATA}; \text{Parameters})$$

(d) Statistics is probability in reverse

How do we incorporate "prior information"

idea: Consider θ as being a random quantity.

· In this way we can define $\mathbb{P}(\theta)$ a prior over Θ

that captures your prior belief.

Moreover we can formally define

$$\overbrace{\mathbb{P}(\theta \mid X)}^{\text{Posterior}} = \frac{\mathbb{P}(X \mid \theta)\, \mathbb{P}(\theta)}{\mathbb{P}(X)} \qquad \text{(Bayes Rule)}$$

$$\propto \underbrace{\mathbb{P}(X \mid \theta)}_{\text{Likelihood}} \underbrace{\mathbb{P}(\theta)}_{\text{Prior}}$$

<u>Ex</u>: $\theta \in \{0.25, 0.5, 0.75\}$

$$\mathbb{P}(X \mid \theta = 0.25) = \binom{10}{9}\left(\tfrac{1}{4}\right)^9 \left(\tfrac{3}{4}\right)^1 = 10\ \tfrac{3}{4^{10}}$$

$$\mathbb{P}(X \mid \theta = 0.5) = \dots = 10\ \tfrac{1}{2^{10}}$$

$$\mathbb{P}(X \mid \theta = 0.75) = \dots = 10\ \tfrac{3^9}{4^{10}}$$

If $\mathbb{P}(\theta = 0.25) = \mathbb{P}(\theta = .5) = \mathbb{P}(\theta = .75)$ noninformative prior

$$\mathbb{P}(\theta=0.75) = \frac{\mathbb{P}(X|\theta=.75)\,\mathbb{P}(\theta=.75)}{\sum_{t\in\Theta}\mathbb{P}(X,\theta=t)}$$

$$= \frac{10\,3^9/4^{10}\,\cancel{p}}{10\cdot\frac{3}{4^{10}}\cancel{p} + 10\frac{1}{2^{10}}\cancel{p} + 10\frac{3^9}{4^{10}}\cancel{p}} \simeq 0.95$$

Similar calculations show $\mathbb{P}(\theta=.5|X) \simeq 0.05$

<u>Rmk</u>: Also could calculate odds $\frac{\mathbb{P}(\theta=.75|X)}{\mathbb{P}(\theta=.5|X)} \simeq 19$ doesn't require marg. calculation

Alternatively we could weight the prior

$\mathbb{P}(\theta=.5) = 20p$, $\mathbb{P}(\theta=.25) = \mathbb{P}(\theta=.75) = p$

$$\frac{\mathbb{P}(\theta=.75|X)}{\mathbb{P}(\theta=.5|X)} = \frac{\mathbb{P}(X|\theta=.75)}{\mathbb{P}(X|\theta=.5)} \times \frac{\mathbb{P}(\theta=.75)}{\mathbb{P}(\theta=.5)} = \frac{10\,3^9/4^{10}\,p}{10\,1/2^{10}\,20p} \simeq 0.96$$

That is $\mathbb{P}(\theta=0.5|X) > \mathbb{P}(\theta=0.75|X)$