Exercise 6.1 Given as example of $\{\mathcal{F}_1^n\}_{n\geq 1}$ such that $\bigcup_{n=1}^\infty \mathcal{F}_1^n$ is a field but not a sigma-field.

Solution: Let $\Omega = \mathbb{N}$ and $X_k = k$ be the constant random variable. Then $\sigma(X_k) = \{\emptyset, \{k\}, \mathbb{N} \setminus \{k\}, \mathbb{N}\}$ and $\mathcal{F}_1^k = \sigma(X_1, \dots, X_k) = \sigma(1, 2, \dots, k)$. Let $\mathcal{F} = \bigcup_{k=1}^\infty \mathcal{F}_1^k$. We will now show that $\mathcal{F}$ is a field.

- Notice that $\mathbb{N} \in \mathcal{F}_1^k$ for all k. So $\mathbb{N} \in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A \in \mathcal{F}_1^k$ for some k. $\mathcal{F}_1^k$ is a sigma-field by definition so $A^C \in \mathcal{F}_1^k$ and $A^C \in \mathcal{F}$
- Let $A, B \in \mathcal{F}$, then there exists k_1 and k_2 such that $A \in \mathcal{F}_1^{k_1}$ and $B \in \mathcal{F}_1^{k_2}$. Npw, let $k = \max\{k_1, k_2\}$. Then, by construction $A \in \mathcal{F}_1^{k_1} \subset \mathcal{F}_1^k$ and $B \in \mathcal{F}_1^{k_2} \subset \mathcal{F}_1^k$. Since $\mathcal{F}_1^k$ is a sigma-field, $A \cup B \in \mathcal{F}_1^k$. Hence $A \cup B \in \mathcal{F}$

To see why $\mathcal{F}$ is not a *sigma*-field, we will show that it is not closed under countable unions. Let $2\mathbb{N}$ be the set of all even natural numbers. First note that for any given even number, $\{2n\} \in \mathcal{F}$. But notice that there does not exist a k such that $2\mathbb{N} = \bigcup_{n=1}^\infty 2n \in \mathcal{F}_1^k$. Therefore, $2\mathbb{N} \notin \mathcal{F}$ although it can be written as a countable union of elements in $\mathcal{F}$. Therefore $\mathcal{F}$ is not a sigma-field.

Exercise 6.2 Give an example of a sequence of nonindependent events in which the Borel-Cantelli Lemma II fails.

Solution: Consider $\Omega = [0,1]$ with the Lebesgue measure λ. Consider the sequence of events given by $A_n = (0, 1/n]$. To see why these events are not independent, notice that $\lambda(\bigcap_{n=1}^{\infty} A_n) = \lambda((0,1]) = 1$ and that

$$\prod_{i=1}^{\infty}\lambda(A_i) = \prod_{n=1}^{\infty} 1/n = \lim_{k\to\infty}\prod_{n=1}^{k} 1/n = \lim_{k\to\infty} 1/k! = 0$$

Therefore the events are not independent. Moreover, notice that

$$\sum_{i=1}^{\infty}\lambda(A_n) = \sum_{i=1}^{\infty}\frac{1}{n} = \infty$$

Having this, we look to calculate $\lambda(A_n, i.o)$. Recall that an event happening infinitely often corresponds to the $\limsup$ of sets. Measuring these sets we have

$$\lambda(A_n, i.o) = \lambda(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_n) = \lambda(\bigcap_{n=1}^{\infty} A_n) = \lambda(\emptyset) = 0$$

where the second equality is justified because for any given k, $A_k \supset A_{k+1} \supset A_{k+2} \supset \ldots$. Hence, if the events are not independent BC II may fail.

Exercise 6.3 Show that if an event happens infinitely often then it must be in $\mathcal{F}_{tail}$.

Solution: Let B be an event that occurs infinitely often. This means that the pre-image of B under the random variable X_k is in $\mathcal{F}$ for infinitely many X_k. Formally, for every $n \geq 1$ there exists $k \geq n$ such that $X_k^{-1}(B) \in \mathcal{F}$. Moreover we see $B \in \sigma(X_k) = \{A : X_k^{-1}(A) \in \mathcal{F}\}$. If $B \in \sigma(X_k)$, then it must be in $\sigma(X_n, X_{n+1}, \dots, X_{k-1}, X_k, X_{k+1}, \dots)$. This is due to the fact that $\sigma(W) \subset \sigma(W, Z)$ for any random variables W and Z. Recall that this was true for any given n. Hence we see

$$B \in \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots) = \mathcal{F}_{tail}$$

Exercise 6.4 Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(0) = 0$, $\lim_{x\to\infty} h(x) = \infty$, and h is strictly increasing. Define $k(x) = h^{-1}(x)$ to be a pointwise inverse of h. Show for any $a, b \in \mathbb{R}^+$

$$ab \leq \int_0^a h(x)dx + \int_0^b k(y)dy = H(a) + K(b)$$

Solution: First notice that we can rewrite the second integral as

$$\int_0^b k(y)dy = bk(b) - \int_0^{k(b)} h(y)dy$$

Using this form, we have

$$H(a) + K(b) = bk(b) + \int_{k(b)}^a h(x)dx$$

Now we break this problem into three case.

1. If $a < k(b)$ then

$$\int_{k(b)}^a h(x)dx = -\int_a^{k(b)} h(x)dx \geq -h(k(b))(k(b) - a) = -b(k(b) - a)$$

   The inequality is due to the fact that h is increasing. Thus, h attains it *maximum* on $(a, k(b))$ at $k(b)$. This implies that

$$bk(b) + \int_{k(b)}^a h(x)dx \geq bk(b) - bk(b) + ab = ab$$

2. If $a > k(b)$ then

$$\int_{k(b)}^a h(x)dx \geq h(k(b))(a - k(b)) = ab - bk(b)$$

   This is due to the fact that h is increasing and attains is *minimum* value at $k(b)$. Putting this together we have

$$bk(b) + \int_{k(b)}^a h(x)dx \geq bk(b) + ab - bk(b) = ab$$

3. If $k(b) = a$ then $\int_{k(b)}^a h(x)dx = 0$. So we have

$$bk(b) + \int_{k(b)}^a h(x)dx = bk(b) = ab$$

   Therefore, we attain equality when $k(b) = a$.

Exercise 6.5 For p, q conjugates, show that for any $a, b \in \mathbb{R}^+$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Solution: Using this result in 5.4, let $h(x) = x^{p-1}$. Then $k(y) = y^{1/(p-1)}$. Using our previous result we have

$$ab \leq \int_0^a x^{p-1}dx + \int_0^b y^{1/(p-1)}dy = \frac{x^p}{p}\Big|_0^a + \frac{y^{p/(p-1)}}{p/(p-1)}\Big|_0^b = \frac{a^p}{p} + \frac{b^{p/(p-1)}}{p/(p-1)}$$

Now notice by definition $\frac{1}{p} + \frac{1}{q} = 1$. Solving for q we see

$$p + q = qp \Longrightarrow p + q - qp = 0 \Longrightarrow q(1-p) = -p \Longrightarrow q = \frac{p}{p-1}$$

Therefore we see

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Exercise 6.6 Show Young's Inequality holds for $p = 1$, $q = \infty$.

Solution:

1. If $b < 1$, then $b^q = 0$ and $\frac{b^q}{q} = 0$. Thus

$$\frac{a^p}{p} + \frac{b^q}{q} = a \geq ab$$

2. if $b = 1$, then $b^q = 1$ and $\frac{b^q}{q} = 0$. Thus

$$\frac{a^p}{p} + \frac{b^q}{q} = a = ab$$

3. If $b > 1$ then $b^q = \lim_{n\to\infty} b^n = \infty$. Then using L'Hospital's Rule we have

$$\frac{b^q}{q} = \lim_{n\to\infty} \frac{b^n}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{nb^{n-1}}{1} = \infty$$

Thus

$$\frac{a^p}{p} + \frac{b^q}{q} = \infty \geq ab$$

is the trivial upper bound on ab.

Exercise 6.7 Use Cauchy-Swartz to prove that $|\rho| \leq 1$.

Solution: Recall that

$$|\rho| = \left|\frac{\mathbb{E}\Big[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\Big]}{\sqrt{\mathbb{E}(X-\mathbb{E}(X))\mathbb{E}(Y-\mathbb{E}(Y))}}\right| \leq \frac{\mathbb{E}\Big|(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\Big|}{\sqrt{\mathbb{E}(X-\mathbb{E}(X))\mathbb{E}(Y-\mathbb{E}(Y))}}$$

We now apply Cauchy-Swartz to $X-\mathbb{E}(X)$ and $Y-\mathbb{E}(Y)$ to attain

$$\begin{aligned}
|\rho| &\overset{C.S.}{\leq} \frac{\sqrt{\mathbb{E}\Big[(X-\mathbb{E}(X))^2\Big]}\sqrt{\mathbb{E}\Big[(Y-\mathbb{E}(Y))^2\Big]}}{\sqrt{\mathbb{E}(X-\mathbb{E}(X))}\sqrt{\mathbb{E}(Y-\mathbb{E}(Y))}}\\
&= \frac{\sqrt{\mathbb{E}\Big[X^2-2X\mathbb{E}(X)+E(X)^2\Big]}\sqrt{\mathbb{E}\Big[Y^2-2Y\mathbb{E}(Y)+E(Y)^2\Big]}}{\sqrt{\mathbb{E}(X-\mathbb{E}(X))}\sqrt{\mathbb{E}(Y-\mathbb{E}(Y))}}\\
&= \frac{\sqrt{\mathbb{E}(X^2)-\mathbb{E}(X)^2}\sqrt{\mathbb{E}(Y^2)-\mathbb{E}(Y)^2}}{\sqrt{Var(X)}\sqrt{Var(Y)}}\\
&= \frac{\sqrt{Var(X)}\sqrt{Var(Y)}}{\sqrt{Var(X)}\sqrt{Var(Y)}}\\
&= 1
\end{aligned}$$