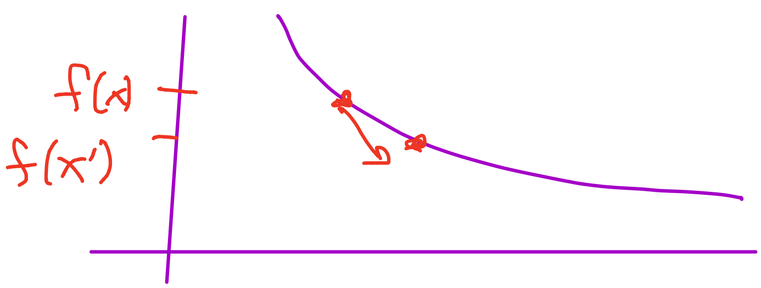
How to Solve LPs & SDPs

(First order MWU)

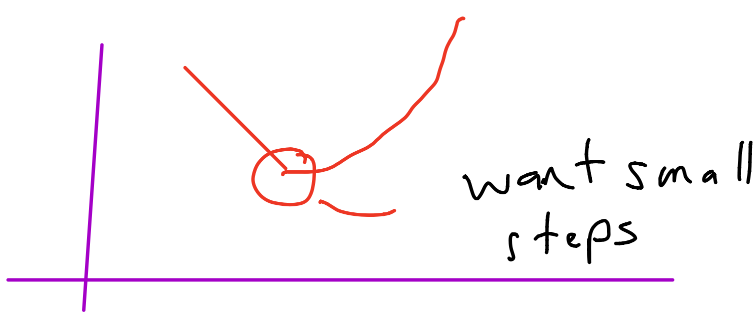
Gradient Descent

- Useful when f is convex and diff. and is "smooth"

$$\|\nabla f(x) - \nabla f(y)\| \leq L \cdot \|x - y\|$$



$$x' = x - \eta \nabla f(x)$$



Discretization of

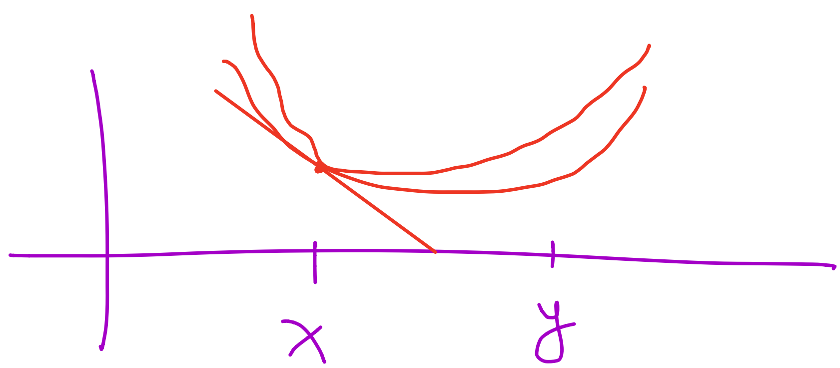
$$\dot{X} = -\nabla f(x) \quad \begin{bmatrix} \text{gradient} \\ \text{flow} \end{bmatrix}$$

If $\nabla f(x)$ is Lipsitz then there exits a unique $X : [0, \varepsilon) \to \mathbb{R}^n$ satisfying the above.

When the gradient is Lipschidz we can integrate to show

$L \Rightarrow \forall\, x, y$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2$$



Ex: $x' = x - \frac{1}{L}\nabla f(x)$ (for ℓ_2)

$$f(x') \le f(x) - \frac{\|\nabla f(x)\|_2^2}{2L}$$

Pf: For a general norm

$$x' = \min_y \{ \langle \nabla f, y - x \rangle + \frac{L}{2}\|x-y\| \}$$

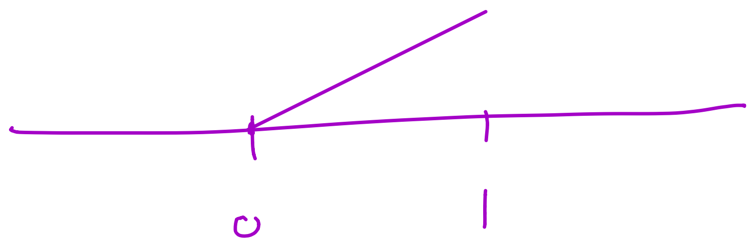
$$= \frac{1}{2L}\|\nabla f\|_*^2$$

Pf: $f(x') \le f(x) + \langle \nabla f, x'-x \rangle + \frac{L}{2}\|x'-x\|^2$

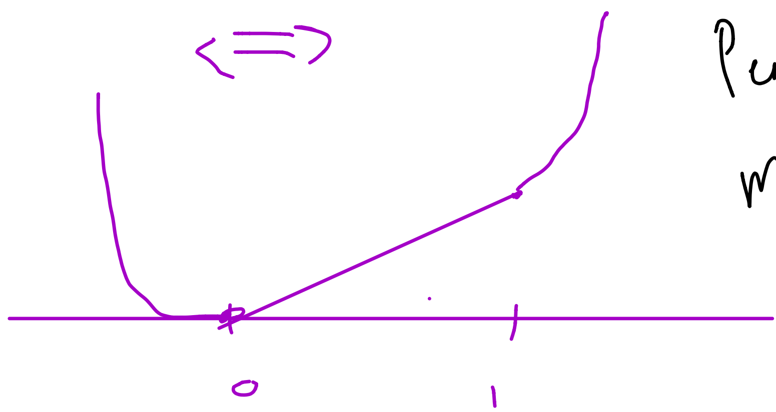
Constrained: $\underbrace{f : X}_{\text{polytope}} \longrightarrow \mathbb{R}$

In convex analysis

$$f(z) = \begin{cases} +\infty & z \notin X \\ f(z) & \text{o.w.} \end{cases}$$



$\Longleftrightarrow$



Penalty method

LP - feasibility

feasibility if $b < 0$ $\langle a/b, x \rangle \geq 1$

$Ax \leq 1$ (in general $Bx \geq 1$)

Dual Problem:

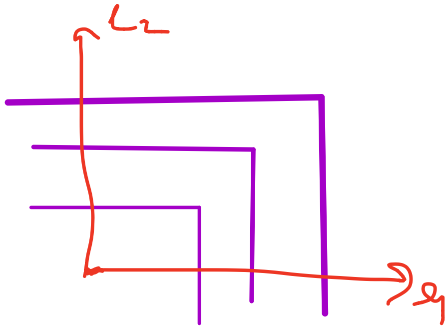
$$\min_{x \in X} \max_{1 \le i \le n} \langle a_i, x \rangle$$

$$=$$

$$\min_{x \in X} \underbrace{\max_{p \in S_m} p^T A x}_{\text{nonsmooth problem.}} = f(x)$$

Subgradient.

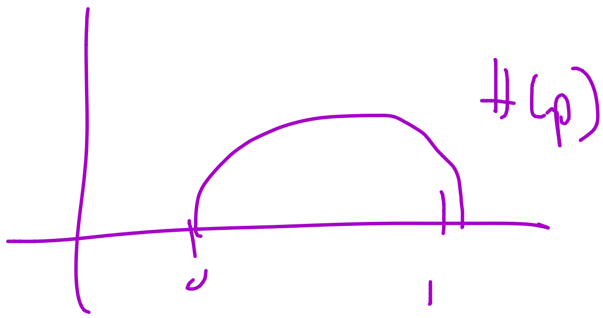
$$\partial f(x) = A^T \operatorname*{argmax}_{p \in \Delta_m} p^T A x$$



Regularization / Smoothing

Define a "smooth f given by

$$f_\mu(x) = \max_{p \in S_n} pA^T x + \mu H(p)$$



$$\underbrace{= \mu \log \sum_{i=1}^{m} e^{-\frac{(Ax)_i}{\mu}}}$$

$$\max (Ax)_i \leq \text{Softmax}_\mu(Ax) \leq \max(Ax)_i + \mu \log n$$

softmax function is μ smooth wrt to $\|\cdot\|_\infty$