Lecture: 9/13/18

Todays Topic: Bounds on covergence of random walks + averaging process through the laplacian quadratic form.

Continuous Random Walk

Characterized by the diff eq.

$$\frac{d\, p(t)}{dt} = -(I - W)\, p(t)$$

Ex: Regular Graph $W = \frac{A}{d}$

let $W = U \Lambda U^T$ then

$$\frac{d\, U^T p(t)}{dt} = -\Lambda \left(U^T p(t)\right)$$

Or in other notation

$$\forall i \quad \frac{d\,(U^T p_i(t))}{dt} = -\lambda_i\,(U^T p_i(t))$$

Recall $\quad \frac{dy}{dt} = -\lambda y \Rightarrow y = y(0)e^{-\lambda t}$

So $\forall i$

$$U^T p_i(t) = U^T p_i(0)\, e^{-\lambda_i t}$$

or in matrix notation

$$U^T p(t) = \mathrm{diag}(e^{-\lambda_i})\, U^T p(0)$$

$$\boxed{p(t) = U\, \mathrm{diag}(e^{-\lambda_i})\, U^T p(0)}$$

Def: For a symmetric $Y = U \Lambda U^T$

then $e^Y \equiv U \mathrm{diag}(e^{\lambda_i}) U^T$ is

the matrix exponential

Rmk: Keep eigenvectors with eigenvalues

$\{e^{\lambda_i} : 1 \leq i \leq n\}$

So the RW is characterized by $p(t) = U e^{-(I - \frac{A}{d})} U^T p(0)$

Def: We can also write the matrix exponential

$$e^Y = U^T \operatorname{diag}(e^{-\lambda_i}) U$$

$$= U^T \operatorname{diag}\left(\sum_{n=0}^{\infty} \frac{(\lambda_i)^n}{n!}\right) U$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} U^T \operatorname{diag}(\lambda_i^n) U$$

$$= \sum_{n=0}^{\infty} \frac{Y^n}{n!}$$

For a regular graph $W = \frac{A}{d}$

CRW: $e^{-(I-W)t} \overset{*}{=} e^{-t} e^{tW}$

* not true in general just here

$$= e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} W^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{t^n e^{-t}}{n!}}_{\text{Poisson Process probabilities}} \underbrace{W^n}_{n\text{-steps in a RW with ptm } W}$$

Now considering non regular graphs

$$\frac{d\, p(t)}{dt} = -(I - AD^{-1})p(t)$$

$$\frac{d\, D^{-1/2} p(t)}{dt} = -D^{-1/2}(D^{-1/2} - AD^{-1/2})D^{-1/2} p(t)$$

$$= -(I - D^{-1/2} A D^{-1/2})\, p(t)$$

$$= -\underbrace{D^{-1/2}(D-A)D^{-1/2}}_{\mathcal{L} = \text{ normalized Laplacian}}\, p(t)$$

$\mathcal{L}$ is like the symmetrized version of $(I - W)$.

$$\frac{d\, D^{-1/2} p(t)}{dt} = -\mathcal{L} D^{-1/2} p(t)$$

$$D^{-1/2} p(t) = e^{-t\mathcal{L}} D^{-1/2} p(0)$$

$$\boxed{p(t) = D^{1/2} e^{-t\mathcal{L}} D^{-1/2} p(0)}$$

So now if we want to measure distance from stationary.

$$\frac{d}{dt} \|p - \pi\|_{D^{-1}}$$

$$= \frac{d}{dt} \left\| D^{1/2} e^{-t\mathcal{L}} D^{-1/2} \left(p(0) - \pi \right) \right\|_{D^{-1}}^2$$

$$= - \underbrace{\{p(0) - \pi\} D^{-1/2} e^{-t\mathcal{L}}}_{D^{-1/2} p(t)} \mathcal{L} \underbrace{e^{-t\mathcal{L}} D^{-1/2} (p(0) - \pi)}_{D^{-1/2} p(t)}$$

$$= -\left(D^{-1/2} p(t)\right)^T \mathcal{L} \left(D^{-1/2} p(t)\right)$$

$$= -\left(D^{-1}(p(t) - \pi)\right)^T L \left(D^{-1}(p(t) - \pi)\right)$$

Again we see this is the quadratic form of the original graph Laplacian.

Bounding

- Look at how fast we are

decreasing compared to rate until fixed point.

$$\frac{\frac{d}{dt}\|p'(t)-\pi\|^2_{D^{-1}}}{\|p(t)-\pi\|^2_{D^{-1}}} \leq -c$$

then we can bound

$$\frac{d}{dt}\left\{\log \|p(t)-\pi\|^2_{D^{-1}}\right\} \leq e^{-c}$$

So computing

$$\frac{\frac{d}{dt}\|p(t)-\pi\|^2_{D^{-1}}}{\|p(t)-\pi\|^2_{D^{-1}}}$$

$$= \frac{(p(t)-\pi)D^{-1/2}\mathcal{L}D^{-1/2}(p(t)-\pi)}{\quad}$$

$$(p(t)-\pi) D^{-1/2}\, I \, D^{-1/2} (p(t)-\pi)$$

$$\leq - \min_{(D^{1/2}y)^T 1 = 0} \frac{y^T \mathcal{L} y}{y^T y} = \boxed{\text{Spectral Gap}}$$

Rmk: $(p(t)-\pi)1 = \sum p(t) - \sum \pi = 0$

So we require $(D^{1/2}y)^T 1 = 0$

to restrict the space of candidate

y's.

Thrm: If λ_2 is the 2nd smallest eigenvalue of $\mathcal{L}$. Then

$$\|p(t)-\pi\|^2_{D^{-1}} \leq e^{-t\lambda_2} \|p(0)-\pi\|^2_{D^{-1}}$$

I.e. the spectral gap

$$\lambda_2 := \min \quad \frac{y^T \mathcal{L} y}{}$$

$(D^{1/2}y)^T \mathbb{1} = 0 \qquad y^T y$

Ex: What is the spectral gap of K_n.

$L(K_n) = nI - \mathbb{1}\mathbb{1}^T$

$D(K_n) = (n-1)I$

$\mathcal{L} = \frac{n}{n-1} I - \frac{1}{n-1}\mathbb{1}\mathbb{1}^T$ Take off rank 1

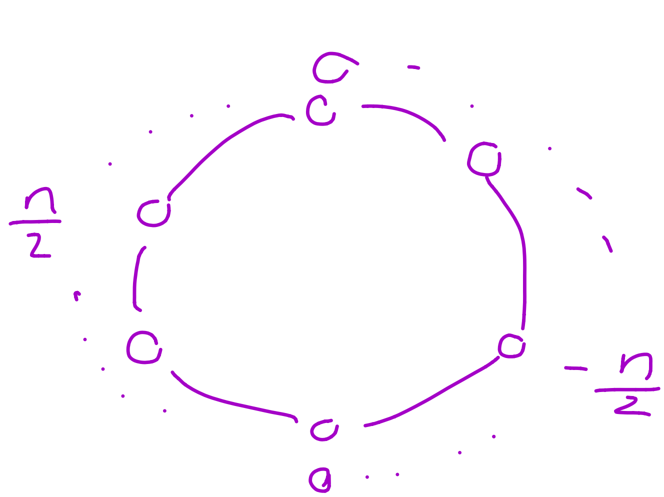
$\Rightarrow \lambda_2 = \frac{n}{n-1} = 1 + \frac{1}{n-1}$

(maximal).

Ex: Cycle of length n

$$\min_{x^T\mathbb{1}=0} \frac{x^T L x}{x^T D x} = \frac{\sum (x_i - x_{i+1})^2 + (x_n - x_1)^2}{2x^T x}$$

$\geq \Omega(n^3)$



$\leq O\left(\frac{1}{n^2}\right)$ worst case Senario for Convergence.