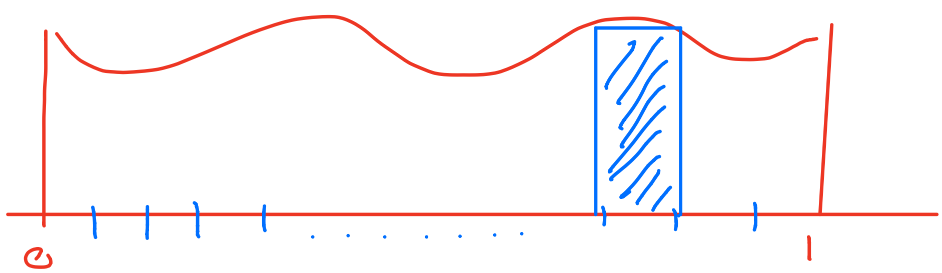
Monte Carlo Simulation

Goal: Suppose we want to compute

$$I = \int_0^1 g(x)\,dx$$

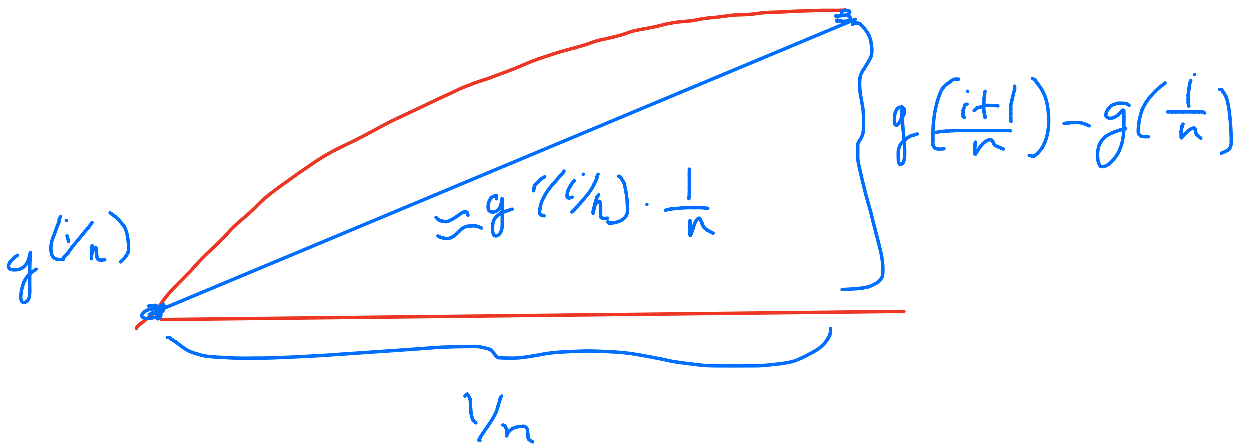
for some complicated function on $[0,1]$.

Simple Approach



$$I \approx \sum_{i=1}^{n} g(i/n) \cdot \frac{1}{n}$$

For each cell



So the error is roughly

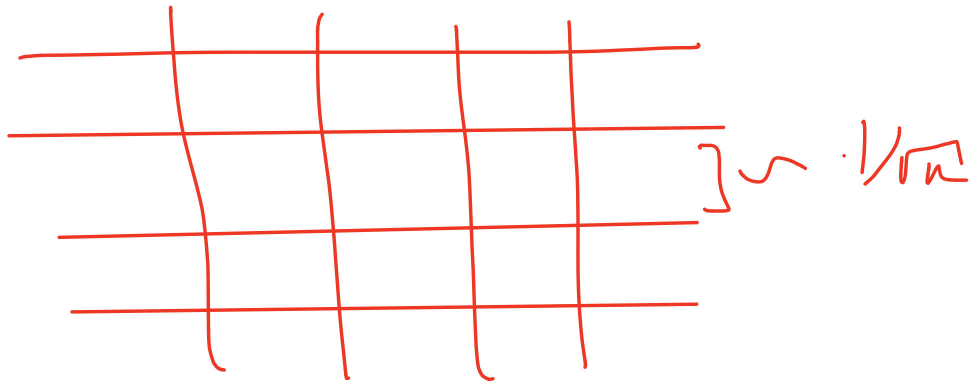
$$\frac{1}{2}\underbrace{g'(i/n)\cdot\frac{1}{n}}_{\text{Height}}\cdot\underbrace{\frac{1}{n}}_{\text{Base}} = O(1/n^2)$$

But if we increase the dimensionality of this problem

$$g : [0,1]^d \longmapsto \mathbb{R}$$

$$I = \int_{[0,1]^d} g(\vec{x})\, d\vec{x}$$

Here the relative error is based on the hyper grid



and in general the numerical rate error is given by

$O(1/n^d)$

Therefore to standardize the error rate we need 2^d more cells to get back to the $O(1/n^2)$ error rate.

What if instead we do

The following

$$X_1, \ldots, X_n \sim \text{iid } U(0,1)^d$$

$$\hat{I} = \frac{1}{n} \sum_{i=1}^{n} g(x_i)$$

$$\mathbb{E}(\hat{I}) = \frac{1}{n} \sum_{i=1}^{n} \int_{[0,1]^d} g(x_i)\,dx$$

$$= \frac{1}{n} \sum_{i=1}^{n} I = I$$

in this case it is "unbiased"

$$\text{(i)} \quad \hat{I} = \frac{1}{n} \sum g(x_i) \xrightarrow{a.s.} I$$

$$\text{(ii)} \quad \frac{\hat{I} - I}{\sqrt{\frac{\mathrm{Var}(g(x))}{n}}} \xrightarrow{D} N(0,1)$$

Thus we can get asym. intervals

$$\hat{I} \pm z_{\alpha/2} \frac{\sqrt{\mathrm{Var}(g(x))}}{\ }$$

$\sqrt{n}$

So the error is $O(1/\sqrt{n})$ independent of d. $\boxed{\text{Monte Carlo Integration}}$

Rmk: There are much better ways of preforming numerical int. by the overall message is we still have issues in high dimensions.

Rmk: We can controll the error

$$e = z_{\alpha/2}\sqrt{\frac{Var(g(x))}{n}} \Rightarrow n = \frac{z_{\alpha/2}^2 \, Var(g(x))}{e^2}$$

$$\widehat{Var(g(x))} = \frac{1}{n}\sum_{i=1}^{n}\left[g(x_i) - \hat{I}\right]^2$$