Review: · The role of the laplacian

- Continuous random walk over the network

  - $e^{-t(I-W)}$ heat kernel

  - $\frac{d\,p(t)}{dt} = -(I-W)p(t)$

- What we showed last time the convergence to station is controlled

$$\|p(t)-\pi\|_{D^{-1}}^2 \leq e^{-t\lambda_2} \|p(0)-\pi\|_{D^{-1}}^2$$

If we have a regular graph we have the nice correspondance

$$p(t)-\pi(t) = e^{-t\frac{L}{a}}(p(0)-\pi)$$

and $L = \sum \lambda_i v_i v_i^T$ then

$$p(t) - \pi = \sum_{i=1}^{n} e^{-\frac{t\lambda_i}{d}} \left[v_i^T (p(0) - \pi) \right]^2$$

$$= \sum_{i=2}^{n} e^{-\frac{t\lambda_i}{d}} \left[v_i^T (p(0) - \pi) \right]^2 + \underbrace{\left[1^T (p(0) - \pi) \right]^2}_{=0}$$

$$= \sum_{i=2}^{n} \underbrace{e^{-\frac{t\lambda_i}{d}}}_{\substack{\text{rate of convergence} \\ \text{along the } v_i \text{ direction.}}} \left[v_i^T (p(0) - \pi) \right]^2$$

What if the graph is disconnected?

<u>Lemma</u>: $\lambda_2 = 0$ iff G is disconnected.

<u>Pf</u>: $\lambda_2 = \min\limits_{x \perp 1} \dfrac{x^T L x}{x^T D x}$

Suppose $G =$ (S) ($\bar{S}$) and let

$$x_i = \begin{cases} \frac{1}{|S|} & i \in S \end{cases}$$

so $x^T 1 = \frac{|S|}{|S|} - \frac{|S|}{|S|}$

$$\left\{ \frac{-1}{|\bar{S}|} \quad \in \bar{S} \right. \qquad = 0$$

So looking at the rayleih qoient.

$$\lambda_2 = \min_{x \perp 1} \frac{x^T L x}{x^T D x} \leq \frac{\sum_{\in S} (x_i - x_j)^2 + \sum_{\in \bar{S}} (x_i - x_j)^2}{x^T D x}$$

$$= 0$$

So $\lambda_2 = 0$. For the back direction

$Lx = 0$, $x^T 1 = 0$ $x \neq 0$ implies

$$G = \quad x \geq 0 \qquad x \leq 0$$

So if there were an edge between the two,

$$0 = \lambda_2 = \frac{x^T L x}{} = \frac{\sum (x_i - x_j)^2}{} > 0$$

$x^T D x$ $\qquad$ $x^T D x$ #

Later: (Cheeger's Inequality)

If $\lambda_2 \leq \varepsilon$ there is a "small cut"

<u>Generalization</u>: If G has k connected components.

$$G = S_1 \quad S_2 \quad S_k$$

if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$

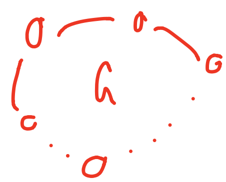
<u>Pf</u>: (tools) (<u>Courant-Fischer Thm</u>)

For the k-th smallest eigenvalue.

$$\lambda_k = \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \max_{x \in S} \frac{x^T L x}{x^T D x}$$

(Topic: Higher-Order Cheeger)

Bounds on λ_2

- Cycle  $\lambda_2 \leq O\left(\frac{1}{n^2}\right)$

- Using a test vector find x, $x^T \mathbb{1} = 0$

$$\min_{x^T \mathbb{1} = 0} \frac{x^T L x}{x^T D x} = \lambda_2 \leq \frac{x^T L x}{x^T D x} = O(1/n^2)$$

<u>Goal</u>: Find lower bounds to understand convergence rates better.

<u>Ex</u>: (path graph).

$$x^T L x = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2$$

given x_1, x_n can we find a lower

bound. So using a CS type bound

$$\left(\sum_{i=1}^{n-1}(x_i - x_{i+1})\right) = \left[\begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{n-1} - x_n \end{pmatrix}^T \mathbb{1}\right]^2$$

$$\| $$

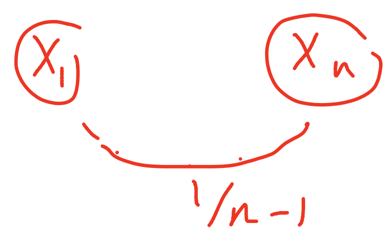
$$(x_1 - x_n)^2 \leq \left(\sum_{i=1}^{n-1}(x_i - x_{i+1})^2\right)(n-1)$$

So the path inequality is given by

$$\boxed{x^T L_P x \geq \frac{(x_1 - x_n)^2}{n-1}}$$

Notice that $\frac{(x_1 - x_n)^2}{n-1}$ is the laplacian

of the graph



Def: For symmetric matrices A, B,

then the psd ordering of A, B

$\forall x \quad x^T A x \geq x^T B x$ which we write

as $A \succcurlyeq B \qquad A - B \succcurlyeq 0$.

Graphic Inequalities

1. $(n-1) L_{path} \succcurlyeq L_{1n}$

Heuristic: think convergence rates of random walks.

- Speeding up the RW by a factor of $n-1$.

2. Q: $\lambda_2(L_{path}) \geq ??$

Compare to a graph with constant

$\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ True for complete graph

$$K_v = nI - 11^T$$

$$= \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix}$$

So $$\frac{L(K_v)}{n} = U \begin{pmatrix} \frac{1}{n} & & & \\ & \ddots & & \\ & & \frac{1}{n} & \\ & & & 0 \end{pmatrix} U^T$$

<u>Lower bounds on λ_2</u>

$$\circ - \circ - \circ - \cdots - \circ$$

$(n-1)L_P \succcurlyeq L_{1n}$ but in general

$$(j-i)L_P \succcurlyeq L_{ij}$$

So summing over all paths

$$\sum_{i<j} (j-i)L_P \succcurlyeq \sum_{i<j} L_{ij} = L(K_v)$$

Now notice

$$\sum_{i=1}^{n} \sum_{i<j} (j-i) L_p \leq \sum_{i=1}^{n} \sum_{i<j} (n-1) L_p$$

$$\lesssim O(n^3)$$

Hence

$$O(n^3)\lambda_2(L_{path}) \geq n$$

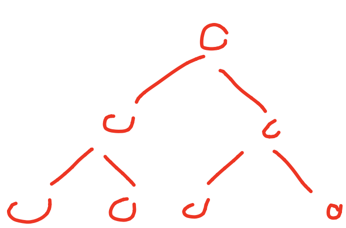
$$\lambda_2(L_{path}) \geq \Omega(1/n^2)$$

And from the test factor

$$\lambda_2(L_{path}) \leq O(1/n^2)$$

$$\lambda_2(L_{path}) \simeq \frac{c}{n^2}$$

<u>Ex:</u> Complete binary tree.



Same edge connection but RWs converge

faster as

Using the path inequality we will show

$$\lambda_2(T_n) \geq \frac{1}{n \log n} \qquad \lambda_2(T_n) \geq \Theta(1/n)$$

So while the tree and path are similar they converge at must different rates.