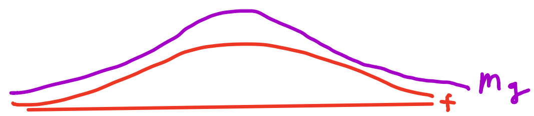
Unbiased MCMC

Rejection Sampling



If f, g are densities such that $f \leq Mg$

then we can translate samples of g to f

Algo: Repeat : $Y \sim g$, $u \sim \mathcal{U}(0,1)$

until $u Mg(Y) \leq f(Y)$

return Y

Maximal Coupling

Let p, q be two densities

Suppose that $q(x) = \min\left(1, \frac{q(x)}{p(x)}\right) p(x) + \left(1 - \int p \wedge q\right) \underbrace{\frac{q(x) - (p \wedge q)(x)}{1 - \int p \wedge q}}_{\text{density } g(x)}$

$(p \wedge q) \equiv \min(p(x), q(x))$

So then $q(x) = \min\left(1, \frac{q(x)}{p(x)}\right) p(x) + \left(1 - \int p \wedge q\right) g(x)$

Alg: Draw $Y \sim p$, $u \sim \mathcal{U}(0,1)$

if $u \leq \min\left(1, \frac{q(Y)}{p(Y)}\right)$, set $X = Y$

else $Z \sim g$ and set $X = Z$

Proposition: $X \sim q$

__Pf:__ $\mathbb{P}(X\in A) = \int \mathbb{P}(X\in A \mid Y=y, U=u)\, p(y)\, du\, dy$

$= \int \Big[\mathbb{1}_{(0,\alpha(y)]}(u)\, \mathbb{1}_A(y) + \mathbb{1}_{(\alpha(y),1)}(u) \int_A g(t)\,dt \Big] p(y)\, du\, dy$

$= \int \Big[\alpha(y)\, \mathbb{1}_A(y) + (1-\alpha(y)) \int_A g(t)\,dt \Big] p(y)\, dy$

$= \int_A \alpha(y)\, p(y)\, dy + \Big(1 - \int \alpha(y) p(y)\, dy\Big) \int_A g(t)\, dt$ $\blacksquare$

__Q:__ How do we sample from g? __A:__ Rejection sampling

$\alpha(x) = \min\left(1, q(x)/p(x)\right)$

$q(x) = \alpha(x)\, p(x) + \left(1 - \int p\wedge q\right) \dfrac{q(x) - (p\wedge q)(x)}{1 - \int p \wedge q}$

$g(x) \le \dfrac{q(x)}{1 - \int p\wedge q}$ $\Rightarrow$ Rejection method using $q(\cdot)$

The rejection condition reads: $\dfrac{U q(Y)}{1 - \int p\wedge q} \le \dfrac{q(Y) - (p\wedge q)(Y)}{1 - \int p\wedge q}$

i.e. $(1-U)\, q(Y) \ge (p\wedge q)(Y)$

__Algo:__ (a) Draw $Y \sim P$, $U \sim \mathcal{U}(0,1)$

(b) If $U \le \min\left(1, \dfrac{q(Y)}{p(Y)}\right)$ stop and return (Y, Y)

(c) otherwise,

Repeat $Z \sim q$, $V \sim \mathrm{Unif}(0,1)$

Until $V \geq \min\left(1, \frac{p(Z)}{q(Z)}\right)$

Return (Y, Z)

$\Rightarrow$ Maximal Coupling of (p, q).

$X \sim p$, $Y \sim q$ $\quad (X, Y)$ is a coupling of p, q

Trivial Coupling: $X \perp\!\!\!\perp Y$

Nontrivial Coupling: p, q "attract"

$$\mathbb{P}(X=Y) \leq \int p \wedge q$$

and this algorithm gives the maximal coupling in the sense

$$\mathbb{P}(X=Y) = \int p \wedge q.$$

<u>Coupled MH</u>: Let P be a MH kernel with proposal $Q(x, \cdot)$

and target distribution π.

We can build a coupled M.C. as follows

$X_0 \sim \mu$, $X_1 | X_0 \sim P(x_0, \cdot)$, $Y_0 \sim \mu$

At the k-th iteration, given $X_0, (Y_0, X_1) \ldots, (Y_{i-1}, X_i)$

* If $Y_{i-1} = X_i$, $X_{k+1} \sim P(x_k, \cdot)$ and set $Y_k = X_{k+1}$

* If $Y_{k-1} \neq X_k$ draw (Y^*, X^*) from the max coupling of

$Q(Y_{k-1}, \cdot)$ and $Q(X_k, \cdot)$

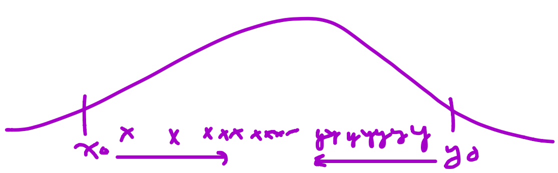
$u \sim \mathrm{Unif}(0,1)$

$u \leq \min\left(1 \quad \frac{\pi(Y^*) Q(Y^*, Y_{k-1})}{\pi(Y_{k-1}) Q(Y_{k-1}, Y^*)}\right)$ set $Y_k = Y^*$

otherwise $Y_k = Y_{k-1}$.

If $u \leq \min\left(1, \frac{\pi(x^*) Q(x^*, x_k)}{\pi(X_k) Q(x_k, X^*)}\right)$

set $X_{k+1} = X^*$
o.w. $X_{k+1} = X_k$



How to use this MC

Fix $h: \mathcal{X} \mapsto \mathbb{R}$ and we want $\pi(h) = \int h(x)\pi(x)dx$

$$\lim_{n \to \infty} \mathbb{E}[h(x_n)] = \pi(h)$$

Set $\Delta_0 = h(X_N)$, $\Delta_1 = h(X_{N+1}) - h(X_1)$

$$\ldots, \Delta_p = h(X_{N+k}) - h(X_{N+k-1})$$

So that

$$\pi(h) = \sum_{k=0}^{\infty} \mathbb{E}\left\{ h(x_{N+1}) - h(x_N) \right\}$$

$$= \sum_{k=0}^{\infty} \left[\mathbb{E}[h(X_{N+1})] - \mathbb{E}[h(Y_{N+k-1})] \right]$$

If we can interchange order

$$\pi(h) = \mathbb{E}\left[\sum_{n=0}^{\infty} h(x_{N+k}) - h(Y_{N+k-1}) \right]$$

$$= \mathbb{E}\left[\Delta_0^+ + \sum_{k=1}^{\tau+1} h(x_{N+k}) - h(Y_{N+k-1}) \right]$$

$$\tau = \inf\{ k \geq 1 : x_k = Y_{k-1} \}$$

$$\boxed{\hat{\pi}(h) = h(X_N) + \sum_{k=1}^{\tau-1} \left[h(X_{N+k}) - (Y_{N+k-1}) \right]}$$