Rounding of LR Relaxation

Distance over graph G given by

length over edges

$$W = \sum_{e \in E} l_e \quad (\mu_i = 1 \text{ for simplicity})$$

$$\sum_V \mu_i \mu_j d_{ij} \geq 1$$

We wanted to round this to

a l_1 metric.

→ d already looks like

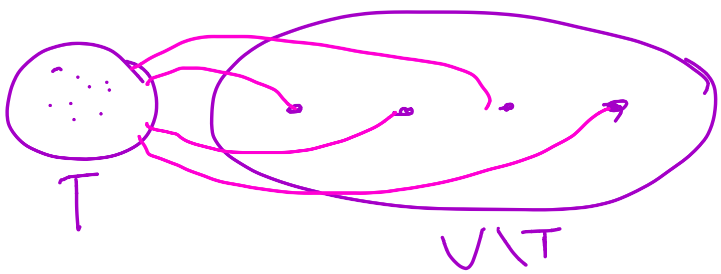
a cut

→ d doesn't.

(i) There exists a component $T \subseteq V$

with radius $1/2n^2$ such that

$|T| \geq \frac{2}{3} n$



Claim $\sum_{u,v \in V\setminus T} d(u,T) \geq \frac{1}{2n}$

Pf: $\forall (u,v)$ $d(u,v) \leq d(T,u) + d(T,v)$ $+ \frac{1}{n^2}$

from radius of T $(i,j) \in T$

$1 \leq \sum_{u,v \in V} d_{uv} = \frac{1}{2} \sum_{(u,v) \in E} d_{uv}$

$\leq 1 \quad \sum \quad (d(T,u) + d(T,u) + \frac{1}{\;})$

$$< \frac{1}{2} \sum_{(u,v)} \ldots \; n^2)$$

$$< n \sum_{w \in V \setminus T} d\,T(w) \quad + \frac{1}{2}$$

So rearranging

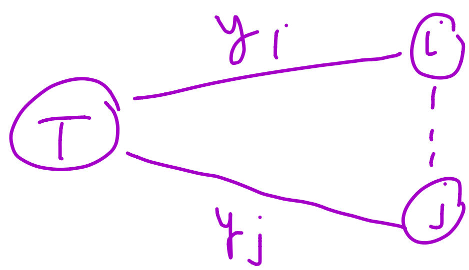
$$\frac{1}{2n} \leq \sum_{w \in V \setminus T} d(T, w)$$

<u>Goal:</u> Come up with ℓ_1 embedding such that

$$\frac{\sum_{E} \|y_i - y_j\|_1}{\sum_{V} \|y_u - y_v\|_1} \leq O(W)$$

The embedding we will use is this Frechet-type. $y_i = d(T, i)$

<u>Fact</u> $|y_i - y_j| \le d_{ij}$



$$\sum_E |y_i - y_j| \le \sum_E d_{ij} = W$$

$$\sum_V |y_u - y_v| \geq \sum_{u \in V \setminus T} \sum_{v \in T} |y_u - y_v|$$

$$= |T| \sum_{u \in V \setminus T} |y_u|$$

$$\geq |T| \cdot \frac{1}{2n} = \frac{1}{3}$$

So together we see

$$\frac{\sum' \|y_i - y_j\|}{\sum \|y_u - y_v\|} \leq 3W$$

We can always change those constants.

<u>Case II</u>: Suppose we don't have this special structure.

<u>Structure Lemma</u>

For any $\Delta > 0$ we can partition G into components of radius $\leq \Delta$ s.t. the number of edges connecting different components is

$\leq \frac{4W \log n}{\Delta}$ for any shortest path. metric.

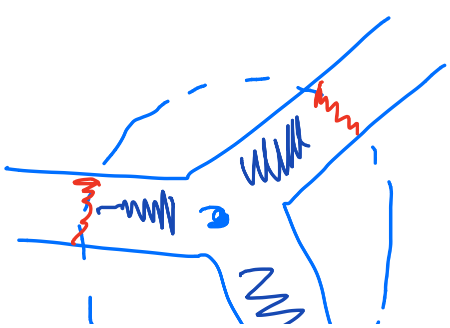
<u>Pf:</u> Vol. from edges W Volume from vertices W

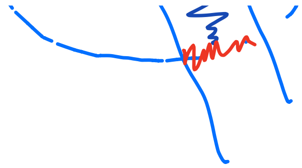
<u>Alg:</u> Pick vertex i grow a ball on this pipe system.

$$B(v,r) = \{u \in V : d_{u,v} \le r\}$$

$$V(s,r) = \frac{W}{n} + \sum_{B(s,r)} \underbrace{c_{uv}}_{\text{cap.}} \underbrace{l_{uv}}_{\text{length.}}$$

$$+ \sum_{uv \in \delta(B(s,v))} c(u,v)(r - d_{us})$$





Ball capacity

$$C(s,r) = \sum_{\delta(B(s,u))\cap E} c_{uv}$$

See boundary
um above

$$\frac{dV(s,r)}{dr} = C(s,r)$$

$$\log \frac{V(s,\Delta)}{V(s,0)} \overset{\text{claim}}{=} \int_0^\Delta \frac{1}{V(s,r)} \frac{dV(s,r)}{dr}\, dr$$

$$= \int_0^\Delta \frac{C(s,r)}{V(s,r)}\, dr$$

$$\geq \Delta \min_{(0,\Delta)} \frac{C(s,r)}{V(s,r)}$$

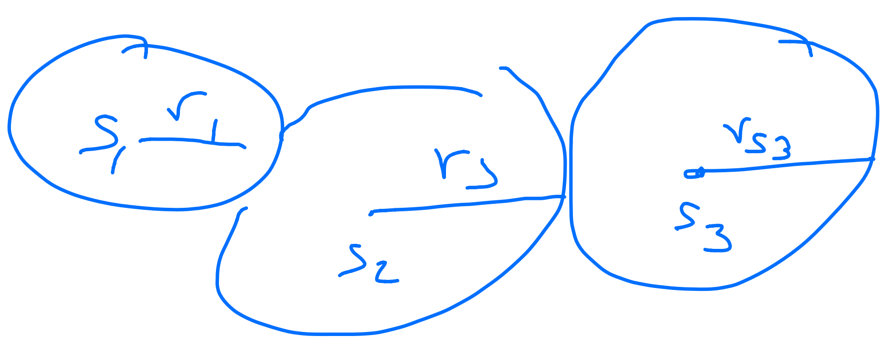
In addition

$$\ln \frac{V(S,\Delta)}{V(S,0)} \le \ln \frac{W}{W/n} = \ln(n)$$

you can find $B(S, r_S)$ such that

$$\frac{C(S,r_S)}{V(S,r_S)} \le \frac{\log n}{\Delta} V(S, r/S)$$

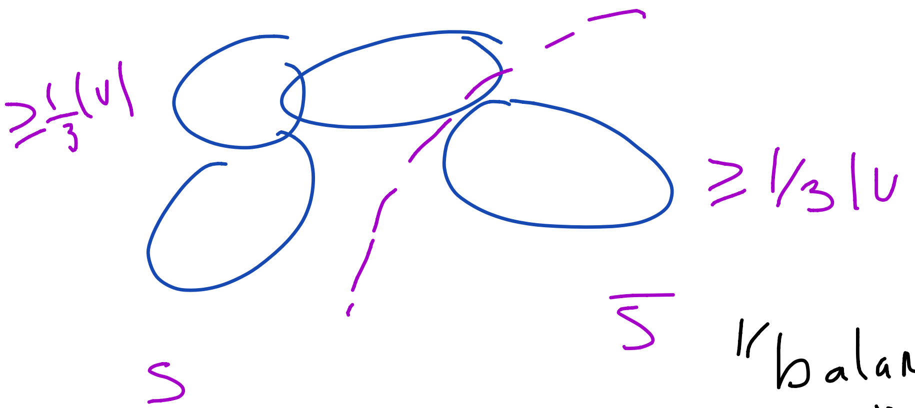
Then iterate



$$\sum C(S_i, r_{S_i}) \le \frac{\log n}{\Delta} 2W$$

∎

Now choosing $\Delta = \frac{1}{2n^2}$ we can get $|E_{cut}| \le 8wn^2 \log n$



"balanced cut"

Then

$$\alpha(S,\bar{S}) = \frac{|E(S,\bar{S})|}{|S|\,|\bar{S}|} \le \frac{4Wn^2 \log n}{n^2/9}$$

$$= O(w \log n)$$