1. (a) Suppose $X \sim N(\theta, \sigma^2)$ then $\theta \sim N(\mu, \tau^2)$. Then $\overline{X} \sim N(\theta, \frac{\sigma^2}{n})$. From here we see the joint distribution is given by

$$f(\overline{x}, \theta) = f(\overline{x}|\theta)\pi(\theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}} exp\Big\{-\frac{(\overline{x}-\theta)^2}{2\sigma^2/n}\Big\}\frac{1}{\sqrt{2\pi\tau^2}} exp\Big\{-\frac{(\theta-\mu)^2}{2\tau^2}\Big\}$$

(b) We know that the marginal of a jointly normal distribution is normal, so we need only find the mean and variance.

$$\mathbb{E}(\overline{X}) = \mathbb{E}[\mathbb{E}(\overline{X}|\theta)] = \mathbb{E}[\theta] = \mu$$
$$Var(\overline{X}) = Var[\mathbb{E}(\overline{X})] + \mathbb{E}[Var(\overline{X})] = Var[\theta] + \mathbb{E}[\sigma^2/n] = \tau^2 + \sigma^2/n$$

Therefore, the marginal distribution is given by $N(\mu, \tau^2 + \sigma^2/n)$.

(c) To derive the posterior distribution we use $\pi(\theta|\overline{X}) = \frac{f(\overline{X}|\theta)\pi(\theta)}{m(\overline{X})}$

$$\pi(\theta|\overline{X}) = \frac{\frac{1}{\sqrt{2\pi\sigma^2/n}} exp\Big\{-\frac{(\overline{x}-\theta)^2}{2\sigma^2/n}\Big\}\frac{1}{\sqrt{2\pi\tau^2}} exp\Big\{-\frac{(\theta-\mu)^2}{2\tau^2}\Big\}}{\frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} exp\Big\{-\frac{(\overline{x}-\mu)^2}{2(\tau^2+\sigma^2/n)}\Big\}}$$
$$= \frac{1}{\sqrt{2\pi\frac{\tau^2\sigma^2/n}{\tau^2+\sigma^2/n}}} \exp\Big\{-\frac{(\overline{x}-\theta)^2}{2\sigma^2/n} - \frac{(\theta-\mu)^2}{2\tau^2} + \frac{(\overline{x}-\mu)^2}{2(\sigma^2/n+\tau^2)}\Big\}$$

$$= \frac{1}{\sqrt{2\pi\frac{\tau^2\sigma^2/n}{\tau^2+\sigma^2/n}}} \exp\Big\{\frac{-(\overline{x}-\theta)^2\tau^2(\sigma^2/n+\tau^2) + -(\theta-\mu)^2\sigma^2/n(\sigma^2/n+\tau^2) + (\overline{x}-\mu)^2\sigma^2/n\tau^2}{2\sigma^2/n\tau^2(\sigma^2/n+\tau^2)}\Big\}$$

We first focus on the numerator of the exponent term. After expanding, we get the following nine terms.

$$-\overline{x}^2(\tau^2)^2(\sigma^2/n+\tau^2) + 2\overline{x}\theta\tau^2(\sigma^2/n+\tau^2) - \theta^2\tau^2(\sigma^2/n+\tau^2)$$
$$-\theta^2(\sigma^2/n)(\sigma^2/n+\tau^2) + 2\theta\mu(\sigma^2/n)(\sigma^2/n+\tau^2) - \mu(\sigma^2/n)(\sigma^2/n+\tau^2)$$
$$+\overline{x}^2(\sigma^2/n)\tau^2 - 2\overline{x}\mu(\sigma^2/n)\tau^2 + \mu^2(\sigma^2)\tau^2$$
$$= -(\overline{x}\tau^2 + \mu\sigma^2/n)^2 - (\sigma^2/n+\tau^2)^2\theta^2 + (\sigma^2/n+\tau^2)(2\overline{x}\theta\tau^2 + 2\theta\mu\sigma^2/n)$$
$$= -\Big(\theta(\sigma^2/n+\tau^2) - (\overline{x}\tau^2 + \mu\sigma^2/n)\Big)^2$$

Now dividing by $(\sigma^2/n+\tau^2)^2$ we see that we have

$$\pi(\theta|\overline{X}) = \frac{1}{\sqrt{2\pi\frac{\tau^2\sigma^2/n}{\sigma^2/n+\tau^2}}} exp\Big\{-\frac{(\theta - \frac{\tau^2}{(\sigma^2/n+\tau^2)}\overline{x} - \frac{\sigma^2/n}{(\sigma^2/n+\tau^2)}\mu)}{2\frac{\tau^2\sigma^2/n}{\sigma^2/n+\tau^2}}\Big\}$$

Which we recognize as a $N(\frac{\tau^2}{(\sigma^2/n+\tau^2)}\overline{x} - \frac{\sigma^2/n}{(\sigma^2/n+\tau^2)}\mu, \frac{\tau^2\sigma^2/n}{\sigma^2/n+\tau^2})$

2. (a) Let $X_1, \dots, X_n \sim Pois(\lambda)$ and $\lambda \sim Gamma(\alpha, \beta)$. Then we have

$$f(X_1, \dots, X_n|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{x_1!x_2!\dots x_n!}$$
$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta}$$

Therefore, we have

$$\pi(\lambda|X_1, \dots, X_n) \propto \lambda^{\alpha+\sum x_i - 1}e^{-(n+1/\beta)\lambda}$$

which we recognize as a Gamma$(\alpha + \sum x_i, \beta/(n\beta+1))$.

(b) The posterior mean and variance is given by the mean and variance of the gamma density.

$$\mathbb{E}(\lambda|X_1, \dots, X_n) = (\alpha + \sum x_i)(\frac{\beta}{n\beta+1}) \qquad Var(\lambda|X_1, \dots, X_n) = (\alpha + \sum x_i)\left(\frac{\beta}{n\beta+1}\right)^2$$

3. Suppose we have $X_1, \dots, X_n \sim N(\theta, \sigma^2)$ with $\theta \sim \frac{1}{2a}e^{-|\theta|/a}$. Then we have

$$\begin{aligned}\pi(\theta|X_1, X_2, \dots, X_n) &\propto f(x_1, \dots, x_n|\theta)\pi(\theta)\\ &\propto e^{-\frac{1}{2}\sum(x_i-\theta)^2}e^{-\frac{|\theta|}{2}}\\ &= \exp\Big\{-\frac{1}{2}\Big(n\theta^2 - 2\theta\sum x_i + 2\frac{|\theta|}{a}\Big)\Big\}\end{aligned}$$

To find the appropriate normalizing constant we split the problem into two parts.

$$C_1 := \int_0^\infty \exp\Big\{-\frac{1}{2}\Big(n\theta^2 - 2\theta\sum x_i + 2\frac{\theta}{a}\Big)\Big\}d\theta$$
$$C_2 := \int_{-\infty}^0 \exp\Big\{-\frac{1}{2}\Big(n\theta^2 - 2\theta\sum x_i - 2\frac{\theta}{a}\Big)\Big\}d\theta$$

Then to calculate the posterior mean we write

$$\mathbb{E}(\theta|\underline{X}) = \frac{\int_{-\infty}^{\infty}(C_1+C_2)\theta\exp\Big\{-\frac{1}{2}\Big(n\theta^2 - 2\theta\sum x_i + 2\frac{|\theta|}{a}\Big)\Big\}d\theta}{\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2}\Big(n\theta^2 - 2\theta\sum x_i + 2\frac{|\theta|}{a}\Big)\Big\}d\theta}$$

4. (a) Let $f(x|\theta) = \frac{1}{2\theta}I_{(-\theta,\theta)}(x)$. Then calculating the joint density we have

$$f(\underline{x}|\theta) = \left(\frac{1}{2\theta}\right)^n\prod_{i=1}^{n} I_{(-\theta,\theta)}(x_i) = \left(\frac{1}{2\theta}\right)^n\prod_{i=1}^{n} I_{[0,\theta)}(|x_i|) = \left(\frac{1}{2\theta}\right)^n I_{[0,\theta)}\max_i\{|x_i|\}$$

Therefore $T = \max_i\{|x_i|\}$ is a sufficient statistic. All we must now show is that is complete and that there exists $\phi(\cdot)$ such that $\phi(T)$ is unbiased. Now notice that T is the maximum order statistic of $|x_1|, \dots, |x_n|$. So $T \sim nt^{n-1}/\theta^n$ Let $g(\cdot)$ be an arbitrary function and consider the following

$$\begin{aligned}
\mathbb{E}(g(T)) &= \int_0^\theta g(t)\frac{nt^{n-1}}{\theta^n}dt \equiv 0 \\
\frac{n}{\theta}g(\theta) - \int_0^\theta n^2\frac{t^{n-1}}{\theta^{n-1}}g(t)dt &= 0 \\
\frac{n}{\theta}g(\theta) - n\theta\mathbb{E}(g(T)) &= 0 \\
g(\theta) &= 0
\end{aligned}$$

Since $g(t) = 0$ is uniformly zero, T is complete sufficient statistic. From here, we need to find our function $\phi(\cdot)$.

$$\mathbb{E}(T) = \int_0^\theta n\frac{t^n}{\theta^n}dt = \frac{n}{\theta^n}\int_0^\theta t^n dt = \frac{n}{\theta^n}\frac{1}{n+1}t^{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$$

Therefore, if we define $\phi(t) = \frac{n+1}{n}t$ we see that $\phi(T)$ is unbiased. Therefore, $\phi(T)$ is a UMVUE. Therefore, the function $\tau(\theta) = 1/\theta$ is a function that has a UMVUE.

5. (a) Consider $f(x|\theta) = \theta x^{\theta-1}$. Then $L(\theta|X_1, \dots, X_n) = \prod_{i=1}^n \theta x^{\theta-1}$ and $l(\theta|X_1, \dots, X_n) = n\log\theta + (\theta-1)\sum\log(x_i)$. Differentiating with respect to θ we see that

$$\frac{\partial}{\partial\theta}l(\theta|X_1, \dots, X_n) = \frac{n}{\theta} + \sum X_i = n\Big[\frac{1}{n}\sum -\log x_i - \frac{1}{\theta}\Big]$$

Now recall from previous homeworks we've shown $-\log(X_i) \sim Exp(1/\theta)$ and $\sum -\log(X_i) \sim Gamma(n, 1/\theta)$. We see that $\frac{1}{n}\sum -\log(X_i)$ is unbiased for $1/\theta$. Therefore, by Corollary 7.3.15, $\frac{1}{n}\sum -\log(X_i)$ attains the CRLB and is a UMVUE.

(b) Suppose $f(x|\theta) = \frac{\log\theta}{\theta-1}\theta^x$. Then by the same process as above, we have $L(\theta|X_1, \dots, X_n) = \left(\frac{\log\theta}{\theta-1}\right)^n \theta^{\sum x_i}$ and $l(\theta|X_1, \dots, X_n) = n\log\log\theta - n\log(\theta-1) + \sum x_i\log\theta$. Therefore, we have

$$\frac{\partial}{\partial\theta}l(\theta|X_1, \dots, X_n) = \frac{n}{\theta\log\theta} - \frac{n}{\theta-1} + \frac{\sum x_i}{\theta} = \frac{n}{\theta}\Big[\overline{x} - \Big(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\Big)\Big]$$

Since $\overline{x}$ is unbiased for the mean, we can again use the Attainment Theorem if the mean of f is given by $\frac{\theta}{\theta-1} - \frac{1}{\log\theta}$. We now calculate the mean of f.

$$
\begin{aligned}
\mathbb{E}(X) &= \int_0^1 x\frac{\log(\theta)}{\theta-1}\theta^x dx \\
&= \frac{\log(\theta)}{\theta-1}\Big[\frac{x\theta^x}{\log\theta}\big|_0^1 - \int_0^1 \frac{\theta^x}{\log\theta}dx\Big] \\
&= \frac{\log(\theta)}{\theta-1}\Big[\frac{\theta}{\log\theta} - \frac{\theta}{\log^2\theta} + \frac{1}{\log^2\theta}\Big] \\
&= \frac{1}{\theta-1}\Big[\theta - \frac{\theta-1}{\log\theta}\Big] \\
&= \frac{\theta}{\theta-1} - \frac{1}{\log\theta}
\end{aligned}
$$

Therefore, the function $\tau(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\log\theta}$ has a UMVUE.

6. We will use the iid case for the CRLB. First we calculate the Fisher information.

$$
\begin{aligned}
I(\theta) &= \mathbb{E}\Big[\Big(\frac{\partial}{\partial\theta}\log f(x|\theta)\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{\partial}{\partial p}(x\log p + (1-x)\log(1-p))\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{x}{p} - \frac{1-x}{1-p}\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{x}{p} + \frac{x-1}{1-p}\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{x-p}{p(1-p)}\Big)^2\Big] \\
&= Var\Big[\frac{x-p}{p(1-p)}\Big] + \Big[\mathbb{E}\Big(\frac{x-p}{p(1-p)}\Big)\Big]^2 \\
&= \frac{1}{p(1-p)}
\end{aligned}
$$

Moreover, note that $\mathbb{E}(\overline{x}) = p$ so $\left(\frac{\partial}{\partial p}p\right)^2 = 1$. Therefore we see the CRBL is given by $\frac{1}{n/[p(1-p)]} = \frac{p(1-p)}{n}$. But note in our case

$$
Var(\overline{x}) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i) = \frac{1}{n^2}\sum_{i=1}^n p(1-p) = \frac{p(1-p)}{n}
$$

Hence $\overline{x}$ is a UMVUE of p

7. Again using the iid version of the CRLB, we calculate Fisher's information for a sin-

gle point.

$$
\begin{aligned}
I(\theta) &= \mathbb{E}\Big[\Big(\frac{\partial}{\partial\theta}\log f(x|\theta)\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{\partial}{\partial p}\Big\{-\frac{1}{2}\log 2\pi + \frac{(x-\theta)^2}{2}\Big\}\Big)^2\Big] \\
&= \mathbb{E}\Big[-(x-\theta)^2\Big] \\
&= Var(x-\theta) + \Big[\mathbb{E}(x-\theta)\Big]^2 \\
&= 1
\end{aligned}
$$

Now, we also have $\mathbb{E}(\overline{X}^2 - \frac{1}{n}) = \theta^2$ so we have $\left(\frac{\partial}{\partial\theta}\theta^2\right)^2 = 4\theta^2$. Therefore, the CRLB is given by $\frac{4\theta^2}{n}$. Now we will show that the variance of this UMVUE does not attain the CRLB.

$$
\begin{aligned}
Var(\overline{x} - \frac{1}{n}) &= \mathbb{E}(\overline{X}^4) - [\mathbb{E}(\overline{X})]^2 \\
&= \mathbb{E}[\overline{X}^3(\overline{X} - \theta + \theta)] - [1/n + \theta^2]^2 \\
&= \mathbb{E}[\overline{X}^3(\overline{X} - \theta)] + \theta\mathbb{E}(\overline{X}^3) - [1/n + \theta^2]^2 \\
&= \frac{3}{n}\mathbb{E}(\overline{X}^2) + \theta\mathbb{E}[\overline{X}^2(\overline{X} - \theta + \theta)] - [1/n + \theta^2]^2 \\
&= \frac{3}{n}[\theta^2 + 1/n] + \theta\mathbb{E}[\overline{X}^2(\overline{X} - \theta)] + \theta^2\mathbb{E}[\overline{X}^2] - [1/n + \theta^2]^2 \\
&= \frac{3}{n}[\theta^2 + 1/n] + \theta/n\mathbb{E}[2\overline{X}] + \theta\mathbb{E}[\overline{X}^2] - [1/n + \theta^2]^2 \\
&= \frac{3}{n}[1/n + \theta^2] + 2\theta^2/n + \theta^2(\theta^2 + 1/n) - (1/n + \theta^2)^2 \\
&= \frac{4\theta^2 + 2/n}{n} > \frac{4\theta^2}{n}
\end{aligned}
$$

8. Recall that the statistic $T = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Now, we consider $T^{p/2}$. (Our hope is to build an unbiased estimate of σ^p through our knowledge of T). First note that

$$
\begin{aligned}
\mathbb{E}[T^{p/2}] &= \int_0^\infty \frac{t^{p/2}}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} t^{(n-1)/2-1}e^{-t/2}dt \\
&= \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})}\int_0^\infty t^{(p+n-1)/2-1}e^{-t/2}dt \\
&= \frac{2^{(p+n-1)/2-1}}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})}\int_0^\infty \frac{t^{(p+n-1)/2-1}e^{-t/2}}{2^{(p+n-1)/2-1}}dt \\
&= \frac{2^{(p+n-1)/2-1}}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})}\int_0^\infty (t/2)^{(p+n-1)/2-1}e^{-t/2}dt
\end{aligned}
$$

Notice that we recognize this as the Gamma density with $\alpha = (p+n-1)/2$ and $\beta = 1$. Therefore, we have

$$C := \mathbb{E}[T^{p/2}] = \frac{2^{p/-1}\Gamma((p+n-1)/2)}{(n-1)/2}$$

Therefore, for we see that $\mathbb{E}[T^{p/2}/C] = 1$ and moreover,

$$\mathbb{E}\Big[\Big(\frac{(n-1)S^2}{C^{2/p}}\Big)^{p/2}\Big] = \sigma^p$$

Therefore, since S^2 is a complete sufficient and for $\phi(t) = \Big(\frac{(n-1)t}{C^{2/p}}\Big)^{p/2}$, $\phi(S^2)$ is unbiased for σ^p. Therefore, $\phi(S^2)$ is a UMVUE for each σ^p.

9. (a) Recall that for the quadratic loss function $R(\theta,\delta) = MSE(\delta)$. So for $\delta(x) = a\overline{X} + b$ we have

$$R(\theta,\delta(x)) = MSE(\delta(x)) = Var(a\overline{X}+b) + [\mathbb{E}[a\overline{X}+b]-\theta]^2 = a^2\frac{\sigma^2}{n} + [b-(1-a)\theta]^2$$

(b) Let $\eta = \frac{\sigma^2}{n\tau^2+\sigma^2}$ then we note that

$$1-\eta = 1 - \frac{\sigma^2}{n\tau^2+\sigma^2} = \frac{n\tau^2+\sigma^2-\sigma^2}{n\tau^2+\sigma^2} = \frac{\tau^2}{\tau^2+\sigma^2/n}$$

So, for the Bayes Estimator $\delta^\pi := \mathbb{E}[\theta|\underline{x}]$. As we've seen in a previous exercise, the posterior mean is given by

$$\delta^\pi = \frac{\tau^2}{\tau^2+\sigma^2/n}\overline{x} + \frac{\sigma^2/n}{\tau^2+\sigma^2/n}\mu$$

Then using the fact that using quadratic risk is just MSE, we see that

$$\begin{aligned}
R(\theta,\delta^\pi) &= Var(\delta^\pi) + [\mathbb{E}[\delta^\pi]-\theta]^2 \\
&= \Big(\frac{\tau^2}{\tau^2+\sigma^2/n}\Big)^2\sigma^2/n + \Big[\frac{\tau^2}{\tau^2+\sigma^2/n}\theta + \frac{\sigma^2/n}{\tau^2+\sigma^2/n}\mu - \theta\Big]^2 \\
&= (1-\eta)^2\sigma^2/n + \Big[\frac{\sigma^2/n\mu - \sigma^2/n\theta}{\tau^2+\sigma^2/n}\Big]^2 \\
&= (1-\eta)^2\sigma^2/n + \Big[\frac{\sigma^2/n(\mu-\theta)}{\tau^2+\sigma^2/n}\Big]^2 \\
&= (1-\eta)^2\sigma^2/n + [\eta(\mu-\theta)]^2
\end{aligned}$$

(c) We now calculate the Bayes Risk.

$$
\begin{aligned}
B(\pi, \delta^\pi) &= \int R(\theta, \delta^\pi)\pi(\theta)d\theta \\
&= (1-\eta)^2\sigma^n + \eta^2\int(\mu-\theta)^2\pi(\theta)d\theta \\
&= (1-\eta)^2\sigma^n + \eta^2\mathbb{E}[(\mu-\theta)^2] \\
&= (1-\eta)^2\sigma^n + \eta^2\Big[Var(\theta) + (\mathbb{E}[\theta]-\mu))^2\Big] \\
&= (1-\eta)^2\sigma^n + \eta^2 Var(\theta) \\
&= (1-\eta)^2\sigma^n + \eta^2\tau^2 \\
&= (\frac{\tau^2}{\tau^2+\sigma^2/n})^2\sigma^2/n + \eta^2\tau^2 \\
&= \tau^2\Big[\frac{n\tau^2\sigma^2}{(n\tau^2+\sigma^2)^2} + \eta^2\Big] \\
&= \tau^2\Big[\frac{n\tau^2\sigma^2+\sigma^4}{(n\tau^2+\sigma^2)^2}\Big] \\
&= \tau^2\sigma^2\Big[\frac{n\tau^2+\sigma^2}{(n\tau^2+\sigma^2)^2}\Big] \\
&= \tau^2\frac{\sigma^2}{n\tau^2+\sigma^2} \\
&= \tau^2\eta^2
\end{aligned}
$$

10. (a) $\mathbb{E}[\overline{X}^2] = Var(\overline{X}) + (\mathbb{E}[\overline{X}])^2 = \frac{\theta(1-\theta)}{n} + \theta^2 \neq \theta^2$

(b) Let $T_n = (\sum_{i=1}^n X_i/n)^2$ and $T_n^{(j)} = \Big(\sum_{i\neq j}^n X_i/(n-1)\Big)^2$ Then we have

$$
JK(T_n) = nT_n - \frac{n-1}{n}\sum_{j=1}^n\left(\sum_{i\neq j}^n\frac{X_i}{n-1}\right)^2
$$

(c)

$$
\begin{aligned}
\mathbb{E}[JK(T_n)] &= n\mathbb{E}[T_n] - \frac{n-1}{n}\sum_{j=1}^n\mathbb{E}[T_n^{(j)}] \\
&= n\mathbb{E}\left[\left(\sum_{i=1}^n\frac{x_i}{n}\right)^2\right] - \frac{n-1}{n}\sum_{j=1}^n\mathbb{E}\left[\left(\sum_{i\neq j}^n\frac{x_i}{n-1}\right)^2\right] \\
&= \theta(1-\theta) + n\theta^2 - \frac{1}{n(n-1)}\sum_{j=1}^n(n-1)\theta(1-\theta) + [(n-1)\theta]^2 \\
&= \theta(1-\theta) + n\theta^2 - \theta(1-\theta) + (n-1)\theta^2 \\
&= \theta^2
\end{aligned}
$$

(d) Recall that a Bernoulli distributions is an exponential family with CSS $\sum x_i$. $JK(T_n)$ is a function of $\sum x_i$ that is unbiased for θ^2. Therefore, $JK(T_n)$ is a UMVUE.