Lecture 1/18

Thursday, January 18, 2018 9:13 AM

# Course Layout

- One Midterm / Final
- Weekly HW – by yourself - but you can get help from others
- Use personal course webpage.

# Course Topics

- Ask loads of questions
- Modeling course first & foremost
- Discussion won't start until 1/31
  - Two weeks from yesterday.
- Recommended Texts
  - Handouts will be provided
- HW
  - mostly from book.

* Read chapter 1 - will not be discussed here - distribution theory.

## Chapter 2: Probability Review

Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

given $P(B) \neq 0$

Conditional PMF: $P_{X|Y}(x|y) = \dfrac{P_{X,Y}[X=x, Y=y]}{P_Y[Y=y]}$

Marginal Conditional: Probability

$$P_X[X=x] = \sum_y P_{X,Y}[X=x, Y=y]$$

Law of Total Probability

$$P(X=x) = \sum_{y=0}^{\infty} P_{X|Y}(x|y) P_Y(y)$$

Ex: $X|N \sim Bin(p, N)$ Hierachial

$N \sim Bin(z, m)$ Model

Q: How is $X$ distributed?

$$P(X=k) = \sum_n P_{X|N}(k|n)\, P_N(n)$$

$$* = \sum_{n=k}^{m} \binom{n}{k} p^k (1-p)^{n-k} \binom{m}{n} z^n (1-z)^{m-n}$$

* $k \le n \le m$ because $P(X=k)$ and $N \sim Binom(z, m)$

$$= \frac{p^k}{k!} (1-p)^{-k} (1-z)^m\, m! \sum_{n=k}^{m} \frac{z^n (1-p)^n}{(n-k)!\,(m-n)!\,(1-z)^n}$$

$$= \left[\frac{p^k}{k!} (1-p)^{-k} (1-z)^m\, m!\right] \sum_{n=k}^{m} \frac{(m-k)!}{(n-k)!\,(m-n)!} \left(\frac{(1-p)z}{1-z}\right)^{n-k}$$

$$\frac{1}{(m-k)!} \left(\frac{(1-p)z}{1-z}\right)^k$$

$$* = \frac{m!}{(m-k)!\,k!} \frac{p^k z^k}{(1-z)^k} (1-z)^m \left(1 + \frac{z(1-p)}{1-z}\right)^{m-k}$$

$$-\frac{\quad}{(m-k)!\,k!}\,\frac{\quad}{(1-q)^k} \quad t) \quad \left( \frac{\quad}{1-q} \right)$$

* Binomial formula

* Finish to show $X \sim \text{Binom}(pq, m)$

**Ex:** $X | N \sim \text{NBin}(p, N)$ $\quad N \sim \text{Geom}(b)$

$$P(X=k) = \sum_{n} P(k|n)\, P_N(n)$$

$$= \sum_{n=1}^{\infty} \binom{n+k-1}{k} p^n (1-p)^k (1-b)\, b^{n-1}$$

$$= (1-b)(1-p)^k \sum_{n=1}^{\infty} \binom{k+n-1}{k} p^n\, b^{n-1}$$

$$= (1-b)(1-p)^k\, p \sum_{n'=0}^{\infty} \binom{n'}{k} (pb)^{n'} \quad \text{for } n' = n-1$$

$$= (1-b)(1-p)^k\, p\, (1-bp)^{-k-1}$$

$$= \frac{p-bp}{1-bp} \left( \frac{1-p}{1-bp} \right)^k \sim \text{Geom}\left( \frac{1-p}{1-bp} \right)$$

Conditional Expectation:

$$E(g(x)|Y) = \sum_{x} g(x)\, P_{X|Y}(x|y)$$

Total Probability Law of Conditional Expectation

$$E(g(x)) = \sum_{y} E(g(x)|Y=y)\, P(Y=y)$$

Properties of Conditional Expectation.

1. $E[g(x)|Y]$ is a R.V. in $Y$.

2. $E[g(x)] = E[E(g(x)|Y)]$

3. $E[c_1 g_1(x) + c_2 g_2(x)|Y]$

$$= c_1 E(g_1(x)|Y) + c_2 E(g_2(x)|Y)$$

4. If $X \perp Y$ $E[g(x)|Y] = E(g(x))$

5. $E(g(x)h(y)|Y) = h(y)\, E(g(x)|Y)$

5. $E(g(X)h(Y)|Y) = h(Y)\,E(g(X)|Y)$

6. $E(h(Y)g(X)) = E[E(h(Y)\,g(X)|Y)]$

$$= E[h(Y)E(g(X)|Y)]$$

7. If $g(X) \geq 0$ $E(g(X)|Y) \geq 0$

8. $Var(g(X)) = E[Var(g(X)|Y)] + Var[E(g(X)|Y)]$

<u>Ex:</u> (Dice Craps game)

(1) Roll two dice, find their sum, S

(2) If $S \in \{2, 3, 12\}$ you lose

(3) If $S \in \{7, 11\}$ you win

(4) If S is o.w. you continue

(5) If you get the o.w then you win

Q: P(winning)?

Q: P(winning)?

Let $Z_n$ = sum of the dice in the $n^{th}$ roll

$A$ = wins

$B_4$ = {a sum 4 before a sum 7}

$$P(Z=2) = P_Z(2) = 1/36$$

$$P(Z=3) = P_Z(3) = P_Z(11) = \frac{2}{36}$$

$$P(Z=4) = P(Z=10) = 3/36$$

$$\vdots \qquad = 4/36$$

$$\vdots \qquad = 5/36$$

$$\qquad = 6/36$$

$$P(A) = \sum_{k=2}^{12} P(A \mid Z_0 = k) P(Z_0 = k)$$

$$P(A \mid Z_0 = k) = \begin{cases} 0 & k = 2, 3, 12 \\ 1 & k = 7, 11 \\ ? & k = 4, 5, 6, 8, 9 \end{cases}$$

Let $\alpha = P(A \mid Z_0 = 4)$

$$\alpha = \sum_{k=2}^{12} P(B_4 \mid Z_1 = k)\, P(Z_1 = k)$$

$$= 1\, P_Z(4) + 0\, P_Z(7) + \alpha \sum_{k \neq 4,7} P_Z(k)$$

$$\Rightarrow \quad \alpha = \frac{P_Z(4)}{P_Z(4) + P_Z(7)}$$

# Lecture 1/23

Tuesday, January 23, 2018 9:08 AM

# Random Sums

A fixed sum is something of the form

$$S_n = \sum_{i=1}^{n} \xi_i, \quad \xi \text{ iid R.V}$$

Random Sum

$$S_n = \sum_{i=1}^{N} \xi_i, \quad \xi_i \text{ iid}$$

$$N \sim F_N$$

Ex: Queue in a bank

Inference is focused on $S_n$.

$$F_{S|N}(s|n) = \frac{P(S_N = s, N = n)}{P(N = n)}$$

$$f_{S|N}(s|n) = \frac{d}{ds} F_{S|N}(s|n)$$

In the cont. case

$$P[a \le S_n \le b, N = n]$$
$$= \int_a^b f_{S|N}(s|n)\, P(N=n)\, ds$$

Moments of $S_n$

Assume that $\mathbb{E}(\xi_i) = \mu$

$\mathrm{Var}(\xi_i) = \sigma^2$, $\mathbb{E}(N) = r$

$\mathrm{Var}(N) = \tau^2$

**Lemma:** $\mathbb{E}(S_N) = \mu r$

and $Var(S_N) = r\sigma^2 + \mu^2 \tau^2$

**Pf:** $\mathbb{E}(S_N) = \mathbb{E}(\mathbb{E}(S \mid N))$

$$= \mathbb{E}\left(\mathbb{E}\left[\sum_{i=1}^{N} \xi_i \mid N\right]\right)$$

$$= \sum_{n=1}^{\infty} \mathbb{E}\left(\sum_{i=1}^{n} \xi_i \mid N=n\right) P(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathbb{E}(\xi_i)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} n\mu\, P(N=n)$$

$$= \mu \sum_{n=1}^{\infty} n\, P(N=n)$$

$$= \mu\, \mathbb{E}(N) = \mu r$$

Now focusing on variance.

$$\mathrm{Var}(S_n) = \mathbb{E}\left[\left(S_n - r\mu\right)^2\right]$$

$$= \mathbb{E}\left[\left(S_n - N\mu + N\mu - r\mu\right)^2\right]$$

$$= \mathbb{E}\left[\left(S_n - N\mu\right)^2\right] + \mathbb{E}\left[\left(N\mu - r\mu\right)^2\right]$$

$$+ 2\,\mathbb{E}\left[\left(S_n - N\mu\right)\left(N\mu - r\mu\right)\right]$$

Then

$$\mathbb{E}\left[\left(N\mu - r\mu\right)^2\right] = \mu^2\, \mathbb{E}\left[\left(N - r\right)^2\right]$$

$$= \mu^2\, \mathrm{Var}(N)$$

$$= \mu^2 \tau^2$$

Then the third term

$$\mathbb{E}\left((S_N - N\mu)(N\mu - r\mu)\right)$$

$$= \mu\, \mathbb{E}\left[(S_N - N\mu)(N - r)\right]$$

$$= \mu\, \mathbb{E}\left[\, \mathbb{E}\left((S_n - N\mu)(N - r) \mid N\right)\right]$$

$$= \mu\, \mathbb{E}\left[(N - r)\, \underbrace{\mathbb{E}\left[(S_n - N\mu) \mid N\right]}_{0}\right]$$

$$= 0$$

Now focusing on the first term.

$$\mathbb{E}\left[(S_N - N\mu)^2\right] = \mathbb{E}\left[\mathbb{E}\left((S_n - N\mu)^2 \mid N\right)\right]$$

$$= \sum^{\infty} \mathbb{E}\left[(S_n - n\mu)^2\right] P(N = n)$$

$$n=1$$

$$= \sum_{n=1}^{\infty} n\sigma^2 P(N=n)$$

$$= \sigma^2 \sum_{n=1}^{\infty} n\, P(N=n) = \sigma^2 \mathbb{E}(N)$$

$$= r\sigma^2$$

Hence

$$Var(S_n) = r\sigma^2 + \mu^2 \tau^2$$

Distribution of Random Sum.

The probability distribution of a fixed sum is the $n^{th}$ fold convolution defined by

$$f^{(1)} = f(z) \sim \text{pdf of } \xi_i$$

$$f_{S_n}(z) = f^{(n)}(z) = \int_{n} f^{(n-1)}(z-\mu) f(\mu)\, d\mu$$

If $N$ and $(\xi_i)_{i=1}^{\infty}$ are independent then

$$f_{S_N}(s) = \sum_{n=1}^{\infty} f_{S_n}(s)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} f^{(n)}(s)\, P(N=n)$$

- The formula about holds if $P(N=0) = 0$
- If $P(N=0) > 0$ then

- If $P(N=0)>0$ then $N$ has a discrete and continuous component.

<u>Ex:</u> An elementary stock price model

Let $\xi_0$ = stock price at closure

$\xi_i$ = price change due to ith transaction

$\xi_i \sim \text{iid } F_{\xi}$ (okay in perfect market)

$\xi_i$ have common variance $\sigma^2$ under the assump. of stationarity.

If we assume the value

If we assume the value of the stock at the end of the day is $S_n = \sum_{i=1}^{n} \xi_i$ (fixed sum) then the CLT says

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1)$$

Some studies empirically confirm this result whereas some don't.

Using a random sum model can we do better?

$$S_N = \sum_{i=1}^{N} \xi_i \quad \text{with } N \sim Pois(\lambda)$$

then $\mathbb{E}(S_N) = r\mu$ $\quad Var(S_N) = \mu^2 \tau^2 + r\sigma^2$

and

$$f_{S|N}(z) = \sum_{n=0}^{\infty} \phi_n(z) \frac{\lambda^n e^{-\lambda}}{n!}$$

when

$$\phi_n(z) \simeq \frac{1}{\sqrt{2\pi(n+1)\sigma}} e^{-\frac{1}{2}\left(\frac{z^2}{(n+1)\sigma^2}\right)}$$

Ex: (Problem 3.1 Chp 2)

Let $N \sim Pois(\lambda)$ and we have $N$ independent Bernoulli trials with prob. $p$. Let $Z$ be the # of successes in $N$ trials. Compute $f_Z$.

$$Z = \sum_{i=1}^{N} \xi_i \qquad \xi_i \sim \text{iid Bern}(p).$$

$$\mathbb{E}(\xi_i) = p \qquad \text{Var}(\xi_i) = p(1-p)$$

$$\mathbb{E}(N) = \text{Var}(N) = \lambda$$

$$\mathbb{E}(Z_N) = \lambda p \qquad \text{Var}(Z_N) = \lambda p$$

If $Z_n = \sum_{i=1}^{N} \xi_i \sim \text{Binom}(n, p)$

with $f_{Z_n}(z=k) = \sum_{n=1}^{\infty} f_{Z_N}(k)\, P(N=n)$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} [\lambda(1-p)]^{n-k}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda} \left[ e^{\lambda(1-p)} \right]$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda p} \sim Pois(\lambda p).$$

Conditioning on Cont. R.V.

$$F_{X|Y}(x|y) = \int f(w|y)\, dw$$

$$E[g(X)|Y] = \int g(x) f_{X|Y}(x|y)\, dx$$

<u>Ex:</u> $X|Y \sim Pois(Y)$ $\quad Y \sim Exp(\theta)$

By LoTP

$$P(X=k) = \int_Y p(X=k|y) f_Y(y)\, dy$$

$$= \int_0^\infty \frac{y^k e^{-y}}{k!} \theta e^{-\theta y} dy$$

$$= \frac{\theta}{k!} \int_0^\infty y^k e^{-(1+\theta)y} dy$$

$$= \frac{\theta}{(1+\theta)^{1+k}}$$

If $Y \sim \text{Gamma}(\alpha, \theta)$. then

$$P(X=k) = \int_0^\infty \frac{y^k e^{-y}}{k!} \frac{\theta}{\Gamma(\alpha)} (\theta y)^{\alpha-1} e^{-\theta y} dy$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha) k!} \int_0^\infty e^{-(1+\theta)y} y^{k+\alpha-1} dy$$

Let $u = (1+\theta)y$. Then

Let $u = (1+\theta)y$. Then

$$= \frac{\theta^{\alpha}}{k!\,\Gamma(\alpha)(1+\theta)^{k+\alpha}} \int_0^\infty u^{k+\alpha-1} e^{-u}\, du$$

$$= \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} \left(\frac{\theta}{1+\theta}\right)^{\alpha} \left(1-\frac{\theta}{1+\theta}\right)^{k}$$

which is Negative Binomial.

Lecture 1/25

Thursday, January 25, 2018 9:11 AM

# Outline

- Finish Review
  - Transformations
- New topics
  - Martingales

## Transformations of R.V.

Ex: Chp 2-4.4

Let $X, Y \overset{iid}{\sim} Exp(2)$

and $Z = \frac{X}{\ }$

and $Z = \frac{X}{Y}$

Q: Find the dist of $Z$.

(1) Find the distribution of $(X, Z)$.

(2) Integrate to find the dist. of $Z$.

For step (1) define

$U = X \quad Z = \frac{X}{Y}$

$\Longleftrightarrow$

$X = U \quad Y = \frac{X}{Z} = \frac{U}{Z}$

Using this we find

$$f_{U,Z}(u,z) = f_{X,Y}\left(u, \frac{u}{z}\right) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix} \right|$$

$$= f_X(u) f_Y\left(\frac{u}{z}\right) \left| \begin{vmatrix} 1 & 0 \\ \frac{1}{z} & -\frac{u}{z^2} \end{vmatrix} \right|$$

$$= \lambda e^{-\lambda u} \cdot \lambda e^{-\lambda u/z} \cdot \frac{u}{z^2}$$

For step (2) integrate $z$.

$$f_Z(z) = \int^{\infty} f_{U,Z}(u,z)\,du$$

$$= \int_0 \lambda^2 e^{-\lambda\left(u + \frac{u}{z}\right)} \frac{u}{z^2}\,du$$

$$= \left(\frac{\lambda}{2}\right)^2 \int_0^\infty u\, e^{-\lambda(1+1/2)u}\, du$$

$$= \left(\frac{\lambda}{2}\right)^2 \int_0^\infty \left( -\frac{1}{\lambda(1+\frac{1}{2})} e^{-\lambda(1+\frac{1}{2})u} \right)' u\, du$$

$$= \left(\frac{\lambda}{2}\right)^2 \left[ \left( -\frac{u}{\lambda(1+\frac{1}{2})} e^{-\lambda(1+\frac{1}{2})u} \right) \Bigg|_0^\infty - \int_0^\infty \frac{-1}{\lambda(1+\frac{1}{2})} e^{-\lambda(1+1/2)u}\, du \right]$$

$$= \left(\frac{\lambda}{2}\right)^2 \left[ 0 + \frac{1}{[\lambda(1+1/2)]^2} \right]$$

$$= \frac{1}{1+z^2}, \quad 0 < z < \infty$$

which is Cauchy $\square$

<u>Ex:</u> Prob. 4.7

$X, Y \sim$ iid Exp(a) and

$Z = X+Y$.

What is the dist of

$X \mid Z$.

Step (1)

$U = X \qquad X = U$

$$u = x \quad \Longleftrightarrow \quad x = u$$
$$z = x + y \qquad\quad y = z - u$$

$$f_{u,z}(u,z) = f_{x,y}(u, z-u) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix} \right|$$

$$= a e^{-au} \, a e^{-a(z-u)} \left| \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \right|$$

$$= a^2 e^{-az}$$

step (2).

$$f_{x|z}(x|z) = \frac{f_{x,z}(x,z)}{f_z(z)}$$

$$= \frac{f_{U,Z}(u,z)}{\int f_{U,Z}(u,z)\,du} = \ldots$$

Finish as exercise

# Change of Variable

**Formula:**

Let $X$ be a R.V. with pdf $f_X$. Let $g_X$ be a monotone increasing function. Let $Y = g(X)$.

Then

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

$$= P(X \le g^{-1}(y)) = F_X[g^{-1}(y)]$$

Which gives

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{d}{dy} F_X(g^{-1}(y))$$

$$= F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

When

$$g(g^{-1}(y)) = y$$

$$\frac{\partial}{\partial y} g(g^{-1}(y)) = \frac{\partial}{\partial y} y$$

$$g'(g^{-1}(y)) \cdot \frac{\partial}{\partial y} g^{-1}(y) = 1$$

So

$$\frac{\partial}{\partial y} g^{-1}(y) = \frac{1}{g'(g^{-1}(y))}$$

Which gives

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{1}{g'(g^{-1}(y))}}$$

# Martingales

<u>Def</u>: A stochastic process $(X_n)_{n=0}^{\infty}$ is called a martingale if $\forall n$ it satisfies:

(i) $\mathbb{E}|X_n| < \infty$

(ii) $\mathbb{E}\left[X_{n+1} \mid X_0, X_1, \ldots, X_n\right] = X_n$

Rmks

(1) $\mathbb{E}(X_k) = \mathbb{E}(X_0) \quad \forall k$

Pf: $\mathbb{E}\left[X_{n+1} \mid X_0, \ldots, X_n\right] = X_n$

$$\mathbb{E}\left[\mathbb{E}\left[X_{n+1} \mid X_0, \ldots, X_n\right]\right] = \mathbb{E}(X_n)$$

$$\mathbb{E}\left[X_{n+1}\right] = \mathbb{E}\left[X_n\right] \quad \forall n$$

Hence

$$\mathbb{E}\left[X_{n+1}\right] = \mathbb{E}\left[X_n\right] = \cdots = \mathbb{E}\left[X_0\right]$$

(2) $\mathbb{E}\left[X_m \mid X_0, \ldots, X_n\right] = X_n \quad \forall m \geq n$

(3) Property (2) implies fairness in a game.

$\sigma$

**Thrm:** (Markov Inequality)

$$P[X \geq \lambda] \leq \frac{E(X)}{\lambda} \quad \text{if } X \geq 0$$

**Pf:**

$$\mathbb{E}(X) = \mathbb{E}\left[X\left(1_{\{X \geq \lambda\}} + 1_{\{X < \lambda\}}\right)\right]$$

$$\geq \lambda\, \mathbb{E}\left[1_{\{X \geq \lambda\}}\right] + \mathbb{E}\left[X 1_{\{X < \lambda\}}\right]$$

$$= \lambda P(X \geq \lambda) + C$$

for some $C \geq 0$. Then

$$\mathbb{E}(X) \geq \lambda P(X \geq \lambda). \qquad \blacksquare$$

**Thrm:** (Martingale maximal

Thrm. (Martingale … inequality). Let $X_n$ be a martingale such that $P(X_n \geq 0) = 1$. Then $\forall \lambda > 0$

$$P\left[\max_{0 \leq n \leq m} X_n \geq \lambda\right] \leq \frac{E(X_0)}{\lambda}$$

Pf:

$$\mathbb{E}(X_0) = \mathbb{E}[X_m]$$

$$= \sum_{n=0}^{m} \mathbb{E}\left(X_m 1_{\{X_0<\lambda, X_1<\lambda, \dots, X_n \geq \lambda\}}\right)$$

$$+ \mathbb{E}\left(X_m 1_{\{X_0<\lambda, \dots, X_m<\lambda\}}\right)$$

$$\geq \sum^{m} \mathbb{E}\left(X_m 1_{\{X_0<\lambda, X_1<\lambda, \dots, X_n \geq \lambda\}}\right)$$

$$= \sum_{n=0}^{m} \mathbb{E}\left( X_m \mathbb{1}\{X_0<\lambda, X_1<\lambda, \dots, X_n \ge \lambda\}\right)$$

$$= \sum_{n=0}^{m} \mathbb{E}\left\{ \mathbb{E}\left(X_m \mathbb{1}(X_0<\lambda, \dots, X_n \ge \lambda)\right) \middle| X_0, \dots, X_n \right\}$$

$$= \sum_{n=0}^{m} \mathbb{E}\left\{ \mathbb{1}_{(X_0<\lambda, \dots, X_n \ge \lambda)} \mathbb{E}\left[X_m \mid X_0, \dots, X_n\right] \right\}$$

$$= \sum_{n=0}^{m} \mathbb{E}\left[ X_n \mathbb{1}\{X_0<\lambda, \dots, X_n \ge \lambda\}\right]$$

$$\ge \lambda \sum_{n=0}^{m} P\left(X_0<\lambda, \dots, X_n \ge \lambda\right)$$

$$= \lambda\, P\left[\max_{0\le n\le m} X_n \ge \lambda\right] \qquad \blacksquare$$

Ex: $X_0 = 1$. and we bet $p \in (0,1)$. We then play a fair game. Then a fair game. Then

$$X_1 = \begin{cases} 1+p & \text{win} \\ 1-p & \text{lose} \end{cases}$$

If we keep playing

$$X_{n+1} = \begin{cases} (1+p)\, X_n & p = 1/2 \\ (1-p)\, X_n & p = 1/2 \end{cases}$$

To see why this is a martingale.

$$\mathbb{E}[X_{n+1} \mid X_0, \ldots, X_n]$$

$$= \frac{1+p}{2} X_n + \frac{1-p}{2} X_n = X_n$$

Now if we look to find

$$P(\text{double my money}) = P\left[\max_{0 \le k \le n} X_n \ge 2\right]$$

$$\le \frac{\mathbb{E} X_0}{2} = \frac{1}{2}.$$

# Lecture 1/30

Tuesday, January 30, 2018 9:28 AM

# Markov Chains (Chp 3)

· Discrete space + time

**Def:** A seq. of R.V. $\{X_n; n \geq 1\}$ has the Markov property

if $P(X_{n+1}=j \mid X_n, \ldots, X_0) = P(X_{n+1}=j \mid X_n)$

**Def:** One step ahead prob.

time

$$P_{i,j}^{n,n+1} = P(X_{n+1}=j \mid X_n=i)$$

space

also called transition probabilities.

**Def:** If $P_{ij}^{n,n+1} = P_{ij}$ (ind of $n$) then the MC has stationary trans. probabilities.

**Def:** The matrix

$$P = (P_{ij})_{1 \le i,j \le n}$$

is called the transition probability matrix.

Properties

1. $0 \le P_{ij} \le 1$

2. $\sum_{i \geq 0} P_{ij} = 1$

3. $P[X_0 = i_0, \ldots, X_n = i_n]$

$$= P[X_0 = i_0, \dots, X_{n-1} = i_{n-1}] \times P[X_n = i_n \mid X_0 = i_0, \dots, X_{n} = i_{n-1}]$$

$$= P[X_0 = i_0, \dots, X_{n-1} = i_{n-1}] P^{n,n+1}_{i_{n-1}, i_n}$$

$$=$$

$$\vdots$$

$$= \underbrace{P[X_0 = i_0]}_{\text{Initial Conditions}} \underbrace{\prod_{k \geq 1} P_{i_{k-1}, i_k}}_{\text{found in } P}$$

Ex: (Pr 3.1.2) Let $X_n$ be a sequnce of binary signals. Let signals. Let

$X_0 = 0$ (signal sent).

$X_n$ signal recieved at stage $n$

$\{X_n\}$ is MC with

$P_{00} = P_{11} = 1-\alpha$

$P_{01} = P_{10} = \alpha$

(a) determine prob of no mistakes up to and including stage 2

(b) Determine the prob. that the correct signal is recieved at stage 2.

$\underline{a}$

$$P[X_0 = 0, X_1 = 0, X_2 = 0]$$

$$= P[X_2 | X_1 = 0]\, P[X_1 = 0 | X_0 = 0]\, P(X_0 = 0)$$

$$= (1-\alpha)(1-\alpha)\, 1$$

$$= (1-\alpha)^2$$

## b

$$P[X_2=0, X_0=0]$$

$$= P[X_0=0, X_1=0, X_2=0]$$

$$= (1-\alpha)^2 + P_{10} P_{01} P[X_0=0]$$

$$= (1-\alpha)^2 + \alpha^2$$

**Def:** Let $P_{ij}^{(n)} = P[X_{m+n}=j \mid X_n=i]$ be the n-step trans. prob.

**Thm:** $$P_{ij}^{(n)} = \sum_{k\geq 0} P_{ik} P_{kj}^{(n-1)}$$

or $\underbrace{P^{(n)}}_{n\text{-step.}} = \underbrace{P^n}_{\text{one step}}$

where $P_{ij}^{(0)} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

Pf: The proof uses first step analysis

$$P_{ij}^{(n)} = P[X_n = j \mid X_0 = i]$$

$$= \sum_{k \geq 0} P[X_n = j, X_1 = k \mid X_0 = i]$$

$$= \sum_{k\geq 0} P[X_n = j \mid X_1 = k, X_0 = i]\, P[X_1 = k \mid X_0 = i]$$

$$= \sum_{k\geq 0} P[X_n = j \mid X_1 = k]\, P[X_1 = k \mid X_0 = i]$$

$$= \sum_{k\geq 0} P_{kj}^{(n-1)} P_{ik}$$

Corollary: $P[X_n = i] = p_i^{(n)}$

The same principle gives

$$p_i^{(n)} = \sum_{j\geq 0} p_j P_{ji}$$

when $P_j = P[X_0 = j]$.

<u>Ex</u>: (Pr 3.2.3)

$X_n$ = quality of $n$th item
produced by a system.

$$X_n = \begin{cases} 0 & \text{good cond.} \\ 1 & \text{bad cond.} \end{cases}$$

Suppose $X_n$ is MC and

$$P = \begin{bmatrix} .99 & .01 \\ .12 & .088 \end{bmatrix}$$

What is the prob.

$$P[X_4 = 1 \mid X_1 = 1]$$

$$= P^{(3)}_{1,1} = P^3_{1,1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= 0.68$$

Ex:

① The Ehrenherst Urn Model

- Diffusion in a membrane.


Let $Y_n$ = # of balls in A at $n^{th}$ stage

$X_n = Y_n - \alpha$

A ball is selected at random from the $2\alpha$ balls and moved to the other container.

$\{X_n; n \geq 1\}$ is a MC with state space $i = -\alpha, -\alpha+1, \ldots, 0, 1, \ldots, \alpha$

$$P_{ij} = \begin{cases} \dfrac{\alpha+i}{2\alpha} & j = i+1 \\ \dfrac{\alpha-i}{2\alpha} & j = i-1 \\ 0 & \text{o.w.} \end{cases}$$

② Inventory Model.

- Commodity is stocked to satisfy a continuing demand.

- $\xi_n$ - demand for period

$n$ with dist.

$$P[\xi_n = k] = \alpha_k, \; k = 0, 1, \ldots$$

- If at the end of the period stock quantities is $s_1$ or less than the amount is increased to $s_2$.

otherwise no action is taken.

- Let $X_n = \#$ quantity at

the end of period $n$

before restocking.

$$X_n \in \{S_2, S_2-1, \dots, 1, 0, -1, \dots\}$$

- $$X_n = \begin{cases} S_2 - \xi_{n+1} & X_n \le S_1 \\ X_n - \xi_{n+1} & S_1 < X_n \le S_2 \end{cases}$$

If $\xi$ are independent then

$\{X_n\}$ is a M.C and

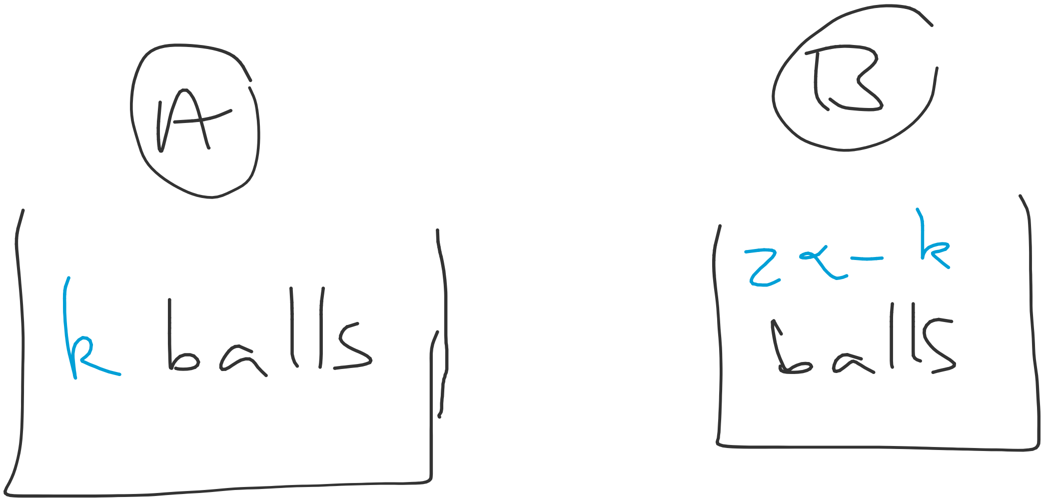
$$P[X_{n+1} = j \mid X_n = i]$$

$$= \begin{cases} P[\xi_{n+1} = i - j] & s_1 < i \leq S_2 \\ P[\xi_{n+1} = S_2 - i] & i \leq S_1 \end{cases}$$

Q: What happens when

$$\lim_{n \to \infty} p_i^{(n)}$$

# The Ehrenfest Urn Model




$Y_n$ = # balls in Urn A at the $n^{th}$ stage

$X_n = Y_n - \alpha$

Goal: Find $P_{ij} = P[X_{n+1} = j \mid X_n = i]$

If there are $k$ balls

in $A$ the probability that at the next stage there are $k-1$ is $\frac{k}{2\alpha}$

So for $P(k \mapsto k-1)$

$= 1 - P(k \mapsto k+1) = 1 - \frac{k}{2\alpha}$

So

$P[Y_{n+1} - Y_n = 1 / Y_n = k] = 1 - \frac{k}{2\alpha}$

$$P[Y_{n+1} - Y_n = -1 \mid Y_n = k] = \frac{k}{\alpha}$$

$$P_{ij} = P[X_{n+1} = j \mid X_n = i]$$

$$= P[Y_{n+1} - \alpha = j \mid Y_n - \alpha = i]$$

$$= P[Y_{n+1} = \alpha + j \mid Y_n = \alpha + i]$$

$$= P[Y_{n+1} - Y_n = j \mid Y_n = \alpha + i]$$

$$= \begin{cases} 0 & \text{if } j \neq i \pm 1 \\ 1 - \dfrac{\alpha + i}{2\alpha} & j = i+1 \\ \dfrac{\alpha + i}{2\alpha} & j = i-1 \end{cases}$$

$$= \begin{cases} 0 & j \neq i \pm 1 \\ \dfrac{\alpha - i}{2\alpha} & j = i+1 \\ \dfrac{\alpha + i}{2\alpha} & j = i-1 \end{cases}$$

# First Step Analysis

<u>Idea</u>: Condition on the event of first

$T_1$

event of first
transition. Then
use total prob.
law.

Ex:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- States 0/3 are absorbing
- States 1/2 are transient

Let $T = \min\{n \geq 0 : X_n \in \{0,3\}\}$

We are interested in

(i) $u_i = P[X_T = 0 \mid X_0 = i]$

(ii) $\nu_i = E[T \mid X_0 = i]$

(i) Trivially $u_0 = 1 \quad u_3 = 0.$

$$u_1 = P[X_T = 0 \mid X_0 = 1]$$

$$= \sum_{k=0}^{3} P[X_T = 0, X_1 = k \mid X_0 = 1]$$

$$= \sum_{k=0}^{3} P[X_T = 0 \mid X_1 = k, X_0 = 1] \, P[X_1 = k \mid X_0 = 1]$$

M.C.

$$= \sum_{k=0}^{3} P[X_T = 0 \mid X_1 = k] \, P_{1k}$$

$$= 1 P_{10} + u_1 P_{11} + u_2 P_{12} + 0 \cdot P_{13}$$

**Rmk:** $u_1 = P[X_T = 0 \mid X_0 = 1]$

But $u_1 = P[X_T = 0 \mid X_1 = 1]$

by the Markov Property.

$$\left.\begin{aligned} u_1 &= P_{10} + P_{11} u_1 + P_{12} u_2 \\ u_2 &= P_{20} + P_{21} u_1 + P_{22} u_2 \end{aligned}\right\} \text{solve this system}$$

**Thrm:** In general

$$u_i = P[\text{Absorb in state } k \mid X_0 = i]$$

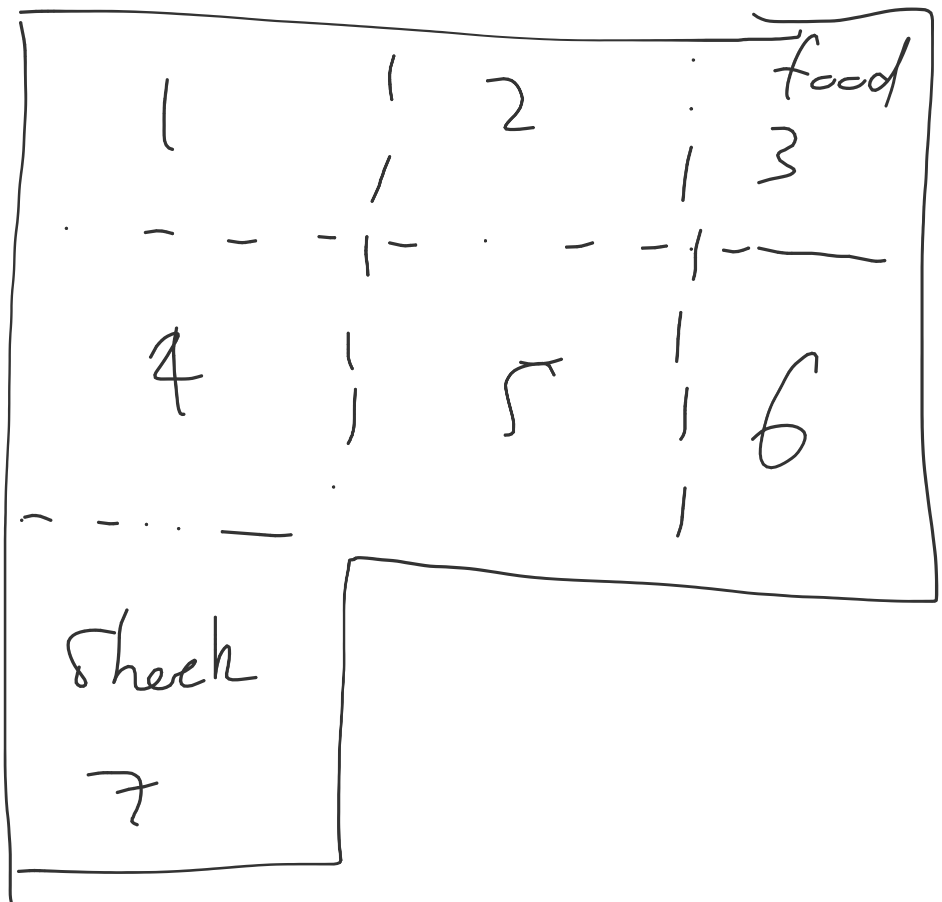
$$= P_{ik} + \sum_{j \text{ transient states}} P_{ij} u_j$$

Ex: $v_i = E[T \mid X_0 = i]$

$$\left.\begin{aligned} v_1 &= 1 + P_{11} v_1 + P_{12} v_2 \\ v_2 &= 1 + P_{21} v_1 + P_{22} v_2 \end{aligned}\right\} \text{Solve this system}$$

* No prob on the $1$ because we are guaranteed to take one step $(p=1)$.

Ex: (Pr 3.4.5)




Assuming the rat moves at random, what is the probability of find food before getting sharked if $X_0 = 4$?

States 3 and 7 are absorbing.

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P[X_T = 3 \mid X_0 = i] = u_i$$

Goal: Find $u_4$.

$$u_1 = 1/2\, u_2 + 1/2\, u_4$$

$$u_2 = 1/3\, u_1 + 1/3\, u_3 + 1/3\, u_5$$

$$u_2 = \tfrac{1}{3}u_1 + \tfrac{1}{3}u_3 + \tfrac{1}{3}u_5$$

$$u_3 = 1$$

$$u_4 = \tfrac{1}{3}u_1 + \tfrac{1}{3}u_5 + \tfrac{1}{3}u_7$$

$$u_5 = \tfrac{1}{3}u_2 + \tfrac{1}{3}u_4 + \tfrac{1}{3}u_6$$

$$u_6 = \tfrac{1}{2}u_3 + \tfrac{1}{2}u_5$$

$$u_7 = 0$$

Ex: Suppose we have a random absorbtion time

$$T = \min\{n \geq 0 : X_n \geq r\}$$

where all states $\geq r$ are absorbing.

absorbing.

Associated with transient state $i$ is cost $g(i)$.

$w_i$ = mean accumulated cost up to absorbtion.

$$w_i = E\left[\sum_{n=0}^{T-1} g(X_n) \mid X_0 = i\right]$$

Rmks: If $g(i) = 1$ then

$$w_i = E[T \mid X_0 = i] = v_i$$

If $g(i) = \begin{cases} 1, & i \text{ transient equal to } k \\ 0, & \text{o.w.} \end{cases}$

visit $k$

then $W_i$ = # times we visit $k$ prior to absorbtion.

By first step anlysis

$$W_i = g(i) + \sum_{j=0}^{r-1} P_{ij} W_j$$

transient

<u>Ex:</u> (Pr3.4.7) $X_n$ M.C. with p.t.m. $P$. If we have a discount $0<b<1$ and cost function $c(i)$.

... discount

Total expected discount cost starting from $i$.

Let $h(i) = \mathbb{E}\left(\sum_{n=0}^{\infty} b^n c(X_n) \mid X_0 = i\right)$

$= c(i) + \mathbb{E}\left(\sum_{n=1}^{\infty} b^n c(X_n) \mid X_0 = i\right)$

$= c(i) + \sum_j \mathbb{E}\left(\sum_{n=1}^{\infty} b^n c(X_n) \mid X_1 = j, X_0 = i\right) P_{ij}$

$= c(i) + \sum_j \mathbb{E}\left(\sum_{n=1}^{\infty} b^n c(X_n) \mid X_1 = j\right) P_{ij}$

$= c(i) + b \sum_j \mathbb{E}\left[\sum_{n=0}^{\infty} b^{n-1} c(X_n) \mid X_1 = j\right] P_{ij}$

$= c(i) + b \sum_j h_j P_{ij}$ $\Big\}$ Solve this

$$= c(i) + b \sum_j h_j P_{ij} \Bigg\}$$ system.

# Lecture 2/6

Tuesday, February 6, 2018 9:27 AM

# Special Markov Chains

## Two State MC

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}$$

Thrm: The n-th transition probability matrix is

$$P^{(n)} = P^n$$

$$= \frac{1}{a+b} \underbrace{\begin{bmatrix} b & a \\ b & a \end{bmatrix}}_{A} + \frac{(1-a-b)^n}{a+b} \underbrace{\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}}_{B}$$

Moreover

where $a+b<1$. Moreover

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

Pf: For $n=1$

$$P^{(1)} = P^1 = P$$

Assume the result holds for $n$. Now

$$P^{(n+1)} = P^n P$$

$$= \left(\frac{1}{a+b} A + \frac{(1-a-b)^n}{a+b} B\right) P$$

$$= \frac{1}{\phantom{a+b}} (AP) + \frac{(1-a-b)^n}{\phantom{a+b}} BP$$

$$= \frac{1}{a+b}(AP) + \frac{(1-a-b)}{a+b} BP$$

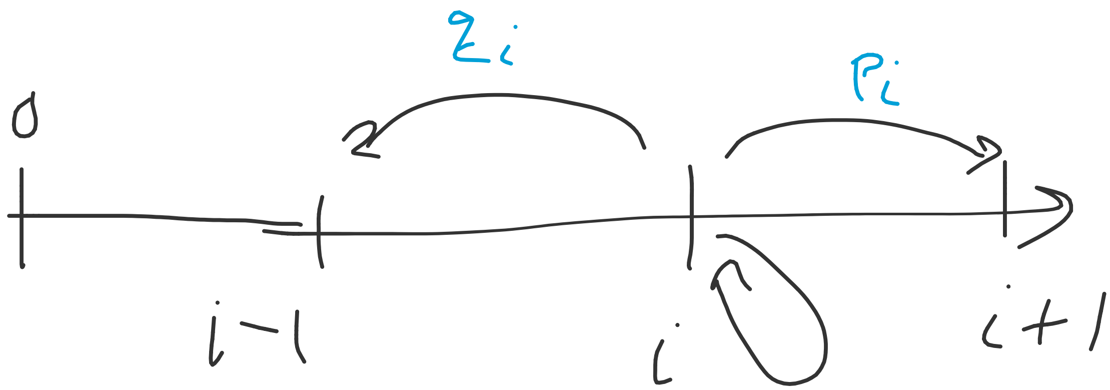
$$= \frac{1}{a+b}\left(A + (1-a-b)^{n+1} B\right)$$

where $AP = A$
$BP = (1-a-b)B$

Rmk: Part (ii) of the theorem says that the prob. of finding the M.C. at state 0 is $\frac{b}{a+b}$ and at state 1 $\frac{a}{a+b}$

One dimensional R.W.

# One dimensional R.W.




$$1 - q_i - p_i := r_i$$

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & & \\
\hline
0 & r_0 & p_0 & 0 & \cdots & & \\
1 & q_1 & r_1 & p_1 & 0 & \cdots & \\
2 & 0 & q_2 & r_2 & p_2 & \cdots & \\
\vdots & \vdots & & \ddots & \ddots & \ddots &
\end{array}
$$

$$P = \begin{bmatrix} r_0 & p_0 & 0 & \cdots & & \\ q_1 & r_1 & p_1 & 0 & \cdots & \\ 0 & q_2 & r_2 & p_2 & \cdots & \\ \vdots & & \ddots & \ddots & \ddots & \end{bmatrix}$$

Now suppose that we have absorbing boundaries

we have "absorbing boundaries"
with $r_i = 0$. Now define

$$T = \min\{n \geq 0 : X_n \in \{0, N\}\}$$

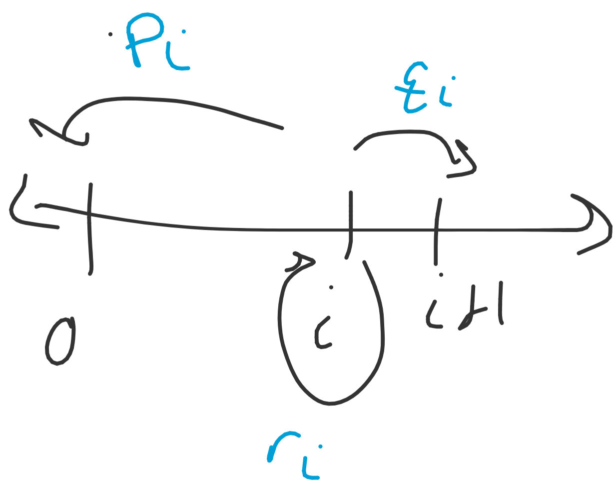
$$u_i = P[X_T = 0 \mid X_0 = i]$$

$$= q_i u_{i-1} + p_i u_{i+1}$$

we can use FSA to solve for $u_i$.

<u>Success Runs</u>

$$P = \begin{bmatrix} p_0 & q_0 & 0 & \cdots & & & \\ p_1 & r_1 & q_1 & 0 & \cdots & & \\ p_2 & 0 & r_2 & q_2 & 0 & \cdots & \\ \vdots & 0 & 0 & r_3 & q_3 & & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \end{bmatrix}$$




Repeated trials with

$P(S) = a \quad P(F) = b$

Success run of length

$n$ is when

$F\,S\,S\,S\,S\,S \cdots S$

$$F \underbrace{SSSSS \cdots S}_{n\text{-times}}$$

$r_n = 0 \qquad q_n = a \qquad p_n = 1 - a$

## Renewal Process

Consider light bulbs whose lifetime is a R.V. $\xi$ with

$$P[\xi = k] = a_k > 0$$

Each bulb is replaced when it dies.

First bulb lasts $\xi_1$

Second bulb lasts $\xi_2 + \xi_1$

Third bulb lasts $\xi_3 + \xi_2 + \xi_1$

.

.

.

Let $X_n$ = age of bulb in service at time $n$

$X_n = 0$ when the bulb fails.

This is a special type of success run with $r_n = 0 \ \forall n$ and

$$p_n = \frac{a_{n+1}}{\sum_{j=n+1}^{\infty} a_j}$$

# Solving 1-D RW

$\tau = \min\{n \geq 0 : X_n \in \{0, N\}\}$

$u_i = P[X_\tau = 0 \mid X_0 = i]$

$r_i = 0, \quad q_i = q, \quad p_i = p$

Then we have

$u_k = p u_{k+1} + q u_{k-1}$

$u_0 = 1, \quad u_N = 0$

Solving this system by the following trick

$$u_k = p u_{k+1} + q u_{k-1}$$

$$(p+q) u_k = p u_{k+1} + q u_{k-1}$$

$$0 = p(u_{k+1} - u_k) + q(u_{k-1} - u_k)$$

Define $x_k = u_k - u_{k-1}$ So

$$0 = p x_{k+1} - q x_k$$

$$x_{k+1} = \frac{q}{p} x_k \quad \forall k$$

$$= \left(\frac{q}{p}\right)\left(\frac{q}{p} x_{k-1}\right)$$

$$\vdots$$

$$= \left(\frac{q}{\phantom{p}}\right)^{k+1}$$

$$= \left(\frac{q}{p}\right)^{k+1} x_0$$

Since this is true for every $k \in \mathbb{N}$. Hence

$$x_N = \left(\frac{q}{p}\right)^{N-1} x_1$$

Reformulating in terms of the $u_k$.

$$u_k = (u_k - u_{k-1}) + (u_{k-1} - u_{k-2}) + \dots$$
$$\dots + (u_1 - u_0) + u_0$$
$$= \sum_{j=1}^{k} x_j + u_0$$
$$= \sum^{k} \left(\frac{q}{p}\right)^{j-1} x \; + u$$

$$= \sum_{j=1} \left(\frac{q}{p}\right)^{j-1} x_1 + u_0$$

$$= x_1 \sum_{j=1}^{k} \left(\frac{q}{p}\right)^{j+1} + 1$$

$x_1$ only remaining unknown

$\Rightarrow$ $u_1$ only remaining unknown.

$u_N = 0$ so

$$x_1 \sum_{j=1}^{N} \left(\frac{q}{p}\right)^{j-1} + 1 = 0$$

$$\Rightarrow \quad x_1 = \frac{-1}{\sum_{j=1}^{N} \left(\frac{q}{p}\right)^{j-1}}$$

Using this for $u_k$ we get

$$u_k = 1 - \frac{\sum_{j=0}^{k-1} \left(\frac{q}{p}\right)^j}{\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j}$$

$$= \begin{cases} 1 - \frac{k}{N} & \text{if } q = p = 1/2 \\ 1 - \frac{1-(q/p)^k}{1-(q/p)^N} & \text{if } q \neq p \end{cases}$$

<u>Ex</u>: If a player $A$ begins with $i$ units and player $B$ with $N-i$ units

player $B$ with $N-i$ units

the probability $A$ loses everything before $B$ goes bankrupt is given by $u_i$.

If Player $B$ is richer than $A$. then

$$u_i = \begin{cases} 1 & q \geq p \\ \left(\frac{q}{p}\right)^i & q < p \end{cases}$$

So player $A$ only has a chance if the game is unfair.

**Ex:** Let $V_i = \mathbb{E}[T \mid X_0 = i]$

$V_i = 1 + pV_{i+1} + qV_{i-1}$

$V_0 = V_N = 0$

$p + q = 1.$

So

$0 = 1 + p(V_{i+1} - V_i) + q(V_{i-1} - V_i)$

Setting $X_k = V_{k+1} - V_k$

which gives

$$X_i = \frac{q X_{i-1}}{p} - \frac{1}{p}$$

$$= \left(\frac{q}{p}\right)^{i-1} X_1 - \frac{1}{p}\sum_{j=0}^{i-2}\left(\frac{q}{p}\right)^{j}$$

But $X_1 = V_1$ so

$$V_N = \sum_{j=0}^{N-1} X_j + V_0$$

$$= \sum_{j=0}^{N-1} X_j$$

$$= \sum_{k=0}^{N-1}\left[\left(\frac{q}{p}\right)^{k} X_0 - \frac{1}{p}\sum_{j=0}^{k-2}\left(\frac{q}{p}\right)^{j}\right]$$

$$= \sum_{k=0}^{N-1}\left[\left(\frac{q}{p}\right)^{k} V_1 - \frac{1}{p}\sum_{j=0}^{k-2}\left(\frac{q}{p}\right)^{j}\right]$$

$$= 0$$

Hence we solve this for $v_1$ which gets $x_0$

So we get

$$v_k = \sum_{j=1}^{k-1} x_j$$

$$= \sum_{j=0}^{k-1} \left[ \left(\frac{q}{p}\right)^j x_0 - \frac{1}{p} \sum_{m=0}^{j} \left(\frac{q}{p}\right)^m \right]$$

Lecture 2/8

Thursday, February 8, 2018 9:27 AM

# Success Runs

$$P = \begin{bmatrix} p_0 & q_0 & 0 & \cdots & & \\ p_1 & r_1 & q_1 & \cdots & & \\ p_2 & 0 & r_2 & q_2 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & & \ddots & \ddots \end{bmatrix}$$

Consider light bulbs whose life time is measured in discrete units of time.

$$\mathbb{P}(\xi = k) = a_k \qquad \sum a_k = 1$$

Each light bulb replaced when it burns out.

Let $X_n$ = age of light bulb in service at time $n$. $X_n = 0$ at time of failure.

$$r_n = 0 \quad \forall n \in \mathbb{N}.$$

$$q_n = \frac{a_{n+1}}{\sum a_{n+k}}$$

$k=1$

$p_n = 1 - q_n$

Lets say $p_0 = 1$

$q_r = 0$. $r_k = 0 \ \forall k$.

The MC has states

$\{0, 1, \ldots, N\}$ $r_N = 1$

$q_N = p_N = 0$

$$T = \min\{n \geq 0 : X_n \in \{0, N\}\}$$

$$u_i = P[X_T = 0 \mid X_0 = i]$$

$$u_k = p_k + q_k u_{k+1}$$

So

$$u_{k+1} = \frac{-p_k}{q_k} + \frac{1}{q_k} u_k$$

$$= \frac{-p_k}{q_k} + \frac{1}{q_k}\left(\frac{-p_{k-1}}{q_{k-1}} + \frac{1}{q_{k-1}} u_{k-1}\right)$$

$$= \frac{-p_k}{q_k} - \frac{p_{k-1}}{q_k q_{k-1}} + \frac{1}{q_k q_{k-1}} u_{k-1}$$

$$= \frac{-p_k}{q_k} - \frac{p_{k-1}}{q_k q_{k-1}} - \frac{p_{k-2}}{q_k q_{k-1} q_{k-2}}$$

$$+ \frac{u_{k-2}}{q_k q_{k-1} q_{k-2}}$$

$$= -\sum_{i=1}^{k} \frac{1}{\prod_{j=i+1}^{k} q_j} \frac{p_i}{q_i} + \frac{1}{\prod_{i=1}^{k} q_i} \cdot u_1$$

# Branching Processes

An organism has $\xi$ offspring with

$$P[\xi = k] = p_k, \quad k = 0, 1, 2, \ldots$$

$$p_k \geq 0 \quad \sum p_k = 1.$$

Then each offspring has the same behavior.

Let $X_n$ = size of population

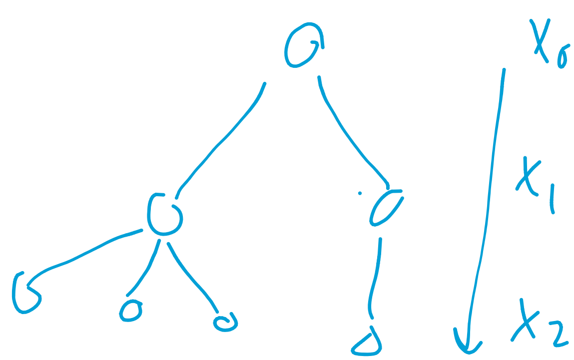
at the $n^{th}$ generation.

In the $n^{th}$ generation each member gives birth to a number of offsprings.

$$\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{X_n}^{(n)}$$

So

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$




Ex: Surival of family names, Neutron chain reactions, and

Neutron chain reactions/ and cell division

$X_{n+1}$ is a random sum.

If $\mathbb{E}(\xi) = \mu$ $Var(\xi) = \sigma^2$.

Let

$$\mu(n) = \mathbb{E}[X_n | X_0 = 1]$$

$$Var(n) = Var[X_n | X_0 = 1]$$

So by properties of R.S.

$$\mu(n+1) = \mu \cdot \mu(n) = \mu^{n+1} \mu(0) = \mu^{n+1}$$

$$Var(n+1) = \sigma^2 \mu(n) + \mu^2 Var(n)$$

$$= \sigma^2 \mu^n + \sigma^2 \left[ \mu^n + \mu^{n+1} + \dots + \mu^{2n-1} \right]$$

$$= \begin{cases} \sigma^2 (n+1) \text{ if } & \mu = 1 \\ \sigma^2 \mu^n \dfrac{1-\mu^{n+1}}{1-\mu} \text{ if } & \mu \neq 1 \end{cases}$$

if $\mu = 1$ variance increasing linearily

$\mu < 1$ —〃— decreases geometrically

$\mu > 1$ —〃— increases geometrically

# Extinction Probabilities

Let $N = \min\{n \geq 0 : X_n = 0\}$

$u_n = \mathbb{P}[N \leq n] = \mathbb{P}[X_n = 0]$

The probability that any particular descendant dies out in the $n-1$ generation is $u_{n-1}$. And the probability that all $k$ subpopulations die out is

$(u_{n-1})^k$

By the total probability law

$$u_n = \sum_{k=0}^{\infty} p_k (u_{n-1})^k$$

$u_0 = 0$

$u_1 = p_0$ = prob that the original member has no offspring.

Let $\xi$ be s.t. $\mathbb{P}(\xi = k) = a_k$. Then the generating function

$$\phi_{\xi}(s) = \mathbb{E}[s^{\xi}] = \sum p_k s^k$$

Recall that

$$p_k = \frac{1}{k!} \frac{d^k \phi(s)}{ds^k}\bigg|_{k=0}$$

$$\phi_{\sum \xi_i}(s) = \prod_{i=1}^{n} \phi_{\xi_i}(s)$$

$$\phi'(s)\big|_{s=1} = \mathbb{E}(\xi) = \mu$$

$$\phi''(s)\big|_{s=1} = \mathbb{E}(\xi^2) - \mathbb{E}(\xi)$$

If $\xi \sim Pois(\lambda)$

$$\phi(s) = \sum s^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda(1-s)} \quad |s|<1$$

If $\xi$ has generating function $\phi(s) = \mathbb{E}(s^{\xi})$

Then

$$u_n = \sum_k p_k (u_{n-1})^k = \phi(u_{n-1})$$

So from here we solve iteratively

$$u_n = \phi(u_{n-1}) = \phi^{(2)}(u_{n-2})$$

$$= \dots = \phi^{(n-1)}(u_1)$$

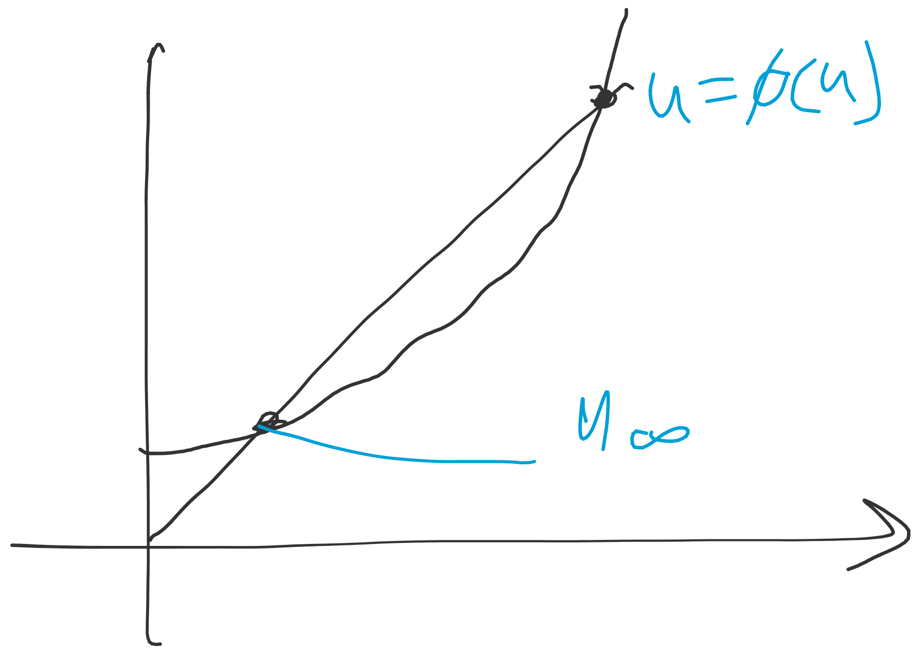
$$= \phi^{(n)}(u_0) = \phi^{(n)}(0)$$

Arguement:

$$u = \lim_n u_n = \lim_n \phi(u_{n-1})$$

gives the formula

$$u = \phi(u)$$




The key question is wheater $\phi$ has a fixed point.

This can be determined via $\phi'(1)=\mu$

If $\phi'(1) = \mu \leq 1 \quad u_\infty = 1$

$\phi'(1) = \mu > 1 \quad u_\infty < 1$

# Lecture 2/13

Tuesday, February 13, 2018 9:29 AM

# Branching Processes

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

$$N = \min\{n \geq 0 : X_n = 0\}$$

$$u_n = \mathbb{P}(N \leq n) = \mathbb{P}(X_n = 0)$$

$$u_n = \sum_{k=0}^{\infty} p_n \, u_{n-1}^{k} = \phi(u_{n-1})$$

$$\phi(s) = \mathbb{E}(s^{\xi})$$

exists

If $\lim_n u_n = u_\infty$ exists

then $u_\infty$ satisfies

$$u_\infty = \phi(u_\infty)$$

If $\phi'(1) \leq 1 \Rightarrow$ no crossing

$\Rightarrow u_\infty = 1$

If $\phi'(1) > 1$ then $u_\infty < 1$

and it will solve $u_\infty = \phi(u_\infty)$

These are true because

$\phi'(1) = \mu = \mathbb{E}(\xi)$.

Thrm: Let $\phi(s)$ be the probability generating function of $\xi$ and let $g_N(s)$ be the PGF of $N$. Let $X_n = \sum_{i=1}^{N} \xi_i$. Assume that $\xi_i$ are iid and $\xi_i \perp\!\!\!\perp N$.

Then $X_n$ has PGF

$$h_{X_n}(s) = g_N(\phi(s)).$$

$\underline{\text{Pf}}$: $$h_{X_n}(s) = \sum_{k=1}^{\infty} \mathbb{P}(X_N = k) s^k$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{N} \xi_i = k\Big) s^k$$

$$= \sum_{k=1}^{\infty} \sum_{n \geq 1} \mathbb{P}\Big[\sum_{i=1}^{n} \xi_i = k \,\Big|\, N = n\Big] \times \mathbb{P}(N = n) s^k$$

$$\boxed{\phi_{\sum Y_i}(s) = \mathbb{E} s^{\sum Y_i} = \prod e^{Y_i} = \phi(s)^n}$$

$$= \sum_{n \geq 1} \left( \sum_{k \geq 1} \mathbb{P}\left[\sum_{i=1}^{n} \xi_i = k \mid N = n\right] s^k \right) P(N=n)$$

$$= \sum_{n \geq 1} [\phi(s)]^n \, \mathbb{P}(N=n) = g_N(\phi(s))$$

For branching processes,

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

So $\phi_{n+1}(s) = \phi_n(\phi(s))$

$$= \phi_{n-1}(\phi(\phi(s))) = \dots = \phi(\phi_n(s))$$

Ex: Consider a pure

Ex: Consider a pure death process. In each period each individual survives with prob $p$ and dies with prob $q$.

$$\phi(s) = \mathbb{E}(s^{\xi}) = q + sp$$

$$\phi_2(s) = q + \phi(s)p$$

$$= q + qp + sp^2$$

$$= 1 - p^2 + sp^2$$

$\vdots$

$$\phi_n(s) = 1 - p^n + s p^n$$

Using this, we can ask extinction type questions.

$$\mathbb{P}[T = n \mid X_0 = k]$$

$$= \mathbb{P}[X_n = 0, X_{n-1} > 0 \mid X_0 = k]$$

$$= \mathbb{P}[X_n = 0 \mid X_0 = k] - \mathbb{P}[X_{n-1} = 0 \mid X_0 = k]$$

$$\boxed{p_k = \frac{1}{k!}\frac{d}{ds^k}\phi(s)\Big|_{s=0}}$$

$$= (\phi_n(0))^k - (\phi_{n-1}(0))^k$$

$$= (1-p^n)^k - (1-p^{n-1})^k$$

# Long Run Behavior of M.C.

**Def:** A M.C. is regular iff all elements of $P^k$ are strictly positive for some $k \geq 1$.

**Rmks:**

1. If $\exists k$ s.t. $P^k$ has nonzero elements then $P^{k+n}$ has nonzero elements.

2. If $P$ is s.t.

   (i) There is at least one state $i$ s.t. $P_{ii} > 0$

   (ii) For every $(i,j)$ there is a path $i \to j$ with probability greater than zero.

then the M.C. is regular.

3. For a regular dist. there exists a limiting probability distribution $\pi$ where $\sum \pi_i = 1$, $\pi_i > 0$ and is independent of the initial state.

$$\lim_{n} \mathbb{P}[X_n = j \mid X_0 = i]$$

$$= \lim_{n} P_{ij}^{n} = \pi_j .$$

Ex: (2-state M.C).

$$\mathbb{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\mathbb{P}^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} \quad \pi = (\pi_0, \pi_1).$$

Thrm: Let $\mathbb{P}$ be a trans. prob. matrix over $\{0, 1, \ldots, N\}$. for a regular M.C. Then $\pi$ the limiting dist. is the unique nonnegative solution of the equations

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj} \iff \pi = \pi \mathbb{P}$$

Rmk: $\pi = \pi \mathbb{P}$ not $\pi = \mathbb{P} \pi$

Solve the eigen system

$\sigma = \pi(I - \mathbb{P})$.

Pf: Because the M.C.

is regular $\exists \pi_j$ s.t.

$$\lim_n P_{ij}^{(n)} = \pi_j$$ (existance)

(characterization)

$$\mathbb{P}^{(n)} = \underbrace{\mathbb{P}^n = \mathbb{P}^{n-1}\mathbb{P}}$$

$$P_{ij}^{(n)} = \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj}$$

$$\lim_{n} P_{ij}^{(n)} = \sum_{k=0}^{N} \lim_{n} P_{ik}^{(n-1)} P_{kj}$$

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}$$

$$= (\pi P)_j$$

(uniqueness).

Suppose that we have another solution $x$.

Both satisfies

$$\pi_j = \sum \pi_k P_{kj} \qquad x_j = \sum x_k P_{kj}$$

We now have

$$x_j = \sum_{k=0}^{N} x_k P_{kj}$$

So we can write

$$x_j P_{jl} = \sum_k x_k P_{kj} P_{jl}$$

Summing over $j$.

$$\sum_{j=0}^{N} x_j P_{jl} = \sum_j \sum_k x_k P_{kj} P_{jl}$$

$$= \sum_k x_k \sum_j P_{kj} P_{jl}$$

$$\sum_{j=0}^{N} x_j P_{jl} = \sum_k x_k P_{kl}^{(2)}$$

$$x_l = \sum_k x_k P_{kl}^{(2)} \quad \forall\, l = 0, 1, \ldots, N$$

Inductively we have

$$x_l = \sum_{k=0}^{N} x_k P_{kl}^{(n)} \quad \forall\, l.$$

Let $n \to \infty$

$$x_l = \sum_{k=0}^{N} x_k \left[\lim_n \left(P_{kl}^{(n)}\right)\right]$$

$$X_\ell = \sum_{k=0}^{N} X_k \pi_\ell$$

$$X_\ell = \pi_\ell \sum_{k=0}^{N} X_k = \pi_\ell .$$

Lecture 2/15

Thursday, February 15, 2018 9:33 AM

If $X_n$ is a regular markov chain.

Then $\pi_j = \lim_{n \to \infty} P_{ij}^{n}$

limiting distribution

Then the vector

$\pi = (\pi_0, \dots, \pi_N)$ satisfies

$$\pi P = \pi$$

$$\Longleftrightarrow$$

$$\pi_j = \sum^{N} \pi_i P_{ij}$$

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}$$

<u>Ex:</u>

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{bmatrix}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 = 0.4\pi_0 + \pi_1 0.05 + 0.05\pi_2$$

$$\pi_1 = 0.5\pi_0 + 0.7\pi_1 + \pi_2 0.5$$

$$\pi_2 = 0.1\pi_1 + 0.25\pi_1 + 0.45\pi$$

From the system

$\pi P = \pi$ not $P\pi = \pi$.

Def: A doubly stochastic matrix is a matrix $A$

s.t. $\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = 1 \quad \forall i,j \in [n]$.

Thrm: For a regular, doubly stochastic matrix

$P$ then $\pi_j = \frac{1}{N}, \; j = 0, 1, \ldots, N-1$.

then $\pi_j = \frac{1}{N}$, ...

Pf: Since it is regular we know $\pi$ exists and $\pi P = \pi$, $\sum \pi_j = 1$ and it is unique. Plug in $1/j$ we have

$$\frac{1}{N} = \sum_k \frac{1}{N} P_{kj}$$

$$\Updownarrow$$

$$1 = \sum P_{kj}$$

$$1 = \sum_k P_{kj}$$

$$\Updownarrow$$

Doubly stochastic matrix.

## Interpretation of $\pi_j$

(i) $\pi_j = \lim_n P_{ij}^{(n)}$

(ii) Probability of find the m.c. at state $j$

after a long period of time.

(iii) Long run mean fraction of time that $X_n$ spends in state $j$.

Thus if each visit to state $j$ incurs a cost $c_j$ then the long run mean cost per unit time is

$$\sum_{j=0}^{N} c_j \pi_j$$

Justifications

(2) If $\pi_j = \lim_{n\to\infty} P_{ij}^{n}$ then

$$\lim_{n} \frac{1}{n} \sum_{k=0}^{N-1} P_{ij}^{(k)} = \pi_j \quad (SLLN) \ldots$$

(3) $$\mathbb{E}\left[ \frac{1}{n} \sum_{k=0}^{n-1} 1\{X=j\} \,/\, X_0 = i \right]$$

mean fraction of time

mean fraction at time at state $j$.

$$= \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(X_k = j \mid X_0 = i)$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{(k)}$$

$$\longrightarrow \pi_j$$

## Including History in Description

We can turn most discrete chains to M.C. by

the statespace.

enlarging the statespace.

Ex: Let's say we are modeling weather on two states $\{\text{Sunny}, \text{Cloudy}\}$. Suppose $W_{n+1}$ depends on $W_n, W_{n-1}$.

So extending the state space we have

$W_n \quad W_{n-1}$

$(S, S)$ Then apply

$(S, S)$ this MC to
$(S, C)$ model $(W_{n+1}, W_n)$.
$(C, S)$
$(C, C)$

$\boxed{\text{Read } 2.2,\ 2.4,\ 2.5}$

# Classification of States

- Not all MC are regular

Ex: $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$P_3 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & \underline{1} \end{bmatrix}$

- Not regular but has a limiting dist.

<u>Def:</u> State $j$ is <u>accessible</u> from state $i$ if $\exists k < \infty$ s.t. $P_{ij}^{k} > 0$.

<u>Def</u>: We say $i$ communicates with $j$ if $\exists k_1, k_2$ s.t.

$$P_{ij}^{k_1} > 0 \quad P_{ji}^{k_2} > 0.$$

Two states do not communicate

if $\nexists k_1$ or $\nexists k_2$ s.t.

$$P_{ij}^{k_1} > 0 \qquad P_{ij}^{k_2} > 0.$$

Communication is a group operator.

(1) $i \longleftrightarrow i$

(2) $i \longleftrightarrow j$ then $j \longleftrightarrow i$

(3) $i \longleftrightarrow j$ $j \longleftrightarrow k$ then

$i \longleftrightarrow k$

Def: A M.C. is irreducible

**Def**: A M.C. is irreducible if all states communicate.

**Def**: A state $i$ has period $d(i)$ is the gcd of all integers $n \geq 1$ for which

$$P_{ii}^{(n)} > 0.$$

**Proporties**

① $i \longleftrightarrow j$ then $d(i) = d(j)$

② If state $i$ has

② If stat... period $d(i)$ then
$\exists N(i)$ s.t. $P_{ii}^{(N(i)d(i))} > 0$

③ If $P_{ji}^{(m)} > 0$ then
$P_{ji}^{(m+nd(i))} > 0$ $\forall n$ large

Ex:

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

$P \qquad \qquad P \quad {}^2 = 0 \quad P_{00}^3 = 0$

$P_{00} = 0 \quad P_{00}^{2} = 0 \quad P_{00}^{3} = 0$

$P_{00}^{4} = 1/2 \quad P_{00}^{5} = 0 \quad P_{00}^{6} = 1/2$

$P_{00}^{7} = 0 \quad P_{00}^{8} = 1/2 \quad \ldots$

So $P_{00}^{(n)} > 0$ for $n \in \{4, 6, 8, \ldots\}$

and $d(0) = 2$.

As all states communicate

$d(i) = 2 \quad i = 0, 1, 2, 3.$

# Lecture 2/22

Thursday, February 22, 2018 9:34 AM

**Def:** A mc is irreducible iff $i \leftrightarrow j \;\; \forall i,j$

**Def:** State $j$ is accessible from state $i$ if

$$P_{ik_1} P_{k_1 k_2} \cdots P_{k_{n-1} j} > 0$$

**Def:** Periodicity is given by

$$d(i) = \gcd\{n : P_{ii}^{(n)} > 0\}$$

**Def:** A M.C. is aperiodic

if $d(i) = 1 \ \forall i.$

## Characterizing States

Let $f_{ii}^{(n)} = \mathbb{P}[X_n = i, X_k \neq i \mid X_0 = i$

Trivially $f_{ii}^{(1)} = P_{ii}$

**Lemma:** $P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}$

for $f_{ii}^{(0)} \equiv 0$

**Pf:** Let

$E_k = \{X_0 = X_n = i,$ first return to $i$ is at $k^{th}$ trans.$\}$

$\mathbb{P}[E_k] = \mathbb{P}($first $\mid X_0 = i)\,\mathbb{P}(X$ …

is at ]

$$\mathbb{P}[E_k] = \mathbb{P}(\text{first}\ldots \mid X_0 = i)\, \mathbb{P}(X_n = i \mid X_k = i)$$

$$= f_{ii}^{(k)}\, p_{ii}^{(n-k)}$$

So

$$p_{ii}^{(n)} = \mathbb{P}[X_n = i \mid X_0 = i]$$

$$= \sum_{k=0}^{n} \mathbb{P}(E_k)$$

$$= \sum_{k=0}^{n} f_{ii}^{(k)}\, p_{ii}^{(n-k)} \qquad \blacksquare$$

Let $f_{ii} = \mathbb{P}[$ will return to state i at some $n < \infty \mid X_0 = i]$

$$f_{ii} = \sum_{n=0}^{\infty} f_{ii}^{(n)} = \lim_{n \to \infty} \sum_{k=0}^{n} f_{ii}^{(k)}$$

Def: A state is recurrent iff $f_{ii} = 1$

Def: A state is transient iff $f_{ii} < 1$

Thrm: A state is recurrent iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$. A state is transient iff $\sum^{\infty} P_{ii}^{(n)} < \infty$

$n \geq 1$

<u>Pf:</u> (transient $\Rightarrow$)

Assume $i$ is transient.

Define $m$ = # of returns to state $i$.

$$m = \sum_{k=1}^{\infty} 1\{X_k = i\}$$

Now note

$$\mathbb{P}(\text{return to state } i \text{ at least } k \text{ times}) = (f_{ii})^k$$

$$\mathbb{P}(m \geq k \mid X_0 = i) = (f_{ii})^k$$

So $m \sim \text{Geom}(f_{ii})$

$$\mathbb{E}(m) = \frac{f_{ii}}{1-f_{ii}} < \infty \quad \text{b.c. } f_{ii} < 1$$

$$\mathbb{E}(m) = \sum_{n=1}^{\infty} \mathbb{P}(X_n = i \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P_{ii}^{(n)} = \frac{f_{ii}}{1-f_{ii}} < \infty$$

($\Leftarrow$) If $\sum P_{ii}^{(n)} < \infty$

then $\mathbb{E}(m) < \infty$

Since $m \geq 0$ $\quad |m| < \infty$

So $m$ is a finite R.V.

So the number of returns

is finite. So starting

from $i$, the chain returns

to $i$ a finite number of

times $1-f_{ii}>0$

$\Longrightarrow f_{ii}<1$

Cor: $i \leftrightarrow j$ and $i$ recurrent

implies $j$ is recurrent.

Pf: Since $i \leftrightarrow j$ $\exists\, m,n>0$

s.t. $P_{ij}^{(n)}>0 \quad P_{ji}^{(m)}>0$

Let $r>0$

$$p_{jj}^{(m+n+r)} \geq p_{ji}^{(m)} \, p_{ii}^{(r)} \, p_{ij}^{(n)}$$

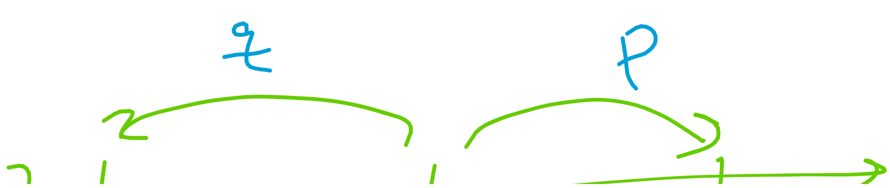
$$\sum_{r\geq 0} p_{jj}^{(m+n+r)} \geq p_{ji}^{(m)} \, p_{ij}^{(n)} \sum_{r} p_{ii}^{(r)}$$

$$= \infty$$

So $\displaystyle\sum_{r\geq 0} p_{jj}^{(m+n+r)} = \infty$

$\Rightarrow$ $j$ recurrent. $\blacksquare$

Ex: 1-D R.W.




- First note this is irreducible. Only need to consider $i=0$, say.

Need to analyze $\sum_{n \geq 0} P_{00}^{(n)}$

$$P_{00}^{(2n+1)} = 0$$

$$P_{00}^{(2n)} = \binom{2n}{n} p^n q^n$$

$$= \frac{2n!}{n!\,n!} (pq)^n$$

$$\frac{\quad}{n!\,n!}$$

$$\approx (pq)^n \frac{2^{2n}}{\sqrt{\pi n}}$$ Sterling's Formula

$$= \frac{(4pq)^n}{\sqrt{\pi n}}$$

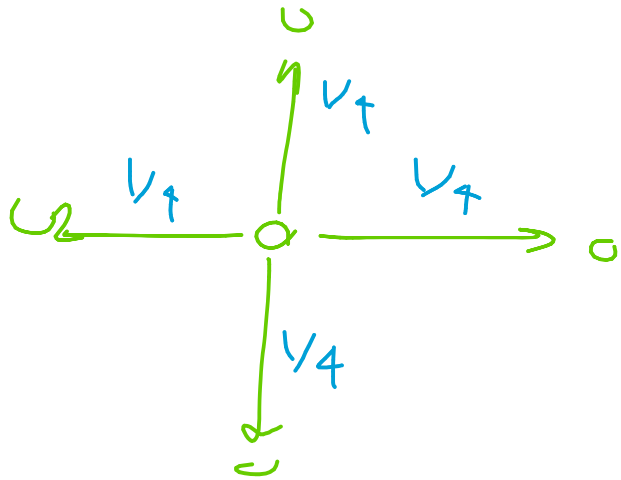
$$\sum_n P_{00}^{(n)} = \sum_n P_{00}^{(2n)}$$

$$= \sum_n \frac{(4pq)^n}{\sqrt{\pi n}} = \begin{cases} \infty & p = 1/2 \\ < \infty & p \neq 1/2 \end{cases}$$

So $p = 1/2 \Rightarrow$ recurrent

$p \neq 1/2 \Rightarrow$ transient

Ex: 2-D R.V.




$$P_{00}^{(2n+1)} = 0$$

$$P_{00}^{(2n)} = \sum_{2i+2j=2n} \frac{(2n)!}{i!\,i!\,j!\,j!} \left(\tfrac{1}{4}\right)^{2n}$$

$$\approx \frac{1}{\pi n}$$

$$\sum_{n \geq 1} P_{00}^{2n} = \sum \frac{1}{\pi n} = +\infty$$

So all states recurrent.

So all states recurrent.

Ex: 3-d R.W. with prob $\frac{1}{6}$

$$p^{(2n+1)}_{(0,0,0),(0,0,0)} = 0$$

$$p^{(2n)}_{000,000} = \sum_{i+k+j=n} \frac{(2n)!}{i!i!j!j!k!k!} \left(\frac{1}{6}\right)^{2n}$$

$$\Rightarrow \sum_{n=1}^{\infty} p^{2n}_{000,000} < \infty$$

So in three dimensions the symmetric R.W. is transient.

# Exam Notes

- Material up through today
- Exam based on H.W.
- Know proofs
- Learn distributions

# Lecture 2/27

Tuesday, February 27, 2018 9:34 AM

Review of generating functions

$u_n = \phi(u_{n-1})$ $\quad u_n = \mathbb{P}[X_n = 0]$

as $n \to \infty$ $\quad u_n \to u_\infty$

$u_\infty = \phi(u_\infty)$

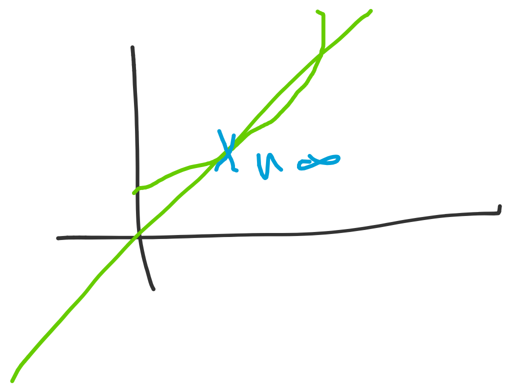
$\phi(s) = \mathbb{E}(s^{\xi})$

$\phi'(s) = \mu = \mathbb{E}(\xi)$

If $\phi'(1) < 1$ $\quad u_\infty = 1$

$\phi'(1) \geq 1$ $\quad u_\infty < 1$

We find $u_\infty$ by solving the equation $u_\infty = \phi(u_\infty)$




## Basic L.T. on M.C.

Define $R_i = \min\{n \geq 1 : X_n = i\}$

then $f_{ii}^{(n)} = \mathbb{P}[X_n = i, X_r \neq i, 1 \leq r \leq n-1]$

$$= \mathbb{P}[R_i = n \mid X_0 = i]$$

Define $m_i = \mathbb{E}[R_i \mid X_0 = i]$

$$= \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

"mean duration between visits to state $i$"

<u>Thrm:</u> Let $X_n$ be a

recurrent, irreducible, aperiodic M.C. This means $f_{ii}=1$

$\sum_{n=1}^{\infty} P_{ii}^{(n)} = +\infty$. Then

(i) $\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{m_i} = \frac{1}{\sum_n n f_{ii}^{(n)}}$

(ii) $\lim_{n\to\infty} P_{ii}^{(n)} = \lim_{n\to\infty} P_{ji}^{(n)} = \Pi_i = \frac{1}{m_i}$

Rmks:

1. Let $X_n$ be a M.C. and let $C \subseteq S$. be a recurrent communication class.

Then $P_{ij}^{(n)} = 0$ $i \in C$ $j \notin C$.

for all $n$.

IF $\lim_n P_{ii}^{(n)} > 0$ for $i \in C$

which is recurrent and

aperiodic, then $\pi_j > 0$

$\forall j \in C$.

Def: A communication class is <u>positive recurrent</u> or <u>ergodic</u> iff $\pi_i > 0 \ \forall i \in C$ or

$m_i < \infty$

Def: A communication class is called <u>null recurrent</u> if

$\pi_i = 0 \quad \forall i \in C$.

Thrm: In a positive recurrent, aperiodic class

$$\lim_n P_{jj}^{(n)} = \pi_j = \sum_{i \in C} \pi_i P_{ij}$$

then the limiting/stationary

then the limiting/stationary dist. is given by the $\{\pi_i\}$

Rmks:

(i) If $\mathbb{P}[X_0 = \pi_i]$ then

$$\mathbb{P}[X_1 = i] = \sum_{k \geq 0} \mathbb{P}[X_1 = i \mid X_0 = k]\, \mathbb{P}[X_0 = k]$$

$$= \sum_{k \geq 0} P_{ki} \pi_k = \sum_{k \geq 0} \pi_k P_{ki} = \pi_i .$$

(ii) If $\mathbb{P}[X_0 = i]$ then

$$\mathbb{P}[X_n = i, X_{n+1} = j] = \pi_i \cdot P_{ij}$$

(iii) If a limiting dist exists it is the stationary distribution

* Converse not true

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j = \sum_{i\in C} \pi_i P_{ij}$$

$$1 \Longrightarrow 2$$

Ex: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$P^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & n \text{ even} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & n \text{ odd} \end{cases}$$

$\lim P^n$ d.n.e.

But

$$\pi = \pi P \Rightarrow \pi = (1/2, 1/2)$$

Stationary exists limiting does not.

Lemma: Let $\{a_n\}$ be

s.t. $a_n \longmapsto a$ then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \longmapsto a$$

Apply this result to

$$a_n = P_{ii}^{(n)}$$

$P_{ii}^{(n)} \longmapsto \pi_i$ so

$$\lim_{n} \frac{1}{n}\sum_{k=1}^{n} P_{ii}^{(k)} = \pi_i = \frac{1}{m_i} > 0$$

Let $M_{ii}^{(n)} = \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = n\}}$

$$\mathbb{E}(M_{ii}^{(n)} \mid X_0 = i)$$

$$= \sum_{k=0}^{n-1} \mathbb{P}[X_k = i \mid X_0 = i]$$

$$= \sum_{k=0}^{n-1} P_{ii}^{(k)}$$

So $\lim_{n} \frac{1}{n} \mathbb{E}[M_{ii}^{(n)} | X_0 = i]$

$$= \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{(k)}$$

$$= \frac{1}{m_i} = \pi_i$$

Let $r(i)$ be the cost associated upon each visit to state $i$.

$$R^{(n-1)} = \sum_{k=0}^{n-1} r(X_k)$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} 1_{\{X_k = i\}} r(i)$$

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} 1_{\{X_k = i\}} r(i)$$

$$= \sum_{i=0}^{\infty} r(i) M_{ii}^{(n)}$$

$$i=0$$

Taking limits over $n$

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[R^{(n.1)} | X_0 = i]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} r(i)\, \mathbb{E}[m_i^{(n)} | X_0 = i]$$

$$= \sum_{i=0}^{\infty} r(i) \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[m_i^{(n)} | X_0 = i]$$

$$= \sum_{i=0}^{\infty} r(i)\, \pi_i \quad \text{if } |r(i)| < \infty$$

So for $n$ large

$$\mathbb{E}[R^{(n-1)} | X_0 = i] \simeq n \cdot \sum_{i=0}^{\infty} \pi_i\, r(i)$$

## Periodic Case

(i) $i \in C$ recurrent class but

$C$ periodic irreducible class

C periodic irreducible class with period $d$ then

$$\lim_{n\to\infty} P_{ii}^{n\cdot d} = \frac{d}{m_i} \qquad P_{ii}^{(m)} = 0 \quad m \neq nd$$

(ii) If the chain is positive recurrent then stationary dist. exists and

$$\lim_{n} \sum_{m=0}^{n-1} P_{hi}^{m} = \pi_i = \frac{1}{m_i}$$

# Lecture 3/15

Thursday, March 15, 2018 9:30 AM

## Exam Comments

70% ~ average

5 + curve

## Chapter 4 Sec 5 - Reducible MC

- We aren't guarenteed to have a limiting dist.
- What if we have a combination of transient + recurrent states?

Properties:

1. $j$ transients $\lim_{n} P_{ij}^{(n)} = \lim_{n} P_{ij}^{(n)} = 0$

2. If $i,j \in C$ and $C$ is an aperiodic recurrent class

aperiodic recurrent class

then $\lim_{n} P_{ij}^{(n)} = \frac{1}{m_j} = \pi_j$

If $C$ is periodic and

recurrent $\lim \frac{1}{n} \sum_{m=0}^{n-1} P_{ij}^{(m)} = \frac{1}{m_j}$

3. What if $i \in C_1$ transient

and $j \in C_0$ recurrent then

$$\boxed{\lim_{n} P_{ij}^{(n)} = \pi_i(C_0)\,\pi_j}$$

absorbtion prob. of class $C_0$

stationary

To find $u_i$ we do

$$\pi_i(C_0) = \sum_{j \in C_0} P_{ij} + \sum_{k \in C_1} P_{ik}\,(C_0)$$

4. If $|S| < \infty$ then then

4. If $|S|<\infty$ then there can be no null recurrent states and not all states can be transient.

Reason: $\displaystyle\sum_{j=0}^{m-1} P_{ij}^{(n)} = 1 \quad \forall n$

So it is impossible to have

$$\lim_{n} P_{ij}^{(n)} = 0 \quad \forall j.$$

Ex:

$$\mathbb{P} = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}$$

$C_1 = \{0,1\}$ recurrent

$C_2 = \{2,3\}$ recurrent

$C_2 = \{2,3\}$ recurrent

$C_3 = \{4,5\}$ transient.

$$\lim_n P_{55}^{(n)} = \lim_n P_{54}^{(n)} = 0$$

$$\lim P_{50}^{(n)} = \pi_5(C_1)\,\pi_1$$

$$\lim P_{51}^{(n)} = \pi_5(C_1)\,\pi_0$$

$$\left.\begin{aligned} \pi_0 + \pi_1 &= 1 \\ 1/3\,\pi_0 + \tfrac{2}{3}\pi_1 &= \pi_0 \\ 2/3\,\pi_0 + 1/3\,\pi_1 &= \pi_1 \end{aligned}\right\}$$ Solve for $(\pi_0, \pi_1)$.

$$\pi_5(C_1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\pi_4(C_1) + \frac{1}{6}\pi_4(C_1)$$

$$\pi_4(C_1) = \frac{1}{4} + 1/4\,\pi_4(C_1) + 1/4\,\pi_5(C_1)$$

Solve for $(\pi_4(C_1), \pi_5(C_1))$

# Poisson Processes

- State - discrete

  Time - Continuous

Poisson Dist. $X \sim Pois(\lambda)$

$$\mathbb{P}(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\mathbb{E}(X) = Var(X) = \lambda$$

<u>Thrm:</u> If $X \sim Pois(\mu)$ $Y \sim Pois(\nu)$

and $X \perp Y$ then

$$X+Y \sim Pois(\mu+\nu)$$

<u>Pf:</u>

$$\mathbb{P}(X+Y=n) = \sum_k \mathbb{P}(X=k, Y=n-k)$$

$$= \sum_k \mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)$$

$$= \sum_k \frac{\mu^k e^{-\mu}}{k!} \frac{\nu^{n-k} e^{-\nu}}{(n-k)!}$$

$$= \frac{e^{-(\mu+r)}}{n!} \sum_k \frac{n!}{k!(n-k)!} \mu^k r^{n-k}$$

$$= \frac{(\mu+r)^n e^{-(\mu+r)}}{n!}$$ Poisson with rate $\mu+r$.

<u>Thrm:</u> $N \sim Pois(\mu)$ $M|N \sim Bin(N,P)$

then $M \sim Pois(NP)$.

$$\mathbb{P}(M=m) = \sum_n \mathbb{P}(M=m|N=n)\mathbb{P}(N=n)$$

$$= \sum_n \binom{n}{m} p^m (1-m)^{n-m} \frac{\mu^n e^{-\mu}}{n!}$$

$$= \frac{e^{-\mu p}(\mu p)^k}{k!}$$

## Poisson Process

Def: A Poisson Process with intensity $\lambda > 0$. is an integer valued process $\{X(t) : t \geq 0\}$ s.t.

1. $\forall \quad t_0 < t_1 < \dots < t_n$ then

$X_{t_1} - X_{t_0} \perp X_{t_2} - X_{t_1} \perp \dots$

$\perp X_{t_n} - X_{t_{n-1}}$ (independent increments)

2. $\mathbb{P}(X_{t+s} - X_t = k) = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$

3. $X(0) = 0$

Rmk: $\mathbb{E}(X_t) = \mathrm{Var}(X_t) = \lambda t$

Ex: Customers arrive at a store with rate $\lambda = 4/hr$. If the store

$\lambda = 4/hr$. If the store opens at 9:00 a.m. Find

$$\mathbb{P}(X(1/2) = 1,\ X(5/2) = 5)$$

$$\mathbb{P}(X(1/2) = 1,\ X(5/2) - X(1/2) = 4)$$

$$= \mathbb{P}(X(1/2) = 1)\,\mathbb{P}(X(5/2) - X(1/2) = 4)$$

$$= \left(\frac{(1/2 \cdot 4)^1 e^{-4/2}}{1!}\right)\left(\frac{(2 \cdot 4)^4 e^{-4 \cdot 2}}{4!}\right)$$

Constant $\lambda$ usually naive

<u>Nonhomogeneous Poisson Process</u>

Observe that

$$\mathbb{P}(X(t+h) - X(t) = 1) = \frac{(\lambda h) e^{-\lambda h}}{1!}$$

$$= \lambda h \sum_{k=0}^{\infty} \frac{(\lambda h)^k}{k!}$$

$$= \lambda h \sum_{k=0} \frac{\quad}{k!}$$

$$= \lambda h - (\lambda h)^2 + \frac{1}{2}(\lambda h)^3 - \dots$$

$$= \lambda h + o(h) \text{ as } h \searrow 0$$

So $\lambda$ is the proportionality constant in probability of an event occurring in an arb. small time interval.

If $\lambda = \lambda(t)$. Then the process is called non-homogeneous.

In this case

$$\boxed{X(t) - X(s) \sim \text{Pois}\left(\int_s^t \lambda(u)\, du\right)}$$

Rmk: If $\lambda(t) = \lambda \ \forall t$, then

$$\int_s^t \lambda(u)\,du = \lambda(t-s).$$

Connecting Homogeneous to non-homogeneous.

Let $\Lambda(t) = \int_0^t \lambda(u)\,du$

$$d\Lambda(t) = \lambda(t)\,dt + o(dt)$$

So

$$P(X(t+dt) - X(t) = 1)$$

$$= \lambda(t)\,dt + o(dt) = d\Lambda(t) + o(dt)$$

define $Y(s) = X(t)$ $s = \Lambda(t)$

$$Y(s) \sim \mathrm{Pois}(1)$$

# Lecture 3/20

Tuesday, March 20, 2018 9:22 AM

## Poisson Processes

- (In)homogeneous
- Cox Process: nonhomogeneous Poisson Process $X(t)$ with parameter $\lambda(t)$ which is also stochastic.

(Sometimes called double stochastic).

Ex: Mixed Poisson Process

$X(t) \sim Pois(1)$

$Y(t) = X(\theta t)$ where $\theta$ is a random variable

If $\theta$ is cont. with pdf

$f(\theta)$ then

$$\mathbb{P}(Y(t)=k) = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} f(\theta)\, d\theta$$

## The Law of Rare Events

Consider $N$ independent

$\mathrm{Bern}(p)$ trials for $p$ small.

$X_{n,p} =$ # of successe in $N$ trials

$X_{n,p} \sim \mathrm{Bin}(N,p)$

Assume $N \to \infty$ $p \to 0$

$Np \to \mu > 0$

Then $X_{N,p} \xrightarrow{D} X_\mu \sim \mathrm{Pois}(\mu)$

Thrm: Assume each event has different probability of success. $\varepsilon_i \sim \text{Bern}(p_i)$

Let $S_n = \sum_{i=1}^{n} \varepsilon_i$ If

$p_i = p \ \forall i$ then $S_n \sim \text{Bin}(N, p)$

If $p_i \neq p$ then $S_n \not\sim \text{Binom}(N, p)$

$$\mathbb{P}(S_n = k) = \sum^{(k)} \prod_{i=1}^{n} p_i^{x}(1-p_i)^{1-x_i}$$

$\sum^{(k)}$ denotes sum over all $x_i \in \{0, 1\}$ s.t. $\sum_{i=1}^{n} x_i = k$.

Let $\varepsilon_i \sim_{ind} \text{Bern}(p_i)$ then

$$\left| \mathbb{P}(S_n = k) - \frac{\mu^k e^{-\mu}}{k!} \right| \leq \sum_{i=1}^{n} p_i^2$$

for $\mu = \sum_{i=1}^{n} p_i$

Rmk: If $p_i = \mu/n$ then

$$\sum_{i=1}^{n} p_i^2 = \mu^2/n$$

## Poisson Motivation

Let $N((a,b])$ = # of events that occur in $(a,b]$ with timestamps $\{t_i, i \geq 1\}$

Assumptions:

1. The number of events happen in disjoint intervals and are ind
2. $N(t,t+h]$ depends only on $h$
3. $\mathbb{P}(N(t,t+h] \geq 1) = \lambda h + o(h)$, $h \to 0$
4. $\mathbb{P}(N(t,t+h] \geq 2) = o(h)$, $h \to 0$

Rmk:

- 3 + 4 implies events can't happen at the same time.

- 1 + 2 implies

$$N(s,t] \overset{D}{=} N(0,t-s]$$

Thrm: Under (1)–(4)

$$P_k(t) = \mathbb{P}[N(0,t] = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Pf: Divide $(0,t]$ into $n$ subintervals of equal length $t/n$. Let

$$\varepsilon_i = \begin{cases} 1 & \text{event in } \left(\frac{i-1}{n}, \frac{i}{n}\right] \\ 0 & \text{o.w.} \end{cases}$$

$$S_n = \sum_{i=1}^{n} \varepsilon_i \quad \text{\# events in } (0,t]$$

By (3) $P_i = \mathbb{P}(\varepsilon_i = 1) = \frac{\lambda t}{n} + o(t/n)$

$$\left| \mathbb{P}(S_n = k) - \frac{\mu^k e^{-\mu}}{k!} \right|$$

$$\leq n\left(\frac{\lambda t}{n} + o(t/n)\right)^2$$

$$\leq \frac{(\lambda t)^2}{n} + 2\lambda t\, o(t/n) + n\,(o(t/n))^2$$

Let $\mu = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \lambda t/n + o(t/n)$

$= \lambda t + n\, o(t/n)$

Notice $n\, o(t/n) = \dfrac{t\, o(t/n)}{t/n} \to 0$

Finally, we need to show

$$\lim_n \mathbb{P}(S_n = k) = \lim_n \mathbb{P}(N(0,t] = k)$$

Well $S_n$, $N((0,t])$ only differ if there is a subinterval contains two or more events.

But by (4) this never occurs.

$$\left|P_k(t) - \mathbb{P}(S_n = k)\right| \le \mathbb{P}(N(0,t] \ne S_n)$$

$$\le \sum_{i=1}^{n} \mathbb{P}\left(N\left(\left(\tfrac{i-1}{n}t, \tfrac{i}{n}t\right]\right) \ge 2\right)$$

$$\le n\,o(t/n) = t\,\frac{o(t/n)}{t/n} \longrightarrow 0$$

As we know

$$\mathbb{P}(S_n = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

We see

$$\mathbb{P}(N(0,t] = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Recall $N(s,t] \overset{D}{=} N(0,t-s] \sim \text{Pois}(\lambda)$

Let's write $X(t) = N(0,t]$

Define

$W_n$ = time of occurrence of $n^{th}$ event

"waiting time"

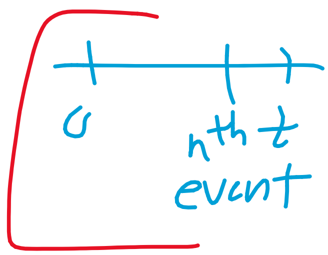
$$\boxed{S_n = W_{n+1} - W_n \quad \text{sojourn time}}$$

**Thrm:** $W_n \sim \text{Gamma}(n, \lambda)$

$$f_{W_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{\Gamma(n)}$$

**Pf:** $F_{W_n}(t) = \mathbb{P}(W_n \le t)$

$$= \mathbb{P}(X(t) \ge n)$$




$$= \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$f_{W_n}(t) = \frac{d}{dt} \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \ldots = \frac{\lambda^n t^{n-1}}{\Gamma(n)} e^{-\lambda t} \qquad \blacksquare$$

**Thrm:** $S_i \sim \text{iid } \text{Exp}(\lambda)$

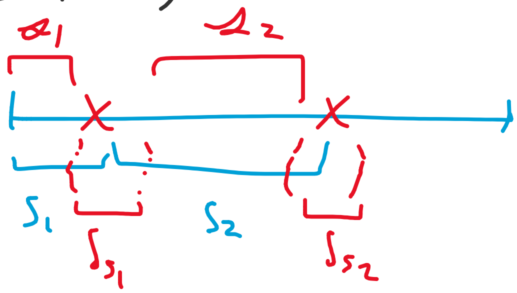
**Pf:** We want to show

$$f_{S_1, \ldots, S_n} = \prod_{i=1}^{n} f_{S_i} = \prod_{i=1}^{n} \lambda e^{-\lambda s_i}$$

$$f_{S_0,\dots,S_{n-1}} = \prod_{i=1}^{n} f_{S_i} = \prod_{i=1}^{n} \lambda e^{-\lambda s_i}$$

(Induction)

$n=2$




No event in $(0, s_1)$

1 event in $(s_1, s_1+\delta_{s_1})$

no event in $(s_1+\delta_{s_1}, s_1+\delta_{s_1}+s_2)$

1 event in $(s_1+\delta_{s_1}+s_2, s_1+\delta_{s_1}+s_2+\delta_{s_2})$

$$f_{S_1,S_2}(s_1,s_2) = \frac{\mathbb{P}(s_1 < S_1 < s_1+\delta_{s_1},\ s_1+\delta_{s_1}+s_2 < S_2 < s_2+\delta_{s_2}+s_1+\delta_{s_1})}{\delta_{s_1}\delta_{s_2}} + o(\delta_{s_1},\delta_{s_2})$$

$$= \frac{\mathbb{P}(N(0,s_1)=0)\,\mathbb{P}(N(s_1,s_1+\delta s_1)=1) \times \cdots}{\delta s_1\,\delta s_2}$$

$$= \frac{e^{-\lambda s_1}\,\lambda \cancel{\delta s_1}\, e^{-\lambda \delta s_1}\; e^{-\lambda s_2}\,\lambda \cancel{\delta s_2}\, e^{-\lambda \delta s_2}}{\cancel{\delta s_1}\,\cancel{\delta s_2}}$$

$$= \lambda e^{-\lambda s_1}\,\lambda e^{-\lambda s_2}\, e^{-\lambda \delta s_1}\, e^{-\lambda \delta s_2}$$

$$\xrightarrow[\delta s_2 \to 0]{\delta s_1 \to 0} \left(\lambda e^{-\lambda s_1}\right)\left(\lambda e^{-\lambda s_2}\right)$$

- Looks like a mess, but just write everything in terms of nonoverlapping intervals.

Can argue for arb.

$n$.

# Lecture 3/22

Thursday, March 22, 2018 9:38 AM

Let $X_t$ be a Poisson Process with rate $\lambda$. Let

$0 < u < t \qquad 0 \le k \le n$

Then

$$\mathbb{P}(X_u = k \mid X_t = k) = \binom{n}{k}\left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$$

Pf:

$$\mathbb{P}(X_u = k \mid X_t = n)$$

$$= \frac{\mathbb{P}(X_t - X_u = n-k)\,\mathbb{P}(X_u = k)}{\mathbb{P}(X_t = n)}$$

$$= \text{see HW} = \binom{n}{k}\left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$$

- . . . . . $\ldots (k) \le t \} \ldots t \}$

## Poisson Process + Unif

Conditioned on a fixed number of events in an interval, the location of those events are distributed uniformly.

**Thrm:** Let $\{W_i\}$ be the occurrence time in a Poisson process with rate $\lambda$. Then

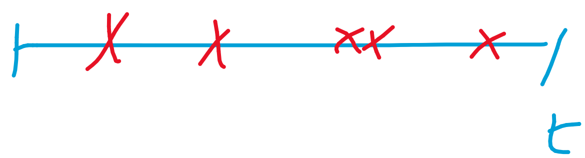
$$f_{W_1, W_2, \dots, W_n | X_t = n}(w_1, \dots, w_n) = n! \, t^{-n}$$

for $0 < w_1 < w_2 < \dots < w_n < t$

**Rmk:** $n!$ ways to arrange

Rmk: $n!$ ways to arrange $u_1, \dots, u_n \sim U(0,t)$ that lead to the same ordered values $w_1 < w_2 < \dots < w_n$

Pf:




Create $\varepsilon$-neighborhoods around the $w_i$. The event

$$A_i = \left\{ w_i \le W \le w_i + \varepsilon_i \,,\ i = 1..n \,,\ X(t) = n \right\}$$

all events occur in $\cup A_i$

no events occur in $(\cup A)^c$

exactly one event in

$A_i$. These intervals are disjoint so

$$P(N(A_1^c) = 0, N(A_1) = 1,$$

$$\vdots$$

$$N(A_n^c) = 0, N(A_n) = 1)$$

$$= \prod_{j=1}^{n} P(N(A_i^c) = 0)\, P(N(A_i) = 1)$$

$$= \prod_{i=1}^{n} e^{-\lambda \varepsilon_i} \frac{e^{-\lambda \varepsilon_i} (\lambda \varepsilon_i)^1}{1!} \qquad (*)$$

So $f_{W_1, \ldots, W_n}(w_1, \ldots, w_n \mid X(t) = n)$

$$= \frac{P(w_i \le W_i \le w_i + \varepsilon_i \mid X(t) = n)}{\varepsilon_1 \cdots \varepsilon_n}$$

$$= \frac{\mathbb{P}(w_i \le W_i \le w_i + \varepsilon_i ; X(t) = n)}{\mathbb{P}(X_t = n)\, \varepsilon_1 \cdots \varepsilon_n}$$

$$\overset{(*)}{=} \frac{\prod_{i=1}^{n} e^{-\lambda \varepsilon_i} \quad \lambda \cancel{\varepsilon_i}\, e^{-\lambda \varepsilon_i}}{\mathbb{P}(X_t = n)\, \cancel{\varepsilon_1 \cdots \varepsilon_n}}$$

telescoping

$$= \frac{\cancel{\lambda^n}\, \cancel{e^{-\lambda t}}}{\frac{\cancel{e^{-\lambda t}} (\cancel{\lambda} t)^n}{n!}}$$

$$= n!\, t^{-n}$$

Let $\varepsilon_i \longmapsto 0$ so

$$\longmapsto n!\, t^{-n}$$

∎

Ex: Customers arrive at a facility according to a Poisson Process with rate $\lambda$. Each customer pays $\$1$ on arrival. Expected value of total sum during $(0,t]$ discounted back to time $0$ with discount $\beta>0$.

\# events

Let $m = \mathbb{E}\left(\sum_{k=1}^{X(t)} 1\, e^{-\beta W_k}\right)$

arrival time

$$= \sum_{n=0}^{\infty} \mathbb{E}\left( \sum_{k=1}^{X(t)} e^{-\beta W_k} \,\middle|\, X(t)=n \right)$$

$$\mathbb{P}(X(t)=n$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left( \sum_{k=1}^{n} e^{-\beta W_k} \right) \mathbb{P}(X_t=n)$$

uniform iid

$$= \sum_{n=0}^{\infty} \mathbb{E}\left( \sum_{k=1}^{n} e^{-\beta U_k} \right) \mathbb{P}(X_t=n)$$

$$= \sum^{\infty} n \, \mathbb{E}\left( e^{-\beta U_1} \right) \mathbb{P}(X_t=n)$$

$n=0$

$$= \sum_{n=0}^{\infty} n \int_0^1 \frac{e^{-\beta u}}{t}\, dt \; \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \frac{1}{\beta t}\left(1-e^{-\beta t}\right) \sum_{n=1}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= \frac{\lambda}{\beta}\left(1-e^{-\beta t}\right)$$

Ex: (sum quota sampling)

Sample size not fixed

Let

Let

$$\overline{Y}_{X(t)} = \frac{W_{X(t)}}{X(t)}$$

$$= \frac{Y_1 + \cdots + Y_{X(t)}}{X(t)} = \frac{\sum_{i=1}^{X(t)} Y_i}{X(t)}$$

for $X(t) = \max\{n : X_1 + \cdots + X_n \leq t\}$

Stopping rule

Consider the special case

$Y_i \sim \mathrm{Exp}(\lambda)$ $\quad X_t \sim \mathrm{Pois}(t)$

$$\mathbb{E}\left(\overline{Y}_{X(t)} \mid X(t) > 0\right)$$

$$= \sum_{n=1}^{\infty} \mathbb{E}\left(\bar{Y}_n \mid X_t = n\right) \mathbb{P}(X_t = n)$$

$$= \sum_{n=1}^{\infty} \mathbb{E}\left(\frac{W_n}{n} \,\middle|\, X_t = n\right) \mathbb{P}(X_t = n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(W_n \mid X_t = n\right) \mathbb{P}(X_t = n)$$

max order

stat of $\{U_1, \ldots, U_n\}$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\max\{U_1, \ldots, U_n\}\right) \mathbb{P}(X_t = n)$$

$$= \sum^{\infty} \frac{1}{\ } \, t \, \frac{n}{\ } \quad e^{-\lambda t} \, \lambda t^{n}$$

$$= \sum_{n=1} \frac{1}{n} \, t \, \frac{n}{n+1} \, \frac{e^{\lambda t}}{n!}$$

$$= \frac{e^{-\lambda t}}{\lambda} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!}$$

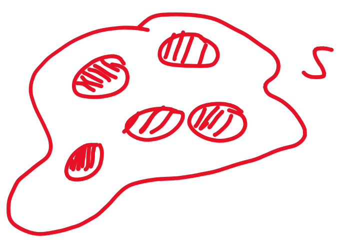
$$= \ldots = \frac{1}{\lambda}\left(1 - \frac{e^{\lambda t}}{e^{\lambda t} - 1}\right)$$

Lecture 3/27

Tuesday, March 27, 2018 9:35 AM

# Spatial Poisson Process

Let $S$ be an $n$-dimensional space and define $\mathcal{A} \subseteq \mathcal{P}(S)$. Let $N(\mathcal{A})$ be a point process over $S$




Assume

1. $A \cap B = \emptyset$ $N(A \cup B) = N(A) + N(B)$
2. Let $|A|$ be the size of $A$.

Def: $\{N(A) : A \in \mathcal{A}\}$ is a spatial poisson process if

(i) $\forall A_1, \ldots, A_k$ s.t.

$A_i \cap A_j = \emptyset \ (i \neq j)$

then $N(A_1) \perp\!\!\!\perp N(A_2) \perp\!\!\!\perp \cdots \perp\!\!\!\perp N(A_k)$

(ii) $\forall A \in \mathcal{A}$ $N(A)$ has a Poisson dist. with paramter

$\lambda |A|$

Rmk: (i) $N(A)$ depends only on $|A|$ and not on the shape/location

(ii) Disjoint regions do not affect each other

(iii) No two points occupy the same location.

$$\mathbb{P}(N(A)=k) = \frac{e^{-\lambda |A|} (\lambda |A|)^k}{k!}$$

**Property:** If $B \subseteq A$ then

$$\mathbb{P}(N(B)=1 \mid N(A)=1) = \frac{|B|}{|A|}$$

**Pf:**

$$\mathbb{P}(N(B)=1 \mid N(A)=1)$$

$$= \frac{\mathbb{P}(N(B)=1,\ N(A)=1)}{\mathbb{P}(N(A)=1)}$$

$$= \frac{\mathbb{P}(N(B)=1)\,\mathbb{P}(N(A\setminus B)=0)}{\mathbb{P}(N(A)=1)}$$

$$= \frac{\cancel{e^{-\lambda|B|}}\,\cancel{\lambda}|B|\ \cancel{e^{-\lambda(|A|-|B|)}}}{\cancel{e^{-\lambda|A|}}\,\cancel{\lambda}|A|}$$

$$= \frac{|B|}{|A|} \qquad \boxed{\beta}$$

# Compound Poisson Process

$X_t$ - Poisson Process with intensity $\lambda > 0$

- Each event is associated with a R.V. $Y_i$ with $X_t \perp\!\!\!\perp Y_i$

The compound Poisson process is given by

$$Z_t = \sum_{i=1}^{X_t} Y_i$$

<u>Ex:</u> Insurance claims

$X_t$ - claims

$Y_k$ - magnitude of $k^{th}$ claim

$Z_t$ - cumulative amount claimed

Transactions in Stocks

Transactions in Stocks

$X_t$ - transactions

$Y_k$ - change in market price of stock

$Z_t$ - Cummulative change of stock price

## Marked Poisson Process

$\{(W_i, Y_i)\}_{i=1}^{n}$

waiting times of Poisson

"cost"/"reward" of $k^{th}$ event

More on this later...

## Continuous Time M.C.

1. Pure birth process

(e.g. Poisson process)

2. Pure death process

3. Birth & death.

Def: The transition probability function is given by

$$P_{ij}(t) = \mathbb{P}\left[ X(t+u) = j \mid X(u) = i \right]$$

and is independent of $u$.

An example for poisson processes

$$\lim_{h \to 0} \frac{\mathbb{P}(X(t+h) - X(t) = 1 \mid X(t) = k)}{h} = \lambda$$

$$\mathbb{P}(X(t+h) - X(t) = 1 \mid X(t) = k) = \lambda h + o(h)$$

$h \to 0$

(ii) $\mathbb{P}(X(t+h) - X(t) = 0 \mid X_t = k)$

$= 1 - \lambda h + o(h) \quad h \to 0$

In a poisson process
the chance of an event
is independent of the num.
of events

- Let's relax this
assumption

Ex: Population genetics

$\mathbb{P}(\text{birth}) = f(\text{size of pop})$

Rmk: We want to keep
Markovian properties.

Consider a sequence of
positive numbers $\{\lambda_k\}$

A pure birth process is

a markov process satisfying

(i) $\mathbb{P}(X(t+h)-X(t)=1 \mid X_t=k)$

$= \lambda_k h + o(h)$ as $h \to 0$

(ii) $\mathbb{P}(X_{t+h}-X_t=0 \mid X_t=k) = 1-\lambda_k h + o_k(h)$

as $h \to 0$

(iii) $\mathbb{P}(X_{t+h}-X_t<0 \mid X_t=k) = 0$

(iv) $X_0 = 0$

Thrm: Let $P_n(t) = \mathbb{P}(X_t = n)$

Under the previous def.

$P_n(t)$ satisfies the system

of diff. eq.

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

with initial conditions

$$P_0(0) = 1$$

$$P_n(0) = 0 \quad \forall n > 0$$

Pf: Using the Markov Property we will write down

$$P_n(t+h) - P_n(h)$$

Then divide by $h$ and let

$$h \to 0$$

First note

$$P_n(t+h) = \mathbb{P}(X_{t+h} = n)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_{t+h} = n \mid X_t = k)\mathbb{P}(X_t = k)$$

$$= \sum^{\infty} \mathbb{P}(X_{t+h} - X_t = n-k \mid X_t = k)\, P_k(t)$$

$k = 0$

as $h \mapsto 0$ nonzero

for $k = n$, $k = n-1$

$$= P(X_{t+h} - X_t = 0 \mid X_t = n) P_n(t)$$

$$+ P(X_{t+h} - X_t = 1 \mid X_t = n-1) P_{n-1}(t)$$

$$+ \sum_{\substack{k \neq n \\ k \neq n-1}} P(X_{t+h} - X_t = n-k \mid X_t = n) P_n(t)$$

$$= \left( \left(1 - \lambda_n h + o_n(h)\right) \overset{P_n(t)}{\big)} + \left(\lambda_{n-1} h + o_n(h)\right) P_{n-1} + o(h) \right)$$

$$= P_n(t) - \lambda_n P_n(t) h + \lambda_{n-1} P_{n-1}(t) h$$

$$+ \left( o_n(h) P_n(t) + o_{n-1}(h) P_{n-1}(t) + o(h) \right)$$

$\Rightarrow$

$$P_n(t+h) - P_n(t)$$

$$= -\lambda_n P_n(t)h + \lambda_{n-1} P_{n-1}(t)h + o(h)$$

$\Rightarrow$ divide by $h$

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

$\Rightarrow$ Take $h \to 0$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

# Lecture 3/29

Thursday, March 29, 2018 9:31 AM

Consider a sequence of positive numbers $\{\lambda_k\}$

Assume that $X(t)$ satisfies

1. $\mathbb{P}(X_{t+h} - X_t = 1 \mid X_t = k) = \lambda_k h + o(h)$ as $h \searrow 0$

2. $\mathbb{P}(X_{t+h} - X_t = 0 \mid X_t = k) = 1 - \lambda_k h + o(h)$ as $h \searrow 0$

3. $\mathbb{P}(X_{t+h} - X_t < 0 \mid X_t = k) = 0$

4. $X_0 = 0$

__Thrm:__ Define $P_n(t) = \mathbb{P}(X_t = n)$

Then $P_n(t)$ satisfies

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

$$P_0(0) = 1$$

$$P_n(0) = 0$$

Solving the system of diffy Q's

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$P_0(0) = 1$$

$$\Rightarrow \boxed{P_0(t) = e^{-\lambda_0 t}}$$

Next we solve

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

$$P_n(0) = 0$$

Proceeding recursively...

Let $Q_n(t) = e^{\lambda_n t} P_n(t)$

$$Q_n'(t) = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t)$$

$$= e^{\lambda_n t}\left(\lambda_n P_n(t) + P_n'(t)\right)$$

$$= e^{\lambda_n t} \lambda_{n-1} P_{n-1}(t)$$

$\Rightarrow$

$$Q_n(t) = \int_0^t e^{\lambda_n s} \ldots$$

$$Q_n(t) = \int_0^t e^{\lambda_n s} \lambda_{n-1} p_{n-1}(s)\,ds + C$$

$\Rightarrow$

$$e^{\lambda_n t} p_n(t) = \int_0^t e^{\lambda_n s} \lambda_{n-1} p_{n-1}(s) + C$$

$\Rightarrow$

$$p_n(t) = e^{-\lambda_n t} \int_0^t e^{\lambda_n s} \lambda_{n-1} p_{n-1}(s)\,ds$$

$$+ e^{-\lambda_n t} c$$

Using the initial condition

$c = 0$ so

$$\boxed{p_n(t) = e^{-\lambda_n t} \int_0^t e^{\lambda_n s} \lambda_{n-1} p_{n-1}(s)\,ds}$$

Clearly $p_n(t) \geq 0$

$$\sum_{n=0}^{\infty} p_n(t) \Longleftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = +\infty$$

Restriction on the rate at which the population becomes infinite.

Let $S_k$ be the time between $k$ and $k+1$ birth.

$$P_n(t) = \mathbb{P}\left[\sum_{i=0}^{n-1} S_i \le t < \sum_{i=0}^{n} S_i\right]$$

$S_j$ are the sojourn times.

$$\mathbb{P}(S_0 \le t) = 1 - \mathbb{P}(X_t = 0)$$

$$= 1 - e^{-\lambda_0 t}$$

$$S_0 \sim Exp(\lambda_0)$$

Also one can show

$$S_k \sim Exp(\lambda_k)$$

So $1/\lambda_k$ is the expected

waiting time to the next birth

$$\sum_{k=0}^{\infty} 1/\lambda_k = \sum_{k=0}^{\infty} \mathbb{E}(S_k)$$

↳ mean time <u>before</u> the population becomes infinite

If $\sum 1/\lambda_k < \infty$ then we have infinitly many births in a finite period of time.

$$\mathbb{P}(X_t = \infty) > \infty$$

for $t < \infty$.

Then

$$\sum 1/\lambda_n < \infty \iff P(X_t = \infty) > 1$$

$$\iff 1 - \mathbb{P}(X_t = \infty) < 1$$

$$\Leftrightarrow \sum_{n=0}^{\infty} P_n(t) < 1$$

Ex: For $\lambda_k = \lambda$ (Poisson)

$$\sum 1/\lambda_k = \frac{1}{\lambda} \sum 1 = \infty$$

Rmk: The recursive formula can be really hard to solve in general

Yale Process

It describes the growth of a population where each member has a probability $bh + o(h)$ $h \searrow 0$ of giving birth to a new member in $h$ units of time.

$$\mathbb{P}(X_{t+h} - X_t = 1 \mid X_t = n)$$

$$= \binom{n}{1}\left(bh + o(h)\right)^1 \left(1 - bh + o(h)\right)^{n-1}$$

$$= \underbrace{n(bh)}_{\lambda_n} + o(h)$$

So for the Yule Process

$\lambda_n = nb$ hence

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$$

$$= -bn\, p_n(t) + b(n-1)\, p_{n-1}(t)$$

$$= \cdots \quad (\text{HW})$$

$$\Rightarrow p_n(t) = e^{-bt}\left(1 - e^{bt}\right)^{n-1}$$

So $X_t \sim \text{Geom}(e^{-bt})$

$$\sum 1/\lambda_n = \frac{1}{b} \sum \frac{1}{n} = +\infty$$

So this gives a proper probability model.

# Pure Death Process

If we have a stochastic process that moves through the states

$N \mapsto N-1 \mapsto \cdots \mapsto 0$

with

1. $\mathbb{P}(X_{t+h} - X_t = -1 \mid X_t = n) = \mu_k h + o(h)$
2. $\mathbb{P}(X_{t+h} - X_t = 0 \mid X_t = n) = 1 - \mu_k h + o(h)$
3. $\mathbb{P}(X_{t+h} - X_t > 0) = 0$
4. $X_0 = N$

See book for derivation

$$P_N(t) = e^{-\mu_n t}$$

$$P_n(t) = \mu_{n+1} \cdots \mu_N \sum_{k=n}^{N} a_{k,n} e^{-\mu_k t}$$

$$a_{k,n} = \frac{1}{\prod_{j=n}^{N} (\mu_j - \mu_k)} \qquad \mu_j \neq \mu_k$$

The linear death

process has $\mu_n = k\alpha$

which corresponds to

$$p_n(t) = \binom{N}{n}\left(e^{-\alpha t}\right)^n\left(1-e^{-\alpha t}\right)^{N-n}$$

Let $T = \min\{t \geq 0 : X_t = 0\}$

$$\mathbb{P}(T \leq t) = \mathbb{P}(X_t = 0) = \left(1-e^{-\alpha t}\right)^N$$

linear death process

Lecture 4/3

Tuesday, April 3, 2018 9:31 AM

# Birth & Death M.P.

$$P_{ij}(t) = \mathbb{P}(X_{t+s} = j \mid X_s = i)$$

Assumptions: (as $h \to 0$)

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$
2. $P_{i,i-1}(h) = \mu_i h + o(h)$
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i) h + o(h)$
4. $\mu_0 = 0,\ \lambda_0 > 0$ & $\mu_i, \lambda_i > 0$
5. $P_{ij}(0) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

The infintesimal generator

of this process is given by

for this process is given by

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Note: $\displaystyle\sum_{j=1}^{\infty} A_{ij} = 1 \quad \forall i$

$A$ denotes the infinitesimal rates of movement of the chain.

We wish to derive solutions to $P_n(t) = \mathbb{P}(X_t = n)$

<u>Chapman Kolmogorov Equation:</u>

$$P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s)$$

$$= \sum_k P_{ik}(s) P_{kj}(t)$$

Intuition: Given that a transition will occur at time $t$ we transition to $i+1$ with prob. $\frac{\lambda_i}{\mu_i+\lambda_i}$

to $i-1$ with prob $\frac{\mu_i}{\mu_i+\lambda_i}$

## Sojourn times

$S_i$ – Sojourn time for $X_t$ at state $i$.

Let $F_i(t) = \mathbb{P}(S_i \le t)$

$G_i(t) = \mathbb{P}(S_i > t)$ By M.P.

$$G_i(t+h) = G_i(t)\, G_i(h)$$

$$G_i(t+h) = G_i(t)G_i(h)$$

If $h$ is small

$$G_i(h) = \mathbb{P}(S_i > h) \overset{*}{=} P_{ii}(h) + o(h)$$

Then

$$G_i(t+h) = G_i(t)G_i(h)$$

$$= G_i(t)\{P_{ii}(h) + o(h)\}$$

$$= G_i(t)\{1 - (\lambda_i + \mu_i)h + o(h)\}$$

So

$$\frac{G_i(t+h) - G_i(t)}{h} = -(\lambda_i + \mu_i)G_i(t) + \frac{o(h)}{h}G_i(t)$$

Take $h \mapsto 0$

$$G_i'(t) = -(\lambda_i + \mu_i)G_i(t)$$

$$G_i(0) = 1$$

$$\boxed{G_i(t) = e^{-(\lambda_i + \mu_i)t}}$$

$$\boxed{\begin{aligned} F_i(t) &= 1 - G_i(t) \\ &= 1 - e^{-(\lambda_i + \mu_i)t} \end{aligned}}$$

$$S_i \sim \text{Exp}(\lambda_i + \mu_i)$$

## Deriving the Diff. Eq. $P_n(t)$

$$P_{ij}(t+h) = \sum_k P_{ik}(h) P_{kj}(t)$$

$$= P_{i\,i-1}(h) P_{i-1\,j}(t) +$$

$$P_{ii}(h) P_{ij}(t) +$$

$$P_{i\,i+1}(h) P_{i+1\,j}(t) + o(h)$$

$$= \mu_i h P_{i-1\,j}(t) + [1 - (\lambda_i + \mu_i)h] P_{ij}(t)$$

$$+ \lambda_i h P_{i+1\,j}(t) + o(h)$$

Now rearranging

$P_{ij}(t+h) - P_{ij}(t) =$

$$h\left[\mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t)\right] + o(h)$$

divide by $h$ and take

$h \to 0$

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) + \frac{o(h)}{h}$$

This gives the

Backward Kolmogorov Eq.

$$P'_{ij}(t) = \mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t)$$

$$P_{ij}(0) = \delta_{ij}$$

The forward Kol. eq.

$$P_{ij}(t+h) = \sum_k P_{ik}(t) P_{kj}(h)$$

$k$

$$\boxed{\begin{aligned} P'_{ij}(t) = {} & \lambda_{j-1} P_{ij-1}(t) - (\lambda_i + \mu_i) P_{ij}(t) \\ & + \mu_{j-1} P_{ij+1}(t) \end{aligned}}$$

<u>Rmk</u>: Backward - starting changes

Forward - ending changes.

Backward ~ Rows of $A$

Forward ~ Columns of $A$.

<u>Ex</u>: Let $X(t)$ be a birth and death process with linear imigration

$$\lambda_n = n\lambda + \alpha$$

and linear death rate

$$\mu_n = n\mu$$

Assume $\mu, \lambda, \alpha > 0$.

Goal: Find

$$M_i(t) = \mathbb{E}\left( X(t) \mid X(0) = i \right)$$

Given $X(t)$ for small $h$

$$X(t+h) = \begin{cases} X(t)+1 & (X(t)\lambda + \alpha)h + o(h) \\ X(t) & 1 - [X(t)(\lambda+\mu) - \alpha]h \\ X(t)-1 & hX(t)\mu + o(h) \end{cases}$$

$$m(t+h) = \mathbb{E}(X_{t+h} \mid X_t)$$

$$= (X_t+1)\left[(X_t\lambda+\alpha)h\right] +$$

$$(X_t)\left\{ 1 - [X(t)(\lambda+\mu) - \alpha]h \right\}$$

$$+ (X_t - 1)[X(t)\mu]h + o(h)$$

This gives

$$\frac{\mathbb{E}(X_{t+h} \mid X_t) - X_t}{h} = \frac{(\alpha + (\lambda-\mu)X_t)\cancel{h}}{\cancel{h}} + \frac{o(h)}{h}$$

$$\boxed{\begin{aligned} m'(t) &= \alpha + (\lambda - \mu)\, m(t) \\ m(0) &= i \end{aligned}}$$

By solving this O.D.E.

$$m(t) = \begin{cases} \alpha t + i & \lambda = \mu \\ \dfrac{\alpha}{\lambda - \mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t} & \end{cases}$$

$$\lim_{t \to \infty} m(t) = \begin{cases} \infty & \lambda \geq \mu \\ \dfrac{\alpha}{\mu - \lambda} & \lambda < \mu \end{cases}$$

<u>Ex</u>:

$$A = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

$$\lambda_0 = \alpha \qquad \mu_0 = 0$$

$$\lambda_1 = 0 \qquad \mu_1 = \beta$$

$$P_{00}'(t) = -\alpha P_{00}(t) + \beta P_{01}(t)$$

$$P_{01}(t) = 1 - P_{00}(t) \qquad (*)$$

$$\Longrightarrow$$

$$P_{00}'(t) = \beta - (\alpha+\beta) P_{00}(t)$$

Let $Q_{00}(t) = e^{(\alpha+\beta)t} P_{00}(t)$

$$Q_{00}'(t) = (\alpha+\beta) Q_{00}^{t} + e^{(\alpha+\beta)t} P_{00}'(t)$$

$$= (\alpha+\beta) Q_{00}^{t} + e^{(\alpha+\beta)t}\left(\beta - (\alpha+\beta) P_{00}(t)\right)$$

$$= \cancel{(\alpha+\beta) Q_{00}(t)} + \beta e^{(\alpha+\beta)t} - \cancel{(\alpha+\beta) Q_{00}(t)}$$

$$\Longrightarrow$$

$$Q_{00}(t) = \int Q_{00}'(t)\, dt$$

$$= \beta \int e^{(\alpha+\beta)t} \, dt$$

$$= \frac{\beta}{\alpha+\beta} e^{(\alpha+\beta)t} + \frac{\alpha}{\alpha+\beta}$$

from initial $Q_{00}(0) = 1$

So

$$P_{00}(t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}$$

From here we can say

$$P_{00}(t) \xrightarrow{t \to \infty} \frac{\beta}{\alpha+\beta}$$

$$P_{01}(t) \xrightarrow{t \to \infty} 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}.$$

# Lecture 4/5

Thursday, April 5, 2018 9:36 AM

# Limiting Behavior of B&D

Interested in $P_n(t)$

which can be found through a system of O.D.E.s

If $n$ large, hard to solve. In this case we can approximate by

$$\pi_j = \lim_{t \to \infty} P_{ij}(t)$$

$\boxed{\begin{array}{l}\underline{\text{Prop}}\text{: If there are no abs. states then} \\ \text{(i) } \pi_j \text{ exists } \forall j \\ \text{(ii) } \sum \pi_j = 1 \\ \text{(iii) } \pi \text{ is stationary with} \\ \qquad \pi_j = \sum_i \pi_i P_{ij}(t)\end{array}}$

What we know is the Kol $\rightleftarrows$ equation.

$$P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t)$$

$$P_{ij}'(t) = \lambda_{j-1} P_{i\,j-1} - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i\,j+1}(t)$$

Taking $t \to \infty$ these eq. become

$$0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

$$0 = \lambda_{j-1} \pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1} \pi_{j+1}$$

which can be solved iteratively.

$$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0 := \theta_1 \pi_0$$

By induction set

$$\theta_j = \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}}$$

So that $\pi_j = \theta_j \pi_0$

Then by the second eq.

$$\mu_{j+1} \pi_{j+1} = (\lambda_j + \mu_j)\pi_j - \lambda_{j-1} \pi_{j-1}$$

$$= (\lambda_j + \mu_j)\theta_j \pi_0 - \lambda_{j-1} \theta_{j-1} \pi_0$$

$$= \left(\lambda_j + \mu_j \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}} - \lambda_{j-1} \prod^{j-2} \frac{\lambda_k}{\mu_{k+1}}\right) \pi_0$$

$\Rightarrow$

$\Rightarrow$

$$\mu_{j+1}\pi_{j+1} = \lambda_j \theta_j \pi_0$$

$$\boxed{\begin{gathered}\pi_{j+1} = \theta_{j+1}\pi_0 \\ \theta_j = \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}}\end{gathered}}$$

With the additional eq.

$$\sum \pi_j = 1 \Rightarrow \pi_0 + \sum \theta_j \pi_0 = 1$$

$$\Rightarrow \boxed{\pi_0 = \frac{1}{1 + \sum_{j\geq 1} \theta_j}}$$

Ex: Linear growth with immigration

$$\lambda_n = n\lambda + \alpha \qquad \mu_n = n\mu$$

$$\theta_j = \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}} = \frac{\alpha(\alpha+\lambda)(\alpha+2\lambda)\cdots}{j!\,\mu^j}$$

$$= \frac{\left(\frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}+1\right)\cdots\left(\frac{\alpha}{\lambda}+j-1\right)}{j!}\left(\frac{\lambda}{\mu}\right)^j$$

$$= \binom{\frac{\alpha}{\lambda}+j-1}{j}\left(\frac{\lambda}{\mu}\right)^j$$

Then by the Binomial thm.

$$\sum_{j=0}^{\infty}\binom{\alpha/\lambda+j-1}{j}\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{\alpha/\lambda} = 1$$

So we can solve for $\vec{\pi}$.

Ex: (Logistic Process) $X(t) \in [N, m]$

$$\left.\begin{aligned} \lambda &= \alpha(m - X(t)) \\ \mu &= \beta(X(t) - N) \end{aligned}\right\}$$ ensures the pop. is of size $N \le X(t) \le M$

Members act independently

with the birth/death rates

with these birth/death rates.

$$\lambda_n = n\alpha(M-n)$$

$$\mu_n = n\beta(n-N)$$

$$\pi_{N+m} = \theta_{N+m}\,\pi_N$$

$$\theta_{N+m} = \frac{\lambda_n \lambda_{n+1} \cdots \lambda_{n+m-1}}{\mu_{n+1} \cdots \mu_{n+m}}$$

$$= \frac{N}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m$$

## Absorbing States

Consider a B&D s.t. $\lambda_0 = 0$

(abs. state at zero).

We are interested in absorbtion probabilities.

Define

$$u_i = \mathbb{P}[\text{Abs. into } 0 \mid X_0 = i]$$

$$i \longmapsto i+1 \qquad \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$i \longmapsto i-1 \qquad \frac{\mu_i}{\lambda_i + \mu_i}$$

By first step

$$u_i = \frac{\lambda_i}{\lambda_i + \mu_i} u_{i-1} + \frac{\mu_i}{\lambda_i + \mu_i} u_{i+1}$$

which gives

$$(\lambda_i + \mu_i) u_i = \lambda_i u_{i-1} + \mu_i u_{i+1}$$

So

$$\underbrace{u_{i+1} - u_i}_{v_i} = \frac{\mu_i}{\lambda_i} (\underbrace{u_i - u_{i-1}}_{v_{i-1}})$$

$V_i$ $\qquad\qquad\qquad\qquad$ $V_{i-1}$

and iteratively we have

$$V_i = \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\underbrace{\lambda_i \lambda_{i-1} \cdots \lambda_1}_{\rho_i}} V_0$$

Then to find $V_0$

$$u_{i+1} - u_i = \rho_i V_0 = \rho_i (u_1 - 1)$$

Then applying a telescoping sum we have

$$u_m - u_1 = (u_1 - 1) \sum_{i=1}^{m-1} \rho_i$$

letting $m \to \infty$

If

$$\sum^{\infty} \rho_i = \infty \Rightarrow \left\{ u_1 = 1 \right.$$

$$\sum_{i=1}^{\infty} f_i = \infty \Rightarrow \begin{cases} u_1 = 1 \\ u_m = 1 \quad \forall m \end{cases}$$

Hence absorbition is guaranteed.

If $0 < u_1 < 1$ then

$\sum f_i < \infty$ which implies

$u_m$ decreases as $m$ increases. (as $\sum f_i < \infty$ $(u_1 - 1) < 0$)

So being monotone & bounded we have

$$\lim_{m \to \infty} u_m = C$$

So

$$\boxed{u_1 = \frac{\sum_{i=1}^{\infty} \rho_i}{1 + \sum_{i=1}^{\infty} \rho_i}}$$

which gives

$$\boxed{u_m = \frac{\sum_{i=m}^{\infty} \rho_i}{1 + \sum_{i=m}^{\infty} \rho_i}}$$

Mean time until absorb.

$$w_i = \mathbb{E}[T \mid X_0 = i]$$

$$T = \min\{t > 0 : X(t) = 0\}$$

Abs. will have for sure if

$$\sum \rho_i = \infty.$$

$$w_0 = 0$$

waiting time to move from $S_i$

$$w_i = \frac{1}{\quad} + \frac{\lambda_i}{\quad} w_{i+1}$$

$$w_i = \frac{1^{\tau}}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} w_{i+1}$$

$$+ \frac{\mu_i}{\lambda_i + \mu_i} w_{i-1}$$

# Lecture 4/10

Tuesday, April 10, 2018 9:34 AM

## Mean time to Abs.

If $\sum_{i=1}^{\infty} \gamma_i = \infty$ for

$$\gamma_i = \prod_{j=1}^{i} \frac{\mu_i}{\lambda_i}$$

then abs. is certain. We are interested in

$$w_i = \mathbb{E}[T \mid X_0 = i]$$

$$T = \inf\{t \geq 0 : X(t) = 0\}$$

then we have the equations

$$w_0 = 0$$

$$w_i = \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} w_{i+1} + \frac{\mu_i}{\lambda_i + \mu_i} w_{i-1}$$

Define $z_i = w_i - w_{i+1}$ then

$$z_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} z_{i-1}$$

$$= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\left\{\frac{1}{\lambda_{i-1}} + \frac{\mu_{i-1}}{\lambda_{i-1}} z_{i-2}\right\}$$

$$= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\left\{\frac{1}{\lambda_{i-1}} + \frac{\mu_{i-1}}{\lambda_{i-1}} z_{i-2}\right\}$$

$$= \vdots$$

$$= \sum_{j=1}^{i} \frac{1}{\lambda_j} \prod_{k=j+1}^{i} \frac{\mu_k}{\lambda_k} + \prod_{j=1}^{i} \frac{\mu_j}{\lambda_j} z_0$$

$$= \sum_{j=1}^{i} \frac{1}{\lambda_j} \frac{\rho_i}{\rho_i} + \rho_i z_0$$

$$\Longrightarrow$$

$$\frac{z_i}{\rho_i} = \sum_{j=1}^{i} \frac{1}{\lambda_j \rho_j} + z_0$$

But $z_0 = w_0 - w_1 = -w_1$

So $\forall i$

$$\frac{1}{\rho_i} z_i = \sum_{j=1}^{i} \left(\frac{1}{\lambda_j \rho_j} - w_1\right)$$

For $i \mapsto \infty$ we get

the hueristic is that

The alternative is

"$$\frac{1}{\rho_\infty} z_\infty = \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_i \rho_i} - w_1 \right)$$"

If

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_i \rho_i} = \infty \Rightarrow w_1 = \infty$$

If $\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_i} < \infty$ then we have

$$w_1 = \sum_{j=1}^{\infty} \frac{1}{\lambda_i \rho_i} - \underbrace{\lim_{i \to \infty} \frac{1}{\rho_i} z_i}$$

$$\sum \rho_i = \infty$$

$$\Rightarrow \frac{1}{\rho_i} \to 0$$

$$z_i = \underbrace{w_i - w_{i+1}}_{w_i \nearrow}$$

So we argue

$$\lim_{i \to \infty} \frac{1}{\rho_i} z_i = 0$$

and

$$\boxed{w_1 = \sum_{j=1}^{n} \frac{1}{\lambda_j \rho_j}}$$

So

$$w_m = \begin{cases} \infty & \text{if } \displaystyle\sum_{i=1}^{\infty} \frac{1}{\lambda_i \rho_i} = \infty \\[2ex] \displaystyle\sum \frac{1}{\lambda_i \rho_i} + \sum_{k=1}^{m-1} \rho_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j \rho_j} & \text{if } \displaystyle\sum_{i=1}^{\infty} \frac{1}{\lambda_i \rho_i} < \infty \end{cases}$$

<u>Ex:</u> Suppose we have a population model

$$\lambda_n = n\lambda, \quad \mu_n = n\mu$$

$$\rho_i = \prod_{j=1}^{i} \frac{\mu_j}{\lambda_j} = \left(\frac{\mu}{\lambda}\right)^i$$

$$\sum_{i=1}^{n} \rho_i = \sum_{j=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^j = \frac{\lambda}{\lambda - \mu}$$

for $\mu < \lambda$

Now using our formula

$$u_m = \begin{cases} (\mu/\lambda)^m & \mu < \lambda \\ 1 & \mu \geq \lambda \end{cases}$$

Also

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i \beta_i} = \sum_{i=1}^{\infty} \frac{1}{i \cdot \lambda \left(\frac{\mu}{\lambda}\right)^i}$$

$$= \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{\lambda}{\mu}\right)^i$$

$$= \frac{1}{\lambda} \sum_{i=1}^{\infty} \int_0^{\lambda/\mu} x^{i-1}\, dx$$

$$= \frac{1}{\lambda} \int_0^{\lambda/\mu} \sum_{i=1}^{\infty} x^{i-1}\, dx$$

$$= \frac{1}{\lambda} \int_0^{\lambda/\mu} \frac{1}{1-x}\, dx$$

for $\lambda < \mu$

$$= -\frac{1}{\lambda} \log(1-x) \Big|_0^{\lambda/\mu}$$

$$= \begin{cases} \frac{1}{\lambda} \log\left(\frac{\mu}{\mu - \lambda}\right) & \mu > \lambda \\ \infty & \mu \le \lambda \end{cases}$$

## Finite State Cont. M.C.

$$P_{ij}(t) = \mathbb{P}\left(X(t+s) = j \mid X(s) = i\right)$$

$$\sum_{j=0}^{N} P_{ij}(t) = 1 \qquad 0 \le P_{ij}(t) \le 1$$

The Kolmogorov-Chapman equations become

$$P_{ik}(t+s) = \sum_{j=0}^{N} P_{ij}(s) P_{jk}(t)$$

$$P(t) = \left[ P_{ik}(t) \right]_{i,k \in \{0, \dots, N\}}$$

Then C-K become

$$P(t+s) = \underbrace{P(t)P(s)}_{\text{matrix}}$$

$$\underbrace{P(t+s)}_{\text{matrix}} = \underbrace{P(t)P(s)}_{\text{matrix}}$$

So we have

$$P(t+h) = P(t)P(h) \quad (h \searrow 0)$$

$$\frac{P(t+h) - P(t)}{h} = \frac{P(t)P(h) - P(t)}{h}$$

$$= \frac{P(t)\{P(h) - I\}}{h}$$

Letting $h \to 0$

$$P'(t) = P(t) \underbrace{\lim_{h \to 0} \left( \frac{P(h) - I}{h} \right)}_{\text{def: } A}$$

$$= P(t)A$$

1. If $x' = ax$ then

$x(t) = e^{ax}$ where $x(0) = 1$

So the solution to this is (as $P(0) = I$)

$$P(t) = e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

def.

here $A$ is the infinitesimal generator.

- $$A = \lim_{h \to 0} \frac{P(h) - I}{h}$$

$$= \begin{bmatrix} -q_0 & q_{01} & \cdots & q_{0N} \\ q_{10} & -q_1 & \cdots & q_{1N} \\ \vdots & & \ddots & \vdots \\ & & & -q_N \end{bmatrix}$$

where $\sum_{j=1}^{N} A_{ij} = 0$

Ex: $A = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}$

$$A^n = [-(a+b)]^{n-1} A$$

So $N$

So

$$p(t) = \sum_{n=0}^{N} \frac{(At)^n}{n!}$$

$$= \sum_{n=0}^{N} \frac{t^n}{n!} \left\{-(a+b)\right\}^{n-1} A$$

$$= I - \frac{1}{\alpha+b} \sum_{n} \frac{(-(a+b)t)^n}{n!} A$$

$$= I - \left( \frac{e^{-(\alpha+b)t} - 1}{\alpha+b} \right) A$$

Notice that $t \to \infty$

$$\to \quad I + \frac{1}{\alpha+b} A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{-a}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{-b}{a+b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

# Renewal Phenomena

Def: A renewale process $\{N(t), t \geq 0\}$ is a non neg. integer valued stochastic process counting events in $(0, t]$

Assump: The time between successive events are iid RV denoted by $X_n = 1, 2, 3, \ldots$

Let $F(x) = \mathbb{P}(X_k \leq x)$

$F(0) = 0$ as $X_k \geq 0$

Interests:

· Waiting times

$$W_n = \sum_{k=1}^{n} X_k$$

Def: The associated renewale counting process

$N(t) =$ # of times $n$ s.t.

$$0 < W_n \le t$$

# Lecture 4/12

Thursday, April 12, 2018 9:33 AM

## Renewale Processes

Let $\{X_k, k \geq 1\}$ denote the successive duration between events.

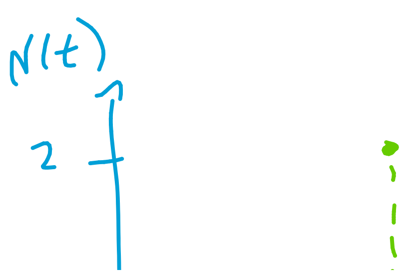
Define

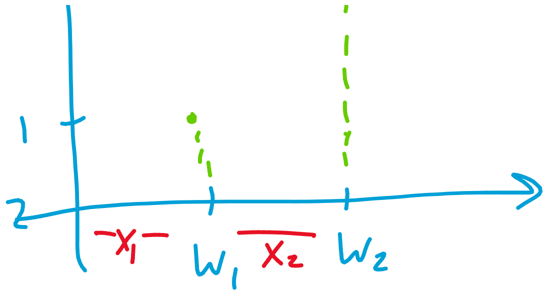
$W_n = \sum_{i=1}^{n} X_i$ be the waiting time for the $k^{th}$ event.

$N(t) =$ # of indicies $n$

s.t. $0 \leq W_n \leq t$

is a renewale process






## Observations

(i) $\mathbb{P}(W_n \le t)$

$$= F_n(x) = \int_0^x F_{N-1}(x-y)\,dF(x)$$

where $F(x) = \mathbb{P}(X_1 \le x)$

(ii) $\mathbb{P}(W_1 \le x) = \mathbb{P}(X_1 \le x)$

(iii) $N(t) \ge k$ iff $W_k \le t$

So

$$\mathbb{P}(N(t) \ge k) = \mathbb{P}(W_n \le t)$$

$$= F_k(t)$$

Hence

$$\mathbb{P}(N(t)=k) = \mathbb{P}(N(t)\geq k)$$

$$- \mathbb{P}(N(t) \geq k+1)$$

$$= \mathbb{P}(W_k \leq t) - \mathbb{P}(W_{k+1} \leq t)$$

<u>Quantities of Interest</u>

1. $M(t) = \mathbb{E}(N(t))$

$$= \sum_k \mathbb{P}(N(t) \geq k)$$

$$= \sum_k F_k(t)$$

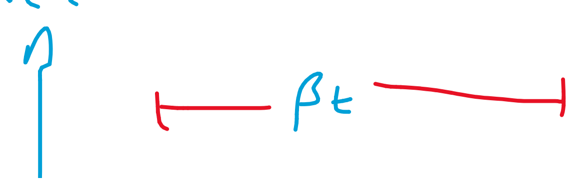
2. Excess, Current, total lifetime

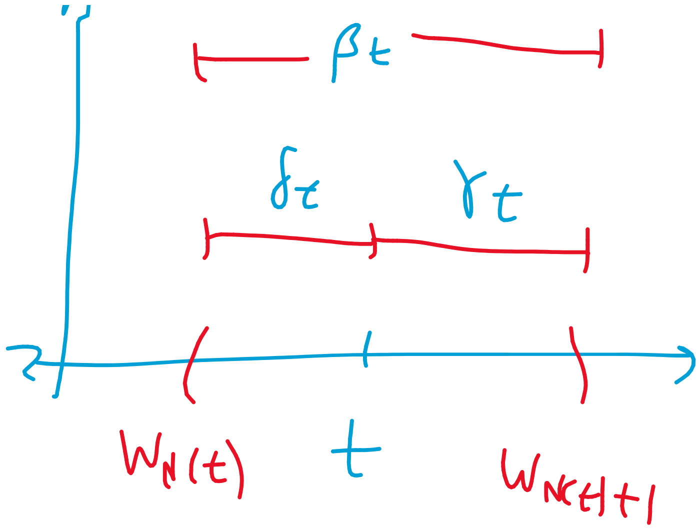
Excess: $\gamma_t = W_{N(t)+1} - t$

Current: $\delta_t = t - W_{N(t)}$

Total: $\beta_t = \gamma_t + \delta_t$

$N(t)$







**Prop**: $\mathbb{E}(W_{N_t+1}) = \mathbb{E}(X_1)\mathbb{E}(N_t+1)$

$= \mathbb{E}(X_1)(M(t)+1)$

**Rmk**: $W_{N(t)+1} = \sum_{i=1}^{N(t)+1} X_i$

Note $N(t) \not\perp X_i$ So

not a stand random sum.

**Rmk**: $\mathbb{E}(W_{N(t)}) \neq \mathbb{E}(X_1)\mathbb{E}(N(t))$

**Pf**: $\{N(t) \geq j-1\} = \{W_{j-1} \leq t\}$

So

$$\mathbb{1}_{\{N(t) \geq j-1\}} = \mathbb{1}_{\{W_{j-1} \leq t\}}$$

$$\mathbb{1}_{\{N(t) \geq j-1\}} = \mathbb{1}_{\{W_{j-1} \leq t\}}$$

$$X_j \mathbb{1}_{\{N(t) \geq j-1\}} = X_j \mathbb{1}_{\{W_{j-1} \leq t\}}$$

$$\mathbb{E}\left\{X_j \mathbb{1}_{\{N(t) \geq j-1\}}\right\} = \mathbb{E}\left\{X_j \mathbb{1}_{\{W_{j-1} \leq t\}}\right\}$$

Well, as

$X_j \perp\!\!\!\perp W_{j-1}$ we have

$$\mathbb{E}(X_j \mathbb{1}_{\{W_{j-1} \leq t\}})$$

$$= \mathbb{E}(X_j)\, \mathbb{P}(W_{j-1} \leq t)$$

$$= \mathbb{E}(X_1)\, F_{j-1}(t)$$

Now

$$\mathbb{E}(W_{N(t)+1}) = \mathbb{E}\left(\sum_{i=1}^{N(t)+1} X_i\right)$$

$$= \mathbb{E}(X_1) + \mathbb{E}\left(\sum_{i=2}^{N(t)+1} X_i\right)$$

$$= \mathbb{E}(X_1) + \mathbb{E}\left(\sum_{i \geq 2} X_i \mathbb{1}_{\{i \leq N(t)+1\}}\right)$$

$$= \mathbb{E}(X_1) + \mathbb{E}\left(\sum_{i \geq 2} X_i \mathbb{1}_{\{N(t) \geq i-1\}}\right)$$

$$= \mathbb{E}(X_1) + \mathbb{E}\left(\sum_{i \geq 2} X_i \mathbb{1}_{\{W_{i-1} \leq t\}}\right)$$

$$= \mathbb{E}(X_1) + \sum_{i \geq 2} \mathbb{E}(X_1) F_{i-1}(t)$$

$$= \mathbb{E}(X_1)\left\{1 + \sum_{i \geq 1} F_i(t)\right\}$$

$$= \mathbb{E}(X_1)\left\{1 + \mathbb{E}(N(t))\right\}$$

$$= \mathbb{E}(X_1)\left\{1 + m(t)\right\} \quad \blacksquare$$

Ex: (Poisson)

$N(t) \sim \text{Pois}(\lambda t)$

$X \sim \text{Exp}(\lambda)$

Ex: Counter Process

· Successive times between electrical impulses / signals

Ex: Traffic Flow

· Time duration between consecutive cars passing a fixed point.

Ex: Queueing

Poisson as Renewal

Assume that $N(t) \sim Pois(\lambda)$

then $\mathbb{E}(N(t)) = \lambda t$

Excess life:

$$\{\gamma_t \geq x\} = \left\{\begin{array}{c}\text{no events in} \\ (t, t+x]\end{array}\right\}$$

$$\mathbb{P}(\gamma_t \geq x) = \mathbb{P}\begin{pmatrix}\text{no events} & \text{in} \\ (t, t+x] & \end{pmatrix}$$

$$= \mathbb{P}\left(N(t+x) - N(t) = 0\right]$$

$$= e^{-\lambda x}$$

So

$$F(x) = \mathbb{P}(\gamma_t \leq x) = 1 - e^{-\lambda x}$$

$$\gamma_t \sim \text{Exp}(\lambda)$$

Current time

$$\{\delta_t \leq x\} = \begin{cases} t > x & \text{N.R. } (t-x, t] \\ t \leq x & \{\delta_t \leq x\} \text{ is certain} \end{cases}$$

$$\mathbb{P}(\delta_t \leq x) = \begin{cases} 1 - e^{-\lambda x} & t > x \\ 1 & t \leq x \end{cases}$$

Total Life:

$$\mathbb{E}(\beta_t) = \mathbb{E}(\gamma_t) + \mathbb{E}(\delta_t)$$

$$= \frac{1}{\lambda} + \int_0^t \mathbb{P}(\delta_t > x)\,dx$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right)$$

Joint dist of $(\delta_t, \gamma_t)$

$\{\gamma_t > x, \delta_t > y\}$ $\quad 0 < y < t$

$= \{\text{no renewals in } (t-y, t+x)\}$

$$\mathbb{P}(\gamma_t > x, \delta_t > y)$$

$$= \mathbb{P}(N(t+x) - N(t-y) = 0)$$

$$= \mathbb{P}(N(x+y) - N(0))$$

$$= e^{-\lambda(x+y)} = e^{-\lambda x} e^{-\lambda y}$$

So for $y > t$

$\mathbb{P}(\delta_t > y) = 0 = \mathbb{P}(\gamma_t > x, \delta_t > y)$

So in any case

$$\delta_t \perp\!\!\!\perp \gamma_t .$$

Ex: (Pr. 7.3.1)

Sum Quota Sampling.

Observe iid R.V. until some quota $t$ is exceeded.

The sample size is

$N(t)+1$

The experiment we run up to $W_{N(t)+1}$ "time"

$$\mathbb{E}(W_{N(t)+1}) = \mathbb{E}(X_1)(1+\mathbb{E}(N(t)))$$

$$\widehat{\frac{W_{N(t)+1}}{N(t)+1}} = \frac{1}{N(t)+1} \sum_{i=1}^{N(t)+1} X_i$$

Q: is this estimator unbiased?

for $X_1$?

for $X_1$?

S: $X \sim Exp(\lambda)$ $\quad \mathbb{E}(X) = \frac{1}{\lambda}$

- $\mathbb{E}\left[\frac{1}{N(t)+1}\right]$

$$= \sum \frac{1}{k+1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \cdots = \frac{1}{\lambda t} e^{-\lambda t}\left(e^{\lambda t} - 1\right)$$

- $N(t) \perp\!\!\!\perp W_{N(t)+1}$

$$\mathbb{E}\left(\frac{1}{N(t)+1} \sum^{N(t)+1} X_i\right)$$

$$= \mathbb{E}(W_{N(t)+1})\, \mathbb{E}\left(\frac{1}{N(t)+1}\right)$$

$$= \frac{1}{\lambda}\left[\left(1 - \frac{1}{\lambda t}\right)\left(1 - e^{-\lambda t}\right)\right]$$

# Lecture 4/17

Tuesday, April 17, 2018 9:29 AM

Recall

$N(t) = $ #events in $(0,t]$

$N(t) \geq 0$

The inter-arrival times are defined by

$X_i \sim \text{iid } F(x)$

The waiting times are denoted

$$W_n := \sum_{i=1}^{n} X_i$$

Lastly recall

$$\{N(t) \geq k\} = \{W_k \leq t\}$$

# Asymptotic Behavior of Renewale Processes

Define $m(t) = \mathbb{E}(N(t))$

We say $m(t)$ is the

renewale function

Finite time - hard to analyze.

Letting $t \to \infty$ then much easier.

Thrm: All renewale functions are asymptotically linear i.e.

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

where $\mu = \mathbb{E}(X_i)$

**Rmk:** Poisson Process only renewale process where $m(t) = \lambda t$ (exactly linear)

**Intuition around Result:**

$\frac{m(t)}{t} \leftarrow$ long run mean number of events per unit time

$\frac{1}{\mu} \leftarrow$ reciprocal of mean life of a component

**Ex:** Age replacement policies

- the theorm tells us to replace items over the long run at a mean [illegible] per unit time

Long run ...

time of $1/\mu$ per unit time

- If we replace this item prior to the failure we will use more than $1/\mu$ items per unit time.

## Properties

- $X_i$ cont. with pdf $f(x)$

- $\lim_{t \to \infty} m(t, t+h] = \frac{h}{\mu} \quad \forall h > 0$

$$* \quad \frac{m(t)}{t} \simeq \frac{1}{\mu}$$

$$m(t) \simeq \frac{t}{\mu}$$

$$m(t+h) - m(t) \simeq \frac{t+h}{\mu} - \frac{t}{\mu}$$

$$= \frac{h}{\mu}$$

- $m(t) := \frac{d}{dt} M(t)$

$\infty$ ...

$$= \sum_{n=1}^{\infty} f_n(x)$$

$f_n$ pdf of waiting times

$W_n$

$$* \ \frac{d}{dt} \mathbb{E}(N(t)) = \frac{d}{dt} \sum_{n=0}^{\infty} \mathbb{P}(N(t) \geq n)$$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} \mathbb{P}(W_n \leq t)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dt} F_n(t)$$

$$= \sum_{n=0}^{\infty} f_n(t).$$

- $$\frac{m(t+h) - m(t)}{h} \xrightarrow{t \to \infty} \frac{1}{\mu}$$

- $$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \frac{d}{dt} m(t)$$

$$= \lim \lim \frac{m(t+h) - m(t)}{1}$$

$$= \lim_{t\to\infty} \lim_{h\to 0} \frac{\quad}{h}$$

$$= \lim_{t\to\infty} \frac{1}{\mu} = \frac{1}{\mu}$$

$$\boxed{\lim_{t\to\infty} \left( m(t) - \frac{t}{\mu} \right) = \frac{\sigma^2 - \mu^2}{2\mu^2}}$$

$$\sigma^2 = Var(X_1)$$

Ex: Suppose $X_i \sim xe^{-x}$

$$W_n = \sum^{n} X_i \sim f_n(x) = \frac{x^{2n-1}}{(2n-1)!} e^{-x}$$

$$m(t) = \sum_{n=0}^{\infty} f_n(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

$$= e^{-x} \left( \frac{e^{x} - e^{-x}}{2} \right)$$

$$= \frac{1}{2}\left(1 - e^{-2x}\right)$$

$t$

$$M(t) = \int_0^t m(u)\, du$$

$$= \int_0^t \frac{1}{2}\left(1-e^{-2u}\right) du$$

$$= \frac{1}{2}t - \frac{1}{4}\left(1-e^{-2t}\right)$$

Now

$$\lim_{t\to\infty} \frac{1}{2}\left(1-e^{-2t}\right) = \frac{1}{2} = \frac{1}{\mu}$$

$$\lim_{t\to\infty}\left(\frac{1}{2}t - \frac{1}{4}\left(1-e^{-2t}\right) - \frac{t}{\mu}\right)$$

$$\to \frac{\sigma^2-\mu^2}{2\mu^2} \qquad \sigma^2 = \mu = 2$$

# Asymptotic Distribution of $N(t)$

$$\frac{\mathbb{E}(N(t))}{t} \longrightarrow \frac{1}{\mu}$$

$$\frac{Var(N(t))}{t} \longrightarrow \frac{\sigma^2}{\mu^3}$$

$$\boxed{\frac{N(t) - t/\mu}{\sqrt{\sigma^2 t/\mu^3}} \xrightarrow{D} N(0,1)}$$

Ex:

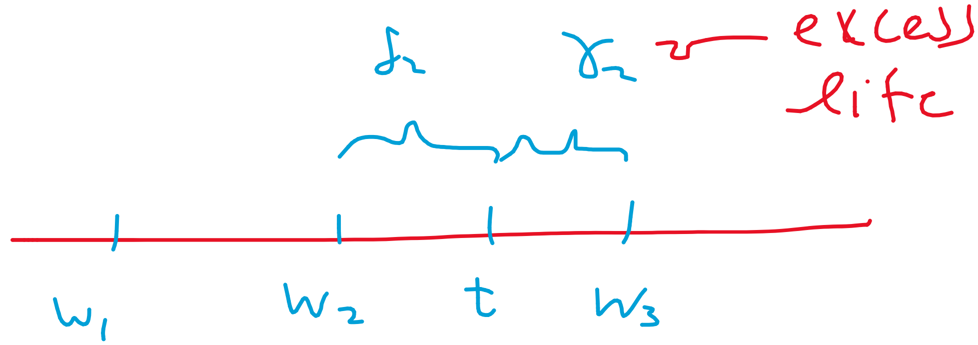
$$\mathbb{P}(W_k \le t) = \mathbb{P}(N(t) > k)$$

$$= \mathbb{P}\left[\frac{N(t) - t/\mu}{\sqrt{\frac{\sigma^2 t}{\mu^3}}} > \frac{k - t/\mu}{\sqrt{\frac{\sigma^2 t}{\mu^3}}}\right]$$

$$= 1 - \Phi\left(\frac{k - t/\mu}{\sqrt{\sigma^2 t/\mu^3}}\right)$$

$$\ldots \left(\sqrt{\sigma^2 t/\mu^3}\right)$$

# Limiting Result for Excess Life




$$\lim_{t\to\infty} \mathbb{P}(\gamma_t \le x) = \underbrace{\frac{1}{\mu}\int_0^x (1-F(y))\,dy}_{H(x)}$$

$H(x)$ – valid CDF function

$$h(x) = \frac{d}{dx} H(x)$$

$$= \frac{1}{\mu}\left(1 - F(x)\right)$$

# Brownian Motion

Suppose we have a density function $\rho(y,t|x)$. Then (given some thing) we have

$$\frac{d\rho}{dt} = \frac{1}{2}\sigma^2 \frac{d^2\rho}{dx^2}$$

If $\sigma^2 = 1$

$$\rho(y,t|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2t}(y-x)^2}$$

In general

$$\rho \sim C e^{-\frac{1}{2t\sigma^2}(y-x)^2}$$

Def: A seq. of R.V. $\{B(t)\}_{t\geq 0}$

is called Brownian motion with diffusion coefficient $\sigma^2$ if the following hold.

(a) $B(t+s) - B(t) \sim N(0, \sigma^2 t)$

(b) $B(t+s) - B(t) \perp\!\!\!\perp B(\tilde{t}+\tilde{s}) - B(\tilde{t})$

if $(t, t+s] \cap (\tilde{t}, \tilde{t}+\tilde{s}] = \emptyset$

(independent increments

(c) $B(0) = 0$.

(d) $B(t)$ is a continuous function

<u>Def</u>: The covariance function is given by the following.

First notice

$$B(t) = B(t) - B(0) \sim N(0, \sigma^2 t)$$

So

$$\mathbb{E}(B(t)) = \mathbb{E}(B(s)) = 0$$

From here for $s < t$

$$\operatorname{Cov}(B(t), B(s)) = \mathbb{E}\big(B(t)B(s)\big)$$

$$= \mathbb{E}\Big[\big(B(s) + B(t) - B(s)\big)B(s)\Big]$$

$$= \mathbb{E}\big(B(s)^2\big) + \mathbb{E}\Big(\big(B(t) - B(s)\big)B(s)\Big)$$

$$= \mathbb{E}\big(B(s)^2\big) + \mathbb{E}\big(B(t) - B(s)\big)\,\mathbb{E}\big(B(s)\big)$$

$$= \sigma^2 s + (0)(0)$$

$$= \sigma^2 s$$

So

$$\boxed{\operatorname{Cov}(B(s), B(t)) = \sigma^2 \{s \wedge t\}}$$

$$\boxed{\operatorname{Cov}\left(B(s), B(t)\right) = \sigma^2 \{s \wedge m\}}$$

# Lecture 4/19

Thursday, April 19, 2018 9:32 AM

## Brownian Motion

Simulation:

$\{B(t)\}_{t\geq 0}$ want to simulate

path on $[0,T]$

Properties of $B(t)$

- $B(0)=0$
- $B(t+s)-B(s) \sim N(0,\sigma^2 t)$
- Independent increments
- $B(t)$ continuous.

Let $0=t_0<t_1<\cdots<t_n=T$

in particular $t_j=j\Delta t$ $\Delta t=\frac{T}{n}$

Firstway: (wrong way!)

$B(t)\sim N(0,\sigma^2 t)$ $\forall t$

So $\forall t_j \quad B(t_j) \stackrel{D}{=} \sqrt{\sigma^2 t_j}\, Z_j$

for $j = 0, \dots, n$

- $Z_j \sim N(0,1)$
- $t_j = j\Delta t$
- $B_j = \sqrt{\sigma^2 t_j}\, Z_j$

Second Way: (Correct way)

$$B(j\Delta t) = \sum_{k=1}^{j} \Big\{ \underbrace{B(k\Delta t) - B((k-1)\Delta t)}_{\Delta B_k} \Big\}$$

$\Delta B_k \sim N(0, \sigma^2 \Delta t)$

for $j = 0, \dots, n$

- $t_j = t_{j-1} \Delta t$
- $Z_i \sim N(0,1)$
- $B_j = B_{j-1} + \sqrt{\sigma^2 \Delta t}\, Z_i$

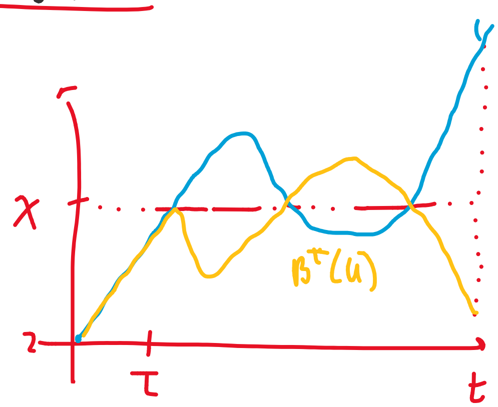
# Maximum Variable &

# Reflection Principle

Define

$$B^*(u) = \begin{cases} B(u), & u \le \tau \\ x-(B(u)-x), & u > \tau \end{cases}$$

as the <u>reflected Brownian motion</u>




The law of the path for $u > \tau$ is symmetric wrt $y > x$, $y < x$

So $\left\{ \max_{0 \le u \le t} B(u) \ge x \right\}$

happens either one of two equally likely sample paths one of which satisfies

$\{B(t) > x\}$

So

$$\mathbb{P}\left(\max_{0 \le u \le t} B(u) > x\right)$$

$$= 2\,\mathbb{P}(B(t) > x)$$

$$= 2\left(1 - \Phi_t(x)\right)$$

$$\Phi(t) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2 t}\, dy$$

Let $\tau_x = \min\{u \ge 0 : B(u) = x\}$

$$\mathbb{P}(\tau_x \le t) = \mathbb{P}\left(\max_{0 \le u \le t} B(u) > x\right)$$

$$= 2\left(1 - \Phi_t(x)\right)$$

So diff. to get the pdf we have

$$f_{\tau_x}(x) = 2\,\varphi_t(x)$$

## Reflected BM

$$R(t) = |B(t)|$$

$$\mathbb{E}[R(t)] = \int_{-\infty}^{+\infty} |x| \, \varphi_t(x) \, dx$$

$$= 2 \int_0^{\infty} x \, d\Phi(x) = \sqrt{\frac{2t}{\pi}}$$

$$\operatorname{Var}(R(t)) = \mathbb{E}(B(t)^2) - \frac{2t}{\pi}$$

$$= t - \frac{2t}{\pi}$$

## Absorbed BM

$$A(t) = \begin{cases} B(t) & t \le \tau \\ 0 & t > \tau \end{cases}$$

$$\tau = \inf\{t > 0 : B(t) = 0\}$$

## BM w Drift

$$X(t) = \mu t + \sigma B(t)$$

$$B(t) \sim N(0, t)$$

$B(t) \sim N(0, t)$

$X(t) \sim N(\mu^2, \sigma^2 t)$

$\mathbb{P}(X(t) \le y \mid X(0) = x)$

$= \mathbb{P}\left(B(t) \le \frac{y - xt}{\sigma} \,\middle|\, B(0) = \frac{x}{\sigma}\right)$

$$= \int_{-\infty}^{\frac{y-xt}{\sigma}} \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t} \, dz$$

## Gambler's Ruin

$T = T_{ab} = \inf\{t \ge 0 : X(t) = a, b\}$

Thrm:

$u(x) = \mathbb{P}(X(T_{ab}) = b \mid X(0) = x)$

$$= \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}$$

Pf: Within time $\Delta t$

$X(t)$ will be at position

$$X(0) + \underbrace{X(t+\Delta t) - X(t)}_{\Delta X}$$

$$= X(0) + \Delta X$$

By L.O.T.P.

$$u(x) = \mathbb{P}\left(X(T) = b \mid X(0) = x\right)$$

$$= \mathbb{E}\left[\mathbb{P}\left(X(T) = b \mid X(0) = x,\ X(\Delta t) = x + \Delta x\right) \mid X(0) = x\right]$$

$$= \mathbb{E}\left[\mathbb{P}\left(X(T) = b \mid X(\Delta t) = x + \Delta x\right) \mid X(0) = x\right)$$

$$= \mathbb{E}\left(u(\Delta X + x)\right)$$

Now by Taylor

$$u(x + \Delta x) = u(x) + u'(x)\Delta x$$

$$+ \frac{1}{2} u''(x)(\Delta x)^2 + o\left((\Delta x)^3\right)$$

So

$$\mathbb{E}\left(u(x + \Delta x)\right)$$

$$= u(x) + u'(x)\,\mathbb{E}(\Delta x) + \frac{1}{2}u''(x)$$

$$\mathbb{E}((\Delta x)^2) + \mathbb{E}((\Delta x)^3)$$

Hame all together

$$u(x) = u(x) + \cancel{\quad}$$

and we see

$$u'(x)\,\mathbb{E}(\Delta x) + \frac{1}{2}u''(x)\,\mathbb{E}(\Delta x^2)$$

$$+ \mathbb{E}((\Delta x)^3) = 0$$

$$\mathbb{E}(\Delta x) = \mathbb{E}(\mu\Delta t + \sigma\Delta B)$$

$$= \mu\Delta t$$

$$\mathbb{E}(\Delta x^2) = \mathbb{E}((\mu\Delta t)^2 + (\beta\Delta t)^2$$

$$+ 2\mu\Delta t\beta\Delta t)$$

$$= (\mu\Delta t)^2 + \sigma^2\Delta t$$

and lastly

$$\mathbb{E}((\Delta t)^3) = o(\Delta t)$$

So

$$0 = u'(x)\mu\Delta t + \frac{1}{2}u''(x)(\mu^2\Delta t^2 + \sigma^2\Delta t) + o(\Delta t)$$

Now dividing by $\Delta t$

and letting $\Delta t \to 0$

$$\boxed{\begin{aligned} &0 = u'(x)\mu + \frac{1}{2}u''(x)\sigma^2 \\ &u(b) = 1 \quad u(a) = 0 \end{aligned}}$$

The solution to this

is given by the Thrm.

<u>Thrm</u>:

$$V(x) := \mathbb{E}[T_{ab} \mid X(0) = x]$$

$$= \frac{1}{\mu}\left[u(x)(b-a) - (x-a)\right]$$

<u>Pf</u>: After time $\Delta t$ the

BM with drift is at

position $X(0) + \Delta X$.

and the conditional

mean time to exit is

$$\Delta t + V(x + \Delta X)$$

So by conditioning on the first step

$$v(x) = \mathbb{E}[T \mid X(0) = x]$$

$$= \mathbb{E}\big[\Delta t + \mathbb{E}[T - \Delta t \mid X_0 = x, X(\Delta t) = x + \Delta x) \mid X_0 = x\big]$$

$$= \Delta t + \mathbb{E}[v(X + \Delta X)]$$

So by Taylor

$$= \Delta t + v(x) + v'(x)\mathbb{E}(\Delta X)$$
$$+ \frac{1}{2} v''(x)\mathbb{E}(\Delta x^2)$$
$$+ o(\Delta X^3)$$

use model

$$= \Delta t + v(x) + v'(x)\mu \Delta t$$
$$+ \frac{1}{2} v''(x)(\sigma^2 \Delta t + \mu \Delta t^2)$$
$$+ o(\Delta t)$$

So putting it together

$$\boxed{\begin{array}{l} 0 = 1 + v'(x)\mu + \frac{1}{2}\sigma^2 v''(x) \\ v(a) = 0 \qquad v(b) = 0 \end{array}}$$

The sol. is given by the thrm. ∎

If we define infinitesimal generator

$$\mathcal{L}(f(x)) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x)$$

then $u(x)$ solves

$$\mathcal{L}\, u(x) = 0$$

and $v(x)$ solves

$$1 + \mathcal{L}\, v(x) = 0$$

# Lecture 4/24

Tuesday, April 24, 2018 9:32 AM

Last time we studied

$$X(t) = \mu t + \sigma B(t)$$

the brownian motion with drift.

We also defined stopping times

$$T = \min\{t > 0 : X(t) = a, b\}$$

$$u(x) = \mathbb{P}(X(T) = b \mid X(0) = x)$$

$$v(x) = \mathbb{E}[T \mid X(0) = x]$$

Then we define the operator

$$\mathcal{L}f = \mu f'(x) + \tfrac{1}{2}\sigma^2 f''(x)$$

Then we showed

$u(x)$ solves

$$\mathcal{L}u = 0 \qquad u(a) = 0$$
$$u(b) = 1$$

and $v(x)$ solves

$$\mathcal{L}v = -1 \qquad v(a) = v(b) = 0$$

Rewritting for general $(\sigma, \mu)$

$$X(t) = \int_0^t \mu(X(t))\,ds + \int_0^t \sigma(X(s))\,dB_s$$

So we can solve a Stochastic diffy Q to get estimates of the drift and diffusion.

Ex: Price of the stock

Ex.: Price of the stock today is

$X(0) = 100$ $\quad a = 95$

$b = 105$

## Geometric Brownian Motion

Define

$X(t) = \log Z(t)$

and

$$Z(t) = z e^{X(t)}$$
$$= z e^{(a - \frac{1}{2}\sigma^2)t + \sigma B(t)}$$

Black-Scholles Model

1. $Z(t) \geq 0$

2. Random during large periods have exp

growth & decay.

3. Rate of return ind. of interval.

$$\frac{Z(t_k)}{Z(t_{k-1})} \perp\!\!\!\perp \frac{Z(t_{k-1})}{Z(t_{k-2})} \quad \forall k.$$

i.e. changes over nonoverlapping intervals are ind.

Pf: (3)

$$\frac{Z(t_k)}{Z(t_{k-1})} = e^{(a-\frac{1}{2}\sigma^2)(t_k-t_{k-1})+\sigma\underbrace{(B(t_k)-B(t_{k-1}))}}$$

ind of $B(t_{k-1})-B(t_{k-2})$

. Properties:

$\cdot\ \mathbb{E}(Z(t) \mid Z(0) = z)$

MGF

$$= z e^{(a - \frac{1}{2}\sigma^2)t} \, \mathbb{E}\left(e^{\sigma B(t)}\right)$$

$$= z e^{(a - \frac{1}{2}\sigma^2)t} \, e^{\frac{1}{2}\sigma^2 t}$$

$$\boxed{= z e^{at}}$$

$$Var(Z(t)) = \mathbb{E}(Z(t)^2) - \mathbb{E}(Z)^2$$

$$\mathbb{E}(Z(t)^2 \mid Z(0) = z)$$

$$= z^2 \, \mathbb{E}\left\{ e^{2(a - \frac{1}{2}\sigma^2)t + 2\sigma B(t)} \right\}$$

$$= z^2 e^{2at - 2\sigma^2 t}$$

So the variance is given by

$$\boxed{Var(Z(t)) = z^2 e^{2at} \left\{ e^{\sigma^2 t} - 1 \right\}}$$

$$\mathrm{Var}[Z(t)] = z e \left\{ e - 1 \right\}$$

Rmk: $X(t) = \underbrace{(a - \tfrac{1}{2}\sigma^2)}_{\mu} t + \sigma B(t)$

If $a < \tfrac{1}{2}\sigma^2$ then

$$\lim_{t \to \infty} X(t) = -\infty$$

bc $\mu t = O_p(t)$

$$\sigma B(t) = O_p(\sqrt{t})$$

this then implies

$$Z(t) \longrightarrow e^{-\infty} = 0$$

But

$$\mathbb{E}(Z(t)) = z e^{ta} \longrightarrow +\infty$$

because the variance is

$$\mathrm{Var}(Z(t)) \longrightarrow +\infty$$

Note: $Z(t) = z e^{(a - \frac{1}{2}\sigma^2)t + \sigma B(t)}$

is the unique sol. to

the SDE

$$Z(t) = z + \int_0^t a Z(s)\,ds + \int_0^t \sigma Z(s)\,dB(s)$$

Thm: Let

$$T = \min\left\{ t > 0 : \frac{Z(t)}{Z(0)} = A \text{ or } B \right\}$$

then

$$\mathbb{P}\left( \frac{Z(T)}{Z(0)} = B \right) = \frac{1 - A^{1 - 2a/\sigma^2}}{B^{1-2a/\sigma^2} - A^{1-2a/\sigma^2}}$$

Pf: $\dfrac{Z(T)}{Z(0)} = B$

$\iff$

$$e^{X(T)-X(0)} = B$$

$$\Longleftrightarrow$$

$$X(T) = X(0) + \log B$$

then take

$$b = X(0) + \log(B)$$

$$\mu = (a - 1/2\,\sigma^2)$$

and solve via sol. to

$\mathcal{L}u = 0$ from above.

# Option Pricing

Call option: right to purchase

a certain # of stocks

at a given time $T$

at a given price $K$

called the strike price

called the strike price

Put option: the right to sell .....

The time $T$ is called the maturity.

Today we pay a premium in order to buy the option.

Assume that the price of the stock follows Geometric BM.

$$Z(t) = z e^{(a - \frac{1}{2}\sigma^2)t + \sigma B(t)}$$

Let $F(\underbrace{z}_{\text{current price}}, \underbrace{\tau}_{\text{remaining time until maturity}})$ be the price time until maturity

Value of the option

Exercise if $z - K > 0$ (if $z > 0$)

So

$$F(z,0) = (z-k)^+$$

Black-Scholes proved

$$F(z,t) = e^{-rt}\,\mathbb{E}\left\{ (z(T)-K)^+ \,/\, z(0) = z \right\}$$

If $z(t)$ is GBM then we can calculate it to get

$$F(z,\tau) = z\,\Phi\left(\frac{\log(z/a) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{T}}\right)$$

$$- a e^{-rT}\,\Phi\left(\frac{\log z/a + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{T}}\right)$$

$(r, T, \sigma, a)$ are observable.

$\sigma$ is not.

- estimate via historical data
- implied volatility

Pr 8.3.3

a. Show that $B(t) - tB(1)$ for $t \in (0,1)$ is ind of $B(1)$.

$$\mathbb{E}(B(1)) = 0$$

$$\mathbb{E}(B(t) - t(B(1))) = 0$$

$$\mathbb{E}\big(B(1)(B(t) - t(B(1)))\big)$$

$$= \mathbb{E}(B(1)B(t)) - t\,\mathbb{E}(B(1)^2)$$

$= (1 \wedge t) - t \cdot 1$

$= t - t = 0$

So

$B(t) - t\,B(1) \perp\!\!\!\perp B(1).$

Pr: P.1.7

Show $B^2(n) - n$ is a mart.

$\mathbb{E}\left( B^2(n+1) - (n+1) \mid B^2(n) - n \right)$

$= \mathbb{E}\left\{ \left(B(n+1) - B(n) + B(n)\right)^2 - (n+1) \mid - \right)$

$= \mathbb{E}\left\{ \left(B(n+1) - B(n)\right)^2 \right\}$

$+ B^2(n)$

$+ 2\,\mathbb{E}\left( \left(B(n+1) - B(n)\right) B(n) \mid \right)$

$-(n+1)$

$= 1 + B^2(n) - (n+1)$

$= B^2(n) - n$

# Lecture 4/25

Wednesday, April 25, 2018 3:36 PM

## Brownian motion

← particle →

$B(t)$ = position of particle at time $t$

· Continuous func.

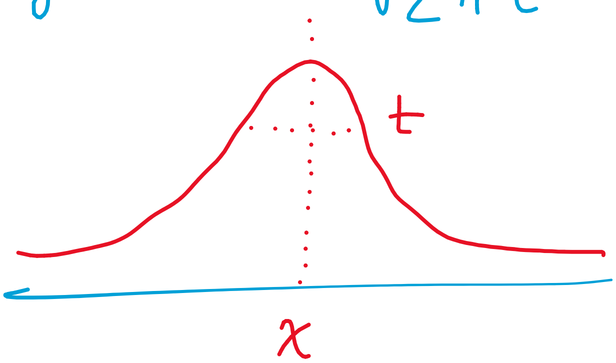
$p(y, t \mid x)$ = pdf of position $y$ at time $t$ with initial sition $x$

Einstein: $\frac{\partial P}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x^2}$

Einstein: $\dfrac{\partial p}{\partial t} = \dfrac{1}{2}\underbrace{\sigma^2}_{\text{diffusion}}\dfrac{\partial^2 p}{\partial x^2}$.

the solving with $\sigma^2 = 1$

$$p(y,t \mid x) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2t}(y-x)^2}$$




Properties:

- $B(s+t) - B(s) \sim N(0, \sigma^2 t)$
- $B(0) = 0$ $\quad B(t)$ cont.
- Independent increments

$$\mathrm{Cov}(B(s), B(t)) = \sigma^2 \min(s, t)$$

# BM w Drift

$B(t)$ standard BM

$$X(t) = \underbrace{\mu}_{\text{drift}} t + \underbrace{\sigma}_{\text{variance parameter}} B(t)$$

$$\mathbb{P}(X(t) \leq y \mid X(0) = x)$$

$$= \mathbb{P}\left(B(t) \leq \frac{y - \mu t}{\sigma} \;\middle|\; B(0) = \frac{x}{\sigma}\right)$$

$$= \mathbb{P}\left(B(t) - B(0) \leq \frac{y - \mu t - x}{\sigma}\right)$$

$$= \Phi_t\left(\frac{y - \mu t - x}{\sigma}\right)$$ cdf of $N(0, t)$

$$= \mathbb{P}\left( \frac{B(t) - B(0)}{\sqrt{t}} \le \frac{y - \mu t - x}{\sigma \sqrt{t}} \right)$$

$$= \Phi\left( \frac{y - \mu t - x}{\sigma\sqrt{t}} \right)$$

Let $X(0) = x \quad a < x < b$

$$T = T_{ab} = \{ t \geq 0 : X(t) = b, a \}$$

$$u(x) = \mathbb{P}\left( X(T) = b \mid X(0) = x \right)$$

$$= \frac{\exp\left(\frac{-2\mu x}{\sigma^2}\right) - \exp\left(\frac{-2\mu a}{\sigma^2}\right)}{\exp\left(\frac{-2\mu b}{\sigma^2}\right) - \exp\left(\frac{-2\mu a}{\sigma^2}\right)}$$

$$\mathbb{E}(T) = \frac{1}{\mu}\left( u(x)(b-a) - (x-a) \right)$$

# Geometric BM

$$Z(t) = z e^{X(t)}$$

BM with drift.

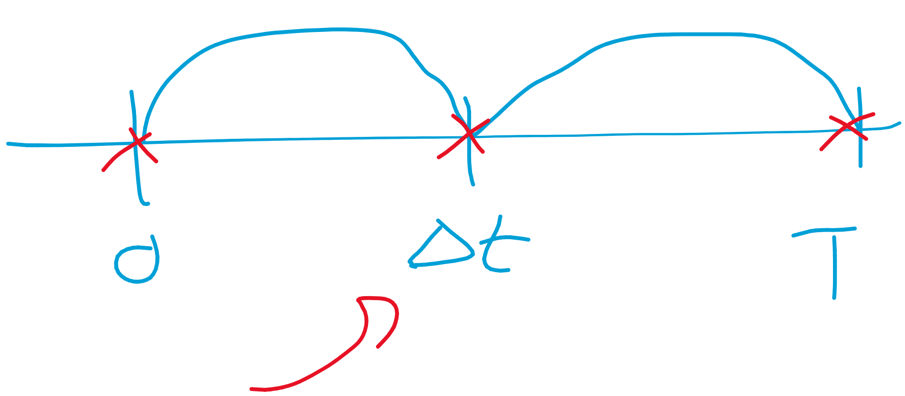
$$\mathbb{E}\left(Z(t) \mid Z(0) = z\right) = z e^{\alpha t}$$

drift param.

$$\mathbb{E}\left(Z^2(t) \mid Z(0) = z\right)$$

$$= z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$$

"First step analysis"



Condition on

this the let

$\Delta t \longrightarrow 0$

Pr 8.4.10

$0 = t_0 < t_1 < \cdots$

$$X_n = Z(t_n) e^{-r t_n}$$

$Z(t_n) =$ GBM with drift $r$

and variance $\sigma^2$

Show $X_n$ is a martingale.

$$\mathbb{E}|X_n| = \mathbb{E}\left| Z(t_n) e^{-r t_n} \right|$$

$$= \mathbb{E}\left| z e^{\alpha t} e^{-r t_n} \right|$$

$$= \begin{cases} \mathbb{E}(Z(t_n)e^{-t_n} & Z > 0 \\ -\mathbb{E}(Z(t_n)e^{-rt_n}) & Z < 0 \end{cases}$$

$$\mathbb{E}\left(Z(t_n)e^{-rt_n}\right)$$

$$= e^{-rt_n}\,\mathbb{E}(Z(t_n))$$

$$= e^{-rt_n}\,\lambda e^{\alpha t} < \infty$$

Left to show the mart. property

$$\mathbb{E}\left(Z(t_{n+1})e^{-rt_{n+1}} \,\middle|\, Z(t_n)e^{-rt_n}, \ldots\right)$$

$$= e^{-rt_{n+1}}\,\mathbb{E}\left(Z(t_{n+1}) \,\middle|\, Z(t_n)e^{-rt_n}, \ldots\right)$$

$$= e^{-r t_{n+1}} \mathbb{E}\left( Z(t_{n+1}) \,\middle|\, Z(t_n), \dots \right)$$

$$= e^{-r t_{n+1}} \mathbb{E}\left( Z(t_{n+1}) \,\middle|\, Z(t_n) \right)$$

$$= e^{-r t_{n+1}} \mathbb{E}\left[ Z(t_{n+1} - t_n) \,\middle|\, Z_0 = Z(t_n) \right]$$

$$= e^{-r t_{n+1}} Z(t_n) e^{r(t_{n+1} - t_n)}$$

$$= Z(t_n) e^{-r t_n} = X_n$$

# Lecture 4/26

Thursday, April 26, 2018 9:35 AM

## Queueing Systems

Customers arrive at random times to a facility and recieve a service.

Important Questions

1. Utilization of servers?
2. Long run number of customers
3. Long run waiting time.

Let

$L =$ ave. # of customers in system

$\lambda =$ rate of arrivals

$W =$ ave time spent in the system

Assume that the system operates long enough so it has reached some stationary state. Then in the long run

$$L = \lambda W$$

# Queueing Models

$A \mid B \mid C$

interarrival dist

service time dist

number of servers.

When $A, B = M$ then arrivals follow a Poisson dist. and the service times are Expon'l.

Ex: $M \mid M \mid 1$

Let $X(t)$ be the number

of customers in the system at time $t$.

Interarrival $\sim \text{Exp}(\mu)$

Arrival $\sim \text{Pois}(\lambda)$

$$\mathbb{P}(X(t+h) = k+1 \mid X(t) = k)$$

$$= (\lambda h + o(h))(1 - \mu h + o(h))$$

$$= \lambda h + o(h)$$

$$\mathbb{P}(X(t+h) = k-1 \mid X(t) = k)$$

$$= (1 - \lambda h + o(h))(\mu h + o(h))$$

$$= \mu h + o(h)$$

$= \mu h + o(h)$

$\mathbb{P}(X(t+h)=k \mid X(t)=k)$

$= 1 - (\lambda+\mu)h + o(h)$

This $X(t)$ is a birth and death M.C. with parameters $\lambda_k = \lambda$ $\mu_k = \mu$

then the limit. dist.

$$\pi_k = \theta_k \pi_0 = \frac{\theta_k}{\sum \theta_k}$$

$$\theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \left(\frac{\lambda}{\mu}\right)^k$$

We can also derive the traffic intensity

$$\rho = \frac{\text{arrival rate}}{\text{system service time}} = \frac{\lambda}{\mu}$$

Ex: $M|M|\infty$

then $\lambda_k = \lambda$

$\mu_k = k\mu$

Ex: $M|M|S$ then

$\lambda_k = \lambda \quad \mu_k = \mu \min\{k, s\}$

$$\rho = \frac{\lambda}{s\mu}$$

# Final Notes

- Focused on post midterm questions

# Lecture 5/1

Tuesday, May 1, 2018 9:37 AM

# Final Exam Review

Chp 5: Sec 1, 2, …, 6

Only compound P.P from 6

Chp 6: Sec 1, …, 6

Chp 7: Sec 1, …, 4

Chp 8: Sec 1, 2, 3.1, 4

Chp 9: Sec 1, 2

Chp 6 Postulates for

B & D.

B & D.

$$\mathbb{P}(X(t+h) - X(t) = k+1 \mid X(t)=k)$$
$$= \lambda_k h + o(h)$$

$$\mathbb{P}(X(t+h) - X(t) = k-1) \mid = k)$$
$$= \mu_k h + o(h)$$

$$\mathbb{P}(\# = k \mid k) = 1 - (\lambda_k + \mu_k)h$$
$$+ o(h)$$

$$P_{ij}(0) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & o.w. \end{cases}$$

Thrms & Proofs:

Chp 5: 5.1, 5.2, , 5.6

Chp 6: · Postulates of BED

· Derivation of ODE

· Derivation of

$\mu_i, w_i$

Chp 7: Proof of

$\mathbb{E}(W_{N(t)+1}) = \mu(1 + m(t))$

Sect 7.1

Chp 8: · Idea of reflection principle

· PDF of hitting time

$T_x$

· Thms 8.1 – 8.2

· Thrms 7.1 – 8.2

Ex: M | M | 1

arrivals — service — # servers

arrivals are Poisson Process $\Rightarrow$ Interarrival are Exp.

Service times are exponential

This is a birth and death process with $\lambda_k = \lambda$, $\mu_k = \mu$

$$\pi_k = \lim_{t \to \infty} \mathbb{P}(X(t) = k)$$

As in section 4 of chapter 6

As in section 4 of chapter 6

$\pi_k = \theta_k \qquad \pi_0 = \dfrac{\theta_k}{\sum_{j=0}^{\infty} \theta_j}$

$\theta_j = \dfrac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}$

In our case $\lambda_k = \lambda$

$\mu_k = \mu$

If $\sum \theta_j = \infty$ then

$\pi_k = 0 \ \forall k$ and the

queue grows without

limit. As in our case

$\theta_j = \left(\dfrac{\lambda}{\mu}\right)^j$

$$\theta_j = \left(\frac{\lambda}{\mu}\right)$$

$$\sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \begin{cases} \dfrac{1}{1-\frac{\lambda}{\mu}} & \frac{\lambda}{\mu} < 1 \\ \infty & \frac{\lambda}{\mu} \geq 1 \end{cases}$$

So the queue grows without bound when

$$\lambda \geq \mu$$

birth

death

Or we have a limiting distribution when $\lambda < \mu$.

$$\pi_k = \frac{\theta_k}{\sum \theta_j} = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k$$

or

$$\pi \sim \text{Geom}(\lambda/\mu)$$

<u>Rmks</u>

1.

$$\mathbb{E}(\pi) = L = \frac{\lambda}{\mu - \lambda}$$

long run length of queue

2. Traffic intensity

$$\rho = \frac{\text{arrival rate}}{\phantom{x}} = \frac{\lambda}{\phantom{x}}$$

$$\rho = \frac{\text{arrival rate}}{\text{service rate}} = \frac{\lambda}{\mu}$$

3. $L = \dfrac{\rho}{1-\rho} \longrightarrow +\infty$

if $\rho \longrightarrow 1$

4. Long run probability of being served immediatly

$$\pi_0 = 1 - \frac{\lambda}{\mu}$$

5. Dist of waiting time

$(\lambda < \mu)$ $n$ customers

$T =$ my waiting $+$ $n$ waiting

$$T = \sum_{i=1}^{n} E_i \quad E_i \sim \mathrm{Exp}(\mu)$$

$$T/n \sim \mathrm{Gamma}(n+1, \mu)$$

$$\mathbb{P}(T \le t \mid N \text{ ahead})$$

$$= \int_0^t \frac{\mu^{n+1} \tau^n e^{-\mu\tau}}{\Gamma(n+1)} \, d\tau$$

So unconditionally

$$\mathbb{P}(T \le t) = \sum_{n \ge 0} \mathbb{P}(T \le t \mid n \text{ ahead}) \mathbb{P}(n \text{ ahead})$$

LR

$$= \sum_{n=0}^{\infty} \mathbb{P}(T \le t \mid n \text{ ahead}) \pi_n$$

$$= \sum_{n \geq d} \int_d^t \frac{\mu^n \tau^n e^{-\mu\tau}}{\Gamma(n+1)} d\tau \, \pi_n$$

$$= 1 - e^{-t(\mu - \lambda)}$$

$$\sim \mathrm{Exp}(\mu - \lambda)$$

So long run dist

$$\boxed{T \sim \mathrm{Exp}(\mu - \lambda)}$$

Mean waiting time

$$W = \frac{1}{\mu - \lambda}$$

Thus the queueing formula

$L = W\lambda$ hold.

# M/M/∞

- Unlimited # servers

$\lambda_k = \lambda$ ← arrival

$\mu_k = k\mu$ ← departure rate

$$\pi_k = \frac{\theta_k}{\sum_{j \geq 0} \theta_j}$$

$$\theta_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k$$

$$\sum_{k=0}^{\infty} \theta_k = e^{\lambda/\mu}$$

and

$$= \left(\lambda\right)^k e^{-\lambda/\mu}$$

$$\pi_k = \frac{(\lambda/\mu)^k e^{-\lambda/\mu}}{k!}$$

and

$$\pi \sim \mathrm{Pois}(\lambda/\mu)$$

So

$$L = \lambda/\mu \quad W = \frac{1}{\mu}$$

from Exp. waiting times

## M|M|S

$$\lambda_k = \lambda$$

$$\mu_k = \mu \min\{S, k\}$$

$$\theta_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n & 0 \le n \le s \\ \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{\lambda}{s\mu}\right)^{n-s} & n > s \end{cases}$$