1. (6.3)

   (a) First let T_1 and T_2 be UMP tests for corresponding to α_1 and α_2 respectively. Then for the sake of contradiction, assume that $c(\alpha_1) < c(\alpha_2)$. By Neyman-Person our two tests are given by the following

$$T_1 = \begin{cases} 1 & f_1(x) > c(\alpha_1)f_0(x) \\ \gamma_1 & f_1(x) = c(\alpha_1)f_0(x) \\ 0 & f_1(x) < c(\alpha_1)f_0(x) \end{cases} \qquad T_2 = \begin{cases} 1 & f_1(x) > c(\alpha_2)f_0(x) \\ \gamma_1 & f_1(x) = c(\alpha_2)f_0(x) \\ 0 & f_1(x) < c(\alpha_2)f_0(x) \end{cases}$$

   Define $LR(x) = \frac{f_1(x)}{f_0(x)}$. We now argue by cases

   - $LR(x) < c(\alpha_1)$: In this case $T_1(x) = 0$ and $T_2(x) =$ so $T_1(x) \geq T_2(x)$
   - $LR(x) = c(\alpha_1)$: In this case $T_1(x) = \gamma_1$ and $T_2(x) = 0$ so $T_1(x) \geq T_2(x)$
   - $c(\alpha_1) < LR(x) < c(\alpha_2)$: In this case $T_1(x) = 1$ and $T_2(x) = 0$ so $T_1(x) \geq T_2(x)$
   - $LR(x) = c(\alpha_2)$: In this case $T_1(x) = 1$ and $T_2(x) = \gamma_2$ so $T_1(x) \geq T_2(x)$
   - $c(\alpha_2) < LR(x)$: In this case $T_1(x) = 1$ and $T_2(x) = 1$ so $T_1(x) \geq T_2(x)$

   In any case we note that $T_1(x) \geq T_2(x)$. Hence $\mathbb{E}_0(T_1) \geq \mathbb{E}_0(T_2)$ but recall this, by definition, is just the type I error corresponding to α_i. That is we see that $\mathbb{E}_0(T_1) = \alpha_1 \geq \alpha_2 = \mathbb{E}_0(T_2)$. But recall $\alpha_1 < \alpha_2$ so we have a contradtiction. Therefore, we conclude that $c(\alpha_1) \geq c(\alpha_2)$

   (b) We look to show that $1 - \mathbb{E}_1(T_1) \geq 1 - \mathbb{E}_1(T_2)$. Hence, it suffies to show that $\mathbb{E}_1[T_1] \leq \mathbb{E}_1[T_2]$. Well recall that $\mathbb{E}_1[T_*]$ is maximized at $T_* = T_2$ where T_* are all tests such that $\mathbb{E}_0[T_*] \leq \alpha_2$. But $\mathbb{E}_0[T_1] = \alpha_1 < \alpha_2$. Hence $\mathbb{E}_1[T_1] \leq \mathbb{E}_1[T_2]$ by definition of UMP.

2. (6.4) Let T_* be UMP for size α with power $\beta < 1$ for testing H_0 vs H_1. Define $T_{**} = 1 - T_*$. We look to show that this test is UMP for testing H_1 vs H_0 with size $1 - \beta$. We must show two things (i) T_{**} is of size β and (ii) for any test $\tilde{T}_{**}$ with $\mathbb{E}_1(\tilde{T}_{**}) \leq \beta$ we have that $\mathbb{E}_0[\tilde{T}_{**}] \leq \mathbb{E}_0[T_{**}]$.

   (i) First note that $\mathbb{E}_1[T_{**}] = 1 - \mathbb{E}_1[T_*]$. Note this is just the power of T_* when H_1 is true which we defined to be β. Hence $\mathbb{E}_1[T_{**}] = 1 - \mathbb{E}_1[T_*] = 1 - \beta$. Hence T_{**} is of size $1 - \beta$.

   (ii) Let $\tilde{T}_{**}$ be a test with $\mathbb{E}_1[\tilde{T}_{**}] \leq 1 - \beta$. Then there exists a test $\tilde{T}_*$ of H_0 vs H_1 with $\tilde{\alpha} = \mathbb{E}_0[\tilde{T}_*] \geq \alpha$ such that we can write $\tilde{T}_{**} = 1 - \tilde{T}_*$. Using this we see that

$$\mathbb{E}_0[T_{**}] = 1 - \mathbb{E}_0[T_*] = 1 - \alpha \geq 1 - \tilde{\alpha} = 1 - \mathbb{E}_0[\tilde{T}_*] = \mathbb{E}_0[\tilde{T}_{**}]$$

Hence T_{**}, being of the form in the Neyman-Pearson Lemma, is UMP for H_1 vs H_0.

3. (6.7) Suppose that $\phi_* \in \mathcal{T}_0$ is given by equation (6.9). We derive a useful result for later in the proof.

$$\begin{aligned}
\int (\phi_* - \phi)\sum_{i=1}^m c_i f_i d\nu &= \int \sum_{i=1}^m c_i f_i \phi_* d\nu - \int \sum_{i=1}^m c_i f_i \phi d\nu \\
&= \sum_{i=1}^m c_i \left\{ \int \phi_* f_i d\nu - \int \phi f_i d\nu \right\} \\
&= \sum_{i=1}^m c_i \left\{ 0 - 0 \right\} \\
&= 0
\end{aligned}$$

Where the thrid line justified by the definition of $\mathcal{T}_0$. Now, define the following quantity

$$g(x)h(x) = (\phi_* - \phi)(f_{m+1} - \sum_{i=1}^m c_i f_i)$$

Now notice by defintion of ϕ_* if $h(x) > 0$ then $g(x) > 0$ and $g(x)h(x) \geq 0$. Moroever, if $h(x) < 0$, then $g(x) < 0$ as $\phi_* = 0$ and $0 \leq \phi \leq 1$. In either cases we have that

$$(\phi_* - \phi)(f_{m+1} - \sum_{i=1}^m c_i f_i) \geq 0$$

Using this, we can write

$$\begin{aligned}
(\phi_* - \phi)(f_{m+1} - \sum_{i=1}^m c_i f_i) &\geq 0 \\
(\phi_* - \phi) f_{m+1} &\geq (\phi_* - \phi)\sum_{i=1}^m c_i f_i \\
\int (\phi_* - \phi) f_{m+1} d\nu &\geq \int (\phi_* - \phi)\sum_{i=1}^m c_i f_i d\nu \\
\int \phi_* f_{m+1} d\nu - \int \phi f_{m+1} d\nu &\geq 0 \\
\int \phi_* f_{m+1} d\nu &\geq \int \phi f_{m+1} d\nu
\end{aligned}$$

Hence, ϕ_* maximizes this quantity over all possible $\phi \in \mathcal{T}_0$.

Now, consider $\phi \in \mathcal{T}$. Then, by the same argument as above, we have that

$$\begin{aligned}
\int (\phi_* - \phi) f_{m+1} d\nu &\geq \int (\phi_* - \phi) \sum_{i=1}^{m} c_i f_i d\nu \\
\int (\phi_* - \phi) f_{m+1} d\nu &\geq \sum_{i=1}^{m} c_i \left\{ \int \phi_* f_i d\nu - \int \phi f_i d\nu \right\} \\
\int (\phi_* - \phi) f_{m+1} d\nu &\geq \sum_{i=1}^{m} c_i \left\{ t_i - \int \phi f_i d\nu \right\}
\end{aligned}$$

Recall, by definition of $\mathcal{T}$ that $s_i := \int \phi f_i d\nu \leq t_i$. Therefore, along with the assumption that $c_i \geq 0$ we have

$$\sum_{i=1}^{m} c_i \left\{ t_i - \int \phi f_i d\nu \right\} \geq \sum_{i=1}^{m} c_i \{t_i - s_i\} \geq 0$$

Thus

$$\begin{aligned}
\int (\phi_* - \phi) f_{m+1} d\nu &\geq 0 \\
\int \phi_* f_{m+1} d\nu &\geq \int \phi_* f_{m+1} d\nu
\end{aligned}$$

Therefore, if $c_i \geq 0$ for all i then ϕ_* maximizes this quantity over all $\phi \in \mathcal{T}$.