Graph Cuts & Spectral Gap

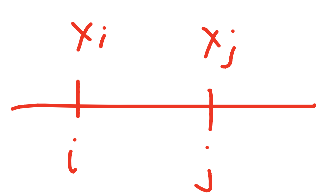
Recall: $\lambda_2 = 0 \iff G$ disconnected

Robust version: $\lambda_2 \lesssim \varepsilon \iff G$ "almost" disconnected

Given $v \perp \mathbb{1}$

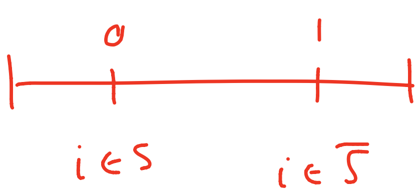
$$\lambda_2 \leq \frac{v^T L v}{v^T D v} \leq \varepsilon \overset{}{\underset{\text{rounding}}{\iff}} \text{"small cuts" in } G$$

Think of v as assignment



which we can extend to the

"cut assignment"



"Small" Cuts

Conductance

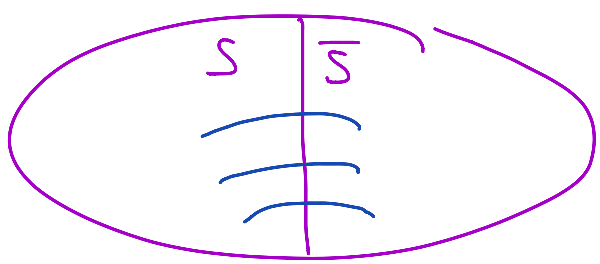
$G = (V, E)$

Def: the conductance of a cut $S \subseteq V$

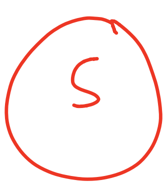
$$\phi(S) = \frac{|E(S, \bar{S})|}{\min\{Vol(S), Vol(\bar{S})\}}$$

(isoperimetric ratio)

$$Vol(S) = \sum_{i \in S} d_i$$



Random Walk Interpretation



$$(\pi_S)_i = \begin{cases} \frac{d_i}{Vol(S)} & i \in S \\ 0 & o.w. \end{cases}$$

uniform over ∂

then

$$\mathbb{P}(\text{escaping } S \text{ in one step})$$

$$= \sum_{i \in S} \frac{d_i}{\mathrm{vol}(S)} \frac{|E(i,\bar{S})|}{d_i} = \frac{E(S,\bar{S})}{\mathrm{vol}(S)}$$

Rmk: We could change the denominator as $\min\{|S|, |\bar{S}|\}$ or general ratio cuts.

$$\min\{\mu(S), \mu(\bar{S})\}$$

Def: The conductance of a graph G

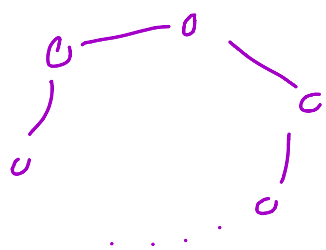
$$\phi_G = \min_{S \subseteq V} \phi(S) \quad \left(\begin{array}{l} \text{NP-Hard} \\ O(\sqrt{\log n}) \text{ approx} \end{array} \right)$$

Rmk: Often used in community detection

Rmk: This problem is called the

minimum conductance cut.

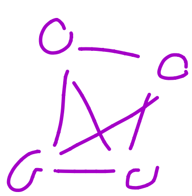
Ex:



n-cycle

$$\phi_{C_n} = \frac{2}{n/2} = \frac{1}{n}$$

Ex: Complete graph



$$\phi_{K_n} = \frac{(k)(n-k)}{(k)(n-1)} = \frac{n-k}{n-1}$$

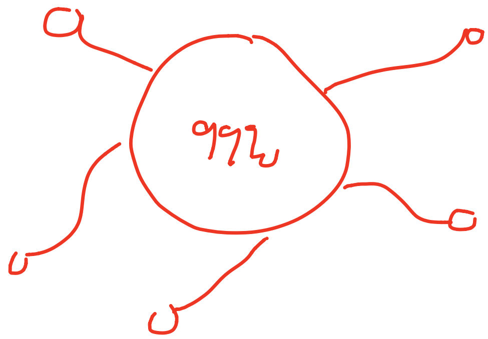
which is minimal at

$$k = n/2$$

$$\phi_{K_n} \approx \frac{1}{2}$$

Rmk: $0 \le \phi_G \le 1/2 \qquad 0 \le \phi(S) \le 1$

Rmk: Real world graph



So min conductance cuts here will cut off the "tentacles" which doesn't really tell us about the global structure.

Layout

$\lambda_2 \longleftrightarrow v_2 \longleftrightarrow$ small conductance cuts

Rmk: We're going to work with

$$\phi(S) \le \bar{\phi}(S) = \frac{|E(S,\bar{S})|\,\mathrm{Vol}(G)}{} \le 2\phi(S)$$

$$\mathrm{Vol}(S)\ \mathrm{Vol}(\bar{S})$$

Rmk: Spectral gap is a relaxation of conductance.

$$\lambda_2 = \min_{x^T 1 = 0} \frac{x^T L x}{x^T D x} = \min \frac{x^T L x}{x^T L(K_G) x}$$

where $L(K_G) = D - \dfrac{D 1 1^T D}{\mathrm{Vol}(G)}$

Complete randomized version of G

$$\bar{\phi}(S) = \frac{E_G(S,\bar{S})}{\dfrac{\mathrm{Vol}(S)\,\mathrm{Vol}(\bar{S})}{\mathrm{Vol}(V)}}$$

$$|E_{K_G}(S,\bar{S})|$$

$$= \frac{E_G(S,\bar{S})}{E_{K_G}(S,\bar{S})}$$

K_n

$$\sum_{i\in S} \frac{d_i \,\mathrm{Vol}(\bar S)}{\mathrm{Vol}(V)} = \frac{\mathrm{Vol}(S)\,\mathrm{Vol}(\bar S)}{\mathrm{Vol}(V)}$$

So

$$\bar\phi_S(S) = \frac{|E_G(S,\bar S)|}{|E_{K_n}(S,\bar S)|}$$

So minimizing gives the greatest discrepancy between the complete and the actual graph.

<u>Cut vector</u>: $\vec{1}_S$ $\quad \vec{1}_S =$

$$1_S^T L 1_S = \sum_{ij\in S} (x_i - x_j)^2$$

$$= |E(S,\bar S)|$$

$$1_S^T L_{K_G} 1_S = |E_{K_G}(S,\bar{S})|$$

$$\bar{\phi}_G = \min_{S \subseteq V} \bar{\phi}(S) = \min_{S \subseteq V} \frac{1_S^T L 1_S}{1_S^T L(K_G) 1_S}$$

So now to relax this problem we extend the span of indicator functions

$$\geq \min_{x \in \mathbb{R}^n} \frac{x^T L x}{x^T L(K_G) x} = \lambda_2$$

<u>Thm:</u> $\bar{\phi}_G \geq \lambda_2$

<u>Rmk:</u> If there is a small conductance cut then the R.W. must converge slowly.

Cheeger's Inequality: $\bar{\phi}_G \leq 2\sqrt{2\lambda_2}$

Rmk: Proof based on rounding