Lecture: 9/11/18

Expect HW after lectures on random walks over graphs.

Random Walk Operator: $W = AD^{-1}$

- Used to describe the trans. matrix of natural MC over a graph
- W^T represents "averaging operator.
- $x \in \mathbb{R}^n$ vertex "opinions"

$\left(\frac{I + W^T}{2}\right) x$ averaging of opinions

- $\lim_{n \to \infty} \left(\frac{I + W^T}{2}\right)^n X(0) \longmapsto \bar{X} \mathbb{1}$

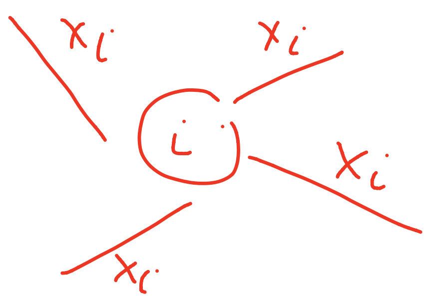
some global averaging $\bar{x} = \frac{\sum x_i d_i}{\ldots}$

Σd_i

Rmk: W_p

$\quad$ ↳ prob. over V

Rmk: Think of x_i as label on edges leaving i



We think that the stationary dist. $\pi_i \propto d_i$ $\quad W\pi = \pi$

In some sense they are the same process under the change of basis $p = Dx$ $\left(Wp = AD^{-1}p \right)$

Mass on edges not

Measuring Convergence

in nodes.

Averaging:

Recall that $\bar{x} = \dfrac{\sum x_i d_i}{\sum d_i}$

So we want to compare

$$\sum_{i=1}^{N} \pi_i \left(x_i - \left(\sum \pi_i x_i \right) \right)^2 = \mathrm{Var}_{\pi_i}(x_i)$$

$$= \|x - \bar{x}\|_{\pi}^2$$

when $\mathrm{Var}_{\pi}(x) \longmapsto 0$ we know we achieve convergence.

Random Walk:

Measuring the discrepancy between $\|p - \pi\|^2$ and $\|X - \mathbb{I}\|_{\pi}^2$

To make thing simplier lets consider

$$\sum d_i (x_i - \bar{x})^2$$

$$= \| x - \bar{x}\vec{1} \|_D^2 = \| D^{-1}p - \bar{x} D^{-1} 1 \|_D^2$$

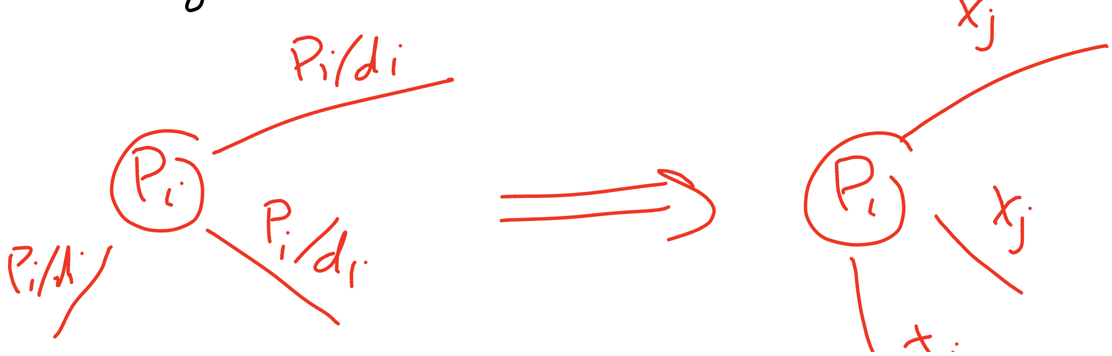
$$= \| p - \bar{x}\vec{1} \|_D^2 = \| p - \pi \|_{D^{-1}}^2$$

(prove this)

The right way to think about measuring convergence is through the norm

$$\| p - \pi \|_{D^{-1}}^2$$

We know the averaging process converges when



Prop: $\underset{\{x:\ \sum d_i x_i = 1\}}{\operatorname{argmin}} \sum d_j x_j^2 = c \cdot \vec{1}$

which arises from minimizing the L^2-norm over the edges leaving i.

Relation to the Laplacian

Def: The Volume of a graph is $Vol(G) = \sum d_i$

Rmk: $Vol(G)\,Var(x) = \sum_{i=1}^{N} \left[d_i \left(x_i - \sum \pi_j x_j\right)^2 \right]$

Claim: $\| x - \bar{x}\vec{1} \|_D^2 = x^T L x$

where L is a Laplacian

Pf: Define the complete graph K_G

with $i \;\underset{d_i d_j}{\rule{3em}{0.4pt}}\; j$ where K_G

$\overline{\qquad\qquad}$
$\mathrm{Vol}(G)$

has the same degree dist. of G.

(Finish this proof). ∎

Laplacian of G

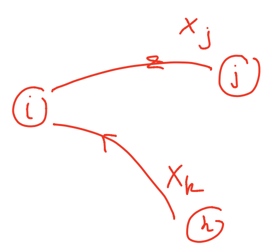
Argue: $x^T L_G x$ measures rate of convergence towards stationary at x

Informal:

$$\frac{d}{dt}\left(x^{T(t)} L(K_a) x^{(t)} \right) = -x^T L_G x$$

when $x^{(t)}$ averaging on G

Continuous time averaging



$$\frac{d}{dt}x(t) = -\left(I - W^T\right)x(t)$$

Recall the discrete version of this problem was characterized by

$$x(t+1) - x(t) = -\frac{1}{2}\left(I - W^T\right)x(t)$$

Now notice that

$$\frac{d}{dt}\left(\frac{1}{2}x(t)^T L(K_G)\, x(t)\right)$$

$$= \left(\frac{d}{dt}x(t)\right)^T L(K_G)\, x(t)$$

$$= -\left(\left(I - W^T\right)x(t)\right)^T L(K_G)\, x(t)$$

$$= -x^T \underbrace{\left(I - W\right) L(K_G)}_{=\, L_G} x(t)$$

Need to show

Recall $L(K_G) = D - A$

D_G

$$a_{ij} = \frac{d_i d_j}{Vol(G)}$$

$$A = \frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)}$$

<u>Rmk:</u> $D\left(I - \frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)}\right)x$ serves as the rank 1 projection.

$$(I-W)L(K_G) = (I - AD^{-1})\left(D - \frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)}\right)$$

$$= D - A - \frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)} + AD^{-1}\frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)}$$

$$= D - A - \frac{D\mathbb{1}\mathbb{1}^T D}{Vol(G)} + \frac{A\mathbb{1}\mathbb{1}^T D}{Vol(G)}$$

but notice $A\mathbb{1} =$ row sums $= D$

$$= D - A = L(G)$$