Semi-Definite Programming

- Extension of linear programming

- SDP, LP > Conic Programs

In graph partioning we saw

a spectral relaxation

$\lambda_2 \leq \overline{\phi_G}$

Metric Relaxation

$$\min_{\text{dmetr.}} \frac{d(G)}{d(K_G)} \leq \overline{\phi_G}$$

Solved with:

eigenvector computation

SDP

Linear programming

Linear Programming (Canonical Form)

-

$$\min \; c^T x$$
$$\text{s.t.} \quad Ax = b, \; x \geq 0$$

$x \geq 0$ — conic constraint

$x \in K$ = non. vectors

$A \in \mathbb{R}^{m \times n}$ — constraint matrix

c — objective vector.

Dual

$$\max \; y^T b$$
$$\text{s.t.} \quad A^T y \leq c$$
$$y \in \mathbb{R}^m$$

Review of duals

$$\max_{\substack{y \\ s \geq 0}} \min \; \underbrace{c^T x - y^T(Ax - b) - s^T(x - 0)}_{\text{Lagrangian}}$$

y^T lagrange s^T

$\Longleftrightarrow$

$$\min \max_{\substack{y \\ s \geq 0}} \; x^T(c - A^T y - s) + y^T b$$

$\Longleftrightarrow$

dual written

above

<u>SDP</u> — Generalization of eigenvectors

$A^T = A$ $\qquad \max_{x \in \mathbb{R}^n} \; x^T A x \qquad x^T x = 1$

optimization over a nonconvex set

changing the parameterization of the sol. space,

$$X_{ij} \simeq x_i x_j$$

if we could do this

$$x^T A x = \sum A_{ij} x_i x_j = \sum A_{ij} X_{ij}$$

$$= \langle A, X \rangle \equiv A \bullet X$$

and $\quad x^T x = I \bullet X$

$$\boxed{A \bullet B = \sum_{ij} A_{ij} B_{ij} = \mathrm{Tr}(A^T B)}$$

Useful because

$$x^T A x = \mathrm{Tr}(x^T A x) = \mathrm{Tr}(A x x^T)$$

$$= A \bullet x x^T$$

Want to think about $X = xx^T$

The eigenvector problem then becomes

$$\max A \bullet X \qquad X \in \mathbb{R}^{n \times n}$$
$$I \bullet X = 1$$

bad because we need $X = xx^T$

So X needs to be

(i) Symmetric

(ii) ideally rank 1 ← nonconvex

So we relax to

$$X = \sum_{i=1}^{d} p_i x_i x_i^T$$

↖ convex combination of rank 1 things

this set we can optimize over.

→ Each scaling needs to be between $[0, 1]$ so

$$0 \preceq X = \sum p_i x_i x_i^T \text{ is PSD}$$

→ Using the eigen decomp can be

used to go back

$$X = \sum \lambda_i u_i u_i^T \qquad X \cdot I = 1$$

$$\Rightarrow$$

$$\sum \lambda_i = 1$$

<u>Claim:</u> $PSD_n = \{ X \in S^{n \times n} : X \cdot I = 0 \}$

<u>Pf:</u> $X, Y \in PSD_n$

$\forall r \quad r^T X r \geq 0$

$\forall s \quad s^T Y s \geq 0$

$$z^T (aX + bY) z = a\, z^T X z + b\, z^T Y z \geq 0$$

$\blacksquare$

SDPs as solutions to Eigenvector Problem

$$\max A \cdot X$$

$$I \cdot X = 1$$

$$X \succeq 0 \longleftarrow X = \sum_{i=1}^{n} p_i \cdot \underbrace{x_i x_i^T}_{\text{eigenbasis of } A}$$

then assign a prob to which is the most important vector of A.

Ex:

$$\min \; C \cdot X$$

$$\text{s.t. } i \in [m] \quad A_i \cdot X = b_i \quad X \succeq 0$$

Dual: (Lagrange)

$$\max_{y \in \mathbb{R}^n} \; \min \; C \cdot X - \sum_i y_i (A_i \cdot X - b_i) - S \cdot (X - I)$$

$S \in \mathbb{R}^{n \times n}$

$S \succeq 0$

$$= \min \max \; X \cdot \left(C - \sum y_i A_i - S\right) + y^T b$$

$$\Longleftrightarrow$$

$$\max \; y^T b$$

$$\text{s.t.} \quad C = \sum y_i A_i + S$$

$$\sum y_i A_i \preceq C$$

$$y \in \mathbb{R}^m$$

$$S \succeq 0$$

SDPs Vector Embeddings

$$X \succeq 0 \longleftrightarrow X = V^T V$$

$$V = (v_1 \cdots v_n) \in \mathbb{R}^{d \times n}$$

$$X_{ij} = \langle v_i, v_j \rangle$$

Ex:

$$\min \sum_{E} \|v_i - v_j\|^2 \iff L \cdot X$$

$$\text{s.t.} \sum_{i<j} \|v_i - v_j\|^2 = 1 \iff L(K_n) \cdot X = 1$$

to see why

$$\|v_i - v_j\|^2 = \langle v_i, v_i \rangle + \langle v_j, v_j \rangle - 2\langle v_i, v_j \rangle$$

$$= L_{ij} \cdot X$$