Let π be a prob. measure on

$(\mathcal{X}, \mathcal{B})$

$\llcorner$ Typically $\mathbb{R}^p$

We build a M.C. $\{X_n, n \geq 0\}$

$X_0 \sim \mu$ with transition kernel P

s.t.

$$\sup_{A \in \mathcal{B}} |\mathbb{P}(X_n \in A) - \pi(A)| = \|\mu P^n - \pi\|_{TV} \xrightarrow{n \to \infty} 0$$

where

$$\|\mu - \nu\|_{TV} \stackrel{def}{=} \sup_{A} |\mu(A) - \nu(A)|$$

$$= \sup_{A \in \mathcal{B}} \left(\mu(A) - \nu(A)\right)$$

<u>Notation</u> $P : \mathcal{X} \times \mathcal{B} \longmapsto [0,1]$

$P(x, \cdot)$ is a p.m $\forall x$

$P(\cdot, A)$ is a measurable map.

μ a p.m.

$$\underbrace{\mu P}_{\text{as a single measure}} (A) \stackrel{def}{=} \int \mu(d\xi)\, P(\xi, A)$$

If $X_0 \sim \mu$, $X_1 | X_0 = x \sim P(x, \cdot)$

$$\mathbb{P}(X_1 \in A) = \int \mathbb{P}(X_1 \in A \,|\, X_0 = x)\, \mu_{X_0}(dx)$$

$$= \int \mu(dx)\, P(x, A)$$

Similarly define

$$P^n(x,A) = \int P^{n-1}(x,dy)\,P(y,A)$$

Also given $f : \mathcal{X} \mapsto \mathbb{R}$

define $Pf(x) \stackrel{\text{def}}{=} \int P(x,dy)\,f(y)$

Suppose

$$X_1 \mid X_0 = x \sim P(x,\cdot)$$

$$\mathbb{E}[f(X_1) \mid X_0 = x] = \int f(y)\,P(x,dy)$$

$$= Pf(x)$$

average val of f over one iteration of the MC.

<u>Three Requirements</u> $\|\mu P^n - \pi\|_{TV} \longrightarrow 0$

(1) $\pi P = \pi$

(2) P is φ-irreducible → finite travel times → continuous analogue.

(3) P is aperiodic

Def: φ-irreducibility,

$$\varphi(A)>0 \implies P^n(x, A) > 0, \quad n<\infty.$$

Theorem If P satisfy (1) – (3) then

$$\| P^n(x,\cdot) - \pi \|_{TV} \xrightarrow[n\to\infty]{} 0$$

for π-almost x.

How big should n be?

$\implies$ Rate of convergence of the chain

Rmk: We say that P is π reversible if

$$\int_A \pi(dx) P(x, B) = \int_B \pi(dx) P(x, A)$$

$\forall A, B \in \mathcal{B}$.

Intution: Suppose $X_0 \sim \pi$

$X_1 | X_0 = x \sim P(x, \cdot)$

$$\mathbb{P}(X_0 \in A, X_1 \in B) = \int_A \mathbb{P}(X_1 \in B \mid X_0 = x) \mu_{X_0}(dx)$$

$$\mathbb{P}(X_0 \in B, X_1 \in A) = \int_B \mathbb{P}(X_1 \in A \mid X_0 = x) \mu_{X_0}(dx)$$

If we take $B = \mathcal{X}$

$\pi(A) = (\pi P)(A)$

Def: P is <u>lazy</u> if $P(x,x) \geq 1/2$

Throughout we assume P is π-reversible and lazy.

Def: Let $L^2(\pi)$ be the set of square int. functions.

$$\|f\|_2^2 = \int |f(x)|^2 \pi(dx)$$

$$\langle f, g \rangle = \int f(x) g(x) \, \pi(dx)$$

Rmk: On T.V. metric. if

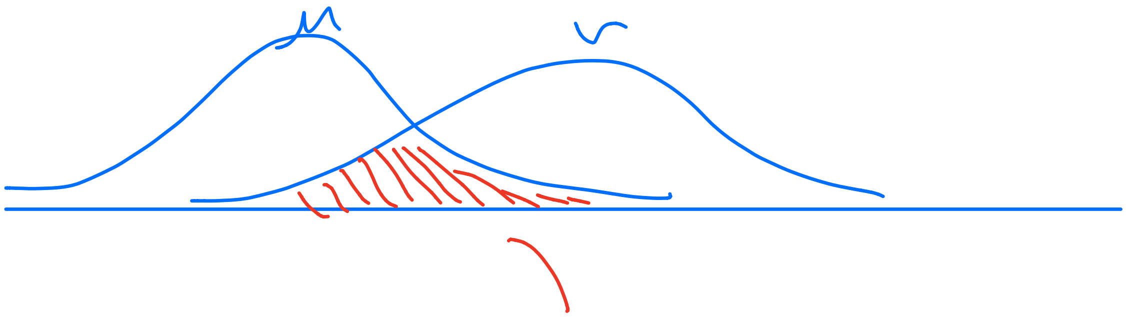
$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(B)|$$

Ex: $\|\mu - \nu\|_{TV} = \frac{1}{2} \int |f_\mu(x) - f_\nu(x)| \, dx$

$$= 1 - \int \min\{f_\mu(x), f_\nu(x)\}\, dx$$

if $\mu(dx) = f_\mu(x)dx$

$\nu(dx) = f_\nu(x)dx$



$$\|\mu-\nu\|_{TV} = 1 - \text{Area}$$

Given $f \in L^2(\pi)$

$$\mathcal{E}(f,f) \overset{def}{=} \frac{1}{2} \int\int (f(y)-f(x))^2 \pi(dx) P(x,dy)$$

$$\text{SpecGap}(P) = \inf\left\{ \frac{\mathcal{E}(f,f)}{\text{Var}_\pi(f)} ; f \in L^2(\pi), \text{Var}_\pi(f) > 0 \right\}$$

$$\mathrm{Var}_\pi(f) = \langle f,f\rangle - \pi(f)^2$$

$$= \langle f,f\rangle - \left(\int f(x)\,\pi(dx)\right)^2$$

We say P satisfies Poincare inequalitie with constant c_P

if $\mathcal{E}(f,f) \geq c_P \mathrm{Var}_\pi(f)$

$\Rightarrow$ $\mathrm{SpecGap}(P) \geq c_P$.

Thm: Suppose that $X_0 \sim \pi_0$ s.t.

$\pi_0(dx) = f_0(x)\,\pi(dx)$, $f_0 \in L^2(\pi)$

then for all $n \geq 1$

$$\| \pi_0 P^n - \pi \|_{TV}^2 \leq \mathrm{Var}_\pi(f_0)\left(1 - \mathrm{SpecGap}(P)\right)^n$$

Ex. Independent MH

$\mathcal{X} \in \mathbb{R}^p \quad \pi(dx) = \pi(x)dx.$

Choose Q, a density on $\mathbb{R}^p$,

Algo: Given $X_n = x$

· Draw $Y \sim Q$

$$X_{n+1} = \begin{cases} Y & \alpha(x,Y) \\ X_n & 1-\alpha(x,y) \end{cases}$$

$$\alpha = \min\left(1, \frac{\pi(y)\, Q(x)}{\pi(x)\, Q(y)}\right)$$

The kernel is then

$$P(x,A) = \mathbb{P}(X_{n+1} \in A \mid X_n = x)$$

$$= \int_A \alpha(x,y)\, Q(y)\, dy + (1-a(x))\, \mathbb{1}_A(x).$$

Bounding the eigengap

$$2\mathcal{E}(f,f) = \iint (f(y)-f(x))^2 \pi(x) P(x,dy)\,dx$$

$$\geq \iint (f(y)-f(x))^2 \pi(x) \alpha(x,y) \pi(x) Q(y)\,dy\,dx$$

Now notice

$$\alpha(x,y) = \min\left(\frac{Q(x)}{\pi(y)}, \frac{Q(y)}{\pi(x)}\right) \frac{\pi(y)}{Q(y)}$$

So

$$= \iint (f(y)-f(x))^2 \min\left(\frac{Q(x)}{\pi(x)}, \frac{Q(y)}{\pi(y)}\right) \pi(x)\pi(y)\,dx\,dy$$

Let $m = \sup_x \frac{\pi(x)}{Q(x)} \Rightarrow \frac{Q(x)}{\pi(x)} \geq \frac{1}{m} \; \forall x$

So

$$2\mathcal{E}(f,f) \geq \frac{1}{m} \underbrace{\iint (f(x)-f(y))^2 \pi(x)\pi(y)\,dx\,dy}_{2\,\mathrm{Var}_\pi(f)}$$

$$\Rightarrow \mathrm{SpecGap}(P) \geq \frac{1}{m}$$

and a bound could be given by

$$\|\pi_c P^n - \pi\|_{TV}^2 \leq \mathrm{Var}_\pi(f)\left(1-\frac{1}{m}\right)^{1/n}$$