Overview

- Take Home final
- Two HWs
- Will need to scribe one

Topics

- Spectral Graph Theory I
  - LA Review
  - Graphs & Matrices
- Spectral Graph Theory II
  - Random Walks & Alg. App
  - Graph Partioning
  - Spetral Clustering
- Iterative Algorithms Laplacian It. Systems
  - $Lx=b$ where L is

a Laplacian

- Graph Sparsification
  - Is there a graph $G' \subseteq G$ such that

    G $\binom{n}{2}$ edges

    G' $O(n)$ edges

Lecture:

Def: Undirected Graphs

$$G = (V, E)$$

unordered pairs of vertices $E \subseteq \binom{V}{2}$

Def: Weighted Graph has an assoc. weight vector $w \in \mathbb{R}^E$

Goal: - Study graphs as $G \mapsto M_G$

The action of $M_G : \mathbb{R}^n \mapsto \mathbb{R}^n$

$f(x) := M_G x$ is connected to random walks & averaging

- Study the quadratic forms

$$\left.\begin{array}{l} g(x) = x^T M_G x \\ g : \mathbb{R}^n \mapsto \mathbb{R} \end{array}\right\} \begin{array}{l} \text{Convergence of R.W.} \\ \text{Graph Clustering} \end{array}$$

<u>Key Insight</u>: Move graphs from combinitorial $\longrightarrow$ algebraic structure.

— Good math / good code.

<u>Graph Matrices</u>

$\boxed{\text{Adjacency matrix}}$

$$(A_G)_{ij} = \begin{cases} w_{ij} & \{i,j\} \in E \end{cases}$$

$\quad \{ 0 \quad o.w.$

- zeros on the diagonal
- Symmetric
- Fact: $[A_G^k]_{ij} = \sum_{\substack{k\text{-len. walks} \\ \text{from } i \to j}}$ product of weights of each path
- Rmk: If $w_{ij} = 1 \; \forall \, ij$ then this quantity is just the number of k walks from $i \mapsto j$

$\boxed{\text{Degree Matrix}}$

$$d_i = \sum_{j \sim i} w_{ij}$$

$$D_G = \text{diag}(d_1, \dots, d_n)$$

Consider: $W = A_G D_G^{-1}$ and

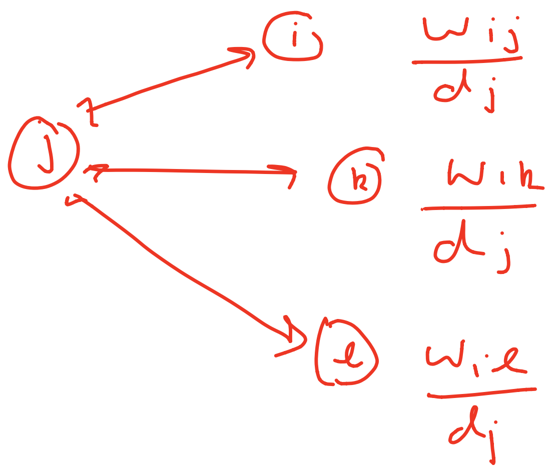
Let $\vec{p} \in \mathbb{R}^n$ be a pmf and

Let $\vec{p} \in \mathbb{R}^n$ be a pdf and

Consider $\boxed{\text{Random Walk Matrix}}$

$$(W\vec{p})_i = \sum_{i \sim j} \frac{w_{ij}}{d_j} p_j$$

new prob distribution



Defines a new pdf in terms of a Markov Chain

Fact: So W describes the transition probability matrix of a random walk on the graph

$\boxed{\text{Laplacian Matrix}}$

Def: $L = D - A$

Def: $L_G := D_G - A_G$

- symmetric
- eigenvalues are positive
- D_G in some sense serves as an identity

Ex: Laplacian with one edge

$$\underset{i}{\circ} \overset{w_{ij}}{\text{———}} \underset{j}{\circ} \qquad L = w \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= w \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

We call these x_{ij}

Thm $\operatorname{rank}(L) \leq \#\text{edges}$.

Prop: $L_G, L_H \quad V(G) = V(H)$

then $L_{G \oplus H} = L_G + L_H$

$$V(G \oplus H) = V(H) \cup V(G)$$

$$E(G \oplus H) = E(H) \cup E(G)$$

$$\vec{w}_{G \oplus H} = \vec{w}_G + \vec{w}_H$$

A different way to think of the laplacian

$$L_G = \sum_{e \in E} w_e L_e$$

Rank 1 matrices

Then the quadratic form also looks like

$$\boxed{x^T L_G x = \sum_{e \in E} w_e x^T L_e x = \sum_{e \in E} w_e (x_i - x_j)^2}$$

<u>Rmk:</u> $w_e \geq 0 \implies x^T L_G x \geq 0$

<u>Spectral Theorem</u>

[illegible] $M \in \mathbb{R}^{n \times n}$ $M^T = M$ [illegible]

Thm. If $M \in \mathbb{R}^{n \times n}$ $M^T = M$ then it has an orthonormal basis of eigenvectors

$$M = V \Lambda V^T$$