1. Suppose we have a sequence of independent random variables $\{F_n, n \geq 1\}$ that converges in the Kolmogorov metric to a random variable F. That is

$$\sup_{h\in H_{Kol}} |\mathbb{E}[h(F_n)] - \mathbb{E}[h(F)]| \to 0$$

This then implies

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(F_n \leq z) - \mathbb{P}(F \leq z)| \to 0$$

Now, let $C(F)$ be the set of points where F is continuous. As $C(F) \subseteq \mathbb{R}$ we can then write

$$\sup_{z\in C(F)} |\mathbb{P}(F_n \leq z) - \mathbb{P}(F \leq z)| \leq \sup_{z\in\mathbb{R}} |\mathbb{P}(F_n \leq z) - \mathbb{P}(F \leq z)| \to 0$$

But notice that the left most quantity is just the definition of convergence in distribution. Having shown this quantity converges to zero, we conclude $F_N \xrightarrow{D} F$ as desired.

To see why Kolmogorov convergence is strictly stronger than convergence in distribution, consider the constant sequence $F_n = \frac{1}{n}$ and $F = 0$. Clearly, $F_n \xrightarrow{D} F$. That is, for $\epsilon, \delta > 0$ there exists an N such that $\mathbb{P}(|n^{-1} - 0| > \delta) < \epsilon$ for all $n \geq N$. Hence $F_n \xrightarrow{P} F$ and $F_n \xrightarrow{D} F$. Now, consider the following.

$$\begin{aligned} d_{Kol}(F_n, F) &= \sup_{z\in R} |\mathbb{P}(F_n \leq z) - \mathbb{P}(F \leq z)| \\ &= \sup_{z\in R} |\mathbb{P}(n^{-1} \leq z) - \mathbb{P}(0 \leq z)| \\ &= 1 \end{aligned}$$

That is the Kolmorogov distance between these two distributions is 1 for all n. Taking $z = \frac{1}{n+1}$, say, we see that $\mathbb{P}(F_n < z) = 0$ and $\mathbb{P}(F < z) = 1$. Hence this sequence converges in distribution by not in the Kolmorogov metric.

2. Suppose that N is a random variable and let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $\mathbb{E}[h'(N)] < \infty$ and $\mathbb{E}[Nh(N)] < \infty$. Then we look to show that $N \sim N(0, 1)$ iff $\mathbb{E}[h'(N)] - \mathbb{E}[Nh(N)] = 0$.

($\Longrightarrow$) Suppose that $N \sim N(0, 1)$ and let $\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/x}$ be the standard normal density. Then by using the Gaussian integration by parts formula gives

$$\mathbb{E}[h'(N)] = \int_{-\infty}^{\infty} h'(N)\gamma(dN) \overset{GIP}{=} \int_{-\infty}^{\infty} nh(n)\gamma(dn) = \mathbb{E}[Nh(N)]$$

From here it follows that

$$\mathbb{E}[h'(N)] - \mathbb{E}[Nh(N)] = 0$$

($\Longleftarrow$) Suppose that $\mathbb{E}[h'(N)] - \mathbb{E}[Nh(N)] = 0$ and take $h(x) = \frac{1}{t} e^{tx}$ (we can do this as $h(x)$ is differentiable for all $x \in \mathbb{R}$). Then the above gives $\mathbb{E}[e^{tN}] = \frac{1}{t}\mathbb{E}[Ne^{tN}]$ we can rewrite as

$$t\mathbb{E}[e^{tN}] = \mathbb{E}[\frac{d}{dt} e^{tN}] = \frac{d}{dt}\mathbb{E}[e^{tN}]$$

Notice that we can interchange the differentiation with the expectation due to the assumption that $\mathbb{E}[Nh(N)] < \infty$ and use of the dominated convergence theorem. Let $M_N(t) = \mathbb{E}[e^{tN}]$ be the moment generating function of N. Then our form reduces to the first order differential equation

$$0 = M_N'(t) - tM_N(t)$$

This general separable ODE has a solution of the form $M_N(t) = e^{x^2/2+c}$. But recall that we also have an initial condition of $M_N(0) = \mathbb{E}[e^0] = 1$. So $M_n(0) = 1 = e^c$ which corresponds to $c = 0$. Hence $M_N(t) = e^{t^2/2}$ which is just the moment generating function of the standard normal. As the normal is characterized by its moments, we have shown that $N \sim N(0, 1)$.

3. (a) Let $N \sim N(0, 1)$ and let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function with $h(N) \in L^1(\Omega)$. We look to solve the first order ordinary differential equation given by

$$f'(x) - xf(x) = h(x) - \mathbb{E}(h(N))$$

First, we define the integrating factor $u(x) = \exp\left\{\int -xdx\right\} = e^{-x^2/2}$. By defining this quantity in this way, we have that

$$\frac{d}{dx}u(x)f(x) = \frac{d}{dx}e^{-x^2/2}f(x) = e^{-x^2/2}[f'(x) - xf(x)]$$

Applying this in our situation

$$\begin{aligned}
f'(x) - xf(x) &= h(x) - \mathbb{E}(h(N)) \\
e^{-x^2/2}[f'(x) - xf(x)] &= e^{-x^2/2}[h(x) - \mathbb{E}(h(N))] \\
\frac{d}{dx}e^{-x^2/2}f(x) &= e^{-x^2/2}[h(x) - \mathbb{E}(h(N))] \\
\int_{\infty}^{x}\frac{d}{dx}e^{-y^2/2}f(y)dy &= \int_{-\infty}^{x} e^{-y^2/2}[h(y) - \mathbb{E}(h(N))]dy + c
\end{aligned}$$

Now, by the fundamental theorem of calculus we have

$$\begin{aligned}
\int_{\infty}^{x}\frac{d}{dx}e^{-y^2/2}f(y)dy &= \int_{-\infty}^{x} e^{-y^2/2}[h(y) - \mathbb{E}(h(N))]dy + c \\
e^{-x^2/2}f(x) &= c + \int_{-\infty}^{x}[h(y) - \mathbb{E}(h(N))]e^{-y^2/2}dy \\
f(x) &= ce^{x^2/2} + e^{x^2/2}\int_{-\infty}^{x}[h(y) - \mathbb{E}(h(N))]e^{-y^2/2}dy
\end{aligned}$$

(b) Define the following solution corresponding to $c = 0$

$$f_h(x) = e^{x^2/2}\int_{-\infty}^{x}[h(y) - \mathbb{E}(h(N))]e^{-y^2/2}dy$$

We first show it satisfies the statement. We start with the case for $x \to -\infty$

$$\lim_{x\to-\infty} e^{-x^2/2} f_h(x) = \lim_{x\to-\infty} \int_{-\infty}^{x} [h(y) - \mathbb{E}(h(N))] e^{-y^2/2} dy = 0$$

Now we consider the case for $x \to \infty$.

$$\begin{aligned}
\lim_{x\to\infty} e^{-x^2/2} f_h(x) &= \lim_{x\to\infty} \int_{-\infty}^{x} [h(y) - \mathbb{E}(h(N))] e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} [h(y) - \mathbb{E}(h(N))] e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy - \mathbb{E}(h(N)) \int_{-\infty}^{\infty} e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy\right) \left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) \\
&= \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy\right) \\
&= \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy \,(1 - 1) \\
&= 0
\end{aligned}$$

To see why that this is the unique solution with this property, notice that for a general solution to this equation that

$$e^{-x^2/2} f(x) = c + \int_{-\infty}^{x} [h(y) - \mathbb{E}(h(N))] e^{-y^2/2} dy$$

Using the result we just proved above

$$\lim_{x\to\pm\infty} c + \int_{-\infty}^{x} [h(y) - \mathbb{E}(h(N))] e^{-y^2/2} dy = c$$

Therefore, for this limit to be 0 corresponds to the solution with $c = 0$. That is $f_h(x)$ is the unique solution that has this property.