Graphical Models

Up to this point we've only analyzed

$X_1, \dots, X_n \overset{iid}{\sim} F_\theta$

In general we can have

$\mathbb{P}(X_1, \dots, X_n)$ with some conditional independence. One way to visualize this is graphically.

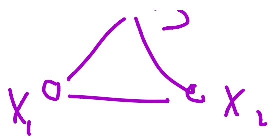
$$X_i \perp\!\!\!\perp X_j \mid X_k , \quad k \neq i,j$$

$$\Longleftrightarrow$$

$$A_{ij} = 0$$

Ex:

$(X_1, X_4) \in G$



$X_i \perp\!\!\!\perp X_j \mid X_2, X_3$

Def: A <u>clique</u> is a maximal complete subgraph of G that is

$C = (S, F) = G[S]$, C is complete

and $G[S + v]$ is not complete for $v \in V \setminus S$.

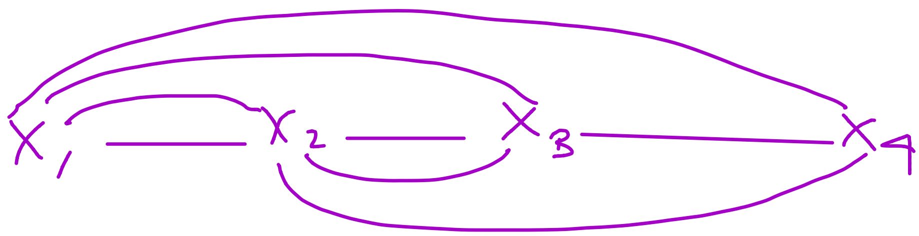
Ex: We can always write

$$\mathbb{P}(X_1, \dots, X_n) = \mathbb{P}(X_1)\,\mathbb{P}(X_2 \mid X_1) \cdots \mathbb{P}(X_n \mid X_1, \dots, X_{n-1})$$

If $\mathbb{P}$ is such that

$$= \mathbb{P}(X_1)\,\mathbb{P}(X_2 \mid X_1)\,\mathbb{P}(X_3 \mid X_2) \cdots \mathbb{P}(X_n \mid X_{n-1})$$

then we have markov chain.



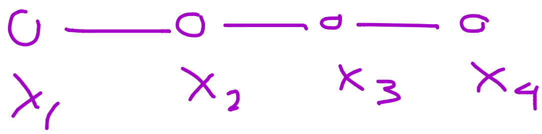
$$\mathbb{P}(X_2, X_4 \mid X_1, X_3) = \frac{\cancel{\mathbb{P}(x_1)}\,\mathbb{P}(x_2 \mid x_1)\,\mathbb{P}(x_3 \mid x_2)\,\mathbb{P}(x_4 \mid x_3)}{\sum_{\tilde{x}_2 \tilde{x}_4} \cancel{\mathbb{P}(x_1)}\,\mathbb{P}(\tilde{x}_2 \mid x_1)\,\mathbb{P}(x_3 \mid \tilde{x}_2)\,\mathbb{P}(\tilde{x}_4 \mid x_3)}$$

$$= \frac{\mathbb{P}(x_2 \mid x_1)\,\mathbb{P}(x_3 \mid x_2)}{\sum_{\tilde{x}_2} \mathbb{P}(\tilde{x}_2 \mid x_1)\,\mathbb{P}(x_3 \mid \tilde{x}_2)} \times \frac{\mathbb{P}(x_4 \mid x_3)}{\sum_{\tilde{x}_4} \mathbb{P}(\tilde{x}_4 \mid x_3)}$$

$$= \mathbb{P}(X_2 \mid x_1, x_3)\,\mathbb{P}(x_4 \mid x_3, x_1)$$

Hence $X_2 \perp\!\!\!\perp X_4 \mid X_1, X_3$

By a similar argument we just get



Ex: $\vec{X} \sim N(\vec{\mu}, \Sigma)$

$$\mathbb{P}(x) \propto \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} (x_i-\mu_i)^2 \Sigma^{-1}_{ii}\right.$$

$$\left. + \sum_{i \neq j} (x_i - \mu_i)(x_j - \mu_j)\, \Sigma^{-1}_{ij} \right\}$$

* $\Sigma^{-1}_{ij} = 0 \iff$ No cross terms

So conditionally independent.

$$\mathrm{Var}\left[(x_i, x_j) \mid x_{[\cdot i, \cdot j]}\right]$$

$= \left((\Sigma^{-1})_{(i,j)(i,j)} \right)$ from Schur complement.

"Gaussian Graphical Models"

Identify $X = (X_1, \ldots, X_n)$ with a vertex in a graph G

$$X_A = \{ X_v : v \in A \subseteq V \}$$

<u>Def:</u> We say that $\mathbb{P}$ is a <u>markov random field</u> (MRF) wrt G iff

$$\mathbb{P}(X_v \mid X_{[-v]}) = \mathbb{P}(X_v \mid X_{N_v})$$

<u>Ex:</u> Suppose 

$$\mathbb{P}(x_i / x_{[-i]}) = \underline{\mathbb{P}(x_i)\,\mathbb{P}(x_j \mid x_{i-1}) \cdots \mathbb{P}(x_n \cancel{\mid x_{n-1}})}$$

$$\sum_{\tilde{x}_i} \mathbb{P}(x_i) \cdots \mathbb{P}(x_i | x_{i-1}) \mathbb{P}(x_{i+1} | \tilde{x}_i)$$

$$\cancel{\cdots} \; \mathbb{P}(\cancel{x_n | x_{n-1}})$$

$$= \frac{\mathbb{P}(x_i)\,\mathbb{P}(x_i | x_{i-1})\,\mathbb{P}(x_{i+1} | x_i)}{\sum_{\tilde{x}_i} \mathbb{P}(\tilde{x}_i | x_{i-1})\,\mathbb{P}(x_{i+1} | \tilde{x}_i)\,\mathbb{P}(\tilde{x}_i)}$$

$$= \frac{\mathbb{P}(x_i | x_{i-1}, x_{i+1})}{\mathbb{P}(x_i | x_{N_i})}$$

So MC is a MRF.