Midterm Review

- Chp 1 – Chp. 5 (Hierarchical Models)
- Open Note

Q 1:

$\mathbb{P}(Y|\beta,\phi) \propto \phi^{-n/2} \exp\left\{-\frac{D(y,\beta)}{2\phi}\right\}$

(a) Conjugate prior for ϕ given β?

$\boxed{\phi^{-n/2}} \exp\left\{-\boxed{\frac{D(y,\beta)}{2\phi}}\right\}$

depends on data

$\mathbb{P}(\phi|\beta) \propto \phi^{-\nu/2-1} \exp\left\{\frac{-\nu\tau^2}{2\phi}\right\}$

$\sim \text{Inv-}\chi^2(\nu,\tau^2)$

(b) Posterior

$$\mathbb{P}(\phi|Y,\beta) \propto \phi^{-\frac{n+\nu}{2}-1} \exp\left\{-\frac{1}{2\phi}\left[D(y,\beta)+\nu\tau^2\right]\right\}$$

$$\sim \text{Inv-}\chi^2\left(\nu_n := n+\nu,\ \tau_n^2 := \frac{D(y,\beta)+\nu\tau^2}{n+\nu}\right)$$

(c) Point estimate of $\phi \mid \beta = \hat{\beta}_{MLE}$?

$$\ell(\phi) = -\frac{n}{2}\log\phi - \frac{D(y,\hat{\beta})}{2\phi}$$

$$\frac{d\ell(\phi)}{d\phi} = -\frac{n}{2\phi} + \frac{D(y,\hat{\beta})}{2\phi^2} \overset{set}{=} 0 \implies \boxed{\hat{\phi}_{MLE} = \frac{D(y,\hat{\beta})}{n}}$$

Well the posterior mode is given by

$$\phi^* = \frac{v_n \tau_n^2}{v_n + 2} \qquad \mathbb{E}[\phi \mid y, \beta] = \frac{v_n \tau_n^2}{v_n - 2}$$

$$\phi^* = \frac{v_n \tau_n^2}{v_n + 2} = \frac{\overbrace{\boxed{\frac{D(y,\hat{\beta})}{n}}}^{\hat{\phi}_{MLE}} + \frac{r\tau^2}{n}}{1 + \frac{r+2}{n}} \xrightarrow{n \nearrow \infty} \hat{\phi}_{MLE}$$

$\boxed{2}$ (a)

$Y_i \mid \theta \sim Po(\theta)$ — # trains/hr.

$n = 5$

$\sum y_i = 8$

$\mathbb{E}[\theta] = 1/\beta = 0.5 \Rightarrow \beta = 2.$

(b) Posterior?

$$\mathbb{P}(\theta \mid y) \propto \mathbb{P}(Y \mid \theta)\mathbb{P}(\theta) \propto \prod_{i=1}^{n} \frac{\theta^{y_i} e^{-\theta}}{y_i!} \cdot \beta e^{-\beta\theta}$$

$$= \theta^{\sum y_i} e^{-(n+\beta)\theta} \Rightarrow \theta \mid Y \sim \text{Gamma}(n\bar{y}+1, n+\beta)$$

(c) $\sum y_i, n$ are sufficient statistics

(d) Give an asymptotic 95% CI.

$$\theta | y \approx N(\theta^*, I^{-1}(\theta^*)) \qquad \theta^* = \text{post. mode}$$

$$\ell(\theta) = \log \mathbb{P}(\theta|y) = (\Sigma y_i) \log \theta - (n+\beta)\theta$$

$$\frac{d\ell(\theta)}{d\theta} = \frac{\Sigma y_i}{\theta} - (n+\beta) \implies \theta^* = \frac{\Sigma y_i}{n+\beta}$$

$$\frac{d^2\ell(\theta)}{d\theta^2} = -\frac{n\bar{y}}{\theta^2}$$

So $I^{-1}(\theta^*) = \dfrac{(\theta^*)^2}{n\bar{y}}$

So $\theta | y \approx N\left(\dfrac{n\bar{y}}{n+\beta}, \dfrac{(\theta^*)^2}{n\bar{y}}\right)$

and an approximate int. is $\theta^* \pm 2\dfrac{\theta^*}{\sqrt{n\bar{y}}}$

(e) $\tilde{y} | \theta \sim \text{Pois}(\tilde{x}\theta)$

$$\tilde{y} | y \sim NB\left(1+n\bar{y}, \frac{\tilde{x}}{\tilde{x}+\beta+n}\right)$$

$$\mathbb{P}(\tilde{y}|y) = \frac{\Gamma(1+n\bar{y}+\tilde{y})}{\tilde{y}!\,\Gamma(1+n\bar{y})} \left(\frac{\tilde{x}}{\tilde{x}+\beta+n}\right)^{\tilde{y}} \left(\frac{\beta+n}{\tilde{x}+\beta+n}\right)^{1+n\bar{y}}$$

$$\mathbb{P}(\tilde{y}=0 \mid y) = \left(\frac{\beta+n}{\bar{x}+\beta+n}\right)^{1+n\bar{y}} = \left(\frac{5+2}{3+5+2}\right)^{1+8} = 0.04$$

<u>Exercise</u>

Observe $y_i \mid \theta \overset{iid}{\sim} N(\theta, \theta)$, $\theta > 0$

Conjugate prior + post?

$$\mathbb{P}(y \mid \theta) = \prod_{i=1}^{n} (\theta)^{-1/2} \exp\left\{\frac{-1}{2\theta}(y_i - \theta)^2\right\}$$

$$= (\theta)^{\boxed{-n/2}} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n}(y_i - \theta)^2\right\}$$

$$= (\theta)^{-n/2} \exp\left\{-\frac{1}{2\theta}\boxed{\sum y_i^2} + \frac{2\theta\left(\sum_{i=1}^{n} y_i\right)}{2\theta} - \frac{n\theta^2}{2\theta}\right\}$$

$$= (\theta)^{-n/2} \exp\left\{-\frac{1}{2\theta}\sum y_i^2 + \sum y_i - \frac{n}{2}\theta\right\}$$

$$\propto (\theta)^{\boxed{-n/2}} \exp\left\{-\frac{1}{2\theta}\boxed{\sum y_i^2} - \boxed{\frac{n}{2}}\theta\right\}$$

$$\boxed{\mathbb{P}(\theta) \propto \theta^{k} \exp\left\{-\frac{1}{2\theta}s^2 - k\theta\right\}}$$ prior $f(k, s^2)$

$$\mathbb{P}(\theta \mid y) \propto \mathbb{P}(y \mid \theta)\,\mathbb{P}(\theta)$$

$$\boxed{\propto \theta^{-n/2+k} \exp\left\{-\frac{1}{2\theta}\left(\Sigma y_i^2 + s^2\right) - \left(-\frac{n}{2}+k\right)\theta\right\}}$$

$$f\left(-\frac{n}{2}+k,\ \Sigma y_i^2 + s^2\right)$$

Sol: $\mathbb{P}(Y|\theta) \propto \theta^{-n/2} \exp\left\{\frac{-1}{2\theta}\left[\Sigma(y_i - \bar{y})^2 + n(\bar{y}-\theta)^2\right]\right\}$

$$\propto \theta^{\boxed{-n}/2} \exp\left\{\frac{-1}{2}\left[\frac{\overbrace{\boxed{\Sigma(y_i-\bar{y})^2 + n\bar{y}^2}}^{S_y}}{\theta} - 2n\bar{y} + n\theta\right]\right\}$$

$$\mathbb{P}(\theta) \propto \theta^{-r/n} \exp\left\{\frac{-1}{2}\left[\frac{\tau^2}{\theta} + r\theta\right]\right\} \sim GIG\left(r, \tau^2, 1-\frac{r}{2}\right)$$

$$\mathbb{P}(\theta|y) \propto \mathbb{P}(\theta)\,\mathbb{P}(Y|\theta)$$

$$\propto \theta^{\left(-\frac{(n+r)}{2}\right)} \exp\left\{-\frac{1}{2}\left[\frac{\Sigma(y_i-\bar{y})^2 + n\bar{y}^2 + \tau^2}{\theta} + (n+r)\theta\right]\right\}$$

$$\sim GIG\left(n+r,\ \Sigma(y_i-\bar{y})^2 + n\bar{y}^2 + \tau^2,\ 1-\frac{n+r}{2}\right)$$

Exercise

	died	Total
control	40	500
treatment	25	500

θ_c = prob of dying in control

θ_T = " ________ " treatment.

$Y_c | \theta_c \sim \text{Binom}(\theta_c, n)$

$Y_t | \theta_t \sim \text{Binom}(\theta_T, n)$

Using Jeffreys prior, calc. an approx. 95% post. for $\theta_T - \theta_c$

$\theta_T, \theta_c \sim Beta(1/2, 1/2)$

$\theta_c | y_c \sim Beta(1/2 + y_c, 1/2 + n_c - y_c)$

$\theta_T | y_T \sim Beta(1/2 + y_T, 1/2 + n - y_T)$

Use a Laplace $\theta_c | y_c \simeq N(\theta_c^*, I(\theta_c^*)^{-1})$

mimic + covariance $\Rightarrow$ rough interval

$$\theta_T - \theta_c | y_c, y_T \simeq N\left(\theta_T^* - \theta_c^*, I(\theta_c^*)^{-1} + I(\theta_T^*)^{-1}\right)$$

$$I(\theta_c^*)^{-1} = \frac{\theta_c^*(1 - \theta_c^*)}{1/2 + 1/2 + n - 2}$$

$$\theta_c^* = \frac{1/2 + y_c - 1}{1/2 + 1/2 + n_c - 2}$$

$$\theta_T^* - \theta_c^* \pm 2\sqrt{I^{-1}(\theta_c^*) + I(\theta_T^*)^{-1}}$$

(c) $\dfrac{P(\tilde{y} = 0 | y)}{P(\tilde{y} = 1 | y)} \sim BetaBin(\cancel{\ })$

$$P(\tilde{y}|y) \propto \int P(\tilde{y}|\theta)\, P(\theta|y)\, d\theta = \int \binom{n}{\tilde{y}} \theta^{\tilde{y}} (1-\theta)^{n-\tilde{y}} \frac{\theta^{1/2+y-1}\, \theta^{1/2+n-y-1}}{B(1/2+y,\ 1/2+n-y)}\, d\theta$$

$$= \binom{n}{\tilde{y}} \frac{B(1/2+\tilde{y}+y,\ 1/2+n-y+n-\tilde{y})}{B(1/2+y,\ 1/2+n-y)}$$

$$\frac{P(\tilde{y}=0|y)}{P(\tilde{y}=1|y)} = \frac{B(1/2+y,\ 1/2+2-y)}{B(1/2+y\cancel{,\ 1/2}+n-y)} \Bigg/ \binom{n}{1} \frac{B(3/2+y,\ 2n-1/2+y)}{B(1/2+\cancel{y,\ 1/2}+n-y)}$$

$$= \frac{B(1+y,\ 1/2+2n-y)}{n\, B(3/2+y,\ 2n-1/2+y)}$$

- Choose y from 40 for control and 25 from treatment.