Suppose that we have $X_1, X_2, \ldots, X_n \mid \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$

Goal: deduce a conjugate prior

$$\mathbb{P}(X \mid \mu, \sigma^2) \propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}$$

$$\propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \Big[\underbrace{\sum_i (x_i - \bar{x})^2}_{S_x^2} + n(\bar{x} - \mu)^2 \Big] \right\}$$

$$\propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{S_x^2}{\sigma^2} - \frac{(\bar{x} - \mu)^2}{2\sigma^2/n} \right\}$$

Idea: define $\mathbb{P}(\mu, \sigma^2) = \mathbb{P}(\mu \mid \sigma^2)\,\mathbb{P}(\sigma^2)$ prior as a hierarchy

$$\mathbb{P}(\mu, \sigma^2) = \mathbb{P}(\sigma^2 \mid \mu)\,\mathbb{P}(\mu)$$

Conditional on σ^2: $\mathbb{P}(\mu \mid \sigma^2)$ and we know $\bar{x} \mid \mu, \sigma^2 \sim N(\mu, \sigma^2/n)$

which suggests $\mathbb{P}(\mu \mid \sigma^2) \propto \exp\left\{ \frac{-(\mu - \mu_0)^2}{2\sigma^2/k_0} \right\}$

and $\mu \mid \sigma^2 \sim N(\mu_0, \sigma^2/k_0)$

Thus the conditional likelihood is

$$\mathbb{P}(X, \mu \mid \sigma^2) \propto (\sigma^2)^{-n/2} \exp\left\{ \frac{-S_x^2}{2\sigma^2} - \frac{(\bar{x} - \mu)^2}{2\sigma^2/n} \right\} (\sigma^2)^{-1/2} \exp\left\{ -\frac{(\mu - \mu_0)^2}{2\sigma^2/k_0} \right\}$$

$$= (\sigma^2)^{-n/2} (\sigma^2)^{-1/2} \exp\left\{ -\frac{1}{2\sigma^2} \left(S_x^2 + (\bar{x} - \mu)^2 + k_0 (\mu - \mu_0)^2 \right) \right\}$$

With $n(\bar{x}-\mu)^2 + k_0(\mu-\mu_0)^2 = (k_0+n)(\mu-\mu_n)^2 + \frac{k_0 n}{k_0+n}(\bar{x}-\mu_0)^2$

where $\mu_n = \frac{n\bar{x}+k_0\mu_0}{n+k_0} = \frac{n}{n+k_0}\bar{x} + \frac{k_0}{n+k_0}\mu_0$ Acts as "prior sample size"

$$P(X,\mu|\sigma^2) \propto (\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\left[S_x^2 + \frac{k_0 n}{n+k_0}(\bar{x}-\mu_0)^2\right]\right\}(\sigma^2)^{-1/2}\exp\left\{-\frac{(\mu-\mu_n)^2}{2\sigma^2/(n+k_0)}\right\}$$

Thus $\mu|X,\sigma^2 \sim N(\mu_n, \sigma^2/(n+k_0))$.

But we want

$$P(X|\sigma^2) = \int P(X,\mu|\sigma^2)\,d\mu = (\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\left[\underline{S_x^2+\frac{k_0 n}{k_0+n}(\bar{x}-\mu_0)^2}\right]\right\}$$

Thus for conjugacy

$$P(\sigma^2)\propto(\sigma^2)^{-\nu_0/2+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \sim \text{Inv.}\chi^2(\nu_0,\sigma_0^2)$$

$$P(X,\sigma^2)\propto(\sigma^2)^{-\left(\frac{n+\nu_0}{2}+1\right)}\exp\left\{-\frac{1}{2\sigma^2}\left(S_x^2+\frac{k_0 n}{n+k_0}(\bar{x}-\mu_0)^2+\nu_0\sigma_0^2\right]\cdot\frac{n+\nu_0}{n+\nu_0}\right\}$$

$$\sigma^2|X \sim \text{Inv}\chi^2\left(n+\nu_0, \frac{1}{n+\nu_0}\left(S_x^2+\frac{k_0 n}{n+k_0}(\bar{x}-\mu_0)^2+\nu_0\sigma_0^2\right)\right)$$

We write $\mu, \sigma^2 \sim N\text{-}\mathrm{Inv}\chi^2(\mu_0, k_0; \nu_0, \sigma_0^2)$

meaning $\mu \mid \sigma^2 \sim N(\mu_0, \sigma^2/k_0)$

$\sigma^2 \sim \mathrm{Inv}\chi^2(\nu_0, \sigma_0^2)$

with posterior

$\mu, \sigma^2 \mid X \sim N\text{-}\mathrm{Inv}\chi^2(\mu_n, k_n; \nu_n, \sigma_n^2)$

It can be shown $\dfrac{\mu - \mu_n}{\sigma_n/\sqrt{k_n}} \sim t_{\nu_n}$, $\mathbb{P}(\mu \mid X) = \int \mathbb{P}(\mu \mid X, \sigma^2)\, \mathbb{P}(\sigma^2 \mid X)\, d\sigma^2$

Consider the following study:

log(Dose)	# Animals	# deaths
x_1	n_1	y_1
x_2	n_2	y_2
x_3	n_3	y_3
$\vdots$	$\vdots$	$\vdots$
x_m	n_m	y_m

We can assume $y_i | \theta_i \sim \text{Binom}(n_i, \theta_i)$

$$\theta_i = f(x_i)$$

Choose to model - logistic regression

$$\text{logit}(\theta_i) = \log \frac{\theta_i}{1-\theta_i} = \alpha + \beta x_i$$

Under this setup $y_i | \alpha, \beta \overset{ind}{\sim} \text{Binom}(n_i, \text{logit}^{-1}(\alpha + \beta x_i))$

$$\mathbb{P}(y|\alpha,\beta) \propto \prod_i \text{logit}^{-1}(\alpha+\beta x_i)^{y_i} \left[1 - \text{logit}^{-1}(\alpha+\beta x_i)\right]^{n_i - y_i}$$

Conjugacy hard $\Longrightarrow$ numerical methods

Two common choices

(a) $\mathbb{P}(\alpha,\beta) \propto 1$ (b) $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \sim N\left(\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}, \Sigma\right)$

Then, gridding over Θ and numerically calculate

$$\mathbb{P}(\alpha,\beta|y) \approx \frac{\mathbb{P}(y|\alpha,\beta)\mathbb{P}(\alpha,\beta)}{\sum_{\tilde{\alpha},\tilde{\beta}} \mathbb{P}(y|\tilde{\alpha},\tilde{\beta})\mathbb{P}(\tilde{\alpha},\tilde{\beta})}$$