1. Let $X_1, X_2, \dots, X_n$ be a random sample from $X \sim f(x) = e^{-(x-\theta)}I_{(\theta,\infty)}(x)$. Let $Y_n = \min\{X_1, X_2, \dots, X_n\}$. Then Y_n is the first order statistic of the random sample. Recall that the density of Y_n is given by $f_{Y_n}(y) = 1-(1-F_X(y))^n$. First we find $F_x(y)$.

$$F_X(y) = \int_{\infty}^{y} e^{-(x-\theta)}I_{(\theta,\infty)}dx = \int_{\theta}^{y} e^{-(x-\theta)}dx = \frac{1}{\theta}e^{-(x-\theta)}\Big|_{\theta}^{y} = -e^{-(y-\theta)} = 1-e^{-(y-\theta)}$$

Using this we see

$$f_{Y_n}(y) = n(1-1+e^{-(y-\theta)})^{n-1}e^{-(y-\theta)}I_{(\theta,\infty)}(y) = ne^{-n(y-\theta)}I_{(\theta,\infty)}(y)$$

Now, we are ready to show that Y_n is a consistent point estimator of θ.

$$\begin{aligned}
P(|Y_n-\theta|<\epsilon) &= P(\theta-\epsilon \le Y_n \le \theta+\epsilon)\\
&= P(\theta-\epsilon \le Y_n \le \theta) + P(\theta \le Y_n \le \theta+\epsilon)\\
&= 0 + \int_{\theta}^{\theta+\epsilon} f_{Y_n}(y)dy\\
&= \int_{\theta}^{\theta+\epsilon} ne^{-n(y-\theta)}I_{(\theta,\infty)}(y)dy\\
&= n\int_{\theta}^{\theta+\epsilon} e^{-n(y-\theta)}dy\\
&= -e^{-n(y-\theta)}\Big|_{\theta}^{\theta+\epsilon}\\
&= 1-e^{-n\epsilon}
\end{aligned}$$

Now, letting $n\to\infty$ shows that $P(|Y_n-\theta|<\epsilon) \to \lim_{n\to\infty} 1-e^{-n\epsilon} = 1$. Thus our point estimator is consistent.

2. Let $f(x) = \frac{1}{2}(1+\theta x)I_{(-1,1)}(x)$. Then for $X \sim f(x)$ the mean is given by

$$\mathbb{E}(X) = \int_{\mathbb{R}} \frac{1}{2}x(1+\theta x)I_{(-1,1)}(x)dx = \int_{-1}^{1}\frac{1}{2}(x+\theta x^2)dx = x^2+\theta\frac{x^3}{6}\Big|_{-1}^{1} = 1+\frac{\theta}{6}-1+\frac{\theta}{6} = \frac{\theta}{3}$$

Moreover we can find variance by the following

$$\mathbb{E}(X^2) = \int_{\mathbb{R}} \frac{1}{2}x^2(1+\theta x)I_{(-1,1)}(x)dx = \int_{-1}^{1}\frac{1}{2}(x^2+\theta x^3)dx = \frac{x^3}{6}+\theta\frac{x^4}{8}\Big|_{-1}^{1} = \frac{1}{6}+\frac{\theta}{8}+\frac{1}{6}-\frac{\theta}{8} = \frac{1}{3}$$

This implies

$$Var(X) = E(X^2)-E(X)^2 = \frac{1}{3}-\frac{\theta^2}{9} = \frac{3-\theta^2}{9}$$

Therefore, our candidate point estimate is $\widehat{\theta} = 3\overline{X}$. Recall that $\overline{X}$ is unbiased for the mean. Therefore $\mathbb{E}(\widehat{\theta}) = \mathbb{E}(3\overline{X}) = 3\frac{\theta}{3} = \theta$. So our estimate is unbiased. To see why our estimate is consistent in mean squared error, consider the following

$$
\begin{aligned}
E|\widehat{\theta}-\theta|^2 &= \mathbb{E}(\widehat{\theta}^2) - 2\mathbb{E}(\widehat{\theta})\theta + \theta^2\\
&= Var(\widehat{\theta}) + \left[\mathbb{E}(\widehat{\theta})\right]^2 - 2\mathbb{E}(\widehat{\theta})\theta + \theta^2\\
&= Var(\widehat{\theta}) + \left[\mathbb{E}(\widehat{\theta}) - \theta\right]^2\\
&= 9Var(\overline{X})\\
&= \frac{9Var(X)}{n}\\
&= \frac{3-\theta^2}{n}
\end{aligned}
$$

The fourth equality is justified by $\widehat{\theta}$ being unbiased estimate for θ. Letting $n \to \infty$ we see that $\lim_{n\to\infty} E|\widehat{\theta}-\theta|^2 = 0$ and $\widehat{\theta}$ is mean squared - consistent.

3. (a) Recall that by the Central Limit Theorem (CLT)

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \overset{D}{\to} Z \sim (0,1)$$

In our case $\mu = \theta$ and $\sigma^2 = \theta(1-\theta)$. First note that the sequence $\theta(1-\theta) \to \theta(1-\theta)$ in probability (it is a constant sequence). So by Slutsky's Theorem we see

$$\sqrt{n}(\overline{X}_n - \theta) = \sqrt{\theta(1-\theta)}\frac{\sqrt{n}(\overline{X}_n - \theta)}{\sqrt{\theta(1-\theta)}} \overset{D}{\to} \sqrt{\theta(1-\theta)}Z$$

Now notice that the random variable $Y = \sqrt{\theta(1-\theta)}Z$ is normally distributed with

$$\mathbb{E}(Y) = \mathbb{E}(\sqrt{\theta(1-\theta)}Z) = \sqrt{\theta(1-\theta)}\mathbb{E}(Z) = 0$$

Moreover we see

$$Var(Y) = Var(\sqrt{\theta(1-\theta)}Z) = \theta(1-\theta)Var(Z) = \theta(1-\theta)$$

Therefore, we see

$$\sqrt{n}(\overline{X}_n - \theta) \xrightarrow{D} Y \sim N(0, \theta(1-\theta))$$

(b) Let $g(t) = t(1-t)$. Then $g(t)$ is differentiable everywhere with $g^{'}(t) = 1-2t$. Moreover we note that $g'(\theta) \neq 0$ for $\theta \neq 0.5$. Having shown these properties, we can apply the first order Delta method to the result in a. Specifically, for

$$\sqrt{n}(\overline{X}_n - \theta) \xrightarrow{D} Y \sim N(0, \theta(1-\theta))$$

we note that

$$\sqrt{n}(g(\overline{X}_n) - g(\theta)) \xrightarrow{D} V \sim N(0, \theta(1-\theta)g^{'}(\theta)^2)$$

Here we see that $\theta(1-\theta)g^{'}(\theta)^2 = \theta(1-\theta)(1-2\theta)^2$. Simplifying the form above we see

$$\sqrt{n}(\overline{X}_n(1-\overline{X}_n) - \theta(1-\theta)) \xrightarrow{D} Y \sim N(0, \theta(1-\theta)(1-2\theta)^2)$$

(c) To apply the Delta method we required that $\theta \neq 0.5$ so $g'(\theta) \neq 0$. If $\theta = 0.5$ we instead need to look at higher orders of the Delta method. Namely, the second order Delta method. Specifically, note that for $g(t) = t(1-t)$, $g^{''}(t) = -2 \neq 0$ for all values of θ. This along with $g^{'}(0.5) = 0$ allows us to use the second order Delta Method. Specifically, we have

$$\begin{aligned} n\big[g(\overline{X}_n) - g(\theta)\big] &\xrightarrow{D} \frac{\theta(1-\theta)g^{''}(\theta)}{2}Z^2 \\ n\big[\overline{X}_n(1-\overline{X}_n) - 1/2(1-1/2)\big] &\xrightarrow{D} \frac{-2(1/2(1-1/2))}{2}Z^2 \\ n\big[\overline{X}_n(1-\overline{X}_n) - \frac{1}{4}\big] &\xrightarrow{D} -\frac{1}{4}Z^2 \end{aligned}$$

Seeing this, we can apply Slutsky's Theorem to see

$$-4n\big[\overline{X}_n(1-\overline{X}_n) - \frac{1}{4}\big] = 4n\big[\frac{1}{4} - \overline{X}_n(1-\overline{X}_n)\big] \xrightarrow{D} Z^2 = \chi^2(1)$$

4. First consider the following equality.

$$\begin{aligned} \frac{\sqrt{n}(\overline{X}_n - \mu_n)}{\sigma_n} &= \frac{\sqrt{n}(\overline{X}_n - \overline{\mu}_n + \overline{\mu}_n - \mu_n)}{\sigma_n} \\ &= \frac{\sqrt{n}(X_n - \overline{\mu}_n)}{\sigma_n} + \sqrt{n}\frac{(\overline{\mu}_n - \mu_n)}{\sigma_n} \\ &= \frac{\sqrt{n}(X_n - \overline{\mu}_n)}{\overline{\sigma}_n}\frac{\overline{\sigma}_n}{\sigma_n} + \sqrt{n}\frac{(\overline{\mu}_n - \mu_n)}{\sigma_n} \end{aligned}$$

Assume $X \sim AN(\mu_n, \sigma_n^2)$, $\overline{\sigma}_n/\sigma_n \to 1$ and $\frac{\overline{\sigma}_n - \mu_n}{\sigma_n} \to 0$. Then we see

$$\lim_{n\to\infty} \frac{\sqrt{n}(\overline{X}_n - \mu_n)}{\sigma_n} = \lim_{n\to\infty} \frac{\sqrt{n}(\overline{X}_n - \overline{\mu}_n)}{\overline{\sigma}_n} = Z \sim N(0,1)$$

Hence $\overline{X}_n \sim AN(\overline{\mu}_n, \overline{\sigma}_n^2)$. Now, assuming that $\overline{X}_n \sim AN(\overline{\mu}_n, \overline{\sigma}_n^2)$ and $X_n \sim AN(\mu_n, \sigma_n^2)$. Then we see that

$$\lim_{n\to\infty} \Big(\frac{\sqrt{n}(X_n - \overline{\mu}_n)}{\overline{\sigma}_n}\frac{\overline{\sigma}_n}{\sigma_n} + \sqrt{n}\frac{(\overline{\mu}_n - \mu_n)}{\sigma_n}\Big) = \lim_{n\to\infty} \frac{\sqrt{n}(\overline{X}_n - \mu_n)}{\sigma_n} = Z \sim N(0,1)$$

Since $\overline{X}_n \sim AN(\overline{\mu}_n, \sigma_n^2)$ we see that the left hand side limit is given by

$$\lim_{n\to\infty} \Big(\frac{\sqrt{n}(X_n - \overline{\mu}_n)}{\overline{\sigma}_n}\frac{\overline{\sigma}_n}{\sigma_n} + \sqrt{n}\frac{(\overline{\mu}_n - \mu_n)}{\sigma_n}\Big) \xrightarrow{D} aZ + b$$

where $a = \lim_{n\to\infty}\overline{\sigma}_n/\sigma_n$ and $b = \lim_{n\to\infty}\frac{\overline{\mu}_n - \mu_n}{\sigma_n}$ (if they exist). But note that as we see above this limit is equal $Z \sim (0,1)$ and hence $a = 1$ and $b = 0$. Thus

$$\lim_{n\to\infty}\frac{\overline{\sigma}_n}{\sigma_n} = 1 \qquad \lim_{n\to\infty}\frac{\overline{\mu}_n - \mu}{\sigma_n} = 0$$

.

5. (a) First notice that

$$Z_n = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} = \frac{1/n}{1/n}\frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} = \frac{\overline{X}_n - E(\overline{X}_n)}{\sqrt{Var(\overline{X}_n)}} = \frac{X_n - \mu}{\sqrt{\sigma^2/n}} = \frac{\sqrt{n}(X_n - \mu)}{\sigma}$$

We note that this is just a statement of the CLT for $X \sim Pois(\lambda)$. We require that $\mu < \infty$ and $0 < \sigma^2 < \infty$. Here, $\mu = \sigma^2 = \lambda$ and by assumption $0 < \lambda < \infty$. Thus we apply the CLT and see that

$$Z_n = \frac{S_n - E(S_n)}{\sqrt{Var(S_n}} \xrightarrow{D} Z \sim N(0,1)$$

(b) Using the above fact, and the fact that $\cos(\cdot)$ is infinity differentiable, we can use the Delta method to find the limiting distribution of $\cos(\overline{X}_n)$.
Let $g(t) = \cos(t)$. Then assuming that $g'(\lambda) = -\sin(\lambda) \neq 0$ we have

$$\sqrt{n}(\cos(\overline{X}_n) - \cos(\lambda)) \xrightarrow{D} N(0, -\lambda\sin^2(\lambda))$$

If $-\sin(\lambda) = 0$ then we use the second order delta method (this is sufficient because $-\sin(\lambda)$ and $-\cos(\lambda)$ cannot both be zero simultaneously). Specifically, we see

$$n(\cos(\overline{X}_n) - \cos(\lambda)) \xrightarrow{D} \frac{-\lambda\cos(\lambda)}{2}\chi_1^2$$

(c) By the Delta Method of the first order

$$\sqrt{n}(g(\overline{X}_n) - g(\lambda)) \xrightarrow{D} N(0, \sigma^2 g'(\lambda)^2)$$

Thus, we seek a function $g(\cdot)$ such that the variance of this asymptotic distribution is 1. This corresponds to solving the equation the following $\sigma^2 g'(\lambda)^2 = 1$. Recall that $\sigma^2 = \lambda$. This yields

$$g(\lambda) = \int \frac{d\lambda}{\sqrt{\lambda}} = 2\sqrt{\lambda}$$

Using this transformation along with the Delta Method we have

$$\sqrt{n}(2\sqrt{\overline{X}_n} - 2\sqrt{\lambda}) \xrightarrow{D} Z \sim N(0, \lambda(1/\sqrt{\lambda})^2) = N(0,1)$$

Thus we see that $\sqrt{n}(2\sqrt{\overline{X}_n} - 2\sqrt{\lambda}) \sim AN(0,1)$

6. For $\mu \neq 0$, then the first derivative of $\log(x)$ is nonzero at μ. That is $1/\mu \neq 0$. Therefore, we can apply the first order delta method to see, $Y_n = \log|X_n| \sim AN(\log|\mu|, (\sigma/\mu)^2)$. Now notice for $\mu = 0$, we note that *all derivatives of* $\log(\cdot)$ *are not defined*. That is $\frac{d}{d^n}\log(x)|_{x=0}$ does not exist for all n. Therefore, the asymptotic distribution for $\mu = 0$ does not exist.