Flow based Graph Partioning

Goal: find the min. cond cut

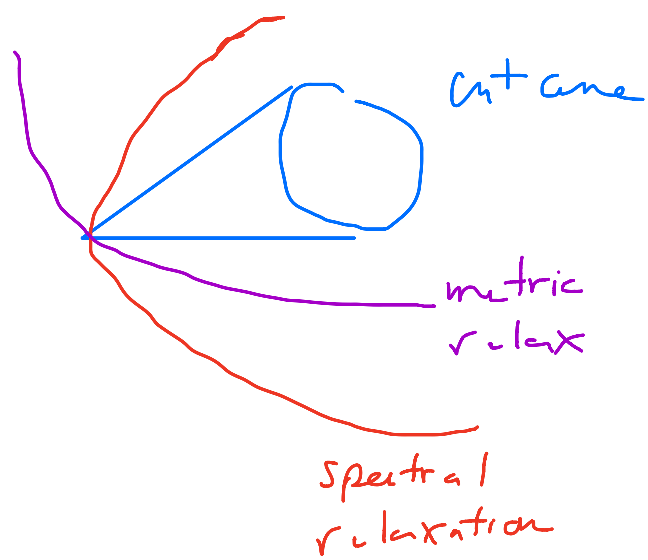
$$\bar{\phi}_G = \min_{S \subseteq V} \bar{\phi}(S)$$

Approach metric relaxation

Set of all ℓ_1-embeddable metrics

$\cong$

Convex(cut-metrics)



Solved via linear programming

$$\min \sum_{e \in G} \delta_e \quad \text{s.t.} \quad \sum \frac{d_i d_j}{Vol(G)} \underset{\alpha}{\delta_{ij}} = 1$$

$$\forall ij \quad \forall p \in P_{ij} \quad \delta_{ij} \le \sum_{e \in p} \delta_e \, f_p$$

We can generalize this flow prob. to solve more complex problems

$$\alpha_\mu(S) = \frac{E(S,\bar{S})}{\mu(S)\,\mu(\bar{S})}$$ any measure.

__Ex:__ $\mu = 1$, $\alpha(S) = \dfrac{|E(S,\bar{S})|}{|S|\,|\bar{S}|}$

In our case we are looking at the problem

$$\min_{\delta_S} \frac{\delta_S(G)}{\delta_S(K_\mu)} \qquad K_\mu : \; i \underset{\mu_i \mu_j}{\rule{2cm}{0.4pt}} j$$

__Dual__

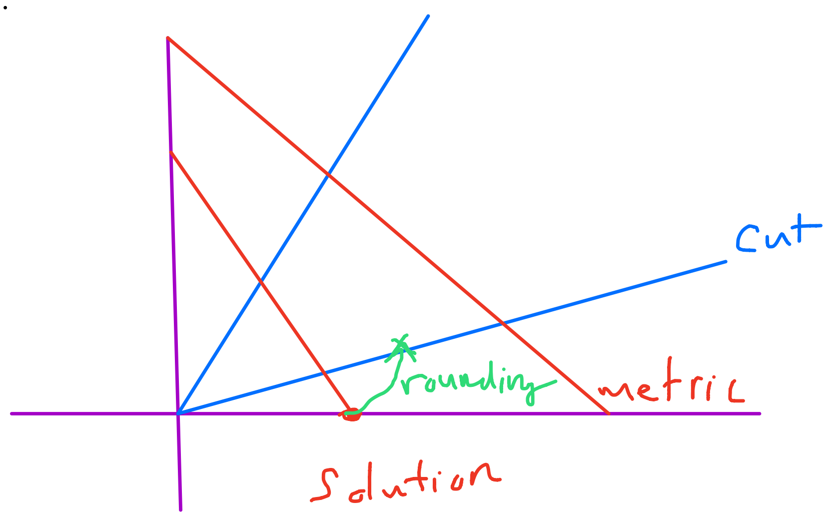
$\max \alpha$ s.t. $\forall e \in E \sum_{p \ni e} f_p \leq 1$ (capacity)

$\forall ij \sum_{p \in P_{ij}} f_p \geq \alpha \mu_i \mu_j$ total flow requirements

$\forall p \; f_p \geq 0$

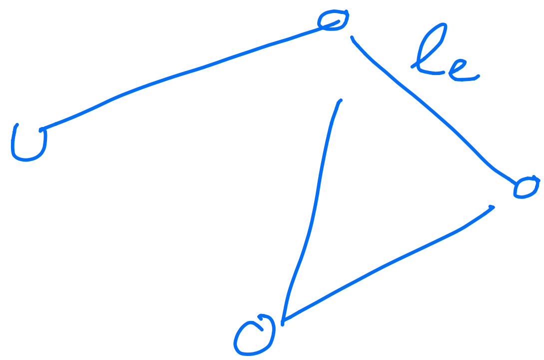
Suppose that we have a max flow.

Now we want to turn this into a cut.



Rounding Sol. to cut cone

Suppose we have the sol. to this flow prob.



<u>Lemma</u>: Assume there exists $S \subseteq V$

with $\mu(S) \geq \frac{\mu(V)}{c} = \frac{m}{c}$

s.t. $\forall\, i, j \in S \quad \delta_{ij} \leq \alpha$

$$\sum_{j \in V \setminus S} \mu_j \, d(j, S)$$

closes distance from $j \mapsto i \in S$

$$\leq \sum \mu_j \sum \mu_k \left(d_{jk} + d(k, S) \right)$$

$j \in V \setminus S$ $\quad k \in V$.

$$= \underbrace{\sum_{(j,k) \in (V \setminus S, V)} \mu_j \mu_k d_{jk}}_{\geq \frac{1}{2}(1-\mu(S))^2 \alpha} + \mu(V \setminus S) \sum_{k \in V} \mu_i d(k,S)$$

bc:

$$\sum_{i,j \in S \times S} \mu_i \mu_j d_{ij} \leq (\mu(S))^2 \alpha$$

Frechet Embedder

$y_i = d(j, S)$

$|y_i - y_j| \leq |d(j,S) - d(i,S)| \leq d_{ij}$

$$\sum \mu_i \mu_j |y_i - y_j| = \sum \mu_i \mu_j |d(i,S) - d(j,S)|$$

$$\geq \sum_{i \in V} \mu_i \, d(i,S)$$

So these ℓ_1 embeddings are interpreted as restrictions of the distances.