Suppose

offsets

$$Y_i \mid \beta \sim \text{Bern}\left[\underbrace{\text{logit}^{-1}(x_i\beta + \phi_i)}_{\mu_i = \mathbb{E}[Y_i \mid \beta]}\right]$$

$$\beta \sim N(\beta_0, \omega^{-1})$$

precesion or information

Goal: $\beta \mid Y$. No analytical solution

$$\mathbb{P}(\beta \mid Y) \propto \mathbb{P}(Y \mid \beta)\,\mathbb{P}(\beta)$$

$$\propto \prod_{i=1}^{n} \mu_i^{Y_i}(1-\mu_i)^{1-Y_i} \exp\left\{-\frac{\omega}{2}(\beta-\beta_0)^2\right\}$$

$$\Pi(\beta) \equiv \log \mathbb{P}(\beta \mid Y)$$

$$= \sum_{i=1}^{n} \left\{ Y_i \log\left(\frac{\mu_i}{\quad}\right) + \log(1-\mu_i) \right.$$

$$= \sum_{i=1} \Big(\cdots \log\left(\frac{\cdots}{1-\mu_i}\right) + \log(1-\mu_i) \Big)$$

$$- \frac{w}{2}(\beta - \beta_0)^2 \Big\}$$

$$= \sum_{i=1}^{n} \Big\{ y_i(x_i\beta + \phi_i) - \log\left(1 + \exp\{x_i\beta + \phi_i\}\right)$$

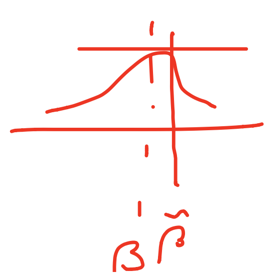
$$- \frac{w}{2}(\beta - \beta_0)^2 \Big\}$$

Using 2-order Taylor

$$\pi(\beta) \simeq \pi(\tilde{\beta}) + \pi'(\beta)(\beta - \tilde{\beta}) + \frac{\pi''(\beta)}{2}(\beta - \tilde{\beta})^2$$

Assume $\mathbb{P}(\beta \mid y)$ is roughly unimodal with mass centered around the mode, with $|\tilde{\beta} - \beta| < \delta$ then

$$\pi'(\tilde{\beta}) < \varepsilon$$



and we see

$$\pi(\beta) \simeq \pi(\check\beta) - \frac{1}{2}\left(-\pi''(\check\beta)\right)(\beta-\check\beta)^2$$

$$\propto \underbrace{\frac{\pi''(\check\beta)}{2}(\beta-\check\beta)^2}_{\text{Kernel of a normal.}}$$

$$\pi(\beta) \mathrel{\dot\sim} N\left(\check\beta, \left(-\pi''(\check\beta)\right)^{-1}\right)$$

$$\pi'(\beta) = \sum_{i=1}^{n}\left(y_i x_i - \frac{\exp\{x_i\beta + \phi_i\}}{1+\exp(x_i\beta+\phi_i)} x_i - w(\beta-\beta_0)\right)$$

$$= \sum_{i=1}^{n}\left\{y_i x_i - \mu_i x_i - w(\beta-\beta_0)\right\}$$

$$\pi''(\beta) = -\sum_{i=1}^{n} \frac{\exp(x_i\beta+\phi_i)}{(1+\exp(x_i\beta+\phi_i))^2} x_i^2 - w$$

Hence for the proposal

$$Q(\beta^*|\beta^{(t)}) = N\left(\beta^{(t)}, \left(\sum_{i=1}^{n} \mu_i(\beta^{(t)})(1-\mu_i(\beta^{(t)})) + w\right)^{-1}\right)$$

So the actual algorithm looks like the following

<u>Laplace - Proposal MH Step:</u>

Current: $\beta^{(t)}$

1. Compute $\mu_i(\beta^{(t)}) = \text{logit}^{-1}(x_i\beta^{(t)} + \phi_i)$

$$\sigma^2(\beta^{(t)}) = \left(\sum_{i=1}^{n} \mu_i(\beta^{(t)})(1-\mu_i(\beta^{(t)})) + w\right)^{-1}$$

2. Sample Candidate

$$\beta^* \sim N\left(\beta^{(t)}, \sigma^2(\beta^{(t)})\right)$$

3. Compute $\mu_i(\beta^*)$, $\sigma^2(\beta^*)$

4. Compute acceptance ratio

$$R = \frac{\mathbb{P}(\beta^* | y)\, Q(\beta^{(t)} | \beta^*)}{\mathbb{P}(\beta^{(t)} | y)\, Q(\beta^* / \beta^{(t)})}$$

$$\log R = \pi(\beta^*) - \pi(\beta^{(t)}) + \left(\phi(\beta^{(t)}; \beta^*, \sigma^2(\beta^*)) - \phi(\beta^* | \beta^{(t)}, \sigma^2(\beta^{(t)}))\right)$$

5. If $\log R \geq 0 \;||\; \log(\text{runif}) < \log R$

$\beta^{(t+1)} \quad \beta^*$

$\beta \quad = \beta$

else

$$\beta^{(t+1)} = \beta^{(t)}$$

Ex: $Y_i = \# \; I(i\text{th ball is blue})$

$\sim \text{Bern}(\mu_i)$ $\quad$ $W = \#$ white balls

$B = e^{\beta}$ Blue balls

$$\mu_i = \frac{B}{B+W}$$

$$= \frac{e^{\beta}}{e^{\beta}+W} = \frac{e^{\beta - \log W}}{e^{\beta - \log W}+1}$$

$$= \text{logit}^{-1}(\beta - \log W)$$