1. (a) Using the fact that $h : \mathbb{R} \to [0,1]$ we have

$$\begin{aligned}
|f_h| &= \left| e^{x^2/2} \int_{-\infty}^{x} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| \\
&\leq e^{x^2/2} \int_{-\infty}^{x} \Big| h(y) - \mathbb{E}(h(N)) \Big| e^{-y^2/2} dy \\
&\leq e^{x^2/2} \int_{-\infty}^{x} 1 \cdot e^{-y^2/2} dy \\
&\leq e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} dy
\end{aligned}$$

In a similar fashion we also have

$$\begin{aligned}
|f_h| &= \left| e^{x^2/2} \int_{-\infty}^{x} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| \\
&= \left| e^{x^2/2} \int_{-\infty}^{\infty} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy - e^{x^2/2} \int_{x}^{\infty} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| \\
&\leq \left| e^{x^2/2} \int_{-\infty}^{\infty} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| + \left| e^{x^2/2} \int_{x}^{\infty} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| \\
&\leq \left| e^{x^2/2} \left(\int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy - \mathbb{E}(h(N)) \int_{-\infty}^{\infty} e^{-y^2/2} dy \right) \right| \\
&+ \left| e^{x^2/2} \int_{x}^{\infty} \left(h(y) - \mathbb{E}(h(N))\right) e^{-y^2/2} dy \right| \\
&\leq \left| e^{x^2/2} \left(\sqrt{2\pi}\mathbb{E}(h(N)) - \sqrt{2\pi}\mathbb{E}(h(N)) \right) \right| + e^{x^2/2} \int_{x}^{\infty} \Big| h(y) - \mathbb{E}(h(N)) \Big| e^{-y^2/2} dy \\
&\leq e^{x^2/2} \int_{x}^{\infty} e^{-y^2/2} dy
\end{aligned}$$

Therefore we see that

$$|f_h| \leq e^{x^2/2} \min \left\{ \int_{-\infty}^{x} e^{-y^2/2} dy, \int_{x}^{-\infty} e^{-y^2/2} dy, \right\}$$

(b) asdf

(c) ???

(d) By Stein's equation we know that $f_h'(x) = x f_h(x) + h(x) - \mathbb{E}[h(N)]$

2. (a)

$$\begin{aligned}
f_z(x) &= e^{x^2/2}\int_{-\infty}^{x}\left(\mathbf{1}_{(-\infty,z]}(y) - \mathbb{E}[\mathbf{1}_{(-\infty,z]}(N)]\right)e^{-y^2/2}dy \\
&= e^{x^2/2}\left(\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathbf{1}_{(-\infty,z]}(y)e^{-y^2/2}dy - \mathbb{P}(N\leq z)\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy\right) \\
&= e^{x^2/2}\left(\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathbf{1}_{(-\infty,z]}(y)e^{-y^2/2}dy - \Phi(z)\sqrt{2\pi}\Phi(x)\right)
\end{aligned}$$

Now, the first integral depends on $x \leq z$ or $x \geq z$. That is the integral's top limit will be given by $\min\{x, z\}$. Hence we can write f_z in general as

$$f_z(x) = \begin{cases} \sqrt{2\pi}e^{x^2/2}\Phi(x)(1-\Phi(z)) & z \geq x \\ \sqrt{2\pi}e^{x^2/2}\Phi(z)(1-\Phi(x)) & z \leq x \end{cases}$$

(b) Recall the useful fact that $\Phi(-z) = 1 - \Phi(z)$ due to the symmetry of $\Phi(\cdot)$. With this fact we have

$$f_{-z}(-x) = \begin{cases} \sqrt{2\pi}e^{x^2/2}\Phi(-x)(1-\Phi(-z)) & -z \geq -x \\ \sqrt{2\pi}e^{x^2/2}\Phi(-z)(1-\Phi(-x)) & -z \leq -x \end{cases}$$

$$= \begin{cases} \sqrt{2\pi}e^{x^2/2}(1-\Phi(x))(\Phi(z)) & z \leq x \\ \sqrt{2\pi}e^{x^2/2}(1-\Phi(z))(\Phi(x)) & z \geq x \end{cases} = f_z(x)$$

Here we may assume without loss of generality that $z \geq 0$.

(c) Now, we take the derivative of $xf_z(x)$ in an attempt to show the function is increasing in x. Let φ be the density of a standard normal random variable. First we calculate the derivative of f_h.

$$\begin{aligned}
f_h'(x) &= \begin{cases} \sqrt{2\pi}xe^{x^2/2}(1-\Phi(x))(\Phi(z)) - \sqrt{2\pi}e^{x^2/2}\varphi(x)\Phi(z) & z \leq x \\ \sqrt{2\pi}xe^{x^2/2}(1-\Phi(z))(\Phi(x)) + \sqrt{2\pi}e^{x^2/2}(1-\Phi(z))\varphi(x) & z \geq x \end{cases} \\
&= \begin{cases} \sqrt{2\pi}e^{x^2/2}\Phi(z)\left[x(1-\Phi(x)) - \varphi(x)\right] & z \leq x \\ \sqrt{2\pi}e^{x^2/2}(1-\Phi(z))\left[x\Phi(x) + \varphi(x)\right] & z \geq x \end{cases}
\end{aligned}$$

Therefore, in general

$$\frac{d}{dx}[xf_h(x)] = f_h(x) + xf_h'(x)$$