Bayesian Topics

Posterior: $\mathbb{P}(\theta | X) \propto \underbrace{\mathbb{P}(X|\theta)}_{\text{Likelihood}} \underbrace{\mathbb{P}(\theta)}_{\text{Prior}}$

Post-Pred:
$$\mathbb{P}(\tilde{X}|X) = \int \mathbb{P}(\tilde{X}, \theta | X) \, d\theta$$
$$= \int \mathbb{P}(\tilde{X} | X, \theta) \mathbb{P}(\theta | X) \, d\theta$$
$$= \int \underbrace{\mathbb{P}(\tilde{X}|\theta)}_{\text{one unit likelihood.}} \underbrace{\mathbb{P}(\theta | X)}_{\text{post.}} \, d\theta$$

Post-Summaries:

- $\hat{\theta}_{pm} = \mathbb{E}[\theta | X]$ $\quad \theta_{MAP} = \underset{\tilde{\theta} \in \Theta}{\operatorname{argmax}} \, \mathbb{P}(\tilde{\theta} | X)$

- Credibility intervals

  $\mathbb{P}(\theta \in S_\alpha | X) = 1 - \alpha$

  – Percentage credibility interval

  $$\mathbb{P}\left(\theta \in \left(F^{-1}_{\alpha/2}, F^{-1}_{1-\alpha/2}\right)\right) = 1 - \alpha$$

  – HDP

  $\{\theta : \mathbb{P}(\theta | X) > k_\alpha\}$ with

  $\mathbb{P}(\{\theta : \mathbb{P}(\theta|X) > k_\alpha\}) = 1 - \alpha$

Bayesian Topics

Ex: $\Theta \sim \text{Beta}(\alpha, \beta) \quad X \mid \Theta \sim \text{Binom}(n, \theta)$

$\Theta \mid X \sim \text{Beta}(\alpha + X, n - X + \beta)$

post mode: $\dfrac{X + \alpha - 1}{n + \alpha + \beta - 2}$

mean: $\dfrac{X + \alpha}{n + \alpha + \beta}$

var: $\dfrac{\mathbb{E}[\theta \mid X](1 - \mathbb{E}[\theta \mid X])}{n + \alpha + \beta + 1} \xrightarrow{\mathbb{P}} 0$

Noninformative Priors

Jeffrey's Prior: main prop is invariance to reparametrization

$$\mathbb{P}(\theta) \propto \underbrace{I(\theta)}_{\text{fisher's info}}^{-1/2} \qquad I(\theta) = \mathbb{E}_{X \mid \theta}\left[\left(\frac{\partial l}{\partial \theta}\right)^2\right]$$

$$= -\mathbb{E}_{X \mid \theta}\left[\frac{\partial^2 l}{\partial \theta^2}\right]$$

Obs Fisher's info $\qquad l = \log \mathbb{P}(\theta)$

$$I^{obs}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log \mathbb{P}(X_i \mid \theta)$$

Bayesian Topics

Examples from Exp. families.

· Jeffreys for $X|\theta \sim Binom(n,\theta)$ is

$$\boxed{\theta \sim Beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right)}$$

· Conjugate Prior for $N(\mu, \underbrace{\sigma^2}_{\text{known}})$

$$\mathbb{P}(\mu) \propto \exp\left(-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2\right) = \boxed{N(\mu_0, \tau_0^2)}$$

Conjugate post: $\boxed{N(\mu_1, \sigma_1^2)}$

$$\mu_1 = \frac{\mu_0/\tau_0^2 + \bar{X}/(\sigma^2/n)}{\frac{1}{\tau_0^2} + \frac{1}{(\sigma^2/n)}} \qquad \sigma_1^2 = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2/n}$$

Post predictive:

$\tilde{X}|\mu \sim N(\mu, \sigma^2)$ $\quad \tilde{X} = \mu + e \sim N(0, \sigma^2)$

$\mu|X \sim N(\mu_1, \sigma_1^2)$ $\quad \mu = \mu_1 + f \sim N(0, \sigma_1^2)$

notice $e \perp\!\!\!\perp f$ so

$$\boxed{\tilde{X}|X \sim N(\mu_1, \sigma^2 + \sigma_1^2)}$$

Bayesian Topics

Examples from Exp. families

model: $X_i \mid \sigma^2 \sim N(\underbrace{\mu}_{\text{known}}, \sigma^2)$

Conj prior: $\mathbb{P}(\sigma^2) \propto (\sigma^2)^{-(\nu_0/2+1)} \exp\left(\frac{-\nu_0 \tau_0^2}{2\sigma^2}\right)$

$$\sim \text{Inv}\chi^2(\nu_0, \tau_0^2)$$

Post: $\sigma^2 \mid X \sim \text{Inv}\chi^2\left(n+\nu_0, \frac{S(x)+\nu_0\tau_0^2}{n+\nu_0}\right)$

for $S(x) = \sum_{i=1}^{n} (x_i - \mu)^2$

Model: $Y \mid \theta \sim P_o(x\theta)$

Prior: $\mathbb{P}(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta} \sim \text{Gamma}(\alpha, \beta)$

Post: $\theta \mid Y \sim \text{Gamma}(\alpha+n\bar{y}, \beta+n\bar{x})$

Post Pred: $\tilde{Y} \mid y \sim NB\left(\alpha+n\bar{y}, \frac{\tilde{x}}{\tilde{x}+\beta+n\bar{x}}\right)$

Bayesian Topics

model: $X_1, \dots, X_n \mid \mu, \sigma^2 \sim N(\mu, \sigma^2)$

prior: $\mathbb{P}(\mu, \sigma^2) = \mathbb{P}(\mu \mid \sigma^2)\,\mathbb{P}(\sigma^2)$

$$\mathbb{P}(\mu \mid \sigma^2) \propto \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma^2/k_0}\right\}$$

$$\mathbb{P}(\sigma^2) \propto (\sigma^2)^{-\frac{\nu_0}{2}-1} \exp\left(-\frac{\nu_0 \sigma_0^2}{2\sigma^2}\right)$$

$$\mu, \sigma^2 \sim N\text{-Inv}\chi^2(\mu_0, k_0; \nu_0, \sigma_0^2)$$

post: $\mu, \sigma^2 \mid X \sim N\text{-Inv}\chi^2(\mu_n, k_n; \nu_n, \sigma_n^2)$

$$\mu_n = \frac{n\bar{X} + k_0\mu_0}{n + k_0} \quad \text{(See text.)}$$

Laplace Approximation.

Set $\theta^* = \underset{\Theta}{\arg\max}\, \pi(\theta)$ for $\pi(\theta) = \log \mathbb{P}(\theta \mid X)$

then $\theta \mid X \,\dot\sim\, N\left(\theta^*, I^{obs}(\theta^*)^{-1}\right)$

Berstein von Mises: $\theta^* \approx \hat{\theta}_{MLE}$ so

$$\theta \mid X \,\dot\sim\, N\left(\hat{\theta}_{MLE}, I^{obs}(\hat{\theta}_{MLE})^{-1}\right)$$