Control Variates

Suppose we want to estimate

$$I = \mathbb{E}(h(x))$$

and we want to know

$$J = \mathbb{E}[c(Y)]$$ analytically.

If $h(x), c(Y)$ are correlated then we could use MC estimates of J to improve estimates of I

$$\hat{I}_{cv} = \hat{I}_{mc} - \lambda(\hat{J}_{mc} - J)$$

"control variate"

Still unbiased → want lower variance.

$$Var(\hat{I}_{cv}) = Var(\hat{I}_{mc}) + \lambda^2 Var(\hat{J}_{mc})$$

$$-2\lambda\,\mathrm{Cov}(\hat{I}_{mc}, \hat{J}_{mc})$$

So the minimizing λ is

$$\lambda^* = \frac{\mathrm{Cov}(\hat{I}_{mc}, \hat{J}_{mc})}{\mathrm{Var}(\hat{J}_{mc})}$$

This is just the regression coefficient $h(X_i) \sim (c(Y_i) - J)$ the slope is λ^* and the intercept is $\hat{I}_{cv}$.

<u>Digression</u>: We often want to compute $p = p_1 + p_2$. To avoid underflow we work on log-space

$$l_1 = \log p_1 \quad l_2 = \log p_2 \quad l = \log p$$

$$\ell = \log(\exp(\ell_1) + \exp(\ell_2))$$

assume $\ell_1 \geq \ell_2$

$$= \log\left\{e^{\ell_1}\left(1 + e^{\ell_2 - \ell_1}\right)\right\}$$

$$= \ell_1 + \underbrace{\log\left(1 + e^{\ell_2 - \ell_1}\right)}_{\text{this we know directly}}$$

"soft max"

Suppose we have $p = \sum_{i=1}^{3} p_i$

$$\ell = \log\left(e^{\ell_1} + e^{\ell_2} + e^{\ell_3}\right)$$

$$= \log\left(\exp\left(\underbrace{\log\left(e^{\ell_1} + e^{\ell_2}\right)}_{lse(\ell_1, \ell_2)}\right) + e^{\ell_3}\right)$$

$$= lse\left(lse(\ell_1, \ell_2), \ell_3\right)$$

So in general

$$\ell = \mathrm{lse}(\cdots \mathrm{lse}(\ell_1, \ell_2), \cdots, \ell_n)$$

this is what we call a reduce.

```
lse2 <- function(x, y){
    m <- max(x, y)
    d <- -abs(x - y)
    ifelse(d < log(ε), m,
           m + log(1 + exp(d)))
}

lse <- function(x⃗){
    Reduce(lse2, x)
```

Project 2 Notes