1. (4.44) Theorem 4.5.6, with $a = b = 1$, serves as the base case of our inductive argument. Assume that the statement holds for $n > 1$. That is, assume

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\leq i<j\leq n} Cov(X_i, X_j)$$

Now for the $n+1$ case we have

$$\begin{aligned}
Var\left(\sum_{i=1}^{n+1} X_i\right) &= Var\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)\\
&= Var\left(\sum_{i=1}^{n} X_i\right) + Var(X_{n+1}) + 2Cov\left(\sum_{i=1}^{n} X_i, X_{n+1}\right) && (4.5.6)\\
&= \sum_{i=1}^{n+1} Var(X_i) + 2\sum_{1\leq i<j\leq n} Cov(X_i, X_j) + 2Cov\left(\sum_{i=1}^{n} X_i, X_{n+1}\right) && (Assumption)
\end{aligned}$$

Now, first notice that we can rewrite the second term as

$$2\sum_{1\leq i<j\leq n} Cov(X_i, X_j) = 2\sum_{j=2}^{n}\sum_{i=1}^{j-1} Cov(X_i, X_j)$$

It should be clear that we simply need third term above proves the $j = n + 1$ case in this sum. Consider the following.

$$\begin{aligned}
Cov(\sum_{i=1}^{n} X_i, X_{n+1}) &= \mathbb{E}\left(X_{n+1}\sum_{i=1}^{n} X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} X_i\right)\mathbb{E}\left(X_{n+1}\right)\\
&= \mathbb{E}\left(\sum_{i=1}^{n} X_{n+1}X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} X_i\right)\mathbb{E}\left(X_{n+1}\right)\\
&= \sum_{i=1}^{n}\mathbb{E}\left(X_{n+1}X_i\right) - \sum_{i=1}^{n}\mathbb{E}\left(X_i\right)\mathbb{E}\left(X_{n+1}\right)\\
&= \sum_{i=1}^{n}\Big[\mathbb{E}\left(X_{n+1}X_i\right) - \mathbb{E}\left(X_i\right)\mathbb{E}\left(X_{n+1}\right)\Big]\\
&= \sum_{i=1}^{n} Cov(X_i, X_{n+1})\\
&= \sum_{j=n+1}^{n+1}\sum_{i=1}^{j-1} Cov(X_i, X_j)
\end{aligned}$$

Using this, we see

$$2\sum_{1\leq i<j\leq n} Cov(X_i, X_j) + 2Cov(\sum_{i=1}^{n} X_i, X_{n+1}) = 2\sum_{j=2}^{n+1}\sum_{i=1}^{j-1} Cov(X_i, X_j) = 2\sum_{1\leq i<j\leq n+1} Cov(X_i, X_j)$$

Hence

$$Var\left(\sum_{i=1}^{n+1} X_i\right) = \sum_{i=1}^{n+1} Var(X_i) + 2\sum_{1\leq i<j\leq n+1} Cov(X_i, X_j)$$

2. (4.63) Let $X = \log Z$. Then $X = \exp(Z)$. Recall that $\exp(\cdot)$ is a convex function so by Jensen's Inequality, we have

$$\mathbb{E}(X) = \mathbb{E}(\exp(Z)) \geq \exp(\mathbb{E}(Z)) = \exp(0) = 1$$

Therefore, $\mathbb{E}(X) \geq 1$

3. (5.3) First note that $Y_i = 0$ with probability $P(X_i \leq \mu) = F_X(\mu)$ and $Y_i = 1$ with probability $P(X_i > \mu) = 1 - F_X(\mu)$. This holds for all $1 \leq i \leq n$ so assuming that we consider 1 as a "success" we have $Y_i \sim Bern(1 - F_X(\mu)$. Hence for $Z = \sum_{i=1}^n Y_i$ we have that $Z \sim Binom(n, 1 - F_X(\mu))$.

4. (a) For $0 < t < h$, the function e^{tx} is nondecreasing and nonegative on $(0, \infty)$. Thus, using the Markov-Inequality, we have

$$P(X \geq a) \leq \frac{1}{e^{ta}}\mathbb{E}(e^{tX}) = e^{-ta}M_X(t)$$

   (b) For $-h < t < 0$, the function e^{-tx} is nondecreasing and nonnegative on $(0, \infty)$. Again, using the Markov-Inequality, we have

$$P(X \leq a) = P(-X \geq -a) = \frac{1}{e^{ta}}\mathbb{E}(e^{tX}) = e^{-ta}M_X(t)$$

5. (a) Suppose $g(t) = t^2 + \sigma^2$. Then $g(t)$ is nonegative and nondecreasing on $(0, \infty)$, using the Markov-Inequality we have

$$P(X \geq a) \leq \frac{1}{a^2 + \sigma^2}\mathbb{E}(X^2 + \sigma^2) = \frac{\sigma^2}{a^2 + \sigma^2}$$

   (b) asdf

6.

7. (a) Yes. Consider $X \sim N(0, 1)$ and $Y = X^2$. Then

$$Cov(X, Y) = Cov(X, X^2) = \mathbb{E}(X^3) - \mathbb{E}(X^2)\mathbb{E}(X) = \mathbb{E}(X^3)$$

   Then using the moment generating function of a standard normal, we see

$$\frac{\partial^3}{\partial t^3}\exp(1/2t^2)\Big|_{t=0} = t\exp(1/2t^2) + 2t\exp(1/2t^2) + t^3\exp(1/2t^2)\Big|_{t=0} = 0$$

   (b) No.

8. (a) Let $Y_1, Y_2 \overset{iid}{\sim} F_Y(y)$. Let $M = \max(Y_1, Y_2)$ and let m be the median of $F_Y(y)$. Then M is the largest order statistic and has cumulative distribution function $G_M(t) = (F_Y(t))^2$. Using this, we can calculate the desired probability

$$P(M > m) = 1 - P(M \leq m) = 1 - G_M(m) = 1 - (F_Y(m))^2 = 1 - (1/2)^2 = \frac{3}{4}$$

Here, the fourth equality used the fact that m was the median of $F_Y(y)$.

(b) Now, let $Y_1, Y_2, \dots, Y_n \sim F_Y(y)$, $M = \max(Y_i)_{i=1}^n$, and m be the median of $F_Y(y)$. Then M has CDF $G_M(t) = (F_y(t))^n$. With this, we can compute the desired probability.

$$P(M > m) = 1 - P(M \leq m) = 1 - G_M(m) = 1 - (F_Y(m))^n = 1 - (1/2)^n$$

9. Recall that if Y_k is the kth order statistic of $X_1, X_2, \dots, X_n$ then $U_k = F_X(Y_k)$ is the kth order statistic of a sample of size n from a Uniform on $[0, 1]$. Now recall that the distribution of the kth order statistic is given by the following

$$\begin{aligned} g_{U_k}(y) &= \frac{n!}{(k-1)!(n-k)!} F_U(y)^{k-1}(1 - F_U(y))^{n-k} \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} y^{k-1}(1-y)^{n-k} \end{aligned}$$

which we recogonize as the Beta density function with parameters $(k, n-k+1)$. Hence, we can use the form $\mathbb{E}\big[F(Y_k)^2\big] = Var(F(Y_K)) + \mathbb{E}(F(Y_k))^2$ to find the desired value.

$$\begin{aligned} \mathbb{E}\big[F(Y_k)^2\big] &= \frac{k(n-k+1)}{(n+1)^2(n+2)} + \frac{k^2}{(n+1)^2} \\ &= \frac{k(n-k+1) + k^2(n+2)}{(n+1)^2(n+2)} \end{aligned}$$