Partial Correlation Networks

Recall problem: - Infer edge set from vertex level attributes

- One idea: testing perspective $E = \{(i,j) : \rho_{ij} \neq 0\}$

and estimate via $\hat{E} = \{(i,j) : \text{Reject } H_{0,ij}\}$

Problem: correlation among $x_i \; x_j \Longrightarrow$ false edge

$$x_i \leftarrow x_k \rightarrow x_j$$

Sol: Use partial correlation based on set S_m.

$$\rho_{ij} | S_m = \frac{\sigma_{ij} | S_m}{\sqrt{\sigma_{ii} | S_m \;\; \sigma_{jj} | S_m}}$$

Suggests defining $E = \{(i,j) : \rho_{ij} | S_m \neq 0 \;\; \forall S_m \in V^{(m)} \setminus \{i,j\}\}$

then defining $H_0' : \rho_{ij} | S_m = 0$ has a nominal $N(0, 1/(n-m-3))$

Still need to combine p-values

One idea: $p_{ij,max} = \{p_{ij} | S_m : S_m \in V^{(m)}_{\setminus \{i,j\}}\}$

Typically can do up to order $m=3$. based on dynamic program.

Association networks from Time Series

- Oftentimes, vertex attributes are Time Series
- Need to deal with temporal correlation
- Maximal cross-correlation – Pearson Corr.

→ Cross corr. multiple time series, lag dependence across time series

$$s_{ij} = \max_{\tau} |c_{ij}(\tau)| \qquad \tau - \text{lag}$$

$$c_{ij}(\tau) = \frac{1}{\sigma_i \sigma_j (n-2\tau)} \sum_{t=1}^{n-\tau} [x_i(t) - \bar{x}][x_j(t+\tau) - \bar{x}_j]$$

→ Assess association between i,j by understanding coupling between x_i, x_j

→ Take Fisher transform $c_{ij}^F(\tau)$, $s_{ij}^F = \max_{\tau} c_{ij}^F(\tau)$, then

$$z_{ij}^F = \frac{s_{ij}^F}{\sqrt{\mathrm{var}}^{1/2}(c_{ij}^F)}$$

→ Large deviation approx.

$$\mathbb{P}_{H_0}(z_{ij}^F \leq z) \simeq \exp\{-2\exp(-a_n(z-b_n))\}$$

Gaussian Graphical Models

Let $m = N_v - 2$, $X \sim N(0, \Sigma)$ and denote $\rho_{ij|V \setminus \{i,j\}}$

Rmk: $\rho_{ij|V \setminus \{i,j\}} = 0$ iff $X_i \perp X_j \mid X \setminus X_i, X_j$

We can write $\rho_{ij|V \setminus \{i,j\}} = \dfrac{-w_{ij}}{\sqrt{w_{ii} w_{jj}}}$ $\quad W = \Sigma^{-1}$ under Gaussian assump.

Typically encode graphical models to incode cond. independence

$\mathbb{E}[X_i \mid X^{(-i)} = x^{(-i)}] = \beta^{T(-i)} x^{(-i)}$ $\qquad \beta_j^{(-i)} = -\dfrac{w_{ij}}{w_{ii}} = 0$ iff $\rho_{ij|V \setminus \{i,j\}} = 0$

Idea: use penalized regression to infer $\beta_j^{(-i)}$ (lasso)