# ERGMs

A discrete random vector $Z$ belongs to the exponential family

$$\mathbb{P}_\theta(Z=z) = \exp\{\underbrace{\theta^T}_{\text{param.}} \underbrace{g(z)}_{\text{sufficient stat}} - \psi(\theta)\}$$

In ERGMs $Y$ random adj. matrix

$$\mathbb{P}_\theta(Y=y) = \frac{1}{K}\exp\Big(\underbrace{\sum_H}_{\text{configurations}} \underbrace{\theta_H}_{\text{nonzero implies edges in } H \text{ are dependent.}} g_H(y)\Big) \qquad g_H(y) = \begin{cases} 1 & \text{if config. } H \text{ occurs in } y \\ 0 & \text{otherwise} \end{cases}$$

Ex: $Y_{ij} \perp\!\!\!\perp Y_{i'}Y_{j'}$ so $\theta_H = 0 \;\; \forall\, |H| \geq 3$ (ER model)

So $g_H(y) = g_{ij}(y) = y_{ij}$, $\quad \mathbb{P}_\theta(Y=y) = \frac{1}{K}\exp\Big\{\sum_{ij} \theta_{ij}\, y_{ij}\Big\}$

Model is equivalent $p_{ij} = \text{logit}^{-1}(\theta_{ij})$

Assume homogeneity then $\mathbb{P}_\theta(Y=y) = \frac{1}{K}\exp\{\theta L(y)\}$

$L(y) = N_e$ literally a ERGM

Ex: Suppose $Y_{ij} \not\perp\!\!\!\perp Y_{ik} \; \forall k \neq j$. for all $k \neq j$.

Inspires a Markov Random Graph Model

$$\mathbb{P}_\theta(Y=y) = \frac{1}{K}\exp\left\{\sum_{k=1}^{N_v-1}\theta_k S_k(y) + \theta_\tau T(y)\right\}$$

$S_1(y) = N_e, \dots, S_k(y) = \#$ k-stars $\quad 2 \le k \le N_v - 1$

$T(y) = \#$ triangles

Other Extensions

- functions of stars, degree distributions, and k-triangles
- non markovian conditional dependence
- Covariates for edge/vertex attributes

Use ML for $\hat{\theta}$ and LR testing

- use psuedo likelihood for computational issues
- model degeneracy a major concern
  - tons of colinearity in star counts
  - Two solutions: empty/full graphs
    - ↳ lots of use of MGFs + Graphons

• Current state of the art involves MCMC or Robins/Monroe as a way to estimate $\theta$.

· Some initial work on goodness of fit with these MC replicates.

Ways to account for colinearity

$$\mathbb{P}_{\theta,\beta}(Y \mid X = x) = \frac{1}{K(\theta,\beta)} \exp\left(\theta_1 S_1(y) + \theta_2 AKT_\lambda(y) + \beta^T g(y,x)\right\}$$

$N_e$

alternating k-star

vertex attributes

$$AKT_\lambda(y) = 3T_1 + \sum (-1)^{k+1} \frac{T_k(y)}{\lambda^{k-1}}$$

"removes" colinearity

$$g(y,x) = \sum_{i<j} y_{ij}\, h(x_i, x_j)$$