Maxcut SDP

maxcut: Given $G = (V, E)$

find $S, \bar{S}$ s.t. $\max |E(S, \bar{S})|$

Last time: - trivial approx $1/2$

- LP doesn't beat $1/2$

- spectral $9/8$

Vector embedding

$x_i \in \mathbb{R}$ $\quad v_i \in \mathbb{R}^n$

s.t. $\max \frac{1}{4} \sum_{E} \|v_i - v_j\|^2 \quad \|v_i\| \stackrel{2}{=} 1$

Relaxation: SDP $\geq$ maxcut

For any cut S we can build

$$V_i^{(S)} = \begin{cases} 1 & i \in S \\ -1 & i \in \bar{S} \end{cases}$$

$$\max \sum_E \|v_i - v_j\|^2 = |E(S, \bar{S})|$$

Spectral: $\max \frac{1}{4} \sum \|v_i - v_j\|^2$

has n constraints. Could relax to

$$\sum d_i \|v_i\|^2 = 2|E|$$

Goemans Williamson Relaxation

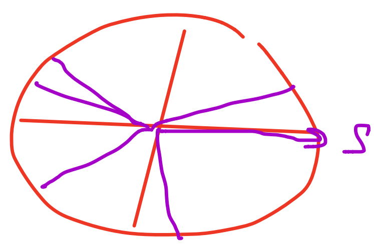
Rounding

Rounding

Claim: GW relaxtion can be rounded to obtain a 0.87856... approx to max cut

Rounding

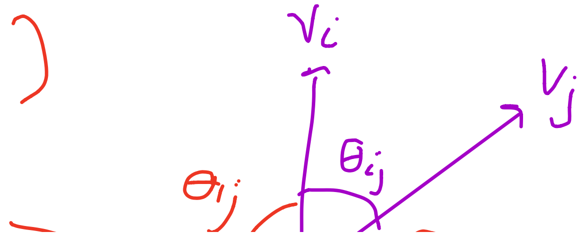
pick a uniform direction at random

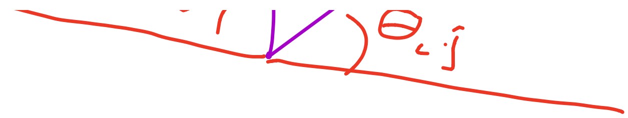


Rule: $i \in S$ if $\langle v_i, s \rangle \geq 0$

Pf: Take any pair i, j.

$\mathbb{P}(ij \text{ cut})$





$$= \frac{2\theta_{ij}}{2\pi} = \frac{\theta_{ij}}{\pi}$$

$$\cos\theta_{ij} = \langle v_i, v_j \rangle$$

$$\theta_{ij} = \arccos \langle v_i, v_j \rangle$$

$$\mathbb{E}[|E(S,\bar{S})|] = \frac{1}{\pi}\sum_{E} \arccos(\langle v_i, v_j \rangle)$$

must be compared to SDP value.

$$\frac{1}{4}\sum_{E} \|v_i - v_j\|^2 = \frac{1}{2}\sum_{E} 1 - \langle v_i, v_j \rangle$$

expand

Consider:

$$\frac{\mathbb{E}[|E(S,\bar{S})|]}{SDP} = \frac{2}{\pi} \frac{\sum \arccos \langle v_i, v_j \rangle}{\sum 1 - \langle v_i, v_j \rangle}$$

$$\min_{-1 \le x \le 1} \frac{2}{\pi} \frac{\arccos x}{1 - x} \geq 0.878\ldots$$

<u>Theorem</u>: There are matching integrality gaps.

<u>Minimum Conductance</u>

$$\bar{\phi}_G = \min_{S \subseteq V} \frac{|E(S,\bar{S})|}{Vol(S)\, Vol(\bar{S})} Vol(G)$$

Cheeger $\lambda_2 \leq \bar{\phi}_G \leq 2\sqrt{2\lambda_2}$

Spectral Relaxation:

$$\min \sum_{E} \| v_i - v_j \|^2$$

$$\sum \frac{d_i d_j}{Vol(G)} \| v_i - v_j \|^2 = 1$$

$$\xLeftrightarrow{SDP}$$

$$\min x^T L x$$

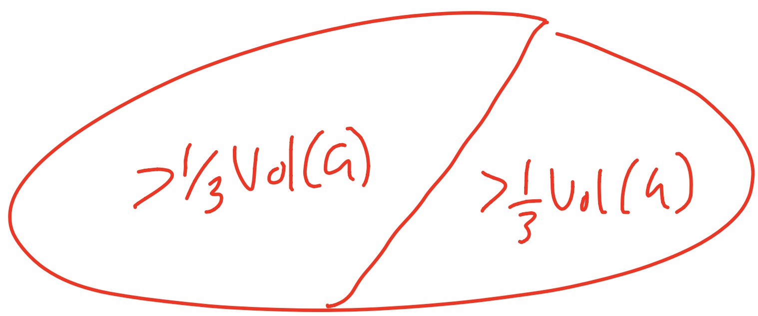
$$x^T L(K_G) = 1$$

Balanced Min Conductance

Divide & Conquer to have

log N depth we want

balanced cut



- Spectral properties won't provide this

<u>b-balanced mincut</u> $(0 \le b < 1/2)$

$$\bar{\phi}_{G,b} = \min_{\substack{S \subseteq V \\ Vol(S) \geq b\,Vol(G)}} \bar{\phi}(S)$$

<u>SDP Relaxation</u>

Integral Solution to Spec. Relax.

$(S, \bar{S})$

$$\hookrightarrow v_i^{(S)} = \begin{cases} c & i \in S \\ 0 & i \in \bar{S} \end{cases}$$

$$\sum \frac{d_i d_j}{\mathrm{Vol}(G)} \| v_i^{(S)} - v_j^{(S)} \|^2 = 1$$

$$\frac{\mathrm{Vol}(S)\,\mathrm{Vol}(\bar{S})}{\mathrm{Vol}(G)} c^2 = 1$$

$$\Rightarrow c = \sqrt{\frac{\mathrm{Vol}(G)}{\mathrm{Vol}(S)\,\mathrm{Vol}(\bar{S})}}$$

also want $\dfrac{\mathrm{Vol}(S)}{\mathrm{Vol}(G)} > b$

$$\frac{\mathrm{Vol}(\bar{S})}{\mathrm{Vol}(G)} < 1 - b$$

Would like to add the constraint

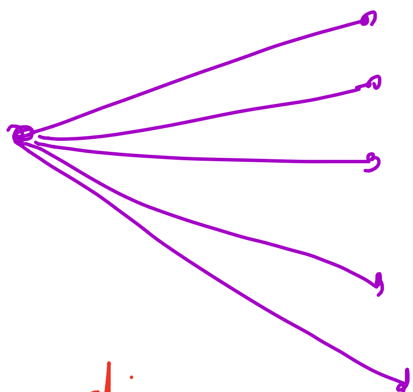
$$\|v_i\|^2 \le \sqrt{Vol(G)}\,\frac{1}{b}$$

* not trans. invariant

perhaps a better way to think about this

$$K_G = \frac{1}{2}\sum_{V} d_i \, S_i^{G}$$

S_i^G :



$$\frac{d_j}{Vol(G)}$$

$$d_i \sum_{j \in V} \frac{d_j}{Vol(G)} \|v_i - v_j\|^2 \; 2 \longrightarrow$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{L(d_i\, S_i)} \qquad \|v_i\|^2 \le \sqrt{Vol(G)}\,\frac{1}{b}$$

SPP Relaxation

$$\min L \cdot X$$

$$\text{s.t. } L(K_G) \cdot X = 1 \qquad (\alpha)$$

$$d_i(L(S_i)) \cdot X \le c = \frac{d_i}{b} Vol(G)$$

Finding c

$$d_i \sum_{F} \frac{d_j}{Vol(G)} \|v_i^{(s)} - v_j^{(s)}\|^2$$

$$= d_i \frac{Vol(\bar{S})}{Vol(G)} \frac{Vol(G)}{Vol(\bar{S})\,Vol(S)}$$

$$= d_i \quad - \quad d_i \quad (\quad) \quad (\quad)$$

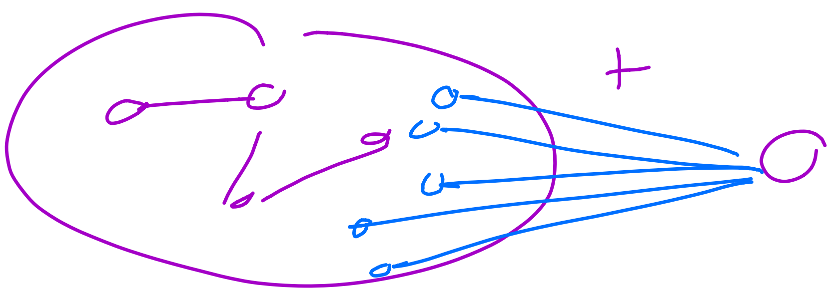
$$\frac{\cdots}{\mathrm{Vol}(S)} \geq \frac{\cdots}{b} \mathrm{Vol}(G) \quad (\beta_i)$$

<u>Dual:</u>

$$\max \quad \alpha \mathrm{Vol}(G) - \sum \frac{d_i \beta_i}{b}$$

s.t.

$$L + \underbrace{\sum d_i \beta_i L(S_i)}_{\text{adding stars}} \geq \alpha \, L(K_G)$$



moves $\lambda_2 \nearrow$ so that

we remove trivial cuts

but wants to maximize

things so we don't

need too many stars.