Alg: Given X_n Do: Compute X_{n+1}

Algo $\Longrightarrow$ $\{X_n, n \geq 0\}$ a M.C. with

initial dist ν_0 and trans. kern. P

(*) Assume P is π-reversible $\pi P = \pi$

when $\mu P(A) \equiv \int \mu(dx)\, P(x, A)$

Note that: $\mathbb{P}(X_n \in A) = \nu_0 P^n(A)$

(*) Assume P is lazy $P(x, \{x\}) \geq 1/2$

Define

$$\text{Spec Gap}(P) = \inf \left\{ \frac{\mathcal{E}(f,f)}{\text{Var}_\pi(f)},\ f \in L^2,\ \text{Var}_\pi(f) > 0 \right\}$$

$$\text{Var}_\pi(f) = \int (f(x) - \pi(f))^2\, \pi(dx)$$

$$\mathcal{E}(f,f) = \frac{1}{2} \iint (f(y) - f(x))^2 \pi(dx) P(x,dy)$$

Thrm: Suppose $\nu_0(dx) = f_0(x)\pi(dx)$: $f_0 \in L^2$

then $\|\nu_0 P^n - \pi\|_{TV}^2 \leq \mathrm{Var}_\pi(f_0)\,(1 - S_{\text{pec gap}})^n$

Goal: Bounds of $\mathcal{E}(f,f)$ gives results on the mixing.

Rmk: $\mathcal{E}(f,f)$ putting constraints on the gap between $\lambda_1 > \lambda_2$.

Pf: Recall $\|\nu - \mu\|_{TV} = \int |f_\nu(x) - f_\mu(x)|\, dx$

$$\nu_0 P(A) = \int \nu_0(dx)\, P(x,A)$$

$$= \iint f_0(x)\, \pi(dx)\, P(x,dy)\, 1_A(y).$$

Time reversibility

$$= \iint 1_A(x) f_0(y)\, \pi(dx)\, P(x,dy)$$

$$= \int_A \pi(dx) \int P(x, dy) f_0(y)$$

So the Radon-Nikodym density w.r.t. π

$$\frac{d(\nu_0 P)}{d\pi}(x) = \int P(x, dy) f_0(y)$$

Similarly, $$\frac{d(\nu_0 P^n)}{d\pi}(x) = \int P^n(x, dy) f_0(y)$$

$$= P^n f_0(x)$$

$$\Rightarrow \| \pi_0 P^n - \pi \|_{TV}^2 = \left(\int | P^n f_0(x) - 1 | \pi(dx) \right)^2$$

Jensens

$$\leq \int (P^n f_0(x) - 1)^2 \pi(dx)$$

$$= \mathrm{Var}_\pi(P^n f_0).$$

$\Big($ b/c $\int P^n f_0(x) \pi(dx) = 1 \Big)$

$$\mathrm{Var}_\pi(Pf) - \mathrm{Var}_\pi(f) \overset{*}{=} \pi((Pf)^2) - \pi(f^2)$$

$$= \langle Pf, Pf \rangle - \langle f, f \rangle$$

$$= \langle f, P^2 f \rangle - \langle f, f \rangle$$

$$= -\langle f, (I - P^2) f \rangle$$

$$* \quad Pf(x) = \int P(x, dy) f(y)$$

Stationarity.

$$\int \pi(dx) Pf(x) = \int \pi(dx) \int P(x, dy) f(y)$$

$$= \int \pi(dx) f(x)$$

Check.

$$= -\frac{1}{2} \iint (f(y) - f(x))^2 \pi(dx) P^2(x, dy)$$

Note that

Choose A, B such that

$$\int_A \pi(dx)\, P^2(x,B) = \int_A \pi(dx) \int P(x,dz)\, P(z,B)$$

$$\geq \int_A \pi(dx) \int_{\{x\}} P(x,dz)\, P(z,B) +$$

$$\int_A \pi(dx) \int_B P(x,dz)$$

$$\geq \frac{1}{2} \int_A \pi(dx)\, P(x,B) + \frac{1}{2} \int_A \pi(dx)\, P(x,B)$$

$$\geq \int_A \pi(dx)\, P(x,B)$$

$$\mathrm{Var}_\pi(Pf) - \mathrm{Var}_\pi(f) \leq -\mathcal{E}(f,f)$$

But

$$\mathcal{E}(f,f) \geq \mathrm{Spec\ Gap}(P)\, \mathrm{Var}_\pi(f)$$

$$\Rightarrow \mathrm{Var}_\pi \leq (1 - \mathrm{SpecGap}(P))\, \mathrm{Var}_\pi(f)$$

So

$$\mathrm{Var}_{\pi}(P^n f) \le \left(1 - \mathrm{SpecGap}(P)\right)^n \mathrm{Var}_{\pi}(f_0)$$

$\blacksquare$

$$\mathrm{SpecGap}(P) = \inf \left\{ \frac{\frac{1}{2}\iint (f(y)-f(x))^2 \pi(dx) P(x,dy)}{\frac{1}{2}\iint (f(y)-f(x))^2 \pi(dx)\pi(dy)} \right\}$$

If $\boxed{P(x,dy) \ge \alpha \pi(dy)}$ then

$$\mathrm{SpecGap}(P) \ge \alpha$$

Uniform minorization condition

Ex: Independent M.H.

Suppose we take $f \equiv 1_A$

$$\Rightarrow \frac{1}{2}\iint (f(y) - f(x))^2 \pi(dx)\pi(dy)$$

$$\pi(A)(1-\pi(A))$$ (denominator)

$$\Rightarrow \frac{1}{2}\left(\pi(A) + \pi(A) - 2\int \pi(dx) P(x,A)\right)$$

A numerator

$$= \int_A \pi(dx)\,(1 - P(x, A))$$

$$= \int_A \pi(dx)\, P(x, A^c)$$

Define the <u>conductance</u> of the kernel.

$$\phi(P) = \inf_A \left\{ \frac{\int_A \pi(dx) P(x, A^c)}{\pi(A)\pi(A^c)},\ 0 < \pi(A) < 1 \right\}$$

Notice that

$$\left(\frac{1}{8}\phi^2(P) \leq\right) \text{SpecGap}(P) \leq \phi(P)$$

Cheeger's Inequality.

So people normally lower bound conductance for Results on Spec Gap.

Rmks: ⊛ Combine minorization + drift to L.B. Spec gap.

⊛ Weaker concept on spectral gap.

⊛ Ind Metropolis.