SDP smax(X)

eigenvector SDP

$$\lambda_{max}(A) = \max A \bullet X$$

$$I \cdot X = 1$$

$$X \succeq 0$$

Create a smooth version

$$\text{smax}_\eta(A) = \max_{\substack{I \cdot X = 1 \\ X \succeq 0}} A \bullet X + \eta \underbrace{H(X)}_{\text{matrix entropy}}$$

$$H(X) = \mathrm{Tr}(X \log X)$$

$$= \sum_{i=1}^{n} \lambda_i \log \lambda_i$$

$$\lambda(X) = \lambda_n \geq \cdots \geq \lambda_2 \geq \lambda_1 \geq 0$$

Error: $\mathrm{smax}_m(A) \geq \lambda_{max}(A)$

$$\geq \mathrm{smax}_m(A) - n \log n$$

Smoothness: Still smooth but complicated to prove

$$\mathrm{argmax}\ \mathrm{smax}_m(A = X_m(A)$$

$$= \frac{e^{A/m}}{I \cdot e^{A/m}}$$

$$= \frac{U \begin{pmatrix} e^{\lambda_1/m} & & \\ & \ddots & \\ & & e^{\lambda_n/m} \end{pmatrix} U^T}{\sum_{i=1}^{n} e^{\lambda_i/m}}$$

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For $A = U \Lambda U'$.

Having this as the maximizer

$$\mathrm{smax}_\eta(A) = \eta \log \mathrm{Tr}\, e^{A/\eta}$$

$$= \eta \log \sum e^{\lambda_i/\eta}$$

$$= \mathrm{smax}_\eta(\vec{\lambda})$$

Best case: $\lambda_i = 0 \quad 1 \le i \le n-1$

Worst case: $\lambda_i = \lambda_j \quad \forall i, j.$

$$\mathrm{smax}_\eta(A + \Delta)$$

$$\mathrm{Tr}\left(e^{\frac{A+\Delta}{\eta}}\right)$$

look up $\boxed{\text{Trotter expansion}}$

Thrm: (Golden-Thompsen)

$$\operatorname{Tr}(e^{A+B}) \leq \operatorname{Tr}(e^{A} e^{B})$$

Caveat: $\operatorname{Tr}(e^{A+B+C}) \not\leq \operatorname{Tr}(e^{A} e^{B} e^{C})$

$$\operatorname{Smax}_{\eta}(A+\Delta) = \eta \log \operatorname{Tr}\left(e^{\frac{A+\Delta}{\eta}}\right)$$

$$\leq \eta \log \operatorname{Tr}\left(e^{A/\eta} e^{\Delta/\eta}\right)$$

$$e^{\Delta/\eta} \leq I + \left(e^{\frac{\|\Delta\|}{\eta}} - 1\right) \frac{\Delta}{\|\Delta\|}$$

$$\leq \eta \log\left(\operatorname{Tr}\left(e^{A/\eta}\right) + \frac{\left(e^{\|\Delta\|/\eta} - 1\right)}{\|\Delta\|} e^{A/\eta} \cdot \Delta\right)$$

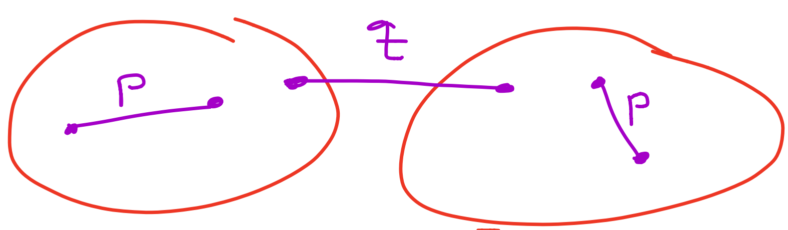
$$= m \log \mathrm{Tr}\, e^{\cdots}$$

$$+ m \log\left(1 + \frac{e^{\frac{\|\Delta\|}{m}}}{\|\Delta\|} \cdot \frac{e^{A/m}}{\mathrm{Tr}(e^{A/m})} \cdot \Delta\right)$$

$$= \mathrm{Smax}_m(A)$$

$$+ m \log\left(1 + \frac{e^{\frac{\|\Delta\|}{m}}}{\|\Delta\|} \cdot \frac{e^{A/m}}{\mathrm{Tr}(e^{A/m})} \cdot \Delta\right)$$

Stochastic Block Model



$|S| = \frac{n}{2}$ $\quad$ $|\bar{S}| = \frac{n}{2}$

Sample graph:

$$Y_{ij} \sim \mathrm{Bern}\left(p\, \mathbb{I}(i,j \in S, \bar{S}) + q\, \mathbb{I}(i \in S, j \in \bar{S})\right)$$

Goal: Recover $(S, \bar{S})$

Suppose $p > q$,

$$p = \frac{a}{\log n} \qquad q = \frac{b}{\log n}$$

$$\mathbb{E}[d_v] = \Theta\left(\frac{n}{\log n}\right)$$

Thrm: If $(\sqrt{a} - \sqrt{b})^2 > 2$, $S, \bar{S}$ can be recovered with prob. $\longrightarrow 1$.

If $(\sqrt{a} - \sqrt{b})^2 < 2$ it is not possible

Rmk: MLE is given by min bisection

Let A be the adjacency matrix

of G.

SDP: $\max \ A \cdot X$

$X_{ii} = \underline{1}$

$11^T \cdot X = 0$

$X \succeq 0$

Relaxation of

$$\max \sum A_{ij} \sigma_i \sigma_j$$

$$\text{s.t. } \sigma_i \in \{-1, 1\}^n$$

$$\sigma^T \underline{1} = 0$$