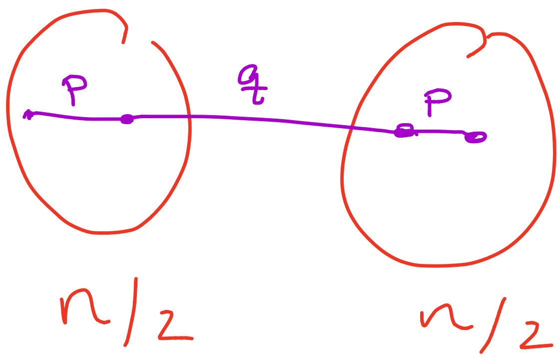
SBM



Goal: What are conditions on (p, q) that allow exact recovery of $(S, \bar{S})$

$$p = \alpha \frac{\log n}{n} \qquad q = \beta \frac{\log n}{n}$$

Q: $\min_{\delta} |\alpha - \beta| > \delta$ s.t. we get exact recovery?

Thrm: $\sqrt{\alpha} - \sqrt{\beta} < 2$ isn't exact recovery.

Thrm: When $\sqrt{\alpha} - \sqrt{b} > 2$, then the exact solution is recovered by an SDP whp.

$$\max A \cdot X, \quad X_{ii} = 1, \quad X \succeq 0$$

Rmk: This is the relaxation of

$$\max x^T A x \qquad x_i \in \{-1, 1\}$$

– max cut problem.

Roadmap: Construct a dual solution whp.

Let $g = 1_S - 1_{\bar{S}}$

$X = gg^T$ is feasible solution.

Dual SDP

Dual SDP

$$\min \sum_i z_i$$

$$\mathrm{diag}(z) \succeq A$$

$$z \in \mathbb{R}^n$$

Solve using complimentary slackness.

We will give a dual solution z such that

(i) z feasible & $Z - A \succeq 0$

(ii) $(Z - A) \bullet gg^T = 0$

(iii) $\lambda_2(Z - A) > 0$

Rmk: (Z, gg^T)

Z: primal sol

gg^T: dual sol.

A way to check if optimal

$$\mathrm{Val}(gg^T) = A \cdot gg^T \underset{d.f.}{\leq} \mathrm{diag}(z) \cdot gg^T$$

$$= \sum_{i=1}^{n} z_i (gg^T)_{ii} \underset{P.f.}{=} \sum z_i$$

$$= \mathrm{val}(z)$$

So only equal when

$$A \cdot gg^T \underset{d.f.}{=} \mathrm{diag}(z) \cdot gg^T$$

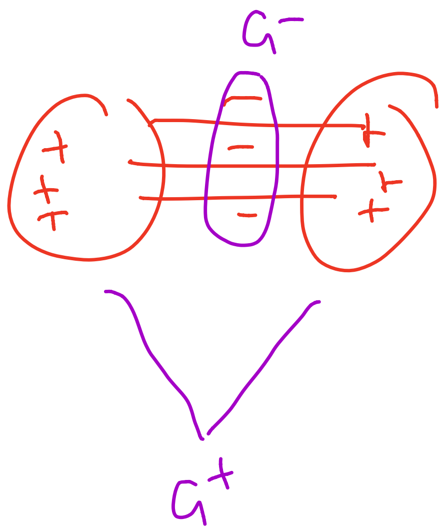
So when weak duality is tight we get optimality.

So

(ii) $\Rightarrow$ (z, gg^T) optimal

(iii) $\Rightarrow$ uniqueness of optimal.

Finding z



$$\min (D_+ - A_+) \cdot X - (D_- + A_-) \cdot X$$

$$= \min \Big[\overbrace{(D_+ - D_-) - \underbrace{A_+ - A_-}_{A}}^{L_{SBM}} \Big] \cdot X$$

$$I \cdot X = 1$$

$$L_{SBM} = \big[(D_+ - D_-) - A \big]$$

$$X^T L_{SBM} = \sum_{i,j \in G_+} (x_i - x_j)^2$$

$$- \sum_{i,j \in G_-} (x_i + x_j)^2$$

(i) $g^T L_{SBM}\, g = 0$

(goal: $Z - A = L_{SBM}$)

Expected Behavior L_{SBM}

$$x_{ij}^- = e_i - e_j$$

$$x_{ij}^+ = e_i + e_j$$

$$L_{SBM} = \sum_{G_+} x_{ij}^- x_{ij}^{-T}$$

$$- \sum_{G_-} x_{ij}^+ x_{ij}^{+T}$$

$$\mathbb{E}[L_{SBM}] = \sum_{S} p\, x_{ij}\, x_{ij}^{-T} +$$

$$\sum_{\bar{S}} p\, x_j^- x_j^{-T}$$

$$- \sum_{\substack{i \in S \\ j \in \bar{S}}} q\, x_{ij}^+ x_{ij}^{+T}$$

$$= pL(K_S) + pL(K_{\bar{S}})$$

$$- q \begin{array}{cc} \left[\begin{array}{c|c} \begin{matrix} n/2 & & \\ & \ddots & \\ & & n/2 \end{matrix} & 1 \\ \hline 1 & \begin{matrix} \ddots & & \\ & & \\ & & n/2 \end{matrix} \end{array}\right] \\ \begin{array}{cc} S \qquad & \qquad \bar{S} \end{array} \end{array}$$

$$= pL(K_S) + pL(K_{\bar{S}}) - nI + L(K_{S,\bar{S}})$$

$$= pL(K_V) + (p-q)L(K_{S,\bar{S}})$$

$$- qL(K_V) + q\,11^T - qnI$$

$$L(K_{S,\bar{S}}) = \frac{L(K_V) + gg^T}{2}$$

So

$$\mathbb{E}[L_{SBM}] = \frac{3}{2}(p-q)\,L(K_V) + (p-q)\,\frac{gg^T}{2}$$

$$- q\,11^T$$