Exercise 3.3.2 Let Y_n be the number of balls in urn A at time n. Now, we have three cases; (1) $Y_{n+1} = Y_n + 1$ when the ball is selected from B and urn A is selected (2) $Y_{n+1} = Y_n - 1$ when the ball is selected from A and urn B is selected, and finally (3) $Y_{n+1} = Y_n$ when the ball is selected from A and urn A is selected *or* the ball is selected from B and urn B is chosen. Choosing a ball from A at time n has probability Y_n/N and choosing a ball from B at time N happens with probability $1 - Y_n/N$. Using this, with the fact that the choice of ball and urn is independent we have

$$\mathbf{P}_{ij} = \begin{cases} (1-i/N)p & j = i+1 \\ (i/N)p + (1-i/N)q & j = i \\ (i/N)q & j = i-1 \\ 0 & |i-j| \geq 2 \end{cases}$$

Exercise 3.4.3 (a) Define $T = \min\{n : T \in \{0,3\}\}$ and let $u_i = \mathbb{P}[X_T = 0|X_0 = i]$. Our goal is to then find u_1. Trivially, $u_0 = 1$ and $u_3 = 0$. Moreover, using the law of total probability

$$u_1 = \mathbb{P}[X_T|X_0 = 1] = \sum_{j=0}^{3}\mathbb{P}[X_T|X_1 = j]P_{1j} = \sum_{j=0}^{3}u_jP_{1j} = P_{10} + P_{11}u_1 + P_{12}u_2$$

$$u_2 = \mathbb{P}[X_T|X_0 = 2] = \sum_{j=0}^{3}\mathbb{P}[X_T|X_1 = j]P_{2j} = \sum_{j=0}^{3}u_jP_{2j} = P_{20} + P_{21}u_1 + P_{22}u_2$$

Solving the second equation for u_2, we see that

$$u_2 = \frac{P_{20} + P_{21}u_1}{1 - P_{22}}$$

Now using the first equation to solve for u_1, we have

$$\begin{aligned} u_1 &= P_{10} + P_{11}u_1 + \frac{P_{20} + P_{21}u_1}{1 - P_{22}} \\ u_1\Big(1 - P_{11} - \frac{P_{12}P_{21}}{1 - P_{22}}\Big) &= P_{10} + \frac{P_{12}P_{20}}{1 - P_{22}} \\ u_1 &= \frac{P_{10}(1 - P_{22}) + P_{12}P_{20}}{(1 - P_{11})(1 - P_{22}) - P_{12}P_{21}} \end{aligned}$$

Finally, using the values from the transition matrix, we have $u_1 \approx 0.3809524$

(b) Let $v_i = \mathbb{E}[T|X_0 = i]$. Seeing that our chain starts in state 1, we seek to find v_1. First note, trivially, that $v_0 = v_3 = 0$. Focusing on the two transient states, we have

$$v_1 = \mathbb{E}[T|X_0 = 1] = \sum_{j=0}^{3}\mathbb{E}[T + 1|X_1 = j]P_{1j} = 1 + \sum_{j=0}^{3}v_jP_{1j} = 1 + P_{11}v_1 + P_{12}v_2$$

$$v_2 = \mathbb{E}[T|X_0 = 2] = \sum_{j=0}^{3}\mathbb{E}[T+1|X_1 = j]P_{2j} = 1 + \sum_{j=0}^{3} v_jP_{2j} = 1 + P_{21}v_1 + P_{22}v_2$$

Solving the second equation for v_2, we get

$$v_2 = \frac{1 + P_{21}v_1}{1 - P_{22}}$$

Plugging this into the first equation we see

$$\begin{aligned}
v_1 &= 1 + P_{11}v_1 + \frac{1}{1-P_{22}} + \frac{P_{21}}{1-P_{22}}v_1\\
v_1\Big(1 - P_{11} - \frac{P_{21}}{1-P_{22}}\Big) &= 1 + \frac{1}{1-P_{22}}\\
v_1 &= \frac{(1-P_{22}) + P_{12}}{(1-P_{22})(1-P_{11}) - P_{12}P_{21}}
\end{aligned}$$

Finally, using the quantities from the transition probability matrix, we have $v_1 \approx 3.33$.

Exercise 3.4.7 Let $\delta_i(X_n) = \mathbf{1}_{\{X_n = i\}}$ and define $T := \min\{n : X_T \in \{0,3\}\}$. Then the value $W_{ik} = \mathbb{E}\left[\sum_{t=0}^{T-1}\delta_i(X_n)|X_0 = k\right]$ is the expected time we visit state i given we start in state k. Our goal is to find W_{11} and W_{21}. First note that $W_{i0} = W_{i3} = 0$ for $i = 1, 2$. Now, by first step analysis we have that

$$\begin{aligned}
W_{11} &= \mathbb{E}\left[\sum_{t=0}^{T-1}\delta_1(X_n)|X_0 = 1\right] = 1 + \mathbb{E}\left[\sum_{t=1}^{T-1}\delta_1(X_n)|X_0 = 1\right]\\
&= 1 + \sum_{j=0}^{3}\mathbb{E}\left[\sum_{t=1}^{T-1}\delta_1(X_n)|X_1 = j\right]P_{1j} = 1 + \sum_{j=0}^{3}W_{1j}P_{1j}\\
&= 1 + W_{11}P_{11} + W_{12}P_{12}
\end{aligned}$$

$$\begin{aligned}
W_{12} &= \mathbb{E}\left[\sum_{t=0}^{T-1}\delta_1(X_n)|X_0 = 2\right] = \mathbb{E}\left[\sum_{t=1}^{T-1}\delta_1(X_n)|X_0 = 2\right]\\
&= \sum_{j=0}^{3}\mathbb{E}\left[\sum_{t=1}^{T-1}\delta_1(X_n)|X_1 = j\right]P_{2j} = \sum_{j=0}^{3}W_{1j}P_{2j}\\
&= W_{11}P_{21} + W_{12}P_{22}
\end{aligned}$$

Now, solving the second equation for $W_{12} = \frac{P_{21}}{1-P_{22}}W_{11}$. Using this in our first equation, we arrive at the following.

$$\begin{aligned} W_{11} &= 1 + W_{11}P_{11} + \frac{P_{21}P_{12}}{1-P_{22}}W_{11} \\ W_{11}\Big(1 - P_{11} - \frac{P_{21}P_{12}}{1-P_{22}}\Big) &= 1 \\ W_{11} &= \frac{(1-P_{22})}{(1-P_{11})(1-P_{22}) + P_{21}P_{12}} \end{aligned}$$

Using the transition probability matrix, we then see that $W_{11} \approx 1.8182$. Proceeding in the same way as above, we have the following two equations

$$\begin{aligned} W_{21} &= \mathbb{E}\left[\sum_{t=0}^{T-1}\delta_2(X_n)|X_0 = 1\right] = \mathbb{E}\left[\sum_{t=1}^{T-1}\delta_2(X_n)|X_0 = 1\right] \\ &= \sum_{j=0}^{3}\mathbb{E}\left[\sum_{t=1}^{T-1}\delta_2(X_n)|X_1 = j\right]P_{1j} = \sum_{j=0}^{3}W_{2j}P_{1j} \\ &= W_{21}P_{11} + W_{22}P_{12} \end{aligned}$$

$$\begin{aligned} W_{22} &= \mathbb{E}\left[\sum_{t=0}^{T-1}\delta_2(X_n)|X_0 = 2\right] = 1 + \mathbb{E}\left[\sum_{t=1}^{T-1}\delta_2(X_n)|X_0 = 2\right] \\ &= 1 + \sum_{j=0}^{3}\mathbb{E}\left[\sum_{t=1}^{T-1}\delta_2(X_n)|X_1 = j\right]P_{2j} = 1 + \sum_{j=0}^{3}W_{2j}P_{2j} \\ &= 1 + W_{21}P_{21} + W_{22}P_{22} \end{aligned}$$

Solving the second equation, we see that $W_{22} = \frac{1+W_{21}P_{21}}{1-P_{22}}$. Now using this in the first equation we see the following.

$$\begin{aligned} W_{21} &= W_{21}P_{11} + \frac{P_{12}}{1-P_{22}} + \frac{P_{12}P_{21}}{1-P_{22}}W_{21} \\ W_{21}\Big(1 - P_{11} - \frac{P_{12}P_{21}}{1-P_{22}}\Big) &= \frac{P_{12}}{1-P_{22}} \\ W_{21} &= \frac{P_{12}}{(1-P_{11})(1-P_{22}) - P_{12}P_{21}} \end{aligned}$$

Using the values from the transition probability matrix, we have $W_{21} \approx 2.2728$.

To find the time until absorption let $w_i = \mathbb{E}[T|X_0 = i]$. Then we have $w_0 = w_3 = 0$ and moreover

$$w_1 = \mathbb{E}[T|X_0 = 1] = \sum_{j=0}^{3}\mathbb{E}[T+1|X_1 = j]P_{1j} = 1 + P_{11}w_1 + P_{12}w_2$$

$$w_2 = \mathbb{E}[T|X_0 = 2] = \sum_{j=0}^{3} \mathbb{E}[T+1|X_1 = j]P_{2j} = 1 + P_{21}w_1 + P_{22}w_2$$

Now, solving the first equation for w_2, we see that $w_2 = \frac{1}{1-P_{22}} + \frac{P_{21}}{1-P_{22}}w_1$. Using this in the first equation we have

$$\begin{aligned} w_1 &= 1 + P_{11}w_1 + \frac{P_{12}}{1-P_{22}} + \frac{P_{12}P_{21}}{1-P_{22}}w_1 \\ w_1\Big(1 - P_{11} - \frac{P_{12}P_{21}}{1-P_{22}}\Big) &= 1 + \frac{P_{12}}{1-P_{22}} \\ w_1 &= \frac{(1-P_{22}) + P_{12}}{(1-P_{11})(1-P_{22}) - P_{12}P_{21}} \end{aligned}$$

Now notice that from above,

$$w_1 = \frac{1-P_{22}}{(1-P_{11})(1-P_{22}) - P_{12}P_{21}} + \frac{P_{12}}{(1-P_{11})(1-P_{22}) - P_{12}P_{21}} = W_{11} + W_{21}$$

Numerically, we see that $w_1 \approx 4.091 \approx 2.2728 + 1.8182$

Problem 3.2.4 First note that $Z_n = (X_{n-1}, X_n)$ and $Z_{n+1} = (X_n, X_{n+1})$ so any transition of the form $(a,b) \to (c,d)$ will have zero probability if $b \neq c$ and positive probability for $b = c$. In particular, this chain can be characterized at considered the transition probabilities for $(a,b) \to (b,c)$ by considering the probability transition matrix of X corresponding to $b \to c$. With this, we arrive at the transition matrix as follows

$$\mathbf{P} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ \alpha & 1-\alpha & 0 & 1 \\ 0 & 0 & 1-\beta & \beta \end{pmatrix} \end{array}$$

Problem 3.3.8 Suppose there are two urns, A and B, and at time t the number of balls in A is k and the number of balls in B is $N-k$. First an urn is selected with probability $p_A = \frac{k}{N}$ and $p_B = 1 - \frac{k}{N}$. Then a ball is selected from either A with probability p or B with probability q and placed in the urn selected above. Let Y_n be the number of balls in urn A at time n. Then using the above we have that $Y_{n+1} = Y_n + 1$ if urn A is chosen and we select a ball from urn B. $Y_{n+1} = Y_n - 1$ if we select urn B then a ball from urn A. Finally, $Y_{n+1} = Y_n$ if we select from urn A and a ball from A or select B and a ball from B. This corresponds to the following transition matrix

$$P_{ij} = \begin{cases} \frac{i}{N}q & j = i+1 \\ \frac{i}{N}p + (1 - \frac{Y_n}{N})q & j = i \\ (1 - \frac{i}{N})p & j = i-1 \\ 0 & |i-j| \geq 2 \end{cases}$$

Problem 3.4.6 First, define $T = \min\{n : X_n = 4\}$ and let $v_i = \mathbb{E}[T|X_0 = i]$. Our goal to find v_0.First note that trivially we have $v_4 = 0$. Moreover, we have

$$v_0 = \mathbb{E}[T|X_0 = 0] = 1 + \sum_{j=0}^{4}\mathbb{E}[T|X_1 = j]P_{0j} = 1 + \sum_{j=0}^{4} v_jP_{0j} = qv_0 + pv_1 + 1$$
$$v_1 = \mathbb{E}[T|X_0 = 1] = 1 + \sum_{j=0}^{4}\mathbb{E}[T|X_1 = j]P_{1j} = 1 + \sum_{j=0}^{4} v_jP_{1j} = qv_0 + pv_2 + 1$$
$$v_2 = \mathbb{E}[T|X_0 = 2] = 1 + \sum_{j=0}^{4}\mathbb{E}[T|X_1 = j]P_{2j} = 1 + \sum_{j=0}^{4} v_jP_{2j} = qv_0 + pv_3 + 1$$
$$v_3 = \mathbb{E}[T|X_0 = 3] = 1 + \sum_{j=0}^{4}\mathbb{E}[T|X_1 = j]P_{3j} = 1 + \sum_{j=0}^{4} v_jP_{3j} = qv_0 + 1$$

Now seeing $p = 1 - q$, $v_0 = qv_0 + pv_1 + 1$ implies $v_0 = v_1 + \frac{1}{p}$. Using this relation recusively, we have

$$\begin{aligned} v_0 &= v_1 + \frac{1}{p} = qv_0 + pv_2 + 1 + \frac{1}{p} = v_2 + \frac{1}{p} + \frac{1}{p^2} = qv_0 + pv_3 + 1 + \frac{1}{p} + \frac{1}{p^2} \\ &= v_3 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} = qv_0 + 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} \\ &= \sum_{k=1}^{4}\frac{1}{p^k} \end{aligned}$$

Problem 3.4.17 Let $T = \min\{n : X_n = 2\}$ be the time of the first failure and define $\phi_i(s) := \mathbb{E}[s^T|X_0 = i]$. Given that our system starts in state 0 (fully operational), we look to evaluate the probability generating function $\phi_0(s)$ for $0 < s < 1$. First note, that trivially $X_0 = 2$ implies $T = 0$ and $\phi_2(s) = 1$. Using this, we can use first step analysis to write

$$\phi_0(s) = \mathbb{E}[s^T|X_0 = 0] = \sum_{j=0}^{2}\mathbb{E}[s^{T+1}|X_1 = j]P_{0j} = s\sum_{j=0}^{2}\phi_j(s)P_{0j} = s\left[P_{00}\phi_0(s) + P_{01}\phi_1(s)\right]$$
$$\phi_1(s) = \mathbb{E}[s^T|X_0 = 1] = \sum_{j=0}^{2}\mathbb{E}[s^{T+1}|X_1 = j]P_{1j} = s\sum_{j=0}^{2}\phi_j(s)P_{1j} = s\left[P_{11}\phi_1(s) + P_{12}\right]$$

Solving the second equation for $\phi_1(s)$, we have $\phi_1(s) = \frac{sP_{12}}{1-sP_{11}}$. Using this in the first equation, we have

$$\begin{aligned}
\phi_0(s) &= sP_{00}\phi_0(s) + sP_{01}\Big(\frac{sP_{12}}{1 - sP_{11}}\Big)\\
\phi_0(s) &= \frac{s^2P_{01}P_{12}}{(1 - sP_{11})(1 - sP_{00})}
\end{aligned}$$

Nowing using the values from the transition probability matrix, we arrive at the following form.

$$\phi_0(s) = \frac{.12s^2}{(1 - .6s)(1 - .7s)}$$