

Sparsification

$$u_i \in \mathbb{R}^n \quad \sum_{i=1}^m u_i u_i^T = I$$

$$\left(\text{Recall this came from} \right. \\ \left. \sum L^{-1/2} x_e x_e^T L^{-1/2} = I \right)$$

Want: $\sum_{i=1}^m \underset{\substack{| \\ \text{Sparse}}}{w_i} u_i u_i^T \approx I$

Strategy: Oblivious sampling at
each iteration $1 \rightarrow T$

- Sample iid. a vector i_t

$$\text{s.t. } P(i_t = i) = P_i$$

- add vector $w_{i_t} = \frac{1}{P_{i_t}}$

w/ weight.

...t

Rml: $\sum \|u_i\|^2 = n$

Thm: If $p_i = \frac{\|u_i\|^2}{n}$ then after

$$T = O\left(\frac{n \log n}{\epsilon^2}\right) \text{ samples}$$

$$\left\| \frac{1}{T} \sum_{t=1}^T w_{i_t} u_{i_t} u_{i_t}^T - I \right\|_2 \leq \epsilon$$

w.c.p.

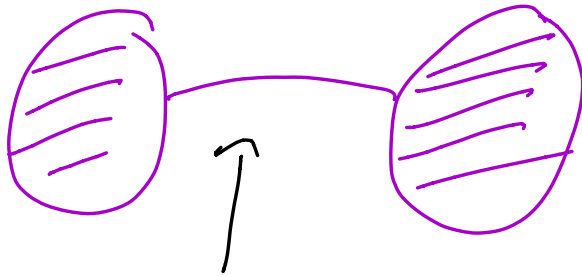
(Matrix Chernoff Bound)

Rmk: $\frac{n \log n}{\epsilon^2}$ optimal for oblivious sampling.

Rmk: Adaptive sampling can achieve

$$O(n/\epsilon^2)$$

Problem w/ Oblivious Samp



e has to be
sampled

For oblivious
 $p_e \approx \frac{1}{n}$

So we need roughly $n \log n$
samples to sample this edge.

Pf:

Claim 1: $\mathbb{E} (w_{it} u_{it} u_{it}^T)$

$$= \sum_{i=1}^m p_i \cdot \frac{1}{p_t} u_i u_i^T = \sum_{i=1}^m u_i u_i^T = \underline{I}$$

Laplacian transform potential function.

$$A_t = \sum_{s=1}^t w_{is} v_{is} u_{is}^T$$

$$\begin{aligned} \lambda_{\max}(A_t) &\leq \frac{1}{n} \log \operatorname{Tr} e^{n A_t} \\ &= \frac{1}{n} \log \left(\sum e^{n \lambda_i(A_t)} \right) \end{aligned}$$

$$\mathbb{I}_t := \mathbb{E} \left[\operatorname{Tr} (e^{n A_t}) \right]$$

Potential Change.

$$\mathbb{E}_{t+1} [\mathbb{I}_{t+1}] = \mathbb{E}_{t+1} [\operatorname{Tr} e^{n A_{t+1}}]$$

$$= \mathbb{E}_{t+1} [\operatorname{Tr} e^{n(A_t + \Delta_{t+1})}] \quad \Delta_{t+1} = w_{i_{t+1}} v_{i_{t+1}} u_{i_{t+1}}^T$$

Using Golden Inequalities:

for X, Y symmetric

$$1 \quad \operatorname{Tr}(e^{X+Y}) \leq \operatorname{Tr}(e^X e^Y).$$

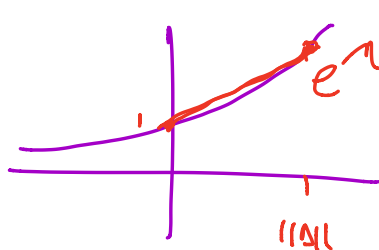
$$\underbrace{\quad \quad \quad}$$

$$\leq \text{Tr} \left(\underbrace{e^{mA_t}}_{\mathcal{F}_t \text{ measurable}} \mathbb{E}_{t+1} (e^{m\Delta_{t+1}}) \right)$$

Thinking of the trace operator as an inner product.

$$= e^{mA_t} \bullet \mathbb{E}_{t+1} [e^{m\Delta_{t+1}}]$$

Claim: $e^{m\Delta} \preceq I + (e^{m\|\Delta\|} - 1) \frac{\Delta}{\|\Delta\|}$



$$e^{mx} \leq 1 + cx$$

$$0 \leq \Delta \leq \|\Delta\| I$$

So $c = (e^{m\|\Delta\|} - 1)$ gives

our result.

Claim: $A \succeq 0$ $B \preceq C$ $A \cdot B \preceq A \cdot C$

So back to our bound

$$\leq \text{Tr}(e^{nA_t}) + e^{nA_t} \cdot \mathbb{E} \left\{ \frac{\Delta}{\|\Delta\|} (e^{n\|\Delta\|} - 1) \right\}$$

So

$$\frac{\mathbb{E}_{t+1}[\Phi_{t+1}]}{\mathbb{I}_t} = 1 + \frac{e^{nA_t}}{\text{Tr}(e^{nA_t})} \cdot \frac{\Delta}{\|\Delta\|} (e^{n\|\Delta\|} - 1)$$

$$\|\Delta_{t+1}\| = \|w_{it} u_{it} n_{it}^T\|$$

$$= \left\| \frac{1}{p_i} u u^T \right\| = \left\| \frac{n}{\|u\|} u u^T \right\|$$

$$= \frac{n}{\|u\|^2} - \|u\|^2 = n$$

and hence $\max_t \|\Delta_{t+1}\| = n$

So the bound becomes

$$\text{Tr}(\mathbb{I}_t) + e^{\eta A_t} \cdot \underbrace{\frac{F(\delta)}{n}}_{=1} (e^{\eta n} - 1)$$

$$= \text{Tr}(\mathbb{I}_t) + e^{\eta A_t} (e^{\eta n} - 1)$$

So

$$\frac{F_{t+1}[\mathbb{I}_{t+1}]}{\mathbb{I}_t} = 1 + \frac{e^{\eta A_t}}{\text{Tr}(e^{\eta A_t})} \cdot F\left[\frac{\eta}{n}(e^{\eta n} - 1)\right]$$

$$\leq 1 + \left(\frac{e^{\eta n} - 1}{n}\right)$$

Recall: $\lambda_{\max}(A) \leq \frac{1}{\eta} \log E \text{Tr}(e^{\eta A})$

$$\mathbb{E}[\mathbb{I}_T] \leq \left(1 + \frac{e^{Mn} - 1}{n}\right)^T \mathbb{I}_0$$

$$= \left(1 + \frac{e^{Mn} - 1}{n}\right)^T n$$

$$\left[\text{Tr}(e^{M0}) = \text{Tr}(I) = n \right]$$

$$\mathbb{P}(\lambda_{\max}(A_+) > T(1+\epsilon))$$

$$\leq \mathbb{P}\left\{ \mathbb{I}_T > e^{n(1+\epsilon)T} \right\}$$

$$\leq \frac{\left(1 + \frac{e^{Mn} - 1}{n}\right)^T n}{e^{n(1+\epsilon)T}}$$

$$\text{Take } \eta = \frac{\epsilon}{2n}$$

$$T > n \log n / \epsilon^2$$

$$\therefore \epsilon/2 \searrow^T$$

$$T \searrow^{\epsilon/2}$$

$$= \frac{(1 + \frac{1}{n})^n}{e^{\frac{\epsilon/2 T/n(1+\epsilon)}}} \leq \frac{e^{-n}}{e^{\frac{\epsilon/2 T/n(1+\epsilon)}}}$$

$$= e^{\frac{\epsilon}{2} \frac{T}{n} (1 - 1 - \epsilon + \log n)}$$

So $T = c \frac{n \log n}{\epsilon^2}$ where

c controls the probability. □

So we know

$$P_e \propto \frac{\|L^{-1/2} \chi_e\|^2}{n} = \frac{\chi_e^T L^{-1} \chi_e}{n}$$

effective resistance
of edge e .