Bayesian Exam Study Guide

$\mathbb{P}(\theta|X) \propto \mathbb{P}(X|\theta)\mathbb{P}(\theta)$

- Summarizing posterior

  $\hat{\theta} = \mathbb{E}[\theta|X]$ $\quad$ $\hat{\theta}_{MAP} = \underset{\Theta}{\operatorname{argmax}}\ \mathbb{P}(\theta|X)$

- Interval summaries

  $(F^{-1}_{\alpha/2}, F^{-1}_{1-\alpha/2})$ "credibility interval"

  $\{\theta : \mathbb{P}(\theta|X) \geq k_\alpha\}$ $\quad$ $k_\alpha : \mathbb{P}(\{\theta : \mathbb{P}(\theta|X) \geq k_\alpha\}) = 1-\alpha$

  "Highest posterior density" credibility interval

$$\boxed{\begin{array}{l} \bullet\ X|\theta \sim \text{Binom}(n,\theta) \quad \theta \sim \text{Beta}(\alpha,\beta) \\ \theta|X \sim \text{Beta}(X+\alpha,\ n-X+\beta) \end{array}}$$

- Noninformative Priors

Jeffrey's Prior: invariant to reparameterization

$$\mathbb{P}(\theta) \propto I(\theta)^{1/2} \qquad I(\theta) = \mathbb{E}_{X|\theta}\left[\left(\frac{\partial \ell}{\partial \theta}\right)^2\right]$$

$$= -\mathbb{E}_{X|\theta}\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]$$

Ex: $X|\theta \sim \text{Binom}(n, \theta)$ ~~Pf~~ $\theta \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$

Examples from Exp Families

(a) $X_i | \mu \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 known

$\mu \sim N(\mu_0, \tau_0^2)$

$$\boxed{\begin{array}{l} \mu | X \sim N\left(\underbrace{\dfrac{\mu_0/\tau_0^2 + \bar{x}/\sigma^2/n}{1/\tau_0^2 + 1/\sigma^2/n}}_{\mu_1}, \underbrace{\dfrac{1}{1/\tau_0^2 + 1/\sigma^2/n}}_{\tau_1^2} \right) \\ \tilde{X} | X \sim N(\mu_1, \tau_1^2 + \sigma^2) \end{array}}$$

(b) $X_i | \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$ μ known

$\sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \tau_0^2)$ $\left[(\sigma^2)^{-(\nu_0/2 - 1)} \exp\left(\dfrac{-\nu_0 \tau_0^2}{2\sigma^2} \right) \right]$

$$\boxed{\sigma^2 | X \sim \text{Inv-}\chi^2\left[n + \nu_0, \frac{S(x) + \nu_0 \tau_0^2}{n + \nu_0} \right]}$$

(c) $Y_i | \theta \overset{ind}{\sim} Po(x_i \theta)$ x_i exposures

$\theta \sim \text{Gamma}(\alpha, \beta)$ $\left[\theta^{\alpha - 1} e^{-\beta\theta} \right]$

$$\boxed{\begin{array}{l} \theta | Y \sim \text{Gamma}(\alpha + n\bar{y}, \beta + n\bar{x}) \\ \tilde{y} | Y \sim NB\left(\alpha + n\bar{y}, \dfrac{\tilde{x}}{\tilde{x} + \beta + n\bar{x}} \right) \end{array}}$$

Multiparameter priors

$$X_1, \dots, X_n \mid \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\mu \mid \sigma^2 \sim N(\mu_0, \sigma^2/k_0)$$

$$\sigma^2 \sim \text{Inv-}\chi^2(v_0, \sigma_0^2)$$

$$\mu, \sigma^2 \mid y \sim N\text{-Inv-}\chi^2(\mu_n, \sigma_n^2/k_n; v_n, \sigma_n^2)$$

$$\mu_n = \frac{k_0\mu_0 + n\bar{y}}{k_0 + n}$$

$$k_n = k_0 + n$$

$$v_n = v_0 + n$$

$$v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)s^2 + \frac{k_0 n}{k_0 + n}(\bar{y} - \mu_0)^2$$

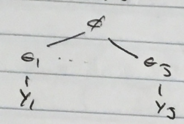
Laplace Approx

$$\theta \mid X \approx N(\theta^*, I_{obs}^{-1}(\theta^*))$$

$$\theta^* = \underset{\Theta}{\arg\max}\, P(\theta \mid X)$$

$$I_{obs}(\theta^*) = \frac{\partial^2 \ell(\theta)}{\partial \theta^2}\Big|_{\theta = \theta^*}$$

Hierarchical Models



$\mathbb{P}(\theta, \phi \mid Y) \propto \mathbb{P}(Y \mid \theta)\, \mathbb{P}(\theta \mid \phi)\, \mathbb{P}(\phi)$

<u>Useful things to remember</u>

- Change of variables:

Let $X \sim f_X$ $\quad Y = h(X)$ with $h(\cdot)$ monotone

Then $f_Y(y) = f_X(h^{-1}(y)) \cdot \left| (h^{-1})'(y) \right|$

<u>Conjugate Distributions</u>

Binomial	$\text{Beta}(\alpha, \beta)$	$\text{Beta}(\alpha + x, \beta + n - x)$
Poisson	$\text{Gamma}(\alpha, \beta)$	$\text{Gamma}(n\bar{x} + \alpha, n + \beta)$
Geometric	$\text{Beta}(\alpha, \beta)$	$\text{Beta}(n + \alpha, n(\bar{x} - 1) + \beta)$
Exponential	$\text{Gamma}(\alpha, \beta)$	$\text{Gamma}(n + \alpha, n\bar{x} + \beta)$
Neg. Binomial	$\text{Gamma}(\alpha, \beta)$	$\text{Gamma}(\alpha + n\bar{x}, rn + \beta)$