Goal: Assess $\mathbb{P}(\theta|Y) = \dfrac{\mathbb{P}(Y|\theta)\mathbb{P}(\theta)}{\int \mathbb{P}(Y|\theta)\mathbb{P}(\theta)d\theta}$ — hard

Want samples from $\mathbb{P}(\theta|Y)$ use MCMC.

Ex: $Y_j|\theta_j \overset{ind}{\sim} \text{Binom}(n_j, \theta_j)$

$\theta_j|\alpha,\beta \overset{iid}{\sim} \text{Beta}(\alpha,\beta)$

$\mathbb{P}(\alpha,\beta) \propto (\alpha+\beta)^{-5/2}$

HW - did numerical integration

Here MCMC — want to compare

Gibbs

(i) $\mathbb{P}(\theta_j|\theta_{-j},\alpha,\beta,Y)$

(ii) $\mathbb{P}(\alpha|\theta,\beta,Y)$

(iii) $\mathbb{P}(\beta|\theta,\alpha,Y)$

$$\mathbb{P}(Y,\alpha,\beta,\theta) = \mathbb{P}(Y|\theta)\mathbb{P}(\theta|\alpha,\beta)\mathbb{P}(\alpha,\beta)$$

$$\propto \prod_{j=1}^{J} \theta_j^{Y_j}(1-\theta_j)^{n_j-Y_j} \prod_{j=1}^{J} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1}(1-\theta_j)^{\beta-1}(\alpha+\beta)^{-5/2}$$

(i) $\mathbb{P}(\theta_j|\theta_{-j},\alpha,\beta,Y) \propto \theta_j^{Y_j}(1-\theta_j)^{n_j-Y_j}\theta_j^{\alpha-1}(1-\theta_j)^{\beta-1} \sim \text{Beta}(Y_j+\alpha, n_j+\beta)$

(ii) $\mathbb{P}(\alpha|\theta,\beta,Y) \propto \left(\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right)^J \left[\prod_{j=1}^{J}\theta_j\right]^{\alpha}(\alpha+\beta)^{-5/2}$

(iii) $\mathbb{P}(\beta|\alpha,\theta,Y) \propto \left(\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right)^J \left[\prod_{j=1}^{J}1-\theta_j\right]^{\beta}(\alpha+\beta)^{-5/2}$

MH within Gibbs

Some MH-strategies (same for β)

(i) • Random Walk: $\alpha^{(t+1)}|\alpha^{(t)} \sim N(\alpha^{(t)}, \sigma_\alpha^2)$

• If we assume $\alpha \gg \beta$ $\quad \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \approx \alpha^{\beta}$, $\alpha+\beta \approx \alpha$

$\alpha \gg \beta$

so $\mathbb{P}(\alpha|\theta,\beta,Y) \approx \alpha^{\beta J-3/2} e^{-\alpha(-\sum \log\theta_j)} \sim \text{Gamma}(\beta J - \frac{3}{2}, -\sum \log\theta_j)$

(independent chain)

- Laplace Approx: (MALA)

  Approximate $\mathbb{P}(\alpha \mid \theta, \beta, Y)$ around $\alpha = \alpha^{(t)}$ → get a mean + variance that depends on $\alpha^{(t)}$ $N(\alpha^{(t)}, I_{obs}^{-1}(\alpha^{(t)}))$

  improved R.W. kernel.

- Trace plots, AR plots,
  - also plot autocorrelation for log-posterior.