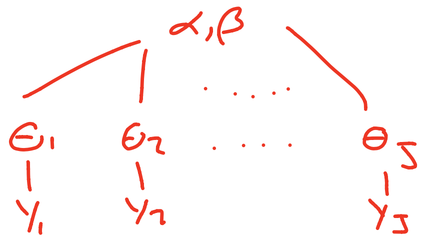


$Y_j \mid \theta_j \sim \text{Binom}(\theta_j, n_j)$

$\theta_j \mid \alpha, \beta \sim \text{Beta}(\alpha, \beta)$

Goal: $\mathbb{P}(\alpha, \beta \mid Y) \propto \prod_j \mathbb{P}(Y_j \mid \alpha, \beta)\, \mathbb{P}(\alpha, \beta)$

$$\mathbb{P}(Y_j \mid \alpha, \beta) = \int_0^1 \mathbb{P}(Y_j, \theta_j \mid \alpha, \beta)\, d(\theta_j) = \int_0^1 \mathbb{P}(Y_j \mid \theta_j)\, \mathbb{P}(\theta_j \mid \alpha, \beta)\, d\theta_j$$

$$= \int_0^1 \binom{n_j}{y_j} \theta^{Y_i} (1-\theta_j)^{n_j - y_j} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1} (1-\theta_j)^{\beta-1}\, d\theta_j$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \binom{n_j}{y_j} \int_0^1 \theta^{Y_i + \alpha - 1} (1-\theta_j)^{n_j - y_j + \beta - 1}\, d\theta_j$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \binom{n_j}{y_j} \times \frac{\Gamma(y_i+\alpha)\,\Gamma(n_j - y_i + \beta)}{\Gamma(y_j + \alpha + n_j - y_j + \beta)}$$

$$\propto \boxed{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(y_i+\alpha)\,\Gamma(n_j - y_i + \beta)}{\Gamma(y_j + \alpha + n_j - y_j + \beta)}}$$

Beta-Binomial

$$\propto \frac{\dfrac{\Gamma(\alpha + y_j)}{\Gamma(\alpha)} \cdot \dfrac{\Gamma(\beta + n - y_j)}{\Gamma(\beta)}}{\dfrac{\Gamma(\alpha + \beta + n_j)}{\Gamma(\alpha + \beta)}}$$

Using this

$$\mathbb{P}(\alpha,\beta|Y) \propto \prod_{j=1}^{n} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(y_j+\alpha)\Gamma(n_j-y_j+\beta)}{\Gamma(y_j+\alpha+n_j-y_j+\beta)} \mathbb{P}(\alpha,\beta)$$

<u>Rmk</u>: $X \sim Beta(\alpha,\beta)$ $\quad \mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$ $\quad Var(X) = \frac{\mu(1-\mu)}{\alpha+\beta-1}$

μ "sample size"

reparameterize in terms of this

One idea: weakly informative priors $\mathbb{P}\left(\frac{\alpha}{\alpha+\beta}, \sqrt{\alpha+\beta}\right) \propto 1$

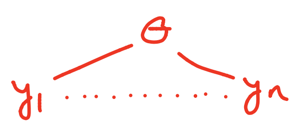
$$\Rightarrow \mathbb{P}(\alpha,\beta) \propto (\alpha+\beta)^{-5/2}$$

They also use a reparameterization in $r = logit\left(\frac{\alpha}{\alpha+\beta}\right)$

$s = \log(\alpha+\beta)$ and use $\mathbb{P}(r,s) \propto \alpha\beta(\alpha+\beta)^{-5/2}$

<u>Ex</u>: $Y_1,\ldots,Y_n|\theta \overset{ind}{\sim} N(\theta,\sigma^2)$

$\theta \sim N(\theta_0,\tau^2)$



$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \Big| \theta \sim N(\vec{\theta}, \sigma^2 I)$$

$$1_n \cdot \theta \sim N(\theta_0, \tau^2 11^T)$$

$$Y \sim N(\vec{\theta}_0, \sigma^2 I + \tau^2 11^T)$$

Rmk: Correlated data.