Recap

Symmetrized Conductance:

$$\bar{\phi}(S) = \frac{|E(S,\bar{S})|}{|Vol(S)\cdot Vol(\bar{S})|} \cdot Vol(V)$$

We argued

$$\bar{\phi}_G = \min_{S \subseteq V} \bar{\phi}(S) = \min_{S \subseteq V} \frac{|E(S,\bar{S})|}{|E_{K_G}(S,\bar{S})|}$$

$$= \min_{S \subseteq V} \frac{1_S^T L_G 1_S}{1_S^T L_{K_G} 1_S}$$

Relaxing cuts $\rightarrow$ all vectors

$$\geq \min_{x \in \mathbb{R}^n} \frac{x^T L_G x}{x^T L_{K_G} x} = \lambda_2$$

Today: $\bar{\phi}_G \leq \sqrt{2\lambda_2}$

Metrics metrics over the vertex set.

Def: A metric $d \in \mathbb{R}^{n \times n}$:

(i) $d_{ii} = 0 \quad i \in V$

(ii) $d_{ij} = d_{ji}$

(iii) $d_{ij} \leq d_{ik} + d_{kj} \quad \forall i, j, k \in V$

Ex: $x, y \in \mathbb{R}^k \quad d(x,y) = \|x - y\|_p$

Suppose $x_1, \ldots, x_n \in \mathbb{R}^k$

$d(x_i, x_j) = \|x_i - x_j\|_2^2$ not a metric in general.

A <u>semi-metric</u> is a metric when (iii) need not hold.

Then if we define

$$d(G) = \sum_{ij \in E} \underbrace{w_{ij} d_{ij}}_{\text{volume of the edge}}$$

Cut Metrics

Def: If $S \subseteq V$ the cut metric $\delta^{(S)}$

$$\delta_{ij}^{(S)} = \begin{cases} 0 & \text{same side of cut} \\ 1 & \text{diff side of cut} \end{cases}$$

Relation to Conductance

$$\delta^{(S)}(G) = |E(S, \bar{S})|$$

$$\delta^{(S)}(K_n) = \frac{Vol(S)\, Vol(\bar{S})}{Vol(V)}$$

$$\bar{\phi}(S) = \frac{\delta^{(S)}(G)}{\delta^{(S)}(K_G)}$$

$$\bar{\phi}_G = \min_{S \in cut} \frac{\delta(G)}{\delta(K_G)} \left(= \min_{S \subseteq V} \frac{1_S^T L 1_S}{1^T L(K_G) 1} \right)$$

<u>Relation to Spectral Gap</u>

$$\lambda_2 = \min_{x} \frac{\sum_{ij \in E} w_{ij} (x_i - x_j)^2}{\sum_{ij \in V} \frac{d_i d_j}{Vol(V)} (x_i - x_j)^2}$$

<u>Def:</u> A semi-metric g is <u>ℓ_2^2-embeddable</u> if there exists some embedding $v_i \in \mathbb{R}^\ell$ s.t.

$$g_{ij} = \|v_i - v_j\|_2^2 \quad \forall ij \in V$$

Claim: $$\lambda_2 = \min_{\substack{g \text{ is } \ell_2^2 \\ \text{embeddable}}} \frac{g(G)}{g(K_G)}$$

Lemma $$\frac{\sum a_i}{\sum b_i} \geq \min_i \frac{a_i}{b_i}$$

Pf (of claim):

$$\min_{g \in \ell_2^2} \frac{g(G)}{g(K)} = \frac{\sum_{i,j \in E} w_{ij} \|v_i - v_j\|^2}{\sum_{i,j \in V} \frac{d_i d_j}{\mathrm{Vol}(V)} \|v_i - v_j\|^2}$$

$$w_{ij} \|v_i - v_j\|^2 = \sum_{k=1}^{\ell} w_{ij} (v_i - v_j)_k^2$$

$$= \frac{\sum_{u=1}^{\ell} \sum_{ij \in E} w_{ij} (v_i - v_j)_u^2}{\sum_{u=1}^{\ell} \sum_{ij \in V} \frac{d_i d_j}{\mathrm{Vol}(V)} (v_i - v_j)_u^2}$$

$$\geq \min_{1 \leq u \leq \ell} \frac{\sum_{i,j \in E} w_{ij} (v_i - v_j)_u^2}{\cdots} \geq \lambda_2$$

$$\sum \frac{d_i d_j}{\mathrm{Vol}(V)} (v_i - v_j)^2$$

Plugging in x_2 then

$$\boxed{\min_{g \in \ell_2^2} \frac{g(G)}{g(K_n)} = \lambda_2}$$

__Rmk:__ A cut metric is ℓ_2^2 embeddable.

$$v_i = \begin{cases} 0 & i \in S \\ 1 & i \in \bar{S} \end{cases}$$

$$f_{ij}^{(S)} = |v_i - v_j|^2$$

__Where we are__

Cut metrics – discrete

ℓ_2^2 embeddable – continuous.

Cut Cone

Def: A set $A \in \mathbb{R}^n$ is a <u>cone</u> if

$$x \in A \implies \alpha x \in A \quad \forall \alpha > 0$$

Def A <u>convex cone</u> generated by a set of points $B = \{v_i \in \mathbb{R}^n\}$

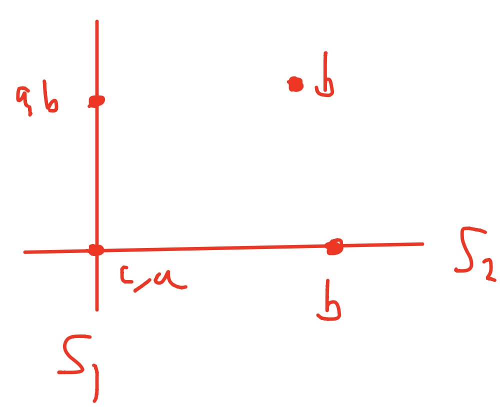
$$\text{Cone}(B) = \{x \in \mathbb{R}^n : x = \sum \alpha_i v_i,\ \vec{\alpha} > 0\}$$

Def: The <u>Cut Cone</u> is $\text{Cone}(\{\delta\}_{S \subseteq V})$

Ex: a, b, c

$S_1 = \{c\}$

$S_2 = \{a, c\}$



$\delta_1 + \delta_2$

Rmk. CUT_V — metrics that are embeddable in ℓ_1.

Def ℓ_1 embeddability: A metric d is ℓ_1 embeddable if $\exists \{v_i \in \mathbb{R}^K\}$ s.t. $d_{ij} = \|v_i - v_j\|_1$

Thrm: The set of ℓ_1-embeddable metrics are in CUT_V

(1) Cut Metrics $\delta^{(S)}$ in ℓ_1 embeddable

(2) Any ℓ_1 embeddable metric d is a conic combination of cut metrics

$$d_{ij} = \|v_i - v_j\|_1 = \sum_S \alpha_S \delta^{(S)}(i,j)$$

Claim It suffices to show this for one dimension

sweep cut $\quad S_t = \{1, 2, \dots, t\}$

$$d = \sum |v_{t+1} - v_t| \, \delta^{(S_t)}$$

$$d_{ij} = \sum_{k=i}^{j-1} |v_{k+1} - v_k| = v_i - v_j$$