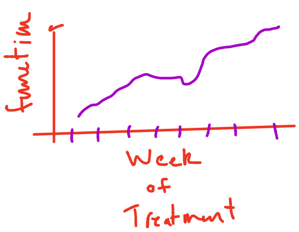
Hierarchical Linear Models

$$Y \mid \sigma^2, \beta = \begin{bmatrix} Y_1 \mid \beta, \sigma^2 \\ \vdots \\ Y_J \mid \beta, \sigma^2 \end{bmatrix} \sim N(X\beta, \sigma^2 I_n), \quad n = \sum_{j=1}^{J} n_j$$

$$\beta \mid \alpha, \tau^2 \sim N(X\alpha, \tau^2 I_{pJ})$$

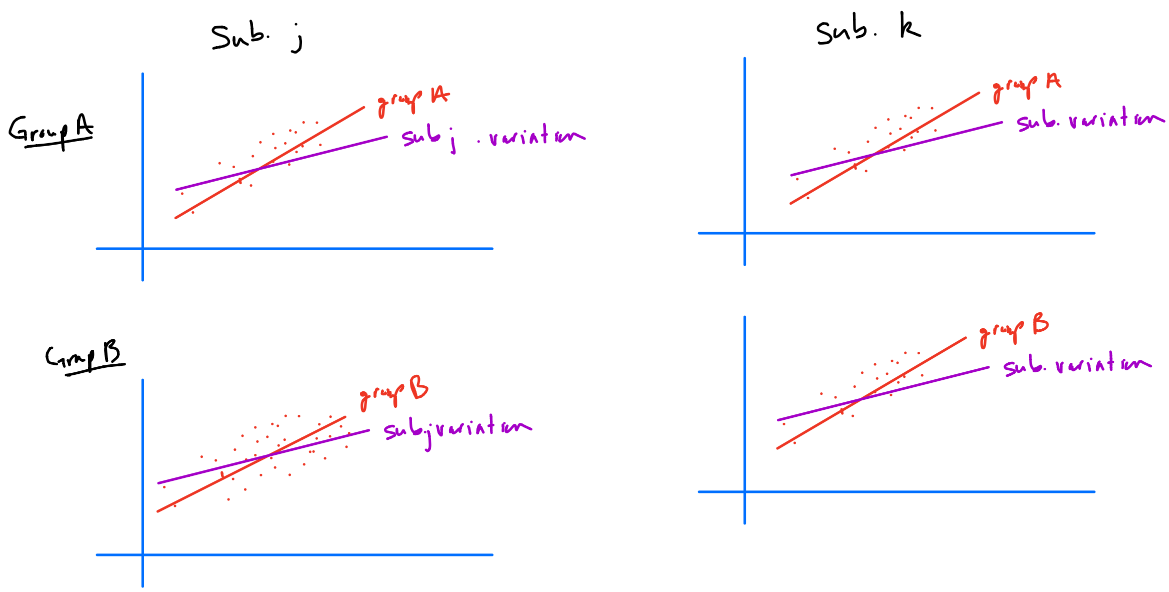
$$\mathbb{P}(\sigma^2, \alpha, \tau^2) \propto \frac{1}{\sigma^2} \cdot \frac{1}{\tau^2}$$

Ex: (Strokes)



$$Y_{ij} = \beta_{0,j} + \beta_{1,j} t_i + e_{ij} \qquad e_{ij} \sim N(0, \sigma^2)$$

week subject



$$\begin{bmatrix} \beta_{0,j} \\ \beta_{1,j} \end{bmatrix} \sim N\left(\begin{bmatrix} \alpha_{0,g(j)} \\ \alpha_{1,g(j)} \end{bmatrix}, \tau^2 I_2 \right)$$

Hypothesis Testing

Suppose we have a parameter $\theta \in \Theta$ and we want to assess if $\theta \in \Theta_0 \subseteq \Theta$

We hope to "test" $H_0: \theta \in \Theta$ against $H_1: \theta \in \Theta_1$ where

$\Theta_0 \cap \Theta_1 = \emptyset$ and $\Theta_0 \cup \Theta_1 \subseteq \Theta$

From a Bayesian perspective: infer $I(\theta \in \Theta_0)$

Formally, regarding this as a decision $\phi(y)$ that depends on data y.

This decision minimizes an expected loss. $L(\theta, \phi)$
"action"

$$\boxed{\phi(y) = \underset{\phi \in \{0,1\}}{\operatorname{argmin}} \; \mathbb{E}_{\theta \mid y}\left[L(\theta, \phi)\right]}$$

$\phi = \begin{cases} 0 & \text{reject} \\ 1 & \underline{\text{accept}} \end{cases}$ — okay here

$I(\theta \in \Theta_0)$

	Loss	0	1
ϕ	0	0	γ
	1	1	0

$$\phi(y) = \underset{\phi \in \{0,1\}}{\operatorname{argmin}}\ \mathbb{E}_{\theta|y}\left[\gamma(1-\phi) I(\theta \in \Theta_0) + \phi I(\theta \notin \Theta_0)\right]$$

$$= \underset{\phi \in \{0,1\}}{\operatorname{argmin}} \left[\gamma(1-\phi)\mathbb{P}(\theta \in \Theta_0 | y) + \phi I(\theta \notin \Theta_0 | y)\right]$$

$$= \underset{\phi \in \{0,1\}}{\operatorname{argmin}} \left[\gamma \mathbb{P}(\theta \in \Theta_1 | y) + \phi\left[\mathbb{P}(\theta \in \Theta_0 | y) - \gamma \mathbb{P}(\theta \in \Theta_0 | y)\right]\right]$$

$$= \begin{cases} 0 & \mathbb{P}(\theta \in \Theta_1 | y) - \gamma \mathbb{P}(\theta \in \Theta_0 | y) \geq 0 \\ 1 & \mathbb{P}(\theta \in \Theta_1 | y) - \gamma \mathbb{P}(\theta \in \Theta_0 | y) < 0 \end{cases}$$

$$= I\left[\mathbb{P}(\theta \in \Theta_1 | y) - \gamma \mathbb{P}(\theta \in \Theta_0 | y) \leq 0\right]$$

Instead compute

$$I\left[\frac{\mathbb{P}(\theta \in \Theta_1 | y)}{\mathbb{P}(\theta \in \Theta_0 | y)} \leq \gamma\right] = I\left[\mathbb{P}(\theta \in \Theta_0 | y) \geq \frac{1}{1+\gamma}\right]$$

Ex: $\gamma = 1$, type I and type II are the same value.

Ex: $Y|\theta \sim N(\theta, \sigma^2)$, $\theta \sim N(\mu, \tau^2)$ $\quad$ $H_0: \theta \leq 0$ $\quad$ $H_1: \theta > 0$

$$\theta | y \sim N\left(\hat{\theta} = \frac{y/\sigma^2 + \theta/\tau^2}{1/\sigma^2 + 1/\tau^2},\ w^2 = \frac{1}{\frac{1}{\sigma^2} + \frac{1}{w^2}}\right)$$

$$\mathbb{P}(\theta \in \Theta_0 | y) = \mathbb{P}(\theta \leq 0 | y) = \Phi\left(-\hat{\theta}/w\right)$$

Which gives the decision $I\left[\Phi(-\hat{\theta}/w) \geq \frac{1}{1+\gamma}\right] = I\left[\hat{\theta} \leq -w\Phi^{-1}\left(\frac{1}{1+\gamma}\right)\right]$