

3.9

$$P(\sigma^2 | y) \propto P(y | \sigma^2) P(\sigma^2) = \int P(y | \mu, \sigma^2) P(\mu | \sigma^2) d\mu P(\sigma^2)$$

$$= \int \prod_{i=1}^n P(y_i | \mu, \sigma^2) P(\mu | \sigma^2) d\mu P(\sigma^2)$$

$$\prod_{i=1}^n P(y_i | \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \mu)^2 \right) \right)$$

$$= \left(\frac{1}{2\pi\sigma^2} \right)^n \exp \left\{ -\frac{1}{2\sigma^2} ((n-1) s_y^2 + n (\bar{y} - \mu)^2) \right\}$$

$$= \left(\frac{1}{2\pi\sigma^2} \right)^n \exp \left\{ -\frac{1}{2\sigma^2} ((n-1) s_y^2) \right\} \exp \left\{ -\frac{n}{2\sigma^2} (\bar{y} - \mu)^2 \right\}$$

So

$$\int \prod_{i=1}^n P(y_i | \mu, \sigma^2) P(\mu | \sigma^2) d\mu P(\sigma^2)$$

$$= \left(\frac{1}{2\pi\sigma^2} \right)^n \exp \left\{ -\frac{1}{2\sigma^2} ((n-1) s_y^2) \right\} \underbrace{\int \exp \left\{ -\frac{1}{2\sigma^2 n} (\bar{y} - \mu)^2 \right\} P(\mu | \sigma^2) d\mu}_{\propto P(\bar{y} | \mu, \sigma^2)} P(\mu | \sigma^2) P(\sigma^2)$$

$$\Rightarrow P(\mu | \bar{y}, \sigma^2) \propto P(\bar{y} | \mu, \sigma^2) P(\mu | \sigma^2)$$

from part (a) we know

$\mu | \gamma, \sigma^2 \sim N($

[3.9]

$$\begin{aligned}
 & p(\sigma^2 | y) \propto \prod_{i=1}^n p(y_i | \sigma^2) p(\sigma^2) \\
 & = \int \prod_{i=1}^n p(y_i | \mu, \sigma^2) p(\mu | \sigma^2) p(\sigma^2) d\mu d\sigma^2 \\
 & = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \frac{1}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2} (\mu - \bar{y})^2\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2} (\mu - \bar{y})^2\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \mu)^2 + (\mu - \bar{y})^2 \right)\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \bar{y})^2 + \bar{y}^2 - 2\bar{y}\mu + \mu^2 \right)\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \bar{y})^2 + (\bar{y} - \mu)^2 \right)\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \bar{y})^2 + (\bar{y} - \mu)^2 \right)\right) \\
 & \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \bar{y})^2 + (\bar{y} - \mu)^2 \right)\right)
 \end{aligned}$$

3.12

(a) choose flat $p(\alpha, \beta) \propto 1$

(b) $\alpha = ?, \beta = -1$ if we know this

(c) $p(\alpha, \beta | y, t) \propto p(y | \alpha, \beta, t) p(\alpha, \beta)$

$$\propto p(\alpha, \beta) \prod_{i=1}^n p(y_i | \alpha, \beta, t_i)$$

$$\propto 1 \cdot e^{n\alpha + \beta \sum t_i} \prod_{i=1}^n (\alpha + \beta t_i)^{y_i}$$

$$\propto \exp \left\{ n\alpha + \beta \sum t_i + \underbrace{\sum_{i=1}^n y_i \log(\alpha + \beta t_i)} \right\}$$

4} (a)

[4a]

$$p(r^2 | \mu) \propto p(r^2 | \sigma^2) p(\mu^2)$$
$$\propto \left(\frac{1}{2\sigma^2} \exp\left(-\frac{1}{2\sigma^2}(r^2 - \mu^2)\right) \right)^{\frac{N-1}{2}} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{(r^2 - \mu^2)^2 + \nu_0 \sigma^2}{\nu_0 + 1}\right)\right)$$
$$\Rightarrow \sigma^2 | \mu \sim \text{Inv-} \chi^2_{\nu_0 + 1} \left(\frac{\nu_0 + 1}{\nu_0 + 1} \cdot \frac{(r^2 - \mu^2)^2 + \nu_0 \sigma^2}{\nu_0 + 1} \right)$$

[4c]

$$p(r^2 | x) = \int p(r^2 | \mu, \sigma^2) d\mu d\sigma^2$$
$$\propto \int p(x | \mu, \sigma^2) p(r^2 | \mu, \sigma^2) d\mu d\sigma^2$$
$$\propto \int (\sigma^2)^{N/2} \exp\left(-\frac{1}{2\sigma^2}(\mu^2 + \nu_0 \sigma^2)\right) \left(\frac{\sigma^2}{\nu_0}\right)^{N/2} \exp\left(-\frac{1}{2\sigma^2}(x^2 + \nu_0 \sigma^2)\right) d\mu d\sigma^2$$
$$\propto \int \left(\frac{\sigma^2}{\nu_0}\right)^{\frac{N+2}{2}} \exp\left(-\frac{1}{2\sigma^2}\left((\nu_0 x^2 + \nu_0 \sigma^2) + \frac{1}{\nu_0}(\mu^2 + \nu_0 \sigma^2)\right)\right) d\mu d\sigma^2$$
$$\propto \left(\frac{\sigma^2}{\nu_0}\right)^{\frac{N+2}{2}} \exp\left(-\frac{1}{2\sigma^2}\left((\nu_0 x^2 + \nu_0 \sigma^2) + \frac{1}{\nu_0}(\mu^2 + \nu_0 \sigma^2)\right)\right) d\sigma^2$$
$$= \left(\frac{\sigma^2}{\nu_0}\right)^{\frac{N+2}{2}} \exp\left(-\frac{1}{2\sigma^2}(\nu_0 x^2 + \nu_0 \sigma^2)\right) = \text{Inv-} \chi^2_{\nu_0 + 1}$$

BIA 3.12

$$Y_i \sim \text{Poisson}(\lambda) \quad \lambda_i = \exp\{\alpha + \beta t_i\}$$

$$\rho(\alpha, \beta) \propto 1.$$

$$\rho(\alpha, \beta | y) \propto \prod_{i=1}^n \exp(-\lambda_i) \lambda_i^{y_i} \rho(\alpha, \beta)$$

$$\propto \prod_{i=1}^n \exp\left(-\exp(\alpha + \beta t_i)\right) \exp(y_i(\alpha + \beta t_i))$$

$$\propto \exp\left\{-\sum_{i=1}^n \exp(\alpha + \beta t_i) + \alpha \sum y_i + \beta \sum y_i t_i\right\},$$

$$\ell(\alpha, \beta | t_i)$$