Cut Metrics

Approximating Alg. & Cut Cones

↳ Relaxation to cut metrics

V abstract metric space

$$\delta_{ij}^{(S)} = \begin{cases} 1 & \text{o.w.} \\ 0 & |\{i,j\} \cap S| \neq 1 \end{cases}$$

Can be embedded in a normed vector space.

$$x_{ij} = \begin{cases} 1 & \text{if} \quad i \in S \\ 0 & \text{if} \quad \text{o.w.} \end{cases}$$

$$\|x_i - x_j\|_1 = \delta_{ij}^{(S)}$$

Optimize ratios

$$\min_{S \subseteq V} \frac{\delta^{(S)}(G)}{\delta^{(S)}(K_n)} = \min \frac{\sum_{E} \delta_{ij}^{(S)}}{\sum_{E(K_n)} w_{ij}\, \delta_{ij}^{(S)}}$$

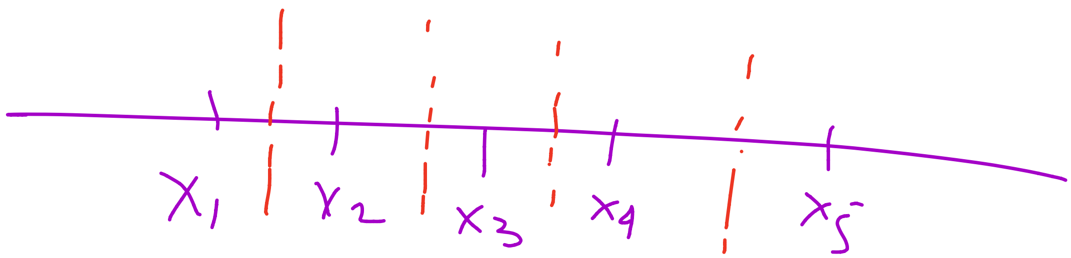
Equivalent to optimizing

Convex Cone

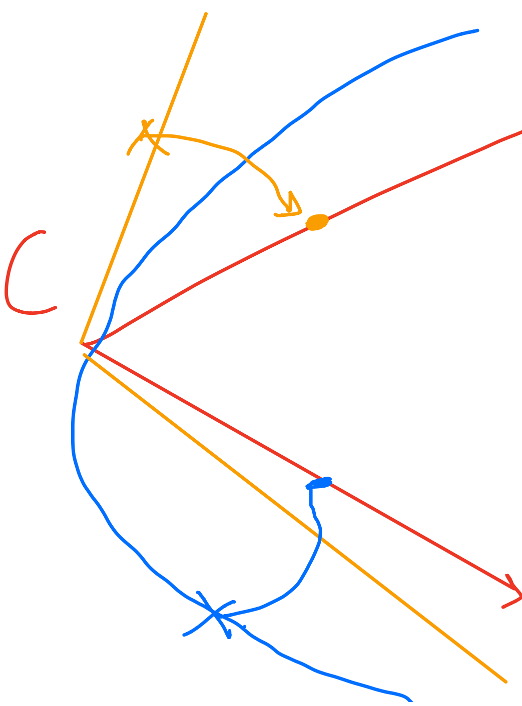
$$CUT_V = \left\{ \sum_{S \subseteq V} \alpha_S \delta^{(S)} : \alpha_S \geq 0 \right\}$$

<u>Thrm</u>: CUT_V is equal to the set of metrics over V that can

isometrically embeddable in ℓ_1.



$$\|x_1 - x_5\|_1 = |x_1 - x_2|\,\delta_{ij}^{\{1\}} + |x_2 - x_3|\,\delta_{ij}^{\{1,2\}}$$
$$+ |x_3 - x_4|\,\delta_{ij}^{\{1,2,3\}}$$
$$+ |x_4 - x_5|\,\delta_{ij}^{\{1-4\}}$$



cut ℓ_1

cone & metrics

spectral relaxation

Bourgain Thm

Any metric over $V, |V|=n$

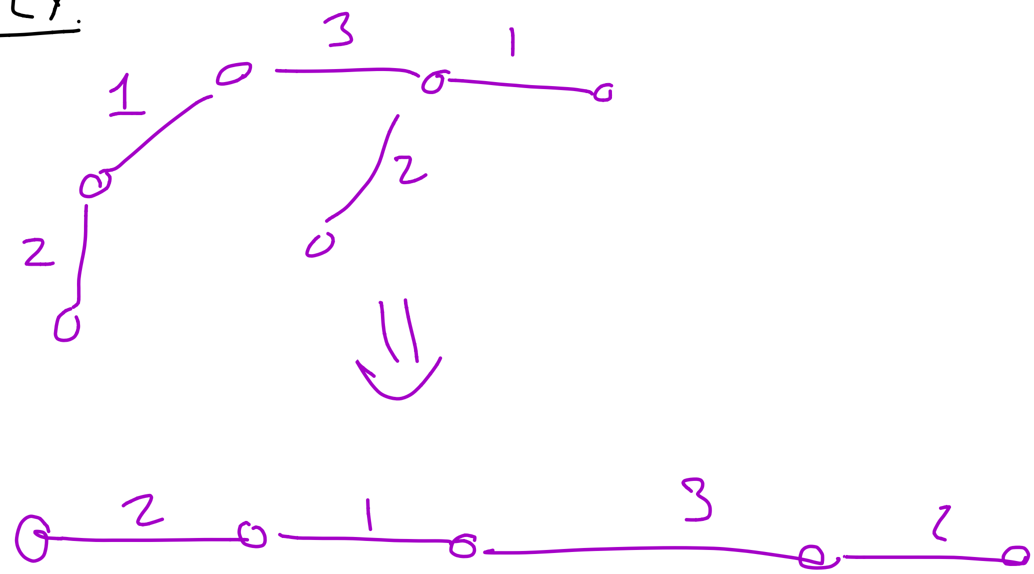
Can be embedded into ℓ_1

with distortion $O(\log n)$.

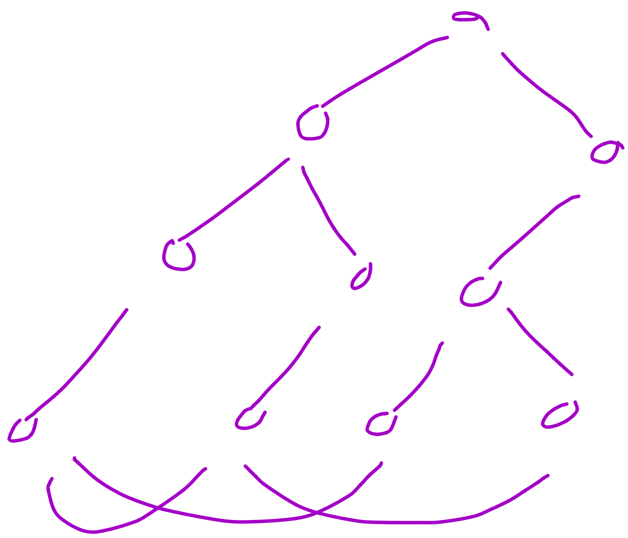
$\exists f: V \longmapsto \mathbb{R}^d$ s.t.

$$d_{ij} \le \|f(v_i)-f(v_j)\|_1 \le O(\log n)\, d_{ij}$$

Ex:



Tight w/ constant degree expander



Graph SDP's

$$\left.\begin{aligned} L_G \cdot X &\le a_1 \\ L_G \cdot X &\le a_2 \\ L_H \cdot X &\ge b_1 \end{aligned}\right\}$$

∃ a $(1+\varepsilon)$ approx. sol. X of rank $\frac{\log n}{\varepsilon^2}$

Computing low rank sol. to SDP's. Burer-Monteiro

$$X = \begin{bmatrix} Z^T \end{bmatrix} \begin{bmatrix} Z \end{bmatrix} \quad \mathrm{Tr}(Z^T L_1 Z) \le a_1,$$

$$\vdots$$

MaxCut SDP $\Rightarrow$ .879 approx to Maxcut.

Rmk: use BM approach with rank k you get a

$1 - \frac{1}{k}$ approx to SDP.