Sparsification

$u_i \in \mathbb{R}^n \qquad \sum_{i=1}^{m} u_i u_i^T = I$

(Recall this came from

$$\sum L^{-1/2} x_e x_e^T L^{-1/2} = I \quad)$$

Want: $\underbrace{\sum_{i=1}^{m} w_i u_i u_i^T}_{\text{Sparse}} \approx I$

Strategy: Oblivious sampling at each iteration $1 \to T$

- Sample i.i.d. a vector i_t
  s.t. $\mathbb{P}(i_t = i) = p_i$

- add vector $w_{i_t} = \frac{1}{p_{i_t}}$

w/ weight.

Rmk: $\sum_i \|u_i\|^2 = n$

Thm: If $p_i = \frac{\|u_i\|^2}{n}$ then after

$T = O\left(\frac{n \log n}{\varepsilon^2}\right)$ samples

$$\left\|\frac{1}{T}\sum_{t=1}^{T} w_{i_t} u_{i_t} u_{i_t}^T - I\right\|_2 \lesssim \varepsilon$$ w.c.p.

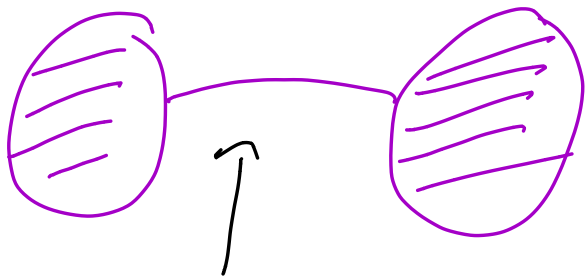
(Matrix Chernoff Bound)

Rmk: $\frac{n \log n}{\varepsilon^2}$ optimal for oblivious sampling.

Rmk: Adaptive sampling can achieve

$O(n/\varepsilon^2)$

Problem w/ Oblivious Samp



e has to be sampled

. For oblivious

$p_e \simeq \frac{1}{n}$

So we need roughly $n \log n$ samples to sample this edge.

Pf:

Claim 1: $\mathbb{E}\left(w_{i_t} u_{i_t} u_{i_t}^T\right)$

$$= \sum_{i=1}^{m} p_i \cdot \frac{1}{p_i} u_i u_i^T = \sum_{i=1}^{m} u_i u_i^T = I$$

Laplacian transform potential function.

$$A_t = \sum_{s=1}^{t} w_{i_s} u_{i_s} u_{i_s}^T$$

$$\lambda_{max}(A_t) \simeq \frac{1}{\eta} \log \operatorname{Tr} e^{\eta A_t}$$

$$= \frac{1}{\eta} \log \left(\sum e^{\eta \lambda_i (A_t)} \right)$$

$$\Phi_t := \mathbb{E}\left[\operatorname{Tr}\left(e^{\eta A_t}\right)\right]$$

Potential Change.

$$\mathbb{E}_{t+1}\left[\Phi_{t+1}\right] = \mathbb{E}_{t+1}\left[\operatorname{Tr} e^{\eta A_{t+1}}\right]$$

$$= \mathbb{E}_{t+1}\left[\operatorname{Tr} e^{\eta(A_t + \Delta_{t+1})}\right] \qquad \Delta_{t+1} = w_{i_{t+1}} u_{i_{t+1}} u_{i_{t+1}}^T$$

Using Golden Inequalities:

For X, Y symmetric

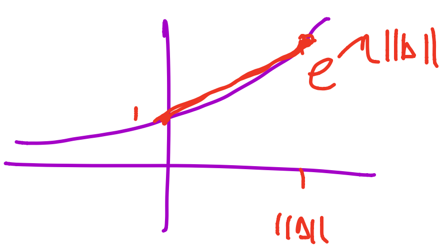
$$\operatorname{Tr}\left(e^{X+Y}\right) \leq \operatorname{Tr}\left(e^X e^Y\right).$$

$$\leq \mathrm{Tr}\Big(\underbrace{e^{\eta A_t}}_{\mathcal{F}_t \text{ measurable.}} \; \mathbb{E}_{t+1}\big(e^{\eta \Delta_{t+1}}\big)\Big)$$

Thinking of the trace operator as an inner product.

$$= e^{\eta A_t} \bullet \mathbb{E}_{t+1}\left[e^{\eta \Delta_{t+1}}\right]$$

<u>Claim</u>: $e^{\eta \Delta} \preceq I + (e^{\eta\|\Delta\|} - 1)\frac{\Delta}{\|\Delta\|}$



$e^{\eta x} \leq 1 + c x$

$0 \preceq \Delta \preceq \|\Delta\| I$

So $c = (e^{\eta\|\Delta\|} - 1)$ gives

our result.

<u>Claim:</u> $A \succeq 0 \quad B \preceq C \quad A \cdot B \le A \cdot C$

So back to our bound

$$\le Tr(e^{\eta A_t}) + e^{\eta A_t} \bullet \mathbb{E}\left\{ \frac{\Delta}{\|\Delta\|}\left(e^{\eta \|\Delta\|} - 1\right)\right\}$$

So

$$\frac{\mathbb{E}_{t+1}\left[\Phi_{t+1}\right]}{\Phi_t} = 1 + \frac{e^{\eta A_t}}{Tr(e^{\eta A_t})} \cdot \frac{\Delta}{\|\Delta\|}\left(e^{\eta\|\Delta\|} - 1\right)$$

$$\|\Delta_{t+1}\| = \|w_{i_t} u_{i_t} u_{i_t}^T\|$$

$$= \left\|\frac{1}{p_i} u u^T\right\| = \left\|\frac{n}{\|u\|} u u^T\right\|$$

$$= \frac{n}{\|u\|^2} \cdot \|u\|^2 = n$$

and hence $\max_t \|\Delta_{t+1}\| = n$

So the bound becomes

$$\mathrm{Tr}(\Phi_t) + e^{\eta A_t} \bullet \underbrace{\frac{E(X)}{n}}_{= I} (e^{\eta n} - 1)$$

$$= \mathrm{Tr}(\Phi_t) + e^{\eta A_t}(e^{\eta n} - 1)$$

So

$$\frac{E_{t+1}[\Phi_{t+1}]}{\Phi_t} = 1 + \frac{e^{\eta A_t}}{\mathrm{Tr}(e^{\eta A_t})} \bullet E_t\left[\frac{\Delta}{n}(e^{\eta n} - 1)\right]$$

$$\leq 1 + \left(\frac{e^{\eta n} - 1}{n}\right)$$

Recall: $\lambda_{max}(A) \leq \frac{1}{\eta} \log E\, \mathrm{Tr}(e^{\eta A})$

$$\mathbb{E}[\Phi_T] \le \left(1 + \frac{e^{\eta n} - 1}{n}\right)^T \Phi_0$$

$$= \left(1 + \frac{e^{\eta n} - 1}{n}\right)^T n$$

$$\boxed{\mathrm{Tr}\left(e^{\eta 0}\right) = \mathrm{Tr}(I) = n}$$

$$\mathbb{P}\left(\lambda_{max}(A_T) > T(1+\varepsilon)\right)$$

$$\le \mathbb{P}\left\{\Phi_T > e^{\eta(1+\varepsilon)T}\right\}$$

$$\le \frac{\left(1 + \frac{e^{\eta n} - 1}{n}\right)^T n}{e^{\eta(1+\varepsilon)T}}$$

Take $\eta = \frac{\varepsilon}{2n}$ $\qquad T > n \log n / \varepsilon^2$

$\left(1 + \varepsilon/2\right)^T \qquad T \underline{\varepsilon/2}$

$$= \frac{\left(1+\frac{\text{[illegible]}}{n}\right)^{n} n}{e^{\varepsilon/2\, T/n\,(1+\varepsilon)}} \leq \frac{e^{\text{[illegible]}}\; n}{e^{\varepsilon/2\, T/n\,(1+\varepsilon)}}$$

$$= e^{\frac{\varepsilon}{2}\frac{T}{n}\left(1-1-\varepsilon+\log n\right)}$$

So $T = c\,\dfrac{n \log n}{\varepsilon^2}$ where

c controlls the probability. $\square$

So we know

$$p_e \propto \frac{\| L^{-1/2} \chi_e \|^2}{n} = \frac{\chi_e^T L^{-1} \chi_e}{n}$$

effective resistence of edge e.