Example: Coin flip example

$$X \mid \Theta \sim \text{Binom}(n, \theta)$$

$$\Theta \sim \text{Beta}(\alpha, \beta)$$

$$\left.\begin{aligned} \mathbb{P}(\Theta) &\propto \Theta^{\alpha-1}(1-\theta)^{b-1} \\ \mathbb{P}(X \mid \Theta) &\propto \theta^{x}(1-\theta)^{n-x} \end{aligned}\right\} \text{Similar shapes}$$

$$\begin{aligned} \mathbb{P}(\Theta \mid X) &\propto \mathbb{P}(X \mid \Theta)\,\mathbb{P}(\Theta) \\ &\propto \Theta^{x+\alpha-1}(1-\theta)^{n-x+b-1} \end{aligned}$$

$$\Theta \mid X \sim \text{Beta}(x+\alpha, n-x+b)$$

So we see that Beta is in the conjugate family of Binom

Independent Chain MH

$$Q(\theta) = \mathbb{P}(\theta)$$

$$R(\theta, \theta^*) = \frac{\mathbb{P}(x \mid \theta^*)}{\mathbb{P}(x \mid \theta)}$$

$$= \frac{(\theta^*)^x (1-\theta^*)^{n-x}}{(\theta^{(t)})^x (1-\theta^{(t)})^{n-x}}$$

Random Walk

$$\theta^* = \theta^{(t)} + \delta \qquad \delta \sim \text{Unif}(-s, s)$$

$$R(\theta^*, \theta^{(t)}) = \frac{\mathbb{P}(\theta^* \mid x)}{\mathbb{P}(\theta^{(t)} \mid x)}$$

$$= \frac{(\theta^*)^{x+a-1} (1-\theta^*)^{n-x+b-1}}{(\theta^{(t)})^{x+a-1} (1-\theta^{(t)})^{n-x+b-1}}$$

__Example:__

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \Sigma \right)$$

__RW:__ $x^* = x^{(t)} + \delta, \quad \delta \sim N(0, \sigma_{rw}^2 I_2)$

$$R(x^*, x^{(t)}) = \frac{\mathbb{P}(x^*)}{\mathbb{P}(x^{(t)})}$$

$$= \frac{\exp\left\{ -\frac{1}{2}(x^* - \mu)^T \Sigma^{-1} (x^* - \mu) \right\}}{\exp\left\{ -\frac{1}{2}(x^{(t)} - \mu)^T \Sigma^{-1} (x^{(t)} - \mu) \right\}}$$

Gibbs Sampling

$$X_1 \mid X_2 \sim N\left(\mu_1 + \frac{\sigma_{12}}{\sigma_2^2}(x_2 - \mu_2), \sigma_1 - \frac{\sigma_{12}^2}{\sigma_2^2} \right)$$

$$X_2 \mid X_1 \sim N\left(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(x_1 - \mu_1),\ \sigma_2 - \frac{\sigma_{12}^2}{\sigma_1^2}\right)$$

update $(\hat{\mu}_1, \hat{\mu}_2)$ iteratively.

<u>Obs:</u> Independent only works well when $Q \simeq \mathbb{P}$.

<u>Ex:</u> "Capture Recapture Models"

fixed population N.

Census	1	2	$\cdots$	i	$\cdots$	k
Captured	c_1	c_2	$\cdots$	c_i		c_k
Marked	0	m_2	$\cdots$	m_i		m_k
newly marked	c_1	$c_2 - m_2$	$\cdots$	$c_i - m_i$		$c_k - m_k$

$$c_i \mid N, \alpha_i \sim \mathrm{Binom}(N, \alpha_i)$$

$$m_i \mid c_i, M_i, N \sim HG(N, M_i, c_i)$$

Use Bayes to get posteriors for N.

$$\mathbb{P}(N) \propto 1, \quad \alpha_i \overset{iid}{\sim} \mathrm{Unif}(0,1)$$

$$\mathbb{P}(m, c \mid N, \alpha) \propto \prod_i \cancel{\binom{N}{c_i}} \alpha_i^{c_i} (1-\alpha_i)^{N-c_i} \frac{\binom{M_i}{m_i}\binom{N-M_i}{c_i-m_i}}{\cancel{\binom{N}{c_i}}}$$

$$\propto \prod_i \alpha_i^{c_i} (1-\alpha_i)^{N-c_i} \binom{N-M_i}{c_i-m_i}$$

So

$$\mathbb{P}(N, \alpha \mid m, c) \propto \mathbb{P}(m, c \mid N, \alpha)\, \mathbb{P}(N)\, \mathbb{P}(\alpha)$$

$$\mathbb{P}(N,\alpha \mid m,c) \propto \mathbb{P}(m,c \mid N,\alpha)\,\mathbb{P}(N)\,\mathbb{P}(\alpha)$$

$$\propto \mathbb{P}(m,c \mid N,\alpha)$$

$$\propto \prod_i \alpha_i^{c_i}(1-\alpha_i)^{N-c_i}\binom{N-m_i}{c_i-m_i}$$

<u>Exercise:</u> Try to simplify

$$\prod_i \binom{N-m_i}{c_i-m_i} = \frac{N!}{\prod_i n_i!\,(N-m_i-n_i)!}$$