Graphical Models

We saw how to encode conditional dependencies using a graph G.

$$X_i \perp\!\!\!\perp X_j \mid X_{[-i,j]} \iff E_{ij} \notin E_G$$

Def: A Markov random field if

$$\mathbb{P}(X_v \mid X_{[-v]}) = \mathbb{P}(X_v \mid X_{N_v})$$

(neighborhood of v

Def: $\mathbb{P}$ is Gibbs w.r.t. a graph G.

if $\mathbb{P}(x) > 0 \ \forall x$ and

$$\mathbb{P}(x) \propto \prod_{c \in \mathcal{C}(G)} F_c(x_c)$$

potentials

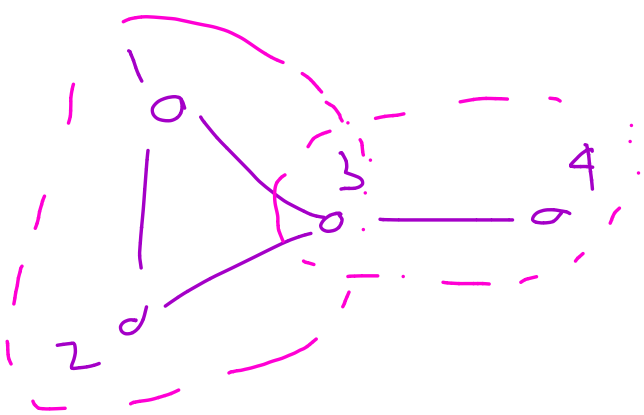
cliques

- only depends on functionals over the cliques in a network.

Ex:

$$\mathbb{P}(x_1, x_2, x_3, x_4) \propto \underbrace{\exp\left\{x_1 x_2 \sin(x_3)\right\}}_{F_{123}} \underbrace{\frac{1}{1+|x_3 - x_4|}}_{F_{34}}$$

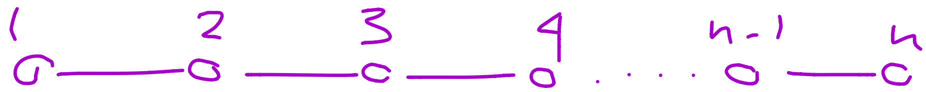
is a Gibbs w.r.t



Ex: $\mathbb{P}(X_1, \dots, X_n) = \mathbb{P}(X_1)\mathbb{P}(X_2 | X_1) \prod_{i=3}^{n} \mathbb{P}(X_i | X_{i-1})$

F_{12} $\qquad$ $F_{i-1\,i}$

Gibbs wrt



So MC is both MRF and Gibbs wrt to the path graph.

<u>Q:</u> MRF = Gibbs ?

<u>Thrm:</u> (Hammersly - Clifford)

$\mathbb{P}$ is a MRF wrt G iff $\mathbb{P}$ is Gibbs wrt G.

<u>Pf:</u> ($\Longleftarrow$)

$$\mathbb{P}(X_v \mid X_{[-v]}) = \frac{\mathbb{P}(X_v, X_{[-v]})}{\mathbb{P}(X_{[-v]})}$$

$$= \frac{P(X_v, X_{[-v]})}{\sum_{\tilde{X}_v} P(\tilde{X}_v, X_{[-v]})} \overset{(G)}{=} \frac{\prod_{\mathcal{C}(G)} F_c(X_c) \,/\, \cancel{Z}}{\sum_{\tilde{X}_v} \prod F_c(X_c) \,/\, \cancel{Z}}$$

$$\left(Z = \sum_{X} \prod_{\mathcal{C}(G)} F_c(x_c) \right)$$

$$= \frac{\cancel{\prod_{\tilde{c} : v \notin \tilde{c}} F_{\tilde{c}}(X_{\tilde{c}})} \prod_{c : v \in c} F_c(X_c)}{\sum_{\tilde{X}_v} \cancel{\prod_{\tilde{c} : v \notin \tilde{c}} F_{\tilde{c}}(X_{\tilde{c}})} \prod_{c : v \in c} F_c(\tilde{X}_v, X_c)}$$

$$= \frac{\prod_{c : v \in c} F_c(X_c)}{\sum_{\tilde{X}_v} \prod_{c : v \in c} F_c(\tilde{X}_v, X_c)}$$

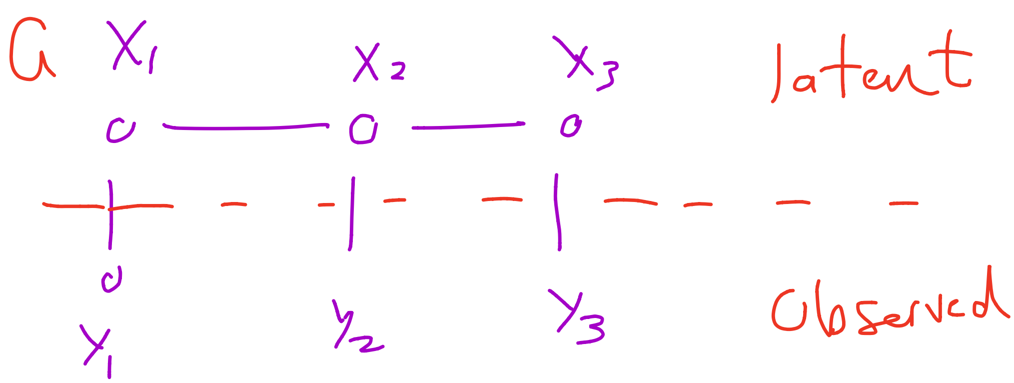
<u>Rmk</u>: All vertices in a clique v must be in the neighborhood of v

Notice that $\bigcup_{v \in C} c = \underbrace{N_S \cup N_v}_{\text{Markov Blanket}}$

$$= \frac{\mathbb{P}(x_v, x_{N_v})}{\sum_{\tilde{x}_v} \mathbb{P}(\tilde{x}_v, x_{N_v})} = \mathbb{P}(x_v \mid x_{N_v})$$

Thus $\mathbb{P}$ is MRF with respect to G. $\boxtimes$

Now consider



$$\mathbb{P}(X, Y) = \mathbb{P}(X)\,\mathbb{P}(Y \mid X)$$

transition

initial $= \mathbb{P}(X_1)\,\mathbb{P}(X_2|X_1)\,\mathbb{P}(X_3|X_2)$

$$\underbrace{\mathbb{P}(Y_1|X_1)\,\mathbb{P}(Y_2|X_2)\,\mathbb{P}(Y_3|X_3)}_{\text{emission}}$$

- Gibbs wrt to the graph above
- $\{X_i\}$ markov chain

Main task to identify $\mathbb{P}(X|Y)$

$$\boxed{\text{Hidden Markov Model}}$$

In general:

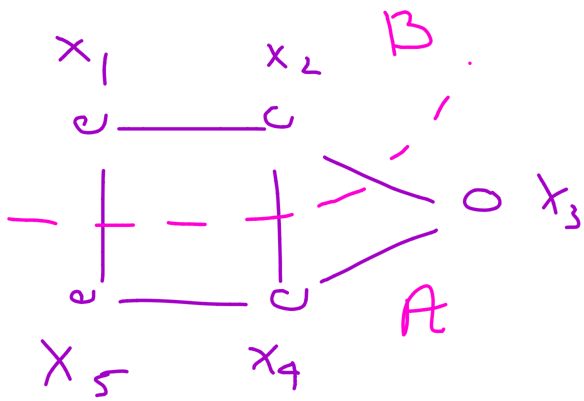
have a joint $\mathbb{P}$ that is MRF

that is observable & latent.
X_A X_B

and want

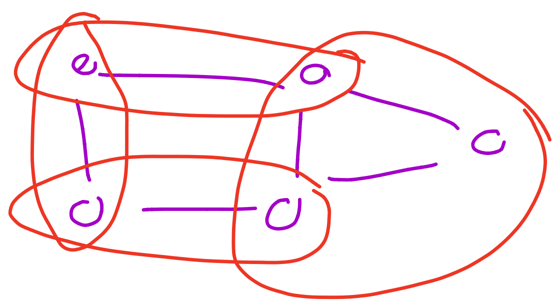
$$\mathbb{P}(X_B|X_A)$$

Ex:



$$\mathbb{P}(X_B | X_A) = \frac{\mathbb{P}(X_A, X_B)}{\sum_{\tilde{x}_B} \mathbb{P}(X_A, \tilde{x}_B)}$$

Cliques

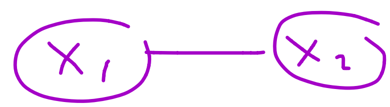


$$= \frac{F_{12}\ F_{15}\ \cancel{F_{45}}\ F_{234}}{\sum_{\tilde{x}_1 \tilde{x}_2} F_{12}(\tilde{x}_1, \tilde{x}_2)\, F_{15}(\tilde{x}_1, x_5)\, \cancel{F_{45}(x_4, x_5)}\, F_{234}(\tilde{x}_2, x_3, x_4)}$$

$$= F_{12}\ F_{15}\ F_{234}$$

$$= \frac{\quad}{\sum_{\tilde{x}_1 \tilde{x}_2} F_{12}(\tilde{x}_1, \tilde{x}_2) F_{15}(\tilde{x}_1, x_5) F_{234}(\tilde{x}_2, x_3, x_4)}$$

$$= \frac{\phi(x_1, x_2)}{\sum_{\tilde{x}_1, \tilde{x}_2} \phi(\tilde{x}_1, \tilde{x}_2)}$$



Thus $\mathbb{P}(X_B \mid X_A)$ is Gibbs wrt the network above