Markov Chain Monte Carlo

(Digression)

Consider a M.C. $\{X_n, n \geq 1\}$ with trans. probabilities

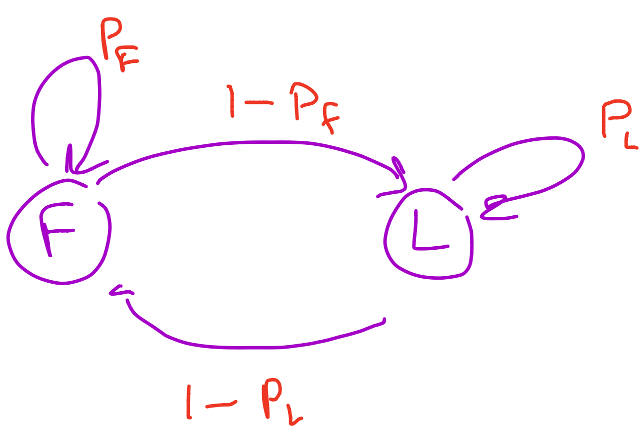
$$p_{ij} = \mathbb{P}(X_k = j \mid X_{k-1} = i) \quad i, j \in S$$

discrete & finite

For simplicity assume $S = \{1, 2, \dots, n\}$

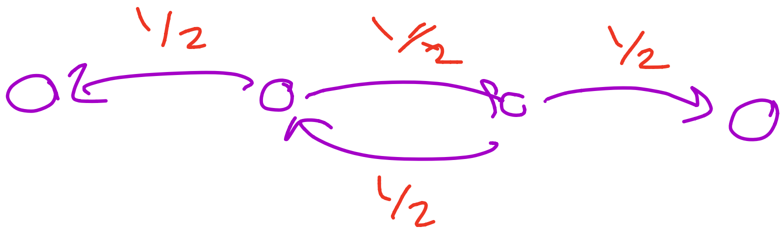
Let $P = (p_{ij})_{1 \leq i \leq j \leq n}$ be the p.t.m.

Ex:



$$P = \begin{bmatrix} p_F & 1-p_F \\ 1-p_L & p_L \end{bmatrix} \qquad S = \{F, L\}$$

Ex:



$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lets compute

$$\mathbb{P}(X_{k+\ell} = j \mid X_k = i) = p_{ij}^{(\ell)}$$

$$p_{ij}^{(\ell)} = \sum_{\alpha \in S} \mathbb{P}(X_{k+\ell} = j \mid X_{k+\ell-1} = \alpha)\, \mathbb{P}(X_{k+\ell-1} = \alpha \mid X_k = i)$$

$(\ell-1)$

$$= \sum_{\alpha \in S} P_{\alpha j} P_{i \alpha}$$

$$P^{(l)} = \left(P_{ij}^{(l)}\right)_{i,j \in S}$$ then

$$= P^{(l-1)} P$$ moreover

$$P^{(1)} = P$$

$$P^{(2)} = P^{(1)} P = P^2$$

$$\vdots$$

$$P^{(m)} = P^m$$

Ergodic Thm: If we have a chain s.t. $\exists n_0$ s.t.

$$\min_{i,j \in S} P_{ij}^{(n_0)} > 0$$ then

$$P_{ij}^{(n)} \longmapsto \pi_j > 0$$ the stationary dist.

$$P_{ij}^{(n)} = \sum_{\alpha \in S} P_{\alpha j} P_{i\alpha}^{(n-1)}$$

$$\pi_j = \sum_{\alpha \in S} P_{\alpha j} \pi_j$$

$$\Longrightarrow \boxed{\pi = P^T \pi}$$

Another version of Ergodic Thrm.

$X_1, \dots, X_n$ is ergodic MC.

$$\frac{1}{n} \sum^{n} f(x_i) \to \mathbb{E}[\text{[illegible]}]$$

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \longrightarrow E_{\pi}[f(Y)]$$

$$Y \sim \pi$$

"time averages converge to space averages"

Suppose we want to sample P to do MC simulation. but we only know P up to a constant.

We could then come up with a MC with stat. dist. P.

Use the Ergodic theorem to compute expectations of functions under P.

<u>Metropolis–Hastings</u>

- Target P
- Proposal distribution $Q(X|Y)$
- Acceptance Ratio

$$R(\xi^{(t)} | \xi^{(*)}) = \frac{\mathbb{P}(\xi^{*})\, Q(\xi^{(t)} | \xi^{(*)})}{\mathbb{P}(\xi^{t})\, Q(\xi^{(*)} | \xi^{(t)})}$$

<u>Alg:</u>

1. Start at $\xi^{(1)}$

2. For $i = 1, 2, \ldots,$ Do

   (i) Sample Candidate

   $$\xi^{*} \sim Q(\cdot | \xi^{(t)})$$

   (ii) $p = \min\{1, R(\xi^{(t)}, \xi^{(*)})\}$

Set $$\xi^{(t+1)} = \begin{cases} \xi^{*} & p \\ \xi^{(t)} & 1-p \end{cases}$$