1. (a) Let $f : \mathbb{N} \to \mathbb{N}$ be a bounded function. Then we have

$$\begin{aligned}
\mathbb{E}(Zf(Z)) &= \sum_{n=0}^{\infty} nf(n)\frac{e^{-\lambda}\lambda^n}{n!}\\
&= \sum_{n=1}^{\infty} nf(n)\frac{e^{-\lambda}\lambda^n}{n!}\\
&= \sum_{k=0}^{\infty}(k+1)f(k+1)\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}\\
&= \sum_{k=0}^{\infty} f(k+1)\frac{e^{-\lambda}\lambda^{k+1}}{k!}\\
&= \lambda\sum_{k=0}^{\infty} f(k+1)\frac{e^{-\lambda}\lambda^{k}}{k!}\\
&= \lambda\mathbb{E}(f(Z+1))
\end{aligned}$$

(b) Let $A \subset \mathbb{N}$ and define the function $f_A : \mathbb{N} \to \mathbb{N}$ by the recursive formula

$$\lambda f_A(k+1) - kf_A(k) = \mathbf{1}_A(k) - \mathbb{P}(Z \in A)$$

and define $f_A(0) = 0$. We show the first form by induction. Consider the following for $f_A(1)$.

$$\begin{aligned}
\lambda f_A(1) - kf_A(0) = \mathbf{1}_A(0) - \mathbb{P}(Z \in A)&\\
f_A(1) &= \frac{1}{\lambda}\left(\mathbf{1}_A(0) - \mathbb{P}(Z \in A)\right)\\
&= \frac{0!}{\lambda}\sum_{n=0}^{0}\left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right)\frac{\lambda^n}{n!}
\end{aligned}$$

Now assume the formula holds for $f_A(k)$, $k \geq 1$. Then we see

$$\begin{aligned}
\lambda f_A(k+1) - kf_A(k) &= \mathbf{1}_A(k) - \mathbb{P}(Z \in A)\\
f_A(k+1) &= \frac{1}{\lambda}\left(kf_A(k) + \mathbf{1}_A(0) - \mathbb{P}(Z \in A)\right)\\
&= \frac{1}{\lambda}\left(k\frac{(k-1)!}{\lambda^k}\sum_{n=0}^{k-1}\left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right)\frac{\lambda^n}{n!} + \mathbf{1}_A(0) - \mathbb{P}(Z \in A)\right)\\
&= \frac{k!}{\lambda^{k+1}}\sum_{n=0}^{k-1}\left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right)\frac{\lambda^n}{n!} + \frac{1}{\lambda}[\mathbf{1}_A(0) - \mathbb{P}(Z \in A)]\\
&= \frac{k!}{\lambda^{k+1}}\sum_{n=0}^{k}\left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right)\frac{\lambda^n}{n!}
\end{aligned}$$

Hence we see this expression is valid. Now using this expression, with the fact that $Z \sim \text{Pois}(\lambda)$ we can write the following

$$\begin{aligned}
f_A(k+1) &= \frac{k!}{\lambda^{k+1}} \sum_{n=0}^{k} \left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right) \frac{\lambda^n}{n!} \\
&= \frac{e^{-\lambda}}{\lambda \mathbb{P}(Z = k)} \left(\sum_{n=0}^{k} \mathbf{1}_A(n) \frac{\mathbb{P}(Z = n)}{e^{-\lambda}} - \sum_{n=0}^{k} \mathbb{P}(Z \in A) \frac{\mathbb{P}(Z = n)}{e^{-\lambda}} \right) \\
&= \sum_{n=0}^{k} \mathbf{1}_A(n) \frac{\mathbb{P}(Z = n)}{\lambda \mathbb{P}(Z = k)} - \frac{\mathbb{P}(Z \in A)}{\lambda \mathbb{P}(Z = k)} \sum_{n=0}^{k} \mathbb{P}(Z = n) \\
&= \sum_{n=0}^{k} \mathbf{1}_A(n) \frac{\mathbb{P}(Z = n)}{\lambda \mathbb{P}(Z = k)} - \frac{\mathbb{P}(Z \in A)\mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)}
\end{aligned}$$

Now notice that the first sum is zero when $n \notin A$ hence we can rewrite this probability as follows

$$\sum_{n=0}^{k} \mathbf{1}_A(n) \frac{\mathbb{P}(Z = n)}{\lambda \mathbb{P}(Z = k)} = \frac{\mathbb{P}(Z \leq k \cap Z \in A)}{\lambda \mathbb{P}(Z = k)}$$

Hence we see that

$$f_A(k+1) = \frac{\mathbb{P}(Z \leq k \cap Z \in A) - \mathbb{P}(Z \in A)\mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)} \tag{1}$$

Using the first expression we see that

$$\begin{aligned}
|f_A(k+1)| &= \left| \frac{k!}{\lambda^{k+1}} \sum_{n=0}^{k} \left(\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right) \frac{\lambda^n}{n!} \right| \\
&= \frac{k!}{\lambda^{k+1}} \sum_{n=0}^{k} \left|\mathbf{1}_A(n) - \mathbb{P}(Z \in A)\right| \frac{\lambda^n}{n!} \\
&\leq \frac{k!}{\lambda^{k+1}} \sum_{n=0}^{k} \frac{\lambda^n}{n!} \\
&\leq \frac{1}{\lambda}
\end{aligned}$$

Therefore, f_A is bounded.

Now, consider the following

$$
\begin{aligned}
&f_A(k+1) + f_{A^c}(k+1)\\
&= \frac{\mathbb{P}(Z \leq k \cap Z \in A) - \mathbb{P}(Z \in A)\mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)} + \frac{\mathbb{P}(Z \leq k \cap Z \in A^c) - \mathbb{P}(Z \in A^c)\mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)}\\
&= \frac{\mathbb{P}(Z \leq k \cap Z \in A) + \mathbb{P}(Z \leq k \cap Z \in A^c) - \mathbb{P}(Z \leq k)[\mathbb{P}(Z \in A) + \mathbb{P}(Z \in A^c)]}{\lambda \mathbb{P}(Z = k)}\\
&= \frac{\mathbb{P}(Z \leq k) - \mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)}\\
&= 0
\end{aligned}
$$

Hence we see that $f_A(k+1) + f_{A^c}(k+1) = 0$ for $k = 0, 1, 2, \ldots$. Specifically, we know that $f_A(k) + f_{A^c}(k) = 0$. Combining these facts, we see that $f_A(k+1) + f_{A^c}(k+1) = f_A(k) + f_{A^c}(k)$ which upon rearranging gives

$$f_A(k+1) - f_{A^c} = f_A(k) - f_{A^c}(k+1)(k)$$

(c) Assume that W has a Poisson distribution. Then by part 1, we see $\mathbb{E}(Wf(W)) = \lambda \mathbb{E}(f(W+1))$ which upon rearranging gives

$$\mathbb{E}\left[\lambda f(W+1) - Wf(W)\right] = 0$$

Now suppose that this equation holds. As it holds for *any* bounded function, it certainly holds for our f_A. Hence we see that

$$
\begin{aligned}
\mathbb{E}\left[\lambda f(W+1) - Wf(W)\right] &= 0\\
\mathbb{E}\left[\lambda f_A(W+1) - Wf_A(W)\right] &= 0\\
\mathbb{E}\left[\mathbf{1}_A(W) - \mathbb{P}(Z \in A)\right] &= 0\\
\mathbb{P}(W \in A) &= \mathbb{P}(Z \in A)
\end{aligned}
$$

Now as $A \subset \mathbb{N}$ was arbitrary, we can take it to be $A = \{0, 1, 2, \ldots, n\}$ and see that $\mathbb{P}(W \leq n) = \mathbb{P}(Z \leq n)$. That is W has the same distribution function as Z which a Poisson random variable and therefore $W \overset{D}{=} Z$ and we see that W has the Poisson distribution.

(d) Suppose that $j = k$. First notice that we can write $f_j(k+1)$ as follows

$$
\begin{aligned}
f_j(k+1) &= f_k(k+1)\\
&= \frac{\mathbb{P}(Z \leq k \cap Z = k) - \mathbb{P}(Z = k)\mathbb{P}(Z \leq k)}{\lambda \mathbb{P}(Z = k)}\\
&= \frac{\mathbb{P}(Z = k)\left[1 - \mathbb{P}(Z \leq k)\right]}{\lambda \mathbb{P}(Z = k)}\\
&= \frac{\mathbb{P}(Z > k)}{\lambda}
\end{aligned}
$$

Moreover we see

$$\begin{aligned} f_j(k) &= f_k(k) \\ &= \frac{\mathbb{P}(Z \leq k-1 \cap Z = k) - \mathbb{P}(Z = k)\mathbb{P}(Z \leq k-1)}{\lambda \mathbb{P}(Z = k-1)} \\ &= -\frac{\mathbb{P}(Z = k)\mathbb{P}(Z \leq k-1)}{\lambda \mathbb{P}(Z = k-1)} \end{aligned}$$

Putting these together we see the following

$$f_j(k+1) - f_j(k) = f_k(k+1) - f_k(k) = \frac{\mathbb{P}(Z > k)}{\lambda} + \frac{\mathbb{P}(Z = k)\mathbb{P}(Z \leq k-1)}{\lambda \mathbb{P}(Z = k-1)} > 0$$

Now, suppose that $j \neq k$ and we will prove the contrapositive. Suppose that $j > k$. Then we see that

$$\begin{aligned} f_j(k+1) &= -\frac{\mathbb{P}(Z \leq k)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k)} \\ f_j(k) &= -\frac{\mathbb{P}(Z \leq k-1)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k-1)} \end{aligned}$$

Hence we see that

$$f_j(k+1) - f_j(k) = -\frac{\mathbb{P}(Z \leq k)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k)} + \frac{\mathbb{P}(Z \leq k-1)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k-1)}$$

For the sake of contradiction $f_j(k+1) - f_j(k) > 0$. Then we see that

$$\frac{\mathbb{P}(Z \leq k-1)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k-1)} > \frac{\mathbb{P}(Z \leq k)\mathbb{P}(Z = j)}{\lambda \mathbb{P}(Z = k)}$$

But notice that this implies the following chain of inequalities

$$f_j(k+1) < f_j(k) < \ldots < 0$$

But as $f_A : \mathbb{N} \to \mathbb{N}$ we see that this is a contradiction. That is $f_j(k+1) - f_j(k) \leq 0$. Now for the case $j < k$ we can write

$$f_j(k+1) - f_j(k) = f_{A \setminus j}(k) - f_{A \setminus j}(k+1)$$

But if we assume that this quantity is greater than 0 then we see that $f_{A\setminus j}(k) > f_{A\setminus j}(k+1)$ and hence f_A is a decreasing function which is not possible. Hence we see that $f_j(k+1) - f_j(k) \leq 0$ which concludes this portion of the proof. With this fact, consider the quantity $|f_A(k+1) - f_A(k)|$. Suppose for the moment that $f_A(k+1) - f_A(k) > 0$. As we just showed we have

$$f_A(k+1) - f_A(k) \leq f_k(k+1) - f_k(k) = \frac{\mathbb{P}(Z > k)}{\lambda} + \frac{\mathbb{P}(Z \leq k-1)\mathbb{P}(Z = k)}{\lambda \mathbb{P}(Z = k-1)}$$

Now notice that
$$\frac{\mathbb{P}(Z=k)}{\lambda\mathbb{P}(Z=k-1)} = \frac{(k-1)!}{k!} = \frac{1}{k}$$
Hence we can write the following
$$\begin{aligned}
f_A(k+1) - f_A(k) &\leq \frac{1}{\lambda}\mathbb{P}(Z>k) + \frac{1}{k}\mathbb{P}(Z \leq k-1)\\
&= \frac{1}{\lambda}\sum_{n=k+1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!} + \frac{1}{k}\sum_{n=0}^{k-1}\frac{e^{-\lambda}\lambda^n}{n!}\\
&= \frac{e^{-\lambda}}{\lambda}\left(\sum_{n=k+1}^{\infty}\frac{\lambda^n}{n!} + \frac{1}{k}\sum_{n=0}^{k-1}\frac{\lambda^{n+1}}{n!}\right)\\
&= \frac{e^{-\lambda}}{\lambda}\left(\sum_{n=k+1}^{\infty}\frac{\lambda^n}{n!} + \frac{1}{k}\sum_{m=1}^{k}\frac{\lambda^{m}}{(m-1)!}\right)\\
&= \frac{e^{-\lambda}}{\lambda}\left(\sum_{n=k+1}^{\infty}\frac{\lambda^n}{n!} + \sum_{m=1}^{k}\frac{\lambda^{m}}{m!}\frac{m}{k}\right)
\end{aligned}$$

Now notice that $\frac{m}{k} \leq 1$ so we can bound from above to get
$$\begin{aligned}
&\frac{e^{-\lambda}}{\lambda}\left(\sum_{n=k+1}^{\infty}\frac{\lambda^n}{n!} + \sum_{m=1}^{k}\frac{\lambda^{m}}{m!}\frac{m}{k}\right)\\
&\leq \frac{e^{-\lambda}}{\lambda}\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} - \sum_{n=0}^{0}\frac{\lambda^{n}}{n!}\right)\\
&= \frac{e^{-\lambda}}{\lambda}\left(e^{\lambda}-1\right)\\
&= \frac{1-e^{-\lambda}}{\lambda}
\end{aligned}$$
Now assume that $f_A(k+1) - f_A(k) < 0$. Then by our complement identity we see that $f_A(k+1) - f_A(k) = f_{A^c}(k) - f_{A^c}(k+1) < 0$ or $f_{A^c}(k+1) - f_{A^c}(k) > 0$. Then by part $f_{A^c}(k+1) - f_{A^c}(k) \leq \frac{1-e^{-\lambda}}{\lambda}$. Hence we see that
$$|f_A(k+1) - f_A(k)| \leq \frac{1-e^{-\lambda}}{\lambda}$$
Lastly, assume that $j > i$ and consider $|f_A(j) - f_A(i)|$. Using the fact from above we can write the following
$$\begin{aligned}
|f_A(j) - f_A(i)| &= |f_A(j) - f_A(j-1) + f_A(j-1) - f_A(i)|\\
&\leq |f_A(j) - f_A(j-1)| + |f_A(j-1) - f_A(i)|\\
&\leq \frac{1-e^{-\lambda}}{\lambda} + |f_A(j-1) - f_A(i)|
\end{aligned}$$

Applying this recursively, we see that

$$|f_A(j) - f_A(i)| \leq \frac{1-e^{-\lambda}}{\lambda} + |f_A(j-1) - f_A(i)| \leq \dots$$

$$\leq (j-i)\frac{1-e^{-\lambda}}{\lambda} + |f_A(i) - f_A(i)| = (j-i)\frac{1-e^{-\lambda}}{\lambda}$$

By as symmetric argument we see that

$$|f_A(j) - f_A(i)| \leq |j-i|\frac{1-e^{-\lambda}}{\lambda}$$

(e) First notice by the Chen-Stein Lemma we have

$$|\mathbb{P}(W\in A) - \mathbb{P}(Z \in A)| = |\mathbb{E}[\mathbf{1}_A(W) - \mathbb{P}(Z\in A)]| = |\lambda f_A(W+1) - Wf_A(W)|$$

Hence we see that

$$\sup_{A\subset\mathbb{N}}|\mathbb{P}(W\in A) - \mathbb{P}(Z\in A)| = \sup_{f_A}|\mathbb{E}(\lambda f_A(W+1) - Wf_A(W))|$$

Now consider the set Ψ of bounded functions that map $\mathbb{N}\to\mathbb{N}$ with the property $|f(j) - f(i)| \leq \frac{1-e^{-\lambda}}{\lambda}|j-i|$. Notice that for all $A\subset\mathbb{N}$ we have $f_A\in\Psi$. Therefore

$$\sup_{A\subset\mathbb{N}}|\mathbb{P}(W\in A)-\mathbb{P}(Z\in A)| = \sup_{f_A}|\lambda f_A(W+1)-Wf_A(W)| \leq \sup_{f\in\Psi}|\lambda f(W+1)-Wf(W)|$$

2. (a)

$$\begin{aligned}
\mathbb{E}(Wf(W)) &= \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}kf(k)\\
&= \sum_{k=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}kf(k)\\
&= \sum_{k=1}^{n}\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}kf(k)\\
&= np\sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!(k-1)!}p^{k-1}(1-p)^{n-k}f(k)\\
&= \lambda\sum_{m=0}^{n-1}\frac{(n-1)!}{(n-1-m)!(m)!}p^m(1-p)^{n-1-m}f(m+1)\\
&= \lambda\sum_{m=0}^{n-1}\binom{n-1}{m}p^m(1-p)^{n-1-m}f(m+1)\\
&= \lambda\mathbb{E}(f(V+1))
\end{aligned}$$

By an identical argument as in exercise 5 we see that

$$\begin{aligned}|\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| &= |\mathbb{E}[\mathbf{1}_A(W) - \mathbb{P}(Z \in A)]| \\ &= |\mathbb{E}[\lambda f_A(W+1) - W f_A(W)]| \\ &= |\mathbb{E}[\lambda f_A(W+1) - \lambda f_A(V+1)]| \\ &= \lambda|\mathbb{E}[f_A(W+1) - f_A(V+1)]|\end{aligned}$$

Hence we see that

$$\sup_{A\subset\mathbb{N}}|\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| = \lambda \sup_{f_A}|\mathbb{E}[f_A(W+1) - f_A(V+1)]|$$

As we argued in the previous section we see that $f_A \in \Psi$ as we have

$$\sup_{A\subset\mathbb{N}}|\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq \sup_{f\in\Psi}|\mathbb{E}[f_A(W+1) - f_A(V+1)]|$$

Moreover, to prove this bound consider the following

$$\begin{aligned}\sup_{A\subset\mathbb{N}}|\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| &\leq \sup_{f\in\Psi}\lambda|\mathbb{E}[f(V+1) - f(W+1)]| \\ &\leq \lambda\sup_{f\in\Psi}\mathbb{E}|f(V+1) - f(W+1)| \\ &\leq \lambda\mathbb{E}|\frac{1-e^{-\lambda}}{\lambda}|V+1-W-1|| \\ &= (1-e^{-\lambda})\mathbb{E}|V+1-W-1| \\ &= (1-e^{-\lambda})\mathbb{E}|X_n| \\ &= (1-e^{-\lambda})\mathbb{E}(X_n) \\ &= p(1-e^{-\lambda})\end{aligned}$$

(b) Let W be a Poisson random variable. Then by definition we have

$$d_{TV}(Y_n, W) = \sup_{B\in\mathcal{B}(\mathbb{R})}|\mathbb{P}(Y_n \in B) - \mathbb{P}(W \in B)|$$

As Y_n and W only take on integer values, for $B, B' \in \mathcal{B}(\mathbb{R})$ that contain the same integer values will produce the same metric. For this reason, we can rewrite the metric as follows

$$\sup_{B\in\mathcal{B}(\mathbb{R})}|\mathbb{P}(Y_n \in B) - \mathbb{P}(W \in B)| = \sup_{A\subset\mathbb{N}}|\mathbb{P}(Y_n \in A) - \mathbb{P}(W \in A)|$$

Now using the bound we just derived we see that

$$d_{TV}(W, Y_n) \leq p_n(1-e^{-np_n}) \to 0(1-e^{-\lambda}) = 0$$

Hence we see that $Y_n \xrightarrow{TV} W$