1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with $\mathbb{E}(|X_1|^r) < \infty$ for some $1 < r < 2$. Let $S_n = \sum_{i=1}^{n} X_i$ and without loss of generality assume $\mathbb{E}(X_1) = 0$. We look to show
$$\frac{S_n}{n^{1/r}} \xrightarrow{P} 0$$
First, let $\epsilon > 0$ and define $c = n^{1/r}\epsilon^{3/(2-r)}$. Next define $Y_{k,n} = X_k \mathbf{1}_{\{|X_k|<c\}}$ for $k = 1, 2, \dots, n$ and $n \geq 1$. Then we can define, $S_n' = \sum_{k=1}^{n} Y_{k,n}$. Now, using the truncated Chebyshev inequality, we have
$$\begin{aligned}
\mathbb{P}(|S_n - \mathbb{E}(S_n')| \geq n^{1/r}\epsilon) &\leq \frac{n\mathrm{Var}(Y_{1,n})}{(n^{1/r}\epsilon)^2} + n\mathbb{P}(|X_1| > c) \\
&\leq \frac{n\mathbb{E}[(Y_{1,n})^2]}{(n^{1/r}\epsilon)^2} + n\mathbb{P}(|X_1| > c) \\
&\leq \frac{n\mathbb{E}[(X_1\mathbf{1}_{\{|X_1|<c\}})^2]}{(n^{1/r}\epsilon)^2} + n\mathbb{P}(|X_1| > c) \\
&\leq \frac{nc}{(n^{1/r}\epsilon)^2}\mathbb{E}[|X_1|\mathbf{1}_{\{|X_1|<c\}}] + n\mathbb{P}(|X_1| > c) \\
&= \frac{nc}{(n^{1/r}\epsilon)^2}\mathbb{E}[|X_1|^r|X_1|^{1-r}\mathbf{1}_{\{|X_1|<c\}}] + n\mathbb{P}(|X_1| > c) \\
&\leq \frac{nc^{2-r}}{(n^{1/r}\epsilon)^2}\mathbb{E}[|X_1|^r\mathbf{1}_{\{|X_1|<c\}}] + n\mathbb{P}(|X_1| > c) \\
&\leq \frac{n^{2/r}\epsilon^3}{n^{2/r}\epsilon^2}\mathbb{E}[|X_1|^r] + n\mathbb{P}(|X_1| > c) \\
&= \epsilon\mathbb{E}[|X_1|^r] + n\mathbb{P}(|X_1| > c)
\end{aligned}$$
Focusing on the second term in this expression, we note that
$$\begin{aligned}
nP(|X_1| > c) = nP(|X_1|^r \geq n\epsilon^{3r/(2-r)}) &= \frac{\epsilon^{3r/(2-r)}}{\epsilon^{3r/(2-r)}} nP(|X_1|^r \geq n\epsilon^{3r/(2-r)}) \\
&= \frac{1}{\epsilon^{3r/(2-r)}}\int_{n\epsilon^{3r/(2-r)}}^{\infty} n\epsilon^{3r/(2-r)} dF_{|X_1|^r}(x) \\
&\leq \frac{1}{\epsilon^{3r/(2-r)}}\int_{n\epsilon^{3r/(2-r)}}^{\infty} x dF_{|X_1|^r}(x)
\end{aligned}$$

We recognize this as the tail of the convergent integral $\mathbb{E}(|X_1|^r) < \infty$. Hence,
$$n\mathbb{P}[|X_1| > c] \xrightarrow[n\to\infty]{} 0$$
Using this result, we have that
$$\limsup_{n\to\infty}\mathbb{P}(|S_n - \mathbb{E}(S_n')| \geq n^{1/r}\epsilon) \leq \epsilon\mathbb{E}[|X_1|^r]$$
As $\epsilon > 0$ was arbitrary, this implies that
$$\frac{S_n - \mathbb{E}(S_n')}{n^{1/r}} \xrightarrow{P} 0$$

It suffices to show that $\frac{\mathbb{E}[S_n']}{n^{1/r}} \longrightarrow 0$. Notice that since the $Y_{k,n}$ independent across the k index, we can write the following

$$\begin{aligned}|\mathbb{E}(S_n')| &= |n\mathbb{E}\Big(X_1\mathbf{1}_{|X_1|\leq c}\Big)| = | - n\mathbb{E}\Big(X_1\mathbf{1}_{|X_1|>c}\Big)|\\ &\leq n\mathbb{E}\Big(|X_1|\mathbf{1}_{|X_1|>c}\Big) = n\mathbb{E}\Big(|X_1|^r|X_1|^{1-r}\mathbf{1}_{|X_1|>c}\Big)\end{aligned}$$

Now, notice that $1-r<0$, so with $|X_1|>c$ we have $|X_1|^{1-r}<c^{1-r}$. Thus,

$$n\mathbb{E}\Big(|X_1|^r|X_1|^{1-r}\mathbf{1}_{|X_1|>c}\Big) \leq nc^{1-r}\mathbb{E}\Big(|X_1|^r\mathbf{1}_{|X_1|>c}\Big) = n^{1/r}\epsilon^{3\frac{1-r}{2-r}}\mathbb{E}\Big(|X_1|^r\mathbf{1}_{|X_1|>c}\Big)$$

Seeing that $\mathbb{E}(|X_1|^r)<\infty$, then $\mathbb{E}\Big(|X_1|^r\mathbf{1}_{|X_1|>c}\Big)\to 0$ as $n\to\infty$. This shows that

$$\frac{|\mathbb{E}(S_n')|}{n^{1/r}} \leq \epsilon^{3\frac{1-r}{2-r}}\mathbb{E}\Big(|X_1|^r\mathbf{1}_{|X_1|>c}\Big) \underset{n\to\infty}{\longrightarrow} 0$$

Finally, we conclude that

$$\frac{S_n}{n^{1/r}} \xrightarrow{P} 0$$

2. (a) First assume that $\mathbb{E}(|X_1|^r)<\infty$ and let $\epsilon>0$. First note that $\frac{1}{\epsilon^r}\mathbb{E}(|X_1|^r)<\infty$. Now consider the following quantity.

$$\begin{aligned}\mathbb{E}(|X_1|^r) &= \int_0^\infty x dF_{|X_1|^r} = \sum_{k=1}^\infty\int_{(k-1)\epsilon^r}^{k\epsilon^r} x dF_{|X_1|^r}\\ &\geq \sum_{k=1}^\infty (k-1)\epsilon^r\int_{(k-1)\epsilon^r}^{k\epsilon^r} dF_{|X_1|^r} = \sum_{k=1}^\infty (k-1)\epsilon^r\mathbb{P}[(k-1)\epsilon^r<|X_1|^r<k\epsilon^r]\\ &= \epsilon^r\sum_{k=1}^\infty\sum_{n=1}^{k-1}\mathbb{P}[(k-1)^{1/r}\epsilon<|X_1|<k^{1/r}\epsilon]\\ &= \epsilon^r\sum_{n=1}^\infty\sum_{k=n+1}^{\infty}\mathbb{P}[(k-1)^{1/r}\epsilon<|X_1|<k^{1/r}\epsilon]\\ &= \epsilon^r\sum_{n=1}^\infty\mathbb{P}[|X_1|>n^{1/r}\epsilon]\end{aligned}$$

Hence

$$\sum_{n=1}^\infty\mathbb{P}[|X_1|>n^{1/r}\epsilon] \leq \frac{1}{\epsilon^r}\mathbb{E}[|X_1|^r]<\infty$$

Conversely, assume that $\sum_{n=1}^\infty \mathbb{P}[|X_1| > n^{1/r}\epsilon] < \infty$. Following the same technique as above, this time bounding from above, we have

$$\begin{aligned}
\mathbb{E}(|X_1|^r) &= \int_0^\infty x dF_{|X_1|^r} = \sum_{k=1}^\infty \int_{(k-1)\epsilon^r}^{k\epsilon^r} x dF_{|X_1|^r}\\
&\leq \sum_{k=1}^\infty k\epsilon^r \int_{(k-1)\epsilon^r}^{k\epsilon^r} dF_{|X_1|^r} = \epsilon^r \sum_{k=1}^\infty k\mathbb{P}[(k-1)\epsilon^r < |X_1|^r < k\epsilon^r]\\
&= \epsilon^r \sum_{k=1}^\infty \sum_{n=1}^k \mathbb{P}[(k-1)^{1/r}\epsilon < |X_1| < k^{1/r}\epsilon]\\
&= \epsilon^r \sum_{n=1}^\infty \sum_{k=n}^\infty \mathbb{P}[(k-1)^{1/r}\epsilon < |X_1| < k^{1/r}\epsilon]\\
&= \epsilon^r \sum_{n=1}^\infty \mathbb{P}[|X_1| > (n-1)^{1/r}\epsilon]\\
&= \epsilon^r \sum_{m=1}^\infty \mathbb{P}[|X_1| > m^{1/r}\epsilon] + \epsilon^r < \infty
\end{aligned}$$

(b) Consider the following reformulation of the definition of almost sure convergence.

$$\begin{aligned}
&\mathbb{P}\{w : \lim_{n\to\infty} \frac{|X_n(\omega)|}{n^{1/r}} = 0\} = 1\\
&\iff \mathbb{P}\{w : \lim_{n\to\infty} |X_n(\omega)| \leq n^{1/r}\epsilon\} = 1\\
&\iff \mathbb{P}\{\omega : \exists N \text{ s.t. } \forall n > N,\ |X_n(\omega)| \leq n^{1/r}\epsilon\} = 1\\
&\iff \mathbb{P}\Big(\liminf_{n\to\infty}\{\omega : |X_n(\omega)| \leq n^{1/r}\epsilon\}\Big) = 1\\
&\iff \mathbb{P}\Big(\limsup_{n\to\infty}\{\omega : |X_n(\omega)| > n^{1/r}\epsilon\}\Big) = 0\\
&\iff \mathbb{P}\{\omega : |X_n(\omega)| > n^{1/r}\epsilon, i.o\} = 0
\end{aligned}$$

Since each of these statements are equivalent, the statement is proved.

(c) It suffices to show one statement in (a) implies a statement in (b) and one statement in (b) implies one statement in (a). First note that $\sum_{n=1}^\infty \mathbb{P}(\frac{|X_n|}{n^{1/r}} > \epsilon) < \infty$ is the hypothesis in Borel Cantelli I. Hence, by taking this assumption, we see that $\mathbb{P}[\frac{|X_n|}{n^{1/r}} > \epsilon, \text{i.o}] = 0$. That is (a) $\Longrightarrow$ (b).
Now, recall by Borel Cantelli II, if $\{A_n\}$ are independent events then

$$\sum_{n=1}^\infty \mathbb{P}(A_n) = \infty \Longrightarrow \mathbb{P}(A_n, i.o) = 1$$

Hence the contrapositive, and use of the zero-one law, can be written as

$$\mathbb{P}(A_n, i.o) = 0 \Longrightarrow \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$$

In our case, seeing the X_n are independent,

$$\mathbb{P}(|X_n| > n^{1/r}\epsilon \text{ i.o}) = 0 \Longrightarrow \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n^{1/r}\epsilon) < \infty$$

Therefore (b) $\Longrightarrow$ (a) and these four statements are equivalent.

3. (a) Suppose that $\sum_{i=1}^{\infty} \text{Var}(X_n) < \infty$. Define $Y_n = X_n - \mathbb{E}[X_n]$ and $S_n = \sum_{i=1}^{n} Y_i$. For $\sum_{n=1}^{\infty} Y_n$ to converge in L^2 is equivalent to showing that S_n is Cauchy in L^2. Let $n, m \in \mathbb{N}$ with $n < m$. Then we wish to consider the quantity

$$\mathbb{E}[|S_n - S_m|^2] = \mathbb{E}\left[\sum_{i=n+1}^{m} Y_i\right]^2$$

Now recall that $\mathbb{E}(Y_i) = \mathbb{E}(X_i) - E(X_i) = 0$. So $\mathbb{E}[(S_n - S_m)^2] = \text{Var}(S_n - S_m)$. From here, we have

$$\mathbb{E}[|S_n - S_m|^2] = \text{Var}\left(\sum_{i=n+1}^{m} Y_i\right) = \sum_{i=n+1}^{m} \text{Var}(X_i)$$

where the last step was due to independence and the fact that $\text{Var}(Y_i) = \text{Var}(X_i - \mathbb{E}(X_i)) = \text{Var}(X_i)$. Letting $m \to \infty$ we have

$$\lim_{m\to\infty} \mathbb{E}[|S_n - S_m|^2] = \sum_{i=n+1}^{\infty} \text{Var}(X_i)$$

Now, by assumption, the $\text{Var}(X_i)$ are summable, so by letting $n \to \infty$ we recognize the above as the tail of a convergence series and hence

$$\lim_{n,m\to\infty} \mathbb{E}[|S_n - S_m|^2] = 0$$

Now, suppose that $\sum_{i=1}^{\infty}(X_i - \mathbb{E}(X_i))$ converges in L^2. This means the sequence $\{S_n, n \geq 1\}$ is L^2 convergent. That is

$$\lim_{n\to\infty} \mathbb{E}[|S_n|^2] = C < \infty$$

Recall, however, that $\mathbb{E}(S_n) = 0$ so

$$\mathbb{E}[|S_n|^2] = \text{Var}(S_n) = \text{Var}\left[\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i))\right] = \sum_{i=1}^{n} \text{Var}(X_i)$$

Now letting $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{\infty} \text{Var}(X_i) = \lim_{n\to\infty} \mathbb{E}[|S_n|^2] < \infty$$

(b) Let $\text{Var}(X_n) = \sigma^2 < \infty$ and define the sequence of iid random variables, Z_n, by $Z_n = a_nX_n$. Notice that $\mathbb{E}(Z_n) = 0$ and $\text{Var}(Z_n) = \sigma^2 a_n^2$. Applying the result from (a) to the sequence Z_n we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\text{Var}(Z_n) < \infty &\iff \sum_{n=1}^{\infty}(Z_n - \mathbb{E}(Z_n)) \quad \text{converges in } L^2\\
\sum_{n=1}^{\infty}\sigma^2 a_n < \infty &\iff \sum_{n=1}^{\infty}a_nX_n \quad \text{converges in } L^2\\
\sum_{n=1}^{\infty}a_n < \infty &\iff \sum_{n=1}^{\infty}a_nX_n \quad \text{converges in } L^2
\end{aligned}$$