**Exercise 2.3.5** Let $N$ the number of accidents at the factory in a week where $N \sim \text{Pois}(2)$. Moreover, let $X_i$ be the number of injuries in accident $i$ of the week where $\mathbb{E}(X_i) = 3$ and $\text{Var}(X_i) = 4$. We assume that $(X_i)_{i=1}^N$ are iid. Since our interest centers on the number of injuries in a week, we define the random sum $Z = \sum_{i=0}^N X_i$ where $Z \equiv 0$ when $N = 0$. Using the results derived in lecture, we see that

$$\begin{aligned}\mathbb{E}(Z) &= \mathbb{E}(N)\mathbb{E}(X_1) = (2)(3) = 6\\ \text{Var}(Z) &= \mathbb{E}(N)\text{Var}(X_1) + [\mathbb{E}(X_1)]^2\text{Var}(N) = (2)(4) + (3)^2(2) = 26\end{aligned}$$

**Problem 2.3.3** Let $Z = \sum_{i=0}^N \xi_i$ where $(\xi_i)_{i=1}^n$ are iid with $\mathbb{P}(\xi_i = \pm 1) = 1/2$ and $N \sim \text{Geom}(\alpha)$.

a $\mathbb{E}(Z) = \mathbb{E}(N)\mathbb{E}(\xi_1) = \left(\frac{1-\alpha}{\alpha}\right)(0) = 0$
$\text{Var}(Z) = \mathbb{E}(N)\text{Var}(\xi_1) + [\mathbb{E}(\xi_1)]^2\text{Var}(N) = \left(\frac{1-\alpha}{\alpha}\right)(1) + [0]^2\left(\frac{1-\alpha}{\alpha^2}\right) = \left(\frac{1-\alpha}{\alpha}\right)$

b First note that $Z^3$ can be written as follows

$$Z^3 = \sum_{i=1}^N \xi_i^3 + \sum_{i\neq j}\xi_i^2\xi_j + \sum_{i\neq j\neq k}\xi_i\xi_j\xi_k = \sum_{i=1}^N \xi_i + \sum_{i\neq j}\xi_j + \sum_{i\neq j\neq k}\xi_i\xi_j\xi_k$$

Now, notice that $\mathbb{E}(\xi_i) = \mathbb{E}(\xi_j) = \mathbb{E}(\xi_i\xi_j\xi_k) = 0$. Hence, regardless of the number of terms in each sum, we have that $\mathbb{E}(Z^3) = 0$. Proceeding in same fashion as above, we have

$$\begin{aligned}Z^4 &= \left(\sum_{i=1}^N \xi_i^2 + \sum_{i\neq j}\xi_i\xi_j\right)\left(\sum_{i=1}^N \xi_i^2 + \sum_{i\neq j}\xi_i\xi_j\right)\\
&= \left(N + \sum_{i\neq j}\xi_i\xi_j\right)\left(N + \sum_{i\neq j}\xi_i\xi_j\right)\\
&= N^2 + 2N\sum_{i\neq j}\xi_i\xi_j + \left(\sum_{i\neq j}\xi_i\xi_j\right)\left(\sum_{k\neq \ell}\xi_k\xi_\ell\right)\\
&= N^2 + 2N\sum_{i\neq j}\xi_i\xi_j + 2\sum_{i\neq j}\xi_i^2\xi_j^2 + \sum_{i\neq j\neq k}\xi_i^2\xi_j\xi_k + \sum_{i\neq j\neq k\neq \ell}\xi_i\xi_j\xi_k\xi_\ell\\
&= N^2 + 2N\sum_{i\neq j}\xi_i\xi_j + 2\sum_{i\neq j}1 + \sum_{j\neq k}\xi_j\xi_k + \sum_{i\neq j\neq k\neq \ell}\xi_i\xi_j\xi_k\xi_\ell\\
&= N^2 + 3N\sum_{i\neq j}\xi_i\xi_j + 2N(N-1) + \sum_{i\neq j\neq k\neq \ell}\xi_i\xi_j\xi_k\xi_\ell\end{aligned}$$

Now, again, regardless of the number of terms in the sum, $\mathbb{E}[\xi_i\xi_j] = \mathbb{E}[\xi_i\xi_j\xi_k\xi_\ell] = 0$. Hence, we can write the expectation of $N$ as follows

$$\begin{aligned}\mathbb{E}(Z^4) &= \mathbb{E}[N^2] + 2\mathbb{E}[N(N-1)] = 3\mathbb{E}[N^2] - 2\mathbb{E}[N] = 3[\text{Var}(N) + \mathbb{E}(N)^2] - 2\mathbb{E}(N)\\
&= 3\left[\frac{1-\alpha}{\alpha^2} - \frac{(1-\alpha)^2}{\alpha^2}\right] - \frac{2-2\alpha}{\alpha} = \frac{1-\alpha}{\alpha}\left(\frac{3-3+3\alpha-2\alpha}{\alpha}\right) = \frac{1-\alpha}{\alpha}\end{aligned}$$

**Exercise 2.4.5** Let $U \sim \text{Unif}(0, L)$ for $L \sim xe^{-x}$. Then we see that the joint distribution can be expressed as

$$f_{U,L}(u, \ell) = f_{U|L}(u|\ell)f_L(\ell) = \frac{1}{\ell}\ell e^{-\ell} = e^{-\ell}$$

Now, let $V = L - U$ and $T = U$. This corresponds to $L = U + T$ and $U = T$. We can then express the joint distribution of $(T, V)$ or equivalently $(U, V)$ as follows.

$$f_{T,V}(t, v) = f_{U,L}(t, v + t)|\det(J)|$$

Where $J = \begin{bmatrix} \frac{\partial T}{\partial U} & \frac{\partial T}{\partial L} \\ \frac{\partial V}{\partial U} & \frac{\partial V}{\partial L} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Using this, we see that $|\det(J)| = 1$. Thus, we can write the final distribution as follows

$$f_{T,V}(t, v) = e^{-(v+t)} \quad \text{where} \quad 0 < u, v < \infty$$

**Problem 2.4.3** Suppose that $X|\lambda = \ell \sim \text{Pois}(\ell)$ where $\lambda \sim \text{Exp}(\theta)$.

(a)

$$\begin{aligned} P(X = n) &= \int_{\ell \in \Lambda} P(X = n|\lambda = \ell)f_\lambda(\ell)d\ell \\ &= \int_0^\infty \frac{\ell^n e^{-\ell}}{n!}\theta e^{-\ell\theta}d\ell \\ &= \frac{\theta}{n!}\int_0^\infty \ell^n e^{-(\theta+1)\ell}d\ell \\ &= \frac{\theta}{n!}\frac{\Gamma(n+1)}{(\theta+1)^{n+1}}\left(\frac{(\theta+1)^{n+1}}{\Gamma(n+1)}\int_0^\infty \ell^n e^{-(\theta+1)\ell}d\ell\right) \\ &= \frac{\theta}{(1+\theta)^{n+1}} \quad \text{for} \quad k = 0, 1, 2, \ldots \end{aligned}$$

(b) Notice here that

$$f_{\lambda|X}(\ell|n) = \frac{f_{\lambda,X}(\ell, n)}{f_X(n)} = \frac{f_{X|\lambda}(n|\ell)f_\lambda(\ell)}{f_X(n)}$$

which are all known quantities. Hence, we can write

$$f_{\lambda|X}(\ell|n) = \frac{e^{-\ell}\ell^n}{n!}\theta e^{-\theta\ell}\left(\frac{\theta}{(\theta+1)^{(n+1)}}\right)^{-1} = \frac{1}{n!}\ell^n e^{-(\theta+1)\ell}(\theta+1)^{n+1} = \frac{\beta^\alpha}{\Gamma(\alpha)}\ell^{\alpha-1}e^{-\beta\ell}$$

for $\beta = \theta + 1$ and $\alpha = n + 1$. This gives that $\lambda|X \sim \text{Gamma}(n + 1, \theta + 1)$

**Problem 2.4.7** We first find the joint distribution of the pair $(X, Y)$. Let $Z = X + Y$ and $U = X$. Then this is equivalent to writing $Y = Z - U$ and $X = U$. Then we can express this joint distribution as

$$f_{X,Z}(u, x) = f_{U,Z}(u, z) = f_{(X,Y)}(u, z - u)|\det(J)| \overset{indep.}{=} f_X(u)f_Y(z - u)|\det(J)|$$

where $J = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial Z}{\partial X} & \frac{\partial Z}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Hence $|\det(J)| = 1$ and we arrive at

$$f_{X,Z}(u,z) = \alpha e^{-\alpha x}\alpha e^{-\alpha(z-x)} = \alpha^2 e^{-\alpha z}$$

Now, notice that $Z = X + Y$ is a sum of iid exponential distributions so $Z \sim \text{Gamma}(2,\alpha)$. Using this fact, we can write the conditional distribution as follows.

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{\alpha^2 e^{-\alpha z}}{\alpha^2/\Gamma(2)z^{2-1}e^{-\alpha z}} = \frac{1}{z}$$

Hence, $X|Z = z \sim \text{Unif}(0,z)$.

**Problem 2.5.1** Suppose that $\{X_n, n \geq 1\}$ is a martingale. By the law of total probability we have

$$\begin{aligned}
\mathbb{E}[X_{n+2}|X_0,\dots,X_n] &= \mathbb{E}[\mathbb{E}\{X_{n+2}|X_0,\dots,X_n,X_{n+1}\}|X_0,\dots,X_n] \\
&= \mathbb{E}[X_{n+1}|X_0,\dots,X_n] \\
&= X_n
\end{aligned}$$

**Problem 2.5.3** Let $S_n = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$ be the sum of $n$ independent random variables. Where $\epsilon_i \overset{iid}{\sim} \text{Exp}(1)$. Let $X_n = 2^n\exp(-S_n)$. We note, we can write this value as $X_n = \prod_{i=1}^n 2\exp(-\epsilon_i)$. To see why $X_n$ is a martingale, we first note that since $\epsilon_i \sim \text{Exp}(1)$ that $S_n > 0$. Thefore, $-S_n < 0$ and $\exp(-S_n) < 1$. Hence we see that

$$\mathbb{E}|X_n| = \mathbb{E}(X_n) = \mathbb{E}[2^n\exp(-S_n)] \leq \mathbb{E}[2^n] = 2^n < \infty$$

Hence $\mathbb{E}|X_n| < \infty$ for all $n$. To prove the second property, we use the independence of $\epsilon_i$ as follows

$$\begin{aligned}
\mathbb{E}[X_{n+1}|X_0,\dots,X_n] &= \mathbb{E}[2^{n+1}\exp(-S_{n+1})|X_0,\dots,X_n] \\
&= \mathbb{E}[2^n\exp(-S_n)2\exp(-\epsilon_{n+1})|X_0,\dots,X_n] \\
&= X_n\mathbb{E}[2\exp(-\epsilon_{n+1})|X_0,\dots,X_n] \\
&= X_n\mathbb{E}[2\exp(-\epsilon_{n+1})] \\
&= X_n
\end{aligned}$$