Exercise 8.1 If $|X_n| \leq |Y_n|$ almost surely and Y_n is uniformly integrable then X_n is uniformly integrable.

Solution $|X_n| \leq |Y_n|$ almost surely, so for any given $a \in \mathbb{R}$,

$$\{\omega : |X_n(\omega)| > a\} \subset \{\omega : |Y_n(\omega)| > a\}$$

This implies

$$I_{|X_n|>a}(x) \leq I_{|Y_n|>a}(x)$$

Moreover since expectation is linear $|X_n| \leq |Y_n|$ implies

$$\mathbb{E}|X_n| \leq \mathbb{E}|Y_n|$$

Now, all inequalities here are between two positive numbers so we can multiple them together to see

$$\mathbb{E}|X_n|I_{|X_n|>a}(x) \leq \mathbb{E}|Y_n|I_{|Y_n|>a}(x)$$

Recall that Y_n is uniformly integrable so $\lim_{a\to\infty} \mathbb{E}|Y_n|I_{|Y_n|>a}(x) = 0$ uniformly in n. Thus

$$\lim_{a\to\infty} \mathbb{E}|X_n|I_{|X_n|>a}(x) = 0 \quad \text{uniformly in } n$$

Therefore, X_n is uniformly integrable.

Exercise 8.2 If X_n is increasing and $X_n \overset{P}{\to} X$ then $X_n \overset{a.s.}{\to} X$.

Solution Let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \epsilon$. Let $A_n = \big\{\omega : |X_n(\omega) - X(\omega)| < \frac{1}{k}\big\}$. Then by monotonicity of X_n we see that $A_n \subset A_{n+1}$. Notice that by monotonicity and convergence in probability we see

$$P(\lim_{n\to\infty} |X_n - X| = 0) = P\Big(\bigcap_{n=1}^{\infty} A_n\Big) \overset{Monont.}{=} \lim_{n\to\infty} P(A_n) \overset{CinP}{=} 1$$

Therefore $X_n \overset{a.s.}{\to} X$

Exercise 8.3 Show that $\mathbb{R}$ has the subsubsequence property.

Solution($\Longrightarrow$) Suppose that $a_n \to a$ pointwise. Then for $\epsilon > 0$, then there exists $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$ we have $|a_n - a| < \epsilon$. Now for any subsequence a_{n_k}, for $n_k > N(\epsilon)$ we have $|a_{n_k} - a| < \epsilon$. Lastly for any further subsequence $a_{n_{k_j}}$, if $n_{k_j} > N(\epsilon)$ we see $|a_{n_{k_j}} - a| < \epsilon$. Hence every subsequence has a subsequence that converges pointwise a.

($\Longleftarrow$) Suppose that for any subsequence a_{n_k} there exists a further subsequence $a_{n_{k_k}} \to a$. Now, for the sake of contradiction, assume that $a_n \stackrel{P}{\not\to} a$. This implies for $\epsilon > 0$ that there exists a_{n_k} such that $|a_{n_k} - a| > \epsilon$ *for all* n_k. Then any further subsequence, $a_{n_{k_j}}$ we have $|a_{n_{k_j}} - a| > \epsilon$ by construction. Therefore, we see there is a sequence such that there does not exist any subsequence that converges to a. This is a contradiction to our initial assumption. Therefore, we must have $a_n \to a$.

Exercise 8.4 Show that convergence in probability has the subsubsequence property.

Solution($\Longrightarrow$) Suppose that $X_n \xrightarrow{P} X$. Then we know that every subsequence of X_n has a further subsequence such that $X_{n_{k_j}} \xrightarrow{a.s.} X$. Recall that almost sure convergence implies convergence in probability. Thus, we see that $X_{n_{k_j}} \xrightarrow{P} X$.

($\Longleftarrow$) Assume that for every subsequence of X_n, there exists a further subsequence such that $X_{n_{k_j}} \xrightarrow{P} X$. Now assume for the sake of contradiction that $X_n \not\xrightarrow{P} X$. Then there exists some X_{n_k} such that for some $\epsilon > 0$ and $\delta > 0$ that $P(|X_{n_k} - X| > \delta) > \epsilon$ for all n_k. Then by construction, we see $\lim_{j\to\infty} P(|X_{n_{k_j}} - X| > \delta) > \epsilon$. Hence there is a subsubsequence that does not converge in probability. Thus we have a contradiction and see that $X_n \xrightarrow{P} X$.

Exercise 8.5 Show that
$$d(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$
gives a metric.

Solution

1. First note that $\frac{|X-Y|}{1+|X-Y|} \geq 0$ so $d(X,Y) \geq 0$. Now if $d(X,Y) = 0$ iff $\int \frac{|X-Y|}{1+|X-Y|} dF(X,Y) = 0$. Notice that the integrand is nonnegative and $F(X,Y) \geq 0$. So this implies $\int \frac{|X-Y|}{1+|X-Y|} dF(X,Y) = 0$ iff $\frac{|X-Y|}{1+|X-Y|} \overset{a.s}{=} 0$ iff $|X-Y| \overset{a.s}{=} 0$ iff $X \overset{a.s}{=} Y$.

2. $$d(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right] = \mathbb{E}\left[\frac{|Y-X|}{1+|Y-X|}\right] = d(Y,X)$$

3. First note that $|X-Y| \leq |X-Z| + |Z-Y|$ by the triangle inequality. Moreover, by the results proved in 8.6 with $\epsilon = |X-Y|$ and $a = |X-Z| + |Z-Y|$ we have
$$\begin{aligned}\frac{|X-Y|}{1+|X-Y|} &\leq \frac{|X-Z|+|Z-Y|}{1+|X-Z|+|Z-Y|} \\ &= \frac{|X-Z|}{1+|X-Z|+|Z-Y|} + \frac{|X-Z|}{1+|X-Z|+|Z-Y|} \\ &\leq \frac{|X-Z|}{1+|X-Z|} + \frac{|X-Z|}{1+|Z-Y|}\end{aligned}$$
Now applying the expectation we see
$$\mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right] \leq \mathbb{E}\left[\frac{|X-Z|}{1+|X-Z|+|Z-Y|}\right] + \mathbb{E}\left[\frac{|X-Z|}{1+|X-Z|+|Z-Y|}\right]$$
Thus $d(X,Y) \leq d(X,Z) + d(Z,Y)$

Exercise 8.6 Show that if $a > 0$ that $a > \epsilon$ iff $\frac{a}{1+a} > \frac{\epsilon}{1+\epsilon}$

Solution

$$\begin{aligned} a &> \epsilon \\ a + a\epsilon &> \epsilon + a\epsilon \\ a(1+\epsilon) &> \epsilon(1+a) \\ \frac{a}{1+a} &> \frac{\epsilon}{1+\epsilon} \end{aligned}$$

Exercise 8.7 Show that $X_n \xrightarrow{P} X$ is equivalent to $d(X_n, X) \to 0$.

Solution First let $Z_n = \frac{|X_n - X|}{1+|X_n-X|}$. Then $d(X_n, X) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right] = \mathbb{E}(Z_n)$. ($\Longleftarrow$) Now if $d(X_n, X) \to 0$ then $\mathbb{E}(Z_n) \to 0$ and $\lim_{n\to\infty} \int Z_n dF(Z_n) = 0$. Since $Z_n \geq 0$ and $F(Z_n) \geq 0$ this implies that $\lim_{n\to\infty} Z_n \overset{a.s}{=} 0$ or $Z_n \xrightarrow{P} 0$. This implies for any $\epsilon > 0$

$$\lim_{n\to\infty} P\Big(\frac{|X_n - X|}{1+|X_n - X|} < \epsilon\Big) = 1$$
$$\lim_{n\to\infty} P\Big(|X_n - X| < \epsilon + \epsilon|X_n - X|\Big) = 1$$
$$\lim_{n\to\infty} P\Big(|X_n - X| < \frac{\epsilon}{1-\epsilon}\Big) = 1$$

ϵ was arbitrary so $X_n \xrightarrow{P} X$. ($\Longrightarrow$) If $X_n \xrightarrow{P} X$ then by Slutksy $Z_n = \frac{|X_n-X|}{1+|X_n-X|} \xrightarrow{P} 0$. This implies that

$$\lim_{n\to\infty} \mathbb{E}(Z_n) = \lim_{n\to\infty} \int Z_n dF(Z_n) = 0$$

But notice that

$$\lim_{n\to\infty} \mathbb{E}(Z_n) = \lim_{n\to\infty} d(X_n, X) = 0$$