Example:

$Y_i \mid \theta \overset{ind}{\sim} Po(x_i \theta)$

$\theta \sim Gamma(\alpha, \beta)$

$\theta \mid y \sim Gamma(\alpha + \sum_i y_i, \beta + \sum x_i)$

$\tilde{y} \mid y \sim NB\left(\alpha + \sum y_i, \frac{\tilde{x}}{\tilde{x} + \beta + \sum x_i}\right)$

Data

X	Y
1	2
2	4
4	16

Prior

- On average 4 tr/hr
- Most likely between $1-6$/hr.

$\mathbb{E}[\theta] = 4 = \frac{\alpha}{\beta} \implies \boxed{\alpha = 4\beta}$

$\boxed{\mathbb{P}(\theta \in [1,6]) = 0.95}$

System of Eqs for prior elicitation

Solving this system numerically yields $\beta \approx 3, \alpha \approx 12$

- Can then do post inference using $Gamma(\alpha + \sum y_i, \beta + \sum x_i)$
- Rmk: variance should always decrease in prior $\longrightarrow$ post comparison

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Suppose that $z = \{1, 2, \dots, k\}$ has distribution

$\mathbb{P}(z = j) = \theta_j : \sum_{j=1}^{k} \theta_j = 1$ then $z \sim Categorical(k, \theta = (\theta_1, \dots, \theta_k))$

$z_i \mid \theta \overset{iid}{\sim} Cat(k, \theta)$ and we summarize them with $Y_j = \sum_{i=1}^{n} I(z_i = j)$

where $\sum_j y_j = n$ generalization of the binomial

In this case we say $Y|\theta \sim \text{Multinomial}(n, \theta)$

$$\mathbb{P}(Y|\theta) \propto \prod_{j=1}^{k} \theta_j^{y_j} \xrightarrow[\text{prior}]{\text{conjugate}} \mathbb{P}(\theta) \propto \prod_{j=1}^{k} \theta_j^{\alpha_j - 1}$$

looks like beta

$(\theta)^{\alpha-1}(1-\theta)^{\beta-1}$

different value for each class

$$\xrightarrow{\text{post.}} \mathbb{P}(\theta|Y) = \prod_{j=1}^{k} \theta_j^{y_j + \alpha_j - 1}$$

In this case $\theta \sim \text{Dirichlet}(\alpha)$ and

$\theta|Y \sim \text{Dirichlet}(\alpha + y)$

- Generalization of Beta