Recall

Census	i
Captured	c_i
marked	m_i
newly marked	$n_i = c_i - m_i$

Total newly marked $\quad m_i = \sum_{j=1}^{i-1} n_j$

$$c_i \mid N, \alpha_i \overset{ind}{\sim} \text{Binom}(N, \alpha_i)$$

$$m_i \mid c_i, N, m_i \sim HG(N, m_i, c_i)$$

$$P(N) \propto 1 \qquad \alpha_i \overset{iid}{\sim} U(0,1)$$

$$\Downarrow$$

$$\mathbb{P}(N=0)=\mathbb{P}(N=1)=\cdots=\mathbb{P}(N=\underbrace{i}_{\text{any value}})$$

$$\mathbb{P}(N,\alpha \mid c,m) \propto \prod_{i=1}^{k} \binom{N-m_i}{c_i-m_i} \alpha_i^{c_i} (1-\alpha_i)^{N-c_i}$$

$$\propto \mathbb{P}(c,m \mid N,\alpha)\, \mathbb{P}(N,\alpha)$$

$$\overset{\text{ind}}{=} \mathbb{P}(c,m \mid N,\alpha)\, \mathbb{P}(N)\, \mathbb{P}(\alpha)$$

$$\propto \mathbb{P}(c,m \mid N,\alpha)$$

So

$$\mathbb{P}(N,\alpha \mid c,m) \propto \mathbb{P}(c,m \mid N,\alpha)$$

$$\prod_{i=1}^{k} \binom{N-m_i}{c_i-m_i}$$

$$= \frac{N - m_1}{(N - m_1 - n_1)!\, n_1!} \times \cdots \times \frac{N - m_k}{(N - m_k - n_k)!\, n_k!}$$

$$m_i = \sum_{k=1}^{i-1} n_k = m_{i-1} + n_{i-1}$$

hence

$$= \frac{(N - m_1)!}{(N - \cancel{m_2})!\, n_1!} \times \cdots \times \frac{\cancel{(N - m_k)!}}{(N - m_k - n_k)!\, n_k!}$$

$$= \frac{(N - m_1)!}{n_1!\, n_2! \cdots n_k!\, (N - m_k - n_k)!}$$

$$= \frac{N!}{\prod_{i=1}^{k} n_i!\, (N - (\underbrace{m_k + n_k}_{\sum^{k} n_i}))!}$$

$$= r = \sum_{i=1} n_i$$

Therefore

$$P(N, \alpha \mid c, m) \propto \frac{N!}{(N-r)!} \prod_{i=1}^{k} \alpha_i^{c_i} (1-\alpha_i)^{n-c_i}$$

Hard to sample from, use MCMC (Gibbs) to achieve samples.

We can construct a gibbs sampler based on the conditionals

$$P(\alpha_i \mid c, m, N, \alpha_{[-i]}) \propto \alpha_i^{c_i} (1-\alpha_i)^{N-c_i}$$

$$\Longrightarrow \alpha_i \mid m, c, N, \alpha_{[-i]} \sim \text{Beta}(c_i+1, N-c_i+1)$$

$$P(N \mid c, m, \alpha) \propto \frac{N!}{} \prod_{i=1}^{n} (1-\alpha_i)^N$$

$$(N-r)!$$

$$\Longrightarrow \quad X \sim NB(r,p) \quad \mathbb{P}(X=x) = \binom{x+r-1}{x} p^r (1-p)^x$$

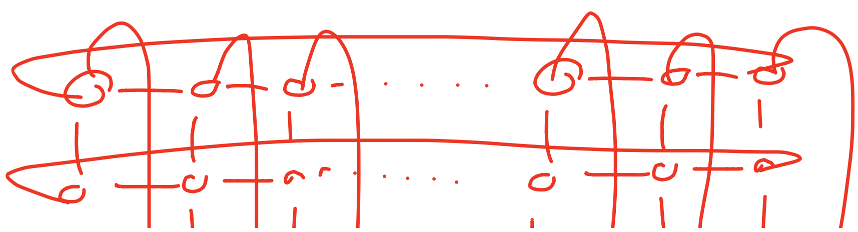
$$\mathbb{P}(N \mid \cdot) \propto \frac{((N+r)-(r+1)-1)!}{(N-r)!} \times$$

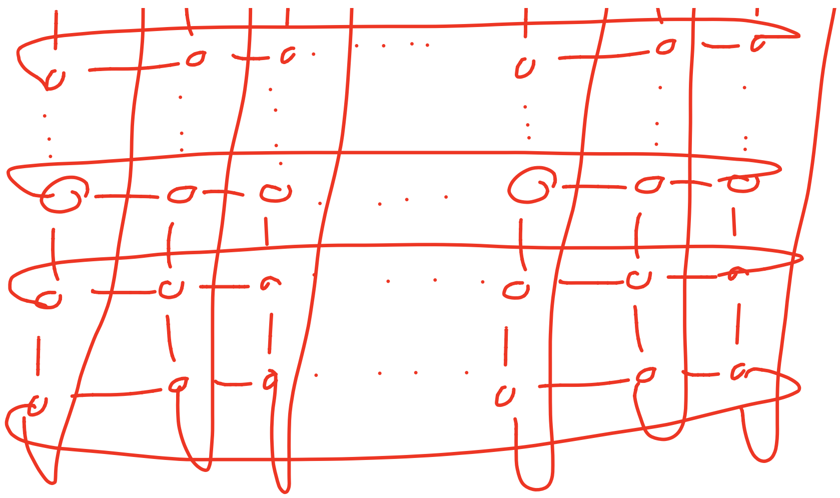
$$\left[\prod_{i=1}^{k}(1-\alpha_i)\right]^{N-r} \left[1-\prod_{i=1}^{k}(1-\alpha_i)\right]^{r+1}$$

Therefore we see

$$N-r \mid m, c, \alpha \sim NB\left(r+1,\ 1-\prod_{i=1}^{k}(1-\alpha_i)\right)$$

Another example





n×n lattice with periodic boundary $X \in \{-1, 1\}$

define the energy as

$$E(x) = -\sum_{i,j \in L} x_i x_j$$

$$P(x) \propto \exp\left\{-\frac{1}{T} E(x)\right\}$$

T is temperature

“T [illegible]”

Ising Model

Finding the normalizing constant is hard

$$Z = \sum_{x} \exp\left\{-\frac{1}{T} E(x)\right\}$$

$$= O\left(n^2 2^{n+1}\right)$$

Last resort MCMC

Note: $P(x)$ is Gibbs wrt L.

$\Rightarrow$ When sampling we only need to consider a nodes 4 neighbors.

$$P(x_i \mid x_{[-i]}) = \frac{P(x)}{\sum P(\dots)}$$

$$\sum_{\tilde{x}_i \in \{-1,1\}} \pi(x_i, x_i)$$

$$= \frac{\exp\left\{-\frac{1}{T} E(x)\right\} / \cancel{Z}}{\sum_{\tilde{x}_i} \exp\left\{-\frac{1}{T} E(\tilde{x}_i, x_{[-i]})\right\} / \cancel{Z}}$$

$$= \frac{\exp\left\{\frac{1}{T} \sum_{u,v \in L} x_u x_v\right\}}{\sum_{\tilde{x}_i} \exp\left\{\frac{1}{T} \sum_{L} x_v x_i\right.}$$

$$= \frac{\exp\left\{\cancel{\frac{\sum x_i x_v}{T}}\right\} \exp\left\{\frac{1}{T} \sum_{j \in N_i} x_i x_j\right\}}{\sum_{\tilde{x}_i} \exp\left\{\cancel{\frac{\sum x_v x_u}{T}}\right\} \exp\left\{\frac{1}{T} \sum_{N_i} \tilde{x}_i x_j\right\}}$$

Hence

$$\mathbb{P}(X_i = 1 \mid X_{[-i]}) =$$

$$\frac{\exp\left\{ \sum_{j \in N_i} \frac{x_j}{T} \right\}}{\exp\left\{ \sum_{j \in N_i} \frac{x_j}{T} \right\} + \exp\left\{ -\sum_{j \in N_i} \frac{x_j}{T} \right\}}$$

$$= \left(1 + \exp\left\{ -2 \sum x_j / T \right\} \right)^{-1}$$

$$\overset{\log}{\Longrightarrow} \; -\log\left(1 + \exp\left\{ -2 \sum_{j \in N_j} x_j / T \right\} \right)$$

$$= -\log 1pe\left(-2 \sum_{N_j} x_j / T \right)$$