Recall: We want to conduct a MC simulation using P but it is hard to sample from P

Idea: Build MC X_t with stat. dist. P.

Then by Ergodic Theory

$$X_1, \ldots, X_t, \ldots \sim MC$$

$$\frac{1}{T} \sum_{t=1}^{T} f(\xi^{(t)}) \longrightarrow \mathbb{E}_P(f(X))$$
$$X \sim P$$

We build the chain using Metropolis-Hastings

We need

- P - Proposal: $Q(\xi_i \mid \xi_i)$

- Acceptance Ratio

$$R(\xi^*, \xi^{(t)}) = \frac{P(\xi^*)\, Q(\xi^{(t)} \mid \xi^*)}{P(\xi^{(t)})\, Q(\xi^* \mid \xi^{(t)})}$$

<u>Alg:</u>

1. Start arb. $\xi^{(t)}$

2. For $i = 1, 2, \ldots$ until convergence

   (i) Sample $\xi^* \sim Q(\cdot \mid \xi^{(t)})$

   (ii) $\xi^{(t+1)} = \xi^*$ with prob $\min\{1, R(\xi^*, \xi^{(t)})\}$

   $\xi^{(t+1)} = \xi^{(t)}$ o.w.

<u>How to choose Q</u>

<u>Independent Chain M.H.</u>

$$Q(\xi_1 \mid \xi_2) = Q(\xi_1)$$

then

$$R(\xi^*, \xi^{(t)}) = \frac{P(\xi^*) / Q(\xi^{(*)})}{P(\xi^{(t)}) / Q(\xi^{(t)})}$$

If $Q(\cdot) \simeq P$ then $R \simeq 1$

<u>Ex</u>: Bayesian Inference

$\mathbb{P}(\theta \mid x)$ posterior

$Q \equiv \mathbb{P}(\theta)$ prior

$$R(\theta^*, \theta^{(t)}) = \frac{P(\theta^*) / Q(\theta^*)}{P(\theta^{(t)}) / Q(\theta^{(t)})}$$

$$= \frac{\mathbb{P}(\theta^* \mid x) / \mathbb{P}(\theta^*)}{\quad}$$

$$\mathbb{P}(\theta^{(*)}|x) \,/\, \mathbb{P}(\theta^{(t)})$$

$$= \frac{\mathbb{P}(x|\theta^*)/\mathbb{P}(x)}{\mathbb{P}(x|\theta^{(t)})/\mathbb{P}(x)}$$

$$= \frac{\mathbb{P}(x|\theta^*)}{\mathbb{P}(x|\theta^{(t)})}$$ likelihood ratio

<u>Ex</u>: θ = prob of heads

X = # heads of n coin flips

$$\theta \sim \text{Beta}(\alpha, \beta)$$

$$X|\theta \sim \text{Binomial}(n, \theta)$$

$$R(\theta^*, \theta^{(t)}) = \frac{\binom{n}{x}(\theta^*)^x(1-\theta^*)^{1-x}}{\binom{n}{x}(\theta^t)^x(1-\theta^t)^{1-x}}$$

$$= \left(\frac{\theta^*}{\theta^{(t)}}\right)^x \left(\frac{1-\theta^*}{1-\theta^{(t)}}\right)^{1-x}$$

Random Walk MH

Proposal $\xi^* = \xi^{(t)} + \delta \qquad \delta \sim g(\cdot \mid \xi^{(t)})$

$\hookleftarrow$ pertubation

$$Q(\xi^* \mid \xi^{(t)}) = g\left(d(\xi^*, \xi^{(t)}) \mid \xi^{(t)}\right)$$

If $\delta \sim g(\cdot)$ "location independence"

$$R(\xi^*, \xi^{(t)}) = \frac{P(\xi^*)\, \cancel{g(d(\xi^{(t)}, \xi^*))}}{P(\xi^{(t)})\, \cancel{g(d(\xi^*, \xi^{(t)}))}}$$

$$= \frac{P(\xi^*)}{P(\xi^{(t)})}$$

"Metropolis Algorithm"

Ex: $\theta^* = \theta^{(t)} + \delta \quad \delta \sim \text{Unif}(-s, s)$

Issues: Hard to tune how large the perturbations should be.

Gibbs Sampling

MH with a proposal Q. s.t.

$R(\xi^*, \xi^*) = 1$

Suppose $X = (x_1, \dots, x_n)$

Alg:

1. Start at $\xi^{(1)}$ arbitrarily

2. For $t = 1, 2, \ldots$ until con

(i) Sample $\xi_1^{(t+1)} \sim P(X_1 \mid \xi_2^{(t)}, \ldots, \xi_n^{(t)})$

$\xi_2^{(t+1)} \sim P(X_2 \mid \xi_1^{(t+1)}, \ldots, \xi_n^{(t)})$

Cycle through.

$\vdots$

$\xi_n^{(t+1)} \sim P(X_n \mid \xi_1^{(t+1)}, \ldots, \xi_{n-1}^{(t+1)})$