1. We begin with stating the theorem we are trying to prove. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $\mathbb{E}(X_k) = \mu_k$ and $\text{Var}(X_k) = \sigma_k^2 < \infty$. Define $s_n^2 = \sum_{i=1}^n \sigma_k^2$ and for some $\delta > 0$ define the following quantity

$$L_3(n) := \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E}[|X_i - \mu_i|^{2+\delta}]$$

The Lyapunov condition is satisfied if $\lim_{n\to\infty} L_3(n) = 0$. We will call this condition L_3. The Lyapunov CLT states that if L_3 holds then

$$\frac{1}{s_n}\sum_{i=1}^n (X_i - \mu_i) \xrightarrow{D} N(0,1)$$

Recall by the Lindeberg CLT, we showed that $L_2 \iff L_1 + CLT$. Therefore it suffices to show that $L_3 \implies L_2$. Consider the following

$$\begin{aligned}
L_2(n) &= \frac{1}{s_n^2}\sum_{i=1}^n \mathbb{E}[|X_i - \mu_i|^2 \mathbf{1}_{|X_i - \mu_i| > \epsilon s_n}] \\
&= \frac{1}{s_n^2}\sum_{i=1}^n \mathbb{E}[|X_i - \mu_i|^{2+\delta}|X_i - \mu_i|^{-\delta} \mathbf{1}_{|X_i - \mu_i| > \epsilon s_n}] \\
&\leq \frac{1}{s_n^2}\sum_{i=1}^n (\epsilon s_n)^{-\delta}\mathbb{E}[|X_i - \mu_i|^{2+\delta} \mathbf{1}_{|X_i - \mu_i| > \epsilon s_n}] \\
&\leq \frac{1}{\epsilon^\delta}\frac{1}{s_n^{2+\delta}}\sum_{i=1}^n \mathbb{E}[|X_i - \mu_i|^{2+\delta}] \\
&= \frac{1}{\epsilon^\delta} L_3(n)
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} L_2(n) \leq \frac{1}{\epsilon^\delta} L_3(n)$$

Therefore, if $L_3(n) \underset{n\to\infty}{\longrightarrow} 0$ then $L_2(n) \underset{n\to\infty}{\longrightarrow} 0$ and by the Lindeberg CLT

$$\frac{1}{s_n}\sum_{i=1}^n (X_i - \mu_i) \xrightarrow{D} N(0,1)$$

2. Using Boole's inequality we can write the following

$$\begin{aligned}
\mathbb{P}\left[\max_{1\leq k\leq n} |X_k - \mu_k| \geq \epsilon s_n\right] &= \mathbb{P}\left[\bigcup_{k=1}^n \{|X_k - \mu_k| \geq \epsilon s_n\}\right] \\
&\leq \sum_{k=1}^n \mathbb{P}\left[|X_k - \mu_k| \geq \epsilon s_n\right] = \sum_{k=1}^n \mathbb{E}\left[\mathbf{1}_{|X_k - \mu_k| \geq \epsilon s_n}\right]
\end{aligned}$$

Now, in this expectation we have

$$|X_k - \mu_k| > \epsilon s_n \implies (X_k - \mu_k)^2 > \epsilon^2 s_n^2 \implies \frac{|X_k - \mu_k|}{\epsilon^2 s_n^2} > 1$$

Using this fact we see that we can write

$$\sum_{k=1}^{n} \mathbb{E}\left[\mathbf{1}_{|X_k-\mu_k|\geq \epsilon s_n}\right] \leq \sum_{k=1}^{n} \mathbb{E}\left[\frac{|X_k - \mu_k|}{\epsilon^2 s_n^2}\mathbf{1}_{|X_k-\mu_k|\geq \epsilon s_n}\right] = \frac{1}{\epsilon^2}L_2(n)$$

Hence,

$$\limsup_{n\to\infty}\mathbb{P}\left[\max_{1\leq k\leq n}|X_k - \mu_k| \geq \epsilon s_n\right] \leq \frac{1}{\epsilon^2}\limsup_{n\to\infty} L_2(n)$$

Therefore, under the Lindeberg conditions, $L_2(n) \underset{n\to\infty}{\longrightarrow} 0$ and we see that

$$\mathbb{P}\left[\max_{1\leq k\leq n}|X_k - \mu_k| \geq \epsilon s_n\right] \underset{n\to\infty}{\longrightarrow} 0$$

This shows that no single variable's variance can dominate s_n under the Lindeberg conditions.

3. (a) Suppose $X \sim f$ where $f(x) = |x|^{-3}\mathbf{1}_{(\infty,-1]\cup[1,\infty)}(x)$. Using the second order Taylor approximation, and the fact that f is symmetric we can write the following

$$\phi(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[1 + itX + \frac{(itX)^2}{2} + \mathcal{O}(t^2)] = 1 + \frac{(itX)^2}{2} + \mathcal{O}(t^2)$$
$$= 1 - t^2\left(\frac{\mathbb{E}(X^2)}{2} + \mathcal{O}(1)\right)$$

Now notice that

$$\mathbb{E}(X^2) = \int_{(\infty,-1]\cup[1,\infty)} \frac{x^2}{|x|^3}dx = 2\int_1^\infty \frac{x^2}{x^3}dx = 2\lim_{t\to 0}\int_1^{1/t}\frac{1}{x}dx = 2\lim_{t\to 0}\log\left(\frac{1}{|t|}\right)$$

Using this representation we see

$$\phi(t) = 1 - t^2\left(\frac{1}{2} * 2\log(\frac{1}{|t|}) + \mathcal{O}(1)\right) \quad \text{as } t \to 0$$
$$= 1 - t^2\left(\log\frac{1}{|t|} + \mathcal{O}(1)\right) \quad \text{as } t \to 0$$

(b) To show the result, we will show that $\phi_{S_n/\sqrt{n\log(n)}}(t) \longrightarrow e^{-t^2/2}$ for all $t \in \mathbb{R}$. First consider the following,

$$\phi_{S_n/\sqrt{n\log(n)}}(t) = \mathbb{E}\left[\exp\left\{\frac{it}{\sqrt{n\log(n)}}\sum_{k=1}^n X_k\right\}\right] \overset{ind.}{=} \prod_{k=1}^n \mathbb{E}\left[\exp\left\{\frac{it}{\sqrt{n\log(n)}}X_k\right\}\right]$$
$$= \prod_{k=1}^n \phi_{X_k}\left(\frac{t}{\sqrt{n\log(n)}}\right) \overset{i.d.}{=} \left[\phi_{X_1}\left(\frac{t}{\sqrt{n\log(n)}}\right)\right]^n$$

Using the expression from part (a) above, we can continue to write

$$\begin{aligned}\phi_{S_n/\sqrt{n\log(n)}}(t) &= \left[1 - \frac{t^2}{n\log(n)}\left(\log\frac{\sqrt{n\log(n)}}{|t|} + \mathcal{O}(1)\right)\right]^n \\ &= \left[1 - \frac{1}{n}\times\frac{t^2\log(\sqrt{n\log(n)}/|t|)}{\log(n)} + \frac{\mathcal{O}(t^2)}{n\log(n)}\right]^n\end{aligned}$$

Therefore, in the limit we see that

$$\lim_{n\to\infty}\phi_{S_n/\sqrt{n\log(n)}}(t) = \lim_{n\to\infty}\left[1 - \frac{t^2}{n}\times\frac{\log(\sqrt{n\log(n)}/|t|)}{\log(n)}\right]^n$$

Now, recall that if $c_n \to c$ then $(1+\frac{c_n}{n})^n \to e^c$. Therefore, it suffices to show that

$$\frac{\log(\sqrt{n\log(n)}/|t|)}{\log(n)} \longrightarrow 1/2$$

Seeing that the numerator and denominator go to infinity as $n \to \infty$ we use L'Hopital's Rule to write

$$\begin{aligned}\lim_{n\to\infty}\frac{\log(\sqrt{n\log(n)}/|t|)}{\log(n)} &\overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{|t|}{\sqrt{n\log(n)}}\frac{1}{2|t|}(n\log(n))^{-1/2}[\log(n)+1]}{1/n} \\ &= \lim_{n\to\infty}\frac{1}{2\log(n)}[\log(n)+1] \\ &= \lim_{n\to\infty}\frac{1}{2} + \frac{1}{2\log(n)} \\ &= \frac{1}{2}\end{aligned}$$

Thus, we conclude $\lim_{n\to\infty}\phi_{S_n/\sqrt{n\log(n)}}(t) = e^{-t^2/2}$ and as $n \to \infty$

$$\frac{S_n}{\sqrt{n\log(n)}} \xrightarrow{D} N(0,1)$$