Graph Matching

Suppose we have two adjacency matrices A, B

$$\underset{P \in \mathcal{P}_n}{\operatorname{argmin}} \| A - PBP^T \|_F^2$$

$$= \underset{P}{\operatorname{argmin}} \| AP - PB \|_F^2 \qquad \|X\|_F = \operatorname{tr} X^T X$$

$$= \underset{P}{\operatorname{argmax}} \operatorname{tr} APB^T P^T \qquad \text{NP-Hard}$$

Relax problem to double stochastic matrix. $\mathcal{D}_n$ Convex hull of $\mathcal{P}_n$

$$\underset{D \in \mathcal{D}_n}{\operatorname{argmin}} \|AD - DB\|_F^2$$ convex so

$\exists$ polynomial

$$\underset{D \in \mathcal{D}_n}{\operatorname{argmax}} \underbrace{\operatorname{tr} ADB^T D^T}_{f}$$ no polynomial

FAQ algorithm to solve this second problem

— Frank Wolfe gradient descent

Input: A, B $D_0 \in \mathcal{D}_n$ $k = 0$

while ! convergence

which dir $$P_k = \underset{P \in \mathcal{P}_n}{\operatorname{argmax}} \operatorname{tr} \underbrace{\nabla f(D_k)}_{= ADB} P$$

how

for $\quad \alpha_k = \underset{\alpha \in [0,1]}{\arg\max} \; \mathrm{tr} A \underbrace{D_{\alpha_k}}_{\| \atop \alpha D_k + (1-\alpha) P_k} B D_{\alpha_k}$

$$D_{k+1} = D_k$$

<u>Percolated</u>

Input: A, B, A_0 - seeds, r threshold

for $(i,j) \in A_0$

- add a mark to all neighboring pairs of (i,j)

- If the score of a pair $\geq r$ add it to M the match set

$Z \leftarrow A_0$ if $M/Z \neq \emptyset$

Randomly choose $(i, j) \in M/Z$

and use it as a new seed

$\rightarrow$ leave mark unchanged.

Issues: no way to correct

for mismatched

edges.

<u>Goal</u>: Correct this irreversibility

Modify just adding it to

M

if score of a pair is $\geq r \notin M$

if $[i, j]$ not a <u>conflict</u>

add it to M

else (i,j) conflict $[i', j]$

then if the score is higher.

replace $[i', j]$ by

$[i, j]$

and adjust for marks.

<u>First problem</u>

1. $(i,j) \in A_U$ incorrect

what is prob of correcting?

Under the correlated ER models

$G(n, p, s)$

p — prob of edge

"being in both"

$G(n; \Lambda, R)$ Λ = Prob

R = corr.

Denote $I_{ij}(t)$ = $[i,j]$ remains

match at t.

So if $i_t = j_t$

$$I_{ij}(t) = \begin{cases} 1 & ps^2 \quad i=j \\ 0 & p^2s^2 \quad i \neq j \end{cases}$$

$$M_{ij}(t) = \sum_{s=1}^{t} I_{ij}(s)$$

$X_{ij}(t) = (i,j)$ not corrected at t

$Y_{ij}(t) = (i,j)$ corrected at t.

$$X_{ij} = \bigcap_t X_{ij}(t)$$

$$Y_{ij} = \bigcup_t Y_{ij}(t)$$

$$\underbrace{\mathbb{P}(X_{ij})}_{\to 0} + \underbrace{\mathbb{P}(Y_{ij})}_{\to 1} = 1$$

wts.

$$\mathbb{P}(X_{ij}(t)) = \mathbb{P}\Big(\underbrace{\max\{M_{ii}(t)\ \ m_{jj}(t)\} < r}_{A}$$

or

$$\underbrace{r \le \max(\#) \le m_{ij}(t)}_{B}\Big)$$

$$\le \mathbb{P}(A) + \mathbb{P}(B)$$

$$\mathbb{P}(B)$$

$$= \mathbb{P}\left(\max \le M_{ij}(t) \cap r \le \max \right)$$

$$= \mathbb{P}\left(\bigcup_{a=d}^{t} \max \le a \quad M_{ij} = a \cap r \le \max \right)$$

$$= \mathbb{P}\left(\bigcup \{ \max \le a, M_{ij}t = a, \max \ge r \} \right)$$

$$= \sum_{a=r}^{t} \mathbb{P}\underbrace{\left(\max \le a, M_{ij} = a \right)}_{\le (ps)^{2a}}$$

$$= \sum_{a=r}^{t} (ps)^{2a} = \frac{ps^{2r}\left(1 - (ps)\right)^{2(t-r+1)}}{1-(ps)^2}$$

$$\le \frac{(ps)^{2r}}{1-(ps)^2}$$

$$P(X_{ij}) \le \frac{n(ps)^2}{} \longrightarrow 0$$

$1-(ps)^2$

1st $\quad ps = O\left(n^{-1/2r}\right)$