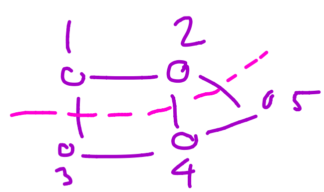
Recall we identify vertices in a graph with $X = (X_1, \ldots, X_n)$ to encode relationships.

$\mathbb{P}$ is MRF $\overset{?}{\Longrightarrow}$ $\mathbb{P}$ is Gibbs

Hammersley – Clifford

Last time we started discussing modeling the conditional distribution of "latent" variables X_B given the observed X_A.

Ex:



$\mathbb{P}(X_B | X_A)$ is Gibbs/MRF wrt

In general if $\mathbb{P}$ is Gibbs/MRF wrt G, what can we say?

$$\mathbb{P}(x_B|x_A) = \frac{\mathbb{P}(x_B, x_A)}{\sum_{\tilde{x}_B} \mathbb{P}(\tilde{x}_B, x_A)}$$

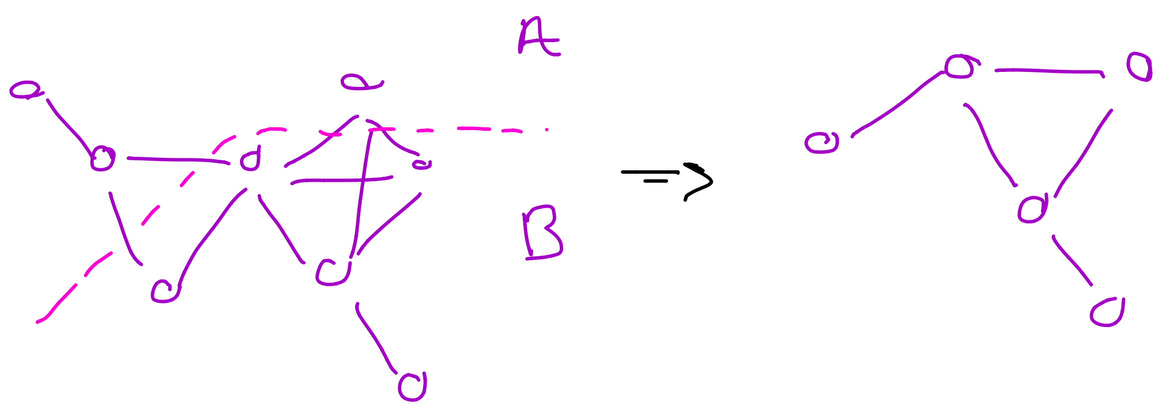
$$= \frac{\prod_c F_c(x_c)}{\sum_{\tilde{x}_B} \prod_c F_c(\tilde{x}_c)}$$

$$= \frac{\cancel{\prod_{\tilde{c}:\tilde{c}\cap B=\phi} F_{\tilde{c}}(x_{\tilde{c}})} \prod_{c:c\cap B\neq\phi} F_c(x_c)}{\sum \cancel{\prod_{\tilde{c}:\tilde{c}\cap B=\phi} F_{\tilde{c}}(x_{\tilde{c}})} \prod_{c\cdot c\cap B\neq\phi} F_c(\tilde{x}_c)}$$

$$= \frac{\prod_{c : c \cap B \neq \emptyset} F_c(x_c)}{\sum \prod_{c : c \cap B \neq \emptyset} F_c(\tilde{x}_c)}$$

only a function of nodes in B

So $\mathbb{P}(X_B | X_A)$ is Gibbs/MRF with respect to the subgraph enduced by B.



What if we want $\mathbb{P}(X_B)$?

<u>Ex:</u> 1 — 3   2 $\mathbb{P}(x_1, x_2, x_3)$?

$X_1, X_2 \mid X_3$ is Gibbs wrt

$$\mathbb{P}(X_1, X_2) = \sum_{X_3} \mathbb{P}(x_1, x_2, x_3)$$

$$\propto \underbrace{\sum_{x_3} F_{13}(x_1, x_3)\, F_{32}(x_3, x_2)}_{= \phi_{12}(x_1, x_2)}$$

Thus since in general

$$\phi_{12}(x_1, x_2) \neq \phi_1(x_1)\, \phi_2(x_2)$$

$\mathbb{P}(x_1, x_2)$ is Gibbs wrt 

In general

$$\mathbb{P}(X_{[-v]}) = \sum_{X_v} \mathbb{P}(X)$$

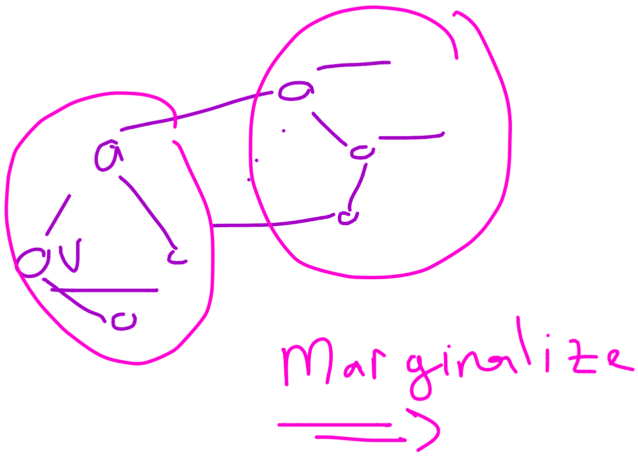
$$\propto \sum_{X_v} \prod_{c} F_c(X_c)$$

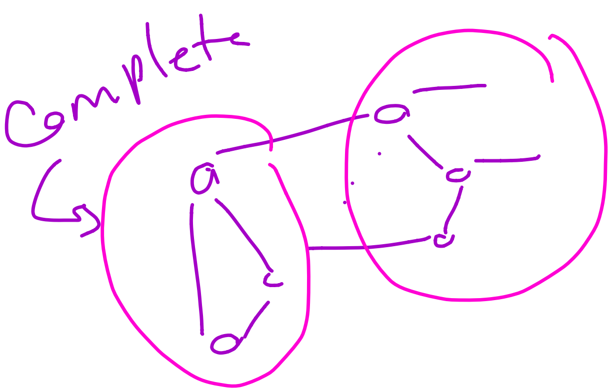
$$= \sum_{X_v} \prod_{\tilde{c}: v \notin \tilde{c}} F_{\tilde{c}}(X_{\tilde{c}}) \prod_{c: v \in c} F_c(X_c)$$

$$= \prod_{\tilde{c}: v \notin \tilde{c}} F_{\tilde{c}}(X_{\tilde{c}}) \sum_{X_v} \prod_{c: v \in c} F_c(X_c)$$

$$= \underbrace{\prod_{\tilde{c}: v \notin \tilde{c}} F_{\tilde{c}}(X_{\tilde{c}})}_{\text{potentials over all cliques}} \quad \underbrace{\phi_{N_v}(X_{N_v})}_{\text{potential over neighborhood.}}$$

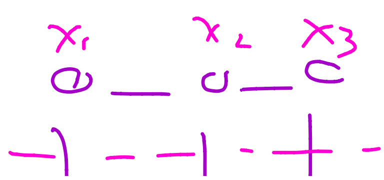
w/o v

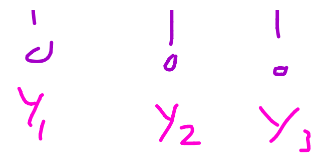




Rmk: Thus marginalizing X_v induces a complete subgraph in the neighborhood of v

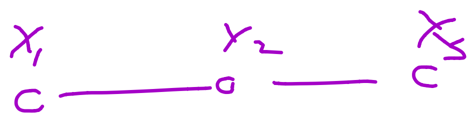
Ex: HMM-3





$$\mathbb{P}(Y|X) = \prod_{i=1}^{3} \mathbb{P}(Y_i|X_i)$$

$$\mathbb{P}(X) = \mathbb{P}(X_1)\mathbb{P}(X_2|X_1)\mathbb{P}(X_3|X_2)$$



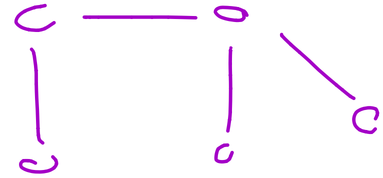
$\mathbb{P}(X|Y)$ is gibbs wrt the same graph.

$$\mathbb{P}(Y) = \sum_{x_1 x_2 x_3} \mathbb{P}(X,Y)$$

$$= \sum_{x_1, x_2} \mathbb{P}(x_1)\mathbb{P}(y_1|x_1)\mathbb{P}(x_2|x_1)\mathbb{P}(y_2|x_2) \underbrace{\sum_{x_3} \mathbb{P}(x_3|x_2)\mathbb{P}(y_3|x_3)}$$

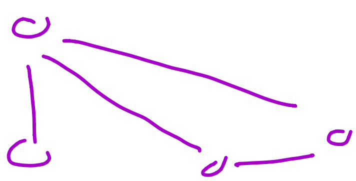
$\phi_{23}(x_2, y_3)$

left with



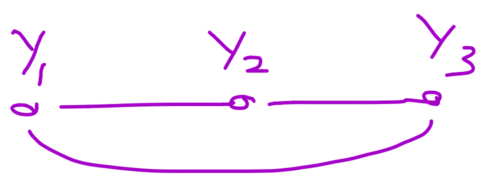
$$= \sum_{x_1} \mathbb{P}(x_1)\mathbb{P}(y_1|x_1) \underbrace{\sum_{x_2} \mathbb{P}(x_2|x_1)\mathbb{P}(y_2|x_2)\phi_{23}(x_2,y_3)}_{\phi_{123}(x_1,y_2,y_3)}$$

left with



$$= \underbrace{\sum_{x_1} \mathbb{P}(x_1)\mathbb{P}(y_1|x_1)\,\phi_{123}(x_1,y_2,y_3)}$$

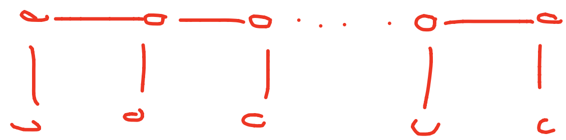
$$= \mathbb{P}(y_1, y_2, y_3)$$



$\mathbb{P}(y)$ is Gibbs wrt the

complete subgraph.

Rmk:



$\Longrightarrow$

Complete