

SDP

- More powerful than LPs for designing algorithms

e.g. max cut

Maxcut Given G find (S, \bar{S})

s.t. $\frac{|E(S, \bar{S})|}{|E|}$ is maximized

Rmk: If G is bipartite,

$$\text{maxcut}(G) = 1$$

relaxations $\left\{ \begin{array}{l} \text{LP} \\ \text{Spectral} \\ \text{SDPs} \end{array} \right\}$ no better than $\frac{1}{2}$ approx.

Theorem 1.1

... ..

initial: assign $v \in V$ to \bar{S}
with prob $1/2$

$$\mathbb{E}(|E(S, \bar{S})|) = \sum_{e \in E} p(e \text{ cut})$$

$$= \frac{|E|}{2}$$

Linear Program

(mincut)

$$\min \frac{1}{2} \sum_{ij \in E} \delta_{ij}$$

For all $\forall i \in V \quad x_i \in [-1, 1]$
 $\bar{S} \quad S$

$$\forall ij \in E \quad \delta_{ij} \geq x_i - x_j$$

$$\delta_{ij} \geq x_j - x_i$$

better to look at mincut

$$\min \frac{1}{2} \sum |x_i + x_j| = \begin{cases} 0 & \text{cut} \\ 2 & \text{uncut} \end{cases}$$

with the relaxation

$$\min \frac{1}{2} \sum \delta_{ij}$$

$$\text{s.t.} \quad \delta_{ij} \geq x_i + x_j$$

$$\delta_{ij} \geq -x_i - x_j$$

which we can reformulate as
the max cut through

$$\max |E| - \frac{1}{2} \sum \delta_{ij}$$



$\text{opt}_{\text{maxcut}}$



≤ 1
LP maxcut

Spectral Relaxation

$$\min \sum_E x_i x_j$$

$$\sum x_i^2 = |V| \quad \text{ideally want } x_i = \pm 1$$

$\forall i$

or

$$\max \frac{1}{4} \sum_i (x_i - x_j)^2$$

$$\sum d_i x_i^2 = 2|E|$$

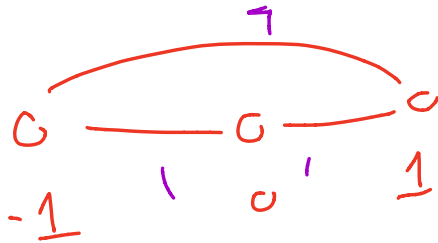
relaxation of Rayleigh Quotient

$$\max \frac{\frac{n}{4}}{\dots} \frac{x^T L x}{x^T x} \dots$$

$$= \frac{|E|}{2} \lambda_{\max}(L)$$

Rmk: $\lambda_{\max}(L) = 2$ iff bipartite

Integrality Gap (for Spectral)



$$x^T D x = 2(1 + 0 + 1) = 4$$

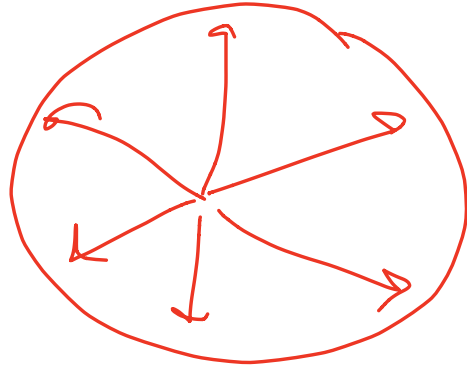
$$x^T L x = 6$$

Using the spectral version of the problem allows us to use

$$\max \frac{1}{4} \sum \|v_i - v_j\|^2$$

$$\text{s.t. } \|v_i\|^2 = 1$$

embedding in \mathbb{R}^n . Convex
optimization over the sphere



Goemans-Williams shows that
the integrability gap is

0.878.

Rounding: Choose a random
hyperplane cut.