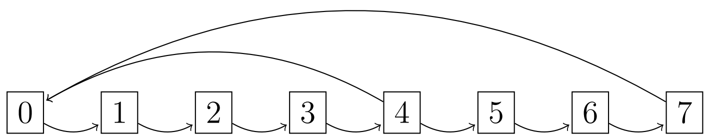
Exercise 4.3.1 This Markov Chain can be described by the following diagram.



Here, we see that every state communicates and hence their is only one equivalence class $\{0, 1, \ldots, 7\}$. Therefore, any class property (e.g. periodicity) applies to the entire Markov Chain. Starting at state 0, we see we can return in 5 steps corresponding to $0 \to 1 \to 2 \to 3 \to 4 \to 0$ and 8 steps corresponding to $0 \to 1 \to \cdots \to 7 \to 0$. Hence

$$\{n \leq 20 : P_{00}^{(n)} > 0\} = \{5, 8, 10, 13, 15, 16, 18, 20\}$$

Note that for $n \geq 40$ we have $P_{00}^{(n)} > 0$. So for $n = 41$ and $n = 43$, say (both primes), we have $gcd(41, 43) = 1$. As $41, 43 \in \{n \in \mathbb{N} : P_{00}^{(n)} > 0\}$ we see that the period of this state is $d(0) = 1$. As there is only one communication class, this Markov Chain is aperiodic.

Exercise 4.3.2 Recall that we need only consider the sum $\sum_{n=1}^{\infty} P_{ii}^{(n)}$ to classify states as either recurrent or transient. First note that we can decompose this Markov Chain's state space S into its communication classes as $S = \{0\} \cup \{1\} \cup \{2, 4\} \cup \{3\} \cup \{5\}$. As transient/recurrent are class properties, we need only compute the above quantity for one state in each class.

- State 0: $\sum_{n=1}^{\infty} P_{00}^{(n)} = \sum_{n=1}^{\infty}(1/3)^n = 3/2 - 1 = 1/2 < \infty$ **Transient**
- State 1: $\sum_{n=1}^{\infty} P_{11}^{(n)} = \sum_{n=1}^{\infty}(1/4)^n = 4/3 - 1 = 1/3 < \infty$ **Transient**
- State 2: $\sum_{n=1}^{\infty} P_{22}^{(n)} = \sum_{n=1}^{\infty}(1)^{2n} = \infty$ **Recurrent**
- State 3: $\sum_{n=1}^{\infty} P_{33}^{(n)} = \sum_{n=1}^{\infty} 0 = 0$ **Transient**
- State 5: $\sum_{n=1}^{\infty} P_{55}^{(n)} = \sum_{n=1}^{\infty}(1) = \infty$ **Recurrent**

Hence, we see that $\{0, 1, 3\}$ are transient states and $\{2, 4, 5\}$ are recurrent states.

Problem 4.3.2 First recall that if a Markov Chain is irreducible then all its states must communicate by definition. That is for each (i, j) there exists $k_{(i,j)} \in \mathbb{N}$ such that $P_{ij}^{k_{ij}} > 0$. Let $k = \max_{(i,j) \in S \times S} k_{ij}$. We know that $k < \infty$ as it is a maximum over a finite set of finite elements. This then implies that$P_{ij}^{k} > 0$ for all $i, j \in S$. That is, P is regular.

Again, since all states communicate, there is only one communication class. As a result, we need only show that there exists $j \in S$ such that j is recurrent. Well, with

$|S| = m$ we can write

$$\sum_{j=1}^{m} P_{ij}^{n} = 1$$
$$\sum_{n=1}^{\infty}\sum_{j=1}^{m} P_{ij}^{n} = \infty$$
$$\sum_{j=1}^{m}\sum_{n=1}^{\infty} P_{ij}^{n} = \infty$$

Note we can change the order of summation as it is a finite sum of positive elements. As this is a finite sum, there exists $j*$ such that $\sum_{n=1}^{\infty} P_{ij*}^{n} = \infty$. With this in mind, we can also condition on the arrival of the chain to state $j*$ as follows

$$\infty = \sum_{n=1}^{\infty} P_{ij*}^{n} = \sum_{n=1}^{\infty}\sum_{m=1}^{n} P_{j*j*}^{(n-m)} f_{ij*}^{(m)} = \sum_{m=1}^{\infty} f_{ij*}^{(m)} \sum_{n=m}^{\infty} P_{j*j*}^{(n)}$$

Now, notice that $\sum_{m=1}^{\infty} f_{ij*}^{(m)}$ is just the probability of going $i \to j*$ or f_{ij*}. Moreover, we know that $f_{ij*} \leq 1$ hence

$$\infty = \sum_{m=1}^{\infty} f_{ij*}^{(m)} \sum_{n=m}^{\infty} P_{j*j*}^{(n)} = f_{ij*} \sum_{n=1}^{\infty} P_{j*j*}^{(n)}$$

Therefore, we see that $\sum_{n=1}^{\infty} P_{j*j*}^{(n)} = \infty$ and $j*$ is recurrent. As $j*$ is recurrent, so are all states in S. Therefore this aperiodic, irreducible Markov Chain is recurrent.