Bootstrap Principle

- Approximate F with $\hat{F}$ and use resampling techiques for inference

- We can use this method to build CI's for $\hat{\theta}$

$$\left(\hat{\theta}^*_{(\alpha/2)} , \hat{\theta}^*_{(1-\alpha/2)} \right)$$ percentile C.I.

Why is this correct?

Consider ϕ monotone increasing and H cdf continuous and symmetric about 0. such that

$$\boxed{\phi(\hat{\theta}) - \phi(\theta) \sim H}$$

Then

$$\mathbb{P}\left(H_{\alpha/2} \leq \phi(\hat{\theta}) - \phi(\theta) \leq H_{1-\alpha/2}\right) = 1-\alpha$$

$$\mathbb{P}\left(\phi(\hat{\theta}) - H_{1-\alpha/2} \leq \phi(\theta) \leq \phi(\hat{\theta}) - H_{\alpha/2}\right) = 1-\alpha$$

$$\mathbb{P}\left(\phi^{-1}(\phi(\theta) - H_{1-\alpha/2}) \leq \theta \leq \phi^{-1}(\phi(\theta) - H_{\alpha/2})\right) = 1-\alpha$$

But since $[\phi(\hat{\theta})]_{(\alpha)} = H_\alpha + \phi(\theta)$

$\boxed{\text{B.P.}}$

Big idea

$$[\phi(\hat{\theta}^*)]_{(\alpha)} \approx H_\alpha + \phi(\hat{\theta})$$

Then

$$\phi^{-1}(\phi(\hat{\theta}) + H_\alpha)) \approx \phi^{-1}([\phi(\hat{\theta}^*)]_\alpha)$$

$$\approx \phi^{-1}(\phi(\hat{\theta}^*_\alpha)) = \hat{\theta}^*_\alpha$$

Hence

$$\mathbb{P}\left(\hat{\theta}^*_{\alpha/2} \leq \theta \leq \hat{\theta}^*_{(1-\alpha/2)} \right) \simeq 1-\alpha$$

The percentile method can be extended to account for bias and variances depending on $\phi(\theta)$

Adjustment is called

"Bias-Corrected and Accelerated" BAC CI

$$\boxed{\begin{array}{l}\text{Assume } H = \Phi \text{ and constraints} \\ z_0,\ a,\ \text{such that} \\ \dfrac{\phi(\hat{\theta}) - \phi(\theta)}{1 + a\,\phi(\theta)} + z_0 \sim N(0,1)\end{array}}$$

We then follow a similar argument

to build the C.I.

$$\mathbb{P}\left(z_{\alpha/2} \leq \frac{\phi(\hat{\theta}) - \phi(\theta)}{1 + a\,\phi(\theta)} + z_0 \leq z_{1-\alpha/2}\right) = 1-\alpha$$

$$\mathbb{P}\left(\cdots \leq \phi(\theta) \leq \phi(\hat{\theta}) + \frac{z_0 - z_{\alpha/2}}{1 - a(z_0 - z_{\alpha/2})}\left[1 + a\,\phi(\hat{\theta})\right]\right)$$

Notice as well that

$$\phi(\hat{\theta}) \sim N\left(\phi(\theta) - z_0(1 + a\,\phi(\theta)),\ (1 + a\,\phi(\theta))^2\right]$$

$$\left[\phi(\hat{\theta})\right]_{\alpha} = \phi(\theta) - z_0(1 + a\,\phi(\theta)) + z_\alpha(1 + a\,\phi(\theta))$$

$$= \phi(\theta) + (z_\alpha - z_0)(1 + a\,\phi(\theta))$$

Thus

$$\phi(\theta) \leq \phi(\hat{\theta}) + (z_{\alpha_2} - z_0)(1 + a\,\phi(\hat{\theta}))$$

such that

$$z_{\alpha_2} - z_0 = \frac{z_0 - z_{\alpha/2}}{1 - a(z_0 - z_{\alpha/2})}$$

$$\Rightarrow \alpha/2 = \Phi^{-1}\left(z_0 + \frac{z_0 + z_{1-\alpha/2}}{1 - a(z_0 + z_{1-\alpha/2})}\right)$$

Now

$$\phi(\theta) \leq \phi(\hat{\theta}) + (z_{\alpha_2} - z_0)(1 + a\,\phi(\hat{\theta}))$$

$$\boxed{\overset{B.P}{\simeq} \left[\phi(\hat{\theta}^*)\right]_{\alpha/2}}$$

$$\Rightarrow \theta \leq \hat{\theta}^*(\alpha_2)$$

By symmetry find

$$\alpha_1 = \Phi^{-1}\left[z_0 + \frac{z_0 + z_{\alpha/2}}{}\right]$$

$$\left. \frac{\qquad}{1 - a\left(z_0 + z_{\alpha/2}\right)} \right]$$

Hence

$$\mathbb{P}\left(\hat{\theta}^*_{(\alpha_1)} \leq \theta \leq \hat{\theta}^*_{(\alpha_2)} \right) \simeq 1 - \alpha$$

<u>Rmk:</u> Estimate z_0 using bootstrap and a using variance.