Simulation

We've seen how to sample $U(0,1)$

We will discuss how to sample from f, F.

Inverse CDF:

If we know F in closed form then

$$\mathbb{P}(F(X) \leq u) \leq \mathbb{P}(X \leq F^{-1}(u))$$

$$= F_X(F_X^{-1}(u)) = u$$

So $F_X(X) \sim \mathrm{Unif}(0,1)$

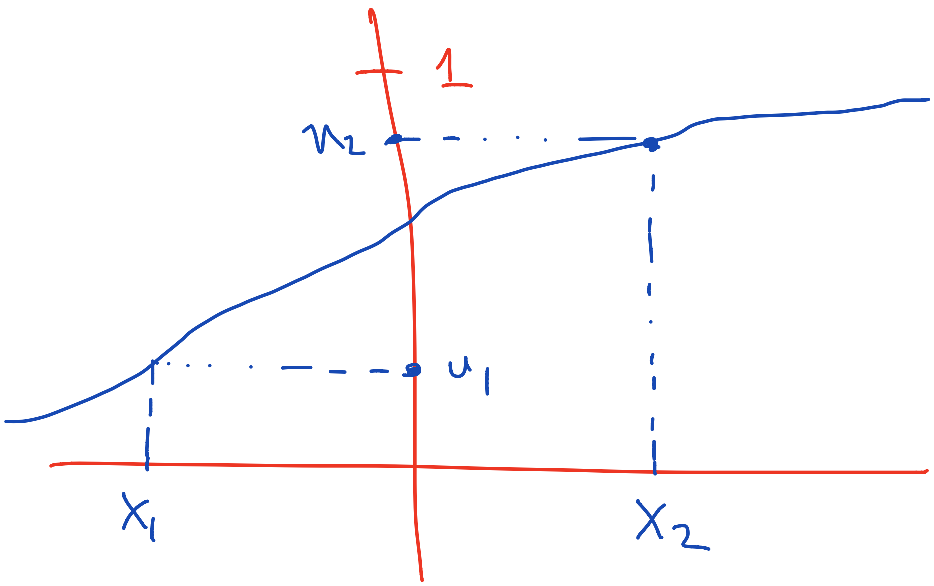
So $X \stackrel{D}{=} F^{-1}(u)$

Therefore to sample a value

of

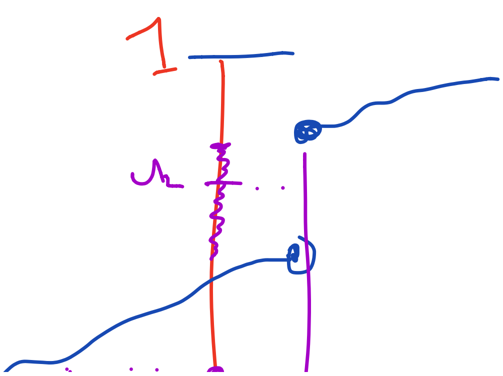
1. Sample from unif u

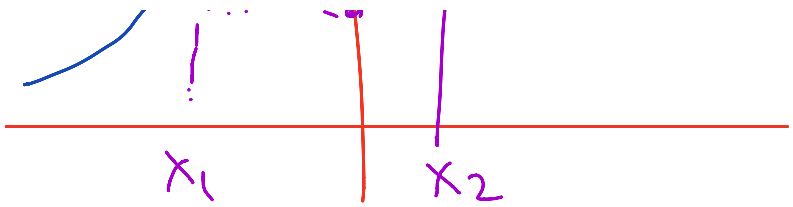
2. Return $F^{-1}(u)$



<u>Rmk</u>: In general we define

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}$$





Ex: $X \sim U(a,b) \qquad F(x) = \dfrac{x-a}{b-a}$

$$F^{-1}(u) = a + (b-a)u$$

$$X = a + (b-a)U \sim U(a,b)$$

$$U \sim U(0,1)$$

Ex: $X \sim \mathrm{Exp}(\lambda)$

$$F(x) = 1 - e^{-\lambda x}$$

$$u = 1 - e^{-\lambda F^{-1}(u)}$$

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda}$$

P

$$= \frac{-\log(u)}{\lambda}$$

$$X \stackrel{D}{=} \frac{-\log u}{\lambda} \sim \mathrm{Exp}(\lambda)$$

Ex: $X \sim \mathrm{Cauchy}(\alpha, \beta)$

$$F(x) = \frac{1}{\pi} \arctan\left(\frac{x-\alpha}{\beta}\right) + \frac{1}{2}$$

$$u = \frac{1}{\pi} \arctan\left(\frac{F^{-1}(u) - \alpha}{\beta}\right) + \frac{1}{2}$$

$$F^{-1}(u) = \alpha + \beta \tan\left(\pi\left(u - 1/2\right)\right)$$

$$U \sim \mathrm{Unif}(0,1)$$

$$X = \alpha + \beta \tan\left(\pi\left(U - 1/2\right)\right) \sim \mathrm{Cauchy}(\alpha, \beta)$$

Ex: Sometimes the R.V. depends

directly on the uniform

$$X \sim Bern(p) \qquad X = \mathbb{I}[U \leq p]$$

$$P(X=1) = \mathbb{P}(U \leq p) = p$$

Mixtures: If the target is such that $f(x) = \lambda f_1(x) + (1-\lambda) f_2(x)$

We say that f is a mixture with weight $0 \leq \lambda \leq 1$ and components f_1, f_2

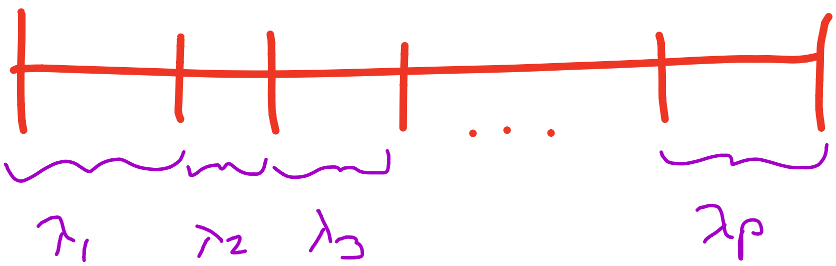
Rmk: We can have p components

$$f = \sum_{i=1}^{p} \lambda_i f_i \qquad \sum_{i=1}^{p} \lambda_i = 1 \qquad \lambda_i \geq 0$$

We can augment the mixture with a latent indicator Z that tells us if the sample came from component $1 \le j \le p$

$$z \sim \text{multinomial}(1, \vec{\lambda})$$

$$X \mid z = i \sim f_i$$



So to sample from a two-mixture

1. Sample $U \sim U(0,1)$ $\quad z = I[U \le \lambda]$
2. If $z = 1$ $\quad X \sim f_1$ $\quad z = 0$ $\quad X \sim f_2$

<u>Ex:</u> $X \sim \text{Laplace}(\lambda)$

Sample from exponential and assign a sign following

$Z \sim \mathrm{Bern}(1/2)$

$X \mid Z \sim (2Z-1)\,\mathrm{Exp}(\lambda)$

$$= \begin{cases} \mathrm{Exp}(\lambda) & Z=1 \\ -\mathrm{Exp}(\lambda) & Z=0 \end{cases}$$

1. Sample $U_1 \sim U(0,1)$ $Z = 1(U_1 \leq 1/2)$

2. $U_2 \sim U(0,1)$ $X = (2Z-1)\,\dfrac{-\log(U_2)}{\lambda}$