Thrm: Given a RKHS, $\mathcal{H}$, of functions on $X \subseteq \mathbb{R}^d$.

then there exists a unique symmetric p.s.d function $K(x,y)$

s.t. $\forall f \in \mathcal{H}$

$$f(x) = \langle f(\cdot), K(\cdot, x) \rangle_{\mathcal{H}}$$

Def: A mercer kernel is p.s.d. kernel which is cont. and bounded.

Example: Take $\mathcal{X} \subseteq \mathbb{R}^p$ closed and bounded

Define $\{\phi_k\}_{k=1}^{\infty}$ on $\mathcal{X}$ an orthonormal basis for

a space of $\mathcal{L}_2$-functions on $\mathcal{X}$.

i.e. $\langle \phi_k(x), \phi_\ell(x) \rangle_{L^2} = \int \phi_k(x)\, \phi_\ell(x)\, dx = \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases}$

Assume $|\phi_k(x)| \leq M$

Define the kernel function $K(x,y) = \sum_{i=1}^{\infty} \gamma_i\, \phi_i(x)\, \phi_i(y)$

$\gamma_i \geq 0 \quad \sum_{i=1}^{\infty} \gamma_i < \infty$

Rmk: If $f(x) = \sum^{\infty} c_i\, \phi_i(x) \quad \tilde{f}(x) = \sum^{\infty} \tilde{c}_i\, \phi_i(x)$

$i=1$ $\qquad$ $i=1$

then $\langle f(x), \tilde{f}(x) \rangle_H = \sum_{i=1}^{\infty} \frac{c_i \tilde{c}_i}{\gamma_i}$

Rmk: If $f = \sum_{i=1}^{\infty} c_i \varphi_i(x)$ $\quad \|f\|_H^2 = \sum_{i=1}^{\infty} \frac{c_i^2}{\gamma_i}$ $\quad \|f\|_{L^2}^2 = \sum_{i=1}^{\infty} c_i^2$

Ex: $\mathcal{X} = [-\pi, \pi] \subseteq \mathbb{R}$ $\quad \varphi_i(x) = \left\{ \frac{\cos nx}{\sqrt{\pi}}, \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{2\pi}} \right\}_{n=1}^{\infty}$

$$\hat{f}(x) = \underset{f \in \mathcal{H}}{\operatorname{argmin}} \sum_{i=1}^{n} V(x_i, f(x_i)) + \lambda \|f\|_{\mathcal{H}}^2$$

$$= \sum_{j=1}^{n} \alpha_j K(x_i, x_j)$$

Rmk: If $\hat{f}(x_i) = (K\alpha)_i$ $\quad K = (K(x_i) K(x_j))_{ij}$