1. This distribution is not in the exponential family. We notice here that the support $\theta < y < \infty$ is dependent on the parameter of the distribution θ. Therefore, regardless of how we can write the density, this distribution is not in the exponential family.

2. (a) Let $Y \sim \text{Geom}(p)$. Then we can write its density as

$$\mathbb{P}(Y = k) = (1-p)^k p = \exp\left\{k\log(1-p) + \log(p)\right\}$$

Let $a(\phi) = 1$, $c(y, \phi) = 0$, $\theta = \log(1-p)$, and lastly $b(\theta) = -\log(1 - \exp(\theta))$.

(b)

$$\mu = \frac{d}{dt}b(t)\Big|_{t=\theta} = \frac{\exp(t)}{1-\exp(t)}\Big|_{t=\theta} = \frac{1-p}{1-1+p} = \frac{1-p}{p}$$

$$V = a(\phi)\frac{d^2}{dt^2}b(t)\Big|_{t=\theta} = \frac{d}{dt}\frac{\exp(t)}{1-\exp(t)}\Big|_{t=\theta} = \frac{(1-\exp(t))\exp(t) + \exp(t)^2}{(1-\exp(t))^2}\Big|_{t=\theta}$$

$$= \frac{p(1-p) + (1-p)^2}{p^2} = \frac{p - p^2 + 1 - 2p + p^2}{p^2} = \frac{1-p}{p^2} = \frac{\mu}{p}$$

(c) The canonical link function is given by the function $g(\mu) = \theta$. Identically, it is the inverse of $\frac{d}{dt}b(t) = \frac{\exp(t)}{1-\exp(t)}$ which is satisfied by the function $g(t) = \log\left(\frac{t}{1+t}\right)$. To check this choice, notice that

$$g(\mu) = g\left(\frac{1-p}{p}\right) = \log\left(\frac{1-p}{p}/1 + \frac{1-p}{p}\right) = \log\left(\frac{1-p}{p}/\frac{1}{p}\right) = \log(1-p) = \theta$$

(d) Here we compare a proposed model to the saturated model where $\tilde{\theta}_i = \log\left(\frac{y_i}{1+y_i}\right)$ and $\hat{\theta}_i = \log\left(\frac{\hat{\mu}_i}{1+\hat{\mu}_i}\right)$

$$D(y_i, \hat{\mu}_i) = 2\sum_{i=1}^{n}\left[y_i(\tilde{\theta}_i - \hat{\theta}_i) - (b(\tilde{\theta}_i) - b(\hat{\theta}_i))\right]$$

$$= 2\sum_{i=1}^{n}\left[y_i\log\left(\frac{y_i(1-\hat{\mu}_i)}{\hat{\mu}_i(1-y_i)}\right) - \log\left(\frac{1+y_i}{1-\hat{\mu}_i}\right)\right]$$

3. (a) Suppose that $Y \sim f(y|\mu) = \left(\frac{\lambda}{2\pi y^3}\right)^{1/2}\exp\left\{\frac{-\lambda(y-\mu)^2}{2\mu^2 y}\right\}$ where λ is known. Notice

that we can write this distribution in canonical form as follows.

$$\begin{aligned}
f(y|\mu) &= \exp\left\{\frac{1}{2}\log(\lambda) - \frac{1}{2}\log(2\pi y^3) - \frac{\lambda(y-\mu)^2}{2\mu^2 y}\right\}\\
&= \exp\left\{\frac{1}{2}\log(\lambda) - \frac{1}{2}\log(2\pi y^3) - \frac{\lambda(y^2-2\mu y+\mu^2)}{2\mu^2 y}\right\}\\
&= \exp\left\{\frac{1}{2}\log(\lambda) - \frac{1}{2}\log(2\pi y^3) - \frac{\lambda y}{2\mu^2} + \frac{\lambda}{\mu} - \frac{\lambda}{2y}\right\}\\
&= \ \exp\left\{\frac{y/\mu^2 - 2/\mu}{-2/\lambda} - \frac{\lambda}{2y} + \frac{1}{2}\log(\lambda) - \frac{1}{2}\log(2\pi y^3)\right\}
\end{aligned}$$

Now setting $a(\phi) = a(\lambda) = -2/\lambda$, $\theta = 1/\mu^2$, $b(\theta) = 2\sqrt{\theta}$, and lasty $c(y,\lambda) = -\frac{\lambda}{2y} + \frac{1}{2}\log(\lambda) - \frac{1}{2}\log(2\pi y^3)$ we see that this in the exponential family.

(b)

$$\begin{aligned}
\mathbb{E}(Y) &= \frac{d}{dt}b(t)\Big|_{t=\theta} = \frac{1}{\sqrt{\theta}} = \frac{1}{\sqrt{1/\mu^2}} = \mu\\
V &= a(\lambda)\frac{d^2}{dt^2}b(t)\Big|_{t=\theta} = \frac{-2}{\lambda}\left(\frac{-1}{2}\theta^{-3/2}\right) = \frac{1}{\lambda(\sqrt{\theta})^3} = \frac{\mu^3}{\lambda}
\end{aligned}$$

(c) The canonical link is given by the function that satisfies $g(\mu) = \theta = 1/\mu^2$. Therefore the link function is given by $g(t) = \frac{1}{t^2}$

(d) Here we compare a proposed model to the saturated model where $\tilde{\theta}_i = \frac{1}{y_i^2}$ and $\hat{\theta}_i = \frac{1}{\hat{\mu}_i^2}$.

$$\begin{aligned}
D(y_i,\hat{\mu}_i) &= 2\sum_{i=1}^{n}\left[y_i(\tilde{\theta}_i - \hat{\theta}_i) - (b(\tilde{\theta}_i) - b(\hat{\theta}_i))\right]\\
&= 2\sum_{i=1}^{n}\left[y_i(\frac{1}{y_i^2} - \frac{1}{\hat{\mu}_i^2}) - (\frac{2}{y_i} - \frac{2}{\hat{\mu}_i})\right]\\
&= 2\sum_{i=1}^{n}\left[\frac{2}{\hat{\mu}_i} - \frac{y_i}{\hat{\mu}_i^2} - \frac{1}{y_i}\right]
\end{aligned}$$