Last time

$$\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h} \frac{I\left(\left|\frac{x_i - x}{n}\right| < 1\right)}{2}$$

$$= \frac{1}{2nh} \sum_{i} I(|x - x_i| < h)$$

wanted:

$$h \longmapsto 0$$

$$nh \longmapsto \infty$$

In general we can use a <u>kernel</u>.

$$\hat{f}_{h,n}(x) = \frac{1}{n} \sum_{i} \frac{1}{h} K\left(\frac{x - x_i}{h}\right)$$

where K is such that

(i) K is a density

(ii) $K(x) = K(-x)$ $\quad 0 = \int x\, K(x)\, dx$

(iii) $\sigma^2 = \int x^2 K(x)\,dx$

Ex:

1. $K(z) = \frac{1}{2} I(|z|<1)$ "Boxcar"

2. $K(z) = \phi(z)$ Gaussian

3. $K(z) = \frac{3}{4}(1-z^2) I(|z|<1)$

   Epanechikov

Rmk: K not so important.

h much more important.

Choosing bandwith

$$ISE(h, X) = \int \left(\hat{f}_{h,X}(t) - f(t)\right)^2 dt$$

Integrated Squared

$$MISE(h) = \mathbb{E}_X(ISE(h,X))$$

$$= \int \mathbb{E}_X\left[\left(\hat{f}_{h,X}(t) - f(t)\right)^2\right] dt$$

$$= \int \mathbb{E}_X\left[\left(\hat{f}_{h,X}(t) - \mathbb{E}_X(\hat{f}_{h,X}(t))\right)^2\right] dt$$

$$+ \int \cancel{2\, \mathbb{E}_X\left[\left(\hat{f}_{h,X}(t) - \mathbb{E}_X(\hat{f}(t))\right)\left(\mathbb{E}(\hat{f}) - f\right)\right]} \rightarrow 0$$

$$+ \int \mathbb{E}_X\left[\left(\mathbb{E}_X(\hat{f}) - f\right)^2\right]$$

$$= \int \mathbb{E}_X\left[(\hat{f} - \mathbb{E}\hat{f})^2\right] + \left[(\mathbb{E}\hat{f} - f)^2\right] dt$$

$$= \int \left(\mathrm{Var}(\hat{f}) + \mathrm{Bias}(\hat{f})^2\right) dt$$

$$= \int \left(\mathrm{Var}(\hat{f}) + \mathrm{Bias}(\hat{f})^2 \right) dx$$

MISE $\simeq$ aggregated MSE.

	Bias	Variance
h small	small	large
h large	large	small

Let's annalyze this more closely.

$$\mathbb{E}\left[\hat{f}_h(x)\right] = \frac{1}{nh} \mathbb{E}\left[\sum_{i=1} K\left(\frac{x-x_i}{h}\right)\right]$$

$$= \frac{1}{h} \mathbb{E}\left(K\left(\frac{x-x_1}{h}\right)\right)$$

$$= \frac{1}{h} \int K\left(\frac{x_1-x}{h}\right) f(x_1)\, dx_1$$

$$x_1 = x + th$$
$$dx_1 = h\,dt$$

$$= \frac{1}{\cancel{h}} \int K(t) f(x+th) \cancel{h}\, dt$$

$$= \int K(t) \left\{ f(x) + th f'(x) + \frac{h^2 t^2}{2} f''(x) + o(h^2) \right\} dt$$

Thus

$$\mathbb{E}\left(\hat{f}_n(x)\right) = f(x) + \frac{h^2 \sigma_K^2 f''(x)}{2} + o(h^2)$$

So the Bias^2 is

$$\text{Bias}^2 \simeq \frac{h^4 (\sigma_K^2)^2}{4} f''(x)^2 + o(h^4)$$

So the MISE is

$$\int \text{Bias}^2 = \frac{h^4(\sigma_K^2)^2}{4} \underbrace{\int f''(x)^2 dx}_{\substack{R(f) \\ \text{roughness}}} + o(h^4)$$

$$\int \text{Var} = \frac{1}{nh} R(K) + o(1/nh)$$

Therefore sending $nh \mapsto \infty$ $h \mapsto 0$

$$\boxed{AMISE = \frac{h^4(\sigma_K^2)^2}{4} R(f'') + \frac{1}{nh} R(K)}$$

$$\frac{d\, AMISE}{dh} = h^3(\sigma_K^2)^2 R(f'') - \frac{R(K)}{nh^2} = 0$$

$$\Rightarrow h^5 = \frac{R(K)}{R(f'')} \cdot \frac{1}{n(\sigma_K^2)^2}$$

$$\left(R(K) \quad 1 \right)^{1/5}$$

$$\boxed{h_{opt} = \left(\frac{R(K)}{R(f'')} \cdot \frac{1}{n(\sigma_K^2)^2} \right)^{1/5}}$$

Silverman's Rule: Using a Guassian kernel and replace f by a normal density with variance $\hat{\sigma}^2 = \hat{\sigma}^2_{unBiased}$.

or $\hat{\sigma} = \min\left\{ \hat{\sigma}_{unbiased}, \frac{IQR}{\Phi^{-1}(.75) - \Phi^{-1}(.25)} \right\}$

Using this

$$\boxed{h_{silverman} = \left(\frac{4}{3n}\right)^{1/5} \hat{\sigma}}$$