Recall the goal of estimating

$$I = \int \underbrace{\cos(\pi x)^2}_{h(x)} \underbrace{1}_{\text{unif density}} dx$$

$$\Longrightarrow \quad \hat{I} = \frac{1}{n} \sum_{i=1}^{n} h(x_i) \qquad X_i \sim \text{Unif}(0,1)$$

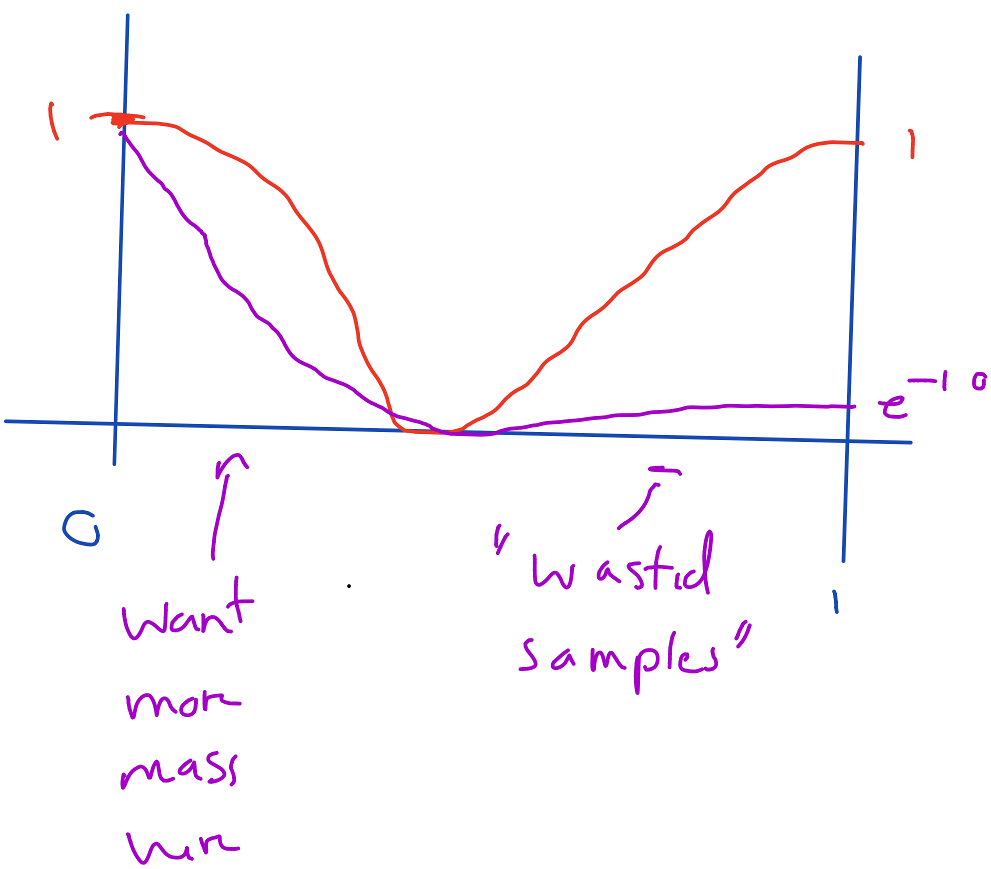
- $\mathbb{E}[\hat{I}] = I$
- $\hat{I} \xrightarrow{a.s.} I$

$$\frac{\hat{I} - I}{\sqrt{\frac{\text{Var}(\hat{I})}{n}}} \xrightarrow{\mathcal{D}} Z$$

$\text{Error} = O(1/\sqrt{n})$ indep. of d

Consider the following example

0 1



$$\int \underbrace{\cos(\pi x)^2}_{h(x)} \underbrace{e^{-10x}}_{f(x) \,\overset{\sim}{\propto}\, \mathrm{Exp}} dx$$

$$= \int \cos(\pi x)^2 \cdot \frac{10 - e^{-10}}{10} \cdot \frac{10e^{-10x}}{1-e^{-10}} dx$$

$$\underbrace{\qquad\qquad}_{h(x)} \qquad \underbrace{\qquad\qquad}_{Exp(\lambda = 10)}$$

truncated exponential

$$= \mathbb{E}_f\left(h(x)\right)$$

with estimate

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} h(x_i) \qquad X_1, \ldots, X_n \overset{iid}{\sim} f$$

In general we want to estimate

$$I = \int h(x)\, f(x)\, dx$$ where f a density

by $$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)$$

$X_1, \ldots, X_n \overset{iid}{\sim} f$ then we get all the same desirable properties as before.

<u>Goal</u>: Find f that reduces the variance.

<u>Ex</u>: $\int_0^{100} e^{-100x}\,dx$

$$= \int_0^{100} \underbrace{\frac{1-e^{-100}}{100}}_{h(x)\ =\ \text{constant}} \; \underbrace{\frac{100\,e^{-100x}}{1-e^{-100}}}_{f(x)}\,dx$$

So $\mathbb{E}_f[h(x)] = \dfrac{1-e^{-100}}{100}$

$\mathrm{Var}_f(h(x)) = 0$ minimized variance

So in general we can consider a

density g and set

$$I = \int h(x) f(x)\, dx = \int h(x) \underbrace{\frac{f(x)}{g(x)}}_{w(x)} g(x)\, dx$$

$$= \int h(x) w(x) g(x)\, dx$$

$$= \mathbb{E}_g(h(x) w(x))$$

So

$$\mathbb{E}_f(h(x)) = \mathbb{E}_g(h(x) w(x))$$

From which

$$\hat{I} = \frac{1}{n} \sum_{i=1}^{n} h(x_i) w(x_i)$$

$$X_i \sim \text{iid } g$$

For this to be advantageous we

need g to be smaller variance.

$$\boxed{\text{Importance Sampling}}$$

and

$$\boxed{\begin{array}{ll} \text{importance weights} & w(x_i) \\ \text{importance function} & g(x_i) \end{array}}$$

$$\mathbb{E}_g(\hat{I}) = \frac{1}{n}\sum \mathbb{E}_g[h(x_i)\, w(x_i)]$$

$$= \frac{1}{n}\sum \int h(x)\, w(x)\, g(x)\, dx$$

$$= \frac{1}{n}\sum \int h(x)\, f(x)\, dx$$

$$= \frac{1}{n}\sum \mathbb{E}_f[h(x_i)]$$

$$= \mathbb{E}_f[h(X_1)] = I$$

<u>Rmk:</u> The trick is to find a g that minimizes variance

<u>Ex:</u> Fair die. Estimate $p_1 = \mathbb{P}(X=1)$

$$\hat{p}_{1,mc} = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 1)$$

$$\mathrm{Var}(\hat{p}_{1,mc}) = 5/36n$$

Let us instead use a loaded die

1	2	3	4	5	6
1	2	3	4	1	1

$$\hat{p} = \frac{1}{n}\sum I(X_i = 1)\frac{1}{\cdot}$$

$$\hat{p}_{1,IS} \quad n \quad i=1 \qquad \underbrace{6 \,/\, 1/2}_{w(x)}$$

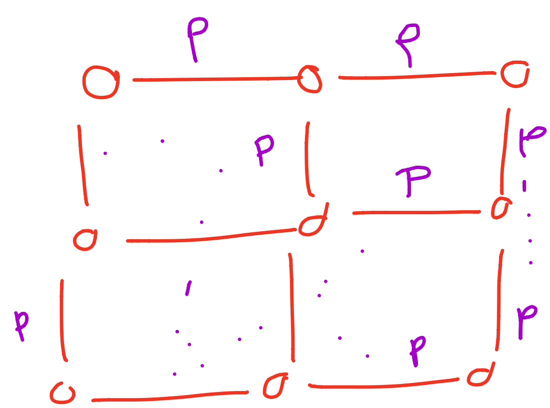
$$= \frac{1}{3n} \sum I(X_i = 1)$$

$$Var(\hat{p}_{1,IS}) = \frac{1}{3^2 n}\left(1/2 \cdot (1 - 1/2)\right)$$

$$= \frac{1}{36n}$$

So $Var(\hat{p}_{IS}) < Var(\hat{p}_{me})$

Ex: Network Reliability



Q: Prob. of this network being disconnected?

$X_i \sim B \quad (p)$

$X_{ij} \sim \text{Bern}(p)$

$$\hat{I} = \frac{1}{n}\sum^{n} I(\text{Network } i \text{ is disconnected})$$

Sample instead from

$$X_{ij} \sim \text{iid Bern}(p^*) \qquad p^* < p \approx 1$$

$$\hat{I}_{IS} = \frac{1}{n}\sum_{i=1}^{n} I(\text{Network } i \text{ is disconnected}) \prod_{k,j \in E} p^{*\,x_{i,kj}} (1-p^*)^{1-x_{i,kj}}$$

$$= \frac{1}{n}\sum^{n} I(\text{disconnected}) \frac{p^{\sum x_{i,kj}} (1-p)^{12-\sum x_{ij}}}{p^{*\,\sum x_{ijk}} (1-p_*)^{12-\sum x_{i,kj}}}$$