SDP with rank 1 solutions.

1. Eigenvector problem

$$\begin{aligned} \max\ & A \bullet X \\ & I \bullet X = 1 \\ & X \succeq 0 \end{aligned}$$

always has an opt. rank 1 sol.

Fact: A, B symm. and $X \succeq 0$ s.t.

$$A \bullet X = a \qquad B \bullet X = b$$

then there exists $x \in \mathbb{R}^n$ s.t.

$$x^T A x = a \qquad x^T B x = b$$

(i.e. $X = xx^T$)

Balanced Cut SDP

$$\min L \bullet X$$

$$L(K_n) \cdot X = \mathrm{Vol}(n)$$

$$L(S_i) \cdot X \leq \frac{1}{b}$$

can be solved up to an

ε-approx in time $\tilde{O}\left(\frac{m}{\varepsilon^2} \log n\right)$

but taking a lower dimensional embedding.

SVD problem

$$\max \quad x^T A y$$

$$\text{s.t.} \quad \|x\| = 1$$

$$\|y\| = 1$$

$\min \ L \bullet X$

$X \succeq 0$

$L_{st} \bullet X = 1$

L laplacian of $G = (V, E)$

L_{st} Laplacian

$\lambda_2 = \min \ L \bullet X$

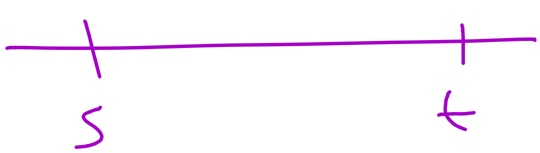
$\quad$ s.t $L(K_G) \bullet X = 1$

same as as

$\min \ x^T L x$

$\quad x^T L_{st} x = 1 \qquad \alpha$

$\quad (x_s - x_t)^2 = 1 \qquad \beta$



Dual:

$$\max \quad \alpha$$

$$\text{s.t.} \quad L \succeq \alpha L_{st}$$

Complementary slackness

Optimal pair (x^*, α^*)

$$x^* \cdot (L - \alpha^* L_{st}) = 0$$

$$L \cdot x^* = \alpha^* x^* \cdot (e_s - e_t)(e_s - e_t)^T$$

Goal: find x s.t.

$$Q(x, L) \qquad L \succeq \alpha^* L_{st}$$

$$L^{-1/2} \, L^{-1/2} \qquad I \succeq \alpha^* L^{-1/2} L_{st} L^{-1/2}$$

$L \quad xL$

$$\Rightarrow \|y\|^2 = \alpha^* \left(y^T L^{-1/2}(e_s - e_t)\right)^2$$

$$\Rightarrow y = \frac{L^{-1/2}(e_s - e_t)}{(e_s - e_t)^T L^{-1}(e_s - e_t)}$$

$$x^* = \frac{L^{-1}(e_s - e_t)}{(e_s - e_t)^T L^{-1}(e_s - e_t)}$$

$$X^* = x^* x^{*T}$$

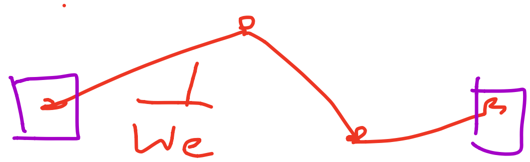
(1) L^+, $L\mathbb{1} = 0$ so we need pseudo-inverses.

$L = U \Lambda U^T \qquad \Lambda^+ = \{ \frac{1}{\lambda_i} \quad \lambda_i \neq 0$

$L = U \Lambda U^T$ $\qquad \Lambda^+ = \begin{cases} \frac{1}{\lambda_i} \\ 0 \end{cases}$

$L^+ = U \Lambda^+ U^T$

Electrical Circuits

$G = (V, E, w)$ 

How much current?

Prob: Given $\begin{cases} V_s = 1 \\ V_t = 0 \end{cases}$ what is voltage?

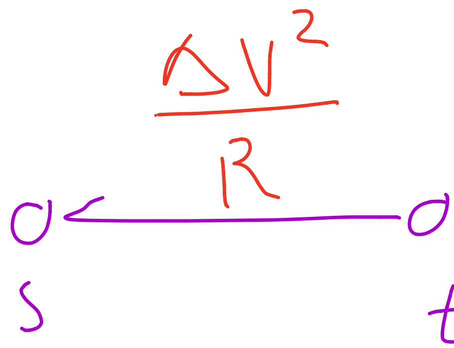
Min Energy

$$R_{st} = \min \sum_{E} w_{ij} (v_i - v_j)^2$$

$$v_s - v_t = 1$$

Same problem as

Laplacian SDP.



$$\text{Energy} = \frac{(\Delta V)^2}{R_{eff}} = \frac{1}{R_{eff}}$$

effective conductance.

Personalized Page Rank

- RW on graph

$$PR = \alpha \sum^{\infty} (1-\alpha)^t W^t$$

$$PR_\alpha = \alpha \sum_{t=0} (1-\alpha) W$$

$(PR_\alpha)s$: (for onestep).

$$\begin{tikzcd} s \arrow[loop above, "\alpha"] \arrow[r, "1-\alpha"] & t \end{tikzcd}$$

Used to measure importance of a website.

Google : $PR_\alpha \underbrace{\vec{1}}_{\text{random web page.}}$

$$(PR)_\alpha = \alpha(I - (1-\alpha)W)^{-1}$$

$$= \alpha D \left(D - (1-\alpha) A \right)^{-1}$$

$$= \alpha D \left(\alpha D + (1-\alpha) L \right)^{-1}$$

$$= D \left(D + \frac{(1-\alpha)}{\alpha} L \right)^{-1}$$

as $\alpha \longmapsto 0$

$$(PR)_0 = D L^{-1}$$

<u>claim</u>: PR is a mixture between eigenvector and Cost.

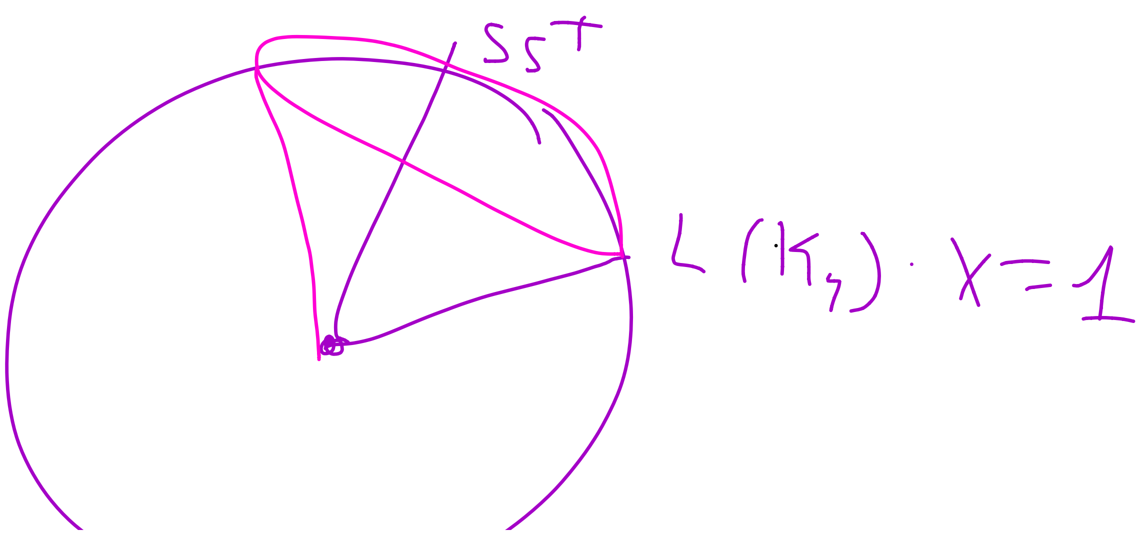
$$\min \sum_{E} \|v_i - v_j\|^2$$

$$\text{s.t.} \sum_{i-} \frac{d_i d_j}{Vol(G)} \|v_i - v_j\|^2 = 1$$

$$\Big(v^T \underbrace{(e_s - e_t)}_{s}\Big)^2 \geq c$$

trying to find a length one vector such that

$$\min L \cdot X \qquad L(K_n) \cdot X = 1$$



So the optimal rank 1 is given by

$$X^{tr} = P R_1 S$$