Exercise 3.5.2 (a) Let A_n be the wealth of player A at time n. Then we can write $A_n = A_{n-1}+1$ with probability p and $A_n = A_{n-1}-1$ with probability $1-p$. We recognize the Gambler's ruin game as a random walk on $\{0, 1, \dots, 100\}$ with absorbing boundaries. From equation (3.4.2) we can calculate the ruin probabilities as follows; let $u_i = \mathbb{P}(A_n \text{ reaches state } 0 \text{ before state } 100|A_0 = i)$. Then in our case (as $p \neq 1-p$) we have

$$u_i = \frac{(\frac{1-p}{p})^i - (\frac{1-p}{p})^{100}}{1-(\frac{1-p}{p})^{100}}$$

Now, for $i = 50$ and $p = 0.49292929$ we have

$$u_{50} = \frac{(\frac{1-0.49292929}{0.49292929})^{50} - (\frac{1-0.49292929}{0.49292929})^{100}}{1-(\frac{1-0.49292929}{0.49292929})^{100}} = 0.804433$$

For $i = 500$ then our random walk is now over $\{0, 1, \dots, 1000\}$ so we have $n = 1000$ and our equation becomes

$$u_i = \frac{(\frac{1-p}{p})^i - (\frac{1-p}{p})^{1000}}{1-(\frac{1-p}{p})^{1000}}$$

In which case for $i = 500$ we have

$$u_{500} = \frac{(\frac{1-0.49292929}{0.49292929})^{500} - (\frac{1-0.49292929}{0.49292929})^{1000}}{1-(\frac{1-0.49292929}{0.49292929})^{1000}} = 0.9999993$$

(b) Following an identical process as above but with $p = 0.5029237$ we have

$$u_{50} = \frac{(\frac{1-0.5029237}{0.5029237})^{50} - (\frac{1-0.5029237}{0.5029237})^{100}}{1-(\frac{1-0.5029237}{0.5029237})^{100}} = 0.3578411$$

$$u_{500} = \frac{(\frac{1-0.5029237}{0.5029237})^{500} - (\frac{1-0.5029237}{0.5029237})^{1000}}{1-(\frac{1-0.5029237}{0.5029237})^{1000}} = 0.002878892$$

Exercise 3.6.4 Let $T = \min\{n \geq 0 : X_n \in \{1, 3\}\}$ and define $v_i = \mathbb{E}[T|X_0 = i]$. Here we have $v_0 = v_3 = 0$. Moreover we see that

$$v_1 = 1 + 0.7v_2 \qquad v_2 = 1 + 0.3v_1 \iff$$
$$v_1 = 1 + 0.7(1 + 0.3v_1) \iff \frac{79}{100}v_1 = \frac{17}{10} \iff v_1 = \frac{170}{79} \approx 2.151899$$

Now, by the equation proceeding equation (3.5.4) we see that

$$v_1 = \frac{1}{p(1-\theta)}\left[N\left(\frac{1-\theta}{1-\theta^N}\right) - 1\right]$$

where $N = 3$, $p = 0.7$, $q = 0.3$, and $\theta = \frac{q}{p} = \frac{3}{7}$. With all this we see that

$$v_1 = \frac{1}{7/10 * (1-3/7)}\left[3\left(\frac{1-3/7}{1-(3/7)^3}\right) - 1\right] = 2.151899$$

Exercise 3.8.1 Let X_n be the number of individuals in generation n and let $\xi_i^{(n)}$ be the number of progeny of individual i from generation n. By assumption, we have $\xi_i^{(n)} = 2$ with probability 1/2 and $\xi_i^{(n)} = 0$ with probability 1/2. From here we see that $\mathbb{E}(\xi_i^{(n)}) = 1$ and $\text{Var}(\xi_i^{(n)}) = \mathbb{E}[(\xi_i^{(n)})^2] - \mathbb{E}[\xi_i^{(n)}]^2 = 2 - 1 = 1$. Using this we can define the size of the $n+1$ generation as the random sum

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

From here we see that

$$\begin{aligned}
\mathbb{E}[X_{n+1}] &= \mathbb{E}[X_n]\mathbb{E}[\xi_1^{(n)}] = \mathbb{E}[X_n] = \ldots = \mathbb{E}[X_0] = 1\\
\text{Var}(X_{n+1}) &= \mathbb{E}[X_n]\text{Var}(\xi_1^{(n)}) + \mathbb{E}[\xi_1^{(n)}]^2\text{Var}(X_n)\\
&= 1 + \text{Var}(X_n) = 2 + \text{Var}(X_{n-1})\\
&\vdots\\
&= (n+1) + \text{Var}(X_0)\\
&= n+1
\end{aligned}$$

where the last equality is due to the fact that $X_0 = 1$ always. Therefore, $\mathbb{E}[X_n] = 1$ and $\text{Var}(X_n) = n$.

Problem 3.5.2 (a) We begin by interpreting the X_n in terms of T.

$$p_i = \mathbb{P}(X_{n+1} = 0|X_n = i) = \mathbb{P}(T = i+1|T > i) = \frac{\mathbb{P}(T = i+1)}{\sum_{n=i+1}^{\infty}\mathbb{P}(T = n)} = \frac{a_{i+1}}{\sum_{n=i+1}^{\infty} a_n}$$

Moreover, seeing that $X_{n+1} \neq X_n$ $r_0 = 0$ which implies that

$$q_i = 1 - \frac{a_{i+1}}{\sum_{n=i+1}^{\infty} a_n}$$

(b) When we enforce a planned replacement policy, we see that $p_N = 1$ and $q_N = 0$. Now, for $0 \leq i < N$, the process is unaffected by planned replacement policy. That is T is independent of N. Hence, for $0 \leq i < N$ the q_i and p_i are given in (a).

Problem 3.5.5 (a) First note that X_n is a random walk on $\{0, 1, 2, \ldots\}$ so $X_n \geq 0$ for all n. Now suppose that $X_0 = k < \infty$. Then $\mathbb{E}|X_n| = \mathbb{E}(X_n) \leq \mathbb{E}(X_0) + n = k + n < \infty$. That is, for each X_n, it has taken at most n 'steps to the right' which is still a finite value. Now, for second martingale property we have

$$\begin{aligned}
\mathbb{E}[X_{n+1}|X_n, \ldots, X_0] &= \mathbb{E}[1/2(X_{n-1}+1) + 1/2(X_{n-1}-1)|X_{n-1}, \ldots, X_0]\\
&= \mathbb{E}[1/2X_{n-1} + 1/2X_{n-1}]\\
&= \mathbb{E}[X_{n-1}]
\end{aligned}$$

Having shown these properties we see that X_n is a nonnegative martingale.

(b) Applying the maximal inequality we have

$$\mathbb{P}(\max_{n\geq 0} X_n \geq N) \leq \frac{\mathbb{E}(X_0)}{N} = \frac{k}{N}$$

As the right side of this inequality is free from n, we have a uniform bound of this quantity for all values in the martingale.

Problem 3.6.7 Let $T = \min\{n \geq 0 : X_n \in \{0,3\}\}$ and define $v_i = \mathbb{E}[T|X_0 = i]$. Then we have $v_0 = v_3 = 0$ and

$$\begin{aligned}
&v_1 = 1 + 0.7v_2 \qquad v_2 = 1 + 0.1v_1 \iff \\
&v_1 = 1 + 0.7(1 + 0.1v_1) \iff \frac{93}{100}v_1 = \frac{17}{10} \iff v_1 = \frac{170}{93} \approx 1.827957
\end{aligned}$$

Now using the results from equation (3.6.6) we have

$$v_1 = \frac{\Phi_1 + \Phi_2}{1 + \rho_1 + \rho_2}$$

where $\rho_1 = q_1/p_1$, $\rho_2 = q_1q_2/p_1p_2$, and $\Phi_1 = \frac{\rho_1}{q_1}$, $\Phi_2 = \frac{\rho_2}{q_1} + \frac{\rho_2}{q_1\rho_1}$ Evaluating these quantities, we see that

$$v_1 = \frac{1.428571 + 1.269841}{1 + 0.4285714 + 0.04761905} = 1.827957$$

Problem 3.6.8 Let $T = \min\{n \geq 0 : X_n = 3\}$ and define $u_i = \mathbb{E}[T|X_0 = i]$. Note that $u_3 = 0$. From a first step analysis, we arrive at the system given below

$$\begin{aligned}
u_0 &= 1 + \alpha u_0 + \beta u_2 \\
u_1 &= 1 + \alpha u_0 \\
u_2 &= 1 + \alpha u_0 + \beta u_1
\end{aligned}$$

First note that

$$u_0 = 1 + \alpha u_0 + \beta u_2 \iff (1-\alpha)u_0 = 1 + \beta u_2 \iff \beta u_0 = 1 + \beta u_2 \iff u_0 = 1/\beta + u_2$$

Moreover, we see that

$$u_0 - 1/\beta = u_2 = 1 + \alpha u_0 + \beta u_1 \iff \beta u_0 = 1 + 1/\beta + \beta u_1 \iff 1/\beta + 1/\beta^2 + u_1$$

Lastly writting u_1 in terms of u_0 we have

$$u_0 = 1/\beta + 1/\beta^2 + 1 + \alpha u_0 \iff \beta u_0 = 1/\beta + 1/\beta^2 + 1 \iff u_0 = 1/\beta + 1/\beta^2 + 1/\beta^3$$

Therefore we see that

$$u_0 = \sum_{k=1}^{3} \frac{1}{\beta^k}$$

Problem 3.8.3 (a) Let N be the number of children this family has and let S_k be the sex of the k-th child. We will begin by conditioning on the first child's sex.

$$\begin{aligned}\mathbb{P}(N=k) &= \sum_{s=F,M}\mathbb{P}(N=k-1|S_1=s)\mathbb{P}(S_1=s)\\ &= \frac{1}{2}\mathbb{P}(N=k-1|S_1=M)+\frac{1}{2}\mathbb{P}(N=k-1|S_1=F)\end{aligned}$$

Now, we note that $N|S_1=F$ is degenerate 2. That is $\mathbb{P}(N=k-1|S_1=F)=\mathbf{1}_{\{k=2\}}$. In a similar way, we see that $N|S_1=M\sim\text{Geom}(1/2)$ where $\mathbb{P}(N=k-1|S_1=M)=(1/2)^{k-2}(1/2)=1/2^{k-1}$. Putting this together, we see that

$$\mathbb{P}(N=k)=\begin{cases}(1/2)^k+1/2 & k=2\\ (1/2)^k & k\geq 3\end{cases}=\begin{cases}3/4 & k=2\\ (1/2)^k & k\geq 3\end{cases}$$

(b) Let B be the number of boys a family has. Then again, conditioning on the sex of the first child we have

$$\mathbb{P}(B=k)=\frac{1}{2}\mathbb{P}(B=k-1|S_1=M)+\frac{1}{2}\mathbb{P}(B=k|S_1=F)$$

Again notice that $B|S_1=F$ is $Bern(1/2)$. That is $\mathbb{P}(B=1|S_1=F)=1/2$ and $\mathbb{P}(B=0|S_1=F)=1/2$ and $\mathbb{P}(B=k|S_1=F)=0$ for all $k\geq 3$. Moreover, we note that $B|S_1=M\sim\text{Geom}(1/2)$ so $\mathbb{P}(B=k-1|S_1=M)=(1-1/2)^{k-1}(1/2)=(1/2)^k$ for $k\geq 1$. Putting this together we arrive at the following

$$\mathbb{P}(B=k)=\begin{cases}(1/2)^2 & k=0\\ (1/2)^{k+1}+(1/2)^2 & k=1\\ (1/2)^{k+1} & k\geq 2\end{cases}=\begin{cases}1/4 & k=0\\ 1/2 & k=1\\ (1/2)^{k+1} & k\geq 2\end{cases}$$

Problem 3.9.3 Here, we introduce terms to see this integral as a Gamma density with parameters $(k+\alpha,\frac{1}{1+\theta})$. Using this, we can derive the distribution as follows.

$$\begin{aligned}p_k &= \int_0^\infty \pi(k|\lambda)f(\lambda)d\lambda=\frac{\theta^\alpha}{k!\Gamma(\alpha)}\int_0^\infty e^{-(1+\theta)\lambda}\lambda^{(k+\alpha)-1}d\lambda\\ &= \frac{\theta^\alpha\Gamma(k+\alpha)(\frac{1}{1+\theta})^{k+\alpha}}{k!\Gamma(\alpha)}\frac{1}{\Gamma(k+\alpha)(\frac{1}{1+\theta})^{k+\alpha}}\int_0^\infty e^{-\frac{\lambda}{1/(1+\theta)}}\lambda^{(k+\alpha)-1}d\lambda\\ &= \frac{\theta^\alpha\Gamma(k+\alpha)}{(1+\theta)^{k+\alpha}k!\Gamma(\alpha)}=\left(\frac{\theta}{1+\theta}\right)^\alpha\left(\frac{1}{1+\theta}\right)^k\frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)}\end{aligned}$$

We recognize this distribution as a negative binomial with parameters $p=\frac{\theta}{1+\theta}$ and $r=\alpha$