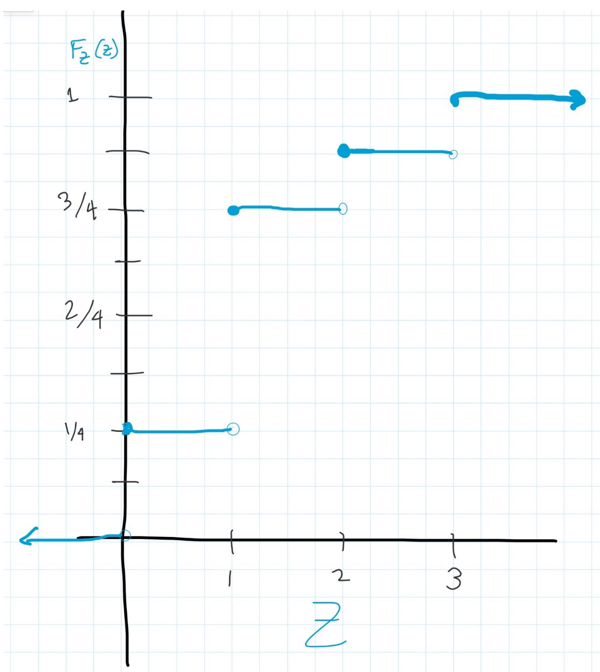
Exercise 1.2.4 (a) The distribution function for Z is given by $F_Z(z) = \mathbb{P}(Z \leq z)$. This gives

$$F_Z(z) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq z < 1 \\ 3/4 & 1 \leq z < 2 \\ 7/8 & 2 \leq z < 3 \\ 1 & 3 \leq z \end{cases}$$



(b)

$$\begin{aligned} \mathbb{E}(Z) &= 0\mathbb{P}(Z=0) + 1\mathbb{P}(Z=1) + 2\mathbb{P}(Z=2) + 3\mathbb{P}(Z=3) \\ &= 0 + 1/2 + 1/4 + 3/8 \\ &= 9/8 \end{aligned}$$

(c)

$$\begin{aligned} \mathbb{E}(Z^2) &= 0\mathbb{P}(Z=0) + 1\mathbb{P}(Z=1) + 4\mathbb{P}(Z=2) + 9\mathbb{P}(Z=3) \\ &= 0 + 1/2 + 1/2 + 9/8 \\ &= 17/8 \end{aligned}$$

$$\mathbb{V}(Z) = \mathbb{E}(Z^2) - [\mathbb{E}(Z)]^2 = 17/8 - 81/64 = 55/64$$

Exercise 1.2.9 (Distribution Function) Suppose that $x \leq 1$. Then we have

$$F_X(x) = \int_{-\infty}^{x} f(y)dy = \int_0^x ydy = \frac{1}{2}x^2$$

Now suppose that $x > 1$. Then

$$F_X(x) = \int_{-\infty}^{x} f(y)dy = \int_0^1 x dy + \int_1^x (2-y)dy = 1/2 + -\frac{1}{2}x^2 + 2x - 3/2 = -\frac{1}{2}x^2 + 2x - 1$$

Therefore, we have

$$F_x(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \leq x \leq 1 \\ -\frac{1}{2}x^2 + 2x - 1 & 1 < x \leq 2 \\ 1 & 2 < x \end{cases}$$

(Mean)

$$\begin{aligned} \mathbb{E}(X) &= \int_0^2 x f(x)dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx \\ &= \frac{1}{3}x^3\Big|_0^1 + x^2\Big|_1^2 - \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3} + 3 - \frac{7}{3} = 1 \end{aligned}$$

(Variance)

$$\begin{aligned} \mathbb{E}(X^2) &= \int_0^2 x^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)dx \\ &= \frac{1}{4}x^4\Big|_0^1 + \frac{2}{3}x^3\Big|_1^2 - \frac{1}{4}x^4\Big|_1^2 = \frac{1}{4} + \frac{14}{3} - \frac{15}{4} = \frac{7}{6} \\ \mathbb{V}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{7}{6} - 1^2 = \frac{1}{6} \end{aligned}$$

Exercise 1.3.5 Let N be the number of bacteria in a portion of a slide where $N \sim \text{Pois}(5)$. Then

$$\begin{aligned} P(N \geq 8) &= 1 - P(N \leq 7) = 1 - \sum_{n=0}^{7} P(N = n) = 1 - \sum_{n=0}^{7} \frac{5^n e^{-5}}{n!} \\ &= 1 - e^{-5}\Big(\frac{5^0}{0!} + \frac{5^1}{1!} + \ldots + \frac{5^7}{7!}\Big) \approx 0.133 \end{aligned}$$

Problem 1.4.2 (a) Let $W \sim \text{Exp}(\theta)$ and denote $\mu = \frac{1}{\theta}$. Then

$$\mathbb{P}(W > \mu) = \int_\mu^\infty \theta e^{-\theta x} dx = -e^{-\theta x}\Big|_\mu^\infty = 0 + e^{-\theta\mu} = e^{-1}$$

Problem 1.5.2 Let $Z = \min(X_1, X_2, \ldots, X_n)$. Then we have

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) = 1 - \mathbb{P}(Z > z) = 1 - \mathbb{P}(X_1 > z, \ldots, X_n > z) \overset{i.i.d}{=} 1 - [\mathbb{P}(X_1 > z)]^n \\ &= 1 - [1 - \mathbb{P}(X_1 \leq z)]^n = 1 - [e^{-\lambda z}]^n = 1 - e^{-n\lambda z} \end{aligned}$$

This shows that $Z \sim \text{Exp}(n\lambda)$

Exercise 2.1.5 Suppose that $X \sim \text{Pois}(\lambda)$. Let $A = \{X = 2n+1 : n \in \mathbb{N}\}$ be the event that X is odd. Then we have

$$\mathbb{E}(X|A) = \sum_{n=0}^{\infty} n\mathbb{P}(X = n|A) = \sum_{n=0}^{\infty} n\frac{\mathbb{P}(X = n, A)}{\mathbb{P}(A)}$$

First we find $\mathbb{P}(A)$.

$$\mathbb{P}(A) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}e^{-\lambda}}{(2k+1)!} = e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} = e^{-\lambda}\sinh(\lambda)$$

Now notice that $\mathbb{P}(X = n, A) = 0$ if n is even. If n is odd, then $\mathbb{P}(X = n, A) = \frac{e^{-\lambda}\lambda^n}{n!}$. This gives a reduction in our expression to

$$\begin{aligned}\mathbb{E}(X|A) &= \sum_{n odd} n\frac{e^{-\lambda}\lambda^n/n!}{e^{-\lambda}\sinh(\lambda)} = \frac{1}{\sinh(\lambda)}\sum_{k=0}^{\infty}(2k+1)\frac{\lambda^{2k+1}}{(2k+1)!}\\ &= \frac{\lambda}{\sinh(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!} = \lambda\frac{\cosh(\lambda)}{\sinh(\lambda)} = \lambda\coth(\lambda)\end{aligned}$$

Problem 2.1.1 (a) Suppose that $M \sim \text{Binom}(N, p)$ and $X|M \sim \text{Binom}(N, \pi)$. Then we have

$$\begin{aligned}\mathbb{P}(X = x) &= \sum_m \mathbb{P}(X = x, M = m) = \sum_m \mathbb{P}(X = x|M = m)\mathbb{P}(M = m)\\ &= \sum_{m=x}^{N}\binom{m}{x}\pi^x(1-\pi)^{m-x}\binom{N}{m}p^m(1-p)^{N-m}\\ &= \frac{N!\pi^x}{x!}\sum_{m=x}^{N}\frac{(1-\pi)^{m-x}p^m(1-p)^{N-m}}{(m-x)!(N-m)!}\\ &= \frac{N!\pi^x}{x!}\sum_{k=0}^{N-x}\frac{(1-\pi)^k p^{k+x}(1-p)^{N-m-k}}{k!(N-m-k)!}\\ &= \frac{N!\pi^x}{x!(N-x)!}\sum_{k=0}^{N-x}\frac{(N-x)!}{k!(N-m-k)!}(1-\pi)^k p^{k+x}(1-p)^{N-m-k}\\ &= \frac{N!(p\pi)^x}{x!(N-x)!}\sum_{k=0}^{N-x}\binom{N-x}{k}[(1-\pi)p]^k(1-p)^{N-m-k}\\ &= \binom{N}{x}(p\pi)^x[(1-\pi)p + (1-p)]^{N-x}\\ &= \binom{N}{x}(p\pi)^x(1-p\pi)^{N-x}\end{aligned}$$

Which we recognize as the Binomial pmf. Hence the marginal distribution of X is given by $X \sim \text{Binom}(N, p\pi)$

(b)

$$\begin{aligned}
Cov(X, M-X) &= Cov(X,M) - Cov(X,X) \\
&= \mathbb{E}(XM) - \mathbb{E}(X)\mathbb{E}(M) - \mathbb{V}(X) \\
&= \mathbb{E}[\mathbb{E}(XM|M)] - N^2p^2\pi - Np\pi(1-p\pi) \\
&= \mathbb{E}[M\mathbb{E}(X|M)] - N^2p^2\pi - Np\pi(1-p\pi) \\
&= \mathbb{E}[M^2\pi] - N^2p^2\pi - Np\pi(1-p\pi) \\
&= \pi[\mathbb{V}(M) + \mathbb{E}(M)^2] - N^2p^2\pi - Np\pi(1-p\pi) \\
&= \pi[Np(1-p) + N^2p^2] - N^2p^2\pi - Np\pi(1-p\pi) \\
&= Np(1-p)\pi + N^2p^2\pi - N^2p^2\pi - Np\pi(1-p\pi) \\
&= Np\pi(1-p-1+p\pi) \\
&= Np\pi(-p+p\pi) \\
&= Np^2\pi(\pi-1)
\end{aligned}$$

Problem 2.1.7 Let A be the event that an airplane crash was diagnosed as due to a structural failure and B be the event that an airplane crash was due to a structural failure. Then we look to find $\mathbb{P}(B|A)$. Using Bayes' Law we have

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^C)\mathbb{P}(B^C)}$$

From the given, we know that $\mathbb{P}(A|B) = 0.85$, $\mathbb{P}(A|B^C) = 0.35$, $\mathbb{P}(B) = 0.3$, $\mathbb{P}(B^C) = 0.7$. Using this information gives our solution,

$$\mathbb{P}(B|A) = \frac{(0.85)(0.3)}{(0.85)(0.3) + (0.35)(0.7)} = 0.51$$