$$\tilde{P} = \begin{pmatrix} Xc^1X^T & \cdots & X\frac{c^1+c^M}{2}X^T \\ & \ddots & \vdots \\ & & Xc^MX^T \end{pmatrix}$$

$$= \sum_{i=1}^{d} H(v_i) \otimes x_i x_i^T$$

$$= \sum_{i=1}^{d} H^+(v_i) \otimes x_i x_i^T + \sum_{i=1}^{d} H^-(v_i) \otimes x_i x_i^T$$

$$H^+(v_i) = \theta_1 w_1 w_1^T \qquad H^-(v_i) = \theta_2 w_2 w_2^T$$

$$\theta_2 \leq 0 \leq \theta_1$$

So $\sum_{i=1}^{d} H^+(v_i) \otimes x_i x_i^T$ is the p.d part of $\tilde{P}$.

$$\sum_{i=1}^{d} \underbrace{H^+(v_i)}_{m\times m} \otimes \underbrace{x_i x_i^T}_{n\times n} = \begin{bmatrix} \sum_{i=1}^{d} H(v_i)_1^+ x_i x_i^T & \cdots & \sum_{i=1}^{d} \left(\frac{H^+(v_i)_1 + H^+(v_i)_m}{2}\right) x_i x_i^T \\ & \ddots & \vdots \\ & & \sum_{i=1}^{d} \left(H^+(v_i)_m\right) x_i x_i^T \end{bmatrix}$$

$$= \sum_{i=1}^{d} \begin{bmatrix} (H(v_i))_{11} x_i x_i^T & \cdots & (h_i)_{1m} x_i x_i^T \\ & & \\ & . & \\ & h & \end{bmatrix}$$

$$\lambda_d(P)+\lambda_n(N) \leq \lambda_d(\tilde{P}) = \lambda_d(P+N) \leq \lambda_d(P)+\lambda_1(N)$$

$$H(v) = \frac{\mathbb{1}v^T + v\mathbb{1}^T}{2} \qquad H^+ = \theta_1 w_1 w_1^T \qquad H^- = \epsilon_2 w_2 w_2^T$$

$$L_S L_S^T = \begin{pmatrix} X(S^1)^2X^T & \cdots & XS^1S^mX^T \\ & \ddots & \vdots \\ & & X(S^m)^2X^T \end{pmatrix}$$

$$S^2 = \operatorname{diag}\left(\alpha_g^{(1)}, \ldots, \alpha_g^{(d)}\right)^2$$

$$(\alpha^{(i)})(\alpha^{(i)})^T = H(v_i)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (x_1 \cdots x_n) = (x_i x_j)$$