1. (a)

$$\begin{aligned}
\int_{-1}^{p} B(w,p)dw &= \int_{-1}^{p} \frac{a_2(p) - a_1(p)w}{a_0(p)a_2(p) - a_1^2(p)} K(w)dw \\
&= \frac{1}{a_0(p)a_2(p) - a_1^2(p)} \left[a_2(p) \int_{-1}^{p} K(w)dw - a_1(p) \int_{-1}^{p} wK(w)dw \right] \\
&= \frac{1}{a_0(p)a_2(p) - a_1^2(p)} \left(a_2(p)a_0(p) - a_1^2(p) \right) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{p} wB(w,p)dw &= \int_{-1}^{p} \frac{a_2(p)w - a_1(p)w^2}{a_0(p)a_2(p) - a_1^2(p)} K(w)dw \\
&= \frac{1}{a_0(p)a_2(p) - a_1^2(p)} \left[a_2(p) \int_{-1}^{p} wK(w)dw - a_1(p) \int_{-1}^{p} w^2K(w)dw \right] \\
&= \frac{1}{a_0(p)a_2(p) - a_1^2(p)} \left(a_2(p)a_1(p) - a_1(p)a_2(p) \right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(\hat{f}_h(x)) &= \frac{1}{nh} \sum_{i=1}^{n} \mathbb{E}\Big[B\left(\frac{x - X_i}{h} \right) \Big] \\
&= \frac{1}{h} \mathbb{E}\Big[B\left(\frac{x - w}{h} \right) \Big] \\
&= \frac{1}{h} \int B\left(\frac{x - w}{h} \right) f(w)dw \\
&\overset{z-sub}{=} \int B(z) f(x - zh)dz \\
&\overset{Taylor}{=} \int B(z) \Big\{ f(x) - zhf'(x) + \frac{z^2h^2}{2} f''(x) + O(h^2) \Big\} dz \\
&= f(x) + f''(x)\frac{h^2}{2}\mu_2(B) + O(h^2)
\end{aligned}$$

Therefore the first term in the bias is given by

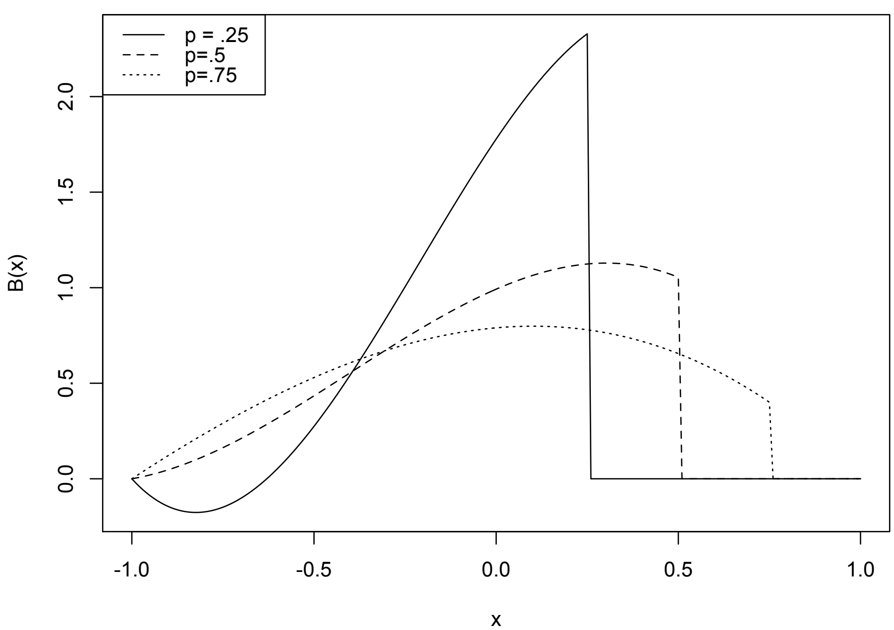
$$f''(x)\frac{h^2}{2}\mu_2(B)$$

(b) We first find the polynomials, $a_0(p)$, $a_1(p)$, and $a_2(p)$. Then we will use these to write the form of the boundary kernel using the Epanechnikov kernel.

$$a_0(p) = \int_{-1}^{p} \frac{3}{4}(1-t^2)dt = \frac{3}{4}\left(p - \frac{p^3}{3} + \frac{2}{3}\right)$$
$$a_1(p) = \int_{-1}^{p} \frac{3t}{4}(1-t^2)dt = \frac{3}{4}\left(\frac{p^2}{2} - \frac{p^4}{4} - \frac{1}{4}\right)$$
$$a_2(p) = \int_{-1}^{p} \frac{3t^2}{4}(1-t^2)dt = \frac{3}{4}\left(\frac{p^3}{3} - \frac{p^5}{5} + \frac{2}{15}\right)$$

This gives the formula for boundary kernel.

$$\begin{aligned} B(t,p) &= \frac{a_2(p) - a_1(p)t}{a_0(p)a_2(p) - a_2(p)^2} K(t) I_{[-1,p]}(t) \\ &= \frac{\frac{p^3}{3} - \frac{p^5}{5} + \frac{2}{15} - t\left(\frac{p^2}{2} - \frac{p^4}{4} - \frac{1}{4}\right)}{\frac{3}{4}\left(\frac{p^3}{3} - \frac{p^5}{5} + \frac{2}{15}\right)\left(p - \frac{p^3}{3} + \frac{2}{3}\right) - \frac{3}{4}\left(\frac{p^2}{2} - \frac{p^4}{4} - \frac{1}{4}\right)^2}(1-t^2)I_{[-1,p]}(t) \end{aligned}$$



2. (a)

$$\begin{aligned}\int K_4(t)dt &= \int \frac{s_4 - s_2 t^2}{s_4 - s_2^2} K(t)dt \\ &= \frac{1}{s_4 - s_2^2}\left[s_4 \int K(t)dt - s_2 \int t^2 K(t)dt\right] \\ &= \frac{1}{s_4 - s_2^2}(s_4 - s_2^2) \\ &= 1\end{aligned}$$

(b) Recall that $K(t)$ is a symmetric function, so $s_{2n+1} = 0$ for any $n = 0, 1, 2, \ldots$. That is $s_k = 0$ for any k odd.

$$\begin{aligned}\int tK_4(t)dt &= \frac{1}{s_4 - s_2^2}\left[s_4 \int tK(t)dt - s_2 \int t^3 K(t)dt\right] = \frac{1}{s_4 - s_2^2}(s_4 s_1 - s_2 s_3) = 0 \\ \int t^2 K_4(t)dt &= \frac{1}{s_4 - s_2^2}\left[s_4 \int t^2 K(t)dt - s_2 \int t^4 K(t)dt\right] = \frac{1}{s_4 - s_2^2}(s_4 s_2 - s_2 s_4) = 0 \\ \int t^3 K_4(t)dt &= \frac{1}{s_4 - s_2^2}\left[s_4 \int t^3 K(t)dt - s_2 \int t^5 K(t)dt\right] = \frac{1}{s_4 - s_2^2}(s_4 s_3 - s_2 s_5) = 0\end{aligned}$$

(c)

$$\int t^4 K_4(t)dt = \frac{1}{s_4 - s_2^2}\left[s_4 \int t^4 K(t)dt - s_2 \int t^6 K(t)dt\right] = \frac{1}{s_4 - s_2^2}(s_4^2 - s_2 s_6) \neq 0$$

(d) First we find the values of s_2 and s_4 for the Epanechnikov kernel.

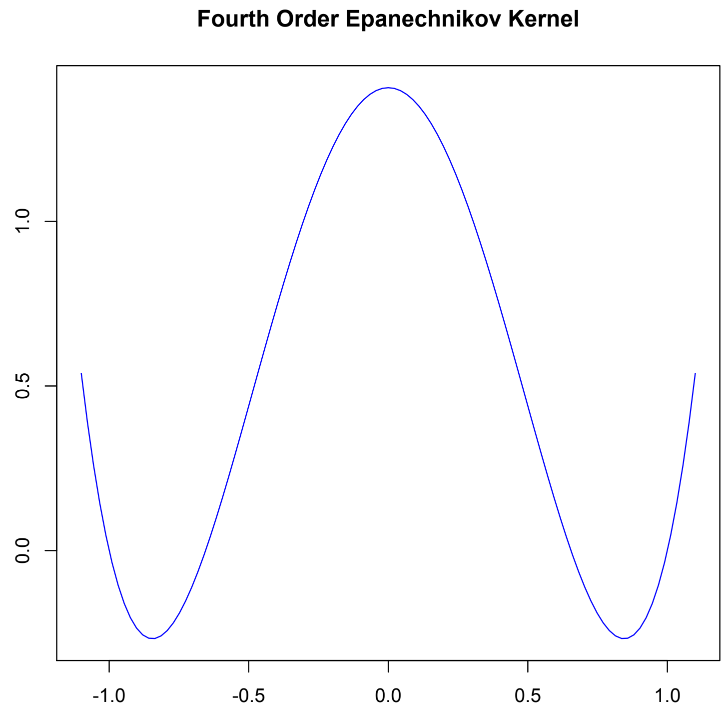
$$\begin{aligned}s_4 &= \int_{-1}^{1} \frac{3}{4} t^4 (1 - t^2) dt = \frac{3}{4}\left[\frac{t^5}{5} - \frac{t^7}{7}\right]_{-1}^{1} = \frac{3}{4}\left[\frac{2}{5} - \frac{2}{7}\right] = \frac{3}{35} \\ s_2 &= \int_{-1}^{1} \frac{3}{4} t^2 (1 - t^2) dt = \frac{3}{4}\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_{-1}^{1} = \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{1}{5}\end{aligned}$$

This yields the fourth order Epanechnikov kernel

$$K_{[4]}(t) = \frac{3/35 - t^2/5}{3/35 - 1/25}\left(\frac{3}{4}(1 - t^2)\right) = \frac{175}{32}\left[\frac{9}{35} - \frac{3}{5}t^2\right](1 - t^2)$$

3. We begin by deriving the bias of $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \kappa\left(\frac{x - X_i}{h}\right)$

$$\begin{aligned}\mathbb{E}[\hat{F}_n(x)] &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\kappa\left(\frac{x - X_i}{h}\right)\right] \\ &= \mathbb{E}\left[\kappa\left(\frac{x - W}{h}\right)\right] \\ &= \int_{-\infty}^{\infty} \kappa\left(\frac{x - w}{h}\right) f(w)dw \\ &\overset{z-sub}{=} h\int_{-\infty}^{\infty} \kappa(z) f(x - zh)dz\end{aligned}$$



We now look integrate by parts with $u = \kappa(z) = \int_{-\infty}^{z} K(x)dx$ and $dv = f(z - zh)dz$. By the Fundamental Theorem of Calculus we see $du = (K(z) - K(-\infty))dz = K(z)dz$ because K is a density function. Moreover, $v = -\frac{1}{h}F_n(x - zh)$. This gives

$$h\int_{-\infty}^{\infty} \kappa(z)f(x - zh)dz = h\Big[-\frac{1}{h}\kappa(z)F_n(x - zh)\Big|_{-\infty}^{\infty} + \frac{1}{h}\int_{-\infty}^{\infty} F_n(x - zh)K(z)dz\Big]$$
$$= -\kappa(\infty)F_n(-\infty) + \kappa(-\infty)F(\infty) + \int_{-\infty}^{\infty} F_n(x - zh)K(z)dz$$

Notice that $F(-\infty) = 0$ and $\kappa(-\infty) = 0$ so the first term drops out completely. This gives

$$\int_{-\infty}^{\infty} F_n(x - zh)K(z)dz \overset{Taylor}{=} \int_{-\infty}^{\infty} K(z)\Big\{F_n(x) - zhf(x) + \frac{z^2h^2}{2}f'(x) + o(h^2)\Big\}dz$$
$$= F_n(x) + \frac{h^2}{2}f'(x)\mu_2(K) + o(h^2)$$

Therefore we see the approximate bias is given by

$$\frac{h^2}{2}\mu_2(K)f'(x) + o(h^2)$$

Now we calculate the variance of our estimator

$$
\begin{aligned}
Var(\hat{F}_n(x)) &= \frac{1}{n^2}\sum_{i=1}^{n} Var\left[\kappa\left(\frac{x-X_i}{h}\right)\right]\\
&= \frac{1}{n}\left[\mathbb{E}\left[\kappa^2\left(\frac{x-W}{h}\right)\right] - \left[\mathbb{E}\left(\kappa\left(\frac{x-W}{h}\right)\right)\right]^2\right]\\
&= \frac{1}{n}\left[\mathbb{E}\left[\kappa^2\left(\frac{x-W}{h}\right)\right] - \left[F_n(x) + O(h^2)\right]^2\right]\\
&= \frac{1}{n}\left[\left(\int_{-\infty}^{\infty}\kappa^2\left(\frac{x-w}{h}\right)f(w)dw\right) - \left[F_n(x) + O(h^2)\right]^2\right]\\
&\overset{z-sub}{=} \frac{1}{n}\left[\left(h\int_{-\infty}^{\infty}\kappa^2(z)f(x-zh)dz\right) - \left[F_n(x) + o(h)\right]^2\right]
\end{aligned}
$$

Focusing on the first integral, we can again integrate by parts with $u = \kappa^2(z)$ and $dv = f(x-zh)dz$. These values correspond to $du = 2\kappa(z)K(z)dz$ (as above) and $v = -\frac{1}{h}F_n(x-zh)$. Using this, we see

$$
\begin{aligned}
h\int_{-\infty}^{\infty}\kappa^2(z)f(x-zh)dz &= -\kappa^2(z)F_n(x-zh)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}2\kappa(z)K(z)F_n(x-zh)dz\\
&= -\kappa(-\infty)F_n(\infty) + \kappa(\infty)F_n(-\infty) + \int_{-\infty}^{\infty}2\kappa(z)K(z)F_n(x-zh)dz\\
&= \int_{-\infty}^{\infty}2\kappa(z)K(z)F_n(x-zh)dz\\
&\overset{Taylor}{=} \int_{-\infty}^{\infty}2\kappa(z)K(z)\Big\{F_n(x) - zhf(z) + o(h)\Big\}dz\\
&= F_n(x)\int_{-\infty}^{\infty}2\kappa(z)K(z)dz - hf(x)\theta + o(h)\\
&= F_n(x) - hf(x)\theta + o(h)
\end{aligned}
$$

Here we used the fact that

$$\int_{-\infty}^{\infty}2\kappa(u)K(u)du = \kappa(u)^2\big|_{-\infty}^{\infty} = \kappa(\infty) - \kappa(-\infty) = 1 - 0 = 1$$

Plugging this into the equation above we see

$$
\begin{aligned}
Var(\hat{F}_n(x)) &= \frac{1}{n}\left[F_n(x) - hf(x)\theta + o(h) - \left[F_n(x) + o(h)\right]^2\right]\\
&= \frac{1}{n}\left[F_n(x) - hf(x)\theta + o(h) - F_n^2(x) + o(h)\right]\\
&= \frac{F_n(x)(1-F_n(x))}{n} - \frac{h}{n}f(x)\theta + o(\frac{h}{n})\\
&= \frac{\sigma_F^2(x)}{n} - \frac{h}{n}f(x)\theta + o(\frac{h}{n})
\end{aligned}
$$

This yields the mean squared error as

$$MSE = \frac{\sigma_F^2(x)}{n} - \frac{h}{n}f(x)\theta + o(\frac{h}{n}) + \frac{h^4}{4}\mu_2^2(K)f'(x)^2 + o(h^4)$$

and now the MISE as

$$\begin{aligned}
MISE &= \int MSE \\
&= \int \frac{\sigma_F^2(x)}{n} - \frac{h}{n}f(x)\theta + o(\frac{h}{n}) + \frac{h^4}{4}\mu_2^2(K)f'(x)^2 + o(h^4)dx \\
&= \frac{1}{n}\int \sigma_F^2(x)dx - \frac{h}{n}\int \theta f(x)dx + o(\frac{h}{n}) + \frac{h^4}{4}\mu_2^2(K)\int f'(x)^2dx + o(h^4) \\
&= \frac{1}{n}C_0 - \frac{h}{n}C_1 + h^4C_2
\end{aligned}$$

Now minimizing MISE with respect to h we can find h_{MISE}.

$$\frac{\partial}{\partial h}MISE = \frac{\partial}{\partial h}\left(\frac{1}{n}C_0 - \frac{h}{n}C_1 + h^4C_2\right) = -\frac{1}{n}C_1 + h^3C_2$$

This corresponds to

$$h_{MISE} = \Big[\frac{C_1}{nC_2}\Big]^{1/3} = \Big[\frac{\int \theta f(x)dx}{n\mu_2^2(K)\int f'(x)^2dx}\Big]^{1/3} = O(n^{-1/3})$$

Therefore, for h_{MISE} we have an improved rate of convergence given by

$$MISE(h_{MISE}) = C_1n^{-4/3} + Cn^{-4/3} = o(n^{-4/3})$$

Now, we calculate θ for the Epanechnikov kernel. First we find $\kappa(\cdot)$.

$$\kappa(u) = \int_{-\infty}^{u}\frac{3}{4}(1-t^2)I_{[-1,1]}(t)dt = \frac{3}{4}\int_{-1}^{u}(1-t^2) = \frac{3}{4}\Big[t - t^3/3\Big]_{-1}^{u} = \frac{1}{4}(-u^3+3u+2)$$

$$\begin{aligned}
\theta &= \int_{-\infty}^{\infty} 2u\kappa(u)K(u)du \\
&= \frac{3}{8}\int_{-1}^{1} u(1-u^2)(-u^3+3u+2)du \\
&= \frac{3}{8}\int_{-1}^{1}(u^6 - 4u^4 - 2u^3 + 3u^2 + 2u)du \\
&= \frac{3}{8}\left(\frac{u^7}{7} - \frac{4u^5}{5} - \frac{u^4}{2} + u^3 + u^2\right)_{-1}^{1} \\
&= \frac{3}{8}\left(\frac{59}{70} - \frac{11}{70}\right) \\
&= \frac{9}{35} > 0
\end{aligned}$$

4. I implemented a CDF kernel estimator using the Epanechnikov. Pictured below is the constant (histogram) estimator of the CDF as well as the kernel estimator with $h = 0.25, 0.5, 0.75$. For implementation details, see the code attached.