Lecture: 9/6/18

Spectral Graph Theory

Undirected graphs $G = (V, E, w)$

think about them as linear operators

- Different types of associated matrices

We want to spend the majority of our time on the Laplacian

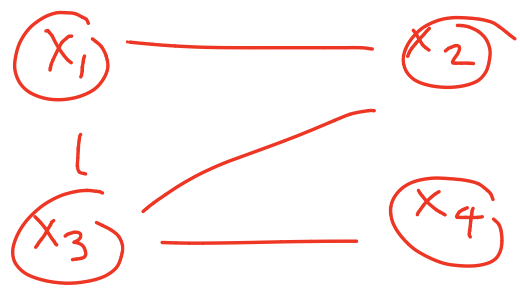
$$L = D - A$$

with quadratic form

$$x^T L_G x = \sum_{\{i,j\} \in E} w_{ij} (x_i - x_j)^2$$

Measures discrepency between

values assigned to vertices



<u>Def</u>: Natural R.W. over G is characterized by the Markov chain with P.T.M. $AD^{-1} = W_G$.

That is $X(0) = i \in V$

$$P(X(t+1) = j \mid X(t) = i) = \frac{w_{ij}}{d_i}$$

<u>Ex</u>: Random Walks can reveal clusters in the network.

- This is the discrete time MC but we could put it in some cont. space

$$p(t+1) = Wp(t)$$

$$p(t+1) - p(t) = -(I-W)p(t)$$

So the chain could be desc. by

$$X(0) = i \in V \qquad \frac{d\,p(t)}{\partial t} = -(I-W)p(t)$$

With solution

$$p(t) = e^{-t(I-W)} p(0)$$

Now notice $I - W = I - AD^{-1} = LD^{-1}$

<u>Spectral Thrm for Sym. Matrices</u>

<u>Thrm</u>: If $M \in \mathbb{R}^{n \times n}$ $\quad M^T = M$

then M has an orthonormal basis of eigenvectors with real eign.

$$M = V \Sigma V^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

$+$

Rmk: $v_i v_i'$ projection map

$$(v_i v_i^T)x = v_i(v_i^T x)$$

Claim 1: Any nonzero symmetric M has an eigenvector $x \neq 0$ with eigenvalue $\lambda \neq 0$.

Pf: Suppose $f_M(x) = \dfrac{x^T M x}{x^T x}$

Goal: $\max\limits_{x \neq 0} f_M(x)$

Finding critical points

$$\nabla f_M(x) = \frac{2Mx(x^T x) - (x^T M x)2x}{(x^T x)^2} = 0$$

Der. of quad form

$$\nabla_x(x^T A x) = (A + A^T)x$$

So maximizing

$$Mx = \frac{x^T M x}{x^T x} x = \underbrace{f_M(x)}_{\in \mathbb{R}} x$$

So all critical values are eigenvectors:

Maximum is achieved. Moreover if we restrict ourselfs to

$$\max_{\|x\|=1} f_M(x)$$ ← cont.

← compact set ⟹ local max achieved.

⟹ critical point.

<u>Remainder Sketch:</u>

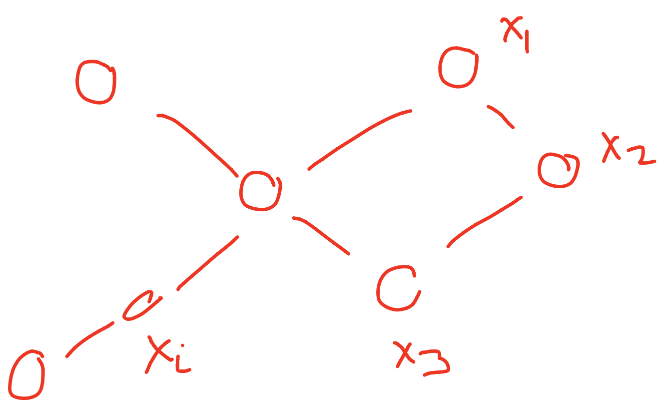
Apply claim I to L. to achieve

$\|v\| = 1 \qquad Lv = \lambda v \qquad \lambda \neq 0$

Iterate until $(I - \lambda v v^T) = 0$ ∎

Consensus / Agreement Problem:

Suppose we have a graph



Task: Compute a weighted average of

x_i

$$\bar{x}_w = \frac{\sum_{i=1}^{N} d_i x_i}{\sum_{i=1}^{N} x_i}$$

Vertices only communicate with neighbors

Trivial: Flooding $\leftarrow$ iterate until each edge hears about

each weight via

neighbors

Averaging: $x_i^{(t+1)} = \frac{1}{2} x_i^{(t)} + \frac{1}{2} \sum_{j \sim i} w_{ij} \frac{x_j^{(t)}}{d_i}$

$$x^{(t+1)} = \left(\frac{1}{2} I + \frac{1}{2} W^T\right) x^{(t)}$$

edge space

$$= \left(\frac{I + W^T}{2}\right) x^{(t)}$$

which in terms of a RW corresponds

to $p^{(t+1)} = \left(\frac{I + W}{2}\right) p^{(t)}$

vertex space

$$q^{(t)} = D x^{(t)}$$

$$q^{(t+1)} = D x^{(t+1)} = D \left(\frac{I + D^{-1} A}{2}\right) x^{(t)}$$

$$= \left(\frac{I + A D^{-1}}{2}\right) q^{(t)}$$

In some sense the Laplacian is informing how quickly we are converging

$$x^T L x = \sum_{e} w_{ij} (x_i - x_j)^2$$