Exercise 9.1 Suppose we have the probability space $(\Omega = [0,1], \mathcal{F} = \mathcal{B}, P = \lambda)$. Define $A_n = [0, 1/2 + 1/n]$ and $A = [1/2, 1]$ and $X_n = 1_{A_n}$ and $X = 1_A$. Show that

$$X_n \xrightarrow{D} X \quad \text{but} \quad X_n \overset{P}{\not\to} X$$

Solution

First notice that

$$\begin{aligned}
\lim_{n\to\infty} P(X_n = 1) &= \lim_{n\to\infty}(1/2 + 1/n) = 1/2 = P(X = 1)\\
\lim_{n\to\infty} P(X_n = 0) &= \lim_{n\to\infty}(1/2 - 1/n) = 1/2 = P(X = 0)
\end{aligned}$$

Hence we see that $X_n \xrightarrow{D} X$. But notice that

$$\begin{aligned}
P(|X_n - X| = 0) &= P(X_n = 0, X = 0) + P(X_n = 1, X = 1)\\
&= (1/2 - 1/n)(1/2) + (1/2 + 1/n)(1/2) = 1/2
\end{aligned}$$

$$\begin{aligned}
P(|X_n - X| = 1) &= P(X_n = 1, X = 0) + P(X_n = 0, X = 1)\\
&= (1/2 + 1/n)(1/2) + (1/2 - 1/n)(1/2) = 1/2
\end{aligned}$$

Therefore, for $\epsilon = 1/2$, say, we see that

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = \lim_{n\to\infty} 1/2 = 1/2 \neq 0$$

Hence $X_n \overset{P}{\not\to} X$.

Exercise 9.2 Suppose that $X_n \to X$ in probability and $\epsilon > 0$. Show that

$$F_X(x - \epsilon) \leq P(X_n \leq x) + P(|X_n - X| > \epsilon)$$

Solution

Following the proof that convergence in probability implies convergene in distribution, let $\epsilon > 0$ and consider the following.

$$\begin{aligned} F_X(x-\epsilon) &= P(X \leq x - \epsilon) \\ &= P(\{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\}) + P(\{X \leq x - \epsilon\} \cap \{|X_n - X| > \epsilon\}) \\ &\leq P(\{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\}) + P(|X_n - X| > \epsilon) \end{aligned}$$

Now notice for $\omega \in \{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\}$ we have

$$X(\omega) \leq x - \epsilon \quad \text{and} \quad X_n(\omega) \leq X(\omega) + \epsilon$$

Combining these two we see

$$X_n(\omega) \leq x - \epsilon + \epsilon = x$$

While $X_n(\omega)$ has a similar lower bound, it is certainly the case that

$$\{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\} \subset \{X_n(\omega) \leq x\}$$

Moreover,

$$P(\{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\}) \leq P(X_n(\omega) \leq x)$$

Using this, we see

$$\begin{aligned} F_X(x-\epsilon) &= P(X \leq x - \epsilon) \\ &\leq P(\{X \leq x - \epsilon\} \cap \{|X_n - X| \leq \epsilon\}) + P(|X_n - X| > \epsilon) \\ &\leq P(X_n \leq x) + P(|X_n - X| > \epsilon) \end{aligned}$$

Exercise 9.3 Show that if $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} c$ then $X_n + Y_n \xrightarrow{D} X + c$

Solution We follow a similar proof structure as the proof that shows that convergence in probability implies convergence in distribution. Let $\epsilon > 0$ and consider a point of contiunity $z \in C(supp(F_{X+a}))$the following

$$\begin{aligned}
F_{X_n+Y_n}(z) &= P(X_n + Y_n \leq z)\\
&= P(\{X_n + Y_n \leq z\} \cap \{|Y_n - a| < \epsilon\}) + P(\{X_n + Y_n \leq z\} \cap \{|Y_n - a| \geq \epsilon\})\\
&\leq P(\{X_n + Y_n \leq z\} \cap \{|Y_n - a| < \epsilon\}) + P(|Y_n - a| \geq \epsilon)
\end{aligned}$$

Now notice, that for $\omega \in \{X_n + Y_n \leq z\} \cap \{|Y_n - a| < \epsilon\}$ we have the following

$$X_n(\omega) \leq z - Y_n(\omega) \quad \text{and} \quad Y_n(\omega) > a - \epsilon$$

Combining these, we see that

$$X_n(\omega) \leq z - (a - \epsilon) \quad \text{or} \quad X_n(\omega) + a \leq z + \epsilon$$

Now, while $X_n(\omega)$ has a similar lower bound, it certainly is true that

$$P(\{X_n + Y_n \leq z\} \cap \{|Y_n - a| < \epsilon\}) \leq P(X_n + a \leq z + \epsilon)$$

Hence we have

$$\begin{aligned}
F_{X_n+Y_n}(z) &\leq P(\{X_n + Y_n \leq z\} \cap \{|Y_n - a| < \epsilon\}) + P(|Y_n - a| \geq \epsilon)\\
&\leq P(X_n + a \leq z + \epsilon) + P(|Y_n - a| \geq \epsilon)\\
&= F_{X_n+a}(z + \epsilon) + P(|Y_n - a| \geq \epsilon)
\end{aligned}$$

This implies that

$$\begin{aligned}
\limsup_{n\to\infty} F_{X_n+Y_n}(z) &\leq \limsup_{n\to\infty} \left(F_{X_n+a}(z + \epsilon) + P(|Y_n - a| \geq \epsilon)\right)\\
&= \limsup_{n\to\infty} F_{X_n+a}(z + \epsilon)
\end{aligned}$$

Where the equality is due the fact that if $Y_n \xrightarrow{D} c \in \mathbb{R}$ then $Y_n \xrightarrow{P} c$. But notice we have that

$$\begin{aligned}
F_{X_n+a}(z) &= P(X_n + a \leq z) = P(X_n \leq z - a) = F_{X_n}(z - a) \to F_X(z - a)\\
&= P(X \leq z - a) = P(X + a \leq z) = F_{X+a}(z)
\end{aligned}$$

Therefore we see that

$$\limsup_{n\to\infty} F_{X_n+Y_n}(z) \leq F_{X+a}(z + \epsilon)$$

We now look to bound the same distribution function from above by the limit infium. Consider

$$\begin{aligned}1 - F_{X_n+Y_n}(z) &= P(X_n + Y_n > z)\\ &= P(\{X_n + Y_n > z\} \cap \{|Y_n - a| < \epsilon\}) + P(\{X_n + Y_n > z\} \cap \{|Y_n - a| \geq \epsilon\})\\ &\leq P(\{X_n + Y_n > z\} \cap \{|Y_n - a| < \epsilon\}) + P(|Y_n - a| \geq \epsilon)\end{aligned}$$

Following the argument above, for $\omega \in \{X_n + Y_n > z\} \cap \{|Y_n - a| < \epsilon\}$ we have the following

$$X_n(\omega) > z - Y_n(\omega) \quad \text{and} \quad Y_n(\omega) < a + \epsilon$$

Combining these, we see that

$$X_n(\omega) > z - (a + \epsilon) \quad \text{or} \quad X_n(\omega) + a > z - \epsilon$$

Hence we have

$$\begin{aligned}1 - F_{X_n+Y_n}(z) &\leq P(\{X_n + Y_n > z\} \cap \{|Y_n - a| < \epsilon\}) + P(|Y_n - a| \geq \epsilon)\\ &\leq P(X_n + a > z - \epsilon) + P(|Y_n - a| \geq \epsilon)\\ &= 1 - F_{X_n+a}(z - \epsilon) + P(|Y_n - a| \geq \epsilon)\end{aligned}$$

From here we can write

$$\begin{aligned}\liminf_{n\to\infty}(1 - F_{X_n+Y_n}(z)) &\leq \liminf_{n\to\infty}\left(1 - F_{X_n+a}(z - \epsilon) + P(|Y_n - a| \geq \epsilon)\right)\\ 1 - \liminf_{n\to\infty} F_{X_n+Y_n}(z) &\leq 1 - \liminf_{n\to\infty} F_{X_n+a}(z - \epsilon)\\ F_{X+a}(z - \epsilon) &\leq \liminf_{n\to\infty} F_{X_n+Y_n}(z)\end{aligned}$$

Now, since we assumed that z was a point of continuty, letting $\epsilon \to 0$ we have

$$F_{X+a}(z) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(z)$$

and

$$F_{X+a}(z) \geq \limsup_{n\to\infty} F_{X_n+Y_n}(z)$$

Hence

$$\limsup_{n\to\infty} F_{X_n+Y_n}(z) \leq F_{X+a}(z) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(z)$$

which implies

$$\lim_{n\to\infty} F_{X_n+Y_n}(z) = F_{X+a}(z)$$

Therefore $X_n + Y_n \xrightarrow{D} X + a$

Exercise 9.4 Generate a sample from $X \sim Exp(5)$ using a sample from $U \sim Unif(0,1)$.

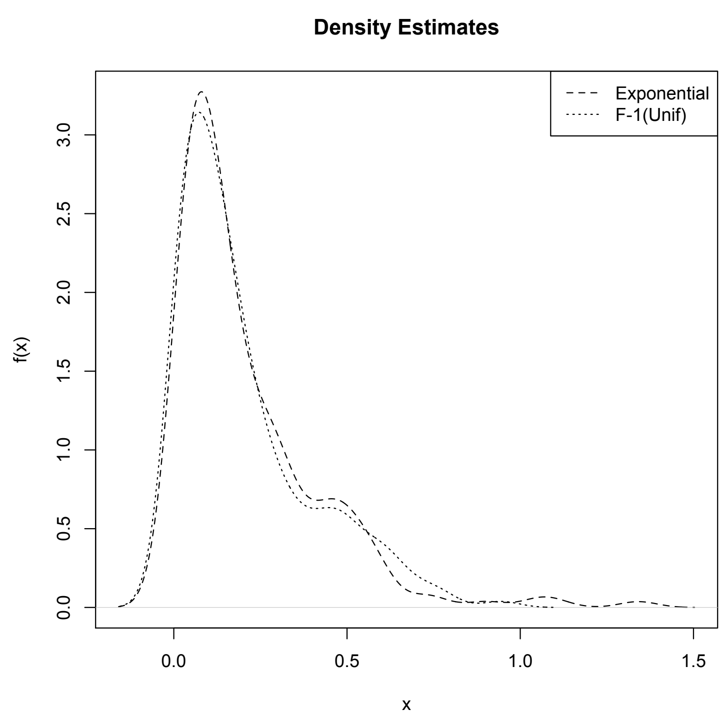
Solution Since the distribution function of X is one to one, we know its inverse must exist. This implies for $Z = F_X^{-1}(U)$ that

$$F_Z(z) = P(Z \leq z) = P(F_X^{-1}(U) \leq z) = P(U \leq F_X(z)) = F_U(F_X(z)) = F_X(z)$$

Therefore we see that $Z \overset{D}{=} X$. Therefore, to generate a sample from X is equivalent to generating a sample from Z which is a functional in terms of a uniform. Notice for $X \sim Exp(\lambda)$ we have $F^{-1}(x) = -\frac{\log(1-x)}{\lambda}$. Hence we can define Z as

$$Z = -\frac{\log(1-U)}{5}$$

In R I generated 200 samples from U, then found the corresponding Z values. Plotted below is the kernel density estimate of a sample from Z and from X. Notice that they are very close, and only deviate from random chance.



Exercise 9.5 Use the Skorohod Representation Theorem to prove the first order delta method.

Solution First note by the Central Limit Theorem

$$Z_n = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \overset{D}{\to} Z \sim \mathcal{N}(0,1)$$

Then by the Skorohod Representation Theorem we now there exists $Z_n' \overset{D}{=} Z_n$ and $Z' \overset{D}{=} Z$ such that $Z_n' \overset{a.s.}{\to} Z'$. Then we have

$$\begin{aligned}
\sqrt{n}\frac{g(\overline{X}_n) - g(\mu)}{\sigma g'(\mu)} &= \sqrt{n}\frac{g(\mu + \sigma Z_n/\sqrt{n}) - g(\mu)}{\sigma g'(\mu)} \\
&\overset{D}{=} \sqrt{n}\frac{g(\mu + \sigma Z_n'/\sqrt{n}) - g(\mu)}{\sigma g'(\mu)} \\
&= \frac{g(\mu + \sigma Z_n'/\sqrt{n}) - g(\mu)}{\sigma Z_n'/\sqrt{n}} \cdot \frac{Z_n'}{g'(\mu)}
\end{aligned}$$

Now notice that $\sigma Z_n'/\sqrt{n} \to 0$. This implies that

$$\frac{g(\mu + \sigma Z_n'/\sqrt{n}) - g(\mu)}{\sigma Z_n'/\sqrt{n}} \to g'(\mu)$$

which is a constant. Moreover, by the representation theorem we see that

$$\frac{g(\mu + \sigma Z_n'/\sqrt{n}) - g(\mu)}{\sigma Z_n'/\sqrt{n}} \cdot \frac{Z_n'}{g'(\mu)} \xrightarrow{a.s.} g'(\mu)\frac{Z'}{g'(\mu)} = Z' \overset{D}{=} Z$$

Exercise 9.6 Show that the characteristic function is uniformly continuous on $\mathbb{R}$.

Solution: First recall that $\cos(\cdot)$, $\sin(\cdot)$ are uniformly continuous on $\mathbb{R}$. Therefore, for $\epsilon > 0$, there exists $\delta_C(\epsilon)$ such that for $|rX - tX| < \delta_C$ we have that $\mathbb{E}|\cos(rX) - \cos(tX)| < \frac{\epsilon}{2}$. Similarly, we have $\delta_S(\epsilon)$ such that for $|rX - tX| < \delta_S$ we have that $\mathbb{E}|\sin(rX) - \sin(rX)| < \frac{\epsilon}{2}$. Let $\delta = \min\{\delta_C, \delta_S\}$. Then, for $|rX - tX| < \delta$ we see

$$\begin{aligned}
|\phi_X(r) - \phi_X(t)| &= |\mathbb{E}(e^{irX}) - \mathbb{E}(e^{itX})| \\
&= |\mathbb{E}(e^{irX} - e^{itX}| \\
&= |\mathbb{E}(\cos(rX) - \cos(tx)) + i\mathbb{E}(\sin(rX) - \sin(tX))| \\
&< |\frac{\epsilon}{2} + i\frac{\epsilon}{2}| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}|i| = \epsilon
\end{aligned}$$

Therefore, we see that ϕ is uniformly continuous on $\mathbb{R}$.