Cheeger Inequality

1. Char. of conductance

$$\bar{\phi}_G = \min_{S \subseteq V} \frac{|E(S,\bar{S})|}{|E_{K_2}(S,\bar{S})|}$$

$$= \min_{d \in cut_V} \frac{d(G)}{d(K_G)}$$

$$= \min_{d \in \ell_1} \frac{d(G)}{d(K_G)} \leq 2\sqrt{2\lambda_2}$$

2. CUT_V = metrics that are ℓ_1 embeddable

Plan:

(a) Start with $v \perp 1 = 0$ s.t.

$$\frac{X^T L X}{X^T L(K_n) X} = \lambda_2$$ and use

V to construct a ℓ_1-metric

$$d_j = \| y_i - y_j \|_1$$

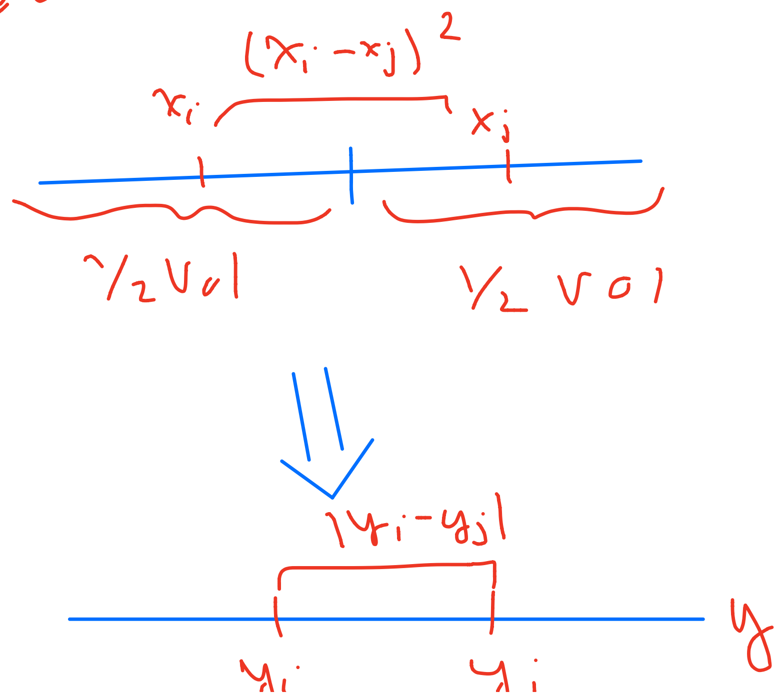
such that

$$\frac{d(G)}{d(K_n)} \leq 2\sqrt{2\lambda_2}$$

Assume that x is translated

such that

$$\sum_{x_i \geq 0} d_i = \frac{1}{2} \mathrm{Vol}(G)$$



Take $y_i = \operatorname{sgn}(x_i)\, x_i^2$

$$d(a) = \sum_{(i,j)\in E} |y_i - y_j|$$

$$= \sum_E \left| s(x_i) x_i^2 - s(x_j) x_j^2 \right|$$

$$\leq \sum_E |x_i - x_j| \, (|x_i| + |x_j|)$$

$$\underset{(CS)}{\leq} \sqrt{\sum_E (|x_i| + |x_j|)^2 \cdot \sum_E (x_i - x_j)^2}$$

$$\leq \sqrt{2 \sum (x_i^2 + x_j^2) \; x^T L x}$$

$$= \sqrt{2 \sum_i d_i x_i^2 \; x^T L x}$$

$$= \sqrt{2 (x^T D x)(x^T L x)}$$

$$d(K_G) = \sum_{i<j} \frac{d_i d_j}{Vol(G)} |y_i - y_j|$$

$$\geq \sum_{x_i > 0 > x_j} \frac{d_i d_j}{Vol(G)} |y_i| + |y_j|$$

$$= \sum_{x_i > 0} \frac{d_i \; Vol(G)/2}{Vol(G)} |y_i|$$

$$+$$

$$\sum_{x_j < 0} \frac{d_j Vol(G)/2}{Vol(G)} |y_j|$$

$$= \frac{1}{2} \sum_{i \in V} d_i |y_i|$$

$$= \frac{1}{2} \sum d_i x_i^2 = \frac{1}{2} x^T D x$$

So together

$$\frac{d(G)}{d(K_G)} \le 2\sqrt{2\,\frac{x^T L x}{x^T D x}} \underset{*}{\le} 2\sqrt{2\lambda_2}$$

$$* \quad \frac{x^T L x}{x^T L_{K_G} x} = \lambda_2 .$$

* higher order Cheeger

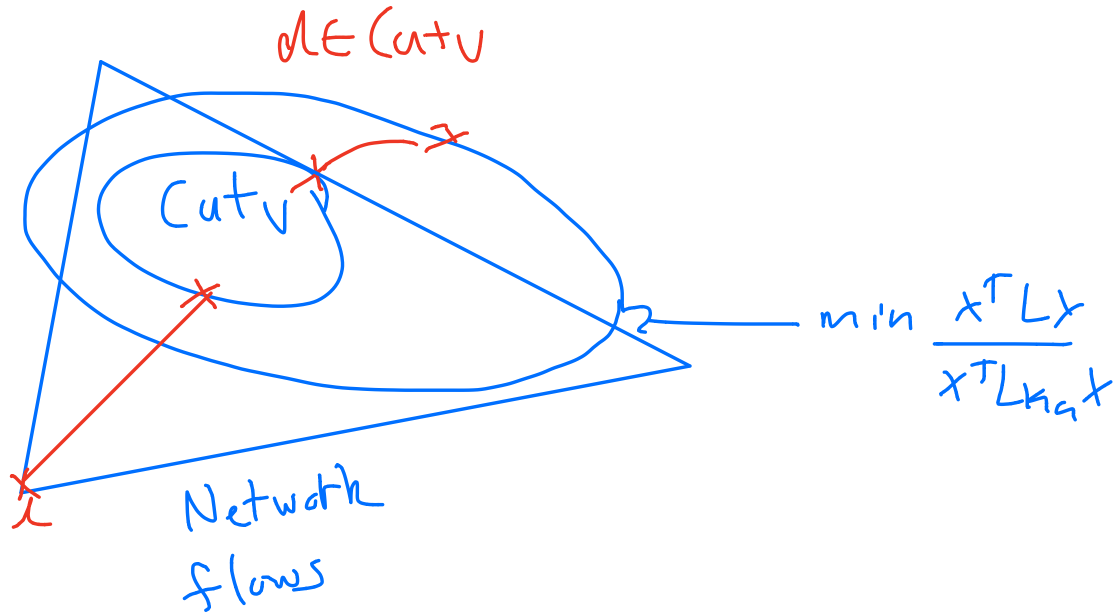
<u>Rmk:</u> Conductance $\Omega(1/n)$

Gap $\Omega(1/n^2)$

So conductance doesn't tell us everything about mixing and convergence problems.

<u>Graph Partioning by Metric Relaxations</u>

Cheeger: $\min d(G)$ s.t. $d(K_G) = 1$



going to relax to $d \in \text{metric}_V$

to use linear programming

Leighton - Roa relaxation

LR Relaxation

$$\min \sum_{E} \delta_{ij} \quad \text{s.t.} \sum_{i<j} \frac{d_i d_j}{\text{Vol}(G)} \delta_{ij} \geq 1$$

such that δ_{ij} is a metric s.t.

for that for all paths

$\forall ij \quad \forall p \in P_{ij} \qquad \delta_{ij} \le \sum_{e \in p} \delta_e$

<u>Claim</u>: In the optimal solution, every δ_{ij}, is given by a path along G.

Hence we can start by thinking about lengths in the graph ℓ_{ij}.

- Look for an example of this

The Dual of this problem,

$\sum d_i d_j$

$$\sum_{i<j} \frac{\quad}{Vol(K_G)} \qquad \delta_{ij} \geq 1 \ : \ \alpha$$

$$\forall i,j \ \forall p \in P_{ij} \subseteq G \quad \delta_{ij} \leq \sum_{e \in p} \ell_e \ : \ f_p$$

So the flow problem

$$\max \ \alpha$$

$$\text{s.t.} \quad \forall e \in E \quad \sum_{e \in p} f_p \leq 1 \qquad \text{(capacity)}$$

$$\forall ij \quad \underbrace{\sum_{p \in P_{ij}} f_p}_{\text{Total flow } i \leftrightarrow j} \geq \alpha \frac{d_i d_j}{Vol(G)} \qquad \text{(demand)}$$

e.g. route αK_G into G as a flow.

$$E_G(S, \bar{S}) \geq \alpha \frac{vol(S)\, vol(\bar{S})}{Vol(G)}$$

$$\longrightarrow \quad \bar{\phi}(S) \geq \alpha$$