1. (a) Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be convergent equivalent sequences of random variables. Then by definition of convergence equivalence this means

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq Y_n) < \infty$$

Note, however, this is exactly the hypothesis of the first Borel-Cantelli Lemma. Hence, using the lemma, we see that convergence equivalence implies $\mathbb{P}(X_n \neq Y_n, i.o.) = 0$

(b) From part (a) we see that almost everywhere X_n and Y_n only differ for a finite number of terms. That is, there exists an n_0 such that for all $n \geq n_0$, $X_n \overset{a.s.}{=} Y_n$. Using this we can write

$$\sum_{n=1}^{\infty}(X_n - Y_n) \overset{a.s.}{=} \sum_{n=1}^{n_0}(X_n - Y_n) < \infty$$

Hence, we see that $\sum_{n=1}^{\infty}(X_n - Y_n)$ converges almost surely.

(c) Using the same argument as in (b) we see that

$$\sum_{n=1}^{\infty}\frac{(X_n - Y_n)}{b_n} \overset{a.s}{=} \sum_{n=1}^{n_0}\frac{(X_n - Y_n)}{b_n} < \infty$$

That is $\sum_{n=1}^{\infty}\frac{(X_n - Y_n)}{b_n}$ converges almost surely. Now, seeing that $b_n \nearrow \infty$, we can use to the Random Kronecker Lemma to conclude

$$\frac{1}{b_n}\sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty$$

2. Define $Y_n = X_n\mathbf{1}_{\{|X_n| \leq A\}}$ for some $A \geq 0$. Recall that convergence of $\sum_{n=1}^{\infty} X_n$ is characterized by the Three-Series Theorem.

First assume that $\{Y_n, n \geq 1\}$ and $\{X_n, n \geq 1\}$ are convergent equivalent. If $\sum_{n=1}^{\infty}\text{Var}(Y_n) = \infty$, then by the Kolmogorov Three-Series Theorem, with probability 1 $\sum_{n=1}^{\infty} Y_n$ diverges. By convergence equivalence, $\sum_{n=1}^{\infty} X_n$ diverges with probability one. If $\sum_{n=1}^{\infty}\text{Var}(Y_n) < \infty$ then by the Kolmorgorov convergence criterion for uniformly bounded random variables $\sum_{n=1}^{\infty}(Y_n - \mathbb{E}(Y_n))$ converges almost surely. Moreover, if $\sum_{n=1}^{\infty}\mathbb{E}(Y_n)$ converges, then $\sum_{n=1}^{\infty} Y_n$ converges almost surely with probability 1 so by convergence equivalence $\sum_{n=1}^{\infty} X_n$ converge almost surely with probability 1. On the other hand, if $\sum_{n=1}^{\infty}\mathbb{E}(Y_n)$ diverges, then by the Kolmorgov Three-Series Theorem $\sum_{n=1}^{\infty} X_n$ diverges.

Now suppose that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are not convergent equivalent. By definition, this implies that $\sum_{n=1}^{\infty}\mathbb{P}(X_n > A) = \infty$. By the Borel-Cantelli lemma, this implies that $\mathbb{P}(X_n > A, i.o) = 1$. That is for each n there exists $m > n$ such that $X_m(\omega) > A$ for almost every $\omega \in \Omega$. This implies that $X_n \overset{a.s.}{\not\longrightarrow} 0$. Seeing this is a necessary condition for convergence of a random series, necessarily we have $\mathbb{P}\left[\sum_{n=1}^{\infty} X_n = \infty\right] = 1$.

Notice that $\mathbb{P}\left[\sum_{n=1}^{\infty} X_n < \infty\right] = 1$ *only when the hypotheses in Kolomorogov Three-Series Theorem were satisfied*. If any are violated $\mathbb{P}\left[\sum_{n=1}^{\infty} X_n < \infty\right] = 0$. This matches our intuition as the Three-Series Theorem is a characterization of convergence of the random series.

3. ($\Longrightarrow$)First suppose that $\sum_{n=1}^{\infty} X_n$ converges almost surely. That is, for all $\epsilon > 0$ and $S_n = \sum_{k=1}^{n} X_k$ we have $\mathbb{P}(\lim_{n,m\to\infty} |S_n - S_m| \geq \epsilon) = 0$. Now notice, that this implies $\mathbb{P}(\lim_{n\to\infty}\sup_{m>n} |S_n - S_m| \geq \epsilon) = 0$. Writing this statement in terms of sets, we have

$$\mathbb{P}\left[\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|S_n - S| \geq \epsilon\}\right] = \lim_{m\to\infty}\mathbb{P}\left[\bigcup_{n=m}^{\infty}\{|S_n - S| \geq \epsilon\}\right]$$

Hence, we see that for any $n > m$ we have

$$\bigcup_{n=m}^{\infty}\{|S_n - S_m| \geq \epsilon\} \supseteq \{|S_m - S_n| \geq \epsilon\}$$

From here we conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}[|S_m - S_n| \geq \epsilon] \leq \lim_{m\to\infty}\mathbb{P}\left[\bigcup_{n=m}^{\infty}\{|S_n - S| \geq \epsilon\}\right] = 0$$

Hence $\{S_n\}_{n=1}^{\infty}$ is Cauchy in probability. This shows that $\sum_{n=1}^{\infty} X_n$ converges in probability.

($\Longleftarrow$) Now suppose that $\sum_{n=1}^{\infty} X_n$ converges in probability. By Ottavianis inequality, for $n, m \in \mathbb{N}$ with $m > n$, we can write

$$\max_{n\leq k\leq m}\mathbb{P}[|S_n - S_k| \geq \epsilon] \geq \frac{1}{3}\mathbb{P}[\max_{n\leq k\leq m}|S_n - S_k| \geq 3\epsilon]$$

Letting $m \to \infty$ we have

$$\max_{n\leq k}\mathbb{P}[|S_n - S_k| \geq \epsilon] \geq \frac{1}{3}\mathbb{P}[\max_{n\leq k}|S_n - S_k| \geq 3\epsilon]$$

Rewriting in terms of sets, we see that we have

$$\sup_{n\leq k}\mathbb{P}[|S_n - S_k| \geq \epsilon] \geq \frac{1}{3}\mathbb{P}[\bigcup_{k=n}^{\infty}\{|S_n - S_k| \geq 3\epsilon\}]$$

Now letting $n \to \infty$ we can write the above inequality as

$$\lim_{n\to\infty}\sup_{k\geq n}\mathbb{P}[|S_n - S_k| \geq \epsilon] \geq \frac{1}{3}\lim_{n\to\infty}\mathbb{P}[\bigcup_{k=n}^{\infty}\{|S_n - S_k| \geq 3\epsilon\}]$$

Now, notice that the left hand side of the inequality is just a characterization of convergence in probability and the right hand side is just a characterization of almost sure convergence. Therefore by assumption

$$0 = \lim_{n\to\infty}\sup_{k\geq n}\mathbb{P}[|S_n - S_k| \geq \epsilon] \geq \frac{1}{3}\lim_{n\to\infty}\mathbb{P}[\bigcup_{k=n}^{\infty}\{|S_n - S_k| \geq 3\epsilon\}]$$

This shows that $\{S_n\}_{n=1}^{\infty}$ is almost surely Cauchy. That is $\sum_{n=1}^{\infty} X_n$ convergences almost surely.