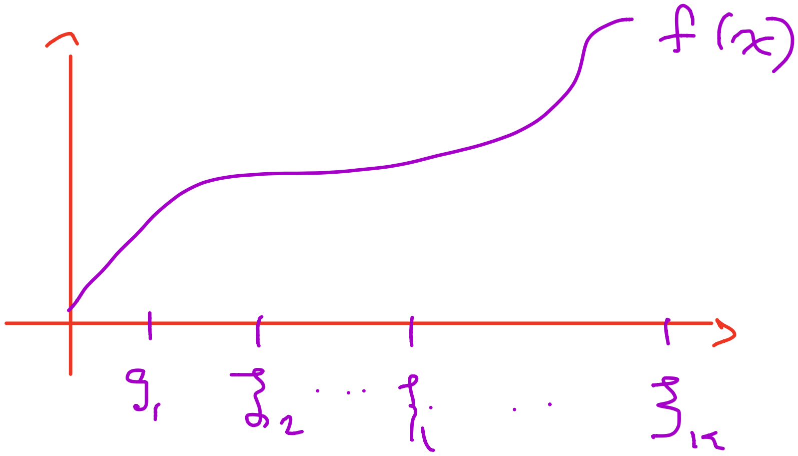
# Splines

$$f(x) = \sum_{i=1}^{m} \beta_i \, h_i(x)$$

spline functions

parameters.




Suppose the knots are fixed and we wish to fit local cubic polynomials (fixed degree $m-1$)

We will require that $\hat{f}$ be $C^2$.

1) $f(\xi_i^-) = f(\xi_i^+)$

2) $f'(\xi_i^-) = f'(\xi_i^+)$

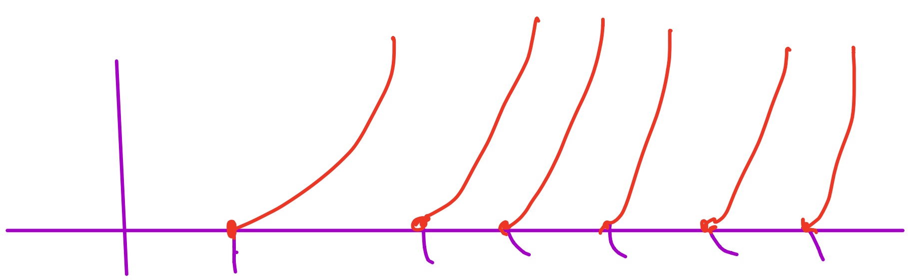
3) $f''(\xi_i^-) = f''(\xi_i^+)$

This class of functions is closed on addition, scalar mult.

$\Longrightarrow$ forms a vector space.

A basis for this span is given by

$$h_1 = x^0 = 1$$
$$h_2(x) = x$$
$$h_3(x) = x^2$$
$$h_4(x) = x^3$$
$$h_{4+\ell}(x) = (x - \xi_\ell)_+^3$$



So the basis is of size $K+4$

## Natural Cubic Splines

- Same as before except we require linear behavior outside the knots

$$N_1(x) = 1 \qquad d_k(x) = \frac{(x-\xi_k)_+^3 - (x-\xi_K)_+^3}{\xi_K - \xi_k}$$

$$N_2(x) = x$$

$$N_{2+k}(x) = d_k(x) - d_{K-1}$$

## Fitting

Define

$$H = \begin{bmatrix} 1 & h_1(x_1) & \cdots & h_m(x_1) \\ 1 & h_1(x_2) & \cdots & \vdots \\ \vdots & & & \vdots \\ 1 & h(x_n) & \cdots & h_m(x_n) \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_0/p & \vec{\beta}_1^T \\ \vdots & \vdots \\ \beta_0/p & \vec{\beta}_p^T \end{bmatrix}$$

$$\hat{f} = [H \circ \beta]\mathbb{1}$$

## Smoothing Splines

**Define:** $W_2^{(m)}[0,1] = \left\{ f : [0,1] \longrightarrow \mathbb{R} \;\middle|\; \int_0^1 (f^{(m)})^2 \, dx < \infty \right\}$

$$\hat{f}(x) = \underset{f \in W_2^{(2)}}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - f(x_i))^2 + \lambda \, \| f'' \|_2^2$$

Penalize roughness. $R(f) = \int_0^1 (f'')^2 dx$

**Theorem:** $\hat{f}$ is a natural cubic spline with knots at the data points $\xi_k = x_k$

## B-Splines