Foxes and Hares

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Table of Contents

Abstract

The Foxes and Hares problem is a graph theory question concerned with investigating the fox number of any given graph. During our research, we focused on interpreting characteristics between one, two, three, and four fox graphs. We determined that prior to any calculations, a graph can be “cleaned” by disregarding all vertices of degree 1 because they ultimately did not impact the fox number of the graph. As a result, any subgraph found in a 1-fox graph has to be a 3-cycle graph. We also discovered the fox number of any graph, if subgraphs exist, is the maximum fox number of any of its subgraphs. Finally, the dimensionality of a graph can be an upper bound for the graph’s fox number. This research primarily focused on defining a method for determining the fox number of any two-dimensional, connected, and undirected graph.

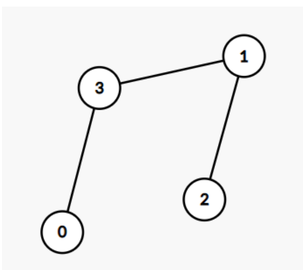
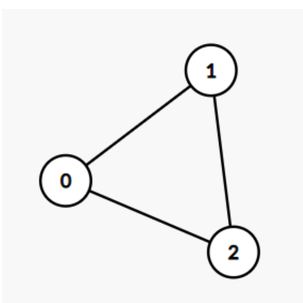
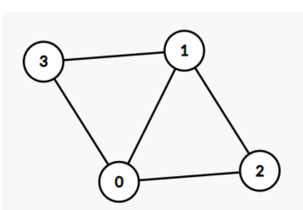
Keywords: Graph theory, foxes and hares

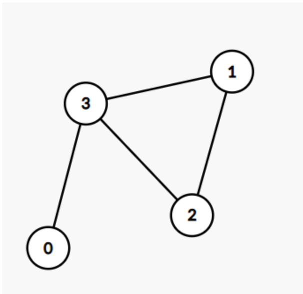
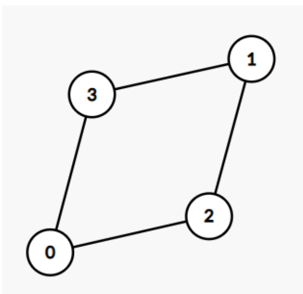
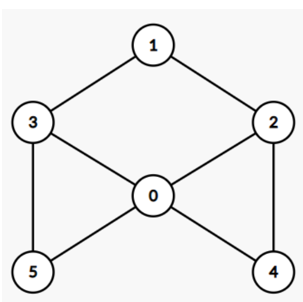
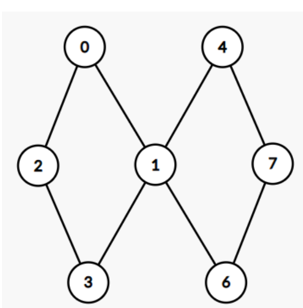
Introduction

Foxes and Hares is a strategy game that can be played on any connected, undirected graph. For each game, the fox(es) and a single hare start on separate vertices; the fox(es) are allowed to move first. On the foxes' turn, each fox can move to an adjacent vertex or remain in place, and multiple foxes may occupy the same vertex. The hare has the same choices of moves on its turn. The foxes win the game if 1 or more foxes occupy the same vertex as the hare, essentially "*catching*" the hare. For a given graph, the minimum number of foxes required to catch the hare when the hare is placed optimally is called the graph's fox number. The goal of this research was to find a method to cleanly determine the fox number for any graph. Given an undirected and connected graph, the fox number of that graph will be equal to the highest fox number of its *subgraphs*. A *subgraph* is defined as a graph where the number of edges is equal to the number of vertices. Please note that this research paper uses the term *subgraph* differently than the traditional mathematical definition of a subgraph. All graphs considered were undirected, connected graphs with n vertices and each vertex labeled as $p_1,\ p_2 ...\ p_n$. For research purposes, the hare was assumed to begin at an optimal starting position, which would be the vertex with the highest degree on the graph and connecting to the maximum fox number subgraph. This vertex increases the number of paths the hare can access in order to avoid the fox(es). If there are multiple vertices with the same degree, the hare should start in the position that creates the greatest number of edges between the hare and the fox(es). For each turn, the hare is assumed to play optimally, which would mean that it considers moving towards the vertex with the most degrees while maximizing the number of edges between the fox and itself.

Experimental Methods

When first conducting our research, we first created many simple 1-fox graphs. Afterward, we developed some more complicated 1-fox graphs and began looking into simple 2-fox graphs. We continued this pattern to find 3 fox and 4 fox graphs. As a result, the table below was created.

Graph	Fox Number	Maximum degree vertex	Minimum degree vertex	Number of vertices
	*1*	*2*	*1*	*4*
	*1*	*2*	*2*	*3*
	*1*	*3*	*2*	*4*

	*1*	*3*	*1*	*4*
	*2*	*2*	*2*	*4*
	*2*	*4*	*2*	*6*
	*2*	*4*	*2*	*7*

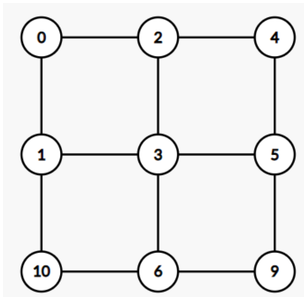
	*2*	*3*	*2*	*9*

Figure 1: Table comparing a graph's fox number vs. characteristics of that graph. Our actual table consisted of more rows consisting of 2 and 3 fox graphs.

While analyzing the graphs, we came up with plausible conjectures, such as the theory that a fox number for a given graph was the minimum degree of any vertex. However, we were able to find graphs that were counterexamples to that theory, which only prompted us to revise and retest newer conjectures. The four conjectures we explored were:

1. The ability to remove nodes of 1 degree without impacting the fox number of a graph.
2. Any subgraph found in a 1-fox graph has to be a 3-cycle graph.
3. The fox number of any given graph is the maximum fox number of any of its subgraphs.
4. The dimensionality[1] of a graph is the highest fox number that any graph could have.

1 - The minimum number of dimensions needed in order to draw an isomorphic graph with no connecting edges occupying the same space

Proofs of Lemmas

1-fox Path Graph Lemma

Lemma: All path graphs have a fox number of 1.

Proof: Start with a path of n vertices ($p_1, p_2, p_3 .... p_n$). We first put the hare at vertex p_i and 1 fox at vertex p_j. On the fox's first turn, its optimal strategy would be to move in the direction that minimizes the distance (number of edges, not Euclidean distance) between the hare and itself. On the hare's turn, its optimal strategy would be to move in the direction that maximizes the distance between the fox and itself. If the hare is on vertex p_1 or p_n, then there is no advantageous route for the hare to maximize the distance between itself and the fox, essentially cornering the hare. Therefore, the fox will use its turns to move closer to the hare, while the hare has no room on the graph to distance itself from the fox.

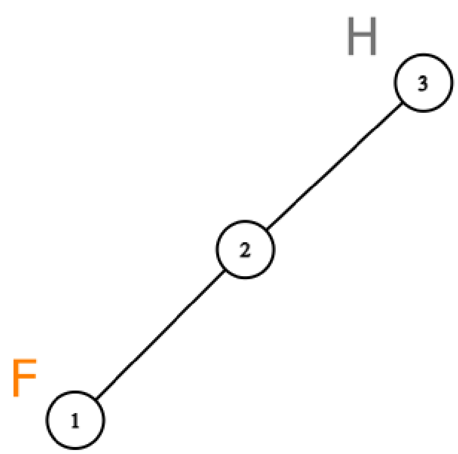


Figure 2: Path graph ($n = 3$) with the fox on p_1 and hare on p_3

1-fox Cycle Graph (n = 3) Lemma

Lemma: All cycle graphs with exactly 3 vertices have a fox number of 1.

Proof: Let G be a cycle graph with $n = 3$, with vertices p_1, p_2, p_3. If the hare begins on p_1 and the fox begins on p_2, as that is optimal placement for both the fox and the hare, then during the next turn, the fox is able to "eat" the hare. Since each vertex on G is connected to all other vertices on the graph, then no matter where the fox is, it will always have a direct path to the vertex the hare is on. Thus, the fox number of a cycle graph with $n = 3$ must be 1 because each vertex within the graph has direct access to all other vertices.

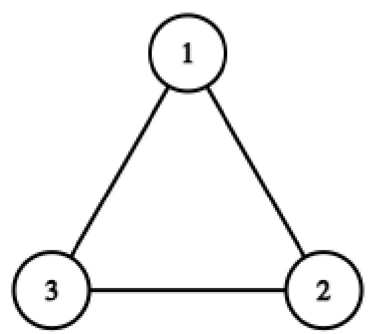


Figure 3: Each vertex {p_1, p_2, p_3} is able to reach all other vertices on the $n = 3$ cycle graph.

2-fox Cycle Graph (n > 3) Lemma

Lemma: All cycle graphs that have 4 or more vertices must have a fox number of 2.

Proof (situation 1): Assume a cycle graph of n vertices (p_1, p_2, p_3.... p_n, where $n > 3$) that contains a single fox and a single hare. The hare begins on p_1 and the fox begins on the vertex that maximizes the initial distance between the fox and the hare, which is p_n in this situation.

When the fox moves, it will move clockwise or counterclockwise to minimize the distance between itself and the hare. In response, the hare will move in the identical direction that the fox moved. Due to the hare continuously maximizing the distance from the fox, the distance (or the number of edges) between the fox and the hare will oscillate between $(n\ /\ 2)$ and $(n\ /\ 2)$ - 1. Due to n being greater than 3, this will never result in the distance being equal to 0. As a result, the fox will never reach the hare.

Proof (situation 2): Assuming 2 foxes are present on the graph of n vertices, the hare may start at the farthest position from the 2 foxes. In this situation, both foxes will begin on the same vertex, which is $floor(n\ /\ 2)$ vertices away from the hare. *Floor* is defined as the division of 2 numbers and truncation down to the nearest integer (ex: 10/3 = 3). If one fox continues moving clockwise, and the other fox continues moving counter-clockwise, the foxes converge on the hare by $floor(n\ -\ 1)\ /\ 2$ cycles. The hare is forced to be stationary because if it moves to any other vertex, the hare would not be following its optimal strategy. When both foxes are at the adjacent vertices to the hare's vertex, the next move that one of the foxes can do is to assume the same position of the hare, which wins the game.

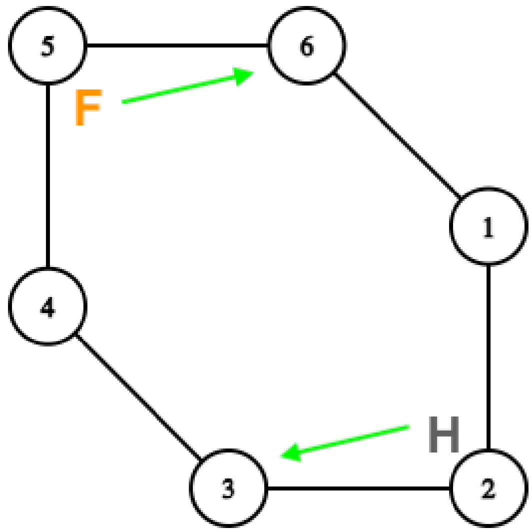


Figure 4: With one fox on a cycle graph (n = 6), the fox will never catch the hare.

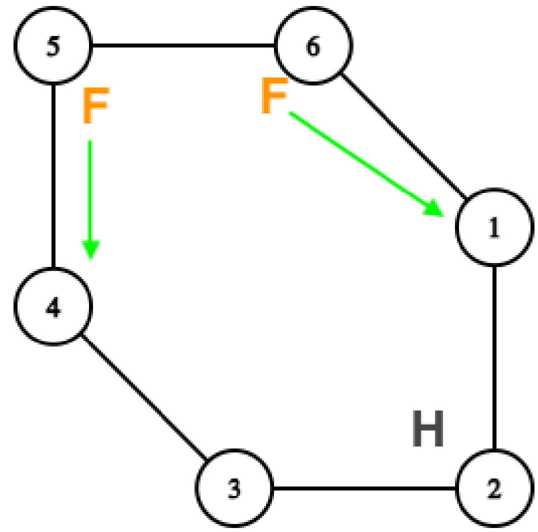


Figure 5: Two foxes are able to "corner" the hare.

Sub-cycle Graph Lemma

Lemma: All *subgraphs* created from a two-dimensional, connected graph must be cycle graphs.

Proof: Recall that the definition of a *subgraph* is any graph where the number of edges is equal to the number of vertices. In any connected and two-dimensional graph, all vertices are connected to at least one other vertex. Thus, for the number of edges to be equal to the number of vertices, all vertices must only have one other connection to another vertex. The only fundamental graphs that can be created when all vertices have one edge linking it to another vertex are cycle graphs.

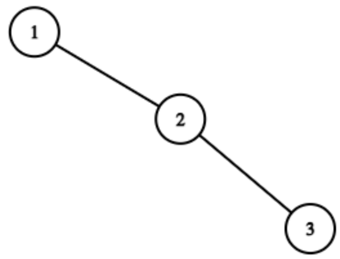


Figure 6: Path graph (n = 3)

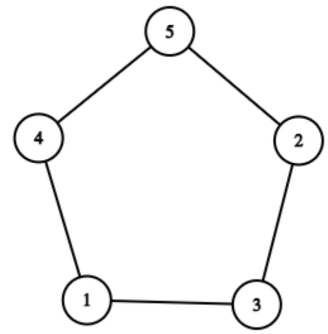


Figure 7: Cycle graph (n = 5)

1 Vertex, 1 Fox Lemma

Conjecture: All graphs with exactly 1 vertex are 1-fox graphs.

Proof: If there is only one vertex for a graph, then the fox and the hare must start on the same vertex. As there are no other vertices for the hare to move to, the fox continues occupying the same vertex as the hare, and the game concludes.

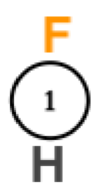


Figure 8: Since the fox and hare occupy the same vertex from the start, the fox number must be one.

Proofs of Theorems

Pre-Calculation Optimization Theorem

Theorem: Before calculating the fox number of a graph, it is possible to remove all vertices of degree 1 without changing the fox number of a graph.

Proof: Consider a graph with a vertex of p_i of degree 1. Considering *optimal play*, the hare would never purposely take a route to which it is *cornered*, if possible. When a hare is *cornered*, it is on a vertex such that no matter which moves the hare makes, the 1 or more foxes still have the ability to occupy the same vertex as the hare. Therefore, the hare will never occupy a vertex with degree 1 (p_i) because if it lands on that vertex, the only options would be to backtrack or to stay put. A fox could easily corner the hare by occupying the adjacent vertex to p_i. As a result, any vertex that has a degree of 1, and its corresponding edge, can be removed from the graph because it would never be a part of the hare's path.

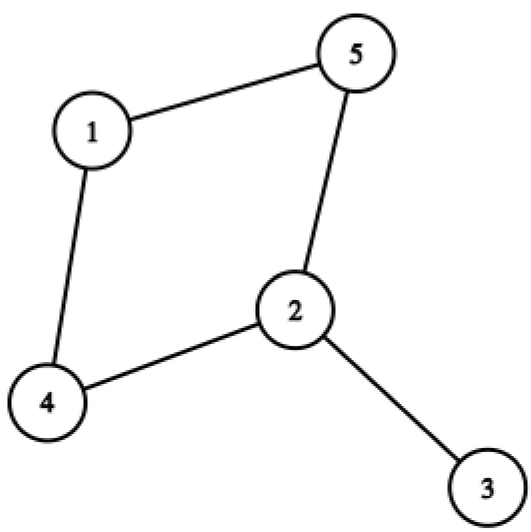


Figure 9: p_3 is able to be removed because it has degree one. Additionally, the edge connecting p_2 to p_3 can also be removed

In figure 9, the graph could be optimized before calculating the fox number. In this example, p_3 would be removed because it has a degree of 1. Assuming optimal play, the hare would never reach p_3. However, if we don't assume optimal play, thus allowing the hare to occupy p_3, the hare is able to be cornered with only one other fox. If the hare stayed on the 4-cycle subgraph, the overall graph would have a fox number of 2. Therefore, the hare gains no advantage whenever it goes on a node with degree 1, allowing the 1-degree nodes to be "removed" for calculation purposes.

Maximum Subgraph Theorem

Theorem: If a graph can be broken down into subgraphs, then the max fox number of all the subgraphs is the fox number of the overall graph.

Proof: According to our definition of a *subgraph*, all subgraphs of an overall graph must be cycle graphs. If the hare plays optimally, then the hare would stay in the cycle graph with the greatest number of vertices and the greatest fox number. According to our previous proof, the lowest fox number of a cycle graph is 1 ($n = 3$), while the highest fox number of a cycle graph is 2 ($n > 3$). Thus, if the hare stays in the cycle graph with the most vertices, then the fox number of the overall graph would be the fox number of the subgraph with the highest fox number.

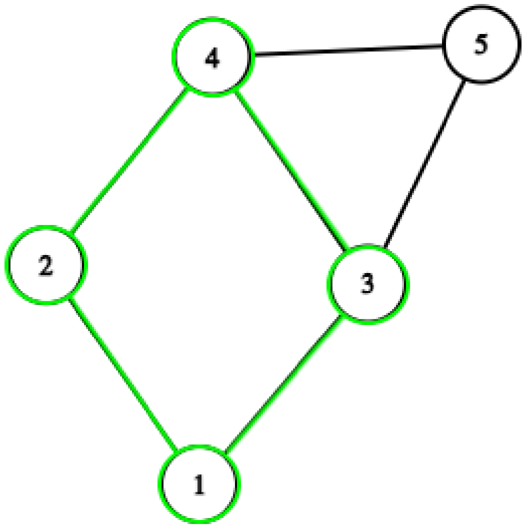


Figure 10: A graph with n = 5

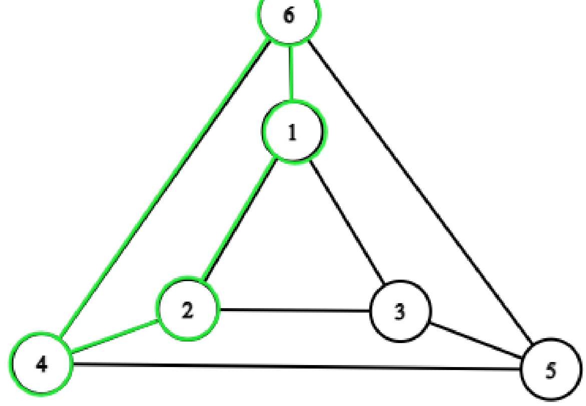


Figure 11: A graph with n = 6

Figure 10 consists of two *subgraphs*, one of which is $n = 3$ and the other is $n = 4$. The fox number of the n = 3 subgraph is 1, while the fox number of the n = 4 fox number is 2. Therefore, the fox number of the overall graph is 2. In figure 11, 4 subgraphs can be formed, with three cycle graphs ($n = 4$) and one cycle graph ($n = 3$). Since the fox number of cycle graphs $n = 4$ is higher than the fox number of $n = 3$cycle graphs, the overall fox number of the graph is the fox number of an $n = 4$cycle graph.

1-fox Cycle Subgraph Theorem

Theorem: If a two-dimensional, 1-fox graph is broken down into all of its subgraphs, then all subgraphs, if they exist, must be cycle graphs with $n = 3$.

Proof (situation 1): Recall from our previous proof that all path graphs have a fox number of one. Thus, no subgraphs are needed to prove that the fox number of a path graph is one.

Proof (situation 2): Let G be a graph with a fox number of 1 containing at least 1 subgraph. Since the fox number of a graph is equal to the greatest fox number of its subgraphs, then the largest fox number of a subgraph in a 1-fox graph is 1. Also, all subgraphs in G must have a fox number of 1 because the subgraphs are cycle graphs ($n = 3$). Only a cycle graph with $n = 3$ has a fox number of 1. Any cycle graph with $n > 3$ has a fox number of 2 while any graph with $n < 3$ is no longer a cycle graph.

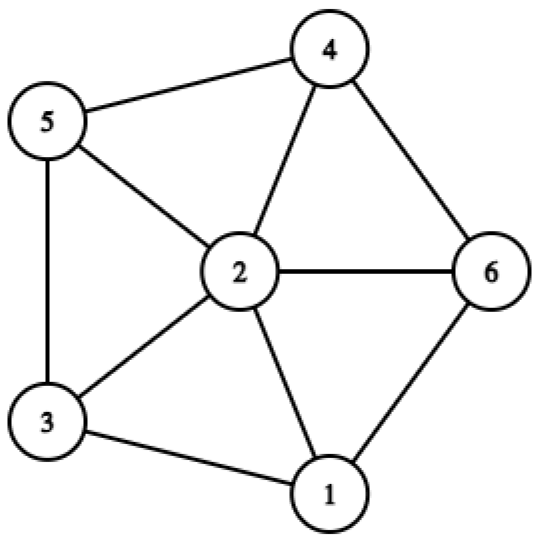
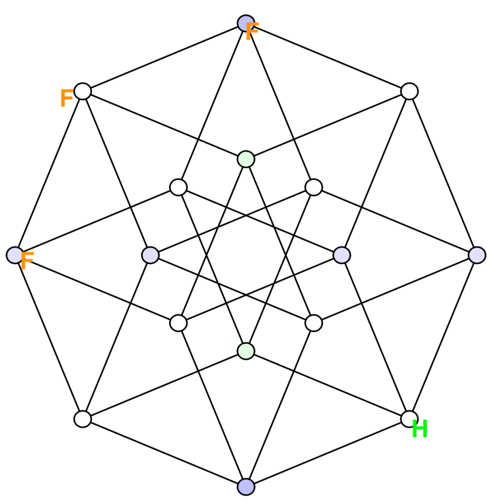


Figure 12: All subgraphs are cycle graphs of n = 3 in a 1-fox graph.

Conjecture

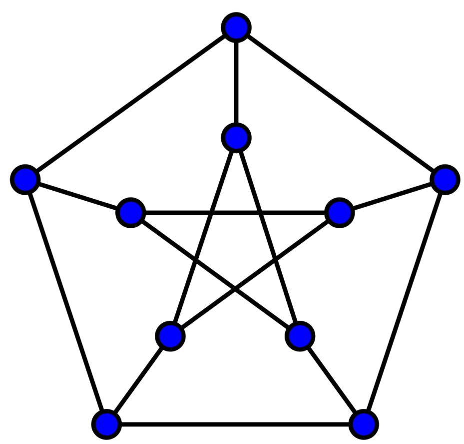
Max Fox Number Conjecture

Conjecture: The dimensionality of a graph is the highest fox number that any graph could have. Recall that the *dimensionality* of a graph is the minimum dimensions of planes needed to make sure no two edges intersect. For example, all planar graphs are considered 2-D graphs, because any planar graph can be drawn as an isomorphic equivalent where no two edges intersect. All planar graphs we have tested so far have had a maximum possible fox number of two, which follows this conjecture because all planar graphs are 2-D. Moreover, all 3-D graphs we tested have had a maximum possible fox number of 3, such as the Petersen graph. We suspect that this behavior is due to the fact that within the Petersen graph, all vertices have degree 3. At any possible point, the hare has three escape routes, meaning that two foxes cannot ever *corner* the hare. Furthermore, in a hypercube graph, which is 4-D, the fox number is 3 if both hares and foxes play optimally, which supports our theory. We have not found a case yet in which the fox number is higher than the *dimensionality* of the graph.



Source: *[Wikipedia](#)*

Figure 13: Hypercube graph
Dimensionality: 4



Source: *[Wikipedia](#)*

Figure 14: Petersen graph
Dimensionality: 3

Fox Number: 3 *Fox Number: 3*

Conclusion

The goal of the research was to determine a method for determining the fox number of any graph in the game Foxes and Hares. The experimental method used to determine our conjectures was focused on making sure most graphs correlated with our theory. Any conjectures that were invalidated were done so by finding graphs that acted as exceptions to our claims.

There are several ways we could expand and vary the game, many of which would allow for avenues of further research. For example, keeping track of the minimum number of moves required for the fox(es) to win was a consideration that our group had, but never fully explored. Additionally, limiting the number of moves each fox could take would be another variation. While time prevented us from further exploration of Foxes and Hares, we analyzed 1-fox and 2-fox graphs to see if there were any patterns in the number of turns for each graph. We developed several rudimentary conjectures that stated that the number of moves was dependent on the number of edges in a graph. Ultimately, our group achieved our goal in finding a strategy to analyze a graph to determine its fox number in our interpretation of Foxes and Hares.

References

1. https://csacademy.com/app/graph_editor
2. Guidelines for Good Mathematical Writing
3. Notes on Writing Mathematics

Appendix

A: Creating Subgraphs

Recall that the definition of a subgraph is a graph in which the number of edges in a graph is equal to the number of vertices in that graph. Vertices of an overall graph can be used in multiple subgraphs. If a graph that is extracted from an overall graph already counts as a subgraph, then we don't need to break down the subgraph anymore.

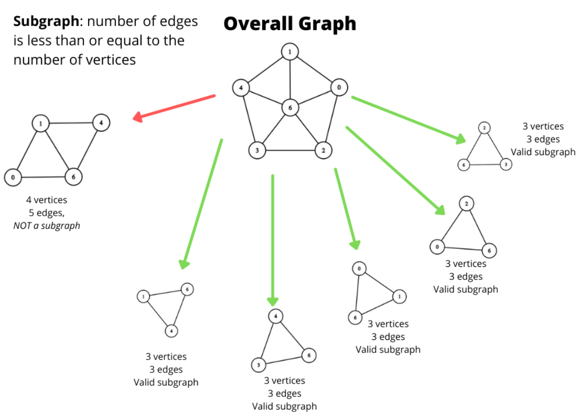


Figure 15: Breaking an overall graph into subgraphs

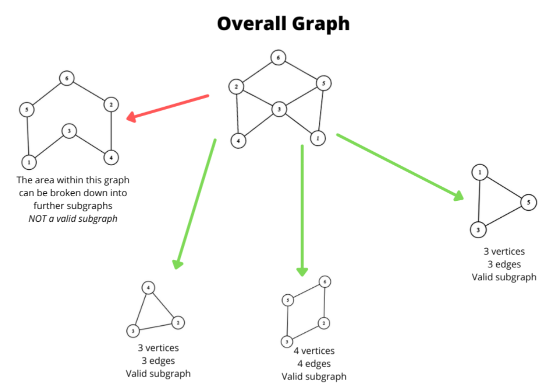


Figure 16: Breaking an overall graph into subgraphs

The process to creating subgraphs is as follows:

1. Find any "closed area" on the graph
2. Find adjacent vertices that the closed area is directly touching
3. Those vertices count as a subgraph if the number of edges within that subgraph is equal to its number of vertices

B: Minimum Degree Theory (Proved Wrong)

Conjecture: The fox number of any graph is equal to the minimum number of degrees of all vertices in the graph.

Contradiction:

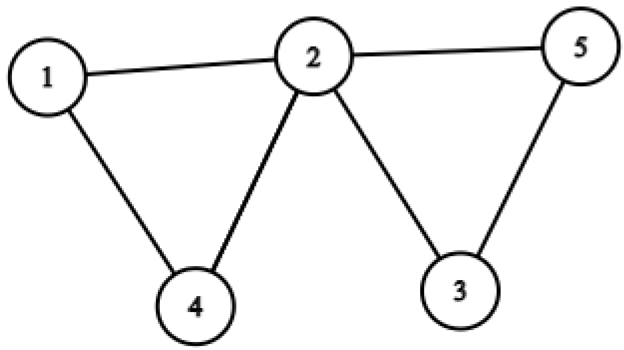


Figure 17: A graph with p_2 having a degree of 4

In this situation, the degrees for the vertices p_1, p_2, p_3, p_4, p_5 are 2, 4, 2, 2 and 2 respectively; the minimum degree in the graph is 2. According to our other definitions, however, the graph in Figure 17 is a 1-fox graph.