APPENDIX B

SOLUTIONS TO PROBLEMS

B.1 Before the student takes the SAT exam, we do not know – nor can we predict with certainty – what the score will be? The actual score depends on numerous factors, many of which we cannot even list, let alone know ahead of time. (The student's innate ability, how the student feels on exam day, and which particular questions were asked, are just a few.) The eventual SAT score clearly satisfies the requirements of a random variable.

B.3 (i) Let Y_{it} be the binary variable equal to one if fund *i* outperforms the market in year *t*. By assumption, $P(Y_{it} = 1) = .5$ (a 50-50 chance of outperforming the market for each fund in each year). Now, for any fund, we are also assuming that performance relative to the market is independent across years. But then the probability that fund *i* outperforms the market in all 10 years, $P(Y_{i1} = 1, Y_{i2} = 1, \ldots, Y_{i,10} = 1)$, is just the product of the probabilities: $P(Y_{i1} = 1)\cdot P(Y_{i2} = 1) \ldots P(Y_{i,10} = 1) = (.5)^{10} = 1/1024$ (which is slightly less than .001). In fact, if we define a binary random variable Y_i such that $Y_i = 1$ if and only if fund *i* outperformed the market in all 10 years, then $P(Y_i = 1) = 1/1024$.

(ii) Let X denote the number of funds out of 4,170 that outperform the market in all 10 years. X has the Binomial (n,θ) distribution with $n = 4{,}170$ and $\theta = 1/1024$. The expected number of funds that will outperform the market in all 10 years is calculated as $n\theta = 4.07 \approx 4$.

(iii) Let X denote the number of funds out of 4,170 that outperform the market in all 10 years. Then $X = Y_1 + Y_2 + \ldots + Y_{4,170}$. If we assume that performance relative to the market is independent across funds, then X has the Binomial (n,θ) distribution with $n = 4{,}170$ and $\theta = 1/1024$. We want to compute $P(X \geq 1) = 1 - P(X = 0) = 1 - P(Y_1 = 0, Y_2 = 0, \ldots, Y_{4,170} = 0) = 1 - P(Y_1 = 0)\cdot P(Y_2 = 0)\cdots P(Y_{4,170} = 0) = 1 - (1023/1024)^{4170} \approx .983$. This means, if performance relative to the market is random and independent across funds, it is almost certain that at least one fund will outperform the market in all 10 years.

(iv) Using the Stata command Binomial(4170,5,1/1024), the answer is about .385. So, there is a nontrivial chance that at least five funds will outperform the market in all 10 years.

B.5 (i) As stated in the hint, if X is the number of jurors convinced of Simpson's innocence, then $X \sim$ Binomial(12,.20). We want $P(X \geq 1) = 1 - P(X = 0) = 1 - (.8)^{12} \approx .931$.

(ii) Above, we computed $P(X = 0)$ as about .069. We need $P(X = 1)$, which we obtain from (B.14) with $n = 12$, $\theta = .2$, and $x = 1$: $P(X = 1) = 12\cdot(.2)(.8)^{11} \approx .206$. Therefore, $P(X \geq 2) \approx 1 - (.069 + .206) = .725$, so there is almost a three in four chance that the jury had at least two members convinced of Simpson's innocence prior to the trial.

B.7 In eight attempts the expected number of free throws is 8(.74) = 5.92, or about six free throws.

B.9 If Y is salary in dollars, then $Y = 1000 \cdot X$, and so the expected value of Y is 1,000 times the expected value of X, and the standard deviation of Y is 1,000 times the standard deviation of X. Therefore, the expected value and standard deviation of salary, measured in dollars, are \$52,300 and \$14,600, respectively.

B.11 (i) Let X be a random variable taking on the values −1 and 1, each with probability 1/2.
$E(X) = x_1 f(x_1) + x_2 f(x_2)$.
$E(X) = 1(1/2) + (-1)(1/2) = 0$.

$E(X^2) = x_1^2 f(x_1) + x_2^2 f(x_2)$.
$E(X^2) = 1(1/2) + 1(1/2) = 1$.

(ii) Let X be a random variable taking on the values 1 and 2, each with probability 1/2.
$E(X) = x_1 f(x_1) + x_2 f(x_2)$.
$E(X) = 1(1/2) + 2(1/2) = 3/2$.

$E(1/X) = (\frac{1}{x_1}) f(x_1) + (\frac{1}{x_2}) f(x_2)$.
$E(1/X) = 1(1/2) + (1/2)(1/2) = 3/4$.

(iii) Form parts (i) and (ii) we observe that
$E(X^2) \neq (E(X))^2$ and $E(1/X) \neq 1/E(X)$.

In general for a nonlinear function $g(\cdot)$,
$E(g(X)] \neq g[E(X)]$.

(iv) To define an F random variable, let $X_1 \approx \chi^2_{k_1}$ and $X_2 \approx \chi^2_{k_2}$. Assume that X_1 and X_2 are independent.
Then, the random variable

$$F = \frac{(X_1/K_1)}{(X_2/K_2)},$$

has an F distribution with (k_1, k_2) degrees of freedom.

We also know that if $X \approx \chi^2_n$, then the expected value of X is n.

$E(F) = E[(X_1/k_1)]E\left[\frac{1}{(X_2/k_2)}\right]$ because we have $E(U/V) = E(U)E(1/V)$ if U and V are independent.
$E(F) = 1/k_1 E(X_1) E\left[\frac{1}{(X_2/k_2)}\right]$
$E(F) = k_1/k_1 \, E\left[\frac{1}{(X_2/k_2)}\right]$ because $E(X_1) = k_1$
$E(F) = E\left[\frac{1}{(X_2/k_2)}\right]$.

$E(F)$ is not equal to one, because the expectation of ratio is not equal to ratio of expectation.