APPENDIX D

SOLUTIONS TO PROBLEMS

D.1 (i) $\mathbf{AB} = \begin{pmatrix} 2 & -1 & 7 \\ -4 & 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 6 \\ 1 & 8 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -6 & 12 \\ 5 & 36 & -24 \end{pmatrix}$

(ii) **BA** does not exist, because **B** is 3 × 3 and **A** is 2 × 3.

D.3 Using the basic rules for transpose, $(\mathbf{X}'\mathbf{X}') = (\mathbf{X}')(\mathbf{X}')' = \mathbf{X}'\mathbf{X}$, which is what we wanted to show.

D.5 (i) The $n \times n$ matrix **C** is the inverse of **AB** if and only if $\mathbf{C}(\mathbf{AB}) = \mathbf{I}_n$ and $(\mathbf{AB})\mathbf{C} = \mathbf{I}_n$. We verify both of these equalities for $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. First, $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{I}_n\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_n$. Similarly, $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AI}_n\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}_n$.

(ii) $(\mathbf{ABC})^{-1} = (\mathbf{BC})^{-1}\mathbf{A}^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

D.7 We must show that, for any $n \times 1$ vector $\mathbf{x}$, $x \neq \mathbf{0}$, $\mathbf{x}'(\mathbf{P}'\mathbf{AP})\,\mathbf{x} > 0$. But we can write this quadratic form as $(\mathbf{P\,x})'\mathbf{A}(\mathbf{P\,x}) = \mathbf{z}'\mathbf{Az}$ where $\mathbf{z} \equiv \mathbf{Px}$. Because **A** is positive definite by assumption, $\mathbf{z}'\mathbf{Az} > 0$ for $\mathbf{z} \neq \mathbf{0}$. So, all we have to show is that $\mathbf{x} \neq \mathbf{0}$ implies that $\mathbf{z} \neq \mathbf{0}$. We do this by showing the contrapositive, that is, if $\mathbf{z} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$. If $\mathbf{Px} = \mathbf{0}$, then, because $\mathbf{P}^{-1}$ exists, we have $\mathbf{P}^{-1}\mathbf{Px} = \mathbf{0}$ or $\mathbf{x} = \mathbf{0}$, which completes the proof.

D.9 To obtain the stated conclusion, first use the fact that $\text{tr}(\mathbf{auu}'\mathbf{a}') = \text{tr}(\mathbf{a}'\mathbf{auu}')$. Next, the expected value passes through the trace operator, because trace is a linear operator. Therefore, $E[tr(\mathbf{a}'\mathbf{auu}')] = tr[E(\mathbf{a}'\mathbf{auu}')]$. Now use the fact that $\mathbf{a}'\mathbf{a}$ is nonrandom, and so the expected value passes through:

$$E(\mathbf{a}'\mathbf{auu}') = \mathbf{a}'\mathbf{a}E(\mathbf{uu}') = \mathbf{a}'\mathbf{a}\mathbf{I}_n = \mathbf{a}'\mathbf{a} = \sum_{i=1}^{n} a_i^2,$$

where we use the assumption that $E(\mathbf{uu}') = \mathbf{I}_n$. Of course the trace of a scalar is just the scalar.

D.11 (i) **X** is $n \times k$ matrix partitioned as $(\mathbf{X_1}\ \mathbf{X_2})$, where $\mathbf{X_1}$ is $n \times k_1$ and $\mathbf{X_2}$ is $n \times k_2$.

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \end{pmatrix}(\mathbf{X_1} \quad \mathbf{X_2}) = \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix}.$$

The dimensions of each of the matrices are

$\mathbf{X}_1'\mathbf{X}_1$ is $k_1 \times k_1$

$\mathbf{X}_2'\mathbf{X}_1$ is $k_2 \times k_1$

$\mathbf{X}_1'\mathbf{X}_2$ is $k_1 \times k_2$

$\mathbf{X}_2'\mathbf{X}_2$ is $k_2 \times k_2$

(ii) Let $\mathbf{b}$ be a $k \times 1$ vector, partitioned as $\mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$, where $\mathbf{b}_1$ is $k_1 \times 1$ and $\mathbf{b}_2$ is $k_2 \times 1$.

$$(\mathbf{X}'\mathbf{X})\mathbf{b} = \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)\mathbf{b}_1 + (\mathbf{X}_1'\mathbf{X}_2)\mathbf{b}_2 \\ (\mathbf{X}_2'\mathbf{X}_1)\mathbf{b}_1 + (\mathbf{X}_2'\mathbf{X}_2)\mathbf{b}_2 \end{pmatrix}.$$