Asymptotic formulas for Vasyunin cotangent sums

Konstantinos A. Draziotis[a], George Fikioris[b]

[a]*Department of Informatics*
Aristotle University of Thessaloniki
54124 Thessaloniki, Greece
[b]*School of Electrical and Computer Engineering,*
National Technical University,
15773 Athens, Greece

Abstract

We study the Vasyunin-type cotangent sum $c_0(h/k) = -\sum_{m=1}^{k-1}(m/k)\cot\big(\pi hm/k\big)$, where h and k are positive coprime integers. This sum is related to Estermann zeta function. By applying the Euler-Maclaurin summation formula to a suitable function, we improve a previous large-k asymptotic approximation of $c_0(h/k)$. We also provide a procedure to compute an arbitrary number of terms of the approximation.

Keywords:
Vasyunin cotangent sums, Euler-Maclaurin Summation formula, digamma function

1. Introduction-Statement of results

We begin with a definition of the cotangent sum that we shall study.

Definition 1.1. *Let $h > 0$ and $k > 1$ be integers with $h < k$ and $\gcd(h, k) = 1$. We define*

$$c_0(h/k) = -\sum_{m=1}^{k-1} \frac{m}{k} \cot\Big(\pi h \frac{m}{k}\Big).$$

We are concerned with the following problem: *Find an asymptotic formula for $c_0(h/k)$ for the case where k is large.* Let us first explain why this problem is useful and interesting.

This cotangent sum has various applications in number theory. First, it is connected with Nyman-Beurling-Baéz-Duarte-Vasyunin approach to Riemann Hypothesis [1, 3, 6, 21, 25] (for an excellent survey of Riemann hypothesis see

Email addresses: `drazioti@csd.auth.gr` (Konstantinos A. Draziotis), `gfiki@ece.ntua.gr` (George Fikioris)

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[7, 8]). Second, it is related to the Estermann zeta function, which is given by the Dirichlet series

$$E\Big(s, \frac{h}{k}\Big) = \sum_{n=1}^{\infty} \frac{d(n)e(nh/k)}{n^s}, \; e(z) = e^{2\pi i z}, \quad d(n) = |\{d \in \mathbb{N} : d|n\}|, \quad \Re(s) > 1,$$

where $h, k \in \mathbb{Z}$ with $k \geq 1$. Estermann in 1930 proved that this function can be analytically continued to $\mathbb{C} - \{1\}$ [10]. For $s = 0$ Ishibashi proved (this is a special case of [20, Theorem 1]),

$$E\Big(0, \frac{h}{k}\Big) = \frac{1}{4} + \frac{i}{2} c_0\Big(\frac{h}{k}\Big).$$

Properties of the function $E(0, h/k)$ are useful for the estimation of a lower bound for those zeros of the Riemann zeta function $\zeta(s)$ that lie on the critical line $1/2 + i\mathbb{R}$, see also [2]. Furthermore, the cotangent sum $c_0(1/k)$ is a special case of the Vasyunin cotangent sum. Vasyunin in [28] considered the following sum,

$$V(q, k) = \sum_{m=1}^{k-1} \Big\{\frac{mq}{k}\Big\} \cot \frac{\pi m}{k},$$

with coprime positive integers q, k, where $\{x\} = x - \lfloor x \rfloor$ ($x \in \mathbb{R}$) (see also [3, 13]). Thus $c_0(1/k) = -V(1, k)$. We provide the following definition.

Definition 1.2. *The following sum is called the Vasyunin cotangent sum,*

$$V\Big(\frac{h}{k}\Big) = \sum_{m=1}^{k-1} \Big\{\frac{mh}{k}\Big\} \cot \frac{\pi m}{k},$$

where h, k are coprime positive integers.

In [4] it was proved that,

$$V\Big(\frac{h}{k}\Big) = -c_0\Big(\frac{h'}{k}\Big),$$

where h' is the inverse of $h \bmod k$ (the inverse exists since $\gcd(h, k) = 1$).

In 1995 Vasyunin [28] provided an asymptotic estimate for $c_0(1/k)$. He proved the following.

Theorem 1.3 (Vasyunin)**.** *For large integer values of k we have*

$$c_0\Big(\frac{1}{k}\Big) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln(2\pi) - \gamma) + O(\ln k), \tag{1}$$

where γ is the Euler-Mascheroni constant.

This result was improved in [26] in the following manner.

Theorem 1.4 (Rassias)**.** *For large integer values of k we have*

$$c_0\Big(\frac{1}{k}\Big) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln(2\pi) - \gamma) + O(1). \tag{2}$$

The proof uses properties of fractional parts for obtaining asymptotic approximation of the sum $S(L;k) = 2k\sum_{1\le a\le L} \frac{1}{a}\lfloor \frac{a}{k}\rfloor$.

A second improvement was provided in [23].

Theorem 1.5 (Maier-Rassias)**.** *Let $k, n \in \mathbb{N}$, $k \geq 6N$, with $N = \lfloor \frac{n}{2} + 1\rfloor$. There exist absolute real constants $A_1, A_2 \geq 1$ and $\{E_l\}$ with $|E_l| \leq (A_1 l)^{2l}$ $(l \in \mathbb{N})$, such that for each $n \in \mathbb{N}$ we have*

$$c_0\Big(\frac{1}{k}\Big) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln(2\pi) - \gamma) - \frac{1}{\pi} + \sum_{l=1}^{n} E_l k^{-l} + R_n^*(k),$$

where

$$|R_n^*(k)| \leq (A_2 n)^{4n} k^{-n+1}.$$

In [13, Corollary 1.2] the authors further improved the previous result.

Theorem 1.6 (Goubi-Bayad-Hernane)**.** *Let $N \geq 2$. For large k we have*

$$c_0\Big(\frac{1}{k}\Big) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln(2\pi) - \gamma) + \frac{1}{\pi} + \frac{\pi}{36k} - \frac{1}{2}\sum_{j=2}^{\lfloor N/2\rfloor} (-1)^j \frac{4^j \pi^{2j-1} B_{2j}^2}{j(2j)!k^{2j-1}} + O(k^{-N}).$$

For the proof the aforementioned authors used continued fractions. What is more, the authors provided asymptotic formulas for $c_0\big(\frac{k+1}{ak+a-1}\big)$, where $a \in \mathbb{Z}_{>1}$.

Remark 1.7. *Our theorem above accounts for a minor typo in Corollary 1.2 of [13]. We have specifically replaced the term $-\frac{\pi}{144k}$ in [13, relation 1.12] with $-\frac{\pi}{36k}$. Our corrected formula follows in a straightforward manner from relation (1.11) of [13].*

For the general case $c_0(h/k)$, we have the following interesting result.

Theorem 1.8 (Maier-Rassias [23])**.** *Let $h, k_0 \in \mathbb{N}$ be fixed, with $\gcd(k_0, h) = 1$. Let k denote a positive integer with $k \equiv k_0 \bmod h$. Then, there exists a constant $C_1 = C_1(h, k_0)$, with $C_1(1, k_0) = 0$, such that*

$$c_0\Big(\frac{h}{k}\Big) = \frac{k}{\pi h}\ln k - \frac{k}{\pi h}\big(\ln(2\pi) - \gamma\big) + C_1 k + O(1). \tag{3}$$

In [3, Corollary 1] it was proved that the computation of $c_0(h/k)$ is polynomial with respect to $\ln k$ (for a given prescribed accuracy). That is, the computation of $c_0(h/k)$ can be done in polynomial bit complexity. Below we provide two figures (Fig. 1 and 2) that contain points of the form[2] $(h/k, c_0(h/k))$, for k fixed and h less than k with $\gcd(h, k) = 1$.

A study of the previous distribution was given in [5] and [23, Theorem 1.5].

[2]Many images were generated by using the computer algebra system Sagemath [27], and stored in https://goo.gl/Kzv5np

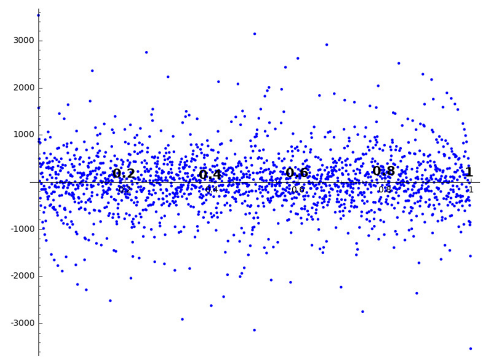

Figure 1: The set of points $\Big(h/k, c_0(h/k)\Big)$ for $1 \leq h < k$, $\gcd(k, h) = 1$ and $k = 1783$ (prime). Remark that the number of points is $\phi(k)$ (the Euler's totient function).

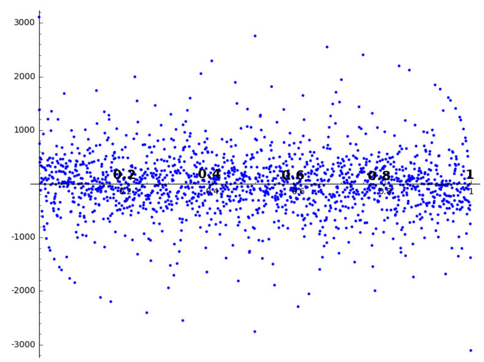

Figure 2: $k = 1597$ (prime).

1.1. Our Contribution

In the present paper we provide a concrete extension of Theorem 1.8. We compute the *hidden* function $C_1(h, k_0)$ and reveal the absolute constant $O(1)$. Furthermore, the proof of our Theorem provides a method to explicitly calculate an arbitrary number of terms in the asymptotic approximation of $c_0(h/k)$ (for fixed h). To the best of our knowledge, these extended formulas do not exist in the literature and cannot be derived via other well-known formulas; this seems to be true even for the case $h = 2$. Our approach relies heavily on the Euler - Maclaurin summation formula.

We now state our Theorem.

Theorem 1.9. *Let $h, k_0 \in \mathbb{N}$ be fixed, with $\gcd(k_0, h) = 1$. Let k denote a positive integer with $k \equiv k_0 \bmod h$. Then, there exists a function $\mathcal{M}_h(k_0)$ such that*

$$c_0(h/k) = \frac{k}{h\pi} \ln k - \frac{k}{h\pi}(\ln(\mu_h \pi) - \gamma) + \frac{1}{\pi h} + k\mathcal{M}_h(k_0) + O\Big(\frac{1}{k}\Big),$$

where for $h > 1$,

$$\mu_h = \frac{2^{1/h} h}{\big((h-1)!\beta(h)\big)^{1/h}} \text{ and } \beta(h) = \frac{\big(1!2!\cdots(h-2)!\big)^2}{2^{h-1}\big((h-1)!\big)^{h-1}}.$$

For $h = 1$ *we get* $\mu_1 = 2$. *Finally,*

$$\mathcal{M}_h(k_0) = \delta(h) - \frac{1}{h} c_0\Big(\frac{k_0}{h}\Big),$$

where

$$\delta(h) = \frac{1}{h^2\pi} \ln\left(\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\right)$$

and $\mathrm{Hyp}(h-1) = \prod_{j=1}^{h-1} j^j$ *is the hyperfactorial function. Also,* $\mathcal{M}_1(k_0) = \mathcal{M}_2(k_0) = 0$.

Remark 1.10. (i). *The function* $C_1(h, k_0)$ *of Theorem 1.8 is* $\mathcal{M}_h(k_0)$ *and the constant term* $O(1)$ *is* $\frac{1}{\pi h} + O(1/k)$.
(ii). *Some values of* μ_h *are,* $\mu_2 = 4, \mu_3 \approx 7.55, \mu_4 \approx 13.85$ *and* $\mu_5 \approx 24.91$.
(iii). *Let* $[k_0]$ *be the class* $k_0 \bmod h$. *Then,* k *goes to* ∞ *and* $k \in [k_0] = k_0 + \mathbb{Z}h$.

Also observe that $\mathcal{M}_h(k) = \mathcal{M}_h(k_0)$. Accordingly, the result of Theorem 1.9 can be re-written as

$$\pi\Big(\frac{h}{k} c_0\Big(\frac{h}{k}\Big) + c_0\Big(\frac{k}{h}\Big) - \frac{1}{k\pi}\Big) = \ln\Big(\frac{k}{\mu_h \pi}\Big) + \gamma + h\pi\delta(h) + O\Big(\frac{1}{k^2}\Big). \tag{4}$$

The left-hand side of equation (4) has been thoroughly studied in [3]; ref. [3] specifically shows that the function (defined in rationals)

$$\mathcal{H} = \pi\Big(\frac{h}{k} c_0\Big(\frac{h}{k}\Big) + c_0\Big(\frac{k}{h}\Big) - \frac{1}{k\pi}\Big)$$

extends to an analytic function on $\mathbb{C}$, with the negative real axis deleted. This is close to what D. Zagier calls a *quantum modular form.*

Since we shall study the sum $c_0(h/k)$ for fixed h and large k, it makes sense to define $F_h(x) = -x \cot \pi h x$, for $x \in (0,1)$ and $h \in \mathbb{N}$. Then

$$c_0(h/k) = \sum_{m=1}^{k-1} F_h(m/k).$$

We provide the following definition.

Definition 1.11.

$$R_h(x) = F_h(x) - \frac{1}{\pi h} \sum_{j=1}^{h} \frac{j}{j - hx},$$

where $F_h(x) = -x\cot(\pi h x)$, *and* $x \in \mathcal{D}_h = (0,1) - \{j/h : j = 1, ..., h-1\}$ $(\mathcal{D}_1 = (0,1))$.

This definition will be discussed and motivated in Subsection 2.1. The idea behind our proof is very simple. We apply the Euler-Maclaurin Summation formula to $R_h(h/k)$ (rather than to $c_0(h/k) = \sum_{m=1}^{k-1} F_h(m/k)$, which is not a smooth function), thus obtaining an asymptotic formula for $R_h(h/k)$. Subsequently, we manage to get a large-k asymptotic approximation for $c_0(h/k)$.

Our methods can further be used for computing an arbitrary number of terms in the asymptotic approximation of $c_0(h/k)$, for large k, see also [12]. Applying Theorem 1.9 for $h = 1$ we get,

$$c_0(1/k) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln 2\pi - \gamma) + \frac{1}{\pi} + O(k^{-1}).$$

If we can compute the numbers,

$$\Delta_{2j-1} = \lim_{x \to 1^-} R_h^{(2j-1)}(x) - \lim_{x \to 0^+} R_h^{(2j-1)}(x).$$

for $2 \le j \le N$, we can approximate $c_0(1/k)$ up to order $2N - 3$ (this can be done by combining Corollary 2.18 and Proposition 3.2; also note that Corollary 2.9 ensures the existence of the numbers Δ_{2j-1}.) For instance, taking $N = 4$ gives

$$c_0(1/k) = \frac{1}{\pi} k \ln k - \frac{k}{\pi}(\ln 2\pi - \gamma) + \frac{1}{\pi} + \frac{\pi}{36k} - \frac{\pi^3}{5400k^3} + \frac{\pi^5}{119070k^5} + O(k^{-7}).$$

Similar for $h = 2$ and $N = 3$ we get,

$$c_0(2/k) = \frac{1}{2\pi} k \ln k - \frac{k}{2\pi}(\ln 4\pi - \gamma) + \frac{1}{2\pi} + \frac{\pi}{18k} - \frac{\pi^3}{675k^3} + O(k^{-5}). \qquad (5)$$

1.2. Some properties of $c_0(x)$ for $x \in \mathbb{Q}$

Let h/k be a rational number with $\gcd(h, k) = 1$ and $1 \le h < k$.

Lemma 1.12. *$c_0(h/k)$ is an algebraic number.*

Proof. Straightforward calculations provide that

$$\cot \pi x = i + i \frac{2}{e^{2i\pi x} - 1}.$$

For $x = m/k \in \mathbb{Q}$, the complex number $e^{2i\pi m/k} - 1$ is algebraic, since it satisfies the equation $(z + 1)^k - 1 = 0$. But the set of algebraic numbers $\overline{\mathbb{Q}}$ is a field and $\cot(\pi x) \in \overline{\mathbb{Q}}$ (for rational integer x). So, $c_0(h/k) \in \overline{\mathbb{Q}}$ as a (finite) linear combination (over $\mathbb{Q}$) of algebraic numbers. $\square$

Lemma 1.13. $c_0(h/k) + c_0((k - h)/k) = 0.$

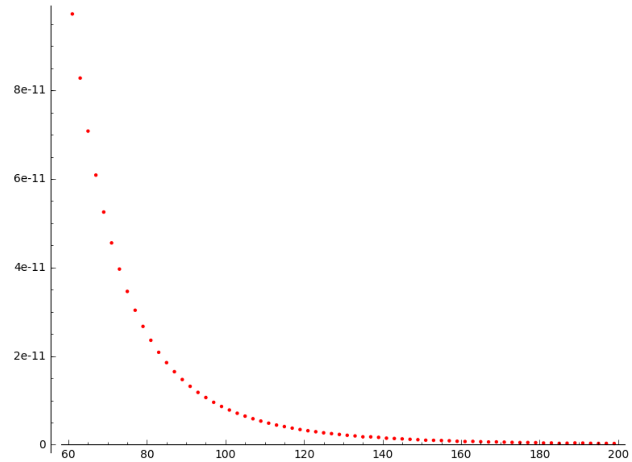

Figure 3: We set $h = 2$ and $g(k) = \frac{1}{2\pi} k \ln k - \frac{k}{2\pi}(\ln 4\pi - \gamma) + \frac{1}{2\pi} + \frac{\pi}{18k} - \frac{\pi^3}{675k^3}$. In this figure we plot the set of points $\Big(k, |g(k) - c_0(2/k)|\Big)$, for $60 \le k \le 200$ with $\gcd(k, 2) = 1$. We note that as k increases the distance between the (real) values of $c_0(2/k)$ and the (approximate) values $g(k)$ from formula (5), tend to zero.

Proof. The Lemma follows from the identity

$$\cot\Big(m\pi(1 - \frac{h}{k})\Big) = -\cot\Big(\frac{m\pi h}{k}\Big),$$

for integers m such that $1 \le m \le k - 1$. Indeed,

$$c_0(h/k) = -\sum_{m=0}^{k-1} \frac{m}{k} \cot\Big(\frac{m\pi h}{k}\Big) = \sum_{m=0}^{k-1} \frac{m}{k} \cot\Big(m\pi \frac{k-h}{k}\Big) = -c_0\big((k-h)/k\big),$$

where the last equality followed from $\gcd(k - h, h) = \gcd(k, h) = 1$. $\square$

We thus proved that the set

$$G = \{(h/k, c_0(h/k)) : \gcd(h, k) = 1,\ h = 1, 2, ..., k - 1\}$$

is symmetric with respect to $(1/2, 0)$.

Roadmap. The next section provides some auxiliary results. In Subsection 2.1, we briefly study the function $R_h(x)$. In Subsection 2.2, we evaluate a definite integral that arises when we apply the Euler-Maclaurin summation formula. Subsection 2.3 gives useful results related to the Euler-Maclaurin summation formula. Subsection 2.4 gives certain useful properties of digamma function and proves an auxiliary lemma. Section 3 proceeds with the proof of Theorem 1.9. Finally, the last section is devoted to some concluding remarks.

2. Auxiliary results

This section gives some definitions and provides proofs of certain elementary lemmas.

Lemma 2.1. *Let the integer $h \geq 2$. Then,*

$$\sum_{j=1}^{h-1} j \ln\left(\frac{j}{h-j}\right) = \ln\left[\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\right],$$

where $\mathrm{Hyp}(h-1) = \prod_{k=1}^{h-1} k^k$ *is the hyperfactorial of* $h-1$ ($\mathrm{Hyp}(0) = 1$).

Proof.

$$\sum_{j=1}^{h-1} j \ln\left(\frac{j}{h-j}\right) = \sum_{j=1}^{h-1} j \ln j + \sum_{j=1}^{h-1} (h-j) \ln(h-j) - \sum_{j=1}^{h-1} h \ln(h-j) =$$

$$2\sum_{j=1}^{h-1} j \ln j - \sum_{j=1}^{h-1} h \ln(h-j) = 2\ln\left(\prod_{j=1}^{h-1} j^j\right) - h \ln \prod_{j=1}^{h-1} j =$$

$$2\ln(\mathrm{Hyp}(h-1)) - h\ln((h-1)!).$$

The Lemma follows. □

Lemma 2.2. *Let $h \in \mathbb{N}$, then* $\lim_{x\to 1}\left(h\pi x \cot(h\pi x) + \frac{1}{1-x}\right) = 1.$

Proof. Let

$$\ell = \lim_{x\to 1}\left(h\pi x \cot(h\pi x) + \frac{1}{1-x}\right).$$

By setting $z = h\pi(1-x)$ we get

$$\ell = \lim_{z\to 0}\left((h\pi - z)\cot(h\pi - z) + \frac{h\pi}{z}\right) = \lim_{z\to 0}\left((z - h\pi)\cot(z) + \frac{h\pi}{z}\right) =$$

$$1 - h\pi \lim_{z\to 0}\left(\cot(z) - \frac{1}{z}\right) = 1.$$

□

Definition 2.3 (*nth* Harmonic number)**.** *For every integer $n \geq 1$, we set*

$$H_n = \sum_{j=1}^{n} \frac{1}{j}, \ H_0 = 0.$$

Lemma 2.4. *For $N \geq 2$ and large n we have*

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{j=1}^{N} \frac{B_{2j}}{2jn^{2j}} + O(n^{-2N-1}),$$

where B_{2j} are the Bernoulli numbers,

$$B_{2n} = 2\frac{(2n)!}{(2\pi)^{2n}}\sum_{j=1}^{\infty}\frac{(-1)^{n+1}}{j^{2n}} \quad (n > 0).$$

Proof. This is a consequence of Euler-Maclaurin summation formula; see, for example, [14, Section 9.6], [29, Chapter I, Section 6], or [15, p. 89]. □

Lemma 2.5. *Let h be a positive integer, then*

$$\sum_{j=1}^{h-1}\frac{j}{j-h} = h(1-H_h).$$

Proof.

$$\sum_{j=1}^{h-1}\frac{j}{j-h} = -\sum_{i=1}^{h-1}\frac{h-i}{i} = -\sum_{i=1}^{h-1}\frac{h}{i} + \sum_{i=1}^{h-1}1 = -hH_{h-1} + h - 1 =$$

$$(-hH_{h-1} - 1) + h = -hH_h + h = h(1-H_h).$$

□

Lemma 2.6. $\lim_{x\to 0^+} R_h(x) = -\frac{h+1}{h\pi}$ *and* $\lim_{x\to 1^-} R_h(x) = \frac{H_{h-1}-1}{\pi}$

Proof. The first limit is an immediate consequence of the definition of $R_h(x)$ (see Definition 1.11). To show the second, observe that

$$\lim_{x\to 1^-} R_h(x) = -\frac{1}{h\pi}\lim_{x\to 1^-}\left(h\pi x\cot(h\pi x) + \frac{1}{1-x}\right) - \frac{1}{\pi}\lim_{x\to 1^-}\sum_{j=1}^{h-1}\frac{j}{h(j-hx)} =$$

$$-\frac{1}{h\pi}\lim_{x\to 1^-}\left(h\pi x\cot(h\pi x) + \frac{1}{1-x}\right) - \frac{1}{\pi}\sum_{j=1}^{h-1}\frac{j}{h(j-h)},$$

so that Lemma 2.2 gives

$$\lim_{x\to 1^-} R_h(x) = -\frac{1}{h\pi} - \frac{1}{\pi}\sum_{j=1}^{h-1}\frac{j}{h(j-h)}.$$

By Lemma 2.5, $\sum_{j=1}^{h-1}\frac{j}{h(j-h)} = 1 - H_h$. Therefore

$$\lim_{x\to 1^-} R_h(x) = \frac{hH_h - h - 1}{h\pi}.$$

Simple calculations yield

$$\frac{hH_h - h - 1}{h\pi} = \frac{H_{h-1}-1}{\pi}.$$

The result follows. □

Lemma 2.7. *Let $0 < \alpha < \beta$. For large x we have,*

$$\frac{\beta}{x(\beta-\alpha)+\beta} = \frac{\beta}{\beta-\alpha}\frac{1}{x} + O\Big(\frac{1}{x^2}\Big)$$

and

$$\ln\Big(\frac{\alpha x}{\beta x + \beta - \alpha x}\Big) = \ln\Big(\frac{\alpha}{\beta-\alpha}\Big) - \frac{\beta}{\beta-\alpha}\frac{1}{x} + O\Big(\frac{1}{x^2}\Big).$$

Proof. The given functions are analytic at $x = \infty$. Thus, the required expansions result by setting $y = 1/x$ and Taylor-expanding about $y = 0$. □

2.1. The function $R_h(x)$

Lemma 2.8. *$R_h(x)$ only has removable singularities in $[0, 1]$.*

Proof. For $z \in \mathbb{C}$ we show, more generally, that the (complex) function $R_h(z)$ only has removable singularities when $|z| \leq 1$. Initially we compute the residues of $F_h(z) = -z \cot h\pi z$ at the points $z = j/h$, $j = 1, 2, ..., h$. From the expansion of $\cot z$ we see that $F_h(z)$ has simple poles at those points. Accordingly,

$$Res(F_h(z), j/h) = \lim_{z \to j/h} F_h(z)(z - j/h) = -\frac{j}{\pi h^2},$$

for $j = 1, 2, ..., h$. Thus, the complex function,

$$R_h(z) = F_h(z) - \sum_{j=1}^{h}\Big(\frac{-j}{\pi h^2}\Big)\frac{1}{z - j/h} = F_h(z) - \frac{1}{\pi h}\sum_{j=1}^{h}\frac{j}{j - hz},$$

has removable singularities at $z = j/h$. The Lemma follows. □

Corollary 2.9. *There is an analytic continuation of $R_h(z)$, $z \in \mathbb{C}$ with $|z| \leq 1$.*

It follows that the restriction of $R_h(z)$ to the interval $[0, 1]$, say $g_h(x)$, has derivatives of all orders in $[0, 1]$.

2.2. A difficult integral

The goal of this subsection is to evaluate the integral $\int_0^1 g_h(x)\, dx$, see Proposition 2.13. To accomplish this, we need three auxiliary lemmas.

Lemma 2.10.

$$I_1 = \int_0^{\pi/2} y \cot y \; dy = \frac{\pi}{2}\ln 2, \quad I_2 = \int_0^{\pi/2}\Big(\cot y - \frac{1}{y}\Big)\; dy = \ln\Big(\frac{2}{\pi}\Big).$$

Proof. $I_1 = \int_0^{\pi/2} y(\ln\sin y)'\, dy = [y \ln\sin y]_0^{\pi/2} - \int_0^{\pi/2}\ln\sin y\, dy = -\int_0^{\pi/2}\ln\sin y\, dy$. We set $J_1 = \int_0^{\pi/2}\ln\sin y\, dy$, so that $I_1 = -J_1$. To compute the integral J_1, first observe that $J_1 = \int_0^{\pi/2}\ln\cos y\, dy$, so that

$$2J_1 = \int_0^{\pi/2}(\ln\sin y + \ln\cos y)\; dy = \int_0^{\pi/2}\ln\Big(\frac{1}{2}\sin(2y)\Big)\; dy =$$

$$-\frac{\pi \ln 2}{2} + \int_0^{\pi/2} \ln \sin(2y)\ dy = -\frac{\pi \ln 2}{2} + \frac{1}{2}\int_0^{\pi} \ln \sin(y)\ dy.$$

Since we also have

$$\int_0^{\pi/2} \ln \sin y\ dy = \int_{\pi/2}^{\pi} \ln \sin y\ dy,$$

it follows that

$$2J_1 = -\frac{\pi \ln 2}{2} + \int_0^{\pi/2} \ln \sin(y)\ dy = -\frac{\pi \ln 2}{2} + J_1.$$

Therefore $J_1 = -I_1 = -\frac{\pi \ln 2}{2}$.

For the integral I_2 we get

$$I_2 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi/2} \Big(\cot(y) - \frac{1}{y}\Big)\ dy =$$

$$\lim_{\varepsilon \to 0} [\ln \sin y - \ln y]_{\varepsilon}^{\pi/2} = \lim_{\varepsilon \to 0} \Big(- \ln \pi/2 - \ln \frac{\sin \varepsilon}{\varepsilon}\Big) = \ln \Big(\frac{2}{\pi}\Big).$$

□

Lemma 2.11. *Let h be a positive integer and let f be a continuous function in $[0, h\pi]$. Then*

$$\int_0^{h\pi} f(y)\ dy = \sum_{m=0}^{h-1} \int_0^{\pi/2} \big(f(m\pi + y) + f((m+1)\pi - y)\big)\ dy.$$

Proof. First we observe that

$$\int_0^{\pi} f(x)\ dx = \int_0^{\pi/2} \big(f(x) + f(\pi - x)\big)dx. \tag{6}$$

Consequently,

$$\int_0^{h\pi} f(y)\ dy = \int_0^{\pi} f(y)\ dy + \sum_{m=1}^{h-1} \int_{m\pi}^{(m+1)\pi} f(y)\ dy =$$

$$\int_0^{\pi} \Big(f(y) + \sum_{m=1}^{h-1} f((m+1)\pi - y)\Big)dy.$$

We set $F(y) = f(y) + \sum_{m=1}^{h-1} f\big((m+1)\pi - y\big)$. By equality (6) we get

$$\int_0^{h\pi} f(y)\ dy = \int_0^{\pi/2} \big(F(y) + F(\pi - y)\big)dy =$$

$$\int_0^{\pi/2} \Big(f(y) + \sum_{m=1}^{h-1} f((m+1)\pi - y) + f(\pi - y) + \sum_{m=1}^{h-1} f(m\pi + y)\Big).$$

The Lemma follows. □

Lemma 2.12. *Let h be a positive integer and let m be an integer with $1 \leq m \leq h$. We set*

$$A_m(h) = \int_0^1 \Big[\frac{1}{x} + \sum_{j=1}^{h} j\Big(\frac{1}{2(m-j-1)+x} - \frac{1}{2(j-m)+x}\Big)\Big]dx.$$

Then,

$$A_m(h) = \ln \frac{(h-m)!(m-1)!}{2(h-m+1)^h}.$$

Proof.

$$A_m(h) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \Big[\frac{1}{x} + \sum_{j=1}^{h} j\Big(\frac{1}{2(m-j-1)+x} - \frac{1}{2(j-m)+x}\Big)\Big]dx =$$

$$\lim_{\varepsilon \to 0^+} \left[\ln\frac{1}{\varepsilon} + \sum_{j=1}^{h} j\left(\ln\left|\frac{2(m-j-1)+1}{2(m-j-1)+\varepsilon}\right| - \ln\left|\frac{2(j-m)+1}{2(j-m)+\varepsilon}\right|\right)\right] =$$

$$\lim_{\varepsilon \to 0^+} \left[\ln\frac{1}{\varepsilon} + \sum_{j=1}^{h} j\ln\left|\frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon}\right|\right].$$

We now consider the two cases $m = 1$ and $m > 1$.
(**i**). When $m = 1$ we have

$$A_1(h) = \lim_{\varepsilon \to 0^+} \left[\ln\frac{1}{\varepsilon} + \sum_{j=1}^{h} j\ln\left|\frac{2(j-1)+\varepsilon}{2j-\varepsilon}\right|\right] =$$

$$\lim_{\varepsilon \to 0^+} \left[\ln\frac{1}{\varepsilon} + \ln\left|\frac{\varepsilon}{2-\varepsilon}\right| + \sum_{j=2}^{h} j\ln\frac{j-1}{j}\right] =$$

$$\lim_{\varepsilon \to 0^+} \left[\ln\left|\frac{1}{2-\varepsilon}\right| + \sum_{j=2}^{h} j\ln\frac{j-1}{j}\right] = \ln\frac{(h-1)!}{2h^h}.$$

(**ii**). When $m > 1$ we have

$$\sum_{j=1}^{h} j\ln\left|\frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon}\right| =$$

$$\sum_{j=1}^{m-2} j\ln\left|\frac{2(j-m)\varepsilon}{2(m-j-1)+\varepsilon}\right| + (m-1)\ln\frac{2-\varepsilon}{\varepsilon} + m\ln\frac{\varepsilon}{2-\varepsilon} + \sum_{j=m+1}^{h} j\ln\left|\frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon}\right| =$$

$$\ln\frac{\varepsilon}{2-\varepsilon} + \sum_{j=1}^{m-2} j\ln\left|\frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon}\right| + \sum_{j=m+1}^{h} j\ln\left|\frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon}\right|.$$

Accordingly,

$$A_m(h) = \lim_{\varepsilon\to 0^+} \left[\ln \frac{1}{2-\varepsilon} + \sum_{j=1}^{m-2} j \ln \left| \frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon} \right| + \sum_{j=m+1}^{h} j \ln \left| \frac{2(j-m)+\varepsilon}{2(m-j-1)+\varepsilon} \right| = \right.$$

$$-\ln 2 + \sum_{j=1}^{m-2} j \ln \left| \frac{j-m}{m-j-1} \right| + \sum_{j=m+1}^{h} j \ln \left| \frac{j-m}{m-j-1} \right| =$$

$$-\ln 2 + \sum_{j=1}^{m-2} j \ln \frac{m-j}{m-j-1} + \sum_{j=m+1}^{h} j \ln \frac{j-m}{j-m+1}.$$

Straightforward calculations provide,

$$\sum_{j=1}^{m-2} j \ln \frac{m-j}{m-j-1} = \ln\big((m-1)!\big),$$

$$\sum_{j=m+1}^{h} j \ln \frac{j-m}{j-m+1} = \ln\Big(\frac{(h-1)!}{(h-m+1)^h} \Big).$$

The Lemma follows. $\square$

Proposition 2.13. *For $h > 1$,*

$$J_h = \int_0^1 g_h(x)dx = -\frac{\ln \pi}{\pi h} + \frac{1}{\pi h^2} \ln\left(\frac{(h-1)!}{2h^h} \right) + \frac{1}{\pi h^2}\alpha(h),$$

where $\alpha(h) = \ln\left(\frac{\big(1!2!\cdots(h-2)!\big)^2}{\big(2(h-1)!\big)^{h-1}} \right)$. *If $h=1$ we get* $J_1 = -\dfrac{\ln(2\pi)}{\pi}$.

Proof.

$$J_h = \int_0^1 g_h(x) = \int_0^1 R_h(x)dx = \int_0^1 \Big(-x\cot(h\pi x) + \frac{1}{\pi h} \sum_{j=1}^{h} \frac{j}{hx-j} \Big) dx.$$

Set $x = 1 - \frac{y}{h\pi}$ to obtain

$$\pi^2 h^2 J_h = \int_0^{h\pi} \Big[(h\pi - y)\cot y + \pi \sum_{j=1}^{h} \frac{j}{\pi(h-j)-y} \Big] dy.$$

Now set $F(y) = (h\pi - y)\cot y + \pi \sum_{j=1}^{h} \frac{j}{\pi(h-j)-y}$. From Lemma 2.11 we get

$$\pi^2 h^2 J_h = \sum_{m=0}^{h-1} \int_0^{\pi/2} \Big(F(m\pi + y) + F\big((m+1)\pi - y\big) \Big)\, dy =$$

$$\sum_{m=0}^{h-1} \int_0^{\pi/2} \left[(\pi - 2y) \cot y + \pi \sum_{j=1}^{h} \left(\frac{j}{\pi(h-j-m) - y} + \frac{j}{\pi(h-j-m-1) + y} \right) \right] dy.$$

Add and subtract $1/y$ to get

$$\pi^2 h^2 J_h = \sum_{m=0}^{h-1} \Bigg[-2 \int_0^{\pi/2} y \cot y \; dy + \pi \int_0^{\pi/2} \left(\cot y - \frac{1}{y} \right) dy +$$

$$\pi \int_0^{\pi/2} \left(\frac{1}{y} + \sum_{j=1}^{h} \left(\frac{j}{\pi(h-j-m) - y} + \frac{j}{\pi(h-j-m-1) + y} \right) \right) dy \Bigg].$$

Lemma 2.10 gives

$$\int_0^{\pi/2} y \cot y \; dy = \frac{\pi}{2} \ln 2$$

and

$$\int_0^{\pi/2} \left(\cot y - \frac{1}{y} \right) dy = \ln \frac{2}{\pi}.$$

Set

$$I(m) = \int_0^{\pi/2} \left(\frac{1}{y} + \sum_{j=1}^{h} \left(\frac{j}{\pi(h-j-m) - y} + \frac{j}{\pi(h-j-m-1) + y} \right) \right) dy,$$

so that

$$J_h = -\frac{\ln \pi}{\pi h} + \frac{1}{\pi h^2} \sum_{m=0}^{h-1} I(m).$$

For $h = 1$ we get $J_1 = -\frac{\ln \pi}{\pi} + \frac{1}{\pi} I(0)$. Since $I(0) = -\ln 2$, we get

$$J_1 = -\frac{\ln 2\pi}{\pi}.$$

Assume that $h > 1$ and set

$$I'(h) = \sum_{m=0}^{h-1} I(m).$$

Furthermore, we set $y = \frac{\pi}{2} z$ to obtain

$$I'(h) = \sum_{m=0}^{h-1} \int_0^1 \left(\frac{1}{z} + \sum_{j=1}^{h} \left(\frac{j}{2(h-j-m) - z} + \frac{j}{2(h-j-m-1) + z} \right) \right) dz.$$

The substitution $h - m = m'$ leads to

$$I'(h) = \sum_{m=1}^{h-1} \int_0^1 \left(\frac{1}{z} + \sum_{j=1}^{h} j \left(\frac{1}{2(m-j-1) + z} - \frac{1}{2(j-m) + z} \right) \right) dz,$$

where we re-set $m' \leftarrow m$. In the notation of Lemma 2.12, the above equality is

$$I'(h) = \sum_{m=1}^{h-1} A_m(h) = A_1(h) + \sum_{m=2}^{h-1} A_m(h).$$

Replacing $A_1(h)$ by $\ln \frac{(h-1)!}{2h^h}$(see Lemma 2.12) gives

$$\sum_{m=2}^{h} A_m(h) = \ln\left(\frac{(1!2!\cdots(h-2)!)^2}{2^{h-1}\big((h-1)!\big)^{h-1}}\right),$$

so that

$$I'(h) = \ln\left(\frac{(h-1)!}{2h^h}\right) + \ln\left(\frac{(1!2!\cdots(h-2)!)^2}{2^{h-1}\big((h-1)!\big)^{h-1}}\right).$$

The Proposition follows. □

The corollary that follows is a simple consequence of Proposition 2.13.

Corollary 2.14. *For $h > 1$,*

$$J_h = -\frac{\ln(\mu_h \pi)}{h\pi}, \text{ with } \mu_h = \frac{2^{1/h} h}{\big((h-1)!\beta(h)\big)^{1/h}},$$

where

$$\beta(h) = \frac{(1!2!\cdots(h-2)!)^2}{2^{h-1}\big((h-1)!\big)^{h-1}}.$$

For $h = 1$, we get $\mu_1 = 2$.

2.3. Euler-Maclaurin summation formula

The Euler-Maclaurin summation formula is a powerful method for estimating sums by using integrals. It can be regarded as an extension of trapezoidal rule for numerical quadrature. It was first published by Euler [11] in 1736 and independently by Colin Maclaurin [22] in 1742. Let us first recall some relevant definitions.

Definition 2.15. *For $x \in \mathbb{R}$, we set*

$$P_{2N+1}(x) = (-1)^{N-1} \sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{(2\pi n)^{2N+1}}, \; N = 0, 1, 2, \ldots$$

From the previous definition we get the following simple lemma.

Lemma 2.16. *If N is a non-negative integer and k a fixed integer, then*

$$|P_{2N+1}(kx)| \le \sum_{n=1}^{\infty} \frac{2}{(2\pi n)^{2N+1}} = 2^{-2N}\pi^{-2N-1}\zeta(2N+1).$$

The following theorem is the Euler-Maclaurin Summation formula as given in [9, relation (2.9.15), p.136]. With $C^n[a,b]$ we denote the set of $n-$times differentiable functions in the interval $[a,b]$.

Theorem 2.17. *(Euler-Maclaurin summation formula). Let $N \geq 1$ and $f \in C^{2N+1}[0,Z]$, where $Z \in \mathbb{N}$. With $h = Z/k$ we have $h\sum_{m=0}^{k} f(hm) =$*

$$h\frac{f(0)+f(Z)}{2} + \int_0^Z f(x)\ dx + \sum_{j=1}^{N} h^{2j}\frac{B_{2j}}{(2j)!}\big(f^{(2j-1)}(Z) - f^{(2j-1)}(0)\big) + r_N(f,Z),$$

where

$$r_N(f,Z) = h^{2N+1}\int_0^Z P_{2N+1}(kx/Z)f^{(2N+1)}(x)dx.$$

Proof. We apply [9, Corollary, p.136] for $a \leftarrow 0, b \leftarrow Z, k \leftarrow N$ and $n \leftarrow k$. $\square$

When $Z = 1$ we have $h = 1/k$, resulting in the following special case.

Corollary 2.18. *Let $k, N \geq 1$. If $f \in C^{2N+1}[0,1]$, then $\sum_{m=1}^{k-1} f(m/k) =$*

$$-\frac{f(0)+f(1)}{2} + k\int_0^1 f(x)dx + \sum_{j=1}^{N}\frac{B_{2j}}{k^{2j-1}(2j)!}\big(f^{(2j-1)}(1) - f^{(2j-1)}(0)\big) + kr_N(f),$$

where for large k we have,

$$|kr_N(f)| = O\Big(\frac{1}{k^{2N}}\Big).$$

Proof. From Theorem 2.17 we get

$$|r_N(f)| = \Big|\frac{1}{k^{2N+1}}\int_0^1 P_{2N+1}(kx)f^{(2N+1)}(x)\ dx\Big| \leq$$

$$\frac{1}{k^{2N+1}}\int_0^1 |P_{2N+1}(kx)||f^{(2N+1)}(x)|\ dx = \frac{1}{k^{2N+1}}|P_{2N+1}(k\xi)|\int_0^1 |f^{(2N+1)}(x)|\ dx,$$

for some $\xi \in [0,1]$. From Lemma 2.16, we get

$$|r_N(f)| \leq 2^{-2N}\pi^{-2N-1}\zeta(2N+1)\frac{1}{k^{2N+1}}\int_0^1 |f^{(2N+1)}(x)|\ dx.$$

So for large k we get

$$|kr_N(f)| = O\Big(\frac{1}{k^{2N}}\Big).$$

$\square$

2.4. Digamma function

In this subsection we present some basic results about the digamma function, which are necessary for the proof of our Theorem. We begin with the following definition.

Definition 2.19. *The digamma (or psi) function $\psi(z)$ is defined by the formula*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the gamma function.

Some useful properties (see [15, section 6.3, p. 58]) are

$$\psi(1) = -\gamma, \tag{7}$$

$$\psi(k) = H_{k-1} - \gamma \quad (k \in \mathbb{N}), \tag{8}$$

which, when combined with Lemma 2.4, gives the large-k formula

$$\psi(k+1) = \ln k + \frac{1}{2k} + O\Big(\frac{1}{k^2}\Big).$$

From [24, section 5.11] we have the following recurrence formula,

$$\psi(z+1) = \psi(z) + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}). \tag{9}$$

From (9) we get an asymptotic approximation of $\psi(k)$ $(k \in \mathbb{N})$,

$$\psi(k) = \psi(k+1) - \frac{1}{k} = \ln k - \frac{1}{2k} + O\Big(\frac{1}{k^2}\Big). \tag{10}$$

The reflection formula for the psi function is

$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z) \quad (z \in \mathbb{C} - \mathbb{Z}), \tag{11}$$

and is the logarithmic derivative of

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

which is Euler's reflection formula for the gamma function (see [15, section 6.3, p. 59] or [24, section 5.5]). Equality (9) and induction show that

$$\sum_{m=1}^{n} \frac{1}{z-m} = \psi(z) - \psi(z-n). \tag{12}$$

In the proof of our Theorem we shall need to study the following sequence, for large values of k,

$$C_h(k) = \frac{1}{h^2\pi} \sum_{j=1}^{h-1} j\Big(\psi(\frac{jk}{h}) - \psi(\frac{jk}{h} - k)\Big).$$

Here, h, k are positive integers with $\gcd(h,k)=1$ and $1<h<k$. We set

$$\mathcal{A}_j(h,k)=\psi\Big(\frac{jk}{h}\Big)-\psi\Big(k+1-\frac{jk}{h}\Big).$$

We shall prove the following Lemma.

Lemma 2.20. *For fixed j, h with $1 \le j < h$, and $k \equiv k_0 \bmod h$ we have* (i).

$$C_h(k)=\frac{1}{h^2\pi}\sum_{j=1}^{h-1} j\Big(\mathcal{A}_j(h,k)+\pi\cot\Big(\frac{\pi jk_0}{h}\Big)\Big),$$

and for large k,

$$\mathcal{A}_j(h,k)=\ln\Big(\frac{kj}{kh+h-kj}\Big)-\frac{h}{2kj}+\frac{h}{2(kh+h-kj)}+O\Big(\frac{1}{k^2}\Big).$$

(ii). *For large k we have*

$$kC_h(k)=k\mathcal{M}_h(k_0)+\frac{1-hH_h}{2\pi h}+O(\frac{1}{k}),$$

where

$$\mathcal{M}_h(k_0)=\delta(h)-\frac{1}{h}c_0(\frac{k_0}{h}),$$

and

$$\delta(h)=\frac{1}{h^2\pi}\ln\Bigg(\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\Bigg).$$

Proof. (i). Set $z=\frac{jk}{h}-k$ in the reflection formula (11) to get

$$\psi\Big(\frac{jk}{h}\Big)-\psi\Big(\frac{jk}{h}-k\Big)=$$

$$\psi\Big(\frac{jk}{h}\Big)-\Big[\psi\Big(k+1-\frac{jk}{h}\Big)-\pi\cot\Big(\pi(\frac{jk}{h}-k)\Big)\Big]=$$

$$\psi\Big(\frac{jk}{h}\Big)-\psi\Big(k+1-\frac{jk}{h}\Big)+\pi\cot\Big(\frac{\pi jk}{h}\Big)=\mathcal{A}_j(h,k)+\pi\cot\Big(\frac{\pi jk}{h}\Big).$$

Since $k\equiv k_0 \bmod h$, there is an integer, say M, such that $k=k_0+Mh$. Accordingly, $\cot\Big(\frac{\pi jk}{h}\Big)=\cot\Big(\frac{\pi jk_0}{h}\Big)$. We thus obtain

$$C_h(k)=\frac{1}{h^2\pi}\sum_{j=1}^{h-1} j\Big(\psi(\frac{jk}{h})-\psi(\frac{jk}{h}-k)\Big)=$$

$$\frac{1}{h^2\pi}\sum_{j=1}^{h-1} j\Big(\mathcal{A}_j(h,k)+\pi\cot\Big(\frac{\pi jk_0}{h}\Big)\Big).$$

We now set $x = \frac{j}{h}$, so that $0 < x < 1$. Thus

$$\mathcal{A}_j(h,k) = \psi(kx) - \psi(k(1-x)+1).$$

From (10) we have

$$\psi(z) - \ln z = -\frac{1}{2z} + O\Big(\frac{1}{z^2}\Big).$$

We set $z = kx$. Then for large k we get

$$\psi(kx) - \ln(kx) = -\frac{1}{2kx} + O\Big(\frac{1}{k^2}\Big).$$

Similar if $z = k(1-x)+1$, for large k we get

$$\psi\big(k(1-x)+1\big) - \ln\big(k(1-x)+1\big) = -\frac{1}{2(k(1-x)+1)} + O\Big(\frac{1}{k^2}\Big).$$

Therefore

$$\mathcal{A}_j(h,k) = \ln\Big(\frac{kx}{k(1-x)+1}\Big) - \frac{1}{2kx} + \frac{1}{2(k(1-x)+1)} + O\Big(\frac{1}{k^2}\Big) =$$

$$\ln\Big(\frac{kj}{kh+h-kj}\Big) - \frac{h}{2kj} + \frac{h}{2(kh+h-kj)} + O\Big(\frac{1}{k^2}\Big).$$

(ii). Set $\alpha = j, \beta = h$, and $x = k$ in Lemma 2.7 to obtain

$$\ln\Big(\frac{kj}{kh+h-kj}\Big) = \ln\big(\frac{j}{h-j}\big) - \frac{h}{h-j}\frac{1}{k} + O\Big(\frac{1}{k^2}\Big)$$

and

$$\frac{1}{2}\frac{h}{k(h-j)+h} = \frac{h}{2(h-j)}\frac{1}{k} + O\Big(\frac{1}{k^2}\Big).$$

Therefore, for large k,

$$kj\mathcal{A}_j(h,k) = kj\ln\big(\frac{j}{h-j}\big) - \frac{hj}{h-j} - \frac{h}{2} + \frac{hj}{2(h-j)} + jO\Big(\frac{1}{k}\Big) =$$

$$kj\ln\big(\frac{j}{h-j}\big) + \frac{hj}{2(j-h)} - \frac{h}{2} + jO\Big(\frac{1}{k}\Big).$$

Sum over j to obtain

$$\sum_{j=1}^{h-1} kj\mathcal{A}_j(h,k) = k\sum_{j=1}^{h-1} j\ln\big(\frac{j}{h-j}\big) + \frac{h}{2}\sum_{j=1}^{h-1}\frac{j}{j-h} - \frac{h}{2}\sum_{j=1}^{h-1} 1 + O\Big(\frac{1}{k}\Big).$$

From Lemma 2.5 we get

$$\sum_{j=1}^{h-1}\frac{j}{j-h} = h(1-H_h).$$

Therefore,

$$\sum_{j=1}^{h-1} kj\mathcal{A}_j(h,k) = k\sum_{j=1}^{h-1} j\ln\big(\frac{j}{h-j}\big) + \frac{h^2}{2}(1-H_h) - \frac{h(h-1)}{2} + O\Big(\frac{1}{k}\Big) =$$

$$k\sum_{j=1}^{h-1} j\ln\big(\frac{j}{h-j}\big) + \frac{h}{2}(1-hH_h) + O\Big(\frac{1}{k}\Big).$$

We thus conclude that

$$kC_h(k) = \frac{k}{h^2\pi}\sum_{j=1}^{h-1} j\bigg(\mathcal{A}_j(h,k) + \pi\cot\Big(\frac{\pi j k_0}{h}\Big)\bigg) =$$

$$k\mathcal{M}_h(k_0) + \frac{1-hH_h}{2\pi h} + O\Big(\frac{1}{k}\Big),$$

where

$$\mathcal{M}_h(k_0) = \frac{1}{h^2\pi}\sum_{j=1}^{h-1} j\Big(\ln\Big(\frac{j}{h-j}\Big) + \pi\cot\Big(\frac{\pi j k_0}{h}\Big)\Big).$$

Furthermore, Lemma 2.1 gives

$$\mathcal{M}_h(k_0) = \frac{1}{h^2\pi}\bigg[\ln\bigg(\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\bigg) + \sum_{j=1}^{h-1} j\pi\cot\Big(\frac{\pi j k_0}{h}\Big)\bigg],\ h = 1, 2, ...$$

where $\mathrm{Hyp}(h-1) = \prod_{k=1}^{h-1} k^k$, is the hyperfactorial of $h-1$ ($\mathrm{Hyp}(0) = 1$). That is,

$$\mathcal{M}_h(k_0) = \delta(h) + \epsilon(h, k_0),$$

with

$$\delta(h) = \frac{1}{h^2\pi}\ln\bigg(\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\bigg),\ \text{and}\ \epsilon(h,k_0) = \frac{1}{h^2}\sum_{j=1}^{h-1} j\cot\Big(\frac{\pi j k_0}{h}\Big).$$

Accordingly,

$$\epsilon(h,k_0) = \frac{1}{h}\sum_{j=1}^{h-1}\frac{j}{h}\cot\Big(\frac{\pi j k_0}{h}\Big) = -\frac{1}{h}c_0\big(\frac{k_0}{h}\big).$$

We conclude that,

$$\mathcal{M}_h(k_0) = \delta(h) - \frac{1}{h}c_0\big(\frac{k_0}{h}\big).$$

The Lemma follows. □

3. Proof of Theorem 1.9

Recall from Definition 1.11 that

$$R_h(x) = F_h(x) - \frac{1}{\pi h}\sum_{j=1}^{h}\frac{j}{j-hx},$$

where $F_h(x) = -x\cot(\pi hx)$, where $x \in \mathcal{D}_h = (0,1) - \{j/h : j = 1, ..., h-1\}$ $(\mathcal{D}_1 = (0,1))$.

Definition 3.1.

$$c(h/k) = \sum_{m=1}^{k-1} R_h(m/k). \tag{13}$$

In accordance with Definition 1.11 we have

$$c_0(h/k) = \sum_{m=1}^{k-1} F_h\Big(\frac{m}{k}\Big) = -\sum_{m=1}^{k-1}\frac{m}{k}\cot\Big(\frac{m\pi h}{k}\Big),$$

so that

$$c(h/k) = c_0(h/k) - \frac{k}{h\pi}\sum_{m=1}^{k-1}\sum_{j=1}^{h}\frac{j}{kj-hm}. \tag{14}$$

The double sum

$$D(h,k) = \sum_{m=1}^{k-1}\sum_{j=1}^{h}\frac{j}{kj-hm},$$

can be simplified by means of digamma function $\psi(z)$. We shall first compute the sum $\sum_{m=1}^{k-1}\frac{j}{jk-hm}$.

CASE 1. $1 \le j \le h-1$. We have

$$\sum_{m=1}^{k-1}\frac{j}{jk-hm} = \sum_{m=1}^{k}\frac{j}{jk-hm} - \frac{j}{k(j-h)} = \frac{j}{h}\sum_{m=1}^{k}\frac{1}{\frac{jk}{h}-m} - \frac{j}{k(j-h)},$$

which, with the aid of equality (12), gives

$$\sum_{m=1}^{k-1}\frac{j}{jk-hm} = \frac{j}{h}\Big(\psi(\frac{jk}{h}) - \psi(\frac{jk}{h}-k)\Big) - \frac{j}{k(j-h)}.$$

CASE 2. $j = h$. For $j = h$, the relation $\psi(1) = -\gamma$ (see (7)) gives

$$\sum_{m=1}^{k-1}\frac{h}{hk-hm} = \psi(k) - \psi(1) = \psi(k) + \gamma.$$

Therefore the double sum for $h > 1$ can be written as

$$D(h,k) = \sum_{m=1}^{k-1} \sum_{j=1}^{h} \frac{j}{kj - hm} =$$

$$\sum_{j=1}^{h} \sum_{m=1}^{k-1} \frac{j}{kj - hm} = \sum_{j=1}^{h-1} \sum_{m=1}^{k-1} \frac{j}{kj - hm} + \psi(k) + \gamma =$$

$$\frac{1}{h} \sum_{j=1}^{h-1} j \Big(\psi\big(\frac{jk}{h}\big) - \psi\big(\frac{jk}{h} - k\big) \Big) - \sum_{j=1}^{h-1} \frac{j}{k(j-h)} + \psi(k) + \gamma. \tag{15}$$

For $h = 1$ (so $j = 1$ in $D(h,k)$) we get,

$$D(1,k) = \sum_{m=1}^{k-1} \frac{1}{k-m} = H_{k-1}.$$

The case $h = 1$ agrees with relation (15). Indeed, for $h = 1$ relation (15) is equal to $\psi(k) + \gamma$, which from (8) is equal to H_{k-1}. Thus, for $h \geq 1$ equality (14) is written as

$$c(h/k) = c_0(h/k) - \frac{k}{h\pi} D(h,k) =$$

$$c_0(h/k) - \frac{k}{h^2\pi} \sum_{j=1}^{h-1} j \Big(\psi\big(\frac{jk}{h}\big) - \psi\big(\frac{jk}{h} - k\big) \Big) + \frac{1}{h\pi} \sum_{j=1}^{h-1} \frac{j}{j-h} - \frac{k}{h\pi} \Big(\psi(k) + \gamma \Big).$$

Finally, from Lemma 2.5 we get $\sum_{j=1}^{h-1} \frac{j}{j-h} = h(1 - H_h)$, so that

$$c(h/k) = c_0(h/k) - \frac{k}{h^2\pi} \sum_{j=1}^{h-1} j \Big(\psi\big(\frac{jk}{h}\big) - \psi\big(\frac{jk}{h} - k\big) \Big) + \frac{1}{\pi}\big(1 - H_h\big) - \frac{k}{h\pi} \Big(\psi(k) + \gamma \Big). \tag{16}$$

The proposition that follows uses the Euler-Maclaurin summation formula to obtain a new asymptotic approximation to $c(h/k)$. (The final goal is to approximate $c_0(h/k)$ by comparing the new approximation to our previous one.)

Proposition 3.2. *For $h \geq 1$ and large k the following relation holds.*

$$c(h/k) = \frac{1}{\pi} + \frac{1}{2\pi h} - \frac{H_{h-1}}{2\pi} - \frac{k}{\pi h} \ln(\mu_h \pi) + O\Big(\frac{1}{k}\Big).$$

Proof. Apply Corollary 2.18 for $N = 1$ to the analytic function $g_h(x)$ in $[0,1]$ (i.e., to the analytic extension of $R_h(x)$) to obtain

$$\sum_{m=1}^{k-1} g_h(m/k) = c(h/k) =$$

$$-\frac{g_h(0) + g_h(1)}{2} + k \int_0^1 g_h(x)dx + \frac{B_2}{2k} \big(g_h'(1) - g_h'(0) \big) + k r_1(g_h).$$

Lemma 2.6 tells us that

$$g_h(0) = -\frac{h+1}{h\pi} \text{ and } g_h(1) = \frac{H_{h-1}-1}{\pi},$$

so that Corollary 2.14 gives

$$\int_0^1 g_h(x)dx = -\frac{\ln(\mu_h \pi)}{\pi h},$$

where μ_h is explicitly given in Corollary 2.14. Since $g_h(x)$ has derivatives of all orders (Corollary 2.9), $g_h'(0)$ and $g_h'(1)$ are real numbers. So, for large k,

$$\frac{B_2}{2k}\big(g_h'(1) - g_h'(0)\big) = O\Big(\frac{1}{k}\Big).$$

Finally, Corollary 2.18 yields,

$$k|r_1(g_h)| = O\Big(\frac{1}{k^2}\Big).$$

The Proposition follows. □

Now substitute $c(h/k)$ from Proposition 3.2 to relation (16) to obtain

$$\begin{gathered}\frac{1}{\pi} + \frac{1}{2\pi h} - \frac{H_{h-1}}{2\pi} - \frac{k}{\pi h}\ln(\mu_h\pi) + O\Big(\frac{1}{k}\Big) = \\ c_0(h/k) - \frac{k}{h^2\pi}S_h(k) + \frac{1}{\pi}(1-H_h) - \frac{k}{h\pi}\Big(\psi(k)+\gamma\Big),\end{gathered} \tag{17}$$

where we have set

$$S_h(k) = \sum_{j=1}^{h-1} j\Big(\psi\big(\frac{jk}{h}\big) - \psi\big(\frac{jk}{h} - k\big)\Big).$$

Observe that, if $h = 1$, then $S_1(k) = 0$. From relation (10) we get,

$$\frac{k}{h\pi}\psi(k) = \frac{k}{h\pi}\ln k - \frac{1}{2h\pi} + O\big(\frac{1}{k}\big).$$

Thus (17) can be written as

$$\begin{gathered}\frac{1}{\pi} + \frac{1}{2\pi h} - \frac{H_{h-1}}{2\pi} - \frac{k}{\pi h}\ln(\mu_h\pi) + O\Big(\frac{1}{k}\Big) = \\ c_0(h/k) - \frac{k}{h^2\pi}S_h(k) + \frac{1}{\pi} - \frac{H_h}{\pi} - \frac{k}{h\pi}\ln k + \frac{1}{2h\pi} - \frac{k}{h\pi}\gamma + O\Big(\frac{1}{k}\Big).\end{gathered}$$

Accordingly,

$$c_0(h/k) = \frac{k}{h\pi}\ln k - \frac{k}{h\pi}(\ln(\mu_h\pi) - \gamma) - \frac{H_{h-1}}{2\pi} + \frac{H_h}{\pi} + \frac{k}{h^2\pi}S_h(k) + O\Big(\frac{1}{k}\Big).$$

Since

$$\frac{H_h}{\pi} - \frac{H_{h-1}}{2\pi} = \frac{1}{2\pi h} + \frac{H_h}{2\pi},$$

we get

$$c_0(h/k) = \frac{k}{h\pi}\ln k - \frac{k}{h\pi}(\ln(\mu_h\pi) - \gamma) + \frac{1}{2\pi h} + \frac{H_h}{2\pi} + kC_h(k) + O\Big(\frac{1}{k}\Big), \quad (18)$$

where $C_h(k) = \frac{1}{h^2\pi}S_h(k)$. Note that if $h = 1$, then $C_1(k) = 0$. From Lemma 2.20 (ii) we get,

$$kC_h(k) = k\mathcal{M}_h(k_0) + \frac{1 - hH_h}{2\pi h} + O\big(\frac{1}{k}\big),$$

so equation (18) gives

$$c_0(h/k) = \frac{k}{h\pi}\ln k - \frac{k}{h\pi}(\ln(\mu_h\pi) - \gamma) + \frac{1}{2\pi h} + \frac{H_h}{2\pi} + \frac{1 - hH_h}{2\pi h} + k\mathcal{M}_h(k_0) + O\Big(\frac{1}{k}\Big) =$$

$$\frac{k}{h\pi}\ln k - \frac{k}{h\pi}(\ln(\mu_h\pi) - \gamma) + \frac{1}{\pi h} + k\mathcal{M}_h(k_0) + O\Big(\frac{1}{k}\Big). \quad (19)$$

where

$$\mathcal{M}_h(k_0) = \delta(h) - \frac{1}{h}c_0\big(\frac{k_0}{h}\big),$$

and

$$\delta(h) = \frac{1}{h^2\pi}\ln\left(\frac{\big(\mathrm{Hyp}(h-1)\big)^2}{\big((h-1)!\big)^h}\right).$$

For $h = 1$, the value of $\mathcal{M}_h(k_0)$ is zero as we proved previously. For $h = 2$ and $k_0 \equiv k \bmod h$ (so $k_0 = 1$) we get $\mathcal{M}_h(1) = 0$. That is, $\mathcal{M}_1(1) = \mathcal{M}_2(1) = 0$. The Theorem follows.

4. Conclusion

Our main result is Theorem 1.9, which generalizes a theorem by Maier and Rassias [23]. We computed some basic unknown quantities in the asymptotic formula of the Maier-Rassias theorem. Our methods allow one to compute an arbitrary number of terms in the asymptotic approximation of $c_0(h/k)$, for large k and arbitrary positive integer h coprime to k.

Besides using the Euler-Maclaurin summation formula, we applied a number of approximation methods (e.g. Taylor expansion), computed various definite integrals, and also used basic complex analysis. Let us finally note that asymptotic methods such as the Taylor-series method have also been applied to other scientific areas. See, for instance, [16, 17, 18, 19], which suggest a number of asymptotic methods for a variety of problems, including nonlinear problems.

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