THE LJUNGGREN EQUATION REVISITED

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Abstract. We study the Ljunggren equation $Y^2+1 = 2X^4$ using the method "multiplication by 2" of Chabauty [2].

1. Introduction

In [5], Ljunggren proved that the only positive integral solutions of diophantine equation

$$L_2 : Y^2 + 1 = 2X^4$$

are $(X, Y) = (1, 1), (13, 239)$. Since the proof was quite complicated, Mordell asked if one could find a simpler proof. In [8] Tzanakis and Steiner gave a simpler proof using the theory of Baker. A second proof was given by Chen [3], using the Thue-Siegel method combined with Pade approximation on algebraic functions. In this paper we solve this equation with another method. Our approach is inspired by Chabauty [2] and uses the group structure of an elliptic curve and the multiplication by 2-map. This method is used by Poulakis [6] and later by Bugeaud [1] to obtain an upper bound for the height of the integral points.

2. The integral solutions of L_2

The proof consists of two parts. The first uses the group structure of the elliptic curve and the second is a reduction to a unit equation in a certain quartic number field.

To solve the equation L_2 it is enough to solve E_2, where

$$E_2 : F(X, Y) = Y^2 - (X^3 - 2X) = 0.$$

Let $P = (a, b) \in E_2(\mathbb{Z})$. Suppose that a is not zero. Then we set $a = 2x^2$, $b = 2xy$ and we deduce that $(x, y) \in L_2(\mathbb{Z})$. We assume that

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$|a| \geq 2$. Let $R = (s,t)$ be a point of E_2 over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, such that $2R = P$. By [7, chapter 3, p.59], we have

$$a = \frac{(s^2+2)^2}{4s(s^2-2)} \tag{1}$$

and so s is a root of the polynomial

$$\Theta_a(S) = S^4 - 4aS^3 + 4S^2 + 8aS + 4.$$

The roots of $\Theta_a(S)$ are:

$$a \pm \sqrt{a^2-2} \pm \sqrt{2a^2 \pm 2a\sqrt{a^2-2}},$$

where the first $\pm$ coincides with the third. Put $L = \mathbb{Q}(s)$. Since $a = 2x^2$, we have $a^2 - 2 = 4x^4 - 2 = 2y^2$ and so $L = \mathbb{Q}(\sqrt{2x^2 \pm y\sqrt{2}})$. Also, $\mathbb{Q}(\sqrt{2}) \subset L$ and $N_K(2x^2 \pm y\sqrt{2}) = 2$. It follows that the only prime dividing the discriminant of L is 2. So the only prime ramified in L is 2. Furthermore, from [4, Chapter 9, Proposition 9.4.1, p.461] L is a totally real quartic extension of $\mathbb{Q}$. So from Jones list[1] or the database[2] of Jürgen Klüners and Gunter Malle, we conclude that $L = \mathbb{Q}(\sqrt{2+\sqrt{2}})$.

The element $s_{\pm} = \frac{s \pm \sqrt{2}}{2}$ is a root of the polynomial with integer coefficients:

$$\begin{aligned} \lambda(S) &= (1/256) res_W(\Theta_a(2S \mp W), W^2 - 2) \\ &= S^8 - 4aS^7 + \cdots + 1, \end{aligned}$$

where $res_W(\cdot,\cdot)$ denotes the resultant of two polynomials with respect to W. Thus $s_{\pm}$ is a unit in L. So $u = \frac{s+\sqrt{2}}{2}$ and $v = \frac{\sqrt{2}-s}{2}$ satisfy the unit equation $u + v = \sqrt{2}$ in L. The algorithm of Wildanger [9] which is implemented in the computer algebra system Magma[3] V2.10-22, gives the solutions of this unit equation in L, which are listed in table 1 where we have put $[a_1\ a_2\ a_3\ a_4] = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3$, with $\theta = \sqrt{2+\sqrt{2}}$. We substitute to (1) each solution of the unit equation and we check if it gives an integer. Thus, it follows that $a = 2, 338$. So, for $|a| \geq 2$, the solutions of E_2 are $(X,Y) = (2,\pm 2), (338, \pm 6214)$ and for $|a| < 2$, are $(X,Y) = (0,0), (-1,\pm 1)$. So $L_2(\mathbb{Z}) = \{(\pm 1, \pm 1), (\pm 13, \pm 239)\}$.

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[1] Jones, W.J., http://math.la.asu.edu/~jj/numberfields/. Tables of number fields with prescribed ramification.

[2] http://www.mathematik.uni-kassel.de/~klueners/minimum/minimum.html

[3] http://magma.maths.usyd.edu.au/magma

[-1,0,0,0] [-1,0,1,0]	[1,0,0,0] [-3 0,1,0]	[-1,-1,0,0] [-1,-1,1,0]
[-1,1,0,0] [-1,-1,1,0]	[-1,-1,1,0] [-1,1,0,0]	[-3,0,1,0] [1,0,0,0]
[407,533,-119,-156] [-409,-533,120,156]	[-1,1,1,0] [-1,-1,0,0]	[-1,0,1,0] [-1,0,0,0]
[-409,533,120,-156] [407,-533,-119,156]	[5,7,-1,-2] [-7,-7,2,2]	[1,4,0,-1] [-3,-4,1,1]
[-71,39,120,-65] [69,-39,-119,65]	[-1,-1,-1,1] [-1,1,2,-1]	[1,2,-3,-2] [-3,-2,4,2]
[69,39,-119,-65] [-71,-39,120,65]	[-7,7,2,-2] [5,-7,-1,2]	[-3,2,4,-2] [1,-2,-3,2]
[-71,-39,120,65] [69,39,-119,-65]	[-1,2,0,-1] [-1,-2,1,1]	[1,3,0,-1] [-3,-3,1,1]
[11,14,-3,-4] [-13,-14,4,4]	[-1,2,1,-1] [-1,-2,0,1]	[-3,3,1,-1] [1,-3,0,1]
[-1,1,-1,-1] [-1,-1,2,1]	[-1,1,2,-1] [-1,-1,-1,1]	[-3,-4,1,1] [1,4,0,-1]
[11,-14,-3,4] [-13,14,4,-4]	[1,-3,0,1] [-3,3,1,-1]	[-1,-2,0,1] [-1,2,1,-1]
[-13,14,4,-4] [11,-14,-3,4]	[-3,-3,1,1] [1,3,0,-1]	[-1,-2,1,1] [-1,2,0,-1]
[-409,-533,120,156] [407,533,-119,-156]	[1,-2,-3,2] [-3,2,4,-2]	[5,-7,-1,2] [-7,7,2,-2]
[69,-39,-119,65] [-71,39,120,-65]	[-1,-1,2,1] [-1,1,-1,-1]	[1,-4,0,1] [-3,4,1,-1]
[-13,-14,4,4] [11,14,-3,-4]	[-3,-2,4,2] [1,2,-3,-2]	[-3,4,1,-1] [1,-4,0,1]
[407,-533,-119,156] [-409,533,120,-156]	[-7,-7,2,2] [5,7,-1,-2]	

Table 1-The solutions of the unit equation

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