Problem Set 7

David Bunger

November 2022

Problem 1

Theorem:
$(1+\frac{1}{n})^n < n$ for all $n \in \mathbb{N}$ such that $n \geq 3$.
Proof:
Base Case: $n = 3$
$(1+\frac{1}{3})^3 = \frac{64}{27} < 3$
Inductive Step:
Assume that $(1+\frac{1}{k})^k < k$ is true for all $\mathbb{N}$ such that $k \geq 3$.
We will prove that $(1+\frac{1}{k+1})^{k+1} < k+1$
$(1+\frac{1}{k+1})^{k+1} = (1+\frac{1}{k+1})^k \cdot (1+\frac{1}{k+1})$
$< k \cdot (1+\frac{1}{k+1})$ by inductive hypothesis.
$= (k+\frac{k}{k+1})$
$< (k+1)$
Since the base case and the inductive case have been proven, then the statement holds for all $n \in \mathbb{N}$ such that $n \geq 3$.

Problem 2

Theorem:
$(\sum_{i=1}^{n} i)^2 = \sum_{i=1}^{n} i^3$

Proof:
Base Case: $n = 0$
$(\sum_{i=1}^{0} i)^2 = \sum_{i=1}^{0} i^3$
$0 = 0$
Inductive Case:
Assume that the statements holds for $n = k$ where $k \geq 0$.

$(\sum_{i=1}^{k+1} i)^2 = (\sum_{i=1}^{k} i + k + 1)^2$

$= (\sum_{i=1}^{k} i)^2 + 2(k+1)\sum_{i=1}^{k} i + (k+1)^2$

$= \sum_{i=1}^{k} i^3 + 2(k+1)\sum_{i=1}^{k} i + (k+1)^2$ by inductive hypothesis

$= \sum_{i=1}^{k+1} i^3$

Since the base case and the inductive case have been proven, then the statement holds for all positive integers.

Problem 3

Theorem:
$2^n \geq n^2$ for all natural numbers $n \geq 5$.
Proof:
Base Case: $n = 5$
$2^5 \geq 5^2$
$32 \geq 25$
Inductive Case:
Assume that that $2^k \geq k^2$ is true. We will prove that $2^{k+1} \geq (k+1)^2$
$2^{k+1} = 2^k + 2^k$
$\geq k^2 + k^2$ by inductive hypothesis.
$= 2k^2$
$> (k+1)^2$ because $k \geq 5$.
Since the base case and the inductive case have been proven, then the statement holds for all natural numbers $n \geq 5$.

Problem 4

Theorem:
$g_n = \frac{n(n+3)}{2}$ for $n \geq 0$
Proof:
g_0:
$g_0 = \frac{0(0+3)}{2} = 0$ g_n:
Base Case: $n = 1$
$g_0 + 1 + 1 = \frac{1(1+3)}{2}$
$2 = 2$
Inductive Case:
Assume that $g_k = \frac{k(k+3)}{2}$ is true. We will prove that $g_{k+1} = \frac{(k+1)(k+4)}{2}$
$g_{k+1} = g_k + k + 2$
$= \frac{k(k+3)}{2} + k + 2$ by inductive hypothesis.
$= \frac{k(k+3)+2k+4}{2}$
$= \frac{k^2+5k+4}{2}$
$= \frac{(k+1)(k+4)}{2}$
Since the base case and the inductive case have been proven, then the statement holds for all $n \geq 0$.