Problem Set 5

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Problem 1

a)

$R_1 = \{((x,y),(z,t)) : (|x-z| > 1) \wedge (|y-t| > 1)\}$
Theorem: R_1 is not transitive.
Hypothesis: The statement "If a,b, and c are points whose values are real numbers $R_1(a,b)$ and $R_1(b,c)$, then $R_1(b,c)$" false.
Proof:
Let a be $(0,0)$, b be $(2,2)$, and c be $(0,0)$. Suppose that R_1 is transitive. That means if $R_1(a,b)$ and $R_1(b,c)$, then $R_1(a,c)$. $R_1(a,b)$ is true because $(|0-2| > 1) \wedge (|0-2| > 1)$ is true, and $R_1(b,c)$ is true because $((|2-0| > 1) \wedge (|2-0| > 1)$ is true. However, $R_1(a,c)$ is false because $(0-0| > 1) \wedge (|0-0| > 1)$ is false. Therefore, R_1 is not transitive.

b)

$R_2 = \{((x,y),(z,t)) : (x > z) \wedge (y > t)\}$
Theorem: R_2 is transitive.
Hypothesis: If a,b, and c are points whose values are real numbers, $R_2(a,b)$, and $R_2(b,c)$, then $R_2(b,c)$
Proof:
Assume that $R_2(a,b)$ and $R_2(b,c)$ are true. This means that the x value of $a(a_x)$ is greater than the x value of $b(b_x)$, and the y value of $a(a_y)$ is greater than the y value of $b(b_y)$. This also means that $b_x > c_x$ and $b_y > c_y$. If the theorem is true, then $a_x > c_x$ and $a_y > c_y$. We know that ¿ is transitive, because if a number x is greater than another number y, and y is greater than a third number z, x must be greater than z. Because we know this, it must be true that $a_x > c_x$ and $a_y > c_y$, and therefore R_2 is transitive.

c)

$R_3 = \{((x,y),(z,t)) : (x > z) \lor (y > t)\}$
Theorem: R_3 is not transitive.
Hypothesis: The statement "If a,b, and c are points whose values are real numbers $R_3(a,b)$ and $R_3(b,c)$, then $R_3(b,c)$" false.
Proof:
Let a be $(1,2)$, b be $(4,1)$, and c be $(3,3)$. Suppose that R_3 is transitive. That means if $R_3(a,b)$ and $R_3(b,c)$, then $R_3(a,c)$. $R_3(a,b)$ is true because $(1 > 4) \lor (2 > 1)$ is true, and $R_3(b,c)$ is true because $(4 > 3) \lor (1 > 3)$ is true. However, $R_3(a,c)$ is false because $(1 > 3) \lor (2 > 3)$ is false. Therefore, R_3 is not transitive.

Problem 2

Prove that if a and b are integers and $5|a$, then $5|ab$.
Proof:
Assume a and b are integers and $5|a$. This means that a is divisible by 5, meaning there exists an integer k that when multiplied with 5 results in a, or $5k = a$. If both sides of this equation are multiplied by the integer b, the resulting equation would be $5kb = ab$, which would also be true by the laws of algebra. kb can be simplified to k, because the product of two integers is itself an integer, resulting in $5k = ab$. This means that there exists an integer k that when multiplied by 5 results in ab. Therefore, $5|ab$.

Problem 3

Prove that multiples(69) $\subseteq$ multiples(23).
Proof:
The formula to describe multiples(x), $\{y \in \mathbb{Z} : x|y\}$, can be described as the set of all integers that can be divided by x. Because 69 can be divided by 23, 69 is a multiple of 23. This also means 69a is also a multiple of 23, where a is an integer. This shows that every number that can be divided by 69 can also be divided by 23. Therefore, multiples(69) $\subseteq$ multiples(23).

Problem 4

a)

Prove that divisors(b)∩divisors(a-b)⊆divisors(a)∩divisors(b)
Proof:
The function divisors(x) can be defined as $S = \{i \in \mathbb{Z} : i|x\}$. The intersection of divisors(a) and divisors(b) can be defined as $S = \{i \in \mathbb{Z} : i|a \wedge i|b\}$. Suppose that if $i|a$ and $i|b$, then $i|(a-b)$. By the definition of divides, that means there exists an integer k that when multiplied by i results in $(a-b)$. There also exists integers j and l that when multiplied by i, result in a and b respectively. k can be found by subtracting l from j, showing that a, b, and $(a-b)$ can all be divided by the same i. This means that divisors(a-b) includes, but is not limited to, the intersection of divisors(a) and divisors(b). When intersected with divisors(b), the set becomes equivalent to the intersection of divisors(a) and divisors(b). Therefore, divisors(b)∩divisors(a-b)⊆divisors(a)∩divisors(b).

Problem 5

Prove that if 131 doesn't divide 111x then 131 doesn't divide x.
Proof:
If 131 divides 111x, that means there exists an integer k that when multiplied with 131 results in 111x, or $131k = 111x$. By basic algebra, this can be rewritten as $131k/111 = x$. As k can be any integer, $k/111$ can be rewritten as k, so long as 111 divides k. If an integer k does not exist for $131k = 111x$, then it will also not exist for $131k = x$, as 131 itself does not divide 111. Therefore, if 131 doesn't divide 111x then 131 doesn't divide x.

Problem 6

Prove that if $a \in \mathbb{Z}$ and $b \notin \mathbb{Z}$ then $c = a + b \notin \mathbb{Z}$
If b is not an integer, that means $b - \lfloor b \rfloor$ is greater than zero and less than one. If a is an integer, that means $a - \lfloor a \rfloor$ is zero. $a + b$ can be written as $a + \lfloor b \rfloor + (b - \lfloor b \rfloor)$. Since we know that $b - \lfloor b \rfloor$ is greater than zero and less than one and $a - \lfloor a \rfloor$ is zero, $c - \lfloor c \rfloor$ must equal $b - \lfloor b \rfloor$. Because $c - \lfloor c \rfloor$ is not zero, c is not an integer, and therefore if $a \in \mathbb{Z}$ and $b \notin \mathbb{Z}$ then $c = a + b \notin \mathbb{Z}$