Problem Set 6

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Problem 1

Suppose that there exists an element x where $x \in (D \cap E)$. This implies $x \in D$ and $x \in E$ by the definition if intersection. Because $D = C \cap A$ and $E = C \cap B$, this also implies that $x \in A$, $x \in B$, and $x \in C$. Because $x \in A$ and $x \in B$, then $x \in A \cap B$. However, $A \cap B$ is assumed to be empty, so this is a contradiction. Therefore, x cannot exist, meaning $D \cap E = \emptyset$ by the definition of $\emptyset$.

Problem 2

a)

It is unnecessary to declare another variable y. We can achieve the same result by showing $xRx \wedge xRx \rightarrow xRx$. This proof also assumes that every element in the domain has is defined in the relation, which may not be true.

b)

Suppose R is a relation on the set $\{x, y, z\}$, and R is defined as $\{(x, x), (x, y), (y, x), (y, y)\}$. This relation is Lottian, but it is not reflexive because z is not defined in the relation.

c)

Theorem: If a relation is reflexive and Lottian, then it is an equivalence relation.

Lemma 1: If a relation is reflexive and Lottian, then it is also symmetric.

Suppose that the relation R is Lottian and reflexive. Let x and y be elements in the set that R is a relation on. Because R is Lottian, we can use the definition of Lottian to show $xRx \wedge yRx \rightarrow xRy$. Because R is reflexive, we know xRx is always true. This makes the previous statement equivalent to $yRx \rightarrow xRy$, which is the definition of symmetry.

Lemma 2: If a relation is reflexive and Lottian, then it is also transitive.

Suppose that the relation R is Lottian and reflexive. Let x, y, and z be elements in the set that R is a relation on. Because R is Lottian, we can use the definition of Lottian to show $xRy \wedge zRy \to xRz$. Since we know R is symmetric by Lemma 1, we can rewrite this statement as $xRy \wedge yRz \to xRz$, which is the definition of transitivity.

Proof:
Suppose that the relation R is Lottian and reflexive. We know that R is symmetric and transitive by Lemmas 1 and 2. Because R is reflexive, transitive, and symmetric, it is an equivalence relation by the definition of equivalence.

Problem 3

Case 1: $r_1 = 0$
Since r_1 is 0, $n = q_1 d$, meaning n is a multiple of d, and that $q_1 = n/d$. If we substitute $q_1 d$ for n in $n = q_2 d + r_2$, we get $q_1 d = q_2 d + r_2$. If $q_2 \neq q_1$, then q_2 must be less than q_1, and r_2 would be equal to $(q_1 - q_2)d$. However, because $0 \leq r_2 < d$, r_2 must be zero, and therefore $r_1 = r_2$. This leaves $q_1 d = q_2 d$, meaning that $q_1 = q_2$.
Case 2: $r_1 \neq 0$
$q_1 d + r_1$ can be substituted for n in $n = q_2 d + r_2$, giving $q_1 d + r_1 = q_2 d + r_2$. This can be rewritten as $q_1 d = q_2 d + r_2 - r_1$. Similar to the first case, if $q_2 \neq q_1$, then q_2 must be less than q_1, and $r_2 - r_1$ would be equal to $(q_1 - q_2)d$. However, because $0 \leq r_1 < d$ and $0 \leq r_2 < d$, $r_2 - r_1$ must be zero. This can only be accomplished if $r_1 = r_2$. Again, this leaves $q_1 d = q_2 d$, meaning that $q_1 = q_2$.

Problem 4

Theorem: $t = r_3$
Proof:
We want to show that $t = r_3$. We can expand this into $s\%a = b_3\%a$, which can be further expanded to $(r_1 \cdot r_2)\%a = (b_1 \cdot b_2)\%a$, which can again be expanded to get $((b_1\%a) \cdot (b_2\%a))\%a = (b_1 \cdot b_2)\%a$. Because of the distributive property of the modulo operator, the left side can be turned from $((b_1\%a) \cdot (b_2\%a))\%a$ to $((b_1 \cdot b_2)\%a)\%a$. The second modulo operation is redundant because $(b_1 \cdot b_2)\%a$ is already less than a. This leaves us with $(b_1 \cdot b_2)\%a = (b_1 \cdot b_2)\%a$. Since the left side was originally equivalent to t, and the right side was originally equivalent to r_3, this shows that $t = r_3$.