Macro 2 Pset 1

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1 Question 1

Show that Equation (8) implies Equation (5).

1.1 Solution

Note that Equation (8) from Lecture 0 is:

$$V(k) = \max_{c,k'} u(c) + \beta V(k')$$

subject to

$$c + k' \leq f(k) + (1 - \delta)k$$

Consider the Lagrangian for this equation:

$$\mathcal{L} = u(c) + \beta V(k_+) + \lambda(f(k) + (1-\delta)k - c - k_+)$$

Now, we take the FOCs:

$$\frac{\partial \mathcal{L}}{\partial c} = u'(c) - \lambda = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial k_+} = \beta V'(k_+) - \lambda = 0 \tag{2}$$

The envelope condition[1] is:

$$V'(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda f'(k) + \lambda(1-\delta) = \lambda[f'(k) + (1-\delta)]$$

From (9) and (10), we have:

$$\lambda = u'(c) = \beta V'(k_+) \tag{3}$$

Applying them to one period earlier gives:

$$\lambda = u'(c_-) = \beta V'(k) \tag{4}$$

Now, use (11) and (12) to replace λ and $V'(k)$ in the envelope condition:

$$\begin{aligned} \frac{u'(c_-)}{\beta} &= u'(c)(f'(k) + (1-\delta)) \\ \Leftrightarrow u'(c_-) &= \beta u'(c)(f'(k) + (1-\delta)) \end{aligned}$$

By re-arranging terms and moving our time indicators up by 2 periods, this gives:

$$u'(c_t) = \beta[1 + f'(k_{t+1}) - \delta]u'(c_{t+1})$$

which is Equation (5).

[1] Notes on Envelope Theorem; Wikipedia Page on Envelope Theorem

2 Question 2

Show that Equation (14) implies Equation (13).

2.1 Solution

Equation (14) from Lecture 0 is:

$$\rho V(k) = \max_{c} u(c) + V'(k)[f(k) - c - \delta k] \tag{5}$$

with

$$\dot{k} = f(k) - \delta k - c \tag{6}$$

Consider the FOC of (5) wrt c:

$$u'(c(t)) - V'(k(t)) = 0 \Leftrightarrow u'(c(t)) = V'(k(t)) \tag{7}$$

Additionally, note that

$$u''(c(t))\dot{c}(t) = V''(k(t))\dot{k}(t) \qquad \text{Differentiating (7) wrt t} \tag{8}$$

Now, consider the envelope condition for (5):

$$\begin{aligned}
\rho V'(k(t)) &= V''(k(t))[f(k(t)) - c(t) - \delta k(t)] + V'(k(t))[f'(k(t)) - \delta] \\
\Leftrightarrow \rho V'(k(t)) &= V''(k(t))\dot{k}(t) + V'(k(t))[f'(k(t)) - \delta] && \text{Substituting in (6)} \\
\Leftrightarrow \rho u'(c(t)) &= V''(k(t))\dot{k}(t) + u'(c(t))[f'(k(t)) - \delta] && \text{Substituting in (7)} \\
\Leftrightarrow \rho u'(c(t)) &= u''(c(t))\dot{c}(t) + u'(c(t))[f'(k(t)) - \delta] && \text{Substituting in (8)} \\
\Leftrightarrow \frac{\dot{c}(t)}{c(t)} &= -\frac{u'(c(t))}{u''(c(t))c(t)}[f'(k(t)) - \rho - \delta] && \text{Rearranging}
\end{aligned}$$

Thus, we have Equation (13) from Lecture 0.

3 Question 3

Show that Equation (8) implies Equation (5).

3.1 Solution

Note that Equation (8) from Lecture 0 is:

$$V(k) = \max_{c,k'} u(c) + \beta V(k')$$

subject to

$$c + k' \leq f(k) + (1-\delta)k$$

Consider the Lagrangian for this equation:

$$\mathcal{L} = u(c) + \beta V(k_+) + \lambda(f(k) + (1-\delta)k - c - k_+)$$

Now, we take the FOCs:

$$\frac{\partial \mathcal{L}}{\partial c} = u'(c) - \lambda = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial k_+} = \beta V'(k_+) - \lambda = 0 \tag{10}$$

The envelope condition[2] is:

$$V'(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda f'(k) + \lambda(1-\delta) = \lambda[f'(k) + (1-\delta)]$$

From (9) and (10), we have:

$$\lambda = u'(c) = \beta V'(k_+) \tag{11}$$

Applying them to one period earlier gives:

$$\lambda = u'(c_-) = \beta V'(k) \tag{12}$$

Now, use (11) and (12) to replace λ and $V'(k)$ in the envelope condition:

[2]Notes on Envelope Theorem; Wikipedia Page on Envelope Theorem

$$
\begin{aligned}
\frac{u'(c_-)}{\beta} &= u'(c)(f'(k) + (1-\delta)) \\
\Leftrightarrow u'(c_-) &= \beta u'(c)(f'(k) + (1-\delta))
\end{aligned}
$$

By re-arranging terms and moving our time indicators up by 2 periods, this gives:

$$
u'(c_t) = \beta[1 + f'(k_{t+1}) - \delta]u'(c_{t+1})
$$

which is Equation (5).