Lecture 1

Kaldor facts and balanced growth

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See Chapters 2 and 8 in Acemoglu

Neoclassical growth model

- The neoclassical growth model was developed to be a parsimonious model that has natural mapping into key macroeconomic time series and be consistent with several empirical patterns that showed remarkable consistency over long period of time
- Basic accounting definitions

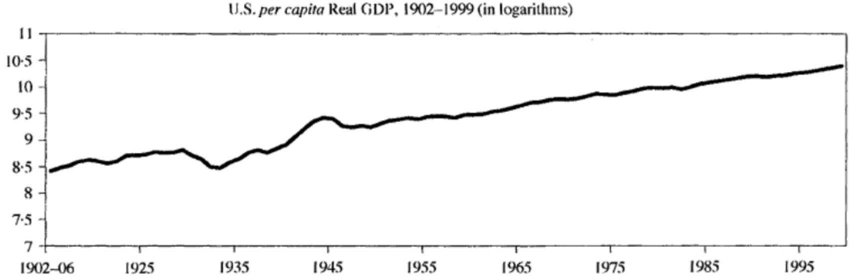
$$C_t + I_t = Y_t$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$Y_t = \text{sum of factor income}$$

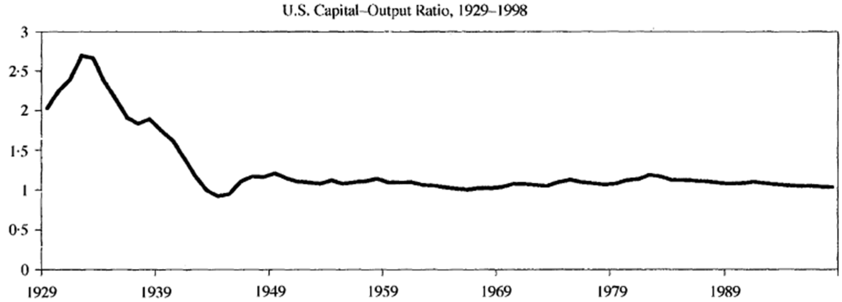
- Further assumptions
  - Y_t is produced by CRS technology F_t (DRS *is* a CRS with a fixed factor, IRS is hard to model parsimoniously)
  - two factors: capital and labor

Fact 1: Historical growth rates



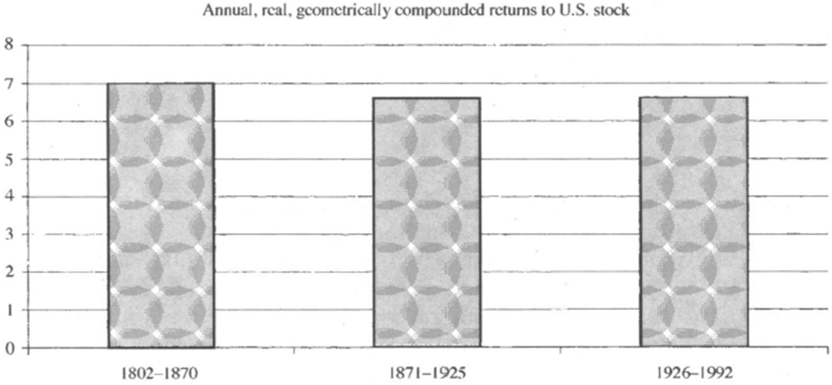
Sources: Historical Statistics of the U.S.; Economic Report of the President; and U.S. Census Bureau.

Fact 2: Historical capital-output ratios



Sources: Fixed assets and consumer durable goods. Survey of current business, May 1997, April 2000. NIPA (B.E.A. website).

Fact 3: Historical interest rates



Source: Siegel (1995)

Fact 3 (con'd): Historical interest rates II

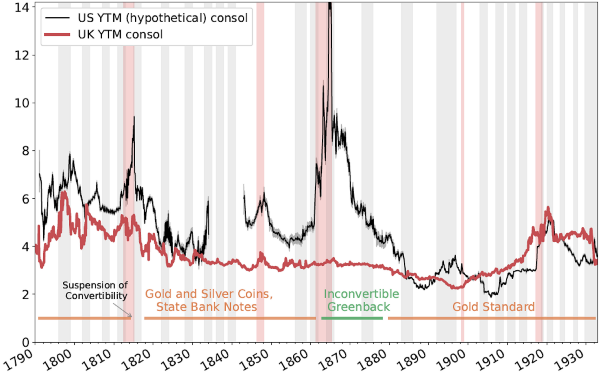


Figure VI: US and UK Consol Yields.

Source: "Costs of Financing US Federal Debt Under a Gold Standard: 1791-1933" Payne et al (2024)

Yield on the 30 year U.S. Treasuries in 2024 is 4.18%

Fact 4: Historical factor shares

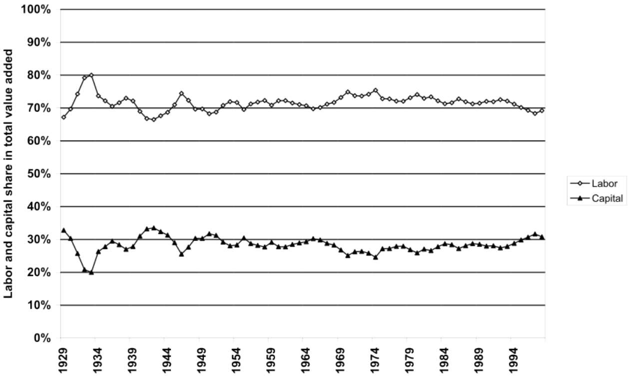


Figure: Capital and Labor Share in the U.S. GDP.

Kaldor facts

1. Output per capital grows at a constant rate
2. Capital-output ratio is roughly constant
3. Interest rate is roughly constant
4. Distribution of income between capital and labor is roughly constant

Technology

- Aggregate production function for the unique final good is

$$Y(t) = \tilde{F}\left(K(t), L(t), \tilde{X}(t)\right)$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$$

- $\tilde{F}$ is CRS in K, L

Implications of growth

- Assume constant growth

$$\begin{aligned} \dot{Y}(t)/Y(t) &= g_Y > 0, \dot{K}(t)/K(t) = g_K > 0, \\ \dot{C}(t)/C(t) &= g_C > 0, \dot{L}(t)/L(t) = n \end{aligned}$$

Theorem

(Uzawa): Constant growth + CRS implies
(1) Balanced growth: $g_Y = g_C = g_K \equiv g$
(2) Labor augmenting technical change: $\tilde{F}$ *can be represented as* $\tilde{F} = F(K(t), X(t)L(t))$ *for some CRS* F *with* $\frac{\dot{X}(t)}{X(t)} = g - n$

Proof (preliminary)

- If any variable Z grows with rate g,

$$\frac{\dot{Z}(t)}{Z(t)} = g \iff Z(t) = e^{(t-\tau)g} Z(\tau) \text{ for all } t, \tau$$

Proof (1)

- Since $\dot{K}(t) = g_K K(t)$ we have

$$(g_K + \delta) K(t) = Y(t) - C(t)$$

or

$$(g_K + \delta) K(0) = e^{(g_Y - g_K)t} Y(0) - e^{(g_C - g_K)t} C(0) \tag{1}$$

- Differentiate w.r.t. t

$$(g_Y - g_K) e^{(g_Y - g_K)t} Y(0) = (g_C - g_K) e^{(g_C - g_K)t} C(0)$$

- Since it holds for all t we must have

$$g_Y - g_K = g_C - g_K$$

- But $g_Y - g_K = g_C - g_K \neq 0$ violates (1) $\Longrightarrow$

$$g_Y = g_C = g_K$$

Proof (2)

- For any $t \geq \tau$ we can write (given part 1)

$$e^{-g(t-\tau)} Y(t) = \tilde{F}\left[e^{-g(t-\tau)} K(t), e^{-n(t-\tau)} L(t), \tilde{X}(\tau)\right].$$

- Equivalently, using CRS

$$Y(t) = \tilde{F}\left[K(t), e^{(g-n)t} L(t) e^{-(g-n)\tau}, \tilde{X}(\tau)\right].$$

- Since this must hold for all τ, there must exist CRS F such that

$$Y(t) = F\left[K(t), e^{(g-n)t} L(t)\right]$$

- Let $X(t) \equiv e^{(g-n)t}$ (obviously $\dot{X}/X = g - n \equiv g_X$) to get

$$Y(t) = F[K(t), X(t) L(t)]$$

Discussion

- CRS implies that constant growth must be balanced (all variables grow at the same rate) and per capital growth is driven by technology.
- Technology must be
  - either purely labor augmenting
  - or elasticity of substitution between capital and labor much be 1:

$$
\begin{aligned}
Y(t) &= \left(\tilde{X}_K(t) K(t)\right)^{\alpha} \left(\tilde{X}_L(t) L(t)\right)^{1-\alpha} \\
&= K(t)^{\alpha} \left(\underbrace{\tilde{X}_K(t)^{\frac{\alpha}{1-\alpha}} \tilde{X}_L(t)}_{\equiv X(t)} L(t) \right)^{1-\alpha}
\end{aligned}
$$

- To connect to the last two Kaldor facts, we need prices $\Longrightarrow$ consider implications of the competitive equilibrium

Constant factor shares

Theorem

Uzawa assumptions + constant factor shares imply that interest rate are constant, $R(t) = R^*$, and that wages grow with rate of technology, $\dot{w}/w = g_X$

- Technology growth fully reflected in growth of wages

Proof

- Under perfect competition, capital income is $R(t)K(t)$ and so

$$\text{Capital share } \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)} \underbrace{=}_{Uzawa\ \ p.1} const \cdot R(t)$$

- Constant factor shares $\Longleftrightarrow$ constant interest rates R^*
- Similarly, with constant factor shares

$$w(t) \equiv \frac{Y(t) - R(t)K(t)}{L(t)} = \frac{Y(t)}{L(t)} \times \left(1 - \frac{K(t)}{Y(t)}R^*\right)$$

- By Uzawa, K/Y is constant, Y/L grows at rate $g - n = g_X$

Constant interest rates

Theorem

Constant interest rates and balanced growth implies that $U(C)$ must be, up to a linear tranformation,

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

Proof

- Euler equation in CE

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma(C(t))}[R(t) - \rho]$$

where

$$\sigma(C) \equiv -\frac{U''(C)\,C}{U'(C)}$$

- Constant interest rates + constant growth $\Longrightarrow \sigma(C) = \sigma$ for all C
- Solving an ode $\sigma U'(C) + U''(C)\,C = 0$ gives

$$U(C) = const_0 \frac{C^{1-\sigma}}{1-\sigma} + const_1$$

Neoclassical growth model

- Infinitely lived representative household with preferences

$$\int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

  and inelastic labor supply (for now)

- Perfectly competitive firms with CRS technology

$$Y(t) = F(K(t), X(t) L(t))$$

- Feasibility

$$\begin{aligned} C(t) + \dot{K}(t) &= Y(t) - \delta K(t) \\ L(t) &= 1 \end{aligned}$$

Neoclassical growth model

- Re-normalize everything per unit of X :

$$
\begin{aligned}
k(t) &\equiv \frac{K(t)}{X(t)}, c(t) \equiv \frac{C(t)}{X(t)}, \\
y(t) &\equiv \frac{Y(t)}{X(t)} = F(k(t), 1), \quad \tilde{\rho} \equiv \rho - (1-\sigma) g_X
\end{aligned}
$$

- This model is isomorphic to neoclassical growth model without growth and therefore we know that
  - competitive equilibrium is efficient
  - $k(t), c(t), y(t)$ converge to the steady state k^{ss}, c^{ss}, y^{ss}

Growth model and Kaldor facts

Theorem

Steady state of the neoclassical growth model is consistent with Kaldor facts

1. $y(t) = y^{ss}$ implies that $Y(t)$ grows at rate g_X
2. Capital-output ratio is constant: $K(t) / Y(t) = k^{ss} / y^{ss}$
3. Since consumption growth rate is constant, so are interest rates
4. Factor shares are constant by labor-augmenting technical change + constant interest rate

Other observations I

- Let's endogenize labor supply
- In XXth century wages increased by a factor of 10
- Work hours decreased relatively little
  - lifetime work hours were 182,100 in 1880 and 122,400 in 1995 (Fogel 2000)
- Income and substitution effect must roughly cancel each other
- Balanced growth path preferences with labor supply

$$U(C,L) = \begin{cases} \frac{C^{1-\sigma}}{1-\sigma} v(L) & \text{if } \sigma \neq 1 \\ \ln C + v(L) & \text{if } \sigma = 1 \end{cases}$$

Other observations II

- Balanced growth requires
  - either no technical progress for capital
  - or unit elasticity of substitution between capital and labor
- Prices of capital goods fell dramatically $\Longrightarrow$ suggests some capital-augmenting technical change
- Direct estimates of the elasticity of substitution suggests that it is less than 1
  - Chirinko 2008 surveys literature and argues for it to be around 0.4-0.6
  - However, long-run elasticity estimates are always controversial
- See “Balanced Growth Despite Uzawa” by Grossman, Helpman, Oberfield, Samson (AER, 2017) how to reconcile with human capital/schooling