Lecture 12

Basic New Keynesian model

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Introduction

- Simplest New Keynesian model
  - Dixit-Stigliz model from Lecture 8 in dynamic settings and *nominal rigidities*
- All prices $P, \{P_i\}_i$ are nominal (e.g., in \$), do not normalize them to 1
- Prices $\{P_i\}_i$ are sticky, it is costly for intermediate firms to change them

Rotemberg cost of price changes

- We consider simplest version of price adjustments, in the spirit of Rotemberg (1982)
  - not the most realistic, but to the first order of approximation we consider it will be equivalent to more realistic costs of price adjustments, e.g. ones proposed by Calvo or Taylor
- Support firm i had price $P_{i,t-1}$ in period t
- The cost of setting price $P_{i,t}$ in period t is

$$\Phi_{i,t} = \frac{\theta}{2} PY \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right)^2$$

  where $\theta \geq 0$ is parameter capturing how costly it is for firm i to change its price
- Costs are rebated lump-sum to consumers

Intermediate firms

- Identical technology

$$Y_{i,t} = A_t L_{i,t}$$

- Potentially stochastic, $a_t := \ln A_t$ follows AR(1) process

$$a_t = \rho a_{t-1} + \varepsilon_t$$

- Cost of price changes

$$\Phi_{i,t} = \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right)^2$$

- Dividends

$$D_{i,t} = P_{i,t} Y_{i,t} - W_t L_{i,t} - \Phi_{i,t}$$

CB policy rule

- Let Π_t be inflation:

$$\Pi_t = P_t / P_{t-1}$$

- Let Q_t be a price of a nominal one period risk-free bond purchased at t
- Let I_t be the nominal risk-free interest rate between period t and $t+1$:

$$I_t := 1/Q_t - 1$$

- Suppose central bank controls nominal interest rate and follows policy rule with parmeter ϕ:

$$1 + I_t = \frac{1}{\beta}\Pi_t^{\phi}$$

  or in log form

$$\iota_t = -q_t = \phi\pi_t - \ln\beta$$

Definition of eqm, I

Stochastic sequences $\{C_t, L_t, B_t, D_{i,t}, \Phi_{i,t}\}_{i,t}$, prices $\{P_t, P_{i,t}, W_t, Q_t, \Pi_t\}$ such that

- Consumers solve

$$\max_{\{C_t, L_t, B_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right]$$

  s.t. $B_{-1} = 0$ and

$$P_t C_t + Q_t B_t = W_t L_t + \int D_{i,t} di + \int \Phi_{i,t} di + B_{t-1}$$

- Final goods firms solve

$$\max_{\{Y_i\}_i, Y} P_t Y_t - \int P_i Y_i di$$

  s.t.

$$Y_t = \left(\int Y_{i,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

Definition of eqm, II

- Easiest to write problem of intermediate firms in recursive form:

$$V\left(A_t, P_{i,t-1}\right) = \max_{P_{i,t}, Y_{i,t}, L_{i,t}} P_{i,t} Y_{i,t} - W_t L_{i,t} - \Phi_{i,t} + Q_t \mathbb{E}_t V\left(A_{t+1}, P_{i,t}\right)$$

s.t.

$$Y_{i,t} = A_t L_{i,t}, \qquad Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t,$$

$$\Phi_{i,t} = \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)^2.$$

- Dividends

$$D_{i,t} = P_{i,t} Y_{i,t} - W_t L_{i,t} - \Phi_{i,t}$$

Definition of eqm, III

- Central Bank sets
$$\frac{1}{Q_t} = \frac{1}{\beta}\Pi_t^{\phi}$$
- Market clearing conditions
$$C_t = Y_t, \; L_t = \int L_{i,t} di, \quad B_t = 0.$$

Consumer optimality

- Consumers intra-temporal FOC is the same as in Lectures 8:

$$C_t^{\gamma} L_t^{\varphi} = \frac{W_t}{P_t}$$

- The inter-temporal optimality conditions (make sure you can derive it!)

$$\begin{aligned} Q_t &= \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \\ &= \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \end{aligned}$$

- All the interesting action is happening on the intermediate firm side

Intermed. firm optimality

- Firm problem

$$V\left(A_t, P_{i,t-1}\right) = \max_{P_{i,t}} P_{i,t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t - \frac{W_t}{A_t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t - \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)^2 + Q_t \mathbb{E}_t V\left(A_{t+1}, P_{i,t}\right)$$

- FOC:

$$(1-\sigma)\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t + \sigma \frac{W_t}{A_t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t \frac{1}{P_{i,t}} - \theta P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)\frac{1}{P_{i,t-1}} + Q_t \mathbb{E}_t \frac{\partial}{\partial P_{i,t}} V\left(A_{t+1}, P_{i,t}\right) = 0$$

Intermed. firm optimality

- Firm problem

$$V\left(A_t, P_{i,t-1}\right) = \max_{P_{i,t}} P_{i,t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t - \frac{W_t}{A_t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t - \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)^2 + Q_t \mathbb{E}_t V\left(A_{t+1}, P_{i,t}\right)$$

- FOC:

$$(1-\sigma)\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t + \sigma \frac{W_t}{A_t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t \frac{1}{P_{i,t}} - \theta P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)\frac{1}{P_{i,t-1}} + Q_t \mathbb{E}_t \frac{\partial}{\partial P_{i,t}} V\left(A_{t+1}, P_{i,t}\right) = 0$$

- Envelope theorem:

$$\frac{\partial}{\partial P_{i,t}} V\left(A_{t+1}, P_{i,t}\right) = \theta P_{t+1} Y_{t+1}\left(\frac{P_{i,t+1}}{P_{i,t}} - 1\right)\frac{P_{i,t+1}}{P_{i,t}^2}$$

Intermed. firm optimality, II

- All firms are identical, which implies that (see Lecture 8) that $P_{i,t} = P_t$ for all i
- This simplifies our previous equation to

$$(1-\sigma) + \sigma \frac{W_t / P_t}{A_t} - \theta \Pi_t \left(\Pi_t - 1\right) + \theta \beta Q_t \mathbb{E}_t \left(\Pi_{t+1} - 1\right) \Pi_{t+1}^2 \frac{Y_{t+1}}{Y_t} \quad = \quad 0.$$

- Combine previous equations and get rid of redundant variables
- $\{C_t, Q_t, L_t, W_t/P_t, \Pi_t\}_t$ are a competitive equilibrium if and only if they solve

$$(1-\sigma)+\sigma\frac{W_t/P_t}{A_t}-\theta\Pi_t\left(\Pi_t-1\right)+\theta\beta Q_t\mathbb{E}_t\left(\Pi_{t+1}-1\right)\Pi_{t+1}^2\frac{C_{t+1}}{C_t} \;=\; 0,$$

$$C_t^\gamma L_t^\varphi=\frac{W_t}{P_t},$$

$$C_t=A_tL_t,$$

$$Q_t=\beta\mathbb{E}_t\frac{1}{\Pi_{t+1}}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma},$$

$$\frac{1}{Q_t}=\frac{1}{\beta}\Pi_t^\phi.$$

Flexible price model

- Consider first the flexible price case, $\theta = 0$
- **Exercise 1**: Suppose $\theta = 0$. Show that competitive equilibrium allocations $\{C_t, L_t\}_t$ and real wages $\{W_t/P_t\}_t$ are independent of the way the central bank sets interest rates $1/Q_t$ and coincide with our solution in Lecture 8
- Thus, a flexible price version of this economy works just like the model in Lecture 8

Deterministic steady state

- To consider the case $\theta > 0$, it is useful to start first with the model in which there are no shocks
- **Exercise 2**: Suppose $\theta > 0$ and $A_t = 1$ for all t. Show that there exists an equilibrium competitive equilibrium allocations and prices are independent of t and satisfy $\{C_t, L_t, W_t/P_t, L_t, Q_t, \Pi_t\}_t = \left(\overline{C}, \overline{L}, \overline{W/P}, \overline{Q}, \overline{\Pi}\right)$ where

$$\overline{\Pi} = 1, \overline{Q} = \beta,$$

$$\overline{C}^{\gamma+\varphi} = \frac{\sigma - 1}{\sigma},\ \overline{L} = \overline{C},\ \overline{W/P} = \frac{\sigma - 1}{\sigma}.$$

- Without shocks, the model with sticky prices looks the same as static / flex price model
  - firms set their prices once and for all as a markup $\sigma / (\sigma - 1)$ over marginal costs
  - no shocks $\Longrightarrow$ no need to ever change those prices

A one time monetary shock

- Let's start with a simple thought experiment
  - economy is in deterministic steady state
  - the central bank has a one time cut in interest rates at $t = 0$
  - central bank reverts to its usual policy rule for $t > 0$
- Formally, equilibrium is described by the same system of equations except

$$Q_0 = \overline{Q} + \Delta \text{ for some } \Delta > 0.$$

A one time monetary shock: analysis

- Note that $\left(\overline{C}, \overline{L}, \overline{W/P}, \overline{Q}, \overline{\Pi}\right)$ still solve all our equations for $t > 0$
- Therefore, period zero allocations $C_0, L_0, W_0/P_0, \Pi_0$ solve

$$\beta + \Delta = \beta \left(\frac{\overline{C}}{C_0}\right)^{-\gamma}, \quad C_0 = L_0, \quad C_0^{\gamma} L_0^{\varphi} = \frac{W_0}{P_0},$$

$$(1-\sigma) + \sigma \frac{W_0}{P_0} - \theta \Pi_0 \left(\Pi_0 - 1\right) = 0$$

- These four equations show main economics of NK models

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  - to satisfy demand, firm need to higher more labor, increases L_0
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  - firms want to charge markups over their marginal costs W_0/P_0 so prices and inflation Π_0 must increase

Responses to MIT shocks

- Let's start with simplest stochastic process $\Longrightarrow$ a response to a one-time unanticipated shock
  - colloquial name is "MIT shock"
- To be specific:
  - we start in the deterministic steady state
  - CB unexpectedly cuts interest rate by v_0 in period 0 below its policy rule
  - in subsequent periods follows deterministic policy

$$-q_t = \phi \pi_t - \ln \beta - \rho^t v_0$$

  for some $\rho \in [0, 1)$.

- This leads to a deterministic sequence $\{(C_t, L_t, W_t/P_t, \Pi_t)\}_t$ that solves a system of non-linear equations given above
  - but they are must be near $\left(\overline{C}, \overline{L}, \overline{W/P}, 1\right)$
- We want to simply analysis by approximating it this system with linear equations

Log-linearization

- Let $x \equiv \ln X$
- First order Taylor expansion around some $\bar{x} = \ln \bar{X}$

$$
\begin{aligned}
F(X) &= F(\exp(x)) \approx F(\exp(\bar{x})) + F'(\exp(\bar{x}))\exp(\bar{x})\underbrace{(x-\bar{x})}_{\equiv \hat{x}} \\
&= F(\bar{X}) + F(\bar{X})\bar{X}\hat{x}
\end{aligned}
$$

Point of approximations in levels

- Our deterministic steady state $(c, q, w-p, \pi)$ in logs ($l = c$ so I just drop t)

$$\bar{\pi} = 0, \quad \bar{q} = \ln \beta,$$

$$(\gamma + \varphi)\, \bar{c} = \overline{w - p},$$

$$\sigma \exp\left(\overline{w - p}\right) = \sigma - 1$$

- Equations $C_t^{\gamma+\varphi} = \frac{W_t}{P_t}$ is trivial to log-linearize:

$$(\gamma + \varphi)\, \hat{c}_t = \widehat{w_t - p_t}$$

- Central bank policy rule can be written as

$$\hat{q}_t = \rho^t v_0 - \phi \hat{\pi}_t$$

Approximation of Euler equation, I

- Take logs of the Euler equation

$$q_t = \ln \beta + \ln \mathbb{E}_t \exp \left\{ -\pi_{t+1} - \gamma \left(c_{t+1} - c_t \right) \right\}$$

- Approximate the last term

$$\begin{aligned} \ln \mathbb{E}_t \exp \left\{ -\pi_{t+1} - \gamma \left(c_{t+1} - c_t \right) \right\} &\approx \ln \mathbb{E}_t \exp \left(\underbrace{-\bar{\pi} - \gamma \left(\bar{c} - \bar{c} \right)}_{=0} \right) \\ &+ \frac{\mathbb{E}_t \exp \left(-\bar{\pi} - \gamma \left(\bar{c} - \bar{c} \right) \right) \cdot \left(-\hat{\pi}_{t+1} - \gamma \left(\hat{c}_{t+1} - \hat{c}_t \right) \right)}{\mathbb{E}_t \exp \left(-\bar{\pi} - \gamma \left(\bar{c} - \bar{c} \right) \right)} \\ &= -\mathbb{E}_t \left(\hat{\pi}_{t+1} + \gamma \left(\hat{c}_{t+1} - \hat{c}_t \right) \right) \end{aligned}$$

- Thus, we have

$$-\hat{q}_t = \mathbb{E}_t \left(\hat{\pi}_{t+1} + \gamma \left(\hat{c}_{t+1} - \hat{c}_t \right) \right)$$

Euler equation, intuition

- Euler eqn can be written as

$$\mathbb{E}_t\left(\hat{c}_{t+1} - \hat{c}_t\right) = \underbrace{\frac{1}{\gamma}}_{\text{elasticity of intertemp subst}} \times \left(\underbrace{\hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1}}_{\text{real interest rate}}\right)$$

- Intuition is very simple:

$$\begin{aligned} &\text{change in consumption growth rate} \\ &= \text{elasticity of subst} \ \times \ \text{change in interest rate} \end{aligned}$$

Approximations of firm optimality

- First stochastic term:

$$\sigma \frac{W_t / P_t}{\overline{A}} \underbrace{=}_{\overline{A}=1} \sigma \exp\left(w_t - p_t\right)$$
$$\approx \sigma \exp\left(\overline{w - p}\right) + \sigma \exp\left(\bar{w} - \bar{p}\right)\left(\widehat{w_t - p_t}\right)$$
$$= (\sigma - 1) + (\sigma - 1)\left(\widehat{w_t - p_t}\right)$$

- Second stochastic term

$$\theta \Pi_t \left(\Pi_t - 1\right) = \theta \exp\left(\pi_t\right)\left(\exp\left(\pi_t\right) - 1\right)$$
$$\underbrace{\approx}_{\bar{\pi}=0} \theta \cdot 0 + \theta \cdot 0 \cdot 1 \cdot \hat{\pi}_t + \theta \cdot 1 \cdot 1 \cdot \hat{\pi}_t = \theta \hat{\pi}_t$$

Intuition for firm optimality

- Proceeding in the same way for all terms, we get

$$\hat{\pi}_t - \underbrace{\frac{\sigma - 1}{\theta}}_{:=\varsigma} \left(\widehat{w_t - p_t} \right) = \beta \mathbb{E}_t \hat{\pi}_{t+1}$$

- We can write it as

$$\hat{\pi}_t = \varsigma \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \underbrace{\widehat{p_{t+s} - w_{t+s}}}_{\text{mark up at } t+s} \right\}$$

- Combine with intertemporal condition

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{(\sigma - 1)(\varphi + \gamma)}{\theta} \underbrace{\hat{c}_t}_{\text{"output gap"}}$$

where

$$\text{"output gap"} = \hat{c}_t - \hat{c}_t^{flex}$$

is difference between consumption (or output) in sticky and flexible price world (recall from Exercise 1 that $c_t^{flex} = \overline{c}$)

3 eqns NK model

- Get rid of $\widehat{w_t - p_t}$ to get canonical 3 eqs NK model:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \varsigma \hat{c}_t$$

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\gamma} \left(\hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1} \right)$$

$$\hat{\imath}_t = \phi \hat{\pi}_t - \rho^t v_0$$

- This is a **linear** dynamic system in three unknowns $\{\hat{c}_t, \hat{\pi}_t, \hat{\imath}_t\}_t$ and one shock, v_0
  - if $v_0 = 0$ then $\{\hat{c}_t, \hat{\pi}_t, \hat{\imath}_t\}_t = \mathbf{0}$ is a solution
- Let's understand what happens if $v_0 \neq 0$

General case

- Make a guess that solution takes the form

$$\hat{\pi}_t = \psi_\pi \rho^t v_0,$$
$$\hat{c}_t = \psi_c \rho^t v_0.$$

- We get

$$\psi_\pi = \beta\rho\psi_\pi + \varsigma\psi_c,$$
$$\psi_c = \rho\psi_c - \frac{1}{\gamma}\left(\phi\psi_\pi - \rho\psi_\pi - \rho^t v_0\right)$$

- Solve it to get

$$\psi_c \frac{\left[(1-\rho)\,\gamma + (\phi - \rho)\right]\varsigma}{1-\beta\rho} = \rho^t v_0,$$
$$\psi_\pi = \frac{\varsigma}{1-\beta\rho}\psi_c.$$

- Assuming $\phi \geq \rho$, we get $\psi_c, \psi_\pi \geq 0$: unexpected cut in interest rates increases consumption and output, but also increases inflation