# Lecture 11

## Business Cycles and RBC model

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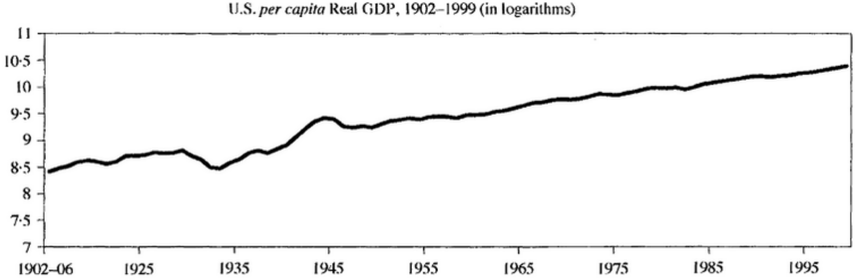
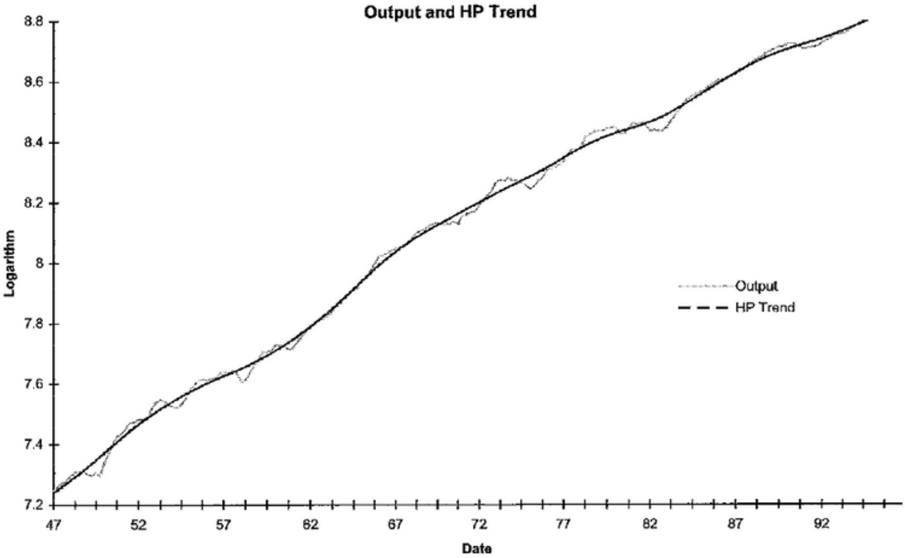
# Back to lecture 1

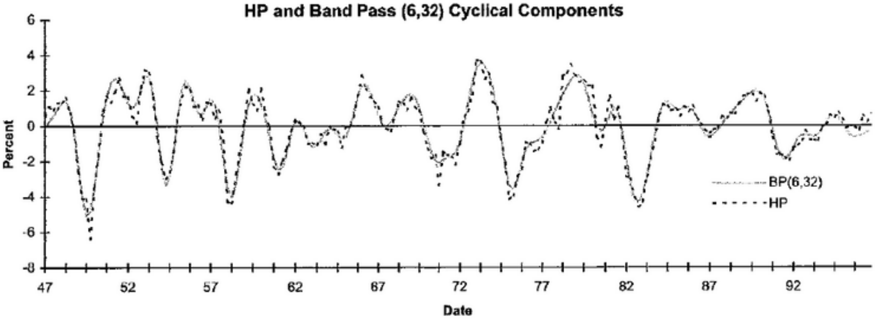



Sources: Historical Statistics of the U.S.; Economic Report of the President; and U.S. Census Bureau.

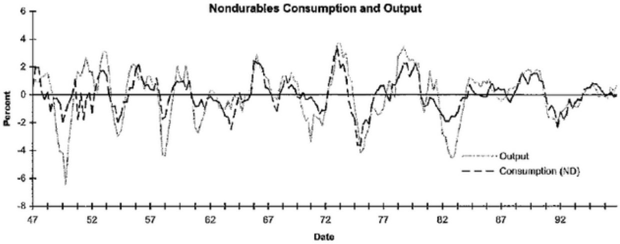
# Trends and cycles

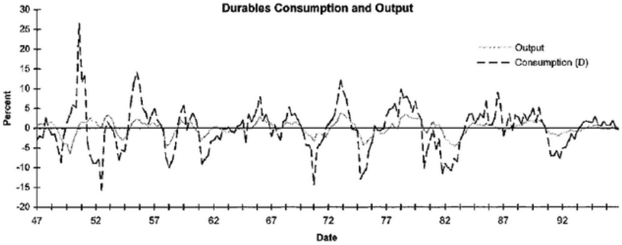


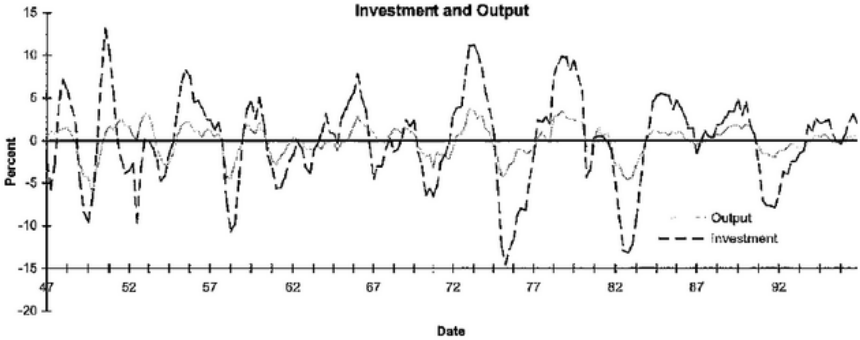
# GDP de-trended



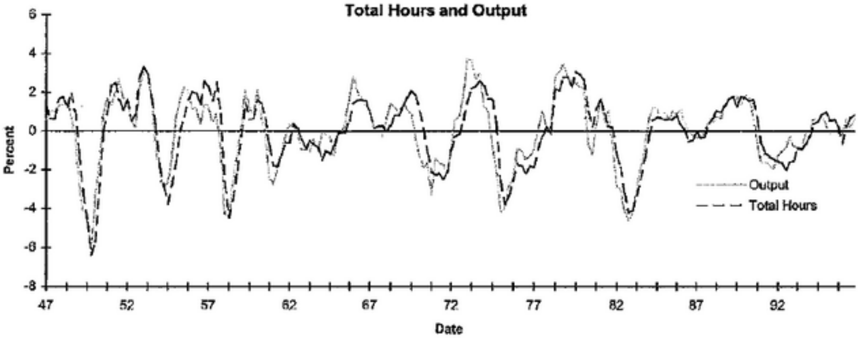
# Stylized facts



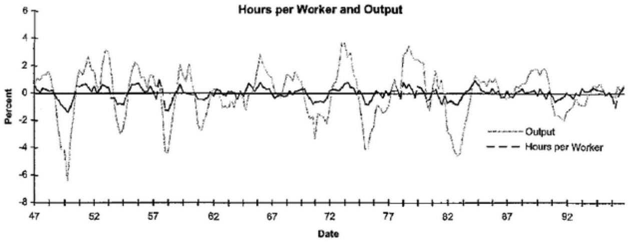



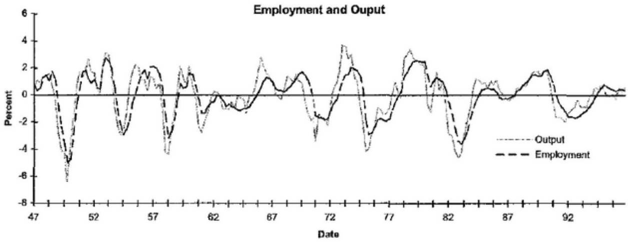

Investment and Output
15
10
5
0
-5
-10
-15
-20
Percent
47
52
57
62
67
72
77
82
87
92
Date
Output
Investment

# Stylized facts

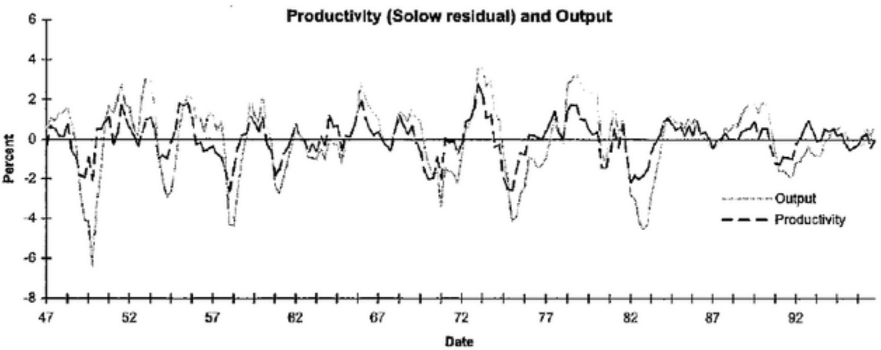


# Stylized facts



# Stylized facts



# Stylized facts

- Output is very correlated with TFP (Solow residual) (corr $\approx 0.8$)
- Output is more volatile than Solow residual
- Investments are much more volatile (x3) than output
- Consumption is much less volatile (x0.5) than output
- Total hours are roughly as volatile as out

## Real Business Cycle model

- Neoclassical growth model did a remarkably good job at explaining growth facts
- Can it also explain business cycle facts?
  - Kydland-Prescott (1982)
- Simplest first cut
  - growth model: we took only trend from Solow residual
  - RBC: let's feed fluctuations of Solow residual as well

# Key elements

- Representative consumer

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

- Representative firm,

$$Y_t = F_t(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Complete markets
- Feasibility

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t$$

- $A_t$ is (the only) exogenous stochastic process

## Some observations

- Let $X_t = (C_t, K_{t+1}, L_t)$ be endogenous variables
- Note that they are *random variables*: they depend on the realizations of $\{A_s\}_{s=0}^t$ shocks
- We need some formalism
  - let $s_t$ be exogenous state ($s_t = A_t$ in our RBC example with one shock, but more generally it is a vector of all exogenous shocks)
  - let $s^t = (s_0, ..., s_t)$ be a history of shocks up to period $t$
  - $\Pr(s^t)$ is the history of realizations
  - $\Pr(s^{t+1}|s^t) = \Pr(s^{t+1}) / \Pr(s^t)$ is the conditional probability
- Endogenous variables in period $t$ are functions of $s^t$: $X_t = X_t(s^t)$.

# Social planner problem

- In this notation, social planner's problem is

$$\max_{C_t, N_t, K_{t+1}} \sum_{t=0}^{\infty} \sum_{s^t} \Pr\left(s^t\right) \beta^t U\left(C_t\left(s^t\right), L_t\left(s^t\right)\right)$$

s.t. $K_0$ given and

$$C_t\left(s^t\right) + K_{t+1}\left(s^{t-1}\right) = A_t K_t\left(s^{t-1}\right)^{\alpha} L_t\left(s^t\right)^{1-\alpha} + (1-\delta) K_t\left(s^{t-1}\right)$$

- Let $\beta^t \Pr\left(s^t\right) \lambda_t\left(s^t\right)$ be the Lagrange multiplier on this constraint
- FOCs

$$\begin{aligned}
U_C\left(s^t\right) &= \lambda_t\left(s^t\right) \\
U_L\left(s^t\right) &= -\lambda_t\left(s^t\right) F_L\left(s^t\right) \\
\lambda_t\left(s^t\right) &= \beta \sum_{s^{t+1}} \Pr\left(s^{t+1} | s^t\right) \left(1 + F_K\left(s^{t+1}\right) - \delta\right) \lambda_{t+1}\left(s^{t+1}\right)
\end{aligned}$$

## Properties

- Combine FOCs

$$U_C\left(s^t\right) = -U_L\left(s^t\right) F_L\left(s^t\right)$$
$$U_C\left(s^t\right) = \beta \sum_{s^{t+1}} \Pr\left(s^{t+1}|s^t\right) U_C\left(s^{t+1}\right)\left(1 + F_K\left(s^{t+1}\right) - \delta\right)$$

- Together with feasibility (and TVC)

$$C_t\left(s^t\right) + K_{t+1}\left(s^{t-1}\right) = A_t K_t\left(s^{t-1}\right)^{\alpha} L_t\left(s^t\right)^{1-\alpha} + (1-\delta) K_t\left(s^{t-1}\right)$$

these three equations pin down (stochastic) solution $\left(C_t\left(s^t\right), K_{t+1}\left(s^t\right), L_t\left(s^t\right)\right)_{t,s^t}$

# Shorthand

- It is common to use shorthand notation for SP problem

$$\max_{C_t, L_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right)$$

s.t. $K_0$ given and

$$C_t + K_{t+1} = A_t K_t^{\alpha} L_t^{1-\alpha} + \left(1-\delta\right) K_t$$

- Short-hand FOCs

$$\begin{aligned} U_{C,t} &= -U_{L,t} F_{L,t}, \\ U_{C,t} &= \beta \mathbb{E}_t U_{C,t+1} \left(1 + F_{K,t+1} - \delta\right). \end{aligned}$$

- While short hand is more elegant and concise, it is easier to make mistakes than if one works with full, explicit form of the problem

# Recursive formulation

- While FOCs are insightful about optimality conditions, they are often hard to work with to fully "solve" the model, since everything is a random variable
- Writing problem recursively is very helpful in such situations
- Suppose $A_t$ is an AR(1) process. Then the SP problem can be written as

$$V(K,A) = \max_{C,L,K_+} U(C,L) + \sum_{A_+} \Pr(A_+|A)\, V(K_+,A_+)$$

s.t.

$$C + K_+ = AK^{\alpha}L^{1-\alpha} + (1-\delta)K$$

# Recursive formulation

- Value function $V$ can be found numerically via value function iteration
- Solution is policy functions $\widetilde{X}(K,A) = \widetilde{C}(K,A), \widetilde{N}(K,A), \widetilde{K}_+(K,A)$ and for any history of shocks $s^t = A^t$ we can find $X_t(s^t)$ by substituting these policy functions recursively, e.g.

$$X_0(s_0) = \widetilde{X}(K_0, A_0),$$
$$X_1(s_0, s_1) = \widetilde{X}\left(\widetilde{K}_+(K_0, A_0), A_1\right),$$
$$\ldots$$

- **Exercise 1**: Use the Bellman equation to derive the optimality conditions

$$U_{C,t} = -U_{L,t}F_{L,t},$$
$$U_{C,t} = \beta \mathbb{E}_t U_{C,t+1}\left(1 + F_{K,t+1} - \delta\right).$$

# Fit for "standard" calibration

Elasticity of labor supply $= 0.5$
all lower case variables are logs of upper cases

| **Volatility (rel. to output)** | | | **Autocorrelation** | | |
|---|---|---|---|---|---|
| *Stat* | *Data* | *RBC* | *Stat* | *Data* | *RBC* |
| $\sigma(y_t)$ | (1.62%) | (1.06%) | $\rho(y_t, y_{t-1})$ | 0.85 | 0.73 |
| $\sigma(c_t)$ | 0.53 | 0.42 | $\rho(c_t, c_{t-1})$ | 0.85 | 0.78 |
| $\sigma(i_t)$ | 2.87 | 2.78 | $\rho(i_t, i_{t-1})$ | 0.88 | 0.72 |
| $\sigma(n_t)$ | 1.12 | 0.31 | $\rho(n_t, n_{t-1})$ | 0.91 | 0.72 |
| **Corr. with Output** | | | **Corr. with TFP** | | |
| *Stat* | *Data* | *RBC* | *Stat* | *Data* | *RBC* |
| $\rho(c_t, y_t)$ | 0.79 | 0.94 | $\rho(c_t, z_t)$ | 0.59 | 0.93 |
| $\rho(i_t, y_t)$ | 0.77 | 0.99 | $\rho(i_t, z_t)$ | 0.44 | 0.99 |
| $\rho(n_t, y_t)$ | 0.87 | 0.98 | $\rho(n_t, z_t)$ | 0.49 | 0.98 |
| | | | $\rho(y_t, z_t)$ | 0.80 | 1.00 |

The fit looks great, except possibly for $\sigma(n_t) \Longrightarrow$ can be improved with higher labor supply elasticity

## Discussion

- With a single exogenous shock – Solow residual or TFP – the RBC model captures remarkably well fluctuations in a wide range of macro aggregates
- If this model is right, it also says that business cycle fluctuations are *efficient*: no scope for policy interventions

# Two issues with RBC models

- While it fits quantities very well, the fit for prices is terrible
- No scope common business cycle concerns and tools
  - monetary policy has no effect on real economy in RBC model while it has some effects in the data
  - no scope for any "inefficient" fluctuations (focus of the next lecture)

# Competitive equilibrium version

- We do the simplest version of competitive equilibrium
  - consumers own all capital
  - rent capital and labor to firms
  - trade risk-free bonds (a security that is bough in period $t$ at price $Q_t$ and pays 1 unit of consumption good in period $t+1$)
  - Risk-free bond is in zero net supply among themselves

# Competitive equilibrium

- Competitive equilibrium are stochastic sequences $\{C_t, K_{t+1}, L_t, B_t\}_t$ and prices $\{W_t, R_t, R_t^{rf}\}_t$ such that
- Consumers solve

$$\max_{C_t, L_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

s.t. $K_0$ given, $B_0 = 0$ and

$$C_t + K_{t+1} + Q_t B_{t+1} = W_t L_t + (1 + R_t - \delta) K_t + B_t$$

- Firms solve

$$\max_{K_t, L_t} A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t K_t$$

- Markets clear

$$B_t = 0$$

- **Exercise 2**: show that competitive equilibrium allocations $\{C_t, K_{t+1}, L_t\}_t$ satisfy the same three optimality conditions as the solution to the planning problem

## Risk-free interest rate

- Let risk-free interest rate $R_t^{rf}$ be defined by

$$R_t^{rf} := 1/Q_t - 1$$

- Note that saving in period $t$ one unit of consumption in
  - bonds pays interest rate $1 + R_t^{rf}$ in period $t+1$
  - capital pays interest rate $1 + R_{t+1} - \delta$ in period $t+1$
- **Excercise 3**: show that the two optimality conditions imply

$$U_{c,t} = \left(1 + R_t^{rf}\right) \beta \mathbb{E}_t U_{c,t+1},$$
$$U_{c,t} = \beta \mathbb{E}_t U_{c,t} \left(1 + R_{t+1} - \delta\right).$$

What is the intuition for why one interest rate is outside of the expectation operator but not the other?

Our RBC model did great matching quantities, but kind of terrible for matching prices

**Data sources** (1979 - 2006)

1. TFP: Solow residual $\implies \rho = 0.98, \sigma_\varepsilon = 0.0065$
2. $r_t$: return on 90-day T-bill, adjusted for realized inflation
3. $r_t^x$: market index from Ken French, adjusted for realized inflation
4. $w_t$: CPS from Haefke, Sonntag, and van Rens (2009)

**Summary statistics** (logged and HP-filtered)

| | $\sigma(r_t)$ | $\sigma(r_t^x)$ | $\sigma(w_t)/\sigma(y_t)$ | $\rho(r_t, z_t)$ | $\rho(r_t^x, z_t)$ | $\rho(w_t, z_t)$ |
|---|---|---|---|---|---|---|
| Data | 2.01% | 32.45% | 0.54 | -0.16 | -0.12 | 0.38 |
| RBC | 0.15% | 0.03% | 0.69 | 0.97 | 0.07 | 0.99 |

# Fit for "standard" calibration revisited

| **Volatility** | | |
|---|---|---|
| *Stat* | *Data* | *GE Model* |
| $\sigma(y_t)$ | 1.62% | 0.86% |
| $\sigma(c_t)/\sigma(y_t)$ | 0.53 | 0.45 |
| $\sigma(i_t)/\sigma(y_t)$ | 2.87 | 2.66 |
| $\sigma(r_t^{\mathrm{rf}})$ | 2.09% | 0.11% |
| $\sigma(rx_t)$ | 32.38% | 0.01% |

| **Autocorrelation** | | |
|---|---|---|
| *Stat* | *Data* | *Model* |
| $\rho(y_t, y_{t-1})$ | 0.85 | 0.74 |
| $\rho(c_t, c_{t-1})$ | 0.85 | 0.78 |
| $\rho(i_t, i_{t-1})$ | 0.88 | 0.73 |
| $\rho\left(r_t^{rf}, r_{t-1}^{rf}\right)$ | 0.40 | 0.73 |
| $\rho(rx_t, rx_{t-1})$ | 0.02 | 0.73 |

| **Cyclicality** | | |
|---|---|---|
| *Stat* | *Data* | *GE Model* |
| $\rho(c_t, y_t)$ | 0.79 | 0.97 |
| $\rho(i_t, y_t)$ | 0.77 | 0.99 |
| $\mathbb{E}[rx_t]$ | 2.67 | -0.00 |
| $\rho\left(r_t^{\mathrm{rf}}, y_t\right)$ | -0.18 | 0.97 |

| **Misc** | | |
|---|---|---|
| *Stat* | *Data* | *GE Model* |
| $\rho(z_t, y_t)$ | 0.80 | 1.00 |
| $\rho\left(z_t, r_t^{rf}\right)$ | -0.15 | 0.99 |
| $\rho(z_t, rx_t)$ | -0.08 | 1.00 |
| $\rho(rx_t, y_t)$ | -0.23 | 0.98 |

## Discussion

- Poor relationship between prices of financial assets and quantities is a puzzle that goes beyond RBC models or macro, and applies to finance field more generally
- Stylized facts:
  - prices of financial assets (e.g., individual stocks, S&P500, exchange rates, interest rates) are very volatile, not very correlated with either current or future quantity variables (e.g., dividends, GDP, consumption, inflation)
  - returns on many financial assets are also predictable $\Longrightarrow$ a risk-neutral / deep-pocket investor can make a fortune
- Active work in finance trying to understand these puzzling phenomena better