Macro 3 Notes

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1 Introduction

2 General Equilibrium Theory

2.1 Terms

- L: The commodity space.
  - Typically, $L = \mathbb{R}^m$ so that there are m goods
  - Sometimes, we let $m = \infty$
- $i = 1, 2, 3, \ldots, I$: The index of agents

- Thus, in the above description, there are I agents
- Sometimes we allow countability infinite agents, $i = 1, 2, 3, \ldots$, or allow a continuum of agents.

- $u^i : X^i \to \mathbb{R}$: Utility function of agent i
- $X^i \subset L$: The consumption possibility set of agent i
  - That is, a household i is able to consume any vector $x \in X^i$.
  - This set may include constraints, such as non-negativity of consumption.
  - Positive entries in x denote quantity demanded; negative entries denote quantity supplied/offered/sold.
- $j = 1, 2, \ldots, J$: The index of firms
  - Thus, in the above description, there are J firms
  - Sometimes we allow countability infinite firms, $j = 1, 2, 3, \ldots$, or allow a continuum of firms.
- $Y^j \subset L$: The production possibility set or technology of firm j
  - $y \in Y^j$: The firm j is able to produce any vector $y \in Y^j$.
  - Positive entries in y denote quantity supplied/sales; negative entries denote quantity demanded/purchased by the firm.
  - As an example, consider the Cobb-Douglas production function:

$$F^j\left(k^j, \ell^j\right) = \left(k^j\right)^{\alpha}\left(\ell^j\right)^{1-\alpha}$$

    could be expressed in terms of Y^j as:

$$\mathbf{Y}^j = \left\{\mathbf{y} = \left(y^j, -k^j, -\ell^j\right) \in \mathbb{R}^3 : y^j = \left(k^j\right)^{\alpha}\left(\ell^j\right)^{1-\alpha}, \left\{y^j, k^j, \ell^j\right\} \mathbb{R}^3_+\right\}$$

- $\theta^i_j \geq 0$: Consumer i's ownership share of firm j.

$$\sum_{i \in I} \theta^i_j = 1 \text{ for all } j \in J$$

- $e^i \in L$: The endowment of agent i.
- p: price vector
  - Supposing m commodities (i.e., $L = \mathbb{R}^m$),

$$p = (p_1, \ldots, p_m)$$

2.2 Feasible Allocations

Definition D.1: Feasible Allocation

A feasible allocation, $\{x^i, y^j\}$, satisfies three conditions:

1. Each consumer can consume x^i,
$$x^i \in X^i, \forall i \in I$$
2. Each firm can produce y^j,
$$y^j \in Y^j, \forall j \in J$$
3. Demand equals supply:
$$\sum_{i \in I} x^i = \sum_{j \in J} y^j + \sum_{i \in I} e^i$$

2.3 Key Concepts for Competitive Equilibrium

Suppose there are m commodities (i.e., $L = \mathbb{R}^m$).

Then the price vector p is given by:

$$p = (p_1, \ldots, p_m)$$

and we have the following equation:

$$p \cdot x = p_1 x_1 + p_2 x_2 + \cdots + p_m x_m \equiv \sum_{s=1}^{m} p_s x_s$$

$p \cdot x$ is the value of the commodity vector x in terms of the numeraire[1]

A quick useful quality to note:

$$p \cdot x^a + p \cdot x^b = p \cdot \left(x^a + x^b\right)$$

[1] By "in terms of the numeraire," I just mean that we are normalizing the price of some good (usually the first element in x) to be 1 and then expressing the price of the other goods relative to that one.

2.4 Competitive Equilibrium

Definition D.2: Competitive Equilibrium

We denote prices by a vector $p \in R^m$.

A competitive equilibrium is a price vector p, and a feasible allocation $\left\{x^i, y^j\right\}$ such that

1. each firm $j \in J$ maximize its profits. We denote the profit of firm j by π^j.

2.5 Welfare Theorems

Definition D.3: Local Non-Satiation

Utility function $u^i : \mathbf{X}^i \to \mathbb{R}$ satisfies the local non-satiation (LNS) property if, for any $\mathbf{x} \in \mathbf{X}^i$ and any neighborhood of $\mathbf{x}$, denote $B_\varepsilon(\mathbf{x})$, there exists $\hat{\mathbf{x}} \in B_\varepsilon(\mathbf{x})$ such that $u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x})$.

Theorem T.1: First Welfare Theorem

Suppose that u^i satisfies local non-satiation for all $i \in \mathbf{I}$. Let $\left\{\mathbf{p}, \overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ be a competitive equilibrium. Then, $\left\{\overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ is a Pareto optimal allocation.

Definition D.4: Convexity

A function $u^i : \mathbf{X}^i \to \mathbb{R}$ is strictly convex if, for any $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^i$ such that $\mathbf{x}' \neq \mathbf{x}$, we have

$$u^i\left(\theta \mathbf{x} + (1-\theta)\mathbf{x}'\right) > \theta u^i(\mathbf{x}) + (1-\theta) u^i\left(\mathbf{x}'\right), \forall \theta \in (0,1).$$

Definition D.5: Quasiconcavity

A function $u^i : \mathbf{X}^i \to \mathbb{R}$ is (strictly) quasiconcave if its upper contour set $\left\{x \in \mathbf{X}^i : u^i(\mathbf{x}) \geq u^i(\overline{\mathbf{x}})\right\}$ is (strictly) convex for all $i \in \mathbf{I}$ and all $\overline{\mathbf{x}} \in \mathbf{X}^i$. Equivalently, for any $\mathbf{x} \neq \overline{\mathbf{x}}$, we must have

$$u^i(\alpha \mathbf{x} + (1-\alpha)\tilde{\mathbf{x}}) > \min\left\{u^i(\mathbf{x}), u^i(\overline{\mathbf{x}})\right\}, \forall \alpha \in (0,1)$$

Theorem T.2: Second Welfare Theorem

Assumption 1. (Assumption HH) Assume that $\mathbf{X}^i$ are convex for all $i \in \mathbf{I}$ and that $u^i : \mathbf{X}^i \to \mathbb{R}$ are continuous and strictly quasiconcave.

Assumption 2. (Assumption FF) Assume that the aggregate production sumset of the economy is convex; i.e.

$$\mathbf{Y} := \left\{\mathbf{y} \in \mathbf{L} : \mathbf{y} = \sum_{j=1}^{J} \mathbf{y}^j, \mathbf{y}^j \in \mathbf{Y}^j, \forall j \in \mathbf{J}\right\}.$$

Let $\left\{\overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ be a Pareto optimal allocation. Then, there exists a price vector $\mathbf{p}$ such that: (i) all firms maximise profits such that, for all $j \in \mathbf{J}$,

$$\mathbf{p} \cdot \overline{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}, \forall \mathbf{y} \in \mathbf{Y}^j$$

(ii) given allocation $\left\{\overline{\mathbf{x}}^i\right\}$, consumers minimise expenditure subject to attaining at least the same utility obtained by consuming $\overline{\mathbf{x}}^i$; i.e.

$$\overline{\mathbf{x}}^i \in \underset{\mathbf{x} \in \mathbf{X}^i}{\arg\min} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u^i(\mathbf{x}) \geq u^i\left(\overline{\mathbf{x}}^i\right).$$

3 Aggregation

3.1 Terms

- U: The set of utilities that are achievable for a feasible allocation

$$U = \left\{u \in R^I : u_i \leq u^i\left(x^i\right) \text{ all } i \in I \text{ for some feasible allocation } \left\{x^i, y^j\right\}\right\}$$

4 Overlapping Generations Economy

4.1 Terms

Baseline Model

- x_i: The set of commodities consumed in period i
- $t = 1, 2, \ldots$: Index for time
- $I = \{0, 1, 2, \ldots\}$: Generation of agents
- v^i: Utility for agents born in generation $i \geq 1$ as a function of only the commodities in their periods they are alive
- u^i: Utility of agent i
  - If $i \geq 1$

$$u^i\left(x_1, x_2, x_3, \ldots\right) = v^i\left(x_i, x_{i+1}\right)$$

  - If $i = 0$

$$u^0\left(x_1, x_2, x_3, \ldots\right) = x_1^0$$

- $e^i = \left(e_1^i, e_2^i, \ldots, e_i^i, e_{i+1}^i, \ldots\right)$: The endowment of agents born in generation i
  - $e_i^i > 0$
  - $e_{i+1}^i > 0$

- $e_t^i = 0$ for all $t \neq i$ or $t \neq i+1$
- $e_1^0 > 0$ and $e_t^0 = 0$ for all $t = 2, 3 \ldots$

- $p = (p_1, p_2, p_3, \ldots)$: Price vector (across generations)
- r_t: The time t interest rate satisfying

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

- c: Consumption
  - c_y: Consumption while young
  - c_o: Consumption while old
  - c_i^i: Consumption for generation i in period i
  - c_{i+1}^i: Consumption for generation i in period $i+1$
  - $\bar{c}_i^j$: Equilibrium consumption for generation i in period j
  - c_i^{*j}: Best symmetric allocation consumption for generation i in period j

Model with the Below Utility Function

$$v^i(c_y, c_o) = (1-\beta)\log c_y + \beta \log c_o$$

for some constant $\beta \in (0,1)$.

- $\beta \in (0,1)$: Weight placed on consumption while old
  - $(1-\beta)$: Weight placed on consumption while young
- $\alpha \in (0,1)$: Share of total lifetime endowment received while old
  - $(1-\alpha)$: Share of total lifetime endowment received while young
- $\bar{e}_t = 1$: Total lifetime endowment for all $t \geq 1$

Terms Added for Social Security

- τ: Taxes on young paid to old

Terms Added in Growing Economy

- N_t: Number of young agents at time t
  - $N_0 = 1$
- n: Growth rate of population

- g: Growth rate of productivity of the endowments. That is, we have

$$e_{t+1}^{t+1} = (1+g)e_t^t$$
$$e_{t+2}^{t+1} = (1+g)e_{t+1}^t$$

  and

$$e_t^t = (1+g)^t(1-\alpha)$$
$$e_{t+1}^t = (1+g)^t\alpha$$

4.2 Pure Exchange Economy

4.2.1 Competitive Equilibrium

The price vector p is an element of R^∞, so that

$$p = (p_1, p_2, p_3, \ldots)$$

The agent problem is

$$\max_x u^i(x)$$

subject to

$$px \leq pe^i$$

which, since generation i neither consume nor has endowments at time $t \neq i$ or $t \neq i+1$, can be specialized as

$$\max_{x_i, x_{i+1}} v^i\left(x_i, x_{i+1}\right)$$

subject to

$$p_i x_i + p_{i+1} x_{i+1} = p_i e_i^i + p_{i+1} e_{i+1}^i$$

and for generation $i = 0$ as

$$\max_{x_1} x_1 \text{ subject to } p_1 x_1 = p_1 e_1^0.$$

4.2.2 No Trade

Proposition P.1

The only competitive equilibrium has

$$x^i = e^i$$

i.e. there is no trade in equilibrium.

4.2.3 Equilibrium Prices

Normalize

$$p_1 = 1$$

We have:

$$\frac{p_{i+1}}{p_i} = \frac{v_2^i\left(e_i^i, e_{i+1}^i\right)}{v_1^i\left(e_i^i, e_{i+1}^i\right)}$$

for all $i \geq 1$

With r_t net interest rate,

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

for all t. From our previous condition we have

$$r_t = \frac{v_1^t\left(e_t^t, e_{i+1}^t\right)}{v_2^t\left(e_t^t, e_{t+1}^t\right)} - 1$$

for all $t > 1$ and

$$p_t = \frac{1}{(1+r_1)(1+r_2)\cdots(1+r_{t-1})}.$$

4.2.4 Equilibrium Prices Under Specified Utility Function

Taking utility function

$$v^i\left(c_y, c_0\right) = (1-\beta)\log c_y + \beta \log c_0$$

we have:

$$r_t \equiv \bar{r} = \frac{(1-\beta)}{\beta}\frac{\alpha}{1-\alpha} - 1 = \frac{\alpha-\beta}{\beta(1-\alpha)}$$

or

$$p_t = \left[\frac{\beta}{(1-\beta)}\frac{1-\alpha}{\alpha}\right]^{t-1} \quad \text{for } t \geq 1$$

4.2.5 Best Symmetric Allocation

We will solve for the best feasible symmetric allocation, where best is for the point of view of the young. In particular, consider the problem

$$\max_{c_y,c_o} v\left(c_y, c_o\right) = \max_{c_y,c_o}(1-\beta)\log c_y + \beta \log c_o$$

subject to

$$c_y + c_o = 1$$

Its sufficient first order condition is given by

$$\frac{\beta}{1-\beta}\frac{c_y}{c_0} = 1$$

so the solution of this f.o.c. that also is feasible, i.e. the solution of the problem is

$$c_y = 1-\beta, c_o = \beta.$$

The best symmetric allocation depends on β in this way because for higher preference parameter β agents give less weight to consumption when young and more weight to consumption when old.

4.2.6 Comparison of CE and Best Symmetric Allocation

We will compare the utility of the unique competitive equilibrium allocation

$$\bar{c}_i^i = 1-\alpha, \quad \bar{c}_{i+1}^i = \alpha \text{ for } i \geq 1 \text{ and } \bar{c}_1^0 = \alpha$$

with the one for the best symmetric allocation

$$c_i^{*i} = 1-\beta, c_{i+1}^{*i} = \beta \text{ for } i \geq 1 \text{ and } c_1^{*0} = \beta$$

Notice that, since the CE allocation has $x^i = e^i$, and since that allocation is a feasible symmetric allocation, then, unless $c^* = \bar{c}$-which happens only when $\alpha = \beta-$ the best symmetric feasible allocation is strictly preferred by the agents of generations $i = 1, 2, \ldots$. It only remains to compare the utility of the initial old, i.e. generation $i = 0$, between the best symmetric and CE allocations.

- Case 1: $\beta > \alpha$: All generations prefer Best Symmetric Allocation
- Case 2: $\beta = \alpha$: Indifferent between CE and Best Symmetric Allocation
- Case 3: $\beta < \alpha$: Original Generation prefers CE

4.3 Social Security

4.3.1 After-Tax Endowments

$$
\begin{aligned}
&e_i^i = (1-\alpha) - \tau \text{ and } e_{i+1}^i = \alpha + \tau \text{ for all } i \geq 1 \\
&e_1^0 = \alpha + \tau
\end{aligned}
$$

Notice that by suitable choice of τ we can make the after-tax endowments equal to the best symmetric allocations, the required τ is

$$\tau = \beta - \alpha$$

4.4 Growing Economy

4.4.1 Setup

We will now consider an economy with population and productivity growth. Let N_t the number of young agents at time t. Let n be the growth rate of population, so that

$$N_{t+1} = (1+n)N_t \text{ for } t \geq 1 \text{ and } N_0 = 1.$$

Let g denote the growth rate of productivity of the endowments of each cohort, so that

$$e_{t+1}^{t+1} = (1+g)e_t^t \text{ and } e_{t+2}^{t+1} = (1+g)e_{t+1}^t$$

so that

$$
\begin{aligned}
e_t^t &= (1+g)^t(1-\alpha) \\
e_{t+1}^t &= (1+g)^t\alpha
\end{aligned}
$$

for all $t \geq 1$.

4.4.2 Feasible Symmetric Allocation

Define the feasible symmetric allocations as those solving

$$N_t c_y^t + N_{t-1} c_o^t = N_t(1-\alpha)(1+g)^t + N_{t-1}\alpha(1+g)^{t-1}$$

where each agent born at time t and young at t consumes

$$c_y^t = \hat{c}_y(1+g)^t,$$

and each agent born at time $t-1$ and old at t consumes

$$c_o^t = \hat{c}_o(1+g)^{t-1}.$$

Notice that this constraint can be written as

$$\hat{c}_y(1+g)(1+n) + \hat{c}_o = (1-\alpha)(1+g)(1+n) + \alpha$$

5 OLG Perpetual Youth Model

5.1 Terms

Baseline Model

- dt: An amount of time
- $p\ dt$: The probability of agent dying in dt
  - $p \in (0, \infty)$
  - $\frac{1}{p}$: Expected lifetime
- $N(s,t)$: Size of cohort born at time s at time t
  - $N(s, t + \Delta) = N(s,t)(1 - p\Delta)$: The size of the cohort born at time s after Δ amount of time is the size of the cohort at time t times the probability of not dying in Δ amount of time
  - $N(s,s) = p$
  - $N(s,t) = pe^{-p(t-s)}$: Expression for the size of the cohort born at time s at time t [2]
- r: Net risk-less interest rate
- v: An investment in period t that pays $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t + \Delta$, and zero if dead.
- $\theta \in (0, \infty)$: Discount rate
  - 1 util at time $t + \Delta$ is worth $\frac{1}{1+\Delta\theta}$ at t.
- z: We use z in the expected utility function as the future time that we integrate over
- $R(t, z)$: price of a good at time z in terms of goods in time t
- $v(t)$: Non-human (financial) wealth at time t
- $y(z)$: Labor income at time z
- $h(t)$: human wealth at time t

$$h(t) = \int_t^\infty y(z)R(t,z)dz$$

- $c(t)$: Consumption at time t
- $pv(t)$: Rate at which insurance company receives payment from those who die, as well as rate that it pays out premia.

Aggregation Section Added Terms

- $Y(t)$: The aggregate labor income of agents alive at time t

[2] Is p playing two roles here or are they connected? Seems they must be connected or the notation would be crazy.

- α: The rate at which the share of $Y(t)$ endowed to generation s falls as s increases
- $y(s,t)$: The labor income in time t of a living agent born at time s

$$y(s,t) = \frac{p+\alpha}{p} Y(t) e^{-\alpha(t-s)}, \quad \text{for } \alpha \geq 0$$

  - Notice that at any point in time t, the share of $Y(t)$ endowed to generation s is falling at the rate α as s increase

- $C(t)$: Aggregate consumption at time t

$$C(t) \equiv \int_{-\infty}^{t} N(s,t) c(s,t) ds$$

  and

$$C(t) = (p+\theta)(H(t) + V(t))$$

- $c(s,t)$: Share of consumption at time t of a living agent born at time s

$$c(s,t) = (p+\theta)(h(s,t) + v(s,t))$$

  aka

$$\frac{dc(s,t)}{dt} = [r(t) - \theta] c(s,t) \quad \text{BC} + \text{ and boundary condition}$$

- $h(t,s)$: Human wealth at time t of a living agent born at time s

$$\begin{aligned} h(t,s) &= \int_{t}^{\infty} aY(z) e^{-\alpha(z-s)} R(t,z) dz \\ &= a \left[\int_{t}^{\infty} Y(z) e^{-\alpha(z-t)} R(t,z) dz \right] e^{-\alpha(t-s)} \end{aligned}$$

- $H(t)$: Aggregate human wealth at time t

$$H(t) \equiv \int_{-\infty}^{t} N(s,t) h(s,t) ds$$

- $V(t)$: Aggregate non-human wealth at time t

$$V(t) \equiv \int_{-\infty}^{t} N(s,t) v(s,t) ds$$

General Equilibrium Section Added Terms

- $F(K) = \mathbb{F}(K,1) - \delta K$: CRTS neoclassical production function
- K: Capital
- δ: Depreciation rate

- $V(t)$ Non-human wealth
  - In one case, we define: $V(t) = K(t)$
  - If we then add government debt, we define: $V(t) = K(t) + B(t)$
- $Y(t)$: Is this labor income or output in this case?
  - $Y(t) = F(K(t)) - F'(K(t))K(t)$
  - $r(t) = F'(K(t))$
- $B(t)$: Government debt

5.2 Setup

Agents that die replaced by newborns. Thus, adding all cohort alive at time t yields:

$$\int_{-\infty}^{t} N(s,t)ds = \int_{-\infty}^{t} pe^{-p(t-s)}ds = 1.$$

5.3 Insurance, Annuities

Invest v at t, gets $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t+\Delta$, and zero if dead.

Continuous time (as $\Delta \downarrow 0$) : $v\frac{1+\Delta r}{1-p\Delta} = v + v(r+p)\Delta + o(\Delta)$

5.4 Household Problem

$$\max \mathbb{E}\left[\int_{t}^{\infty} u(c(z))e^{-\theta(z-t)}dz\right] = \int_{t}^{\infty} \log(c(z))e^{-(p+\theta)(z-t)}dz$$

subject to

$$\int_{t}^{\infty} [c(z) - y(z)]R(t,z)dz = v(t)$$

5.5 Budget Constraint and Human Wealth

The individual's dynamic budget constraint is:

$$\frac{dv(t)}{dt} = (r(t)+p)v(t) + y(t) - c(t)$$

We also have a no-Ponizi-game (NPG) condition:

$$\lim_{z\to\infty} v(z)R(t,z) = 0$$

The price of a good in time z in terms of goods in time t is given by:

$$R(t,z) := \exp\left[-\int_t^z (r(\mu)+p)d\mu\right]$$

We can also integrate the dynamic budget constraint to get the intertemporal budget constraint:

$$v(t) = \int_t^\infty [c(z)-y(z)]R(t,z)dz$$

We define human wealth as the present value of all future income, i.e.,:

$$h(t) = \int_t^\infty y(z)R(t,z)dz$$

with the boundary condition

$$\lim_{z\to\infty} h(z)R(t,z) = 0$$

which is equivalent to:

$$\frac{dh(z)}{dz} = [r(z)+p]h(z) - y(z) \text{ with } \lim_{z\to\infty} R(t,z)h(z) = 0$$

5.6 Optimal Consumption

We find that the solution to our problem is:

$$c(t) = (\theta+p)(v(t)+h(t))$$

The law of motion (or, Euler equation) for consumption is:

$$\frac{dc(t)}{dt} = (r(t)-\theta)c(t)$$

Notes

A few miscellaneous points from Tak's notes:

- We will later find that in the steady state, the interest rate must be higher than our discount rate, i.e., $r > \theta$. We could re-frame this as the interest rate must be higher than our impatience, hence savings will accumulate over time.
- Consumption is independent of the interest rate. This is due to the assumption of log utility, which implies that the substitution and income effect from changes in the interest rate exactly offset each other.

5.7 Aggregation

5.7.1 Distribution of Labor Income

Given aggregate labor income at the time t, $Y(t)$, we have:

$$y(s,t) = \frac{\alpha + p}{p} Y(t) e^{-\alpha(t-s)}, \alpha \geq 0$$

as the expression for the labor income in time t of a living agent born at time s.

This means that, at any particular point in time t, the share of $Y(t)$ endowed to generation s falls at the rate α as s increases.

5.7.2 Aggregate Human Wealth

The aggregate human wealth is defined as

$$\begin{aligned}
H(t) &:= \int_{-\infty}^{t} h(s,t) N(s,t) ds \\
&= \int_{-\infty}^{t} h(s,t) p e^{-p(t-s)} ds \\
&= \int_{-\infty}^{t} ap \left[\int_{t}^{\infty} Y(z) e^{-\alpha(z-t)} R(t,z) dz \right] e^{-\alpha(t-s)} e^{-p(t-s)} ds \\
&= \int_{t}^{\infty} Y(z) \exp\left\{ - \int_{t}^{z} (\alpha + p + r(\mu)) d\mu \right\} dz
\end{aligned}$$

We can then get:

$$\frac{dH(t)}{dt} = (r(t) + p + \alpha) H(t) - Y(t)$$

5.7.3 Aggregate Non-Human Wealth

The aggregate non-human wealth is defined as

$$\begin{aligned} V(t) &:= \int_{-\infty}^{t} N(s,t)v(s,t)ds \\ &= \int_{-\infty}^{t} v(s,t)pe^{-p(t-s)}ds \end{aligned}$$

From which we can get:

$$\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t)$$

5.7.4 Aggregate Consumption

Thus, aggregate consumption is defined as

$$C(t) := \int_{-\infty}^{t} N(s,t)c(s,t)ds.$$

We may also aggregate consumption as

$$C(t) = (p+\theta)(H(t)+V(t)).$$

We can then get:

$$\frac{dC(t)}{dt} = (r(t)+\alpha-\theta)C(t) - (p+\theta)(p+\alpha)V(t)$$

5.7.5 Aggregation Summary

We have now derived the dynamics that describe the aggregate behaviour in the Perpetual Youth Model:

$$\begin{aligned} &\frac{dH(t)}{dt} = (r(t)+p+\alpha)H(t) - Y(t) \\ &\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t) \\ &\frac{dC(t)}{dt} = (r(t)+\alpha-\theta)C(t) - (p+\theta)(p+\alpha)V(t) \\ &C(t) = (p+\theta)(H(t)+V(t)) \end{aligned}$$

We also need a no-Ponzi-game condition, which ensures that agents have finite wealth; i.e.

$$\lim_{T\to\infty} Y(t)\exp\left[-\int_t^T (r(z)+\alpha+p)dz\right] = 0.$$

5.8 Pure Endowment

5.8.1 Labor Income and Consumption

In a pure endowment economy, we have:

$$Y(t) = C(t) = Y$$

5.8.2 Non-human Wealth

In aggregate, we have: $V(t) = 0$, since some agents borrow and others lend:

$$V(t) = \int_{-\infty}^{t} N(s,t)v(s,t)ds = 0.$$

$v(s,t)$ value of cumulated savings (net assets) of cohort born at s at t.

5.8.3 Equilibrium Interest Rate

Equilibrium interest rate $r(t) = \theta - \alpha$

5.8.4 Individual Consumption

$$\frac{dc(s,t)}{dt} = [r(t)-\theta]c(s,t) = -\alpha c(s,t)$$

5.8.5 Equilibrium Outcome

Thus equilibrium is autarky! $c(s,t) = y(s,t)$ and $v(s,t) = 0$

5.8.6 Results Summary

$$
\begin{aligned}
Y(t) &= C(t) = Y, \\
V(t) &= 0 \\
r(t) &= \theta - \alpha, \\
c(s,t) &= y(s,t), \forall t \geq s, \\
v(s,t) &= 0, \forall t \geq s.
\end{aligned}
$$

5.9 Capital Accumulation, Technology

6 Uncertainty

6.1 Terms

- s: State of the economy
- L: Commodity space
- m_1: Number of different values that the state of the economy can take
- m_2: Number of physically different goods
  - r: index of the good
- $m = m_1 \times m_2$: A commodity is indexed by its physical attributes and the state, so this is the number of commodities under that characterization
  - $L = \mathbb{R}^m$
- x: Consumption of each of the physically different goods in each of the states.
  - x_{sr}: Consumption of good r in state s
- i: Index for agents
- e^i: Endowment of agent i, indexed by state s and physical characteristic r
  - e^i_{sr}: Endowment of good r in state s
  - $\bar{e}$: Aggregate endowment of the economy
- Y^j: Production possibility set of each firm
  - $Y^j \subseteq \mathbb{R}^m$
- $u^i : \mathbb{R}^m \to \mathbb{R}$: Utility function of agent i

$$u^i(x) = \sum_{s=1}^{m_1} v^i \left(x_{s1}, x_{s2}, \ldots, x_{sm_2}\right) \pi^i_s$$

- $v^i : R^{m_2} \to R$: Sub-utility function of agent i that only depends on the physical characteristics of the goods

- $\pi^i \in R_+^{m_1}$: Agent i's subjective probabilities over the states of the economy
  - $\sum_{s=1}^{m_1} \pi_s^i = 1$
  - π_s: Common probability of state s
- $(x_{s1}, x_{s2}, \ldots, x_{sm_2})$: Consumption bundle in state s. Can be thought of as a random variable with m_1 possible realizations.
- λ: Vector of weights
- g^i: A strictly increasing function of the aggregate endowment used to characterize a Pareto optimal allocation

$$x_s^i = g^i\left(\bar{e}_s\right) \text{ for all } i \in I$$

- p_s: The state price for state s, i.e., the price of a security that pays one unit of the numeraire in state s and zero otherwise.

Security Market Added Terms

- d_{ks}: The payoff of security k in state s
- K: The number of securities
- q_k: The price of the security k
- h_k^i: The purchases of this security by agent i
- θ_k^i: The endowment of this security by agent i
- D: The matrix with the payoffs of the K securities in the m states
  - $D = \{d_{ks}\}_{k=1,\ldots,K,s=1,\ldots,m}$

Tilde-Economy Added Terms

- $\tilde{u}^i$: Utility function of agent i in the tilde-economy
- $\tilde{x}_k^i$: The number of shares of bought or sold of security k by agent i

Asset Pricing Added Terms

- r_k: The expected gross return of a security k

$$1 + r_k := \frac{\sum_{s=1}^{m} d_{ks}\pi_s}{q_k} = \frac{\mathbb{E}\left[d_k\right]}{q_k}$$

- RP_k: The risk premium of security k[3]

$$RP_k := \frac{1 + r_k}{1 + r_1}$$

[3] Are we saying this is normalized against security 1?

6.2 Introducing Risk Under One Physically Different Good

Suppose there is only one physically different good. Then:

$$u^i(\mathbf{x}) = \sum_{s=1}^{m} v^i(x_s)\,\pi_s^i$$

Definition D.6: Risk Averse

We say that u^i is risk averse if:

$$v^i\left(\sum_{s=1}^{m} x_s \pi_s^i\right) > \sum_{s=1}^{m} v^i(x_s)\,\pi_s^i$$

for any random variable x.

Notice that if $m = 2$, this coincides with the definition of v being strictly concave and the assumption that x, as a random variable, is not degenerate.

Intuition: Logical that risk aversion corresponds to utility over consumption across several possible states of the world being in some sense concave:

Theorem T.3

Fix an arbitrary vector of λ-weights. The corresponding Pareto optimal allocation can be described by a set of strictly increasing functions g^i of the aggregate endowment, i.e. the optimal allocation can be written as

$$x_s^i = g^i(\bar{e}_s) \text{ for all } i \in I$$

Remark R.1

Now we show that the g^i functions are strictly increasing. Consider two states s and s' with

$$\bar{e}_s > \bar{e}_{s'}$$

It must be that for at least some i,

$$x_s^i > x_{s'}^i.$$

Remark R.2

Remark: CE and State Prices. In a CE the budget constraint of agent i is

$$\sum_{s=1}^{m} p_s x_s^i = \sum_{s=1}^{m} p_s e_s^i$$

where p_s are also referred to as state prices, the price of a security that pays one unit of the numeraire in state s and zero otherwise. Thus, agents can buy consumption contingent on the state, and they finance that by selling their endowment contingent on the state.

6.2.1 State Prices

Using the foc of agent i we obtain that in an equilibrium (or in its corresponding Pareto problem)

$$p_s = \frac{\partial v^i\left(x_s^i\right)}{\partial x}\pi_s/\mu_i = \lambda_i \frac{\partial v^i\left(g^i\left(\bar{e}_s\right)\right)}{\partial x}\pi_s$$

so that the state prices reflect the probability that the state s be realized as well as the scarcity of the aggregate endowment in state s.

The state prices are lower if the probability is small or if the aggregate endowment in that state is large, so that goods are relatively plentiful, and hence its marginal value relatively smaller.

6.3 Security Markets

We now consider a securities market. We assume that before the state s is realized agents trade in competitive markets where they buy and sell securities $k = 1, \dots, K$.

6.3.1 Value of Purchases

The first equation is given by

$$\sum_{k=1}^{K} h_k^i q_k = \sum_{k=1}^{K} \theta_k^i q_k$$

This says that the value of purchases is limited by the value of the sales of the securities.

6.3.2 Equation for each State

The second equation, indeed one for each state s, is given by

$$x_s^i = \sum_{k=1}^{K} h_k^i d_{ks} + \hat{e}_s^i$$

for each $s = 1, \dots, m$, which says that consumption in each state s must equal purchases plus endowment.

6.3.3 Summarizing Equations

6.3.3.1 Budget Constraints

$$\sum_{k=1}^{K} h_k^i q_k = \sum_{k=1}^{K} \theta_k^i q_k$$
$$x_s^i = \sum_{k=1}^{K} h_k^i d_{ks} + \hat{e}_s^i \text{ for each } s = 1, \ldots, m$$

6.3.3.2 Market Clearing

$$\sum_{i=1}^{I} h_k^i = \sum_{i=1}^{I} \theta_k^i \text{ for each } k = 1, 2, \ldots K$$
$$\sum_{i=1}^{I} x_s^i = \sum_{i=1}^{I} \left[\hat{e}_s^i + \left(\sum_{k=1}^{K} d_{ks} \theta_k^i \right) \right] \text{ for each } s = 1, 2 \ldots, m$$

6.3.4 More stuff

Definition D.7: Consistent with State Price

We will say that security prices q and payoffs D are consistent with state price p, if

$$q_k = \sum_{s=1}^{m} p_s d_{ks} \quad \text{for all securities } k = 1, \ldots, K.$$

Definition D.8: Equivalent Endowments

The endowment e^i and $\left(\hat{e}^i, \theta^i\right)$ are equivalent if

$$\hat{e}_s^i + \sum_{k=1}^{K} d_{ks} \theta_k^i = e_s^i$$

for all states $s = 1, 2 \ldots, m$.

Proposition P.2

Assume that prices q and payoffs D are consistent with state prices p, as in ($q =$ pv using AD prices). Assume that the endowments are equivalent as in (equivalent endowments). Then:

1. If (x, h) is budget feasible in the security market economy, then x is budget feasible in the A-D economy.

2. If x is budget feasible in the A-D economy, then it must be budget feasible in the security market economy, provided that D has full rank.

Proposition P.3

Assume that the endowments e^i and $\left(\hat{e}^i, \theta^i\right)$ are equivalent, then

1. If $\left(x^i, h^i\right)$ clears the markets in the security market economy, then $\left(x^i\right)$ clears the markets in the A-D economy.
2. Assume also that D has full rank. If $\left(x^i\right)$ clears the markets in the A-D economy, the $\left(x^i, h^i\right)$ clears the markets in the security market economy.

6.4 The Tilde Economy

We will use objects with tildes to denote the A-D economy that corresponds to the security market economy. In this economy, we define the utility as a function of the portfolio shares, so that $\tilde{L} = \mathbb{R}^K$. The utility is given by

$$\tilde{u}^i\left(\tilde{x}_1^i, \tilde{x}_2^i, \ldots, \tilde{x}_K^i\right) := u^i\left(\underbrace{\sum_{k=1}^{K} d_{k1}\tilde{x}_k^i + e_1^i}_{=x_1^i}, \ldots, \underbrace{\sum_{k=1}^{K} d_{ks}\tilde{x}_k^i + e_s^i}_{=x_s^i}, \ldots, \underbrace{\sum_{k=1}^{K} d_{km}\tilde{x}_k^i + e_m^i}_{=x_m^i}\right)$$

The budget constraint is:

$$\sum_{k=1}^{K} \tilde{p}_k \tilde{x}_k^i = \sum_{k=1}^{K} \tilde{p}_k \tilde{e}_k^i,$$

where

$$\tilde{\mathbf{e}}^i = \boldsymbol{\theta}^i$$

LHS: Tilde utility is the utility derived from the K objects that capture the number of shares bought or sold of each of the K securities by agent i.

RHS: Non-tilde utility is the utility derived from the m objects that capture the consumption of each of the m distinguishable commodities, which in this case, since we're only considering one good, reflects the m possible states of the world.

Summation elaboration: Each of the m elements in the RHS contains a summation over the K securities. For each security, we multiply the payoff of that security in state s (corresponding to entry s in the input to u^i) by the number of shares of that security that the agent has bought or sold ($\tilde{x}_k^i$).

This gives the sum of the payoffs of the agent's bought or sold securities in state s. We then add the agent's endowment of the good in the state s. (Notice that there is only one subscript for e, because there is only one good in the simplified model that we're considering.)

6.5 Asset Pricing and the "Equity Premium"

Consider an economy with one good, m states, and complete markets.

We are interested in understanding the price of two securities. Security $k = 1$ is a riskless bond, i.e.

$$d_{1s} = 1, \forall s = 1, 2, \ldots, m$$

Definition D.9: Expected Gross Return of Security k

The expected (gross) return of any security k is denoted as r_k and defined as

$$1 + r_k := \frac{\sum_{s=1}^{m} d_{ks}\pi_s}{q_k} = \frac{\mathbb{E}\left[d_k\right]}{q_k};$$

7 Risk Aversion and Portfolio Choice

7.1 Terms

- u: Utility function of the agent
- $\tilde{x}$: A random variable, which we call "risk"
- σ^2: The variance of $\tilde{x}$
- x: Some realization of $\tilde{x}$
- $ra(x)$: The Arrow-Pratt Coefficient of Absolute Risk Aversion

$$ra(x) = -\frac{u''(x)}{u'(x)}$$

- $rra(x)$: The Coefficient of Relative Risk Aversion

$$rra(x) = -\frac{u''(x)}{u'(x)}x$$

- p: Risk premium. The maximum amount that an agent is willing to pay to avoid a risk $\tilde{x}$[4]

$$u(\mathbb{E}[x] - p) = \mathbb{E}[u(\tilde{x})]$$

  - LHS: The utility of the agent if they receive the expectation of x

[4] Not sure why no tilde on LHS x

Portfolio Choice Added Terms

- N: Number of risky assets
- W: How much the agent chooses to invest (investor wealth)
  - w_i: The fraction of initial wealth that the agent invests in risky asset i
  - w_0: The fraction of wealth invested in the risk-free asset
- $\tilde{R}_i$: The random variable characterizing returns on risky asset i
- $\bar{\mu}$: The fixed return on the risk-free asset
- $\tilde{W}$: Investor wealth after returns are realized

$$\begin{aligned}\tilde{W} &= W\left[\sum_{i=1}^{N} w_i\left(\tilde{R}_i - \bar{\mu}\right) + \bar{\mu}\right] \\ &= w_0 W\bar{\mu} + \sum_{i=1}^{N} w_i W\tilde{R}_i \\ &= W\left(w_0\bar{\mu} + \sum_{i=1}^{N} w_i\tilde{R}_i\right)\end{aligned}$$

One Risky Asset Added/Amended Terms

- w: The fraction of initial wealth that the agent invests in the risky asset
- $\tilde{R}$: Random variable characterizing returns on the risky asset
- w_u^*: The optimal fraction of wealth to invest in the risky asset

7.2 Coefficients of Risk Aversion

Definition D.10: Arrow-Pratt Coefficient of Risk Aversion

This coefficient is a measure of the curvature of the utility function around the point x, and is given by

$$ra(x) = -\frac{u''(x)}{u'(x)}.$$

The higher the coefficient, the greater is the curvature and, hence, the more risk averse the agent is.

Example 7.1. CRRA (Constant Relative Risk Aversion)

$$\begin{aligned}u(x) &= \frac{x^{1-\gamma} - 1}{1-\gamma} \\ \Rightarrow -\frac{u''(x)}{u'(x)} &= -\frac{-\gamma x^{-\gamma-1}}{x^{-\gamma}} = \gamma x^{-1}\end{aligned}$$

End of Example.

Example 7.2. CARA (Constant Absolute Risk Aversion): Is there a utility function $u(x)$ such that $-\frac{u''(x)}{u'(x)} = $ constant? Yes:

$$u(x) = -\frac{1}{a}e^{-ax}$$
$$\Rightarrow -\frac{u''(x)}{u'(x)} = -\frac{-ae^{-ax}}{e^{-ax}} = a$$

See Figure 1.

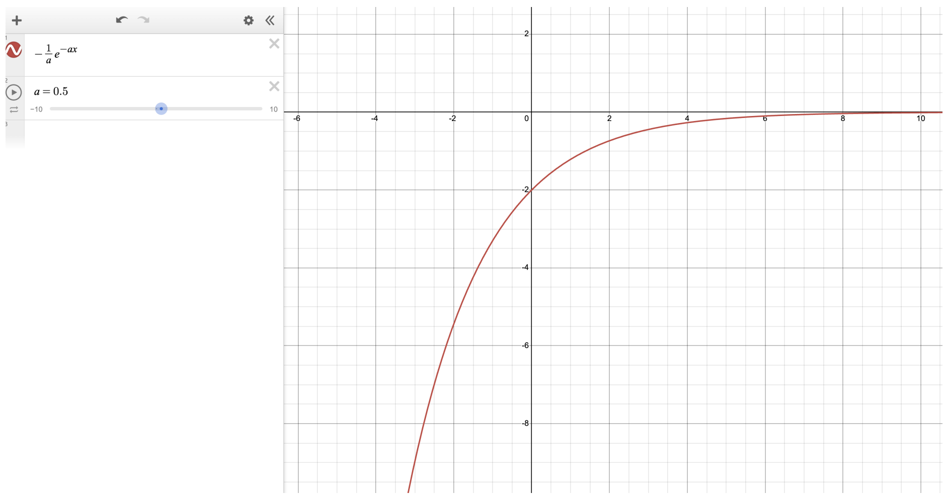

Figure 1: CARA Utility Function

End of Example.

Definition D.11: Absolute Insurance Premium

Absolute insurance premium p is the maximum amount that an agent is willing to pay to avoid a risk $\tilde{x}$ (a random variable); i.e.[a]

$$u(\mathbb{E}[x] - p) = \mathbb{E}[u(\tilde{x})],$$

where p is the premium and $\tilde{x}$ is the risk. The size of p depends on the willingness of the agent to bear risk, as well as the size of the risk.

[a]Not sure why no tilde on LHS x

For small risk, i.e., when the random variable $\tilde{x}$ has a small variance, σ^2, we can express the premium as:

$$p = -\frac{1}{2}\frac{u''(\bar{x})}{u'(\bar{x})}\sigma^2$$

where

$$\bar{x} = E[\tilde{x}]$$

Definition D.12: Coefficient of Relative Risk Aversion

$$rra(x) = -\frac{u''(x)}{u'(x)}x.$$

Definition D.13: Relative Insurance Premium

Relative insurance premium is the the maximum proportion of certain consumption, $\bar{x}$, that an agent is willing to pay to avoid a risk $\tilde{x} = \bar{x}(1+\varepsilon)$; i.e.

$$u((1-\rho)\bar{x}) = \mathbb{E}[u(\bar{x}(1+\varepsilon))]$$

where $\mathbb{E}[\varepsilon] = 0$ and $\mathbb{E}\left[\varepsilon^2\right] = \sigma_\varepsilon^2$

Proposition P.4

Relative insurance premium. Suppose that the risk is small. Then, the relative insurance premium ρ is given by

$$\rho = \frac{1}{2}\left(-\frac{u''(\bar{x})\bar{x}}{u'(\bar{x})}\right)\sigma_\varepsilon^2,$$

where $-u''(\bar{x})\bar{x}/u'(\bar{x})$ measures the agent's willingness to bear relative risk, σ_ε^2 measures the size of the risk, and the utility function is evaluated at the expected value of the risk; i.e. $\bar{x} = \mathbb{E}[\tilde{x}]$.

We can then notice a relationship between the absolute and proportional risk premium:

$$p = \bar{x}\rho$$
$$\sigma^2(x) = (\bar{x})^2\sigma_\varepsilon^2$$

Thus using the expression for p :

$$p = -\frac{u''(\bar{x})}{u'(\bar{x})}\sigma^2(x) = -\frac{u''(\bar{x})}{u'(\bar{x})}(\bar{x})^2\sigma_\varepsilon^2$$

or

$$\frac{p}{x} = \rho = -\bar{x}\frac{u''(\bar{x})}{u'(\bar{x})}\sigma_\varepsilon^2$$

7.3 Certainty Equivalent

Definition D.14: Certainty Equivalent

A certainty equivalent of risk $\tilde{x}$, denoted $c_e(\tilde{x})$, is given by

$$u\left(c_e\right) = \mathbb{E}[u(\tilde{x})].$$

Hence, c_e is the sure (deterministic) amount of consumption that will be equivalent to a given risk $\tilde{x}$. To draw parallel with earlier definitions:

$$c_e = \bar{x} - p = \bar{x}(1-\rho).$$

7.4 Arrow-Pratt Theorem

Theorem T.4

(Arrow-Pratt) Let u and v be utility functions. The following statements are equivalent.

1. If u is an increasing and concave transformation of v; i.e. there exists a function f such that
$$u(x) = f(v(x)), \forall x,$$
and,
$$f'(\cdot) > 0,$$
$$f''(\cdot) < 0.$$

2. u has a higher insurance premium than v; i.e. for all random variables $\tilde{x}$, the insurance premium $p_u(\tilde{x}), p_v(\tilde{x})$ corresponding to the utility functions u and v are:
$$p_u(\tilde{x}) > p_v(\tilde{x}).$$

3. The absolute risk aversion coefficient of u is higher than that of v everywhere; i.e.
$$-\frac{u''(x)}{u'(x)} > -\frac{v''(x)}{v'(x)}, \forall x$$

Intuition: The more concave the increasing utility function, the more risk averse the agent is, and the higher the insurance premium that the agent is willing to pay to avoid risk. (Is that correct?)

7.5 Portfolio Choice Problem

7.5.1 Maximization Problem

$$\max_{\{w_i\}} \mathbb{E}[u(\tilde{W})] = \max_{\{w_i\}} \sum_{s=1}^{S} u\left(W\left[\sum_{i=1}^{N} w_i\left(R_{i,s} - \bar{\mu}\right) + \bar{\mu}\right]\right)\pi_s$$

First order conditions :

$$\begin{aligned}
&\mathbb{E}\left[u'(\tilde{W})\left(\tilde{R}_i-\bar{\mu}\right)\right]\\
=&\mathbb{E}\left[u'\left(W\left[\sum_{j=1}^{N} w_j\left(\tilde{R}_j-\bar{\mu}\right)+\bar{\mu}\right]\right)\left(\tilde{R}_i-\bar{\mu}\right)\right]\\
=&0
\end{aligned}$$

for $i = 1, 2, \ldots, N$.

Lemma L.1

(Properties of concavity).

1. Additive of concavity: let $f(x)$ and $g(x)$ be (strictly) concave, then $h(x) = f(x) + g(x)$ is also (strictly) concave.
2. Preservation of concavity under strictly increasing and strictly concave transformation: let $f(x)$ and $g(x)$ be strictly concave and $f(x)$ be a strictly increasing function, then $h(x) = f(g(x))$ is also strictly concave.

7.5.2 One Risky Asset Case

In this case, we can denote the share of wealth invested in the risky asset as w and the share invested in the risk-free asset as $1 - w$. We can denote the return on the risky asset as $\tilde{R}$.

Then the simplified problem is:

$$\max_{w} \mathbb{E}[u(W[w(\tilde{R}-\bar{\mu})+\bar{\mu}])]$$

and the first order condition becomes:

$$W\mathbb{E}\left[u'\left(W\left[w_u^*(\tilde{R}-\bar{\mu})+\bar{\mu}\right]\right)(\tilde{R}-\bar{\mu})\right]=0$$

Intuition for the below propositions: (i) the investor will always invest in a risky asset with a higher return than the risk-free return, independent of the degree of his risk aversion; and (ii) the more risk averse the agent is, the smaller the size of the investment in the risky asset.

Proposition P.5

The investor will always invest in a risky asset with a higher return than the risk-free return, no matter how risk averse the investor is. That is,

$$w_u^* > 0 \Leftrightarrow \mathbb{E}[\tilde{R}] > \bar{\mu}.$$

Proposition P.6

Suppose $\mathbb{E}[\tilde{R}] > \bar{\mu}, u$ is strictly concave, and that $N = 1$ (i.e. there is only one risky asset). Suppose u is more risk averse than v; i.e.

$$-\frac{u''(x)}{u'(x)} > -\frac{v''(x)}{v'(x)}, \forall x > 0.$$

Then, the proportion of invested into the risky asset, w_u and w_v corresponding to u and v respectively, are such that

$$w_u < w_v.$$