Theory of Income 2 - Pset 1

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1 Question 1

(Marx, Piketty, and the neoclassical growth model)

Consider a one-sector neoclassical growth model. The production side is standard: there is a CRS technology the produces capital and investments:

$$C_t + I_t = F\left(K_t, X_t L_t\right)$$
$$\dot{K}_t = I_t - \delta K_t$$

Technological TFP X grows at a constant growth rate. There are two types of consumers. There is measure π^c of "capitalists" who are born with initial capital stock k_0. They do not work and supply capital to the firms on competitive rental markets. There is also measure π^w of "workers" who have no initial capital stock and supply labor inelastically. To make math easier, each worker supplied $1/\pi^w$ units of labor. Workers and capitalists can freely borrow and lend with each other. Both types of agents have preferences

$$\int_0^\infty \exp(-\rho t) \ln C_t^i dt \text{ for } i \in \{c, w\}.$$

Aggregate consumption C is naturally given by

$$\pi^c C_t^c + \pi^w C_t^w = C_t$$

1. Define competitive equilibrium in this economy.

2. Show that there is the level of capital stock k_0^* such that if $k_0 = k_0^*$ then this economy is on the balanced growth path.

3. What happens to the ratio of wage income of workers to the rental income of capitalists over time on the balanced growth path?

4. Is the assumption that workers can trade assets with capitalists is important for this conclusion?

In his best-selling book "Capital in the Twenty-First Century" Thomas Piketty documents that the growth rate of wages g has been systematically below interest rates r for most of the last century, $r > g$. He argues that this is the reason for widening inequality in many countries between the rich (who rely mainly on interest income) and the poor (who rely mainly on wage income).

5. Show that in the balance growth path we must necessarily have $r > g$.

6. For concreteness, let use Gini as a measure of inequality. What happens to inequality over time on the balanced growth path?

2 Question 2

(Intertemporal optimality conditions)

Lecture 2 Slide 21

Show that optimality conditions for consumers imply

$$\frac{1}{\sigma}(r-\delta-\rho) = \frac{\dot{c}^M(t)}{c^M(t)} = \frac{\dot{c}(t)}{c(t)}$$
$$\frac{\dot{c}^M}{c^M} = \frac{\dot{c}^A}{c^A+\gamma^A} = \frac{\dot{c}^S}{c^S+\gamma^S}.$$

2.1 Solution

The consumer maximizes their preferences:

$$\int_0^\infty \exp(-\rho t)\frac{c(t)^{1-\sigma}-1}{1-\sigma}dt$$
with
$$c(t) = \left(c^A(t)+\gamma^A\right)^{\eta^A} c^M(t)^{\eta^M} \left(c^S(t)+\gamma^S\right)^{\eta^S}$$
$$\eta^i > 0, \sum_{i\in\{A,M,S\}} \eta^i = 1$$
$$\gamma^A < 0, \gamma^S > 0$$

subject to the budget constraint:

$$\sum_{i\in\{A,M,S\}} p^i(t)c^i(t) + \dot{K}(t) = w(t) + (r(t)-\delta)K(t)$$

Thus, the current-value Hamiltonian is:

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda[w + (r-\delta)K - \sum_{i\in\{A,M,S\}} p^i c^i]$$

Our FOCs are then:

$$
\begin{aligned}
\frac{\partial H}{\partial c^A} &= c^{-\sigma}\frac{\partial c}{\partial c^A} - \lambda p^A \\
&= c^{-\sigma}\frac{c}{c^A+\gamma^A}\eta^A - \lambda p^A = 0^1 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^A+\gamma^A}\eta^A = \lambda p^A \qquad (1)\\
\frac{\partial H}{\partial c^S} &= c^{-\sigma}\frac{\partial c}{\partial c^S} - \lambda p^S \\
&= c^{-\sigma}\frac{c}{c^S+\gamma^S}\eta^S - \lambda p^S = 0 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^S+\gamma^S}\eta^S = \lambda p^S \qquad (2)\\
\frac{\partial H}{\partial c^M} &= c^{-\sigma}\frac{\partial c}{\partial c^M} - \lambda p^M \\
&= c^{-\sigma}\frac{c}{c^M}\eta^M - \lambda p^M = 0 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^M}\eta^M = \lambda p^M \qquad (3)\\
\frac{\partial H}{\partial \lambda} &= \dot{K} = w + (r-\delta)K - \sum_{i\in\{A,M,S\}} p^i c^i \\
-\frac{\partial H}{\partial K} &= -\lambda(r-\delta) = \dot{\lambda} - \rho\lambda \\
&\Rightarrow \dot{\lambda} = \lambda(\rho - r + \delta) \qquad (4)
\end{aligned}
$$

First, we will show that

$$\frac{\dot{c}^M}{c^M} = \frac{\dot{c}^A}{c^A+\gamma^A} = \frac{\dot{c}^S}{c^S+\gamma^S}$$

We can begin by attaining the intratemporal rates of substitution. Consider dividing (3) by (1):

$$
\begin{aligned}
\frac{\frac{c}{c^M}\eta^M}{\frac{c}{c^A+\gamma^A}\eta^A} &= \frac{\lambda p^M c^\sigma}{\lambda p^A c^\sigma} \\
\Rightarrow \frac{c^A+\gamma^A}{c^M}\frac{\eta^M}{\eta^A} &= \frac{p^M}{p^A} \\
\Rightarrow \frac{p^A(c^A+\gamma^A)}{\eta^A} &= \frac{p^M c^M}{\eta^M} \\
\Rightarrow \frac{p^A(c^A+\gamma^A)}{\eta^A} &= \frac{c^M}{\eta^M} \qquad \text{since } p^M = 1 \qquad (5)
\end{aligned}
$$

[1] To see how this step works, consider that $\frac{\partial c}{\partial c^A} = \eta^A\left(c^A(t)+\gamma^A\right)^{\eta^A-1}\cdot c^M(t)^{\eta^M}\cdot\left(c^S(t)+\gamma^S\right)^{\eta^S} = \frac{\eta^A\left(c^A(t)+\gamma^A\right)^{\eta^A}\cdot c^M(t)^{\eta^M}\cdot\left(c^S(t)+\gamma^S\right)^{\eta^S}}{\left(c^A(t)+\gamma^A\right)} = \eta^A\frac{c}{c^A+\gamma^A}$

Similarly, dividing (3) by (2) yields:

$$\begin{aligned}
\frac{\frac{c}{c^M}\eta^M}{\frac{c}{c^S+\gamma^S}\eta^S} &= \frac{\lambda p^M c^\sigma}{\lambda p^S c^\sigma} \\
\Rightarrow \frac{c^S+\gamma^S}{c^M}\frac{\eta^M}{\eta^S} &= \frac{p^M}{p^S} \\
\Rightarrow \frac{p^S(c^S+\gamma^S)}{\eta^S} &= \frac{p^M c^M}{\eta^M} \\
\Rightarrow \frac{p^S(c^S+\gamma^S)}{\eta^S} &= \frac{c^M}{\eta^M} && \text{since } p^M = 1 && (6)
\end{aligned}$$

Thus, combining (5) and (6) yields:

$$\frac{p^A(c^A+\gamma^A)}{\eta^A} = \frac{c^M}{\eta^M} = \frac{p^S(c^S+\gamma^S)}{\eta^S}$$

Note that this also gives:

$$c^i + \gamma^i = \frac{\eta^i c^M}{p^i \eta^M} \qquad \text{for } i \in \{A, S\} \qquad (7)$$

Consider then differentiating (5) and (6) with respect to t yields:

$$\begin{aligned}
& d\left(\frac{p^i\left(c^i+\gamma^i\right)}{\eta^i}\right)/dt = d\left(\frac{c^M}{\eta^M}\right)/dt \\
\Rightarrow \quad & \frac{\dot{p}^i\left(c^i+\gamma^i\right)+p^i\dot{c}^i}{\eta^i} = \frac{\dot{c}^M}{\eta^M} \\
\Rightarrow \quad & \frac{p^i\dot{c}^i}{\eta^i} = \frac{\dot{c}^M}{\eta^M} && \text{since } \dot{p^i} = 0 \\
\Rightarrow \quad & \dot{c}^i = \frac{\eta^i\dot{c}^M}{p^i\eta^M} && (8)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\dot{c^i}}{c^i+\gamma^i} &= \dot{c^i}\cdot\frac{1}{c^i+\gamma^i} \\
&= \frac{\eta^i\dot{c}^M}{p^i\eta^M}\cdot\frac{p^i\eta^M}{\eta^i c^M} && \text{by (8) and (7)} \\
&= \frac{\dot{c}^M}{c^M} && \text{cancelling terms}
\end{aligned}$$

Now, we will proceed to show:

$$\frac{1}{\sigma}(r-\delta-\rho)=\frac{\dot{c}^M(t)}{c^M(t)}=\frac{\dot{c}(t)}{c(t)}$$

Recall from (4) that

$$\begin{aligned}
\dot{\lambda} &= \lambda(\rho - r + \delta) \\
\Rightarrow \frac{\dot{\lambda}}{\lambda} &= \rho - r + \delta \qquad\qquad (9)
\end{aligned}$$

Additionally,

$$\begin{aligned}
\lambda &= \frac{\eta^M}{p^M}\frac{c^{1-\sigma}}{c^M} && \text{by (3)} \\
\Rightarrow \lambda c^M p^M &= \eta^M c^{1-\sigma} && (10) \\
\Rightarrow \lambda c^M p^M + \lambda \gamma^M p^M &= \eta^M c^{1-\sigma} \\
\Rightarrow \dot{\lambda} c^M p^M + \lambda \dot{c}^M p^M + \lambda \gamma^M \dot{p}^M &= (1-\sigma)\eta^M c^{-\sigma}\dot{c} && \text{derivative wrt t} \\
\Rightarrow \dot{\lambda} c^M p^M + \lambda \dot{c}^M p^M &= (1-\sigma)\eta^M c^{-\sigma}\dot{c} && \text{since } \dot{p}^M = 0 \\
\Rightarrow \frac{\dot{\lambda} c^M p^M}{\lambda} + \dot{c}^M p^M &= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\lambda} && \text{divide by } \lambda \\
\Rightarrow \frac{\dot{\lambda}}{\lambda} + \frac{\dot{c}^M}{c^M} &= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\lambda c^M p^M} && \text{divide by } c^M \\
&= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\eta^M c^{1-\sigma}} && \text{by (10)} \\
&= \frac{(1-\sigma)\eta^M \dot{c}}{\eta^M c} \\
&= \frac{(1-\sigma)\dot{c}}{c} \\
\Rightarrow (1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}^M}{c^M} = \frac{\dot{\lambda}}{\lambda} &= \rho - r + \delta && \text{by (9)} \qquad (11)
\end{aligned}$$

As a useful interlude, take the log of c and then differentiate with respect to t:

$$\begin{aligned}
\ln c(t) &= \eta^A \ln(c^A(t)+\gamma^A) + \eta^M \ln c^M(t) + \eta^S \ln(c^S(t)+\gamma^S) \\
\Rightarrow \frac{\dot{c}}{c} &= \eta^A \frac{\dot{c}^A}{c^A+\gamma^A} + \eta^M \frac{\dot{c}^M}{c^M} + \eta^S \frac{\dot{c}^S}{c^S+\gamma^S} \\
&= (\eta^A + \eta^M + \eta^S)\frac{\dot{c}^M}{c^M} \\
&= \frac{\dot{c}^M}{c^M} && \text{since } \sum_{i\in\{A,M,S\}} \eta^i = 1
\end{aligned}$$

That is,

$$\frac{\dot{c}}{c} = \frac{\dot{c}^M}{c^M} \tag{12}$$

Now, returning to (11):

$$\begin{aligned}
(1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}^M}{c^M} &= \rho - r + \delta \\
\Rightarrow (1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}}{c} &= \rho - r + \delta && \text{by (12)} \\
\Rightarrow -\sigma\frac{\dot{c}}{c} &= \rho - r + \delta \\
\Rightarrow \frac{\dot{c}}{c} &= \frac{1}{\sigma}(r - \delta - \rho)
\end{aligned}$$

That is,

$$\frac{\dot{c}}{c} = \frac{\dot{c}^M}{c^M} = \frac{1}{\sigma}(r - \delta - \rho)$$

Thus, we have achieved all of our desired results.

3 Question 3

(Intertemporal optimality conditions)

Lecture 3 Slide 14

Let λ be multiplier on the consumer's budget constraint and

$$C(t) \equiv \sum_{i\in\{A,M,S\}} p^i(t)c^i(t)$$

be total consumption expenditures. Show that optimality require

$$\frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} = (1-\theta)\frac{\dot{c}}{c}$$

and

$$\frac{\dot{\lambda}}{\lambda} = -\left(\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} - \delta - \rho\right)$$

3.1 Solution

Preferences are given by:

$$\int_0^\infty \exp(-\rho t)\frac{c(t)^{1-\theta}-1}{1-\theta}dt$$

with

$$c(t) = \left(\sum_{i\in\{A,S,M\}} \eta^i c^i(t)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}$$

and the budget constraint:

$$\sum_{i\in\{A,M,S\}} p^i c^i + \dot{K} = X^M (K)^\alpha (\bar{L})^{1-\alpha} - \delta K$$

Thus, consider the current-value Hamiltonian:

$$H(t) = \frac{c^{1-\theta}-1}{1-\theta} + \lambda\left(X^M K^\alpha \bar{L}^{1-\alpha} - \delta K - \sum_{i\in\{A,M,S\}} p^i c^i\right)$$

Thus, our FOCs are

$$
\begin{aligned}
\frac{\partial H}{\partial c^i} &= c^{-\theta}\frac{\partial c}{\partial c^i} - \lambda p^i = 0 \\
&\Rightarrow c^{-\theta}\frac{\partial c}{\partial c^i} = \lambda p^i \\
\dot{\lambda} &= \rho\lambda - \frac{\partial H}{\partial K} = \rho\lambda - \lambda\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \lambda\delta
\end{aligned}
$$

where $\frac{\partial c}{\partial c^i}$:

$$
\begin{aligned}
\frac{\partial c}{\partial c^i} &= \frac{\sigma}{\sigma-1}\left(\sum_{i\in\{A,S,M\}} \eta^i c^{i(\sigma-1)/\sigma}\right)^{\frac{1}{\sigma-1}} \cdot \left(\frac{\sigma-1}{\sigma}\right)\left(\eta^i c^{i\frac{-1}{\sigma}}\right) \\
&= \left(\sum_{i\in\{A,S,M\}} \eta^i c^{i(\sigma-1)/\sigma}\right)^{\frac{1}{\sigma-1}} \cdot \left(\eta^i c^{i\frac{-1}{\sigma}}\right) \\
&= c^{\frac{1}{\sigma}} \cdot \eta_i(\frac{1}{c^i})^{\frac{1}{\sigma}} \\
&= (\frac{c}{c^i})^{\frac{1}{\sigma}}\eta_i
\end{aligned}
$$

We can multiply our first FOC by c_i:

$$
\lambda p^i c^i = c^{-\theta} c^i \eta^i (\frac{c}{c^i})^{\frac{1}{\sigma}} = c^{-\theta}\eta^i (c^i)^{\frac{\sigma-1}{\sigma}} c^{\frac{1}{\sigma}}
$$

Summing over i, this yields:

$$
\begin{aligned}
\lambda \sum_{i\in\{A,M,S\}} p^i c^i &= c^{-\theta} \sum_{i\in\{A,M,S\}} \eta^i (c^i)^{\frac{\sigma-1}{\sigma}} c^{\frac{1}{\sigma}} \\
&= c^{-\theta} c^{\frac{\sigma-1}{\sigma}} c^{\frac{1}{\sigma}} \\
&= c^{-\theta} c^{\frac{\sigma-1+1}{\sigma}} \\
&= c^{-\theta} c \\
&= c^{1-\theta}
\end{aligned}
$$

Thus,

$$
\lambda C = c^{1-\theta} \qquad \text{since } C = \sum_{i\in\{A,M,S\}} p^i c^i
$$

From here,

$$\begin{aligned}
\lambda C &= c^{1-\theta} \\
\Rightarrow \ln\lambda + \ln C &= (1-\theta)\ln c && \text{take natural log} \\
\Rightarrow \frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} &= (1-\theta)\frac{\dot{c}}{c} && \text{derivative wrt t}
\end{aligned}$$

which is the first condition we wanted to show.

Now, from our second derivative:

$$\begin{aligned}
\dot{\lambda} &= \rho\lambda - \lambda\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \lambda\delta \\
\frac{\dot{\lambda}}{\lambda} &= \rho - \alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \delta \\
&= -(\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} - \delta - \rho)
\end{aligned}$$

which is the second condition we wanted to show.

4 Question 4

(Potatoes and the fall of aristocracy)

Consider a two-sector version of the neoclassical growth model without capital. The output in the manufacturing is given by a constant returns to scale technology

$$Y^M = X^M L^M,$$

where L^M is supply of labor in manufacturing.

The output in the aggricultural sector is given by technology

$$Y^A = X^A \left(L^A\right)^\alpha (\bar{Z})^{1-\alpha},$$

where X is agricultural productivity, L^A is labor supply in agriculture, and $\bar{Z}$ is supply of land, available in fixed supply.

Suppose there are two sets of agents: workers who supply 1 unit of labor inelastically and aristocrats, who supply no labor but own land and receive land rents. Intratemporal preferences of all agents are given by

$$\left[\left(c^A\right)^{(\sigma-1)/\sigma} + \left(c^M\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}$$

with $\sigma < 1$.

Since we abstract from capital, we will focus on the static economy.

1. Define competitive equilibrium in this economy.

2. What happens to labor income if agricultural productivity X^A increases? What happens to land rents? What happens labor allocation L^M and L^A ?

In a series of papers, Nathan Nunn, Nancy Qian and co-authors study an exogenous agricultural technical change: introduction of potatoes to Europe during the Columbian exchange. Potatoes are much superior in their nutritional characteristics to native European staples such as turnips. By exploiting variation in European regional variation in suitability for cultivating potatoes, the authors provide causal evidence that introduction of potatoes in Europe increase urbanization and reduced the incidence of European military conflict.

3. Explain how these findings can be rationalized by the standard two-sector growth model?