Macro 2 Notes

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1 Introduction

Much of this is directly quoted from Golosov's notes, slides, Ragini's notes, or the notes of past students (Jordan Rosenthal-Kay, Jingoo Kwon).

2 Lecture 0: Neoclassical Growth Model without Growth

This section pulls from Golosov's Lecture 0.

2.1 Terms

- t: period
- β: discount factor
- c_t: consumption in period t
- $u(c_t)$: utility derived from consumption in period t
- k_t: capital in period t
- $f(k_t)$: production function
- δ: depreciation rate

2.2 Setup

2.2.1 Preferences

Continuum of identical, infinitely lived consumers with preferences

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right),$$

where $c_t \geq 0$ is consumption in period t.

2.2.2 Technology

Technology Output produced with production function $f(k_t)$, where $k_t \geq 0$ is capital with initial $k_0 > 0$ given. Output can be costlessly transferred between consumption and capital for next period:

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$
$$k_0 > 0 \text{ is given.}$$

for depreciation rate $\delta \in (0,1)$.

2.2.3 Assumptions

1. u, f are strictly increasing, differentiable, u is strictly concave, f is concave;
2. u, f are "nice"[1];
3. u, f satisfy Inada conditions $\lim_{c\to 0} u'(c) = \lim_{k\to 0} f'(k) = \infty$.

2.3 Model

2.3.1 Social Planner Problem

$$\max_{\{c_t,k_t\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t,$$

and $c_t \geq 0, k_t \geq 0, k_0$ is given.

2.3.2 Key Optimality Theorem

Theorem 2.1. *Suppose the assumptions above hold. (necessity) If $\{c_t^*, k_t^*\}_t$ solves (3) then $\{c_t^*, k_t^*\}_t$ satisfies*

$$c_t^* + k_{t+1}^* = f(k_t^*) + (1-\delta)k_t^*,$$
$$u'(c_t^*) = \beta\left[1 + f'\left(k_{t+1}^*\right) - \delta\right] u'\left(c_{t+1}^*\right),$$
$$\lim_{T\to\infty} \beta^T u'(c_T^*)\, k_{T+1}^* \leq 0.$$

(sufficiency) If $\{c_t^*, k_t^*\}_t$ satisfies (4), (5), and (6), then it is a solution to (3).

[1] Notes from earlier in Lecture 0 on niceness: There are multiple ways to assume niceness: bounded $u; u$ bounded from below and F is such that feasible x are bounded; u is CRRA and some assumption on the speed of change in derivatives of F around $x = 0$. The formal arguments are a bit tedious and not that insightful beyond the intuition that I gave here, so we will not talk about them.

3 Lecture 1

3.1 Neoclassical Growth Model

3.1.1 Terms

- t: period
- C_t: consumption in period t
- I_t: investment in period t
- K_t: capital in period t
- Y_t: output in period t, sum of factor income
- F_t: production function
- X_t: (labor-augmenting) technology in period t
- n: growth rate of population
- ρ: discount factor

3.1.2 Basic Accounting Definitions

$$\begin{aligned}
&C_t + I_t = Y_t \\
&K_{t+1} = I_t + (1-\delta)K_t \\
&Y_t = \text{sum of factor income}
\end{aligned}$$

3.1.3 More Relationships

$$\begin{aligned}
&\dot{K}(t) = Y(t) - C(t) - \delta K(t) && \text{Feasibility} \\
&Y(t) = F(K(t), X(t)L(t)) && \\
&L(t) = 1 && \text{Feasibility: inelastic labor}
\end{aligned}$$

3.1.4 Assumptions

- Perfectly competitive firms
- Y_t is produced by CRS technology F_t (DRS is a CRS with a fixed factor, IRS is hard to model parsimoniously).
- Two factors: capital and labor.
- Inelastic Labor

3.1.5 Setup

Household

Infinitely lived representative household with preferences

$$\int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

and inelastic labor supply (for now)

3.1.6 Useful Normalization

Re-normalize everything per unit of X :

$$\begin{aligned} k(t) &\equiv \frac{K(t)}{X(t)} \\ c(t) &\equiv \frac{C(t)}{X(t)} \\ y(t) &\equiv \frac{Y(t)}{X(t)} = F(k(t), 1) \\ \tilde{\rho} &\equiv \rho - (1-\sigma) g_X \end{aligned}$$

In this case, the model becomes isomorphic to the neoclassical growth model without growth. Thus, we have

- Competitive equilibrium is efficient.
- $k(t), c(t), y(t)$ converge to the steady state k^{ss}, c^{ss}, y^{ss}.

3.1.7 Neoclassical Growth Model and Kaldor Facts

Steady state of the neoclassical growth model is consistent with Kaldor facts (presented just below)

1. $y(t) = y^{ss}$ implies that $Y(t)$ grows at rate g_X.
2. Capital-output ratio is constant: $K(t)/Y(t) = k^{ss}/y^{ss}$.
3. Since consumption growth rate is constant, so are interest rates.
4. Factor shares are constant by labor-augmenting technical change + constant interest rate.

3.2 Kaldor Facts

1. Output per capita grows at a constant rate.
2. Capital-output ratio is roughly constant.
3. Interest rate is roughly constant.
4. Distribution of income between capital and labor is roughly constant.

3.3 Constant Growth

- $\frac{\dot{Y}(t)}{Y(t)} = g_Y > 0$
- $\frac{\dot{K}(t)}{K(t)} = g_K > 0$
- $\frac{\dot{C}(t)}{C(t)} = g_C > 0$
- $\frac{\dot{L}(t)}{L(t)} = n$

3.4 Uzawa Theorem

With constant growth and CRS technology, we have

1. Balanced growth: $g_Y = g_C = g_K \equiv g$
2. Labor-augmenting technical change: $\tilde{F}$ can be represented as $\tilde{F}(K(t), L(t), \tilde{X}(t)) = F(K(t), X(t)L(t))$ for some CRS F with $\frac{\dot{X}(t)}{X(t)} = g - n$

3.4.1 Implications of Uzawa

Some implications from Uzawa's Theorem:

- With CRS, all constant growth must be balanced, i.e., all variables grow at the same rate. Moreover, per capita growth is driven by technology.
- Technology must be either purely labor-augmenting or the elasticity of substitution between K and L equals 1.

3.5 Uzawa Theorem - Part 2

With constant growth, CRS technology, and constant factor shares[2], we have

- Constant interest rate: $R(t) = R^* \quad \forall t$
- Constant wage growth rate at the rate of technological growth: $\frac{\dot{w}(t)}{w(t)} = g_X = g_Y - n$

3.6 Constant Interest Rates, Balanced Growth, and U Theorem

Constant interest rates and balanced growth implies that $U(C)$ must be, up to a linear tranformation,

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

[2] Jingoo's notes also mention perfect competition, not sure if that's implicit in Golosov's statement

3.7 Useful Facts

3.7.1 Re-Expressing Growth Rates

If any variable Z grows with rate g, $\frac{\dot{Z}(t)}{Z(t)} = g \iff Z(t) = e^{(t-\tau)g} Z(\tau)$ for all t, τ

4 Lecture 2: Structural Change - Demand Side

4.1 Model

4.1.1 Terms

- t: period
- c_t: aggregate consumption in period t
- $I(t)$: investment at time t
- $K(t)$: capital at time t
- $r(t)$: rental rate of capital at time t
- $w(t)$: wage rate at time t
- ρ: discount factor
- U_0: Utility beginning at period 0
  - $c^A(t) \in [\gamma^A, \infty)$ is the agricultural consumption at time t.
  - $c^M(t) \geq 0$ is the manufacturing consumption at time t.
  - $c^S(t) \geq 0$ is the services consumption at time t.
- $\gamma^A < 0$: constant establishing a subsistence level of agricultural consumption
  - The household must consume at least this much agricultural production (food) to survive
- $\gamma^S > 0$: constant establishing that consumption of services can be zero or negative
- η^i: long-run share of consumption in sector i
- $p^i(t)$ is the price of one unit of $c^i(t)$ for $i \in \{A, M, S\}$
  - In general, we normalize s.t. $p^M(t) = 1$, but we can choose any sector to normalize to 1 if useful
- $Y^i(t)$: Output of sector i at time t
- B^i: Hicks-neutral productivity term for sector $i \in \{A, M, S\}$
- $X(t)$: Labor-augmenting productivity term affecting all sectors.
- $g = \frac{\dot{X}(t)}{X(t)}$: growth rate of labor-augmenting productivity

4.1.2 Model Setup

Preferences

$$
\begin{aligned}
&U_0 = \int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt \\
&\text{with} \\
&c(t) = \left(c^A(t) + \gamma^A\right)^{\eta^A} c^M(t)^{\eta^M} \left(c^S(t) + \gamma^S\right)^{\eta^S} \\
&\eta^i > 0, \sum_{i \in \{A,M,S\}} \eta^i = 1, \\
&\gamma^A < 0, \gamma^S > 0
\end{aligned}
$$

Budget Constraint:

$$
\sum_{i \in \{A,M,S\}} p^i(t) c^i(t) + \dot{K}(t) = w(t) + (r(t) - \delta) K(t)
$$

Technology

Technology F is CRS with

$$
\begin{aligned}
&Y^i(t) = B^i F\left(K^i(t), X(t) L^i(t)\right), \\
&\dot{X}(t)/X(t) = g.
\end{aligned}
$$

with capital goods produced by sector M

4.1.3 Firm's Problem

$$
\max p^i(t) Y^i(t) - w(t) L^i(t) - r(t) K^i(t)
$$

s.t.

$$
Y^i(t) = B^i F\left(K^i(t), X(t) L^i(t)\right)
$$

Optimality Conditions for Firm

Capital:

$$
p^i(t) B^i F_K\left(K^i(t), X(t) L^i(t)\right) = r(t) \tag{1}
$$

Labor:

$$p^i(t)B^iF_L\left(K^i(t), X(t)L^i(t)\right)X(t) = w(t) \tag{2}$$

Interpretation: The rental rate of capital must be the marginal value of capital to production multiplied by price and the productivity term. The wage rate must be the marginal value of labor to production multiplied by price and the productivity term and the labor-augmenting technology.

4.1.4 Market Clearing

Market Clearing for Labor and Capital

$$\begin{aligned}
K^A(t) + K^M(t) + K^S(t) &= K(t) \\
L^A(t) + L^M(t) + L^S(t) &= 1
\end{aligned}$$

Market Clearing for Agricultural and Service Goods

$$\begin{aligned}
c^A(t) &= Y^A(t) \\
c^S(t) &= Y^S(t)
\end{aligned}$$

Manufacturing good is used in production of investment good

$$\begin{aligned}
I(t) + c^M(t) &= Y^M(t) \\
\dot{K}(t) &= I(t) - \delta K
\end{aligned}$$

4.1.5 Competitive Equilibrium

Given initial K_0, collection of prices and quantities, such that

1. Consumers choose their quantities optimally given prices.
2. Firms choose their quantities optimally given prices.
3. All markets clear.

4.1.6 Variable/Parameter Relationships

Nonhomothetic Preferences

Generally, allowing for nonhomothetic preferences:

$$\frac{p^i c^i}{p^M c^M} = \frac{\eta^i}{\eta^M} - \frac{p^i}{p^M}\frac{\gamma^i}{c^M}$$

Note that holding prices fixed, $p^i c^i$ growths faster (slower) than $p^M c^M$ if $\gamma^i > 0\big($ if $\gamma^i < 0\big)$.

- $\gamma^A < 0$: Consumption share of A grows slower than M.
- $\gamma^S > 0$: Consumption share of S grows faster than M.

This is consistent with cross-sectional patterns in spending.

Equalization of capital-labor ratios

Since

$$\frac{r(t)}{w(t)} = \frac{p^i(t)B_i F_K\left(K^i(t), X(t)L^i(t)\right)}{p^i(t)B_i F_L\left(K^i(t), X(t)L^i(t)\right)X(t)}$$

by (1) and (2), we have (by re-arranging and cancellation):

$$X(t)\frac{r(t)}{w(t)} = \frac{F_K\left(K^i(t), X(t)L^i(t)\right)}{F_L\left(K^i(t), X(t)L^i(t)\right)}$$

Notably:

- Since F_K, F_L are HD0, the RHS is a function of $\frac{K^i(t)}{X(t)L^i(t)}$.
  - To be hyper-clear, I am saying that since (1) and HD0 function maps to the same output for inputs with the same ratio and (2) since the RHS is a function of two functions with this quality and with the same inputs, it must be the case that the whole RHS is just a function of the ratio of these two inputs.
- Since $r(t)/w(t)$ does not depend on i, there is some $k(t)$ s.t.

$$\frac{K^i(t)}{X(t)L^i(t)} = k(t) \quad \text{for all } i \tag{3}$$

  - I am not totally sure why they don't reference whole LHS, which also don't depend on i, but essentially, I believe they are just saying that since the LHS doesn't depend on i, there must be some constant capital-labor ratio that characterizes the RHS across sectors.

Constant Relative Prices

From (3), we have that:

$$F_K\left(K^i(t), X(t)L^i(t)\right) = F_K\left(K^j(t), X(t)L^j(t)\right) \equiv \bar{F}(t) \quad \forall i, j \in \{A, M, S\}$$

From (1) with $p^M = 1$, this gives

$$\begin{aligned}
B^M \bar{F}(t) = r(t) &\Rightarrow B_M = \frac{r(t)}{\bar{F}(t)} \\
p^A(t)B^A \bar{F}(t) = r(t) &\Rightarrow p^A = \frac{r(t)}{\bar{F}(t)}\frac{1}{B^A} = \frac{B^M}{B^A} \\
p^S(t)B^S \bar{F}(t) = r(t) &\Rightarrow p^S = \frac{r(t)}{\bar{F}(t)}\frac{1}{B^S} = \frac{B^M}{B^S}
\end{aligned}$$

where the last equality in the second and third lines follows from the first line.

This leads to the conclusion that:

- In CE prices determined by technology, not preferences
  - same growth rate in all sectors $\iff$ same relative prices

Optimality Conditions for Consumers

In Pset 2 or 3, we derive the following from the consumer optimality conditions:

$$\begin{aligned}
\frac{1}{\sigma}(r - \delta - \rho) = \frac{\dot{c}^M(t)}{c^M(t)} &= \frac{\dot{c}(t)}{c(t)} \\
\frac{\dot{c}^M}{c^M} = \frac{\dot{c}^A}{c^A + \gamma^A} &= \frac{\dot{c}^S}{c^S + \gamma^S}
\end{aligned}$$

which implies sectoral reallocation (since $\gamma^A < 0, \gamma^S > 0$)

$$\frac{\dot{c}^A}{c^A} < \frac{\dot{c}^M}{c^M} < \frac{\dot{c}^S}{c^S}$$

Structural Change

Since $c^A(t) = Y^A(t)$ and $c^S(t) = Y^S(t)$, we must have

$$\frac{\dot{Y}^A}{Y^A} < \frac{\dot{Y}^S}{Y^S}$$

Aggregation

We want to aggregate up our economy:

Start with three feasibility conditions

$$
\begin{aligned}
c^A(t) &= B^A F\left(K^A(t), X(t)L^A(t)\right) \\
c^S(t) &= B^S F\left(K^S(t), X(t)L^S(t)\right) \\
c^M(t) + \dot{K}(t) &= B^M F\left(K^M(t), X(t)L^M(t)\right) - \delta K(t)
\end{aligned}
$$

We can then apply (8) to get:

$$
\begin{aligned}
c^A(t) =& B^A \left\{F_K^A(t)K^A(t) + F_L^A X(t)L^A(t)\right\} \\
c^S(t) =& B^S \left\{F_K^S(t)K^S(t) + F_L^S(t)X(t)L^S(t)\right\} \\
c^M(t) + \dot{K}(t) =& B^M \left\{F_K^M(t)K^M(t) + F_L^M(t)X(t)L^M(t)\right\} - \delta K(t)
\end{aligned}
$$

Multiplying each c^i by p^i then gives:

$$
\begin{aligned}
p^A c^A(t) =& p^A B^A \left\{F_K^A(t)K^A(t) + F_L^A X(t)L^A(t)\right\} \\
p^S c^S(t) =& p^S B^S \left\{F_K^S(t)K^S(t) + F_L^S(t)X(t)L^S(t)\right\} \\
p^M c^M(t) + p^M \dot{K}(t) =& p^M B^M \left\{F_K^M(t)K^M(t) + F_L^M(t)X(t)L^M(t)\right\} - \delta p^M K(t)
\end{aligned}
$$

Then, summing gives:

$$
\begin{aligned}
p^A c^A(t) + p^S c^S(t) + p^M c^M(t) =& p^A B^A \left\{F_K^A(t)K^A(t) + F_L^A X(t)L^A(t)\right\} \\
& + p^S B^S \left\{F_K^S(t)K^S(t) + F_L^S(t)X(t)L^S(t)\right\} \\
& + p^M B^M \left\{F_K^M(t)K^M(t) + F_L^M(t)X(t)L^M(t)\right\} - \delta p^M K(t) \\
=& r(t)(K^A(t) + K^S(t) + K^M(t)) + && \text{by (1)} \\
& w(t)(X(t)L^A(t) + X(t)L^S(t) + X(t)L^M(t)) - \delta p^M K(t) && \text{by (2)} \\
=& r(t)K(t) + w(t)\bar{L} - \delta p^M K(t) && \text{by market clearing}
\end{aligned}
$$

We want to get rid of the $r(t)$ and $w(t)$. We will do that in the next subsection.

4.2 CGP Growth Rates

Note that $\frac{K^i(t)}{X(t)L^i(t)} = k(t)$ for all i implies

$$\frac{K(t)}{X(t)\bar{L}} = k(t)$$

Since F_K and F_L are HDO:

$$\begin{aligned} r(t) &= B^M F_K \left(K^M(t), X(t)L^M(t) \right) = B^M F_K(k(t), 1) \\ &= B^M F_K(K(t), X(t)\bar{L}) \end{aligned}$$

and same for $w(t)$

Therefore, the sum of feasibility constraints is

$$p^A c^A(t) + c^M(t) + p^S c^S(t) + \dot{K}(t) = B^M F(K(t), X(t)\bar{L}) - \delta K(t) \tag{4}$$

4.2.1 Constant Capital-Labor Output Ratio

If CGP exists, then $r(t) = r$ and therefore $k(t) = k$ so that all sectorial capital-labor ratios are constant

$$K^i(t) = k \cdot X(t)L^i(t)$$

Sum up across i:

$$K(t) = k \cdot X(t)\bar{L} \tag{5}$$

Therefore

$$\begin{aligned} &\frac{\dot{K}(t)}{K(t)} = g \\ \Rightarrow &\dot{K} = gK(t) \end{aligned} \tag{6}$$

since the only value changing on the RHS of (5) is $X(t)$, and it's multiplied by the other two terms.

Then plugging (6) into (4) gives the aggregate feasibility equation:

$$p^A c^A(t) + c^M(t) + p^S c^S(t) = B^M F(K(t), X(t)\bar{L}) - (\delta + g)K(t) \tag{7}$$

4.2.2 Existence of CGP

Then re-writing (7) as

$$\begin{aligned}
& p^A \left(c^A(t) + \gamma^A\right) + c^M(t) + p^S \left(c^S(t) + \gamma^S\right) \\
& - \left[p^A \gamma^A + p^S \gamma^S\right] \\
= & B^M F(K(t), X(t)\bar{L}) - (\delta + g) K(t)
\end{aligned}$$

implies that $p^A \left(c^A(t) + \gamma^A\right), c^M(t), p^S \left(c^S(t) + \gamma^S\right)$ grow at the same rates.

Apply Uzawa's arguments: can have balanced growth only if

$$p^A \gamma^A + p^S \gamma^S = 0$$

Proposition 4.1. *In the above-described economy a CGP exists if and only if*

$$\frac{\gamma^A}{B^A} + \frac{\gamma^S}{B^S} = 0$$

In a CGP $k(t) = k$ for all t, and moreover

$$\frac{\dot{c}^A}{c^A} = g \frac{c^A + \gamma^A}{c^A}, \frac{\dot{c}^M}{c^M} = g, \frac{\dot{c}^S}{c^S} = g \frac{c^S + \gamma^S}{c^S}$$

- *Growth rate in S starts high and asymptotes to g as $c^S \to \infty$*
- *Growth rate in A starts low and asymptotes to g as $c^A \to \infty$*

4.2.3 Labor Transition

We have

$$c^i(t) = X(t) L^i(t) B^i F(k, 1) \text{ for } i \in \{A, S\}$$

This implies

$$\frac{\dot{c}^i}{c^i} = \frac{\dot{X}}{X} + \frac{\dot{L}^i}{L^i}$$

In Pset 2:

We showed that this + previous equations imply in CGP

$$\frac{\dot{L}^M}{L^M} = 0, \frac{\dot{L}^A}{L^A} < 0, \frac{\dot{L}^S}{L^S} > 0.$$

4.3 Useful Facts

4.3.1 HD1 F

In our context, F is HD1. That is,

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

4.3.2 HD1 F Implications; HD0 Partials

If $F(K, L)$ is HD1 then

$$F(K, L) = F_K(K, L)K + F_L(K, L)L \tag{8}$$

and $F_K(K, L), F_L(K, L)$ are HD0.

That is,

$$\begin{aligned} F_K(\lambda K, \lambda L) &= F_K(K, L) \\ F_L(\lambda K, \lambda L) &= F_L(K, L) \end{aligned}$$

5 Lecture 3: Structural Change - Supply Side

5.1 Terms

- ρ: discount rate
- t: time
- $c(t)$: consumption at time t
- $i \in \{A, M, S\}$: sector
- $c^i(t)$: consumption of good i at time t
- η_i: long-run share of consumption in sector i
- $K^i(t)$: capital in sector i at time t
- $L^i(t)$: labor in sector i at time t
- $Y^i(t)$: output of sector i at time t
- $p^i(t)$: price of one unit of $c^i(t)$ at time t

- $X^i(t)$: labor-augmenting technology in sector i at time t
- g_i: growth rate of labor-augmenting technology, X, in sector i
- σ: price elasticity

5.2 Setup

5.2.1 Preferences

$$\int_0^\infty \exp(-\rho t)\frac{c(t)^{1-\theta}-1}{1-\theta}dt$$

with

$$c(t) = \left(\sum_{i\in\{A,S,M\}} \eta^i c^i(t)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}$$

5.2.2 Technology with Unequal Growth

$$\begin{aligned} Y^i(t) &= X^i(t)K^i(t)^\alpha L^i(t)^{1-\alpha} \\ \dot{X}^i(t)/X^i(t) &= g^i \end{aligned}$$

5.2.3 Additional Assumption

- Inelastic labor
- M produces all capital
- price elasticity, σ, assumed to be same for all goods
- Income elasticity is 1 for all goods

5.2.4 Intratempotal optimality condition

$$\frac{c^i}{c^j} = \left(\frac{\eta^i}{\eta^j}\right)^\sigma \left(\frac{p^i}{p^j}\right)^{-\sigma} \tag{9}$$

If p^i/p^j increases then relatively consumption shares p^ic^i/p^jc^j

- decreases if $\sigma < 1$

- constant if $\sigma = 1$
- increases if $\sigma > 1$

I'm not sure that I understand why this is this way.

5.2.5 Feasibility Constraint

$$\begin{aligned} c^M + \dot{K} &= X^M \left(K^M\right)^\alpha \left(L^M\right)^{1-\alpha} - \delta K \\ c^A &= X^A \left(K^A\right)^\alpha \left(L^A\right)^{1-\alpha} \\ c^S &= X^S \left(K^S\right)^\alpha \left(L^S\right)^{1-\alpha} \end{aligned}$$

Multiply by p^i and sum to get

$$C + \dot{K} = X^M (K)^\alpha (\bar{L})^{1-\alpha} - \delta K$$

where

$$C \equiv \sum_{i \in \{A,M,S\}} p^i c^i$$

To make progress, lets express dynamic conditions in terms of C

5.2.6 More Intertemporal Optimality

Let λ be multiplier on the consumer's budget constraint and

$$C(t) \equiv \sum_{i \in \{A,M,S\}} p^i(t) c^i(t)$$

be total consumption expenditures. Show that optimality requires[3]

$$\frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} = (1-\theta)\frac{\dot{c}}{c}$$

and

$$\frac{\dot{\lambda}}{\lambda} = -\left(\alpha X^M K^{\alpha-1} \bar{L}^{1-\alpha} - \delta - \rho\right)$$

5.2.7 Euler Equation

So we have two conditions

$$\begin{aligned} C + \dot{K} &= X^M K^\alpha \bar{L}^{1-\alpha} - \delta K \\ \frac{\dot{C}}{C} - (1-\theta)\frac{\dot{c}}{c} &= \left(\alpha X^M K^{\alpha-1} \bar{L}^{1-\alpha} - \delta - \rho\right) \end{aligned}$$

[3]Not sure that I understand what little c is with no superscript.

Note that if $\theta = 1$, these are the optimality conditions of the neoclassical growth model

- $\theta = 1$ is a sufficient condition to deliver Kaldor facts
- it also turns out to be a necessary condition

5.3 Results

5.3.1 Production efficiency

Firm's FOCS:

$$p^i(t)X^i(t)\alpha\left(\frac{K^i(t)}{L^i(t)}\right)^{\alpha-1} = r(t)$$
$$p^i(t)X^i(t)(1-\alpha)\left(\frac{K^i(t)}{L^i(t)}\right)^{\alpha} = w(t)$$

From firm's optimization, capital-labor ratios are equalized across sectors

$$\frac{K^i(t)}{L^i(t)} = k(t) \text{ for all } i$$

Relative prices reflect relative productivities

$$\frac{p^i(t)}{p^j(t)} = \frac{X^j(t)}{X^i(t)} \text{ for } i, j \in \{A, S, M\} \tag{10}$$

Notice, relative prices fall in sectors with higher productivity growth.

5.3.2 Consumption Side

Plug these into intratremporal optimality for consumers – that is combine (9) and (10) – to get:

$$\frac{p^i(t)c^i(t)}{p^j(t)c^j(t)} = \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{X^j(t)}{X^i(t)}\right)^{1-\sigma} \tag{11}$$

Consumption share p^ic^i for the more stagnant sector

- increases if $\sigma < 1$ (empirically relevant case)
- constant if $\sigma = 1$

- decreases if $\sigma > 1$

That is, if demand is elastic, then $1-\sigma$ is negative and an increase in the productivity of j means that the RHS is getting smaller, i.e., the share of consumption of i relative to j is decreasing. This is sensible as elastic demand would indicate that the increase in demand for j would be proportionally larger than its decrease in price. The reverse is true for inelastic demand.

5.3.3 Labor Allocation

For $i, j \neq M$,

$$c^i(t) = X^i(t)k(t)^\alpha L^i(t)$$

Then, we have:

$$\begin{aligned}
\frac{c^i(t)}{c^j(t)} &= \frac{X^i(t)k(t)^\alpha L^i(t)}{X^j(t)k(t)^\alpha L^j(t)} \\
&= \frac{X^i(t)}{X^j(t)}\frac{L^i(t)}{L^j(t)} \\
&= \frac{P^j(t)}{P^i(t)}\frac{L^i(t)}{L^j(t)} && \text{by (10)} \\
\Rightarrow \frac{c^i(t)}{c^j(t)}\frac{P^i(t)}{P^j(t)} &= \frac{L^i(t)}{L^j(t)} \\
\Rightarrow \frac{L^i(t)}{L^j(t)} &= \left(\frac{\eta^i}{\eta^j}\right)^\sigma \left(\frac{X^j(t)}{X^i(t)}\right)^{1-\sigma} && \text{by (11)}
\end{aligned}$$

This gives[4]

$$\frac{\dot{L}_i(t)}{L_i(t)} - \frac{\dot{L}_j(t)}{L_j(t)} = (1-\sigma)\left(g^j - g^i\right) \text{ for } i \in \{A, S\}$$

5.3.4 Takeaways

- Suppose demand is inelastic ($\sigma < 1$)
  - prices of faster growing sector fall
  - consumption share of that sector falls
  - labor outflows from that sector
- Same logic extends to arbitrary number of sectors
  - asymptotically, everyone works in the most stagnant sector
  - "Baumol's cost disease"

[4] Would need to think more about this

6 Lecture 8

6.1 Terms

- Y_i: output from firm i
- K_i: capital in firm i
- L_i: labor in firm i
- A_i: technology for firm i
- P_i: price for firm i
- $\bar{K}$: Consumers inelastic endowment of capital
- $\bar{L}$: Consumers inelastic endowment of labor
- $U(C)$: Utility derived from capital
- σ: Elasticity of substitution[5]
- Π_i: Profit for firm i
- r: rental rate of capital
- w: wage rate

6.2 Setup

6.2.1 Intermediate Sector

- Measure one of firms
- Firm $i \in [0, 1]$ produces differentiated product Y_i with technology

$$Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}$$

6.2.2 Final Sector

- Competitive market
- We assume $\sigma > 1$ in the production function as there's no equilibrium otherwise

Production Function:

$$Y = \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

[5] Not sure if this is right

Maximization Problem

The final sector solves:

$$\max_{\{Y_i\}} Y - \int P_i Y_i di$$
$$\text{s.t.}$$
$$Y = \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

FOC for i:

$$Y_i = Y \times P_i^{-\sigma}$$

6.2.3 Household

- Consumers buy final goods
- Inelastic endowment of capital and labor

6.2.4 Equilibrium

Consumers

Prices $\{P_i\}_i, r, w$, allocations $\{Y_i, \Pi_i, K_i, L_i\}_i, C, Y$ such that:

Consumers own firms and get profits $\int \Pi_i di$, supply labor $\bar{L}$ and capital $\bar{K}$ inelastically at w and r and solve

$$\max_{C} U(C)$$
$$\text{s.t.}$$
$$C = w\bar{L} + r\bar{K} + \int \Pi_i di$$

Final Firms

Final goods firms take prices $\{P_i\}$ as given and solve

$$
\begin{aligned}
&\max_{\{Y_i\}_i, Y} Y - \int P_i Y_i di \\
&\text{s.t.} \\
&Y = \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

which produces demand for good i as $Y_i(P_i) = Y P_i^{-\sigma}$

Intermediate Firms

Intermediate firms take w and $Y_i(P_i)$ as given and solve

$$\Pi_i = \max_{P_i, Y_i, L_i, K_i} P_i Y_i - wL_i - RK_i$$

s.t.

$$
\begin{aligned}
Y_i &= Y P_i^{-\sigma} \\
Y_i &= A_i K_i^{\alpha} L_i^{1-\alpha}
\end{aligned}
$$

Market Clearing

$$C = Y, \int K_i di = \bar{K}, \int L_i di = \bar{L}$$

6.2.5 Social Planner's

Social planner chooses allocations subject to feasibilities:

$$\max_{Y, \{Y_i, K_i, L_i\}_i} U(Y)$$

s.t.

$$
\begin{gathered}
Y = \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \\
Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}, \\
\int K_i di = \bar{K}, \quad \int L_i di = \bar{L}
\end{gathered}
$$

6.3 Results

6.3.1 Optimal Price Equation

$$P_i = \underbrace{\frac{\sigma}{\sigma - 1}}_{\text{mark up} > 1} \times \underbrace{\frac{1}{A_i} \left(\frac{R}{\alpha} \right)^{\alpha} \left(\frac{w}{1 - \alpha} \right)^{1-\alpha}}_{\text{marginal cost, } \lambda_i}$$

- All firms charge the same mark up $\frac{\sigma}{\sigma-1}$ over marginal costs
  - perfect competition limit as $\sigma \to \infty$

7 Lecture 9

7.1 Terms

7.2 Setup

7.3 Results

7.3.1 TFP of Intermediate Goods Sector

- Value added of the intermediate good sector is

$$\int P_i Y_i di = Y$$

- We would measure sectoral TFP in the data from

$$Y = TFP \times L$$

- What is this TFP?

$$Y = \left(\int \left(A_i L_i\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} = \underbrace{\left(\int \left[A_i \frac{L_i}{L}\right]^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}}_{=TFP} \times L$$

8 Lecture 9

8.1 Terms

- A_i: technology for firm i
- a_i: $\ln A_i - \mathbb{E} \ln A_i$: Log of technology for firm i minus the expected log of technology across firms
- $\bar{a} = \mathbb{E} \ln A_i$: Expected log of technology across firms
- τ_i: Distortion faced by firm i
- $t_i := \ln(1+\tau_i) - \mathbb{E} \ln(1+\tau_i)$

9 Lecture 10

9.1 Terms

- x_{ij}: inputs from firm i used by firm j

- Y: Output of the final good sector
  - Aggregates all of the N producers
- $\mathcal{D}(c_1, \ldots, c_N)$: The production function for the final good sector
- N: Number of producers
- $p_1, \ldots, p_N$: Prices of products
- F: Number of factors
- $\bar{L}_f$: Inelastic supply of factors
- $w_1, \ldots, w_F$: Prices of factors
- F_i: Production function for producer i
- A_i: Technology for producer i
- y_i: Output of producer i
  - $y_i = A_i F_i \left(\{x_{ij}\}_{j=1}^{N+F} \right)$

9.2 Model

9.2.1 Producer's Problem

Each firm is competitive and operates

$$\pi_i = \max_{y_i, \{x_{ij}\}_{j=1}^{N+F}} p_i y_i - \sum_{j=1}^{N} p_j x_{ij} - \sum_{f=1}^{F} w_f x_{if}$$

s.t.

$$y_i = A_i F_i \left(\{x_{ij}\}_{j=1}^{N+F} \right)$$

where A_i is productivity and F is CRS

9.2.2 Final Good Sector Problem

Final good sector is competitive and solves

$$\Pi = \max_{Y, \{c_j\}_{j=1}^{N}} Y - \sum_{j=1}^{N} p_j c_j$$

s.t.

$$Y = \mathcal{D}(c_1, \ldots, c_N)$$

where $\mathcal{D}$ is CRS.

9.2.3 Consumer Problem

Consumers solve

$$\max_{C} U(C)$$

s.t.

$$C = \sum_{f=1}^{F} w_f \bar{L}_f$$

9.2.4 Competitive Equilibrium

- CE is $C, Y, \{c_j, y_j\}_{j=1}^{N}, \{x_{ij}\}_{i=1\ldots N, j=1\ldots N+F} \{p_i\}_{i=1}^{N}, \{w_f\}_{f=1}^{F}$ such that - Consumers, final good sector, all firms solve their problems - Markets clear

$$C = Y,$$

$$y_i = c_i + \sum_{j=1}^{N} x_{ji} \text{ for all } i = 1, \ldots, N$$

$$\bar{L}_f = \sum_{j=1}^{N} x_{jf} \text{ for all } f = 1, \ldots, F$$

9.2.5 Social Planner's Problem

$$\max_{C, \{c_i, y_i, x_{ij}\}} U(C)$$

s.t.

$$C = \mathcal{D}(c_1, \ldots, c_N),$$

$$y_i = A_i F_i\left(\{x_{ij}\}_{j=1}^{N+F}\right) \text{ for all } i,$$

$$y_i = c_i + \sum_{j=1}^{N} x_{ji} \text{ for all } i = 1, \ldots, N,$$

$$\bar{L}_f = \sum_{j=1}^{N} x_{jf} \text{ for all } f = 1, \ldots, F.$$

Note that $y_i = c_i + \sum_{j=1}^{N} x_{ji}$ for all $i = 1, \ldots, N$ is saying that y_i must equal the amount of good y_i consumed by the final good sector, c_i, and the amount of good i consumed by other firms, x_{ji} for each firm j.

$A_i F_i\left(\{x_{ij}\}_{j=1}^{N+F}\right)$ meanwhile says that y_i must equal the amount of good i produced by firm i under their level of technology and production function.

10 Lecture 11

10.1 Terms

- β: discount factor
- $U(.)$: utility function
- C_t: consumption in period t
- L_t: labor in period t
- Y_t: output in period t
- K_t: capital in period t
- A_t: technology in period t
- s_t: exogenous state in period t
- $s^t = (s_0, s_1, \ldots, s_t)$: history of exogenous states up to period t
- $\Pr\left(s^t\right)$: the probability of the history of realizations comprising s_t
- $\Pr\left(s^{t+1} \mid s^t\right) = \Pr\left(s^{t+1}\right) / \Pr\left(s^t\right)$: the conditional probability of s^{t+1} given s^t

10.2 Setup

10.2.1 Household

We have the representative consumer:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right)$$

10.2.2 Firm

We have the representative firm:

$$Y_t = F_t\left(K_t, L_t\right) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

our feasibility constraint is given by:

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t$$

A_t is the only exogenous stochastic process here.

10.2.3 Market Qualities

- The market is complete (for simplicity)

10.2.4 Conceptualization of Our Exogenous Shocks and Endogenous Variables

Let s_t be exogenous state ($s_t = A_t$ in our RBC example with one shock, but more generally it is a vector of all exogenous shocks)

Recall that we denote $s^t = (s_0, \ldots, s_t)$ to be a history of shocks up to period t

Then:

- $\Pr(s^t)$ is the probability of the history of realizations comprising s_t
- $\Pr\left(s^{t+1} \mid s^t\right) = \Pr\left(s^{t+1}\right) / \Pr(s^t)$ is the conditional probability

Endogenous variables in period t are functions of $s^t : X_t = X_t(s^t)$

10.2.5 Social Planner's Problem

The Social Planner's problem is given by:

$$\max_{C_t, N_t, K_{t+1}} \sum_{t=0}^{\infty} \sum_{s^t} \Pr\left(s^t\right) \beta^t U\left(C_t\left(s^t\right), L_t\left(s^t\right)\right) \tag{12}$$

s.t.

K_0 is given

$$C_t\left(s^t\right) + K_{t+1}\left(s^t\right) = A_t K_t\left(s^{t-1}\right)^{\alpha} L_t\left(s^t\right)^{1-\alpha} + (1-\delta) K_t\left(s^{t-1}\right)$$

- Let $\beta^t \Pr(s^t) \lambda_t(s^t)$ be the Lagrange multiplier on this constraint
  - Note that each Lagrange multiplier is specific to each possible history

Then our FOCs are:[6]

$$U_C\left(s^t\right) = \lambda_t\left(s^t\right)$$
$$U_L\left(s^t\right) = -\lambda_t\left(s^t\right) F_L\left(s^t\right)$$
$$\lambda_t\left(s^t\right) = \beta \sum_{s^{t+1}} \Pr\left(s^{t+1} \mid s^t\right)\left(1 + F_K\left(s^{t+1}\right) - \delta\right) \lambda_{t+1}\left(s^{t+1}\right)$$

Properties of SP's Problem

Combining our FOC's yields:

[6] I'm a bit confused what the full Lagrangian looks like this in case.

$$
\begin{aligned}
U_L\left(s^t\right) &= -U_C\left(s^t\right) F_L\left(s^t\right) \\
U_C\left(s^t\right) &= \beta \sum_{s^{t+1}} \Pr\left(s^{t+1} \mid s^t\right) U_C\left(s^{t+1}\right)\left(1+F_K\left(s^{t+1}\right)-\delta\right)
\end{aligned}
$$

Together with feasibility (and TVC)

$$
C_t\left(s^t\right)+K_{t+1}\left(s^t\right)=A_t K_t\left(s^{t-1}\right)^\alpha L_t\left(s^t\right)^{1-\alpha}+(1-\delta) K_t\left(s^{t-1}\right)
$$

these three equations pin down (stochastic) solution $\left(C_t\left(s^t\right), K_{t+1}\left(s^t\right), L_t\left(s^t\right)\right)_{t, s^t}$

Shorthand Notation for the SP's Problem

It is common to use shorthand notation for the SP problem in (12) and write it as:

$$
\max_{C_t, L_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right)
$$

s.t. K_0 given and

$$
C_t+K_{t+1}=A_t K_t^\alpha L_t^{1-\alpha}+(1-\delta) K_t
$$

The corresponding short-hand FOCs are then:

$$
\begin{aligned}
U_{L,t} &= -U_{C,t} F_{L,t} \\
U_{C,t} &= \beta \mathbb{E}_t U_{C,t+1}\left(1+F_{K,t+1}-\delta\right).
\end{aligned}
$$

Recursive Formulation

Suppose A_t is an AR(1) process. Then the SP problem can be written as

$$
V(K, A)=\max_{C, L, K_{+}} U(C, L)+\sum_{A_{+}} \Pr\left(A_{+} \mid A\right) V\left(K_{+}, A_{+}\right)
$$

s.t.

$$
C+K_{+}=A K^\alpha L^{1-\alpha}+(1-\delta) K
$$

Solution is policy functions $\widetilde{X}(K, A)=\widetilde{C}(K, A), \widetilde{N}(K, A), \widetilde{K}_{+}(K, A)$ and for any history of shocks $s^t=A^t$ we can find $X_t\left(s^t\right)$ by substituting these policy functions recursively, e.g.

$$
\begin{aligned}
& X_0\left(s_0\right)=\widetilde{X}\left(K_0, A_0\right) \\
& X_1\left(s_0, s_1\right)=\widetilde{X}\left(\widetilde{K}_{+}\left(K_0, A_0\right), A_1\right) \\
& \ldots
\end{aligned}
$$

10.2.6 Competitive Equilibrium (Simplest Version)

Setup

- The consumer
  - own all capital
  - rent capital and labor to firms
  - trade risk-free bonds (a security that is bought in period t at price Q_t and pays 1 unit of consumption good in period $t+1$)
- Risk-free bond is in zero net supply

Competitive Equilibrium Characterization

Competitive equilibrium are stochastic sequences $\{C_t, K_{t+1}, L_t, B_t\}_t$ and prices $\left\{W_t, R_t, R_t^{rf}\right\}_t$ such that

Household

Consumers solve

$$\max_{C_t, L_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right)$$

s.t. K_0 given, $B_0 = 0$ and

$$C_t + K_{t+1} + Q_t B_{t+1} = W_t L_t + \left(1 + R_t - \delta\right) K_t + B_t$$

Firm

Firms solve

$$\max_{K_t, L_t} A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t K_t$$

Market Clearing

Markets clear

$$B_t = 0$$

Risk Free Interest Rate

Let risk-free interest rate R_{t+1}^{rf} be defined by

$$R_{t+1}^{rf} := 1/Q_t - 1$$

- Note that saving in period t one unit of consumption in
  - bonds pays interest rate $1 + R^{rf}_{t+1}$ in period $t + 1$
  - capital pays interest rate $1 + R_{t+1} - \delta$ in period $t + 1$
  - excess return of capital $rx_{t+1} := (R_{t+1} - \delta) - R^{rf}_{t+1}$

10.3 Results

11 Lecture 12

11.1 Terms

- $P_{i,t}$: Price of product of firm i in period t
- $A_{i,t}$: Technology of firm i in period t
  - $A = A_{i,t}$: Fix technology initially
  - Later may consider stochastic technology
- $L_{i,t}$: Labor of firm i in period t
- W_t: Nominal wage in period t
- $\frac{W_t}{P_t}$: Real wage in period t
- $D_{i,t}$: Dividends of firm i in period t
  - $D_{i,t} = P_{i,t}Y_{i,t} - W_t L_{i,t} - \Phi_{i,t}$
- $\Phi_{i,t}$: Cost of setting price $P_{i,t}$ in period t
  - $\Phi_{i,t} = \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right)^2$
- $\theta \geq 0$: parameter capturing how costly it is for firm i to change its price
- Q_t: Price of a nominal one period risk-free bond purchased at t.
- I_t: Nominal interest rates in period t (controlled by Central Bank)
  - $I_t = 1/Q_t - 1$
  - Initially, we treat I_t (or equivalently Q_t) as an arbitrary stochastic process
- B_t: Bonds held in period t
- Π_t: Inflation in period t
  - $\Pi_t = P_t / P_{t-1}$

11.2 Model

11.2.1 Equilibrium

Consumers

Stochastic sequences $\{C_t, L_t, B_t, D_{i,t}, \Phi_{i,t}\}_{i,t}$, prices $\{P_t, P_{i,t}, W_t, Q_t, \Pi_t\}$ such that

Consumers solve

$$\max_{\{C_t, L_t, B_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right]$$

s.t. $B_{-1} = 0$ and

$$P_t C_t + Q_t B_t = W_t L_t + \int D_{i,t} di + \int \Phi_{i,t} di + B_{t-1}$$

Final Goods Market

Final goods firms solve

$$\max_{\{Y_i\}_i, Y} P_t Y_t - \int P_i Y_i di$$

s.t.

$$Y_t = \left(\int Y_{i,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

Intermediate Goods Market

Easiest to write problem of intermediate firms in recursive form:

$$V_t\left(P_{i,t-1}\right) = \max_{P_{i,t}, Y_{i,t}, L_{i,t}} P_{i,t} Y_{i,t} - W_t L_{i,t} - \Phi_{i,t} + Q_t \mathbb{E}_t V_{t+1}\left(P_{i,t}\right)$$

s.t.

$$Y_{i,t} = A L_{i,t}, \quad Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} Y_t$$

$$\Phi_{i,t} = \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right)^2$$

where dividends equal $D_{i,t} = P_{i,t} Y_{i,t} - W_t L_{i,t} - \Phi_{i,t}$

Monetary Policy

Q_t is a given stochastic process

Market Clearing

$$C_t = Y_t$$
$$L_t = \int L_{i,t} di$$
$$B_t = 0$$

11.3 Optimality

11.3.1 Consumer Optimality

Intratemporal Optimality

$$C_t^{\gamma} L_t^{\varphi} = \frac{W_t}{P_t}$$

Intertemporal Optimality

The inter-temporal optimality conditions

$$\begin{aligned} Q_t &= \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \\ &= \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \end{aligned}$$

11.3.2 Intermediate Firm Optimality

Firm Problem:

$$V_t \left(P_{i,t-1} \right) = \max_{P_{i,t}} P_{i,t} \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} Y_t - \frac{W_t}{A} \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} Y_t - \frac{\theta}{2} P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right)^2 + Q_t \mathbb{E}_t V_{t+1} \left(P_{i,t} \right)$$

FOC:

$$\begin{gathered} (1-\sigma) \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} Y_t + \sigma \frac{W_t}{A} \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} Y_t \frac{1}{P_{i,t}} \\ - \theta P_t Y_t \left(\frac{P_{i,t}}{P_{i,t-1}} - 1 \right) \frac{1}{P_{i,t-1}} + Q_t \mathbb{E}_t \frac{\partial}{\partial P_{i,t}} V_{t+1} \left(P_{i,t} \right) = 0 \end{gathered}$$

How do we handle it when we have a partial derivative of the value function in our FOC? We use the envelope theorem to take the derivative of V_t wrt our relevant state variable holding our choice variable fixed at the optimal choice.

Envelope Theorem:

$$\frac{\partial}{\partial P_{i,t}} V_{t+1}\left(P_{i,t}\right) = \theta P_{t+1} Y_{t+1} \left(\frac{P_{i,t+1}}{P_{i,t}} - 1\right) \frac{P_{i,t+1}}{P_{i,t}^2}$$

All firms are identical, which implies that (see Lecture 8) that $P_{i,t} = P_t$ for all i This simplifies our equation to

$$(1-\sigma) + \sigma \frac{W_t/P_t}{A} - \theta \Pi_t \left(\Pi_t - 1\right) + \theta Q_t \mathbb{E}_t \left(\Pi_{t+1} - 1\right) \Pi_{t+1}^2 \frac{Y_{t+1}}{Y_t} = 0$$

11.3.3 Equilibrium

Combine previous equations and get rid of redundant variables. Then we get that $\{C_t, Q_t, L_t, W_t/P_t, \Pi_t\}_t$ are a competitive equilibrium if and only if they solve

$$(1-\sigma) + \sigma \frac{W_t/P_t}{A} - \theta \Pi_t \left(\Pi_t - 1\right) + \theta Q_t \mathbb{E}_t \left(\Pi_{t+1} - 1\right) \Pi_{t+1}^2 \frac{C_{t+1}}{C_t} = 0$$

$$C_t^{\gamma} L_t^{\varphi} = \frac{W_t}{P_t}$$

$$C_t = A L_t$$

$$Q_t = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$$\{Q_t\}_t \text{ is given}$$