Macro 3 Notes

Contents

1 Introduction

2 General Equilibrium Theory

2.1 Terms

- L: The commodity space.
  - Typically, $L = \mathbb{R}^m$ so that there are m goods
  - Sometimes, we let $m = \infty$
- $i = 1, 2, 3, \ldots, I$: The index of agents
  - Thus, in the above description, there are I agents
  - Sometimes we allow countability infinite agents, $i = 1, 2, 3, \ldots,$ or allow a continuum of agents.
- $u^i : X^i \to \mathbb{R}$: Utility function of agent i
- $X^i \subset L$: The consumption possibility set of agent i
  - That is, a household i is able to consume any vector $x \in X^i$.
  - This set may include constraints, such as non-negativity of consumption.
  - Positive entries in x denote quantity demanded; negative entries denote quantity supplied/offered/sold.
- $j = 1, 2, \ldots, J$: The index of firms
  - Thus, in the above description, there are J firms
  - Sometimes we allow countability infinite firms, $j = 1, 2, 3, \ldots,$ or allow a continuum of firms.
- $Y^j \subset L$: The production possibility set or technology of firm j
  - $y \in Y^j$: The firm j is able to produce any vector $y \in Y^j$.
  - Positive entries in y denote quantity supplied/sales; negative entries denote quantity demanded/purchased by the firm.
  - As an example, consider the Cobb-Douglas production function:

$$F^j\left(k^j, \ell^j\right) = \left(k^j\right)^{\alpha}\left(\ell^j\right)^{1-\alpha}$$

    could be expressed in terms of Y^j as:

$$\mathbf{Y}^j = \left\{\mathbf{y} = \left(y^j, -k^j, -\ell^j\right) \in \mathbb{R}^3 : y^j = \left(k^j\right)^{\alpha}\left(\ell^j\right)^{1-\alpha}, \left\{y^j, k^j, \ell^j\right\} \mathbb{R}^3_+\right\}$$

- $\theta^i_j \geq 0$: Consumer i's ownership share of firm j.

$$\sum_{i \in I} \theta^i_j = 1 \text{ for all } j \in J$$

- $e^i \in L$: The endowment of agent i.

- p: price vector
  - Supposing m commodities (i.e., $L = \mathbb{R}^m$),
  $$p = (p_1, \ldots, p_m)$$

2.2 Feasible Allocations

Definition D.1: Feasible Allocation

A feasible allocation, $\{x^i, y^j\}$, satisfies three conditions:

1. Each consumer can consume x^i,
$$x^i \in X^i, \forall i \in I$$
2. Each firm can produce y^j,
$$y^j \in Y^j, \forall j \in J$$
3. Demand equals supply:
$$\sum_{i \in I} x^i = \sum_{j \in J} y^j + \sum_{i \in I} e^i$$

2.3 Key Concepts for Competitive Equilibrium

Suppose there are m commodities (i.e., $L = \mathbb{R}^m$).

Then the price vector p is given by:

$$p = (p_1, \ldots, p_m)$$

and we have the following equation:

$$p \cdot x = p_1 x_1 + p_2 x_2 + \cdots + p_m x_m \equiv \sum_{s=1}^{m} p_s x_s$$

$p \cdot x$ is the value of the commodity vector x in terms of the numeraire[1]

A quick useful quality to note:

$$p \cdot x^a + p \cdot x^b = p \cdot \left(x^a + x^b\right)$$

[1]By "in terms of the numeraire," I just mean that we are normalizing the price of some good (usually the first element in x) to be 1 and then expressing the price of the other goods relative to that one.

2.4 Competitive Equilibrium

Definition D.2: Competitive Equilibrium

We denote prices by a vector $p \in R^m$.

A competitive equilibrium is a price vector p, and a feasible allocation $\left\{x^i, y^j\right\}$ such that

1. each firm $j \in J$ maximize its profits. We denote the profit of firm j by π^j.

2.5 Welfare Theorems

Definition D.3: Local Non-Satiation

Utility function $u^i : \mathbf{X}^i \to \mathbb{R}$ satisfies the local non-satiation (LNS) property if, for any $\mathbf{x} \in \mathbf{X}^i$ and any neighborhood of $\mathbf{x}$, denote $B_\varepsilon(\mathbf{x})$, there exists $\hat{\mathbf{x}} \in B_\varepsilon(\mathbf{x})$ such that $u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x})$.

Theorem T.1: First Welfare Theorem

Suppose that u^i satisfies local non-satiation for all $i \in \mathbf{I}$. Let $\left\{\mathbf{p}, \overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ be a competitive equilibrium. Then, $\left\{\overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ is a Pareto optimal allocation.

Definition D.4: Convexity

A function $u^i : \mathbf{X}^i \to \mathbb{R}$ is strictly convex if, for any $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^i$ such that $\mathbf{x}' \neq \mathbf{x}$, we have

$$u^i\left(\theta \mathbf{x} + (1-\theta)\mathbf{x}'\right) > \theta u^i(\mathbf{x}) + (1-\theta) u^i\left(\mathbf{x}'\right), \forall \theta \in (0,1).$$

Definition D.5: Quasiconcavity

A function $u^i : \mathbf{X}^i \to \mathbb{R}$ is (strictly) quasiconcave if its upper contour set $\left\{x \in \mathbf{X}^i : u^i(\mathbf{x}) \geq u^i(\overline{\mathbf{x}})\right\}$ is (strictly) convex for all $i \in \mathbf{I}$ and all $\overline{\mathbf{x}} \in \mathbf{X}^i$. Equivalently, for any $\mathbf{x} \neq \overline{\mathbf{x}}$, we must have

$$u^i(\alpha \mathbf{x} + (1-\alpha)\tilde{\mathbf{x}}) > \min\left\{u^i(\mathbf{x}), u^i(\overline{\mathbf{x}})\right\}, \forall \alpha \in (0,1)$$

Theorem T.2: Second Welfare Theorem

Assumption 1. (Assumption HH) Assume that $\mathbf{X}^i$ are convex for all $i \in \mathbf{I}$ and that $u^i : \mathbf{X}^i \to \mathbb{R}$ are continuous and strictly quasiconcave.

Assumption 2. (Assumption FF) Assume that the aggregate production sumset of the economy is convex; i.e.

$$\mathbf{Y} := \left\{ \mathbf{y} \in \mathbf{L} : \mathbf{y} = \sum_{j=1}^{J} \mathbf{y}^j, \mathbf{y}^j \in \mathbf{Y}^j, \forall j \in \mathbf{J} \right\}.$$

Let $\left\{\overline{\mathbf{x}}^i, \overline{\mathbf{y}}^j\right\}$ be a Pareto optimal allocation. Then, there exists a price vector $\mathbf{p}$ such that: (i) all firms maximise profits such that, for all $j \in \mathbf{J}$,

$$\mathbf{p} \cdot \overline{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}, \forall \mathbf{y} \in \mathbf{Y}^j$$

(ii) given allocation $\left\{\overline{\mathbf{x}}^i\right\}$, consumers minimise expenditure subject to attaining at least the same utility obtained by consuming $\overline{\mathbf{x}}^i$; i.e.

$$\overline{\mathbf{x}}^i \in \underset{\mathbf{x} \in \mathbf{X}^i}{\arg\min} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u^i(\mathbf{x}) \geq u^i\left(\overline{\mathbf{x}}^i\right).$$

3 Aggregation

3.1 Terms

- U: The set of utilities that are achievable for a feasible allocation

$$U = \left\{u \in R^I : u_i \leq u^i\left(x^i\right) \text{ all } i \in I \text{ for some feasible allocation } \left\{x^i, y^j\right\}\right\}$$

4 Overlapping Generations Economy

4.1 Terms

Baseline Model

- x_i: The set of commodities consumed in period i
- $t = 1, 2, \ldots$: Index for time
- $I = \{0, 1, 2, \ldots\}$: Generation of agents
- v^i: Utility for agents born in generation $i \geq 1$ as a function of only the commodities in their periods they are alive
- u^i: Utility of agent i
  - If $i \geq 1$

$$u^i\left(x_1, x_2, x_3, \ldots\right) = v^i\left(x_i, x_{i+1}\right)$$

  - If $i = 0$

$$u^0\left(x_1, x_2, x_3, \ldots\right) = x_1^0$$

- $e^i = \left(e_1^i, e_2^i, \ldots, e_i^i, e_{i+1}^i, \ldots\right)$: The endowment of agents born in generation i
  - $e_i^i > 0$
  - $e_{i+1}^i > 0$

- $e_t^i = 0$ for all $t \neq i$ or $t \neq i+1$
- $e_1^0 > 0$ and $e_t^0 = 0$ for all $t = 2, 3 \ldots$

- $p = (p_1, p_2, p_3, \ldots)$: Price vector (across generations)
- r_t: The time t interest rate satisfying

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

- c: Consumption
  - c_y: Consumption while young
  - c_o: Consumption while old
  - c_i^i: Consumption for generation i in period i
  - c_{i+1}^i: Consumption for generation i in period $i+1$
  - $\bar{c}_i^j$: Equilibrium consumption for generation i in period j
  - c_i^{*j}: Best symmetric allocation consumption for generation i in period j

Model with the Below Utility Function

$$v^i(c_y, c_o) = (1-\beta)\log c_y + \beta \log c_o$$

for some constant $\beta \in (0,1)$.

- $\beta \in (0,1)$: Weight placed on consumption while old
  - $(1-\beta)$: Weight placed on consumption while young
- $\alpha \in (0,1)$: Share of total lifetime endowment received while old
  - $(1-\alpha)$: Share of total lifetime endowment received while young
- $\bar{e}_t = 1$: Total lifetime endowment for all $t \geq 1$

Terms Added for Social Security

- τ: Taxes on young paid to old

Terms Added in Growing Economy

- N_t: Number of young agents at time t
  - $N_0 = 1$
- n: Growth rate of population

- g: Growth rate of productivity of the endowments. That is, we have

$$e_{t+1}^{t+1} = (1+g)e_t^t$$
$$e_{t+2}^{t+1} = (1+g)e_{t+1}^t$$

and

$$e_t^t = (1+g)^t(1-\alpha)$$
$$e_{t+1}^t = (1+g)^t\alpha$$

4.2 Pure Exchange Economy

4.2.1 Competitive Equilibrium

The price vector p is an element of R^∞, so that

$$p = (p_1, p_2, p_3, \ldots)$$

The agent problem is

$$\max_x u^i(x)$$

subject to

$$px \leq pe^i$$

which, since generation i neither consume nor has endowments at time $t \neq i$ or $t \neq i+1$, can be specialized as

$$\max_{x_i, x_{i+1}} v^i\left(x_i, x_{i+1}\right)$$

subject to

$$p_i x_i + p_{i+1}x_{i+1} = p_i e_i^i + p_{i+1}e_{i+1}^i$$

and for generation $i = 0$ as

$$\max_{x_1} x_1 \text{ subject to } p_1 x_1 = p_1 e_1^0.$$

4.2.2 No Trade

Proposition P.1

The only competitive equilibrium has

$$x^i = e^i$$

i.e. there is no trade in equilibrium.

4.2.3 Equilibrium Prices

Normalize

$$p_1 = 1$$

We have:

$$\frac{p_{i+1}}{p_i} = \frac{v_2^i\left(e_i^i, e_{i+1}^i\right)}{v_1^i\left(e_i^i, e_{i+1}^i\right)}$$

for all $i \geq 1$

With r_t net interest rate,

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

for all t. From our previous condition we have

$$r_t = \frac{v_1^t\left(e_t^t, e_{i+1}^t\right)}{v_2^t\left(e_t^t, e_{t+1}^t\right)} - 1$$

for all $t > 1$ and

$$p_t = \frac{1}{\left(1+r_1\right)\left(1+r_2\right)\cdots\left(1+r_{t-1}\right)}.$$

4.2.4 Equilibrium Prices Under Specified Utility Function

Taking utility function

$$v^i\left(c_y, c_0\right) = (1-\beta)\log c_y + \beta \log c_0$$

we have:

$$r_t \equiv \bar{r} = \frac{(1-\beta)}{\beta}\frac{\alpha}{1-\alpha} - 1 = \frac{\alpha-\beta}{\beta(1-\alpha)}$$

or

$$p_t = \left[\frac{\beta}{(1-\beta)}\frac{1-\alpha}{\alpha}\right]^{t-1} \quad \text{for } t \geq 1$$

4.2.5 Best Symmetric Allocation

We will solve for the best feasible symmetric allocation, where best is for the point of view of the young. In particular, consider the problem

$$\max_{c_y,c_o} v\left(c_y, c_o\right) = \max_{c_y,c_o}(1-\beta)\log c_y + \beta \log c_o$$

subject to

$$c_y + c_o = 1$$

Its sufficient first order condition is given by

$$\frac{\beta}{1-\beta}\frac{c_y}{c_0} = 1$$

so the solution of this f.o.c. that also is feasible, i.e. the solution of the problem is

$$c_y = 1-\beta, c_o = \beta.$$

The best symmetric allocation depends on β in this way because for higher preference parameter β agents give less weight to consumption when young and more weight to consumption when old.

4.2.6 Comparison of CE and Best Symmetric Allocation

We will compare the utility of the unique competitive equilibrium allocation

$$\bar{c}_i^i = 1-\alpha, \quad \bar{c}_{i+1}^i = \alpha \text{ for } i \geq 1 \text{ and } \bar{c}_1^0 = \alpha$$

with the one for the best symmetric allocation

$$c_i^{*i} = 1-\beta, c_{i+1}^{*i} = \beta \text{ for } i \geq 1 \text{ and } c_1^{*0} = \beta$$

Notice that, since the CE allocation has $x^i = e^i$, and since that allocation is a feasible symmetric allocation, then, unless $c^* = \bar{c}$-which happens only when $\alpha = \beta-$ the best symmetric feasible allocation is strictly preferred by the agents of generations $i = 1, 2, \ldots$. It only remains to compare the utility of the initial old, i.e. generation $i = 0$, between the best symmetric and CE allocations.

- Case 1: $\beta > \alpha$: All generations prefer Best Symmetric Allocation
- Case 2: $\beta = \alpha$: Indifferent between CE and Best Symmetric Allocation
- Case 3: $\beta < \alpha$: Original Generation prefers CE

4.3 Social Security

4.3.1 After-Tax Endowments

$$
\begin{aligned}
&e_i^i = (1-\alpha) - \tau \text{ and } e_{i+1}^i = \alpha + \tau \text{ for all } i \geq 1 \\
&e_1^0 = \alpha + \tau
\end{aligned}
$$

Notice that by suitable choice of τ we can make the after-tax endowments equal to the best symmetric allocations, the required τ is

$$\tau = \beta - \alpha$$

4.4 Growing Economy

4.4.1 Setup

We will now consider an economy with population and productivity growth. Let N_t the number of young agents at time t. Let n be the growth rate of population, so that

$$N_{t+1} = (1+n)N_t \text{ for } t \geq 1 \text{ and } N_0 = 1.$$

Let g denote the growth rate of productivity of the endowments of each cohort, so that

$$e_{t+1}^{t+1} = (1+g)e_t^t \text{ and } e_{t+2}^{t+1} = (1+g)e_{t+1}^t$$

so that

$$
\begin{aligned}
e_t^t &= (1+g)^t(1-\alpha) \\
e_{t+1}^t &= (1+g)^t\alpha
\end{aligned}
$$

for all $t \geq 1$.

4.4.2 Feasible Symmetric Allocation

Define the feasible symmetric allocations as those solving

$$N_t c_y^t + N_{t-1}c_o^t = N_t(1-\alpha)(1+g)^t + N_{t-1}\alpha(1+g)^{t-1}$$

where each agent born at time t and young at t consumes

$$c_y^t = \hat{c}_y(1+g)^t,$$

and each agent born at time $t-1$ and old at t consumes

$$c_o^t = \hat{c}_o(1+g)^{t-1}.$$

Notice that this constraint can be written as

$$\hat{c}_y(1+g)(1+n) + \hat{c}_o = (1-\alpha)(1+g)(1+n) + \alpha$$

5 OLG Perpetual Youth Model

5.1 Terms

Baseline Model

- dt: An amount of time
- $p\ dt$: The probability of agent dying in dt
  - $p \in (0, \infty)$
  - $\frac{1}{p}$: Expected lifetime
- $N(s,t)$: Size of cohort born at time s at time t
  - $N(s, t+\Delta) = N(s,t)(1-p\Delta)$: The size of the cohort born at time s after Δ amount of time is the size of the cohort at time t times the probability of not dying in Δ amount of time
  - $N(s,s) = p$
  - $N(s,t) = pe^{-p(t-s)}$: Expression for the size of the cohort born at time s at time t [2]
- r: Net risk-less interest rate
- v: An investment in period t that pays $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t+\Delta$, and zero if dead.
- $\theta \in (0,\infty)$: Discount rate
  - 1 util at time $t+\Delta$ is worth $\frac{1}{1+\Delta\theta}$ at t.
- z: We use z in the expected utility function as the future time that we integrate over
- $R(t,z)$: price of a good at time z in terms of goods in time t
- $v(t)$: Non-human (financial) wealth at time t
- $y(z)$: Labor income at time z
- $h(t)$: human wealth at time t

$$h(t) = \int_t^\infty y(z)R(t,z)dz$$

- $c(t)$: Consumption at time t
- $pv(t)$: Rate at which insurance company receives payment from those who die, as well as rate that it pays out premia.

Aggregation Section Added Terms

- $Y(t)$: The aggregate labor income of agents alive at time t

[2] Is p playing two roles here or are they connected? Seems they must be connected or the notation would be crazy.

- α: The rate at which the share of $Y(t)$ endowed to generation s falls as s increases
- $y(s,t)$: The labor income in time t of a living agent born at time s

$$y(s,t) = \frac{p+\alpha}{p} Y(t) e^{-\alpha(t-s)}, \quad \text{for } \alpha \geq 0$$

  - Notice that at any point in time t, the share of $Y(t)$ endowed to generation s is falling at the rate α as s increase

- $C(t)$: Aggregate consumption at time t

$$C(t) \equiv \int_{-\infty}^{t} N(s,t)c(s,t)ds$$

  and

$$C(t) = (p+\theta)(H(t) + V(t))$$

- $c(s,t)$: Share of consumption at time t of a living agent born at time s

$$c(s,t) = (p+\theta)(h(s,t) + v(s,t))$$

  aka

$$\frac{dc(s,t)}{dt} = [r(t) - \theta]c(s,t) \quad \text{BC + and boundary condition}$$

- $h(t,s)$: Human wealth at time t of a living agent born at time s

$$\begin{aligned} h(t,s) &= \int_{t}^{\infty} aY(z)e^{-\alpha(z-s)}R(t,z)dz \\ &= a\left[\int_{t}^{\infty} Y(z)e^{-\alpha(z-t)}R(t,z)dz\right] e^{-\alpha(t-s)} \end{aligned}$$

- $H(t)$: Aggregate human wealth at time t

$$H(t) \equiv \int_{-\infty}^{t} N(s,t)h(s,t)ds$$

- $V(t)$: Aggregate non-human wealth at time t

$$V(t) \equiv \int_{-\infty}^{t} N(s,t)v(s,t)ds$$

General Equilibrium Section Added Terms

- $F(K) = \mathbb{F}(K,1) - \delta K$: CRTS neoclassical production function
- K: Capital
- δ: Depreciation rate

- $V(t)$ Non-human wealth
  - In one case, we define: $V(t) = K(t)$
  - If we then add government debt, we define: $V(t) = K(t) + B(t)$
- $Y(t)$: Is this labor income or output in this case?
  - $Y(t) = F(K(t)) - F'(K(t))K(t)$
  - $r(t) = F'(K(t))$
- $B(t)$: Government debt

5.2 Setup

Agents that die replaced by newborns. Thus, adding all cohort alive at time t yields:

$$\int_{-\infty}^{t} N(s,t)ds = \int_{-\infty}^{t} pe^{-p(t-s)}ds = 1.$$

5.3 Insurance, Annuities

Invest v at t, gets $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t+\Delta$, and zero if dead.

Continuous time (as $\Delta \downarrow 0$) : $v\frac{1+\Delta r}{1-p\Delta} = v + v(r+p)\Delta + o(\Delta)$

5.4 Household Problem

$$\max \mathbb{E}\left[\int_{t}^{\infty} u(c(z))e^{-\theta(z-t)}dz\right] = \int_{t}^{\infty} \log(c(z))e^{-(p+\theta)(z-t)}dz$$

subject to

$$\int_{t}^{\infty} [c(z) - y(z)]R(t,z)dz = v(t)$$

5.5 Budget Constraint and Human Wealth

The individual's dynamic budget constraint is:

$$\frac{dv(t)}{dt} = (r(t)+p)v(t) + y(t) - c(t)$$

We also have a no-Ponizi-game (NPG) condition:

$$\lim_{z\to\infty} v(z)R(t,z) = 0$$

The price of a good in time z in terms of goods in time t is given by:

$$R(t,z) := \exp\left[-\int_t^z (r(\mu)+p)d\mu\right]$$

We can also integrate the dynamic budget constraint to get the intertemporal budget constraint:

$$v(t) = \int_t^\infty [c(z)-y(z)]R(t,z)dz$$

We define human wealth as the present value of all future income, i.e.,:

$$h(t) = \int_t^\infty y(z)R(t,z)dz$$

with the boundary condition

$$\lim_{z\to\infty} h(z)R(t,z) = 0$$

which is equivalent to:

$$\frac{dh(z)}{dz} = [r(z)+p]h(z) - y(z) \text{ with } \lim_{z\to\infty} R(t,z)h(z) = 0$$

5.6 Optimal Consumption

We find that the solution to our problem is:

$$c(t) = (\theta+p)(v(t)+h(t))$$

The law of motion (or, Euler equation) for consumption is:

$$\frac{dc(t)}{dt} = (r(t)-\theta)c(t)$$

Notes

A few miscellaneous points from Tak's notes:

- We will later find that in the steady state, the interest rate must be higher than our discount rate, i.e., $r > \theta$. We could re-frame this as the interest rate must be higher than our impatience, hence savings will accumulate over time.
- Consumption is independent of the interest rate. This is due to the assumption of log utility, which implies that the substitution and income effect from changes in the interest rate exactly offset each other.

5.7 Aggregation

5.7.1 Distribution of Labor Income

Given aggregate labor income at the time t, $Y(t)$, we have:

$$y(s,t) = \frac{\alpha + p}{p} Y(t) e^{-\alpha(t-s)}, \alpha \geq 0$$

as the expression for the labor income in time t of a living agent born at time s.

This means that, at any particular point in time t, the share of $Y(t)$ endowed to generation s falls at the rate α as s increases.

5.7.2 Aggregate Human Wealth

The aggregate human wealth is defined as

$$\begin{aligned}
H(t) &:= \int_{-\infty}^{t} h(s,t) N(s,t) ds \\
&= \int_{-\infty}^{t} h(s,t) p e^{-p(t-s)} ds \\
&= \int_{-\infty}^{t} ap \left[\int_{t}^{\infty} Y(z) e^{-\alpha(z-t)} R(t,z) dz \right] e^{-\alpha(t-s)} e^{-p(t-s)} ds \\
&= \int_{t}^{\infty} Y(z) \exp\left\{ - \int_{t}^{z} (\alpha + p + r(\mu)) d\mu \right\} dz
\end{aligned}$$

We can then get:

$$\frac{dH(t)}{dt} = (r(t) + p + \alpha) H(t) - Y(t)$$

5.7.3 Aggregate Non-Human Wealth

The aggregate non-human wealth is defined as

$$\begin{aligned} V(t) &:= \int_{-\infty}^{t} N(s,t)v(s,t)ds \\ &= \int_{-\infty}^{t} v(s,t)pe^{-p(t-s)}ds \end{aligned}$$

From which we can get:

$$\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t)$$

5.7.4 Aggregate Consumption

Thus, aggregate consumption is defined as

$$C(t) := \int_{-\infty}^{t} N(s,t)c(s,t)ds.$$

We may also aggregate consumption as

$$C(t) = (p+\theta)(H(t) + V(t)).$$

We can then get:

$$\frac{dC(t)}{dt} = (r(t) + \alpha - \theta)C(t) - (p+\theta)(p+\alpha)V(t)$$

5.7.5 Aggregation Summary

We have now derived the dynamics that describe the aggregate behaviour in the Perpetual Youth Model:

$$\begin{aligned} &\frac{dH(t)}{dt} = (r(t) + p + \alpha)H(t) - Y(t) \\ &\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t) \\ &\frac{dC(t)}{dt} = (r(t) + \alpha - \theta)C(t) - (p+\theta)(p+\alpha)V(t) \\ &C(t) = (p+\theta)(H(t) + V(t)) \end{aligned}$$

We also need a no-Ponzi-game condition, which ensures that agents have finite wealth; i.e.

$$\lim_{T\to\infty} Y(t)\exp\left[-\int_t^T (r(z)+\alpha+p)dz\right] = 0.$$

5.8 Pure Endowment

5.8.1 Labor Income and Consumption

In a pure endowment economy, we have:

$$Y(t) = C(t) = Y$$

5.8.2 Non-human Wealth

In aggregate, we have: $V(t) = 0$, since some agents borrow and others lend:

$$V(t) = \int_{-\infty}^{t} N(s,t)v(s,t)ds = 0.$$

$v(s,t)$ value of cumulated savings (net assets) of cohort born at s at t.

5.8.3 Equilibrium Interest Rate

Equilibrium interest rate $r(t) = \theta - \alpha$

5.8.4 Individual Consumption

$$\frac{dc(s,t)}{dt} = [r(t) - \theta]c(s,t) = -\alpha c(s,t)$$

5.8.5 Equilibrium Outcome

Thus equilibrium is autarky! $c(s,t) = y(s,t)$ and $v(s,t) = 0$

5.8.6 Results Summary

$$
\begin{aligned}
Y(t) &= C(t) = Y, \\
V(t) &= 0 \\
r(t) &= \theta - \alpha, \\
c(s,t) &= y(s,t), \forall t \geq s, \\
v(s,t) &= 0, \forall t \geq s.
\end{aligned}
$$

5.9 Capital Accumulation, Technology