Macro 2 Notes

Contents

1 Introduction

Much of this is directly quoted from Golosov's notes, slides, Ragini's notes, or the notes of past students (Jordan Rosenthal-Kay, Jingoo Kwon).

2 Lecture 0: Neoclassical Growth Model without Growth

This section pulls from Golosov's Lecture 0.

2.1 Terms

- t: period
- β: discount factor
- c_t: consumption in period t
- $u(c_t)$: utility derived from consumption in period t
- k_t: capital in period t
- $f(k_t)$: production function
- δ: depreciation rate

2.2 Setup

2.2.1 Preferences

Continuum of identical, infinitely lived consumers with preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $c_t \geq 0$ is consumption in period t.

2.2.2 Technology

Technology Output produced with production function $f(k_t)$, where $k_t \geq 0$ is capital with initial $k_0 > 0$ given. Output can be costlessly transferred between consumption and capital for next period:

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$
$$k_0 > 0 \text{ is given.}$$

for depreciation rate $\delta \in (0,1)$.

2.2.3 Assumptions

1. u, f are strictly increasing, differentiable, u is strictly concave, f is concave;
2. u, f are "nice"[1];
3. u, f satisfy Inada conditions $\lim_{c\to 0} u'(c) = \lim_{k\to 0} f'(k) = \infty$.

2.3 Model

2.3.1 Social Planner Problem

$$\max_{\{c_t,k_t\}_t} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

s.t.

$$c_t + k_{t+1} \leq f\left(k_t\right) + (1-\delta)k_t,$$

and $c_t \geq 0, k_t \geq 0, k_0$ is given.

2.3.2 Key Optimality Theorem

Theorem 2.1. *Suppose the assumptions above hold. (necessity) If $\{c_t^*, k_t^*\}_t$ solves (3) then $\{c_t^*, k_t^*\}_t$ satisfies*

$$c_t^* + k_{t+1}^* = f\left(k_t^*\right) + (1-\delta)k_t^*,$$

$$u'\left(c_t^*\right) = \beta\left[1 + f'\left(k_{t+1}^*\right) - \delta\right] u'\left(c_{t+1}^*\right),$$

$$\lim_{T\to\infty} \beta^T u'\left(c_T^*\right) k_{T+1}^* \leq 0.$$

(sufficiency) If $\{c_t^*, k_t^*\}_t$ satisfies (4), (5), and (6), then it is a solution to (3).

3 Lecture 1

3.1 Neoclassical Growth Model

3.1.1 Terms

- t: period
- C_t: consumption in period t
- I_t: investment in period t
- K_t: capital in period t
- Y_t: output in period t, sum of factor income

[1]Notes from earlier in Lecture 0 on niceness: There are multiple ways to assume niceness: bounded $u; u$ bounded from below and F is such that feasible x are bounded; u is CRRA and some assumption on the speed of change in derivatives of F around $x = 0$. The formal arguments are a bit tedious and not that insightful beyond the intuition that I gave here, so we will not talk about them.

- F_t: production function
- X_t: (labor-augmenting) technology in period t
- n: growth rate of population
- ρ: discount factor

3.1.2 Basic Accounting Definitions

$$
\begin{aligned}
&C_t + I_t = Y_t \\
&K_{t+1} = I_t + (1-\delta)K_t \\
&Y_t = \text{sum of factor income}
\end{aligned}
$$

3.1.3 More Relationships

$$
\begin{aligned}
&\dot{K}(t) = Y(t) - C(t) - \delta K(t) && \text{Feasibility} \\
&Y(t) = F(K(t), X(t)L(t)) && \\
&L(t) = 1 && \text{Feasibility: inelastic labor}
\end{aligned}
$$

3.1.4 Assumptions

- Perfectly competitive firms
- Y_t is produced by CRS technology F_t (DRS is a CRS with a fixed factor, IRS is hard to model parsimoniously).
- Two factors: capital and labor.
- Inelastic Labor

3.1.5 Setup

Household

Infinitely lived representative household with preferences

$$\int_0^\infty e^{-\rho t}\frac{C(t)^{1-\sigma}}{1-\sigma}dt$$

and inelastic labor supply (for now)

3.1.6 Useful Normalization

Re-normalize everything per unit of X :

$$k(t) \equiv \frac{K(t)}{X(t)}$$
$$c(t) \equiv \frac{C(t)}{X(t)}$$
$$y(t) \equiv \frac{Y(t)}{X(t)} = F(k(t), 1)$$
$$\tilde{\rho} \equiv \rho - (1 - \sigma) g_X$$

In this case, the model becomes isomorphic to the neoclassical growth model without growth. Thus, we have

- Competitive equilibrium is efficient.
- $k(t), c(t), y(t)$ converge to the steady state k^{ss}, c^{ss}, y^{ss}.

3.1.7 Neoclassical Growth Model and Kaldor Facts

Steady state of the neoclassical growth model is consistent with Kaldor facts (presented just below)

1. $y(t) = y^{ss}$ implies that $Y(t)$ grows at rate g_X.
2. Capital-output ratio is constant: $K(t)/Y(t) = k^{ss}/y^{ss}$.
3. Since consumption growth rate is constant, so are interest rates.
4. Factor shares are constant by labor-augmenting technical change + constant interest rate.

3.2 Kaldor Facts

1. Output per capita grows at a constant rate.
2. Capital-output ratio is roughly constant.
3. Interest rate is roughly constant.
4. Distribution of income between capital and labor is roughly constant.

3.3 Constant Growth

- $\frac{\dot{Y}(t)}{Y(t)} = g_Y > 0$
- $\frac{\dot{K}(t)}{K(t)} = g_K > 0$
- $\frac{\dot{C}(t)}{C(t)} = g_C > 0$
- $\frac{\dot{L}(t)}{L(t)} = n$

3.4 Uzawa Theorem

With constant growth and CRS technology, we have

1. Balanced growth: $g_Y = g_C = g_K \equiv g$
2. Labor-augmenting technical change: $\tilde{F}$ can be represented as $\tilde{F}(K(t), L(t), \tilde{X}(t)) = F(K(t), X(t)L(t))$ for some CRS F with $\frac{\dot{X}(t)}{X(t)} = g - n$

3.4.1 Implications of Uzawa

Some implications from Uzawa's Theorem:

- With CRS, all constant growth must be balanced, i.e., all variables grow at the same rate. Moreover, per capita growth is driven by technology.
- Technology must be either purely labor-augmenting or the elasticity of substitution between K and L equals 1.

3.5 Uzawa Theorem - Part 2

With constant growth, CRS technology, and constant factor shares[2], we have

- Constant interest rate: $R(t) = R^* \quad \forall t$
- Constant wage growth rate at the rate of technological growth: $\frac{\dot{w}(t)}{w(t)} = g_X = g_Y - n$

3.6 Constant Interest Rates, Balanced Growth, and U Theorem

Constant interest rates and balanced growth implies that $U(C)$ must be, up to a linear tranformation,

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

3.7 Useful Facts

3.7.1 Re-Expressing Growth Rates

If any variable Z grows with rate g, $\frac{\dot{Z}(t)}{Z(t)} = g \Longleftrightarrow Z(t) = e^{(t-\tau)g} Z(\tau)$ for all t, τ

[2] Jingoo's notes also mention perfect competition, not sure if that's implicit in Golosov's statement

4 Lecture 2: Structural Change - Demand Side

4.1 Model

4.1.1 Terms

- t: period
- c_t: aggregate consumption in period t
- $I(t)$: investment at time t
- $K(t)$: capital at time t
- $r(t)$: rental rate of capital at time t
- $w(t)$: wage rate at time t
- ρ: discount factor
- U_0: Utility beginning at period 0
  - $c^A(t) \in [\gamma^A, \infty)$ is the agricultural consumption at time t.
  - $c^M(t) \geq 0$ is the manufacturing consumption at time t.
  - $c^S(t) \geq 0$ is the services consumption at time t.
- $\gamma^A < 0$: constant establishing a subsistence level of agricultural consumption
  - The household must consume at least this much agricultural production (food) to survive
- $\gamma^S > 0$: constant establishing that consumption of services can be zero or negative
- η^i: long-run share of consumption in sector i
- $p^i(t)$ is the price of one unit of $c^i(t)$ for $i \in \{A, M, S\}$
  - In general, we normalize s.t. $p^M(t) = 1$, but we can choose any sector to normalize to 1 if useful
- $Y^i(t)$: Output of sector i at time t
- B^i: Hicks-neutral productivity term for sector $i \in \{A, M, S\}$
- $X(t)$: Labor-augmenting productivity term affecting all sectors.
- $g = \frac{\dot{X}(t)}{X(t)}$: growth rate of labor-augmenting productivity

4.1.2 Model Setup

Preferences

$$
\begin{aligned}
&U_0 = \int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt \\
&\text{with} \\
&c(t) = \left(c^A(t) + \gamma^A\right)^{\eta^A} c^M(t)^{\eta^M} \left(c^S(t) + \gamma^S\right)^{\eta^S} \\
&\eta^i > 0, \sum_{i \in \{A,M,S\}} \eta^i = 1, \\
&\gamma^A < 0, \gamma^S > 0
\end{aligned}
$$

Budget Constraint:

$$
\sum_{i \in \{A,M,S\}} p^i(t) c^i(t) + \dot{K}(t) = w(t) + (r(t) - \delta) K(t)
$$

Technology

Technology F is CRS with

$$
\begin{aligned}
&Y^i(t) = B^i F\left(K^i(t), X(t) L^i(t)\right), \\
&\dot{X}(t)/X(t) = g.
\end{aligned}
$$

with capital goods produced by sector M

4.1.3 Firm's Problem

$$
\max p^i(t) Y^i(t) - w(t) L^i(t) - r(t) K^i(t)
$$

s.t.

$$
Y^i(t) = B^i F\left(K^i(t), X(t) L^i(t)\right)
$$

Optimality Conditions for Firm

Capital:

$$
p^i(t) B^i F_K\left(K^i(t), X(t) L^i(t)\right) = r(t)
$$

Labor:

$$p^i(t)B^iF_L\left(K^i(t),X(t)L^i(t)\right)X(t)=w(t)$$

4.1.4 Market Clearing

Market Clearing for Labor and Capital

$$K^A(t)+K^M(t)+K^S(t)=K(t)$$
$$L^A(t)+L^M(t)+L^S(t)=1$$

Market Clearing for Agricultural and Service Goods

$$c^A(t)=Y^A(t)$$
$$c^S(t)=Y^S(t)$$

Manufacturing good is used in production of investment good

$$I(t)+c^M(t)=Y^M(t)$$
$$\dot{K}(t)=I(t)-\delta K$$

4.1.5 Competitive Equilibrium

Given initial K_0, collection of prices and quantities, such that

1. Consumers choose their quantities optimally given prices.
2. Firms choose their quantities optimally given prices.
3. All markets clear.

4.1.6 Variable/Parameter Relationships

Nonhomothetic Preferences

Generally, allowing for nonhomothetic preferences:

$$\frac{p^ic^i}{p^Mc^M}=\frac{\eta^i}{\eta^M}-\frac{p^i}{p^M}\frac{\gamma^i}{c^M}$$

Note that holding prices fixed, p^ic^i growths faster (slower) than p^Mc^M if $\gamma^i>0\,\big(\text{ if }\gamma^i<0\big)$.

- $\gamma^A < 0$: Consumption share of A grows slower than M.
- $\gamma^S > 0$: Consumption share of S grows faster than M.

This is consistent with cross-sectional patterns in spending.

4.2 Useful Facts

4.2.1 HD1 F

In our context, F is HD1. That is,

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

4.2.2 HD1 F Implications

If $F(K, L)$ is HD1 then

$$F(K, L) = F_K(K, L)K + F_L(K, L)L$$

and $F_K(K, L), F_L(K, L)$ are HD0.

That is,

$$F_K(\lambda K, \lambda L) = F_K(K, L)$$
$$F_L(\lambda K, \lambda L) = F_L(K, L)$$