# Theory of Income 2 - Pset 1

Dylan Baker
Worked with: Meera Mody

## Contents



# 1 Question 1

Show that Equation (8) implies Equation (5).

## 1.1 Solution

Note that Equation (8) from Lecture 0 is:

$$V(k) = \max_{c,k'} u(c) + \beta V(k')$$

subject to

$$c + k' \leq f(k) + (1-\delta)k$$

Consider the Lagrangian for this equation:

$$\mathcal{L} = u(c) + \beta V(k_+) + \lambda(f(k) + (1-\delta)k - c - k_+)$$

Now, we take the FOCs:

$$\frac{\partial \mathcal{L}}{\partial c} = u'(c) - \lambda = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial k_+} = \beta V'(k_+) - \lambda = 0 \tag{2}$$

The envelope condition[1] is:

$$V'(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda f'(k) + \lambda(1-\delta) = \lambda[f'(k) + (1-\delta)]$$

From (1) and (2), we have:

$$\lambda = u'(c) = \beta V'(k_+) \tag{3}$$

Applying them to one period earlier gives:

$$\lambda = u'(c_-) = \beta V'(k) \tag{4}$$

Now, use (3) and (4) to replace $\lambda$ and $V'(k)$ in the envelope condition:

$$\frac{u'(c_-)}{\beta} = u'(c)(f'(k) + (1-\delta))$$
$$\Leftrightarrow u'(c_-) = \beta u'(c)(f'(k) + (1-\delta))$$

[1] Notes on Envelope Theorem; Wikipedia Page on Envelope Theorem

By re-arranging terms and moving our time indicators up by 2 periods, this gives:

$$u'(c_t) = \beta[1 + f'(k_{t+1}) - \delta]u'(c_{t+1})$$

which is Equation (5).

# 2 Question 2

Show that Equation (14) implies Equation (13).

## 2.1 Solution

Equation (14) from Lecture 0 is:

$$\rho V(k) = \max_{c} u(c) + V'(k)[f(k) - c - \delta k] \tag{5}$$

with

$$\dot{k} = f(k) - \delta k - c \tag{6}$$

Consider the FOC of (5) wrt c:

$$u'(c(t)) - V'(k(t)) = 0 \Leftrightarrow u'(c(t)) = V'(k(t)) \tag{7}$$

Additionally, note that

$$u''(c(t))\dot{c}(t) = V''(k(t))\dot{k}(t) \qquad \text{Differentiating (7) wrt t} \tag{8}$$

Now, consider the envelope condition for (5):

$$\begin{aligned}
\rho V'(k(t)) &= V''(k(t))[f(k(t)) - c(t) - \delta k(t)] + V'(k(t))[f'(k(t)) - \delta] \\
\Leftrightarrow \rho V'(k(t)) &= V''(k(t))\dot{k}(t) + V'(k(t))[f'(k(t)) - \delta] && \text{Substituting in (6)} \\
\Leftrightarrow \rho u'(c(t)) &= V''(k(t))\dot{k}(t) + u'(c(t))[f'(k(t)) - \delta] && \text{Substituting in (7)} \\
\Leftrightarrow \rho u'(c(t)) &= u''(c(t))\dot{c}(t) + u'(c(t))[f'(k(t)) - \delta] && \text{Substituting in (8)} \\
\Leftrightarrow \frac{\dot{c}(t)}{c(t)} &= -\frac{u'(c(t))}{u''(c(t))c(t)}[f'(k(t)) - \rho - \delta] && \text{Rearranging}
\end{aligned}$$

Thus, we have Equation (13) from Lecture 0.

# 3 Question 3

(a) Prove that competitive equilibrium in the economy defined above is efficient (i.e., that competitive equilibrium allocation solves social planner's problem). What are the equilibrium dividends in this economy?

(b) Let $\{c_t^*, k_t^*\}_t$ be the solution to the social planner problem. Use these allocations to construct competitive equilibrium prices $\{r_t^{ce}, R_t^{ce}, w_t^{ce}\}_t$.

Hint (listed beside exercise in notes, not in pset): Remember that for any constant return function $G(x_1, ..., x_n)$ we have $G(x_1, ..., x_n) = \sum_{i=1}^{n} G_i x_i$ where $G_i$ is the partial derivative of $G(x_1, ..., x_n)$ wrt $x_i$.

## 3.1 Part A Solution

As our premise, recall several components of building a competitive equilibrium.

First the household/consumer problem:

$$\max_{\{c_t^{ce}, k_{t+1}^{ce}, b_{t+1}^{ce}\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t^{ce})$$

s.t.

$$\begin{aligned} c_t^{ce} + k_{t+1}^{ce} + b_{t+1}^{ce} \leq w_t^{ce} + R_t^{ce} b_t^{ce} + r_t^{ce} k_t^{ce} + (1-\delta) k_t^{ce} + d_t^{ce} \quad \forall t \\ b_{t+1} \text{ is bounded below} \end{aligned} \tag{9}$$

where $c_t^{ce} \geq 0$, $k_{t+1}^{ce} \geq 0$, $k_0$ is given, $b_0 = 0$, and, from the household's perspective, $w_t^[ce]$, $R_t^{ce}$, $r_t^{ce}$, and $d_t^{ce}$ are given.

Next, the firm problem:

$$d_t = \max_{\hat{k}_t^{ce}, \hat{l}_t^{ce}} F(\hat{k}_t^{ce}, \hat{l}_t^{ce}) - w_t^{ce} \hat{l}_t^{ce} - r_t^{ce} \hat{k}_t^{ce} \tag{10}$$

Then, our competitive equlibrium can be characterized as the sequence of prices, $\{r_t^{ce}, R_t^{ce}, w_t^{ce}\}_t$, and allocations, $\{c_t^{ce}, k_t^{ce}, b_t^{ce}, d_t^{ce}, \hat{k}_t^{ce}, \hat{l}_t^{ce}\}_t$, such that:

1. $\{c_t^{ce}, k_t^{ce}, b_t^{ce}\}_t$ solves the household problem, taking $\{r_t^{ce}, R_t^{ce}, w_t^{ce}, d_t^{ce}\}_t$ as given.
2. $\{d_t^{ce}, \hat{k}_t^{ce}, \hat{l}_t^{ce}\}_t$ solves the firm problem, taking $\{r_t^{ce}, w_t^{ce}\}_t$ as given.
3. All markets clear, i.e., $k_t^{ce} = \hat{k}_t^{ce}$, $l_t^{ce} = \hat{l}_t^{ce}$, and $b_t^{ce} = 0$ $\forall t$.

We will begin by looking at the Lagrangian for the household problem. Note that the budget constraint holds with equality, given strictly increasing utility.

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(c_t^{ce}) + \lambda_t \left[w_t^{ce} + R_t^{ce} b_t^{ce} + r_t^{ce} k_t^{ce} + (1-\delta) k_t^{ce} + d_t^{ce} - c_t^{ce} - k_{t+1}^{ce} - b_{t+1}^{ce}\right]\right]$$

The FOCs are then:

$$\frac{\partial \mathcal{L}}{\partial c_t^{ce}} = \beta^t u'(c_t^{ce}) - \lambda_t = 0 \quad \Rightarrow \beta^t u'(c_t^{ce}) = \lambda_t \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^{ce}} = -\lambda_t + \lambda_{t+1}\left[r_{t+1}^{ce} + (1-\delta)\right] = 0 \quad \Rightarrow \lambda_t = \lambda_{t+1}\left[r_{t+1}^{ce} + (1-\delta)\right] \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}^{ce}} = -\lambda_t + \lambda_{t+1} R_{t+1}^{ce} = 0 \quad \Rightarrow \lambda_t = \lambda_{t+1} R_{t+1}^{ce} \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \quad \Rightarrow w_t^{ce} + R_t^{ce} b_t^{ce} + r_t^{ce} k_t^{ce} + (1-\delta) k_t^{ce} + d_t^{ce} - c_t^{ce} - k_{t+1}^{ce} - b_{t+1}^{ce} = 0$$

and the transversality conditions are:

$$\lim_{T\to\infty} \beta^T u'(c_T) l_{T+1} \leq 0$$
$$\lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} \leq 0$$

Note that from (12) and (13), we have:

$$\begin{aligned}\lambda_t = \lambda_{t+1}\left[r_{t+1}^{ce} + (1-\delta)\right] &= \lambda_{t+1} R_{t+1}^{ce} \\ \Rightarrow R_{t+1}^{ce} &= r_{t+1}^{ce} + (1-\delta)\end{aligned} \tag{14}$$

Moreover, consider that

$$\begin{aligned}\lambda_t &= \beta^t u'(c_t^{ce}) && \text{by (11)} \\ \lambda_{t+1} &= \beta^{t+1} u'(c_{t+1}^{ce}) && \text{by (11)}\end{aligned}$$

Then

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\beta^{t+1} u'(c_{t+1}^{ce})}{\beta^t u'(c_t^{ce})} = \beta \frac{u'(c_{t+1}^{ce})}{u'(c_t^{ce})}$$

Additionally,

$$\begin{aligned}
\frac{\lambda_{t+1}}{\lambda_t} &= \frac{\lambda_{t+1}}{\lambda_{t+1} R^{ce}_{t+1}} && \text{by (13)} \\
&= \frac{1}{R^{ce}_{t+1}} \\
&= \frac{1}{r^{ce}_{t+1} + (1-\delta)} && \text{by (14)}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{1}{r^{ce}_{t+1} + (1-\delta)} &= \beta \frac{u'(c^{ce}_{t+1})}{u'(c^{ce}_t)} \\
\Rightarrow u'(c_t) &= \beta u'(c_{t+1})[r^{ce}_{t+1} + (1-\delta)]
\end{aligned} \tag{15}$$

which reflects our standard Euler equation.

Now, we return to the firm's problem in (10).

We have assumed the $F$ is continuous and differentiable and corresponds to positive, diminishing marginal product, and constant returns to scale in $l$ and $k$. Thus, we have that our problem is concave.

Moreover, we have assumed the existence of a representative firm, taking all firms to make identical decisions and maximize dividends paid to their owners.

Then, our FOCS yield:

$$\begin{aligned}
F_k(\hat{k}^{ce}_t, \hat{l}^{ce}_t) &= r^{ce}_t \\
F_l(\hat{k}^{ce}_t, \hat{l}^{ce}_t) &= w^{ce}_t
\end{aligned}$$

Then, by properties of CRS functions, we have

$$\begin{aligned}
F(\hat{k}^{ce}_t, \hat{l}^{ce}_t) &= \hat{k}^{ce}_t F_k(\hat{k}^{ce}_t, \hat{l}^{ce}_t) + \hat{l}^{ce}_t F_l(\hat{k}^{ce}_t, \hat{l}^{ce}_t) \\
&= \hat{k}^{ce}_t r^{ce}_t + \hat{l}^{ce}_t w^{ce}_t
\end{aligned}$$

From this, we have $d_t = 0$, which is logical, since positive dividends would imply that firms should demand arbitrarily large amounts of capital and labor.

Finally, we return to the market clearing conditions and enforce

$$
\begin{aligned}
k_t^{ce} = \hat{k}_t^{ce} &= k^* \\
1 = l_t^{ce} = \hat{l}_t^{ce} &= l^* \\
b_t &= 0
\end{aligned}
$$

Again by CRS properties, we have

$$r_t = f'(k_t)$$

where $k$ reflects the capital-labor ratio. This implies that $r_{t+1} = f'(k_{t+1})$. If we then plug this into our Euler equation, (15), we get:

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)]$$

If we then enforce equality and substitute $f(k_t^*) = r_t^{*ce} k_t^* + w_t^{*ce}$, $d_t = 0$, and $b_t^* = b_{t+1}^* = 0$ into our household budget constraint, (9), we get:

$$c_t^* + k_{t+1}^* = f(k_t^*) + (1 - \delta)k_t^*$$

Recall that the social planner's problem is:

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + k_{t+1} \leq f(k_t) + (1 - \delta)k_t \quad \forall t$$

or, via the Lagrangian,

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) + (1 - \delta)k_t - c_t - k_{t+1}\right]$$

which, via the FOCs, yields:

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + (1-\delta)]$$

Thus, the two maximization problems are the same, and, assuming based on the phrasing the of the question that the social planner's solution is taken to be efficient, the result of the competitive equilibrium is efficient.

### 3.2 Part B Solution

Given the solution to the social planner's problem, $\{c_t^*, k_t^*\}_t$, we will construct the competitive equilibrium prices $\{r_t^{ce}, R_t^{ce}, w_t^{ce}\}_t$.

First, note that

$$\begin{aligned} r_t^{ce} &= f'(k_t^*) \\ w_t^{ce} &= f(k_t^*) - f'(k_t^*)k_t^* = f(k_t^*) - r_t^{ce} k_t^* \end{aligned} \tag{16}$$

The factor prices are equal to the marginal product. The wage has been defined via the CRS production function properties. Moreover, we have:

$$\begin{aligned} R_t^{ce} &= r_t^{ce} + (1-\delta) && \text{by (14)} \\ &= f'(k_t^*) + (1-\delta) && \text{by (16)} \end{aligned}$$

Similar to the final steps of Part (A), substituting these expressions into the budget constraint and the Euler Equation for the competitive equilibrium connects the competitive equilibrium resulting from the firm/household problems to the solution to the social planner's problem.

Thus, $\{R_t^{ce}, r_t^{ce}, w_t^{ce}\}_t$ as described above are the prices that characterize the competitive equilibrium.

# 4 Question 4

(a) Show that any sequences of dividends $\{d_t\}_t$ consumers value as $\sum_{t=0}^{\infty} Q_t d_t$, where $Q_t = R_1^{-1} \times \ldots \times R_t^{-1}$, that is consumers obtain the same utility for any two sequences $\{d'_t\}_t, \{d''_t\}_t$ with the same present value $\sum_{t=0}^{\infty} Q_t d'_t = \sum_{t=0}^{\infty} Q_t d''_t$

[Hint: Note that you can assume that NPG and TVC are satisfied. If useful, you can assume $\lim_{t\to\infty} \left(\prod_{j=0}^{t-1} R_j^{-1}\right) b_t \geq 0$.]

(b) Define a firm optimization problem in which firms own initial capital, make all investment decisions, and hire labor to maximize the present value stream of dividends $\sum_{t=0}^{\infty} Q_t d_t$.

(c) Define competitive equilibrium in this economy and show that it is efficient. How do dividends in this equilibrium compare to the dividends in the equilibrium that was set up in Definition 1?

## 4.1 Part A Solution

First, consider the household's budget set:

$$\begin{aligned} A =& \{\{c_t\}_{t\geq 0} \mid \exists \{b_t\}_{t\geq 0} \text{ s.t.} \\ & c_t + b_{t+1} \leq w_t + R_t b_t + d_t \text{ for all } t \geq 0 & (17) \\ & \text{and } \lim_{t\to\infty} (\prod_{j=0}^{t-1} R_j^{-1}) b_t \geq 0\} & (18) \end{aligned}$$

Next, consider the lifetime budget constraint set:

$$B = \left\{ \{c_t\}_{t\geq 0} \mid \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t} R_j^{-1} \right) c_t \leq \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t} R_j^{-1} \right) w_t + \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t} R_j^{-1} \right) d_t \right\}$$

Our goal is to prove that sets $A$ and $B$ are equivalent.

First, we will show that $A \subseteq B$.

Suppose $\{c_t\}_{t\geq 0} \in A$. Then, by repeatedly applying (17), we have:

$$
\begin{aligned}
b_t &\leq R_{t-1}b_{t-1} + w_{t-1} + d_{t-1} - c_{t-1} \\
&\leq R_{t-1}[R_{t-2}b_{t-2} + w_{t-2} + d_{t-2} - c_{t-2}] + w_{t-1} + d_{t-1} - c_{t-1} \\
&= R_{t-1}R_{t-2}b_{t-2} + R_{t-1}(w_{t-2} + d_{t-2} - c_{t-2}) + w_{t-1} + d_{t-1} - c_{t-1} \\
&\leq R_{t-1}R_{t-2}[R_{t-3}b_{t-3} + w_{t-3} + d_{t-3} - c_{t-3}] + R_{t-1}(w_{t-2} + d_{t-2} - c_{t-2}) + w_{t-1} + d_{t-1} - c_{t-1} \\
&\vdots \\
&\leq \left(\prod_{j=0}^{t-1} R_j\right) b_0 + \sum_{s=0}^{t-2} \left(\prod_{j=0}^{t-2-s} R_{t-1-j}\right) [w_s + d_s - c_s] + (w_{t-1} + d_{t-1} - c_{t-1})
\end{aligned}
$$

That is,

$$
b_t \leq \left(\prod_{j=0}^{t-1} R_j\right) b_0 + \sum_{s=0}^{t-2} \left(\prod_{j=0}^{t-2-s} R_{t-1-j}\right) [w_s + d_s - c_s] + (w_{t-1} + d_{t-1} - c_{t-1})
$$

Multiplying each side by $(\prod_{j=0}^{t-1} R_j^{-1})$, we have:

$$
\begin{aligned}
\left(\prod_{j=0}^{t-1} R_j^{-1}\right) b_t \leq b_0 + \left(\prod_{j=0}^{t-1} R_j^{-1}\right) \sum_{s=0}^{t-2} \left(\prod_{j=0}^{t-2-s} R_{t-1-j}\right) [w_s + d_s - c_s] + \left(\prod_{j=0}^{t-1} R_j^{-1}\right) (w_{t-1} + d_{t-1} - c_{t-1}) \\
= b_0 + \sum_{s=0}^{t-2} \left(\prod_{j=0}^{s} R_j^{-1}\right) [w_s + d_s - c_s] + \left(\prod_{j=0}^{t-1} R_j^{-1}\right) (w_{t-1} + d_{t-1} - c_{t-1}) \\
= b_0 + \sum_{s=0}^{t-1} \left(\prod_{j=0}^{s} R_j^{-1}\right) [w_s + d_s - c_s] \\
= \sum_{s=0}^{t-1} \left(\prod_{j=0}^{s} R_j^{-1}\right) [w_s + d_s - c_s]
\end{aligned}
$$

where the last equality comes from $b_0 = 0$.

Combining this inequality with (18), we have:

$$
\sum_{s=0}^{\infty} \left(\prod_{j=0}^{s} R_j^{-1}\right) [w_s + d_s - c_s] \geq 0
$$

which, if we distribute and re-arrange around the inequality sign, is equivalent to the requirement to be included in $B$. Thus, for any $\{c_t\}_{t\geq 0} \in A$, we have $\{c_t\}_{t\geq 0} \in B$, so $A \subseteq B$.

Next, we will show that $B \subseteq A$.

Suppose $\{c_t\}_{t\geq 0} \in B$.

Define a sequence $\{b_t\}_{t\geq 0}$ as:

$$b_t = R_{t-1}b_{t-1} + w_{t-1} + d_{t-1} - c_{t-1}$$

Then we've met (17) by construction.

Furthermore, from the earlier direction, we have that

$$\lim_{t\to\infty} (\prod_{j=0}^{t-1} R_j^{-1}) b_t = \sum_{s=0}^{\infty} (\prod_{j=0}^{s} R_j^{-1})[w_s + d_s - c_s]$$

Substituting this into our requirement for inclusion in set B immediately gives (18), so $\{c_t\}_{t\geq 0} \in A$ and hence $B \subseteq A$.

Thus, since $A \subseteq B$ and $B \subseteq A$, we have $A = B$.

Thus, we can express the household optimization problem equivalently as either

$$\max_{\{c_t\}_{t\geq 0} \in A} \sum_{t\geq 0} \beta^t u\left(c_t\right)$$

or

$$\max_{\{c_t\}_{t\geq 0} \in B} \sum_{t\geq 0} \beta^t u\left(c_t\right)$$

Finally, note that if we replace $\{d_t\}_t$ with $\{d_t'\}_t$ such that

$$\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} R_j^{-1}\right) d_t = \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} R_j^{-1}\right) d_t'$$

$B$ does not change.

### 4.2 Part B Solution

$$
\begin{aligned}
&\max_{l_t, k_{t+1}} \sum_{t \geq 0} Q_t d_t, \\
&\text{with } d_t = F(k_t, l_t) - w_t l_t - i_t \\
&\text{and } i_t = k_{t+1} - k_t + \delta k_t
\end{aligned}
$$

### 4.3 Part C Solution