Lecture 13

Aggregate shocks in NK model

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Introduction

- Much of business cycle literature focuses on understanding effects of aggregate shocks
- Today we introduce them into our NK model from Lecture 12
- Two goals:
  - learn a simple technique to study stochastic dynamics of macro models (using NK model as an example)
  - derive a canonical "three equations" version of the NK model

Aggregate shocks

- For concreteness, I introduce two shocks to our NK specification
- Productivity shock:
$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t},$$
where $a_t := \ln A_t$
- Monetary policy shock: interest rate $\iota_t := \ln(1 + I_t)$ follows a Taylor rule policy
$$\iota_t = -\ln\beta + \eta\pi_t + \upsilon_t,$$
$$\upsilon_t = \rho_\upsilon \upsilon_{t-1} + \varepsilon_{\upsilon,t}.$$
where $\pi_t := \ln \Pi_t$
- Here, $\varepsilon_{a,t}$, $\varepsilon_{\iota,t}$ are i.i.d. shocks

Equilibrium conditions in levels

- The equilibrium conditions from Lecture 12 (I substituted out for L to have fewer equations)

$$
\begin{aligned}
(1-\sigma) + \sigma \frac{W_t/P_t}{A_t} - \theta \Pi_t (\Pi_t - 1) + \theta Q_t \mathbb{E}_t (\Pi_{t+1} - 1) \Pi_{t+1}^2 \frac{C_{t+1}}{C_t} &= 0, \\
C_t^{\varphi+\gamma} A_t^{-\varphi} &= \frac{W_t}{P_t}, \\
\frac{1}{1+I_t} &= \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}, \\
\{I_t\}_t \text{ is given in previous slide}
\end{aligned}
$$

Equilibrium conditions in logs

- It will be easier to work with logs rather than level
- Same equations in log form:

$$
\begin{aligned}
&(1-\sigma)+\sigma \exp\left((w_t-p_t)-a_t\right)-\theta \exp \pi_t \left(\exp \pi_t - 1\right) \\
&\qquad + \theta \exp\left(-\iota_t\right) \mathbb{E}_t \exp\left(2\pi_{t+1}+c_{t+1}-c_t\right)\left(\exp \pi_{t+1}-1\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
(\varphi+\gamma)\, c_t - \varphi a_t &= w_t - p_t, \\
-\iota_t &= \ln \beta + \ln \mathbb{E}_t \exp\left(-\pi_{t+1}-\gamma\left(c_{t+1}-c_t\right)\right), \\
\iota_t &= -\ln \beta + \eta \pi_t + \upsilon_t
\end{aligned}
$$

- Stochastic processes

$$
\begin{aligned}
a_t &= \rho_a a_{t-1} + \varepsilon_{a,t}, \\
\upsilon_t &= \rho_\upsilon \upsilon_{t-1} + \varepsilon_{\upsilon,t}.
\end{aligned}
$$

Key idea

- Let x_t be the vector of endogenous variables, z_t be the vector of exogenous processes $\{z_{i,t}\}_i$
- Our equilibrium conditions can be written as

$$F\left(x_t, \mathbb{E}_t G\left(x_{t+1}\right), z_t\right) = 0 \text{ at all } z^t$$

- This is a complicated, non-linear dynamic system. We want to simplify it by taking Taylor approximations to convert it to linear system

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- This is a complicated, non-linear dynamic system. We want to simplify it by taking Taylor approximations to convert it to linear system
- Let $(\overline{x}, \overline{z})$ be steady state of the system without aggregate shock. It satisfies

$$F\left(\overline{x}, G\left(\overline{x}\right), \overline{z}\right) = 0$$

- Let $\hat{x}_t := x_t - \overline{x}$, $\hat{z}_t := z_t - \overline{z}$. Then

$$F\left(x_t, \mathbb{E}_t G\left(x_{t+1}\right), z_t\right) \approx \underbrace{F\left(\overline{x}, G\left(\overline{x}\right), \overline{z}\right)}_{=0} + X\hat{x}_t + Y\mathbb{E}_t\hat{x}_{t+1} + \sum_i Z_i \hat{z}_{i,t}$$

where X, Y, $\{Z_i\}_i$ are some **known** matrices pinned down by steady state $(\overline{x}, \overline{z})$

Key idea, II

- So we have a linear system

$$X\hat{x}_t + Y\mathbb{E}_t\hat{x}_{t+1} + \sum_i Z_i\hat{z}_{i,t} = 0 \text{ for all } \hat{z}_t$$

- Linear system $\Longrightarrow$ solution $\hat{x}_t$ must be linear in $\hat{z}_t$:

$$\hat{x}_t = \sum_i \Psi_i\hat{z}_{i,t}$$

  for some **unknown** matrices $\{\Psi_i\}_i$

- Shocks follow AR(1) processes:

$$\mathbb{E}_t\hat{x}_{t+1} = \sum_i \Psi_i\rho_i\hat{z}_{i,t}$$

- Plug it back into first equation and group terms:

$$\sum_i \left[X\Psi_i + Y\Psi_i\rho_i + Z_i\right]\hat{z}_{i,t} = 0 \text{ for all } \hat{z}_t$$

- Since it must hold for all $\{\hat{z}_{i,t}\}_i$, we solve for $\{\Psi_i\}_i$ using

$$X\Psi_i + Y\Psi_i\rho_i + Z_i = 0 \text{ for all } i$$

Deterministic Steady State

- Note that ergodic means of a_t and υ_t are zero so set their SS values as

$$\overline{\upsilon} = \overline{a} = 0$$

- Steady state value of endogenous processes

$$\overline{\pi} = 0, \quad \bar{\imath} = -\ln \beta,$$

$$(1-\sigma) + \sigma \exp\left(\overline{w-p}\right) = 0,$$

$$(\varphi + \gamma)\,\overline{c} = \overline{w-p}.$$

- We want to approximate around this zero inflation steady state

Approximation of Euler equation, I

- The Euler equation is

$$
\begin{aligned}
\underbrace{\iota_t + \ln \beta}_{=\hat{\iota}_t} &= -\ln \mathbb{E}_t \exp\left\{-\pi_{t+1} - \gamma\left(c_{t+1} - c_t\right)\right\} \\
&\approx -\ln \mathbb{E}_t \exp\left(-\bar{\pi} - \gamma\left(\bar{c} - \bar{c}\right)\right) \\
&\quad - \frac{\mathbb{E}_t \exp\left(-\bar{\pi} - \gamma\left(\bar{c} - \bar{c}\right)\right) \cdot \left(-\hat{\pi}_{t+1} - \gamma\left(\hat{c}_{t+1} - \hat{c}_t\right)\right)}{\mathbb{E}_t \exp\left(-\bar{\pi} - \gamma\left(\bar{c} - \bar{c}\right)\right)} \\
&= \mathbb{E}_t\left(\hat{\pi}_{t+1} + \gamma\left(\hat{c}_{t+1} - \hat{c}_t\right)\right)
\end{aligned}
$$

- Alternatively,

$$
\gamma \mathbb{E}_t\left(\hat{c}_{t+1} - \hat{c}_t\right) = \hat{\iota}_t - \mathbb{E}_t \hat{\pi}_{t+1}
$$

Euler equation, intuition

- Euler eqn can be written as

$$\mathbb{E}_t\left(\hat{c}_{t+1}-\hat{c}_t\right)=\underbrace{\frac{1}{\gamma}}_{\text{elasticity of intertemp subst}}\times\left(\underbrace{\hat{\imath}_t-\mathbb{E}_t\hat{\pi}_{t+1}}_{\text{real interest rate}}\right)$$

- Intuition is very simple:

$$\begin{aligned}&\text{change in consumption growth rate}\\ &=\text{elasticity of subst}\ \times\ \text{change in interest rate}\end{aligned}$$

Firm optimality: flex prices

- Before doing general case, consider the case $\theta = 0$
- Firm optimality is

$$\begin{aligned} 0 &= (1-\sigma) + \sigma \exp\left((w_t - p_t) - a_t\right) \\ &\approx 0 + \sigma \exp\left(\overline{w-p}\right)\left(\widehat{w_t - p_t} - \widehat{a}_t\right) \\ &= (\sigma - 1)\left(\widehat{w_t - p_t} - \widehat{a}_t\right) \end{aligned}$$

- So our equilibrium conditions are

$$\widehat{w_t - p_t} - \widehat{a}_t = 0,$$

$$(\varphi + \gamma)\,\widehat{c}_t - \varphi\widehat{a}_t = \widehat{w_t - p_t},$$

$$\gamma\mathbb{E}_t\left(\hat{c}_{t+1} - \hat{c}_t\right) = \hat{\imath}_t - \mathbb{E}_t\hat{\pi}_{t+1},$$

$$\hat{\imath}_t = \eta\hat{\pi}_t + \hat{v}_t.$$

- This gives

$$\widehat{c}_t = \frac{1+\varphi}{\varphi+\gamma}\widehat{a}_t.$$

Output dynamics: flex prices

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- **Excercise 1**: Confirm that this is merely a re-statement of the solution from Exercise 1 in Lecture 12 in the log-deviation form

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- Thus, we found linear policy rule

$$\widehat{c}_t = \psi_a^c \widehat{a}_t + \psi_\upsilon^c \widehat{\upsilon}_t,$$

where $\psi_a^c = \frac{1+\varphi}{\varphi+\gamma} > 0$ and $\psi_\upsilon^c = 0$.

- Observe that

$$\mathbb{E}_t \widehat{c}_{t+1} = \rho_a \frac{1+\varphi}{\varphi+\gamma} \widehat{a}_t$$

- Therefore, $\{\hat{\pi}_t, \hat{\imath}_t\}_t$ inflation solves

$$\gamma\left(\rho_a - 1\right) \frac{1+\varphi}{\varphi+\gamma} \widehat{a}_t = \hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1},$$

$$\hat{\imath}_t = \eta \hat{\pi}_t + \hat{\upsilon}_t$$

- Linear system: make a guess that $\hat{\pi}_t$ satisfies

$$\hat{\pi}_t = \psi_a^\pi \hat{a}_t + \psi_\upsilon^\pi \hat{\upsilon}_t$$

and solve for ψ_a^π,ψ_υ^π.

Inflation dynamics: flex prices

- Under this policy rule,

$$\mathbb{E}_t \hat{\pi}_{t+1} = \psi_a^\pi \rho_a \hat{a}_t + \psi_v^\pi \rho_v \hat{v}_t$$

- Therefore, our optimal pricing equation is

$$(\rho_a - 1) \frac{1+\varphi}{\varphi+\gamma} \widehat{a}_t = \frac{1}{\gamma} \left(\eta \psi_a^\pi \hat{a}_t + \eta \psi_v^\pi \hat{v}_t + \hat{v}_t - \psi_a^\pi \rho_a \hat{a}_t - \psi_v^\pi \rho_v \hat{v}_t \right)$$

- This gives

$$\left[\frac{1}{\gamma} (\eta - \rho_a) \psi_a^\pi - (\rho_a - 1) \frac{1+\varphi}{\varphi+\gamma} \right] \widehat{a}_t + \left[1 + \psi_v^\pi (\eta - \rho_v) \right] \hat{v}_t = 0$$

- Since this holds for all $\widehat{a}_t$, $\hat{v}_t$, we have

$$\psi_a^\pi = -\frac{1}{\gamma} \frac{1-\rho_a}{\eta-\rho_a} \frac{1+\varphi}{\varphi+\gamma}, \quad \psi_v^\pi = -\frac{1}{\eta-\rho_v}.$$

Summary with flexible prices

- Thus,

$$\widehat{c}_t = \psi_a^c \hat{a}_t + \psi_\upsilon^c \hat{\upsilon}_t,$$
$$\hat{\pi}_t = \psi_a^\pi \hat{a}_t + \psi_\upsilon^\pi \hat{\upsilon}_t,$$

  where we solved for ψ_a^c, ψ_υ^c, ψ_a^π, ψ_υ^π in closed form

- With flexible prices we have $\psi_a^c > 0$, $\psi_\upsilon^c = 0$, and (assuming $\eta \geq 1$) $\psi_a^\pi, \psi_\upsilon^\pi < 0$
- Positive productivity shock, $\hat{a}_t > 0$
  - increases output
  - lowest inflation (since it lowers firms marginal costs)
- Positive monetary shock (increase in interest rates), $\hat{\upsilon}_t > 0$
  - has no effect on output
  - lowers inflation

Sticky prices

- When $\theta > 0$, we need to approximate an additional term in firm optimality

$$\theta \exp\left(-\iota_t\right) \mathbb{E}_t \exp\left(2\pi_{t+1} + c_{t+1} - c_t\right)\left(\exp \pi_{t+1} - 1\right) \; - \; \theta \exp \pi_t \left(\exp \pi_t - 1\right)$$

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- It collapses to something very simple around zero inflation SS:

$$\approx \theta\beta\mathbb{E}_t \hat{\pi}_{t+1} - \theta\hat{\pi}_t$$

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- So the firm optimality with sticky prices is

$$\hat{\pi}_t - \underbrace{\frac{\sigma-1}{\theta}}_{:=\varsigma} \underbrace{\left(\widehat{w_t - p_t} - \widehat{a}_t\right)}_{:=\widehat{mc}_t} = \beta\mathbb{E}_t \hat{\pi}_{t+1}$$

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- Alternatively

$$\hat{\pi}_t = \varsigma\mathbb{E}_t\sum_{s=0}^{\infty}\beta^s\widehat{mc}_{t+s}$$

Inflation $\propto$ PDV of MC

Three equations NK model

- Substitute out for $\widehat{w_t - p_t}$ to get canonical 3 equations NK model

$$\hat{\pi}_t - \varsigma (\varphi + \gamma) \left(\widehat{c}_t - \frac{1+\varphi}{\varphi + \gamma} \widehat{a}_t \right) = \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

$$\gamma \mathbb{E}_t (\hat{c}_{t+1} - \hat{c}_t) = \hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1},$$

$$\hat{\imath}_t = \eta \hat{\pi}_t + \hat{\upsilon}_t.$$

- Using flexible price solution, $\hat{c}_t^{flex}$, we can write first eqn as

$$\hat{\pi}_t = \varsigma (\varphi + \gamma) \left(\widehat{c}_t - \hat{c}_t^{flex} \right) + \beta \mathbb{E}_t \hat{\pi}_{t+1}$$

- Inflation today depends on expected inflation tomorrow and "output gap" – difference between observed output and hypothetical output in flexible price world
  - positive output gap $\Longrightarrow$ economy is "overheated" $\Longrightarrow$ inflation is high today
  - a version of the New Keynesian Phillips curve

Solving 3 eqn NK model

- Solve this model using same steps as before
  - guess policy rules of the form

$$\widehat{c}_t = \psi_a^c \hat{a}_t + \psi_v^c \hat{v}_t,$$
$$\hat{\pi}_t = \psi_a^\pi \hat{a}_t + \psi_v^\pi \hat{v}_t.$$

  - plug into previous equations and solve for (ψ_a^c, ψ_a^π) and (ψ_v^c, ψ_v^π)

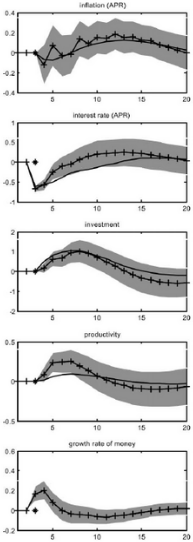
- **Exercise 2**: show that, if $\eta \geq 1$ then
  - positive productivity shock increases output and lowers inflation
  - positive interest rate shock lowest output and lowers inflation

  [To make algebra easy, feel free to show this only for the case $\rho_v = \rho_a = 0$]

Souped-up NK model

- NK models are popular among central banks as they can account for some stylized facts about macroeconomic responses to nominal shocks
- Next picture: Christiano-Eichenbaum-Evans (JPE, 2005)
  - our model + investments + investment adjustment costs + some minor changes
- Impulse responses to unexpected monetary shock in the data and in the model

CEE


inflation (APR)
interest rate (APR)
investment
productivity
growth rate of money

CEE

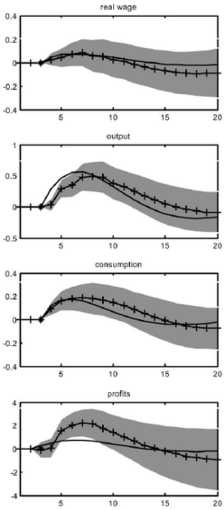


FIG. 1.—*Continued*

Normative implications

- So far, we considered positive implications: how this economy behaves if central bank follows a Taylor rule
- What about normative implications: how should central bank set its monetary policy in response to shocks?
- We saw in Lecture 12 that central bank can improve welfare with price stickiness via inflation even without shocks
- But monetary policy is not the most natural tool to combat inefficiency from monopolies
    - can be done via fiscal policy, regulations

Normative implications, flex prices

- Let's introduce labor subsidy, as in Lecture 8, with

$$\tau_t = -\frac{1}{\sigma - 1} \text{ for all } t$$

- **Exercise 3**: show that flexible price equilibrium is efficient with such tax policy, and responses to the shock is the same as without taxes,

$$\widehat{c}_t^{flex} = \frac{1+\varphi}{\varphi+\gamma}\widehat{a}_t.$$

- What should monetary policy do in this world when prices are sticky?

Normative implications, sticky prices

- Our two equilibrium conditions hold for any monetary policy:

$$\hat{\pi}_t - \varsigma\left(\varphi + \gamma\right)\left(\widehat{c}_t - \widehat{c}_t^{flex}\right) = \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

$$\gamma \mathbb{E}_t \left(\hat{c}_{t+1} - \hat{c}_t\right) = \hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1}.$$

- If we can find monetary policy $\left\{\hat{\imath}_t\right\}_t$ that closes the output gap – i.e., sets $\widehat{c}_t = \widehat{c}_t^{flex}$ – we found the optimal monetary policy

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- If we can find monetary policy $\{\hat{\imath}_t\}_t$ that closes the output gap – i.e., sets $\widehat{c}_t = \widehat{c}_t^{flex}$ – we found the optimal monetary policy
- If such policy exists we would have

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

$$\gamma \mathbb{E}_t \left(\hat{c}_{t+1}^{flex} - \hat{c}_t^{flex}\right) = \hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1}.$$

- Plug previous equation to find that it exists and satisfies

$$\hat{\imath}_t = \gamma(\rho_a - 1)\frac{1+\varphi}{\varphi+\gamma}\widehat{a}_t, \quad \hat{\pi}_t = 0$$

Intuition

- The real rate of interest rate in efficient allocation / flex price economy ("the natural rate of interest") is

$$\hat{r}_t^{flex} = \gamma \mathbb{E}_t \left(\hat{c}_{t+1}^{flex} - \hat{c}_t^{flex} \right)$$

- So central bank sets monetary policy to target that natural real interest rate

$$\hat{r}_t^{flex} = \hat{\imath}_t - \mathbb{E}_t \hat{\pi}_{t+1}$$

- "Divine coincidence": this policy simultaneous keeps allocations efficient and has no inflation
    - one interpretation of the dual mandate of the central bank to keep inflation and unemployment low