Mathematical background for macro models: neoclassical growth model without growth

Mikhail Golosov
University of Chicago

1 Introduction

In microeconomics, you often encounter problems of the sort

$$\sup_{x\in\mathbb{R}_+^n} U(x) \tag{1}$$

s.t.

$$F(x) \leq \mathbf{0},$$

where U is utility function, x are all possible commodities, F is production technology. It is standard to assume that U is strictly concave; F is such that the constraint set is closed, bounded and convex; and that u and F are differentiable.

From mathematical point of view, analyzing such maximization problems is trivial. In finite dimensions (i.e. in $\mathbb{R}_+^n$), any closed and bounded set is compact; a continuous function over compact set always attains maximum, so the maximization problem is well-posed and we can replace sup with max; the maximum can be analyzed by setting up a Lagrangian that in finite dimensions can always be written as

$$\max_{x\in\mathbb{R}_+^n} U(x) - \lambda \cdot F(x); \tag{2}$$

the necessary (first order) conditions to this maximization problem are obvious; and they are also sufficient since the maximum must be unique by strict concavity of u and convexity of the constraint set.

In macroeconomics, it is often natural to study infinite dimensional problems, where there is no pre-determined final date. One would think that we could just take (1), replace $\mathbb{R}_+^n$ with $\mathbb{R}_+^\infty$, and follow exactly the same approach as in micro. This, however, does not have solid mathematical justification: in infinite dimensions closed and bounded sets do not need to be compact so the existence of solution to the maximization problem is not guaranteed; if the maximum exists it can be characterized by a Lagrangian but such Lagrangians may not be possible to write algebraically as in (2); even if it could be written algebraically (as infinite dimensional generalization of (2)) the full set of necessary and sufficient conditions is not obvious.

Vast majority of macro models are infinite dimensional but they are set up in a way that avoids these technical difficulties and ensures that standard techniques familiar from finite dimensional analysis continue to apply in infinite period macro models. This is done by imposing further assumptions on U and F. The most popular way of doing this is to assume discounting, i.e. that $U(x)$ takes a form $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$ for some $\beta \in (0,1)$; and further to assume that F and u functions are sufficiently "nice". Intuitively, this approach works because discounting ensures that your utility is not being affected much by consumption of commodities far away in the future (say, after some large T) unless something wacky happens as either $x_t \to 0$ or $x_t \to \infty$; the "niceness" properties of u and F rule out those wacky possibilities. This ensures that your infinite dimensional problem is "similar" to a finite dimensional T period problem, and standard techniques from finite dimensional analysis apply. There are multiple ways to assume niceness: bounded u; u bounded from below and F is such that feasible x are bounded; u is CRRA and some assumption on the speed of change in derivatives of F around $x = 0$. The formal arguments are a bit tedious and not that insightful beyond the intuition that I gave here, so we will not talk about them.

Discounting + "niceness" gives two things: it ensures that maximum exists, and that it can be analyzed by a Lagrangian that is a simple infinite-dimension generalization of (2). The full set of necessary and sufficient conditions to the maximization of that Lagrangian are not immediate and I will talk a little bit about them next. I will use the simplest version of the neoclassical growth model as concrete example, since one way or another, it lies in the background of most modern macroeconomic models, especially the ones used for quantitative analysis.

2 Neoclassical growth model without growth

The two fundamental ingredients of any macroeconomic model are preferences and technology. You want to start describing any macromodel with these two.

Preferences

Continuum of identical, infinitely lived consumers with preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $c_t \geq 0$ is consumption in period t.

Technology

Output produced with production function $f(k_t)$, where $k_t \geq 0$ is capital with initial $k_0 > 0$ given. Output can be costlessly transferred between consumption and capital for next period:

$$\begin{aligned} c_t + k_{t+1} &\leq f(k_t) + (1-\delta) k_t, \\ k_0 &> 0 \text{ is given.} \end{aligned}$$

for depreciation rate $\delta \in (0,1)$.

Assumptions

1. u, f are strictly increasing, differentiable, u is strictly concave, f is concave;

2. u, f are "nice";

3. u, f satisfy Inada conditions $\lim_{c\to 0} u'(c) = \lim_{k\to 0} f'(k) = \infty$.

Assumption 1 is the basic convexity property that has the same motivation as used in microeconomics. Assumption 2 is the "niceness" assumption discussed above and can take several different forms, for example, bounded u and f. Assumption 3 is not really needed for anything but it makes life easier since it ensures that constraints $c_t \geq 0$ and $k_t \geq 0$ do not bind in the optimum, and we need to worry about fewer things.

3 Social planner's problem

Once you know preferences and technology, you can focus on the Social Planner's problem: what is the most efficient way to allocate resources to maximize efficiency. Formally, social planner problem is defined as following

$$\max_{\{c_t,k_t\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

s.t.

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta) k_t,$$

and $c_t \geq 0, k_t \geq 0, k_0$ is given.

We are interested in characterizing both necessary and sufficient optimality conditions. The key result will be the following theorem

Theorem 1 *Suppose Assumptions 1-3 hold.*

(necessity) If $\{c_t^*, k_t^*\}_t$ solves (3) then $\{c_t^*, k_t^*\}_t$ satisfies

$$c_t^* + k_{t+1}^* = f(k_t^*) + (1-\delta) k_t^*, \tag{4}$$

$$u'(c_t^*) = \beta \left[1 + f'\left(k_{t+1}^*\right) - \delta\right] u'\left(c_{t+1}^*\right), \tag{5}$$

$$\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* \leq 0. \tag{6}$$

(sufficiency) If $\{c_t^*, k_t^*\}_t$ satisfies (4), (5), and (6), then it is a solution to (3).

Equation (4) is just the complementary slackness condition, which under Assumption 1 simply says that it is never optimal for the planner to throw away resources. Equation (5) is the Euler equation. Both of these equations easy to deduce using standard techniques familiar from finite dimensional analysis. Equation (6) is the Transversality condition (TVC), and it is unique to infinite dimensional problems.[1]

In the TA session, you will go through formal derivations of these conditions for simple set of "nice" assumptions. As you will see in the proof, equation (5) makes sure that resources are allocated efficiently between any adjacent periods t, and $t+1$. This also ensures that resources are allocated efficiently between any periods t and $t+T$ for finite T. Equation (6) ensures that you do not save "too much" in perpetuity without ever consuming it.

In practice, the TVC (6) is not used very often explicitly[2] and we never will use it in this

[1] The TVC sometimes is written as equality

$$\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* = 0.$$

Since $k, u' \geq 0$, this is equivalent to (6).

[2] It is implied in any model in which we know that we converge to an interior steady state, for example.

course. However, while I will typically omit it from my slides for brevity, it will be implicitly in the background. For the purposes of the course, it is important that you know how to derive (4) and (6). There are three commonly used approaches that people use in derivations.

Variational approach

With variational approach, you work with constrained maximization problem (3) directly. If you know that $\{c_t^*, k_t^*\}_t$ is an optimum, then any feasible peturbation of these allocations cannot increase welfare. Limit arguments, as peturbation goes to 0, establish (4) and (6). In the TA session, you will use this approach to prove the Theorem above.

Lagrangian approach

Lagrangian is typically the easiest approach, especially in the models that have no uncertainty (such as ours). In particular, under Assumptions 1-3, if $\{c_t^*, k_t^*\}_t$ solves constrained problem (3), then $\{c_t^*, k_t^*\}_t$ also solves an unconstrained maximization problem

$$\max_{\{c_t,k_t\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[\left[f(k_t) + (1-\delta) k_t\right] - (c_t + k_{t+1})\right] \tag{7}$$

and $c_t \geq 0, k_t \geq 0, k_0$ is given, where $\{\lambda_t\}_t$ are Lagrange multipliers with properties

- $\lambda_t \geq 0$
- $\lim_{t\to\infty} \lambda_t = 0$
- $\lambda_t \left[f(k_t^*) + (1-\delta) k_t^* - \left(c_t^* + k_{t+1}^*\right)\right] = 0$

The key observation is that maximization problem (7) is unconstrained, and so it is much simpler then constrained optimization problem (3). Unconstrained maximum is characterized by setting all first order conditions to zero. This immediately gives familiar FOCs

$$\begin{aligned} \beta^t u'(c_t^*) - \lambda_t &= 0, \\ -\lambda_t + \lambda_{t+1}\left[1 + f'\left(k_{t+1}^*\right) - \delta\right] &= 0, \end{aligned}$$

that then imply (5). This simplicity makes the Lagrangian approach very popular in practice.

Recursive approach

In the previous quarter, you learned that (3) can also be written recursively as

$$V(k) = \max_{c,k'} u(c) + \beta V\left(k'\right) \tag{8}$$

subject to

$$c + k' \leq f(k) + (1-\delta) k.$$

First order conditions to this problem + the envelope theorem give you (4) and (5). You typically do not see the TVC condition (6) explicitly under which approach, but that only because in order to prove that you can go back and forth between (3) and (8) you need to assume that u and f are sufficiently nice, which by itself guarantee TVC (for example, various boundedness assumptions in Chapter 4 of the *Recursive methods...* textbook).

Exercise 1 *Show that (8) implies (5)*

3.1 Social planner's problem in continuous time

Sometimes, it is easier to work with macro models when they are set up in continuous rather than discrete time. The continuous time analogue of (3) is

$$\max_{c(\cdot),k(\cdot)} \int_0^\infty \exp(-\rho t)\, u(c(t))\, dt \tag{9}$$

s.t.

$$\dot{k}(t) \leq f(k(t)) - \delta k(t) - c(t) \tag{10}$$

and $c(t) \geq 0, k(t) \geq 0, k(0)$ is given. I used the usual notation here that $\dot{k} \equiv \frac{d}{dt}k$.

To see that this is the correct specification, observes the following. "Time period" in problem (3) is arbitrary (can be years, months, seconds). Let's break time into intervals of length Δ. Suppose output in (discrete) period t, when time split into intervals Δ, where we soon will vary the length of Δ. Let's think about which variables in discrete time feasibility constraint (4) should vary with Δ and which should not. That's really the question of whether we have a stock or flow variable. Capital k is a stock variable - it is accumulated over time. If you capital stock today is 10 machines, it is 10 machines irrespective of whether you measure time in years or seconds. Thus, k should not be scaling with Δ. Depreciating of capital δk_t, output $f(k_t)$, and consumption (which we assume not be non-durable) are all flow variables and should all depend on Δ : the longer the time period, the more capital depreciates, output is produced and consumed. Therefore, the more general, discrete-time version of feasibility (4) should be

$$k_{t+\Delta} \leq [f(k_t) - c_t - \delta k_t]\,\Delta + k_t. \tag{11}$$

All we want to do now is to take a limit as time intervals become small. Subtract k_t from both sides, divide by Δ, and take limit as $\Delta \to 0$ to get (10).

Similar arguments apply to discounting in the objective function but appropriately defining what the discount factor as $\Delta \to 0$. It will be easier to see when I do a recursive version of this problem.

Just like discrete time, this problem can be solved using variational, Lagrangian or recursive techniques. The main advantage of continuous time is that Lagrangian and recursive techniques sometimes become easier (sometimes they also become harder, so it is useful to know both).

Lagrangian approach

The continuous time version of Lagrangians is often called Hamiltonian. You can always look up on Wikipedia the optimality conditions that it implies, but their are very easy to derive if you simply naively adapt the Lagrangian approach from discrete time as is:

$$\max_{c(\cdot),k(\cdot)} \int_0^\infty \left[\exp(-\rho t)\, u(c(t)) + \lambda(t)\,[f(k(t)) - \delta k(t) - c(t)] - \lambda(t)\,\dot{k}(t)\right] dt$$

and $c(t) \geq 0, k(t) \geq 0, k(0)$ is given.

At first sight, it looks like a problem that we try to maximize it with respect to k but we have $\dot{k}$ instead. But all we need to do is to remember the integration by parts

$$\int_0^\infty \lambda(t)\,\dot{k}(t)\, dt = -\int_0^\infty \dot{\lambda}(t)\, k(t)\, dt + \lambda(\infty)\, k(\infty) - \lambda(0)\, k(0),$$

which really follows from a trivial relationship $\int_0^\infty \frac{\partial}{\partial t}\left[\lambda(t)\,k(t)\right]dt = \lambda(t)\,k(t)\,|_{t=0}^{t=\infty}$. So we end up with

$$\max_{c(\cdot),k(\cdot)} \int_0^\infty \left[\exp(-\rho t)\,u(c(t)) + \lambda(t)\left[f(k(t)) - \delta k(t) - c(t)\right] + \dot{\lambda}(t)\,k(t)\right] dt + \lambda(\infty)\,k(\infty) - \lambda(0)\,k(0) \tag{12}$$

and $c(t) \geq 0, k(t) \geq 0, k(0)$ is given.

It is easy to see in this problem that one necessary condition, for $c(t)$, to this problem is

$$\exp(-\rho t)\,u'(c^*(t)) = \lambda(t),$$

so $\lambda(t) \geq 0$.

Now, observe the term $\lambda(\infty)\,k(\infty)$ at the end of (12). If $\lambda(\infty) > 0$, then you can make this max to be arbitrary large by choosing appropriate $k(\infty)$. This is infeasible, so one of the optimality conditions must be $\lambda(\infty)\,k^*(\infty) = 0$, or equivalently

$$0 = \lim_{T\to\infty} \lambda(T)\,k^*(T) = \lim_{T\to\infty} \exp(-\rho T)\,u(c^*(T))\,k^*(T) = 0.$$

This is a continuous time analogue of TVC condition, and just like with TVC formal justification of this intuition is a bit tedious and requires some "niceness" properties on u and f.[3]

The second optimality condition, for $k(t)$, is

$$\lambda(t)\left[f'(k^*(t)) - \delta\right] + \dot{\lambda}(t) = 0$$

We need to know $\dot{\lambda}$ but this is something we can find from differentiating previous optimality condition with respect to t:

$$\begin{aligned}
\dot{\lambda}(t) &= -\rho \exp(-\rho t)\,u(c^*(t)) + \exp(-\rho t)\,u''(c^*(t))\,\dot{c}^*(t) \\
&= -\left[\rho + \frac{-u''(c^*(t))\,c^*(t)}{u'(c^*(t))}\frac{\dot{c}^*(t)}{c^*(t)}\right]\lambda(t) \\
&\equiv -\left[\rho + \sigma(c^*(t))\,\frac{\dot{c}^*(t)}{c^*(t)}\right]\lambda(t),
\end{aligned}$$

where $\sigma(c^*(t))$ is simply the elasticity of intertemporal substitution (EIS).[4] Combine with the optimality for capital to end up with optimality conditions

$$\begin{aligned}
\dot{k}^*(t) &= f(k^*(t)) - \delta k^*(t) - c^*(t), \\
\frac{\dot{c}^*(t)}{c^*(t)} &= \frac{f'(k^*(t)) - \delta - \rho}{\sigma(c^*(t))}, \\
\lim_{T\to\infty} \exp(-\rho T)\,u(c^*(T))\,k^*(T) &= 0.
\end{aligned} \tag{13}$$

[3] You may wonder, looking at (12), why there is no analogous arguments for $\lambda(0)\,k(0)$. The reason for that is that $k(0)$ is not a choice variable, since it is given by the initial condition, so $\lambda(0)\,k(0)$ is simply a constant that does not affect optimization. If $k(0)$ were a choice variable, we indeed would have had an additional TVC condition $\lim_{t\to 0} \lambda(t)\,k^*(t) = 0$.

[4] Without further assumptions, the EIS depends on the consumption level at which it is measured, since the notation $\sigma(c^*(t))$. If preferences are of CRRA form $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, then the EIS is simply a constant σ for any consumption level.

These equations are a continuous time analogue of (4), (5), and (6).

Recursive approach

The recursive version of continuous time model like (9) is known as HJB equation. Once again, you can look it up on Wikipedia, but deriving it heuristically from taking limits of (8) is both easy, and instructive about the underlying economics.

In anticipation of taking limits, let's write a discrete time version of (8), so discrete interval of length Δ. Let's think about objective function, and which variables should scale with Δ. You value of the stock varibles is independent of how you measure time, so $V(k)$ should not depend on Δ. Your utility from the flow variable of consumption should, so $u(c)$ should be scaling with Δ. So should discounting, and in the limit as $\Delta \to 0$ it should disappear. This leads us to the discrete time generalization of the objective function (8)

$$V(k) = \max_{c,k'_\Delta} u(c)\,\Delta + (1-\rho\Delta)\,V\left(k'_\Delta\right).$$

The constraint set followins naturally from (11):

$$k'_\Delta \leq \left[f(k) - c - \delta k\right]\Delta + k.$$

We now want to understand how this maximization problem looks like in the limit as $\Delta \to 0$. Feasibility must bind, so substitute it into V and re-arrange

$$0 = \max_c u(c)\,\Delta - \rho\Delta V\left(\left[f(k) - c - \delta k\right]\Delta + k\right) + V\left(\left[f(k) - c - \delta k\right]\Delta + k\right) - V(k).$$

Now divide by Δ and take limits as $\Delta \to 0$:

$$0 = \max_c u(c) - \rho V(k) + V'(k)\left[f(k) - c - \delta k\right].$$

Now rearrange the terms to obtain a more frequently used form of this equation:

$$\rho V(k) = \max_c u(c) + V'(k)\left[f(k) - c - \delta k\right] \tag{14}$$

with

$$\dot{k} = f(k) - \delta k - c$$

Exercise 2 *Show that (14) implies (13).*

4 (Competitive) equilibrium

One is often interested in understanding not in how fictitious social planner will allocated resources, but rather what outcomes economic agents will achieve in some sort of decentralized way. This leads us to a notion of equilibrium. For the first third of the course, we will (often implicitly) think about competitive equilibrium.

To specify equilibrium, it is useful to describe who owns which resources, how agents trade, etc. As a first step towards building towards competitive equilibrium it will be useful to re-interpret the concave (and therefore decreasing returns to scale) technology $f(k)$ as a constant returns to scale technology $F(k, l)$, where l is labor. We will assume that labor is supplied inelastically at $l = 1$, so *in the aggregate* $f(k) = F(k, 1)$ and our feasibility constraint is exactly the same as before, but this reinterpretation will be useful to think about competitive equilibrium.

Consumers

We assume that consumers own all capital and make investment decisions. Consumers rent capital to firms each period (at rate r_t) supply their inelastic labor at wage w_t, borrow and rent from each other (let b_t be net lending in period t) at rate R_t. Consumers own firms that pay them dividends d_t. Thus, we can write consumer problem as

$$\max_{\{c_t, k_t, b_t\}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{15}$$

s.t.

$$c_t + k_{t+1} + b_{t+1} \leq w_t + R_t b_t + r_t k_t + (1 - \delta) k_t + d_t, \tag{16}$$

$$b_t \text{ is bounded below,} \tag{17}$$

$c_t \geq 0, k_t \geq 0, k_0$ is given, $b_0 = 0$.

The budget constraint (16) should hopefully be self-explanatory. Condition (17) is a No Ponzi Game (NPG) condition. We impose it to rule out the solutions to (15) that we do not like. If we did not impose (17), the optimal behavior of consumers would be to borrow an arbitrary large amount, consumer, and roll over this debt in perpetuity. It is not a very interesting solution, and presumably no very realistic. One can enrich our model to specify why exactly it is unrealistic, but that would make analysis more complicated, and so we simply take a short cut by imposing (17).

One often confuses the role of NPG and TVC conditions. NPG is used to rule out some optimal solutions that we do not like; TVC is a necessary condition for any optimum. One TVC to problem (16) is $\lim_{T\to\infty} \beta^T u'(c_T^*) b_{T+1}^* \leq 0$. This condition is necessary both for the optimum with and without the NPG constraint (17): if consumers are allowed to run Ponzi schemes, they choose optimally $b_T^* \to -\infty$ which satisfied their TVC.

That being said, once the NPG is imposed, the problem start to behave nicely. So in practice, one often does not mention NPG explicitly just like one does not discuss much TVC. So I will typically drop both TVC and NPG conditions from my analysis; similarly, there is no reason to explicitly use them in your problem sets unless I explicitly ask you about them.

Firms

There is a large number of firms, all operating the same technology F. Without loss of generality, we assume that all firms make identical decisions and maximize dividends that they pay to their owners

$$d_t = \max_{k_t, l_t} F(k_t, l_t) - w_t l_t - r_t k_t.$$

We are now ready to define competitive equilibrium. Definition of competitive (really, any type) equilibrium always has the same structure: what each type of economic agents do + aggregate market clearing conditions. You always want to write explicitly the definition of equilibrium if we write or present a paper that has a model. It provides a lot of discipline to your thinking, ensures that everything "add up" in your model, and clarifies what is going on in your model to your audience.

Definition 1 *A competitive equilibrium is a sequence of prices* $\{r_t^{ce}, R_t^{ce}, w_t^{ce}\}_t$ *and allocations* $\left\{c_t^{ce}, k_t^{ce}, b_t^{ce}, d_t^{ce}, \hat{k}_t^{ce}, \hat{l}_t^{ce}\right\}_t$ *such that*

1. $\{c_t^{ce}, k_t^{ce}, b_t^{ce}\}_t$ *solves consumer problem taking* $\{r_t^{ce}, R_t^{ce}, w_t^{ce}, d_t^{ce}\}_t$ *as given;*
2. $\left\{d_t^{ce}, \hat{k}_t^{ce}, \hat{l}_t^{ce}\right\}_t$ *solves firms problem taking* $\{r_t^{ce}, w_t^{ce}\}_t$ *as given;*
3. *All markets clear*
$$k_t^{ce} = \hat{k}_t^{ce}, 1 = \hat{l}_t^{ce}, b_t^{ce} = 0 \text{ for all } t.$$

The first two conditions simply say that no economic agent can affect prices by their actions (hence the term "competitive" in equilibrium); the last simply says that supply is equal to demand in all markets.

Solving for competitive equilibrium is often much harder than for social planners problem. If your model is such that the first welfare theorem holds, you always want to invoke it before solving anything. The social planner problems give you the optimal allocations, that by the first welfare theorem then give you competitive equilibrium allocations. From those you can then back out competitive equilibrium prices.

Whenever it is obvious, I will always invoke it, always implicitly, by focusing on planner's problem. That being said, it is useful to know how to verify it; also you will need to know what optimal allocations imply about competitive equilibrium prices.

Exercise 3 *(a). Prove that competitive equilibrium in the economy defined above is efficient. What are the equilibrium dividends in this economy?*

(b). Let $\{c_t^*, k_t^*\}_t$ *be the solution to the social planner problem (3). Use these allocations to construct competitive equilibrium prices* $\{r_t^{ce}, R_t^{ce}, w_t^{ce}\}_t$.

Hint: Remember for that any constant return function $G(x_1, ..., x_n)$ we have $G(x_1, ..., x_n) = \sum_{i=1}^n G_i x_i$, where G_i is a partial derivative of $G(x_1, ..., x_n)$ with respect to x_i.

Alternative ways of setting up competitive equilibrium

There are many ways of setting up competitive equilibrium that are all equivalent in some sense. For example, instead of having consumers make all investment decisions, we could instead assume that it is firms that own all the capital and make all the investment decisions, and simply pay consumers dividends. Consumer budget constraint in this case would be

$$c_t + b_{t+1} \leq w_t + R_t b_t + d_t.$$

The excercise below will help you to think about what objective function should owners of the firms want those firms to maximize, set up the rest of competitive equilibrium, and show some sense in which it is equivalent to the previous notion of equilibrium.

Exercise 4 *(a) Show that any sequences of dividends* $\{d_t\}_t$ *consumers value as* $\sum_{t=0}^\infty Q_t d_t$, *where* $Q_t = R_1^{-1} \times ... \times R_t^{-1}$, *that is consumers obtain the same utility for any two sequences* $\{d_t'\}_t, \{d_t''\}_t$ *with the same present value* $\sum_{t=0}^\infty Q_t d_t' = \sum_{t=0}^\infty Q_t d_t''$;

(b) Define a firm optimization problem in which firms owns initial capital, makes all investment decisions, and hires labor to maximize the present value stream of dividends $\sum_{t=0}^\infty Q_t d_t$.

(c) Define competitive equilibrium in this economy and show that it is efficient. How do dividends in this equilibrium compare to the dividends in the equilibrium that was set up in Defintion 1?

4.1 Continuous time

Analysis of competitive equilibrium in continuos time is very similar and left as an exercise

Exercise 5 *Provide a continuous time version of Definition 1 of competitive equilibrium and prove that it is efficient.*