Macro 3 Notes

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2 Overlapping Generations Economy

2.1 Terms

Baseline Model

- x_i: The set of commodities consumed in period i
- $t = 1, 2, \ldots$: Index for time
- $I = \{0, 1, 2, \ldots\}$: Generation of agents
- v^i: Utility for agents born in generation $i \geq 1$ as a function of only the commodities in their periods they are alive
- u^i: Utility of agent i
  - If $i \geq 1$

$$u^i\left(x_1, x_2, x_3, \ldots\right) = v^i\left(x_i, x_{i+1}\right)$$

  - If $i = 0$

$$u^0\left(x_1, x_2, x_3, \ldots\right) = x_1^0$$

- $e^i = \left(e_1^i, e_2^i, \ldots, e_i^i, e_{i+1}^i, \ldots\right)$: The endowment of agents born in generation i
  - $e_i^i > 0$
  - $e_{i+1}^i > 0$
  - $e_t^i = 0$ for all $t \neq i$ or $t \neq i+1$
  - $e_1^0 > 0$ and $e_t^0 = 0$ for all $t = 2, 3 \ldots$
- $p = (p_1, p_2, p_3, \ldots)$: Price vector (across generations)
- r_t: The time t interest rate satisfying

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

- c: Consumption
  - c_y: Consumption while young
  - c_o: Consumption while old
  - c_i^i: Consumption for generation i in period i
  - c_{i+1}^i: Consumption for generation i in period $i+1$
  - $\bar{c}_i^j$: Equilibrium consumption for generation i in period j
  - c_i^{*j}: Best symmetric allocation consumption for generation i in period j

Model with the Below Utility Function

$$v^i\left(c_y, c_o\right) = (1-\beta)\log c_y + \beta \log c_o$$

for some constant $\beta \in (0,1)$.

- $\beta \in (0,1)$: Weight placed on consumption while old
  - $(1-\beta)$: Weight placed on consumption while young
- $\alpha \in (0,1)$: Share of total lifetime endowment received while old
  - $(1-\alpha)$: Share of total lifetime endowment received while young
- $\bar{e}_t = 1$: Total lifetime endowment for all $t \geq 1$

Terms Added for Social Security

- τ: Taxes on young paid to old

Terms Added in Growing Economy

- N_t: Number of young agents at time t
  - $N_0 = 1$
- n: Growth rate of population
- g: Growth rate of productivity of the endowments. That is, we have

$$\begin{aligned} e_{t+1}^{t+1} &= (1+g)e_t^t \\ e_{t+2}^{t+1} &= (1+g)e_{t+1}^t \end{aligned}$$

  and

$$\begin{aligned} e_t^t &= (1+g)^t(1-\alpha) \\ e_{t+1}^t &= (1+g)^t\alpha \end{aligned}$$

2.2 Pure Exchange Economy

2.2.1 Competitive Equilibrium

The price vector p is an element of R^∞, so that

$$p = (p_1, p_2, p_3, \ldots)$$

The agent problem is

$$\max_x u^i(x)$$

subject to

$$px \leq pe^i$$

which, since generation i neither consume nor has endowments at time $t \neq i$ or $t \neq i+1$, can be specialized as

$$\max_{x_i, x_{i+1}} v^i\left(x_i, x_{i+1}\right)$$

subject to

$$p_i x_i + p_{i+1} x_{i+1} = p_i e_i^i + p_{i+1} e_{i+1}^i$$

and for generation $i = 0$ as

$$\max_{x_1} x_1 \text{ subject to } p_1 x_1 = p_1 e_1^0.$$

2.2.2 No Trade

Proposition 2.1. *The only competitive equilibrium has*

$$x^i = e^i$$

i.e. there is no trade in equilibrium.

2.2.3 Equilibrium Prices

Normalize

$$p_1 = 1$$

We have:

$$\frac{p_{i+1}}{p_i} = \frac{v_2^i\left(e_i^i, e_{i+1}^i\right)}{v_1^i\left(e_i^i, e_{i+1}^i\right)}$$

for all $i \geq 1$

With r_t net interest rate,

$$\frac{1}{1+r_t} = \frac{p_{t+1}}{p_t}$$

for all t. From our previous condition we have

$$r_t = \frac{v_1^t\left(e_t^t, e_{i+1}^t\right)}{v_2^t\left(e_t^t, e_{t+1}^t\right)} - 1$$

for all $t > 1$ and

$$p_t = \frac{1}{(1+r_1)(1+r_2)\cdots(1+r_{t-1})}.$$

2.2.4 Equilibrium Prices Under Specified Utility Function

Taking utility function

$$v^i\left(c_y, c_0\right) = (1-\beta)\log c_y + \beta \log c_0$$

we have:

$$r_t \equiv \bar{r} = \frac{(1-\beta)}{\beta}\frac{\alpha}{1-\alpha} - 1 = \frac{\alpha-\beta}{\beta(1-\alpha)}$$

or

$$p_t = \left[\frac{\beta}{(1-\beta)}\frac{1-\alpha}{\alpha}\right]^{t-1} \text{ for } t \geq 1$$

2.2.5 Best Symmetric Allocation

We will solve for the best feasible symmetric allocation, where best is for the point of view of the young. In particular, consider the problem

$$\max_{c_y, c_o} v\left(c_y, c_o\right) = \max_{c_y, c_o}(1-\beta)\log c_y + \beta \log c_o$$

subject to

$$c_y + c_o = 1$$

Its sufficient first order condition is given by

$$\frac{\beta}{1-\beta}\frac{c_y}{c_0} = 1$$

so the solution of this f.o.c. that also is feasible, i.e. the solution of the problem is

$$c_y = 1 - \beta, c_o = \beta.$$

The best symmetric allocation depends on β in this way because for higher preference parameter β agents give less weight to consumption when young and more weight to consumption when old.

2.2.6 Comparison of CE and Best Symmetric Allocation

We will compare the utility of the unique competitive equilibrium allocation

$$\bar{c}_i^i = 1 - \alpha, \quad \bar{c}_{i+1}^i = \alpha \text{ for } i \geq 1 \text{ and } \bar{c}_1^0 = \alpha$$

with the one for the best symmetric allocation

$$c_i^{*i} = 1 - \beta, c_{i+1}^{*i} = \beta \text{ for } i \geq 1 \text{ and } c_1^{*0} = \beta$$

Notice that, since the CE allocation has $x^i = e^i$, and since that allocation is a feasible symmetric allocation, then, unless $c^* = \bar{c}$-which happens only when $\alpha = \beta-$ the best symmetric feasible allocation is strictly preferred by the agents of generations $i = 1, 2, \ldots$. It only remains to compare the utility of the initial old, i.e. generation $i = 0$, between the best symmetric and CE allocations.

- Case 1: $\beta > \alpha$: All generations prefer Best Symmetric Allocation
- Case 2: $\beta = \alpha$: Indifferent between CE and Best Symmetric Allocation
- Case 3: $\beta < \alpha$: Original Generation prefers CE

2.3 Social Security

2.3.1 After-Tax Endowments

$$e_i^i = (1 - \alpha) - \tau \text{ and } e_{i+1}^i = \alpha + \tau \text{ for all } i \geq 1$$
$$e_1^0 = \alpha + \tau$$

Notice that by suitable choice of τ we can make the after-tax endowments equal to the best symmetric allocations, the required τ is

$$\tau = \beta - \alpha$$

2.4 Growing Economy

2.4.1 Setup

We will now consider an economy with population and productivity growth. Let N_t the number of young agents at time t. Let n be the growth rate of population, so that

$$N_{t+1} = (1 + n)N_t \text{ for } t \geq 1 \text{ and } N_0 = 1.$$

Let g denote the growth rate of productivity of the endowments of each cohort, so that

$$e_{t+1}^{t+1} = (1+g)e_t^t \text{ and } e_{t+2}^{t+1} = (1+g)e_{t+1}^t$$

so that

$$\begin{aligned} e_t^t &= (1+g)^t(1-\alpha) \\ e_{t+1}^t &= (1+g)^t\alpha \end{aligned}$$

for all $t \geq 1$.

2.4.2 Feasible Symmetric Allocation

Define the feasible symmetric allocations as those solving

$$N_t c_y^t + N_{t-1} c_o^t = N_t(1-\alpha)(1+g)^t + N_{t-1}\alpha(1+g)^{t-1}$$

where each agent born at time t and young at t consumes

$$c_y^t = \hat{c}_y(1+g)^t,$$

and each agent born at time $t-1$ and old at t consumes

$$c_o^t = \hat{c}_o(1+g)^{t-1}.$$

Notice that this constraint can be written as

$$\hat{c}_y(1+g)(1+n) + \hat{c}_o = (1-\alpha)(1+g)(1+n) + \alpha$$

3 OLG Perpetual Youth Model

3.1 Terms

Baseline Model

- dt: An amount of time
- $p\ dt$: The probability of agent dying in dt
  - $p \in (0, \infty)$
  - $\frac{1}{p}$: Expected lifetime
- $N(s,t)$: Size of cohort born at time s at time t
  - $N(s, t+\Delta) = N(s,t)(1-p\Delta)$: The size of the cohort after Δ amount of time is the size of the cohort at time t times the probability of not dying in Δ amount of time
  - $N(s,s) = p$
  - $N(s,t) = pe^{-p(t-s)}$: The size of the cohort at time t given the size of the cohort at time s given that the cohort started of size p[1]

[1] Is p playing two roles here or are they connected? Seems they must be connected or the notation would be crazy.

- r: Net risk-less interest rate
- v: An investment in period t that pays $v\frac{1+\Delta r}{1-p\Delta}$ if alive at $t+\Delta$, and zero if dead.
- $\theta \in (0, \infty)$: Discount rate
  - 1 util at time $t+\Delta$ is worth $\frac{1}{1+\Delta\theta}$ at t.
- z: We use z in the expected utility function as the future time that we integrate over
- $R(t,z)$: price of a good at time z in terms of goods in time t
- $v(t)$: Non-human (financial) wealth at time t
- $y(z)$: Labor income at time z
- $h(t)$: human wealth at time t

$$h(t) = \int_t^{\infty} y(z)R(t,z)dz$$

- $c(t)$: Consumption at time t

3.2 Setup

Agents that die replaced by newborns. Thus, adding all cohort alive at time t yields:

$$\int_{-\infty}^{t} N(s,t)ds = \int_{-\infty}^{t} pe^{-p(t-s)}ds = 1.$$

3.3 Household Problem

$$\max \mathbb{E}\left[\int_t^{\infty} u(c(z))e^{-\theta(z-t)}dz\right] = \int_t^{\infty} \log(c(z))e^{-(p+\theta)(z-t)}dz$$

subject to

$$\int_t^{\infty} [c(z) - y(z)]R(t,z)dz = v(t)$$

We define human wealth as:

$$h(t) = \int_t^{\infty} y(z)R(t,z)dz$$

and find that the solution to our problem is:

$$c(t) = (\theta + p)(v(t) + h(t))$$