Theory of Income 2 - Pset 1

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Contents

1 Question 1

(Marx, Piketty, and the neoclassical growth model)

Consider a one-sector neoclassical growth model. The production side is standard: there is a CRS technology the produces capital and investments:

$$C_t + I_t = F(K_t, X_t L_t)$$
$$\dot{K}_t = I_t - \delta K_t$$

Technological TFP X grows at a constant growth rate. There are two types of consumers. There is measure π^c of "capitalists" who are born with initial capital stock k_0. They do not work and supply capital to the firms on competitive rental markets. There is also measure π^w of "workers" who have no initial capital stock and supply labor inelastically. To make math easier, each worker supplied $1/\pi^w$ units of labor. Workers and capitalists can freely borrow and lend with each other. Both types of agents have preferences

$$\int_0^\infty \exp(-\rho t) \ln C_t^i dt \text{ for } i \in \{c, w\}.$$

Aggregate consumption C is naturally given by

$$\pi^c C_t^c + \pi^w C_t^w = C_t$$

1. Define competitive equilibrium in this economy.

2. Show that there is the level of capital stock k_0^* such that if $k_0 = k_0^*$ then this economy is on the balanced growth path.

3. What happens to the ratio of wage income of workers to the rental income of capitalists over time on the balanced growth path?

4. Is the assumption that workers can trade assets with capitalists is important for this conclusion?

In his best-selling book "Capital in the Twenty-First Century" Thomas Piketty documents that the growth rate of wages g has been systematically below interest rates r for most of the last century, $r > g$. He argues that this is the reason for widening inequality in many countries between the rich (who rely mainly on interest income) and the poor (who rely mainly on wage income).

5. Show that in the balance growth path we must necessarily have $r > g$.

6. For concreteness, let use Gini as a measure of inequality. What happens to inequality over time on the balanced growth path?

1.1 Solution

1.1.1 Part 1

Take the following terms

A_t : households' saving stock
S_t : saving flow
r^s : interest rate for savings
K^s : supply of capital
$L^s = \dfrac{1}{\pi^s}$: supply of labor
K^d : demand of capital
L^d : demand of labor

The competitive equilibrium can be defined as the set of prices $\{r_t^k, r_t^A, w_t\}$ and quantities $\{C_t^c, C_t^w, I_t, K_t^d, K_t^s, L_t^d, L^s, S_t^c, S_t^w\}$ such that

1. $\{C_t^c, S_t^c, I_t\}$ solves the capitalist's problem
2. $\{C_t^w, S_t^w\}$ solves the worker's problem
3. $\{K_t, L_t\}$ solves the firm's problem
4. All markets clear

in which these criteria are clarified below.

The capitalist's problem is:

$$\max_{C_t^c, S_t^c, I_t} \int_0^\infty \exp(-\rho t) \ln C_t^c dt$$

$$\begin{aligned} \text{s.t.} \quad I_t + C_t^c + S_t^c &= r_t^k K_t^c + r_t^A A_t^c && (1) \\ \dot{A}_t^c &= S_t^c \\ \dot{K}_t^c &= I_t - \delta K_t^c && (2) \\ A^c(0) &= A_0^c \\ K^s(0) &= K_0 \\ K_t^s &\geq 0 \end{aligned}$$

The worker's problem is:

$$\max_{C_t^w, S_t^w} \int_0^\infty \exp(-\rho t) \ln C_t^w dt$$

$$\begin{aligned} \text{s.t.} \quad C_t^w + S_t^w &= \frac{w_t}{\pi^w} + r_t^A A_t^w \\ \dot{A}_t^w &= S_t^w \\ A^w(0) &= A_0^w \end{aligned}$$

The firm's problem is:

$$\max_{K_t, L_t} F(K_t^d, X_t L_t^d) - w_t L_t^d - r_t^k K_t^d \tag{3}$$

The market clearing conditions are:

$$\begin{aligned} &\text{Labor market clears:} \quad L_t^d = \pi^w L_t^s = 1 \\ &\text{Capital market clears:} \quad K_t^d = \pi^c K_t^s \\ &\text{Goods market clears:} \quad \pi^c I_t + \pi^c C_t^c + \pi^w C_t^w = Y_t && (4) \\ &\text{Bond market clears:} \quad \pi^w A_t^w + \pi^c A_t^c = 0 && (5) \end{aligned}$$

Let's consider the necessary conditions for each of the above criteria.

For simplicity, we will denote $K^s \equiv K$ and $K^d \equiv \pi^c K$.

The current-value Hamiltonian for the capitalist's problem is:

$$\begin{aligned} \mathcal{H}(K_t, A_t^c, C_t^c, S_t^c, \lambda_t, \mu_t) &= \ln(C_t^c) + \lambda_t (r_t^k K_t + r_t^A A_t^c - C_t^c - S_t^c - \delta K_t) + \mu_t S_t^c \\ &= \ln(C_t^c) + \lambda_t [(r_t^k - \delta) K_t + r_t^A A_t^c - C_t^c - S_t^c] + \mu_t S_t^c \end{aligned}$$

The necessary conditions for the capitalists' problem are then:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial C_t^c} &= \frac{1}{C_t^c} - \lambda_t = 0 \\
&\Rightarrow \lambda_t = \frac{1}{C_t^c} \qquad (6) \\
\frac{\partial \mathcal{H}}{\partial S_t^c} &= -\lambda_t + \mu_t = 0 \\
&\Rightarrow \lambda_t = \mu_t \qquad (7) \\
\dot{\lambda}_t - \rho\lambda_t &= -\frac{\partial \mathcal{H}}{\partial K_t} = -\lambda_t(r_t^k - \delta) \\
&\Rightarrow \frac{\dot{\lambda}_t}{\lambda} = \delta - r_t^k - \rho \qquad (8) \\
\dot{\mu}_t - \rho\mu_t &= -\frac{\partial \mathcal{H}}{\partial A_t^c} = -\lambda_t r_t^A \\
&\Rightarrow \dot{\mu}_t = \rho\mu_t - \lambda_t r_t^A \qquad (9) \\
\dot{K} &= \frac{\partial \mathcal{H}}{\partial \lambda} = (r_t^k - \delta)K_t + r_t^A A_t^c - C_t^c - S_t^c \\
\dot{A}_t^c &= \frac{\partial \mathcal{H}}{\partial \mu} = S_t^c \qquad (10) \\
\lim_{t\to\infty} e^{-\rho t}\lambda_t K_t &= 0 \\
\lim_{t\to\infty} e^{-\rho t}\lambda_t &\geq 0 \\
\lim_{t\to\infty} e^{-\rho t}\mu_t A_t^c &= 0 \\
\lim_{t\to\infty} e^{-\rho t}\mu_t &\geq 0
\end{aligned}
$$

The current-value Hamiltonian for the worker's problem is:

$$
\mathcal{H}(A_t^w, S_t^w, \kappa_t) = \ln(\frac{w_t}{\pi^w} + r_t^A A_t^w - S_t^w) + \kappa_t S_t^w
$$

The necessary conditions for the worker's problem are then:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial S_t^w} &= \frac{-1}{\frac{w_t}{\pi^w} + r_t^A A_t^w - S_t^w} + \kappa_t = 0 \\
&\Rightarrow \kappa_t = \frac{1}{\frac{w_t}{\pi^w} + r_t^A A_t^w - S_t^w} \\
&\Rightarrow \kappa_t = \frac{1}{C_t^w} && (11) \\
\dot{\kappa} - \rho\kappa &= -\frac{\partial \mathcal{H}}{\partial A_t^w} = -\frac{r_t^A}{\frac{w_t}{\pi^w} + r_t^A A_t^w - S_t^w} \\
&\Rightarrow \dot{\kappa} = \rho\kappa - \frac{r_t^A}{\frac{w_t}{\pi^w} + r_t^A A_t^w - S_t^w} \\
&\Rightarrow \dot{\kappa} = \rho\kappa - \frac{r_t^A}{C_t^w} && (12) \\
\dot{A}_t^w &= \frac{\partial \mathcal{H}}{\partial \kappa} = S_t^w \\
\lim_{t\to\infty} e^{-\rho t}\kappa_t A_t^w &= 0 \\
\lim_{t\to\infty} e^{-\rho t}\kappa_t &\geq 0
\end{aligned}
$$

For the firm's problem, the rates should correspond to the marginal product, and we require:

$$
\begin{aligned}
F_K &= r_t^k \\
X_t F_L &= w_t
\end{aligned}
$$

Market clearing conditions:

$$
\begin{aligned}
&\pi^c I_t + \pi^c C_t^c + \pi^w C_t^w = Y_t \\
&\pi^w A_t^w + \pi^c A_t^c = 0
\end{aligned}
$$

1.1.2 Part 2

We will begin by showing that if a BGP exists, then we must have $g = g_{c^w} = g_{c^c} = g_k = g_y = g_{A^w} = g_{A^c} = \frac{\dot{x}}{X}$. We will then proceed to show that a unique k_0^* exists such that the economy begins on this BGP.

We begin with the capitalist's problem to get a value for g_{c^c}.

We can make use of the capitalist's FOCs to get:

$$
\begin{aligned}
\frac{\dot{\lambda}_t}{\lambda} &= \frac{\dot{\mu}_t}{\mu} && \text{by (7)} && (13)\\
\frac{\dot{\mu}_t}{\mu} &= \rho - r_t^A && \text{by (7) and (9)} && (14)\\
\rho + \delta - r_t^k = \frac{\dot{\lambda}_t}{\lambda_t} &= \frac{\dot{\mu}_t}{\mu_t} = \rho - r_t^A && \text{by (8), (13), and (14)} &&\\
\Rightarrow r_t^A &= r_t^k - \delta \quad \forall t &&&&
\end{aligned}
$$

Note that

$$
\begin{aligned}
\lambda_t C_t^c &= 1 && \text{by (6)}\\
\Rightarrow \dot{\lambda}_t C_t^c + \lambda_t \dot{C}_t^c &= 0 && \text{differentiate wrt t}\\
\Rightarrow \frac{\dot{C}_t^c}{C_t^c} &= -\frac{\dot{\lambda}_t}{\lambda_t} &&\\
&= -\left(\rho + \delta - r_t^k\right) && \text{by (8)}
\end{aligned}
$$

Supposing a BGP exists, then $\frac{\dot{C}^c}{C^c} = -\frac{\dot{\lambda}}{\lambda} = -\left(\rho + \delta - r_t^k\right)$ should be constant. Thus, $r_t^k = r^k$ should be constant. Thus, since r_t^A is a function of r_t^k and the constant δ, r_t^A is also constant. Thus, we denote $R = r^k - \delta = r_t^A$

Then, note that

$$
-\left(\rho + \delta - r_t^k\right) = R - \rho
$$

Thus,

$$
g_{C^c} = \frac{\dot{C}_t^c}{C_t^c} = R - \rho
$$

Now, we consider the worker's problem to get a value for g_{C^w}.

Due to the bond market clearing conditions[1], we have that the interest rate earned by workers on saving is R, i.e., $r_A = R$. From there, we have

[1] Note to self: R

$$\begin{aligned}
\dot{\kappa} &= \rho\kappa - r_t^A\kappa && \text{by combining (11) and (12)} \\
\Rightarrow \frac{\dot{\kappa}}{\kappa} &= \rho - r_t^A = \rho - R && (15)
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
&\kappa_t C_t^w = 1 && \text{by (11)} \\
\Rightarrow &\dot{\kappa}_t C_t^w + \kappa_t \dot{C}_t^w = 0 && \text{differentiate wrt t} \\
\Rightarrow &\frac{\dot{C}_t^w}{C_t^w} = -\frac{\dot{\kappa}_t}{\kappa_t} \\
\Rightarrow &\frac{\dot{C}_t^w}{C_t^w} = R - \rho && \text{by (15)}
\end{aligned}$$

Thus, we have

$$g_{C^w} = \frac{\dot{C}_t^w}{C_t^w} = R - \rho = g_{C^c}$$

Now, we consider the firm's problem.

Consider that from (3), we can take the partial derivative wrt to k of

$$F(K_t^d \pi_c, X_t L_t^d) - w_t L_t^d - r_t^k K_t^d \pi_c$$

to get

$$
\begin{aligned}
\pi_c F_k(K_t^d \pi_c, X_t L_t^d) - r_t^k \pi_c = 0 & \\
\Rightarrow \pi_c F_k(K_t^d \pi_c, X_t L_t^d) = \pi_c r_t^k & \\
\Rightarrow F_k(K_t^d \pi_c, X_t L_t^d) = r_t^k & \\
\Rightarrow F_k(\frac{K_t^d \pi_c}{X_t}, \frac{X_t L_t^d}{X_t}) = r_t^k & \qquad \text{since } F_k \text{ is H0} \\
\Rightarrow F_k(\frac{K_t^d \pi_c}{X_t}, L_t^d) = r_t^k & \\
\Rightarrow F_k(\frac{K_t^d \pi_c}{X_t}, 1) = r_t^k & \qquad \text{since } L_t^d = 1 \\
\Rightarrow F_k(\frac{K_t^d \pi_c}{X_t}, 1) = r^k & \qquad \text{since } r_t^k = r^k \text{ is constant}
\end{aligned}
$$

Thus, $\frac{K_t^d \pi_c}{X_t}$ must be constant, since r^k is.

Thus, since $g = \frac{\dot{X}}{X}$ and $\frac{K_t^d \pi_c}{X_t}$ is constant, we must have

$$g_k = g$$

Additionally, note that[2]

$$
\begin{aligned}
Y &= X_t F(\frac{\pi_c K}{X_t}, 1) \\
\Rightarrow & F(\frac{\pi_c K_t}{X_t}, 1) = \frac{Y_t}{K_t}
\end{aligned}
$$

Thus, $\frac{Y_t}{K_t}$ and hence, $\frac{\dot{Y}}{Y}$ must be constant and equal to $\frac{\dot{X}}{X}$.

That is,

$$g_k = g_y = g$$

$$
\begin{aligned}
& Y_t = \pi^c I_t + \pi^c C_t^c + \pi^w C_t^w & \\
\Rightarrow & Y_t = \pi^c(\dot{K} + \delta K_t^s) + \pi^c C_t^c + \pi^w C_t^w & \qquad \text{substituting (2) into (4)} \\
\Rightarrow & Y_t = \pi^c(g + \delta) K_t^s + \pi^c C_t^c + \pi^w C_t^w & \qquad \text{since } \dot{K} = g K_t^s \\
\Rightarrow & Y_0 \exp(gt) = C_0 \exp(g_c t) + \pi^c(g + \delta)\exp(gt) K_0^s & \qquad \text{since } \frac{\dot{x}}{x} = g \Leftrightarrow \exp(gt) x_t
\end{aligned}
$$

[2]Note to self: R

Given $\pi^c(g+\delta), Y_0, C_0 > 0$, and $g_k = g_y = g$ and $g_{C^c} = g_{C^w} = g_C = \frac{\dot{C}_t}{C}$, this gives us the solution $g_c = g$; this follows from Uzawa, but we have provided some direction above.

We will now look at debt.

Consider the capitalist's budget constraints:

$$\begin{aligned}
&\dot{K} = \left(r^k - \delta\right) K_t^s + r_t^A A_t^C - C_t^c - S_t^c && \text{by (1) and (2)} \\
\Rightarrow &\dot{K} = RK_t^s + RA_t^C - C_t^c - S_t^c && \text{since } R = \left(r^k - \delta\right) = r_t^A \\
\Rightarrow &gK^s = R\left(K_t^s + A_t^C\right) - C_t^c - S_t^c && \text{since } \dot{K} = gK_t^s \text{ and } r_t^A = \\
\Rightarrow &gK^s = R\left(K_t^s + A_t^C\right) - C_t^c - \dot{A}_t^c && \text{by (10)}
\end{aligned}$$

From Uzawa's Theorem, we then have $g_{A^c} = g$.

This comes from taking A_t^c to grow at a constant rate, g_{A^c}, on the BGP, then we have from the above:

$$\begin{aligned}
&0 = (R-g)K_0 e^{gt} - C_0 e^{gt} + \left(R - g_{A^c}\right) A_0^c e^{g_A ct} \\
\Rightarrow &\left(g_{A^c} - R\right) A_0^c = e^{(g - g_{A^c})t}\left[(R-g)K_0 - C_0\right] \\
\Rightarrow &g = g_{A^c} && \text{differentiate wrt time}
\end{aligned}$$

To identify the growth rate of worker's assets on this prospective BGP, we have:

$$A_t^c = -\frac{\pi^w}{\pi^c} A_t^w \qquad \text{by (5)}$$

Thus, we need

$$A_0^c = -\frac{\pi^w}{\pi^c} A_0^w$$

and from the BGP, we have:

$$A_0^c = -\frac{\pi^w}{\pi^c} A_0^w e^{(g_A w - g)t}$$

Differentiating wrt time yields:

$$g_{A^w} = g$$

Thus, we have

$$g = g_{c^w} = g_{c^c} = g_k = g_y = g_{A^w} = g_{A^c}$$

Having shown the equivalence of all growth rates under the supposed BGP, we want to show that a k_o^* exists such that the economy begins on this BGP.

Consider that

$$\frac{\dot{C}^c}{C^c} = g = r_t^k - (\rho + \delta)$$

gives

$$r_t^k = g + \delta + \rho$$

which implies that

$$F_k\left(\pi^c \frac{K_t}{X_t}, 1\right) = g + \delta + \rho$$

is fixed.

Moreover, applying aforementioned growth rate facts, we have

$$\begin{aligned} K_t &= k_0 e^{gt} \\ X_t &= x_0 e^{gt} \end{aligned}$$

We define

$$f(k) \equiv F_k(k, 1)$$

Then

$$f\left(\pi^c \frac{k_0}{x_0}\right) = g + \delta + \rho$$

Note that f is strictly positive and monotonically decreasing, and thus, there exists a unique k_0^* to satisfy the above equation. For this k_0^*, the economy begins on the BGP.

1.1.3 Part 3

The ratio is constant on the BGP. At equilibrium,

$$\begin{aligned} w_t &= X_t F_L\left(\pi^c K_t, X_t L_t\right) \\ &= X_t F_L\left(\pi^c \frac{K_t}{X_t}, 1\right) && \text{homogenous degree 0} \end{aligned}$$

which is constant since $\pi^c \frac{K_t}{X_t}$. Therefore, $\frac{\dot{w}}{w} = \frac{\dot{X}}{X} = g$ and the ratio in question is

$$\frac{w}{rK} = \frac{w_0}{k_0 r} e^{(g-g)t} = \frac{w_0}{k_0 r}, \quad \forall t$$

1.1.4 Part 4

No, there is no trading in assets along the BGP, so our conclusion does not depend on the existence of this market. Our conclusion depends only constant interest rates, the FOCs for the firm's problem holding, and the market clearing conditions for L and K.

1.1.5 Part 5

Recall that for the BGP $g = r - \delta - \rho$. We know $\delta, \rho > 0$, so $r > g$.

1.1.6 Part 6

The Gini coefficient is $G = \frac{1}{2\mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x)p(y)|x - y|dxdy$. where $p(x)$ is the density and μ is the mean of income. Income is $\frac{w_t}{\pi^w}$ with probability π^w and $r_t K_t$ with probability π^c, so we can rewrite the expression above to get

$$G_t = \frac{\pi^w \pi^C \left|r_t K_t - (w_t/\pi^w)\right|}{\pi^C r_t K_t + w_t}$$

On the BGP, r_t is constant and K_t and w_t grow at the same rate. Hence, inequality as measured by the Gini coefficient is constant over time.

2 Question 2

(Intertemporal optimality conditions)

Lecture 2 Slide 21

Show that optimality conditions for consumers imply

$$\frac{1}{\sigma}(r-\delta-\rho) = \frac{\dot{c}^M(t)}{c^M(t)} = \frac{\dot{c}(t)}{c(t)}$$
$$\frac{\dot{c}^M}{c^M} = \frac{\dot{c}^A}{c^A+\gamma^A} = \frac{\dot{c}^S}{c^S+\gamma^S}.$$

2.1 Solution

The consumer maximizes their preferences:

$$\int_0^\infty \exp(-\rho t)\frac{c(t)^{1-\sigma}-1}{1-\sigma}dt$$

with

$$c(t) = \left(c^A(t)+\gamma^A\right)^{\eta^A} c^M(t)^{\eta^M} \left(c^S(t)+\gamma^S\right)^{\eta^S}$$
$$\eta^i > 0, \sum_{i\in\{A,M,S\}} \eta^i = 1$$
$$\gamma^A < 0, \gamma^S > 0$$

subject to the budget constraint:

$$\sum_{i\in\{A,M,S\}} p^i(t)c^i(t) + \dot{K}(t) = w(t) + (r(t)-\delta)K(t)$$

Thus, the current-value Hamiltonian is:

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda[w + (r-\delta)K - \sum_{i\in\{A,M,S\}} p^i c^i]$$

Our FOCs are then:

$$
\begin{aligned}
\frac{\partial H}{\partial c^A} &= c^{-\sigma}\frac{\partial c}{\partial c^A} - \lambda p^A \\
&= c^{-\sigma}\frac{c}{c^A+\gamma^A}\eta^A - \lambda p^A = 0^3 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^A+\gamma^A}\eta^A = \lambda p^A \qquad (16) \\
\frac{\partial H}{\partial c^S} &= c^{-\sigma}\frac{\partial c}{\partial c^S} - \lambda p^S \\
&= c^{-\sigma}\frac{c}{c^S+\gamma^S}\eta^S - \lambda p^S = 0 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^S+\gamma^S}\eta^S = \lambda p^S \qquad (17) \\
\frac{\partial H}{\partial c^M} &= c^{-\sigma}\frac{\partial c}{\partial c^M} - \lambda p^M \\
&= c^{-\sigma}\frac{c}{c^M}\eta^M - \lambda p^M = 0 \\
&\Rightarrow \frac{c^{1-\sigma}}{c^M}\eta^M = \lambda p^M \qquad (18) \\
\frac{\partial H}{\partial \lambda} &= \dot{K} = w + (r-\delta)K - \sum_{i\in\{A,M,S\}} p^i c^i \\
-\frac{\partial H}{\partial K} &= -\lambda(r-\delta) = \dot{\lambda} - \rho\lambda \\
&\Rightarrow \dot{\lambda} = \lambda(\rho - r + \delta) \qquad (19)
\end{aligned}
$$

First, we will show that

$$
\frac{\dot{c}^M}{c^M} = \frac{\dot{c}^A}{c^A+\gamma^A} = \frac{\dot{c}^S}{c^S+\gamma^S}
$$

We can begin by attaining the intratemporal rates of substitution. Consider dividing (18) by (16):

$$
\begin{aligned}
\frac{\frac{c}{c^M}\eta^M}{\frac{c}{c^A+\gamma^A}\eta^A} &= \frac{\lambda p^M c^\sigma}{\lambda p^A c^\sigma} \\
\Rightarrow \frac{c^A+\gamma^A}{c^M}\frac{\eta^M}{\eta^A} &= \frac{p^M}{p^A} \\
\Rightarrow \frac{p^A(c^A+\gamma^A)}{\eta^A} &= \frac{p^M c^M}{\eta^M} \\
\Rightarrow \frac{p^A(c^A+\gamma^A)}{\eta^A} &= \frac{c^M}{\eta^M} \qquad \text{since } p^M = 1 \qquad (20)
\end{aligned}
$$

[3] To see how this step works, consider that $\frac{\partial c}{\partial c^A} = \eta^A \left(c^A(t)+\gamma^A\right)^{\eta^A-1} \cdot c^M(t)^{\eta^M} \cdot \left(c^S(t)+\gamma^S\right)^{\eta^S} = \frac{\eta^A\left(c^A(t)+\gamma^A\right)^{\eta^A}\cdot c^M(t)^{\eta^M}\cdot\left(c^S(t)+\gamma^S\right)^{\eta^S}}{\left(c^A(t)+\gamma^A\right)} = \eta^A\frac{c}{c^A+\gamma^A}$

Similarly, dividing (18) by (17) yields:

$$
\begin{aligned}
\frac{\frac{c}{c^M}\eta^M}{\frac{c}{c^S+\gamma^S}\eta^S} &= \frac{\lambda p^M c^\sigma}{\lambda p^S c^\sigma} \\
\Rightarrow \frac{c^S+\gamma^S}{c^M}\frac{\eta^M}{\eta^S} &= \frac{p^M}{p^S} \\
\Rightarrow \frac{p^S(c^S+\gamma^S)}{\eta^S} &= \frac{p^M c^M}{\eta^M} \\
\Rightarrow \frac{p^S(c^S+\gamma^S)}{\eta^S} &= \frac{c^M}{\eta^M} && \text{since } p^M = 1 \qquad (21)
\end{aligned}
$$

Thus, combining (20) and (21) yields:

$$
\frac{p^A(c^A+\gamma^A)}{\eta^A} = \frac{c^M}{\eta^M} = \frac{p^S(c^S+\gamma^S)}{\eta^S}
$$

Note that this also gives:

$$
c^i + \gamma^i = \frac{\eta^i c^M}{p^i \eta^M} \qquad \text{for } i \in \{A, S\} \qquad (22)
$$

Consider then differentiating (20) and (21) with respect to t yields:

$$
\begin{aligned}
& d\left(\frac{p^i\left(c^i+\gamma^i\right)}{\eta^i}\right)/dt = d\left(\frac{c^M}{\eta^M}\right)/dt \\
\Rightarrow \quad & \frac{\dot{p}^i\left(c^i+\gamma^i\right)+p^i\dot{c}^i}{\eta^i} = \frac{\dot{c}^M}{\eta^M} \\
\Rightarrow \quad & \frac{p^i\dot{c}^i}{\eta^i} = \frac{\dot{c}^M}{\eta^M} && \text{since } \dot{p^i} = 0 \\
\Rightarrow \quad & \dot{c}^i = \frac{\eta^i\dot{c}^M}{p^i\eta^M} && (23)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\dot{c^i}}{c^i+\gamma^i} &= \dot{c^i}\cdot\frac{1}{c^i+\gamma^i} \\
&= \frac{\eta^i\dot{c}^M}{p^i\eta^M}\cdot\frac{p^i\eta^M}{\eta^i c^M} && \text{by (23) and (22)} \\
&= \frac{\dot{c}^M}{c^M} && \text{cancelling terms}
\end{aligned}
$$

Now, we will proceed to show:

$$\frac{1}{\sigma}(r-\delta-\rho)=\frac{\dot{c}^M(t)}{c^M(t)}=\frac{\dot{c}(t)}{c(t)}$$

Recall from (19) that

$$\begin{aligned}
&\dot{\lambda}=\lambda(\rho-r+\delta) \\
&\Rightarrow \frac{\dot{\lambda}}{\lambda}=\rho-r+\delta
\end{aligned} \tag{24}$$

Additionally,

$$\begin{aligned}
\lambda &= \frac{\eta^M}{p^M}\frac{c^{1-\sigma}}{c^M} && \text{by (18)} \\
\Rightarrow \lambda c^M p^M &= \eta^M c^{1-\sigma} && (25) \\
\Rightarrow \lambda c^M p^M + \lambda\gamma^M p^M &= \eta^M c^{1-\sigma} \\
\Rightarrow \dot{\lambda} c^M p^M + \lambda \dot{c}^M p^M + \lambda\gamma^M \dot{p}^M &= (1-\sigma)\eta^M c^{-\sigma}\dot{c} && \text{derivative wrt t} \\
\Rightarrow \dot{\lambda} c^M p^M + \lambda \dot{c}^M p^M &= (1-\sigma)\eta^M c^{-\sigma}\dot{c} && \text{since } \dot{p}^M=0 \\
\Rightarrow \frac{\dot{\lambda} c^M p^M}{\lambda} + \dot{c}^M p^M &= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\lambda} && \text{divide by } \lambda \\
\Rightarrow \frac{\dot{\lambda}}{\lambda} + \frac{\dot{c}^M}{c^M} &= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\lambda c^M p^M} && \text{divide by } c^M \\
&= \frac{(1-\sigma)\eta^M c^{-\sigma}\dot{c}}{\eta^M c^{1-\sigma}} && \text{by (25)} \\
&= \frac{(1-\sigma)\eta^M \dot{c}}{\eta^M c} \\
&= \frac{(1-\sigma)\dot{c}}{c} \\
\Rightarrow (1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}^M}{c^M} &= \frac{\dot{\lambda}}{\lambda} = \rho - r + \delta && \text{by (24)} \qquad (26)
\end{aligned}$$

As a useful interlude, take the log of c and then differentiate with respect to t:

$$\begin{aligned}
&\ln c(t) = \eta^A \ln(c^A(t)+\gamma^A) + \eta^M \ln c^M(t) + \eta^S \ln(c^S(t)+\gamma^S) \\
&\Rightarrow \frac{\dot{c}}{c} = \eta^A \frac{\dot{c}^A}{c^A+\gamma^A} + \eta^M \frac{\dot{c}^M}{c^M} + \eta^S \frac{\dot{c}^S}{c^S+\gamma^S} \\
&\quad = (\eta^A+\eta^M+\eta^S)\frac{\dot{c}^M}{c^M} \\
&\quad = \frac{\dot{c}^M}{c^M} && \text{since } \sum_{i\in\{A,M,S\}} \eta^i = 1
\end{aligned}$$

That is,

$$\frac{\dot{c}}{c} = \frac{\dot{c}^M}{c^M} \tag{27}$$

Now, returning to (26):

$$\begin{aligned}
(1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}^M}{c^M} &= \rho - r + \delta \\
\Rightarrow (1-\sigma)\frac{\dot{c}}{c} - \frac{\dot{c}}{c} &= \rho - r + \delta && \text{by (27)} \\
\Rightarrow -\sigma\frac{\dot{c}}{c} &= \rho - r + \delta \\
\Rightarrow \frac{\dot{c}}{c} &= \frac{1}{\sigma}(r - \delta - \rho)
\end{aligned}$$

That is,

$$\frac{\dot{c}}{c} = \frac{\dot{c}^M}{c^M} = \frac{1}{\sigma}(r - \delta - \rho)$$

Thus, we have achieved all of our desired results.

3 Question 3

(Intertemporal optimality conditions)

Lecture 3 Slide 14

Let λ be multiplier on the consumer's budget constraint and

$$C(t) \equiv \sum_{i \in \{A,M,S\}} p^i(t) c^i(t)$$

be total consumption expenditures. Show that optimality require

$$\frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} = (1-\theta)\frac{\dot{c}}{c}$$

and

$$\frac{\dot{\lambda}}{\lambda} = -\left(\alpha X^M K^{\alpha-1} \bar{L}^{1-\alpha} - \delta - \rho\right)$$

3.1 Solution

Preferences are given by:

$$\int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\theta} - 1}{1-\theta} dt$$

with

$$c(t) = \left(\sum_{i \in \{A,S,M\}} \eta^i c^i(t)^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}$$

and the budget constraint:

$$\sum_{i \in \{A,M,S\}} p^i c^i + \dot{K} = X^M (K)^\alpha (\bar{L})^{1-\alpha} - \delta K$$

Thus, consider the current-value Hamiltonian:

$$H(t) = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda \left(X^M K^\alpha \bar{L}^{1-\alpha} - \delta K - \sum_{i \in \{A,M,S\}} p^i c^i \right)$$

Thus, our FOCs are

$$
\begin{aligned}
\frac{\partial H}{\partial c^i} &= c^{-\theta}\frac{\partial c}{\partial c^i} - \lambda p^i = 0 \\
&\Rightarrow c^{-\theta}\frac{\partial c}{\partial c^i} = \lambda p^i \\
\dot{\lambda} &= \rho\lambda - \frac{\partial H}{\partial K} = \rho\lambda - \lambda\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \lambda\delta
\end{aligned}
$$

where $\frac{\partial c}{\partial c^i}$:

$$
\begin{aligned}
\frac{\partial c}{\partial c^i} &= \frac{\sigma}{\sigma-1}\left(\sum_{i\in\{A,S,M\}} \eta^i c^{i(\sigma-1)/\sigma}\right)^{\frac{1}{\sigma-1}} \cdot \left(\frac{\sigma-1}{\sigma}\right)\left(\eta^i c^{i\frac{-1}{\sigma}}\right) \\
&= \left(\sum_{i\in\{A,S,M\}} \eta^i c^{i(\sigma-1)/\sigma}\right)^{\frac{1}{\sigma-1}} \cdot \left(\eta^i c^{i\frac{-1}{\sigma}}\right) \\
&= c^{\frac{1}{\sigma}} \cdot \eta_i(\frac{1}{c^i})^{\frac{1}{\sigma}} \\
&= (\frac{c}{c^i})^{\frac{1}{\sigma}}\eta_i
\end{aligned}
$$

We can multiply our first FOC by c_i:

$$
\lambda p^i c^i = c^{-\theta}c^i\eta^i(\frac{c}{c^i})^{\frac{1}{\sigma}} = c^{-\theta}\eta^i(c^i)^{\frac{\sigma-1}{\sigma}}c^{\frac{1}{\sigma}}
$$

Summing over i, this yields:

$$
\begin{aligned}
\lambda \sum_{i\in\{A,M,S\}} p^i c^i &= c^{-\theta}\sum_{i\in\{A,M,S\}} \eta^i (c^i)^{\frac{\sigma-1}{\sigma}} c^{\frac{1}{\sigma}} \\
&= c^{-\theta}c^{\frac{\sigma-1}{\sigma}}c^{\frac{1}{\sigma}} \\
&= c^{-\theta}c^{\frac{\sigma-1+1}{\sigma}} \\
&= c^{-\theta}c \\
&= c^{1-\theta}
\end{aligned}
$$

Thus,

$$
\lambda C = c^{1-\theta} \qquad\qquad \text{since } C = \sum_{i\in\{A,M,S\}} p^i c^i
$$

From here,

$$
\begin{aligned}
\lambda C &= c^{1-\theta} \\
\Rightarrow \ln\lambda + \ln C &= (1-\theta)\ln c && \text{take natural log} \\
\Rightarrow \frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} &= (1-\theta)\frac{\dot{c}}{c} && \text{derivative wrt t}
\end{aligned}
$$

which is the first condition we wanted to show.

Now, from our second derivative:

$$
\begin{aligned}
\dot{\lambda} &= \rho\lambda - \lambda\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \lambda\delta \\
\frac{\dot{\lambda}}{\lambda} &= \rho - \alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} + \delta \\
&= -(\alpha X^M K^{\alpha-1}\bar{L}^{1-\alpha} - \delta - \rho)
\end{aligned}
$$

which is the second condition we wanted to show.

4 Question 4

(Potatoes and the fall of aristocracy)

Consider a two-sector version of the neoclassical growth model without capital. The output in the manufacturing is given by a constant returns to scale technology

$$Y^M = X^M L^M,$$

where L^M is supply of labor in manufacturing.

The output in the aggricultural sector is given by technology

$$Y^A = X^A \left(L^A\right)^{\alpha} (\bar{Z})^{1-\alpha},$$

where X is agricultural productivity, L^A is labor supply in agriculture, and $\bar{Z}$ is supply of land, available in fixed supply.

Suppose there are two sets of agents: workers who supply 1 unit of labor inelastically and aristocrats, who supply no labor but own land and receive land rents. Intratemporal preferences of all agents are given by

$$\left[\left(c^A\right)^{(\sigma-1)/\sigma} + \left(c^M\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}$$

with $\sigma < 1$.

Since we abstract from capital, we will focus on the static economy.

1. Define competitive equilibrium in this economy.

2. What happens to labor income if agricultural productivity X^A increases? What happens to land rents? What happens labor allocation L^M and L^A ?

In a series of papers, Nathan Nunn, Nancy Qian and co-authors study an exogenous agricultural technical change: introduction of potatoes to Europe during the Columbian exchange. Potatoes are much superior in their nutritional characteristics to native European staples such as turnips. By exploiting variation in European regional variation in suitability for cultivating potatoes, the authors provide causal evidence that introduction of potatoes in Europe increase urbanization and reduced the incidence of European military conflict.

3. Explain how these findings can be rationalized by the standard two-sector growth model?

4.1 Answer

4.1.1 Part 1

The competitive equilibrium is the set of prices $\{p^A, p^M, w, r\}$ and quantities $\{C_w^A, C_w^M, L^A, L^M, C_a^A, C_a^M, Z, Y^A, Y^M\}$ where

1. Workers and aristocrats maximize utility (subject to budget constraint)
2. Firms in M and S maximize profits

3. Markets clear

Given prices, consumers and firms solve the following problems.

1. Consumer

Aristocrat

$$\max_{C_i^A, C_i^M} \left[\left(C_i^A\right)^{\frac{\sigma-1}{\sigma}} + \left(C_i^M\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$
$$\text{s.t. } p^A C_i^A + p^M C_i k M \leq r\bar{Z}$$

Worker

$$\max_{C_w^A, C_w^M} \left[\left(C_w^A\right)^{\frac{\sigma-1}{\sigma}} + \left(C_w^M\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$
$$\text{s.t. } p^A C_w^A + p^M C_w^M \leqslant wL$$

2. Firm

Manufacturing

$$\max_{Y^M, L^M} p^M Y^M - wL^M$$
$$\text{s.t. } Y^M = X^M L^M$$

Agriculture

$$\max_{L^A, Z, Y^A} p^A Y^A - wL^A - rZ$$
$$\text{s.t.} \quad Y^A = X^A \left(L^A\right)^{\alpha} (Z)^{1-\alpha}$$

3. Markets clear

Labor, land, and goods markets clear, so

$$\begin{aligned} L^A + L^M &= 1 \\ Z &= \bar{Z} \\ C_w^M + C_a^M &= Y^M \\ C_w^A + C_a^A &= Y^A \end{aligned}$$

4.1.2 Part 2

We set up the Lagrangian for worker as follows

$$\mathcal{L}_w = \left[\left(C_w^A\right)^{\frac{\sigma-1}{\sigma}} + \left(C_w^M\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \lambda_w \left(wL - p^A C_w^A - p^M C_w^M \right)$$

and the Lagrangian for aristocrat as follows

$$\mathcal{L}_i = \left[\left(C_i^A\right)^{\frac{\sigma-1}{\sigma}} + \left(C_i^M\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} + \lambda_i\left(r\bar{Z} - p^A C_i^A - p^M C_i^M\right)$$

We can then solve the FOCs for workers and aristocrats as follows

$$\begin{aligned}
\frac{\partial \mathcal{L}_w}{\partial C_w^A} &= \frac{\sigma}{\sigma-1}\left[\left(C_w^A\right)^{\frac{1}{\sigma}} + \left(C_w^M\right)^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma-1}}\left(C_w^A\right)^{-\frac{1}{\sigma}} - \lambda_w p^A = 0\\
\frac{\partial \mathcal{L}_w}{\partial C_w^M} &= \frac{\sigma}{\sigma-1}\left[\left(C_w^A\right)^{\frac{1}{\sigma}} + \left(C_w^M\right)^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma-1}}\left(C_w^M\right)^{-\frac{1}{\sigma}} - \lambda_w p^M = 0\\
\frac{\partial \mathcal{L}_w}{\partial \lambda_w} &= wL - p^A C_w^A - p^M C_w^M = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_i}{\partial C_i^A} &= \frac{\sigma}{\sigma-1}\left[\left(C_i^A\right)^{\frac{1}{\sigma}} + \left(C_i^M\right)^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma-1}}\left(C_i^A\right)^{-\frac{1}{\sigma}} - \lambda_i p^A = 0\\
\frac{\partial \mathcal{L}_i}{\partial C_i^M} &= \frac{\sigma}{\sigma-1}\left[\left(C_i^A\right)^{\frac{1}{\sigma}} + \left(C_i^M\right)^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma-1}}\left(C_i^M\right)^{-\frac{1}{\sigma}} - \lambda_i p^M = 0\\
\frac{\partial \mathcal{L}_i}{\partial \lambda_i} &= r\bar{Z} - p^A C_i^A - p^M C_i^M = 0
\end{aligned}$$

The FOCs for firms are as follows

$$\begin{aligned}
\frac{\partial \mathcal{L}_M}{\partial Y^M} &= p^M - wX^M = 0\\
\frac{\partial \mathcal{L}_M}{\partial L^M} &= -p^M + wX^M = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_A}{\partial Y^A} &= p^A - rX^A\left(L^A\right)^{\alpha}(Z)^{1-\alpha} = 0\\
\frac{\partial \mathcal{L}_A}{\partial L^A} &= -rX^A\alpha\left(L^A\right)^{\alpha-1}(Z)^{1-\alpha} = 0\\
\frac{\partial \mathcal{L}_A}{\partial Z} &= -rX^A\left(L^A\right)^{\alpha}(1-\alpha)(Z)^{-\alpha} = 0
\end{aligned}$$

We want to know what happens to labor income if agricultural productivity X^A increases. Notice that from the FOC for the manufacturing firm we have that labor income does not change. From the FOCs for the agriculture firm, L^A and r increase or decrease together. Notice that the consumer FOCs give us that

$$\frac{p^A}{p^M} = \left(\frac{C_i^A}{C_i^M}\right)^{\frac{-1}{\sigma}}$$

which then gives us that $(p^A)^{\sigma}C^A = (p^M)^{\sigma}C^M$

and so a bit of algebra gives us that

$$(X^A)^{1-\sigma}(Z)^{(1-\alpha)(1-\sigma)} = (X^M)^{1-\sigma}\alpha^{\sigma}(1-L^A)(L^A)^{-\sigma(1-\alpha)-\alpha}.$$

Taking the derivative of the RHS with respect to L^A gives us that

$$\frac{\partial}{\partial L^A} = A_0((1-\alpha)(1-\sigma)(1-L^A)-1)(L^A)^{\sigma(1-\alpha)-\alpha-1} < 0$$

and we notice that the LHS of is increasing in X^A.

This gives us that L^A decreases, L^M increases, and r decreases as X^A increases.

4.1.3 Part 3

Improved agricultural productivity (i.e. through the introduction of potatoes and suitable land for their cultivation), leads to a reduction in rents ($r\overline{Z}$) from land ownership. As the value of land ownership decreases, people would be less likely to fight over land.