# Ch. 5 Optimal Experiment Design Notes

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## 1 Terms

- $\omega$: Worker
- $n$: Location where the worker lives and consumes
- $i$: Location where the worker works
- $C_{n\omega}$: Final good consumption

$$C_n = \left[ \sum_{i \in N} \int_0^{M_i} c_{ni}(j)^\rho dj \right]^{\frac{1}{\rho}}, \quad \sigma = \frac{1}{1-\rho} > 1 \tag{1}$$

  - $\sigma$: The elasticity of substitution between different varieties of goods
- $c_{ni}(j)$: Consumption in location $n$ of each variety, $j$, sourced from location $i$
- $H_{n\omega}$: Residential land use
- $b_{ni\omega}$: Idiosyncratic amenities shock
  - This term "captures the idea that individual workers can have idiosyncratic reasons for living and working in different locations."
  - Drawn from an independent Fréchet distribution

$$G_{ni}(b) = e^{-B_{ni} b^{-\epsilon}}, \quad B_{ni} > 0, \epsilon > 1$$

    - $B_{ni}$: This is the "scale parameter," which "determines the average amenities from living in location $n$ and working in location $i$."
    - $\epsilon$: This is the "shape parameter," which "controls the dispersion of amenities."
- $\kappa_{ni}$: Iceberg commuting cost
  - $\kappa_{ni} \in [1, \infty)$
- $U_{ni\omega}$: Utility

$$U_{ni\omega} = \frac{b_{ni\omega}}{\kappa_{ni}} \left( \frac{C_{n\omega}}{\alpha} \right)^\alpha \left( \frac{H_{n\omega}}{1-\alpha} \right)^{1-\alpha}$$

  - $\alpha$: The share of income spent on goods consumption
  - $(1-\alpha)$: The share of income spent on housing

- $X_n$: Aggregate expenditure in location $n$
- $P_n$: The price index dual to (1)
- $p_{ni}(j)$: The "'cost inclusive of freight' price of a variety $j$ produced in location $i$ and consumed in location $n$"
- $\bar{v}_n$: The average labor income of residents across employment locations
- $R_n$: The measure of residents in location $n$
- $Q_n$: Land price in location $n$
- $H_n$: The supply of land in location $n$
- $x_i(j)$: The number of variety $j$ produced in location $i$
- $F$: The fixed cost of producing a variety $j$
- $l_i(j)$: The labor input required to produce an amount of variety $j$, $x_i(j)$, in location $i$

# 2 The Model

## 2.1 Preferences and Endowments

### 2.1.1 Preferences

The preferences of a worker who lives in location $n$ and works in location $i$ is given by the following utility function of the Cobb-Douglas form:

$$U_{ni\omega} = \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{C_{n\omega}}{\alpha}\right)^{\alpha} \left(\frac{H_{n\omega}}{1-\alpha}\right)^{1-\alpha}$$

Idiosyncratic amenities are drawn from an independent Fréchet distribution:

$$G_{ni}(b) = e^{-B_{ni}b^{-\epsilon}}, \quad B_{ni} > 0, \epsilon > 1$$

### 2.1.2 Good Consumption Index

The good consumption index is given the form:

$$C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} dj\right]^{\frac{1}{\rho}}, \quad \sigma = \frac{1}{1-\rho} > 1$$

"The goods consumption index in location $n$ is a constant elasticity of substitution (CES) function of consumption of a continuum of tradable varieties sourced from each location $i$."

Utility maximization will give that "the equilibrium consumption in location $n$ of each variety sourced from location $i$ is given by":

$$c_{ni}(j) = \alpha X_n P_n^{\sigma-1} p_{ni}(j)^{-\sigma} \tag{2}$$

See subsection 4.1 for derivation.

#### 2.1.3 Land and Local Consumption

"We assume that this land is owned by immobile landlords, who receive worker expenditure on residential land as income, and consume only goods where they live."

From there, we get the expression

$$P_n C_n = \alpha \bar{v}_n R_n + (1-\alpha)\bar{v}_n R_n = \bar{v}_n R_n$$

which says that the total expenditure on goods in location $n$, $P_n C_n$ is equal to the total labor income of residents in location $n$, $\bar{v}_n R_n$.

**Questions**

Should I be saying "total labor income" or just "total income" for $\bar{v}_n R_n$?

The middle term can be read as residents total spending on goods in $n$, $\alpha \bar{v}_n R_n$, plus residents total spending on land in $n$, $(1-\alpha)\bar{v}_n R_n$.

We can also get the following expression:

$$Q_n = (1-\alpha)\frac{\bar{v}_n R_n}{H_n}$$

which says that the land price in location $n$, $Q_n$, is equal to the total spending on land in $n$ divided by the supply of land in $n$.

This follows from the land market clearing condition:

$$\underbrace{Q_n \times H_n}_{\text{price} \times \text{quantity of land}} = \underbrace{(1-\alpha)\bar{v}_n R_n}_{\text{total rent paid by residents}}$$

and is useful, because it allows us to express rent as a function of the supply of land.

**Questions**

Why do we only say that it's a function of the supply of land? It seems to be a function of several things, no?

# 3 Production

Firms produce tradable varieties under monopolistic competition and increasing returns to scale using labor as the lone input.

To produce a variety, firms incur a fixed cost, $F$, as well as a variable cost that is determined by the inverse of local productivity, $A_i(j)$: $x_i(j)/A_i(j)$.

Thus, the total amount of labor, $l_i(j)$, required to produce $x_i(j)$ units of variety $j$ in location $i$ is given by:

$$l_i(j) = F + \frac{x_i(j)}{A_i(j)}$$

# 4 Derivations

## 4.1 Derivation of $c_{ni}(j)$ Expression

### 4.1.1 Write the Problem

This is the derivation of (2).

When making consumption decisions surrounding $c_{ni}(j)$, an individual is solving the following problem:

$$\begin{aligned} &\max_{\{c_{ni}(j)\}} C_n \\ \text{s.t. } &\sum_i \int_0^{M_i} p_{ni}(j) c_{ni}(j) dj = \alpha X_n \end{aligned}$$

or expanded out:

$$\begin{aligned} &\max_{\{c_{ni}(j)\}} \left[ \sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} dj \right]^{\frac{1}{\rho}} \\ \text{s.t. } &\sum_i \int_0^{M_i} p_{ni}(j) c_{ni}(j) dj = \alpha X_n \end{aligned}$$

where the $\alpha X_n$ term comes from the fact that people spend $\alpha$ of their total expenditures, $X_n$, on goods.

#### 4.1.2 Lagrangian and FOCs

The Langragian for this problem is then:

$$\mathcal{L} = \left[\sum_{i\in N}\int_0^{M_i} c_{ni}(j)^\rho dj\right]^{\frac{1}{\rho}} + \lambda\left(\alpha X_n - \sum_i \int_0^{M_i} p_{ni}(j)c_{ni}(j)dj\right)$$

which gives the relevant FOC:

$$\begin{aligned}
\{c_{ni}(j)\} \qquad & \frac{\partial}{\partial c_{ni}(j)}\left[\left(\sum_{i\in N}\int_0^{M_i} c_{ni}(j)^\rho dj\right)^{\frac{1}{\rho}}\right] - \lambda p_{ni}(j) = 0 \\
& \Leftrightarrow \frac{1}{\rho}\left[\sum_{i\in N}\int_0^{M_i} \rho c_{ni}(j)^\rho dj\right]^{\frac{1}{\rho}-1} c_{ni}(j)^{\rho-1} - \lambda p_{ni}(j) = 0 \\
& \Leftrightarrow C_n^{1-\rho} c_{ni}(j)^{\rho-1} = \lambda p_{ni}(j) \\
& \Leftrightarrow c_{ni}(j) = \lambda^{\frac{1}{\rho-1}} p_{ni}(j)^{\frac{1}{\rho-1}} C_n \\
& \Leftrightarrow c_{ni}(j) = \lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n \qquad \text{since } \sigma = \frac{1}{1-\rho}
\end{aligned} \tag{3}$$

#### 4.1.3 Solve for $\lambda$

From there, we can define the dual price index

$$P_n \equiv \left(\sum_{i\in N}\int_0^{M_i} p_{ni}(j)^{1-\sigma} d_j\right)^{\frac{1}{1-\sigma}}$$

and revisit to our budget constraint:

$$
\begin{aligned}
\alpha X_n &= \sum_i \int_0^{M_i} p_{ni}(j) c_{ni}(j) dj \\
&= \sum_i \int_0^{M_i} p_{ni}(j) \lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n dj \quad \text{by (3)} \\
&= \lambda^{-\sigma} C_n \sum_i \int_0^{M_i} p_{ni}(j)^{1-\sigma} dj \\
&= \lambda^{-\sigma} C_n P_n^{1-\sigma} \\
\Rightarrow \lambda^{-\sigma} &= \frac{\alpha X_n}{C_n} P_n^{\sigma-1}
\end{aligned}
$$

#### 4.1.4 Plug in $\lambda$

Then, returning to (3), we can plug in our expression for $\lambda^{-\sigma}$:

$$
\begin{aligned}
c_{ni}(j) &= \lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n \\
&= \left( \frac{\alpha X_n}{C_n} P_n^{\sigma-1} \right) p_{ni}(j)^{-\sigma} C_n \\
&= \alpha X_n P_n^{\sigma-1} p_{ni}(j)^{-\sigma}
\end{aligned}
$$

which is what we wanted.