Home Production Notes

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Contents

1 Terms

- ω: Worker
- n: Location where the worker lives and consumes
- i: Location where the worker works
- $C_{n\omega}$: Final good consumption

$$C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^\rho dj \right]^{\frac{1}{\rho}}, \quad \sigma = \frac{1}{1-\rho} > 1 \tag{1}$$

  - σ: The elasticity of substitution between different varieties of goods

- $c_{ni}(j)$: Consumption in location n of each variety, j, sourced from location i
- $H_{n\omega}$: Residential land use
- $b_{ni\omega}$: Idiosyncratic amenities shock
  - This term "captures the idea that individual workers can have idiosyncratic reasons for living and working in different locations."
  - Drawn from an independent Fréchet distribution

$$G_{ni}(b) = e^{-B_{ni}b^{-\epsilon}}, \quad B_{ni} > 0, \epsilon > 1$$

    * B_{ni}: This is the "scale parameter," which "determines the average amenities from living in location n and working in location i."
    * ϵ: This is the "shape parameter," which "controls the dispersion of amenities."

- κ_{ni}: Iceberg commuting cost
  - $\kappa_{ni} \in [1, \infty)$
- $U_{ni\omega}$: Utility

$$U_{ni\omega} = \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{C_{n\omega}}{\alpha} \right)^\alpha \left(\frac{H_{n\omega}}{1-\alpha} \right)^{1-\alpha} s_\omega^\gamma$$

  - α: The share of income spent on goods consumption
  - $(1-\alpha)$: The share of income spent on housing

- s_ω: The share of time an individual spends on home production.
- γ: A term characterizing diminishing returns to home production. γ increasing implies improved returns to home production.
- $1 - s_\omega$: The share of time spent working
- X_n: Aggregate expenditure in location n
- P_n: The price index dual to (1)
- $p_{ni}(j)$: The "'cost inclusive of freight' price of a variety j produced in location i and consumed in location n"

- $\bar{v}_n$: The average wage of residents of n (some of whom may work outside of n)
- R_n: The measure of residents in location n
- Q_n: Land price in location n
- H_n: The supply of land in location n
- $x_i(j)$: The number of variety j produced in location i
- F: The fixed cost of producing a variety j
- $l_i(j)$: The labor input required to produce an amount of variety j, $x_i(j)$, in location i
- d_{ni}: The trade cost of shipping from location i to location n
- M_i: The measure of varieties produced in location i
- L_i: The measure of workers in location i
- $G_{ni}(U)$: The CDF of the indirect utility of living in location n and working in location i
- λ_{ni}: The probability that a worker lives in location n and works in location i
- λ_n^R: The probability that a worker lives in location n and works in location i
- λ_i^L: The probability that a worker works in location i
- $\lambda_{ni|n}^R$: The probability that a worker living in n commutes to i.

2 The Model

2.1 Preferences and Endowments

2.1.1 Preferences

The preferences of a worker who lives in location n and works in location i is given by the following utility function of the Cobb-Douglas form:

$$U_{ni\omega} = \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{C_{n\omega}}{\alpha}\right)^{\alpha} \left(\frac{H_{n\omega}}{1-\alpha}\right)^{1-\alpha} s_{\omega}^{\gamma} \tag{2}$$

Idiosyncratic amenities are drawn from an independent Fréchet distribution:

$$G_{ni}(b) = e^{-B_{ni}b^{-\epsilon}}, \quad B_{ni} > 0, \epsilon > 1$$

2.1.2 Time Allocation

Under the given utility function, we find that the share of time allocated to home production is given by:

$$s_\omega = \frac{\gamma}{1+\gamma} \tag{3}$$

and hence the share of time allocated to work is given by:

$$1 - s_\omega = \frac{1}{1+\gamma} \tag{4}$$

See subsection 3.1 for derivation.

2.1.3 Good Consumption Index

The good consumption index is given the form:

$$C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^\rho dj \right]^{\frac{1}{\rho}}, \quad \sigma = \frac{1}{1-\rho} > 1$$

"The goods consumption index in location n is a constant elasticity of substitution (CES) function of consumption of a continuum of tradable varieties sourced from each location i."

Utility maximization will give that "the equilibrium consumption in location n of each variety sourced from location i is given by":

$$c_{ni}(j) = \alpha X_n P_n^{\sigma-1} p_{ni}(j)^{-\sigma} \tag{5}$$

See subsection 3.2 for derivation.

2.1.4 Land and Local Consumption

"We assume that this land is owned by immobile landlords, who receive worker expenditure on residential land as income, and consume only goods where they live."

From there, we get the expression

$$P_n C_n = \alpha \bar{v}_n R_n + (1-\alpha) \bar{v}_n R_n = \bar{v}_n R_n$$

which says that the total expenditure on goods in location n, $P_n C_n$ is equal to the total labor income of residents in location n, $\bar{v}_n R_n$.

The middle term can be read as residents' total spending on goods in n, $\alpha\bar{v}_n R_n$, plus residents' total spending on land in n, $(1-\alpha)\bar{v}_n R_n$.

We can also get the following expression:

$$Q_n = (1-\alpha)\frac{\bar{v}_n R_n}{H_n}$$

which says that the land price in location n, Q_n, is equal to the total spending on land in n divided by the supply of land in n.

This follows from the land market clearing condition:

$$\underbrace{Q_n \times H_n}_{\text{price} \times \text{quantity of land}} \quad = \quad \underbrace{(1-\alpha)\bar{v}_n R_n}_{\text{total rent paid by residents}}$$

and is useful, because it allows us to express rent as a function of the supply of land.

2.2 Production

2.2.1 Labor Requirement for Production

Firms produce tradable varieties under monopolistic competition and increasing returns to scale using labor as the lone input.

To produce a variety, firms incur a fixed cost, F, as well as a variable cost that is determined by the inverse of local productivity, A_i: $x_i(j)/A_i$.

Thus, the total amount of labor, $l_i(j)$, required to produce $x_i(j)$ units of variety j in location i is given by:

$$l_i(j) = F + \frac{x_i(j)}{A_i} \tag{6}$$

2.2.2 Price: Constant Markup over Marginal Cost

Profit maximization gives us that the equilibrium prices are a constant markup over marginal cost:

$$p_{ni}(j) = \left(\frac{\sigma}{\sigma-1}\right)\frac{d_{ni}w_i}{A_i} \tag{7}$$

The derivation is given in subsection 3.3.

Notice also that $p_{ni}(j)$ is the same for all varieties produced in location i and consumed in location n, i.e., there is no j on the RHS.

2.2.3 Equilibrium Output of j and i

If we then additionally note zero profits, we get an expression for equilibrium output of each variety in each location:

$$x_i(j) = A_i F(\sigma - 1) \tag{8}$$

The derivation is given in subsection 3.4.

2.2.4 Measure of Produced Varieties

(4) and (8), combined with labor market clearing, gives us that the total measure of produced varieties in region i, M_i, is proportional to the measure of employed workers, L_i:

$$M_i = \left(\frac{L_i}{\sigma F}\right)\left(\frac{1}{1+\gamma}\right) \tag{9}$$

The derivation is given in subsection 3.5.

2.3 Goods Trade

2.3.1 Share of n's Expenditure on i's Goods

"The model implies a gravity equation for bilateral trade between locations."

"Using the CES expenditure function, the equilibrium pricing rule, and the measure of firms, the share of location n's expenditure on good's produced in location i is:"

$$\pi_{ni} = \frac{M_i p_{ni}^{1-\sigma}}{\sum_{k\in N} M_k p_{nk}^{1-\sigma}} = \frac{L_i \left(d_{ni} w_i / A_i\right)^{1-\sigma}}{\sum_{k\in N} L_k \left(d_{nk} w_k / A_k\right)^{1-\sigma}} \tag{10}$$

The derivation is given in subsection 3.6.

Notice that trade between locations n and i depends on both their own trade costs, d_{ni}, as well as the trade costs between n and all other locations.

2.3.2 Workplace Income

"Equating revenue and expenditure, and using zero profits, workplace income in each location equals total expenditure on goods produced in that location, namely,"

$$w_i L_i = \sum_{n\in N} \pi_{ni} \bar{v}_n R_n \tag{11}$$

The derivation is given in subsection 3.7.

2.3.3 Price Index Re-Expression

Then, using the equilibrium pricing rule and labor market clearing, we can re-express the price index dual to the consumption index as:

$$
\begin{aligned}
P_n &= \left(\frac{\sigma}{\sigma-1}\right)\left[\left(\frac{1}{1+\gamma}\right)\left(\frac{1}{\sigma F}\right)\right]^{\frac{1}{1-\sigma}}\left[\sum_{i\in N} L_i\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\
&= \left(\frac{\sigma}{\sigma-1}\right)\left[\left(\frac{1}{1+\gamma}\right)\left(\frac{L_n}{\sigma F\pi_{nn}}\right)\right]^{\frac{1}{1-\sigma}}\frac{d_{nn}w_n}{A_n} \qquad (12)
\end{aligned}
$$

The derivation is given in subsection 3.8.

2.4 Labor Mobility and Commuting

2.4.1 Indirect Utility

Workers are geographically mobile and select both the location of their residence and the location of their workplace (possibly different) to maximize their utility.

Building from our baseline utility function, we can then get the indirect utility expression:

$$
U_{ni\omega} = \frac{b_{ni\omega}w_i}{\kappa_{ni}P_n^{\alpha}Q_n^{1-\alpha}}\left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right] \qquad (13)
$$

which is the utility of a worker who lives in location n, works in location i, has idiosyncratic amenities $b_{ni\omega}$, and has made the consumption, housing, and labor supply choices that maximize their utility.

The derivation is given in subsection 3.9.

2.4.1.1 Indirect Utility and the Fréchet distribution

Notice that indirect utility expression follows a Fréchet distribution, since $b_{ni\omega}$ is drawn from an independent Fréchet distribution and the other terms are constant in ω:

$$
U_{ni\omega} = \underbrace{\frac{w_i}{\kappa_{ni}P_n^{\alpha}Q_n^{1-\alpha}}\left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]}_{\text{constant w.r.t. } \omega}\times b_{ni\omega}
$$

Note that $\frac{w_i}{\kappa_{ni}P_n^{\alpha}Q_n^{1-\alpha}}\left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]>0$.

Then, the indirect utility expression is a monotonic transformation of $b_{ni\omega}$ and follows a Fréchet distribution.

The CDF of U is given by:

$$G_{ni}(U) = e^{-\Psi_{ni}U^{-\epsilon}} \tag{14}$$

where

$$\Psi_{ni} = B_{ni}\left(\kappa_{ni}P_n^{\alpha}Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon}\left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]^{\epsilon} \tag{15}$$

The derivation is given in subsection 3.10.

"Each worker selects the bilateral commute that offers her the maximum utility, where the maximum of Fréchet distributed random variables is itself Fréchet distributed."

2.4.2 The Probability of Living in n and Working in i

Based on the above-established distribution of utility, the probability of a worker choosing to live in n and work in i is then given by:

$$\lambda_{ni} = \frac{B_{ni}\left(\kappa_{ni}P_n^{\alpha}Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon}}{\sum_{r\in N}\sum_{s\in N} B_{rs}\left(\kappa_{rs}P_r^{\alpha}Q_r^{1-\alpha}\right)^{-\epsilon} w_s^{\epsilon}} \equiv \frac{\Phi_{ni}}{\Phi} \tag{16}$$

The derivation is given in subsection 3.11.

Thus, the idiosyncratic shocks imply that workers will select different bilateral commutes even when faced with the same set of prices, (P_n, Q_n, w_i), commuting costs, κ_{ni}, and location characteristics, B_{ni}. However, it also implies that, all else equal, workers are more likely to live in location n and work in location i when the consumption goods price index (P_n) is lower, the land prices (Q_n) are lower, wages (w_i) are higher, the amenities (B_n) are nicer, and the commuting costs (κ_{ni}) are lower.

2.4.2.1 The Probability of Living in n

Summing over i, we get the probability of living in n:

$$\lambda_n^R = \frac{R_n}{\bar{L}} = \sum_{i\in N}\lambda_{ni} = \sum_{i\in N}\frac{\Phi_{ni}}{\Phi}$$

where national labor market clearing corresponds to $\sum_n \lambda_n^R = \sum_i \lambda_i^L = 1$.

2.4.3 The Probability of Working in i

Summing over n, we get the probability of working in i:

$$\lambda_i^L = \frac{L_n}{\bar{L}} = \sum_{n \in N} \lambda_{ni} = \sum_{n \in N} \frac{\Phi_{ni}}{\Phi}$$

where national labor market clearing corresponds to $\sum_n \lambda_n^R = \sum_i \lambda_i^L = 1$.

2.5 Worker Income

2.5.1 Probability that a Worker Living in n Commutes to i

The probability that a worker living in n commutes to i is given by:

$$\lambda_{ni|n}^R \equiv \frac{\lambda_{ni}}{\lambda_n^R} = \frac{B_{ni}\left(w_i/\kappa_{ni}\right)^{\epsilon}}{\sum_{s \in N} B_{ns}\left(w_s/\kappa_{ns}\right)^{\epsilon}} \tag{17}$$

(17) implies a "commuting gravity equation," where the elasticity of commuting flows with respect to commuting costs is $-\epsilon$, since

$$\frac{d \ln \lambda_{ni|n}^R}{d \ln \kappa_{ni}} = -\epsilon$$

since

$$\begin{aligned} \ln\left(\lambda_{ni|n}^R\right) &= \ln\left(\frac{B_{ni}\left(w_i/\kappa_{ni}\right)^{\epsilon}}{\sum_{s \in N} B_{ns}\left(w_s/\kappa_{ns}\right)^{\epsilon}}\right) \\ &= \ln(B_{ni}) + \epsilon \ln(w_i) - \epsilon \ln(\kappa_{ni}) - \ln\left(\sum_{s \in N} B_{ns}\left(w_s/\kappa_{ns}\right)^{\epsilon}\right) \end{aligned}$$

Questions

Not totally sure why we're allowed to neglect the term in the denominator. One of you all maybe knows this and could tell me?

2.5.1.1 Labor Market Clearing

We then re-write the labor market clearing condition using the $\lambda_{ni|n}^R$ term:

$$L_i = \sum_{n \in N} \lambda_{ni|n}^R R_n$$

which says that the measure of workers in i must equal the sum across locations n of the share of people living in n and working in i multiplied by the measure of residents in n. Note that we are using the measure of workers and not scaling it by the number of hours that they work, hence this does not change compared to the original paper.

2.5.1.2 Expected Worker Income

Expected worker wage conditional on living in location n can then be written as a weighted average of all possible workplace regions, where the weight is given by the conditional probability of working in the region given living in n:

$$\bar{v}_n = \sum_{i \in N} \lambda^R_{ni|n} w_i$$

Notice that since $\lambda^R_{ni|n}$ increases as commuting costs to i decrease, expected wage is higher in regions with low commuting costs to areas with high wages.

Note again that since I've specified "wage" that this does not change from the origina paper; we would need to scale by share of time spent working to get expected worker income.

2.5.2 Expected Utility

Population mobility implies that expected utility is the same across all work-home location pairs, which is, naturally, the same as the expected utility for the economy as a whole:

$$\bar{U} = \mathbb{E}\left[U_{ni\omega}\right] = \quad \Gamma\left(\frac{\epsilon-1}{\epsilon}\right)\left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]\left[\sum_{r \in N}\sum_{s \in N} B_{ni}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon}\right]^{\frac{1}{\epsilon}} \quad \text{all } n, i \in N \tag{18}$$

The derivation is given in subsection 3.12.

Note that even though expected utility is constant across locations, other details of the locations vary. Wages, for example, must vary to accommodate preference heterogeneity over location.

"Workplaces and residences face upward-sloping supply functions for workers and residents, respectively." In order to attract more workers, workplaces must pay higher wages to raise commuters' real income and induce additional workers with lower idiosyncratic amenities for that workplace to commute there. Similarly, residences must offer lower land prices to raise residents' real income and induce additional workers to live there who have lower idiosyncratic amenities for that residence.

Bilateral commutes with attractive characteristics, e.g., high workplace wage and low residence cost of living, "attract additional commuters with lower idiosyncratic amenities, until expected utility (taking into account idiosyncratic amenities) is the same across all bilateral commutes."

3 Derivations

3.1 Derivation of Time Allocation

This is the derivation of (3).

Note that the individual's problem is:

$$\begin{aligned}
&\max \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{C_{n\omega}}{\alpha}\right)^{\alpha} \left(\frac{H_{n\omega}}{1-\alpha}\right)^{1-\alpha} s_{\omega}^{\gamma} \\
&\text{s.t. } s_{\omega} \in [0,1] \\
&\text{and } P_n C_{n\omega} + Q_n H_{n\omega} = w_i \left(1 - s_{\omega}\right)
\end{aligned}$$

For the earnings that the individual receives, their expenditures on consumption and land will follow the standard Cobb-Douglas results. That is,

$$\begin{aligned}
P_n C_{n\omega} &= \alpha \left[w_i \left(1 - s_{\omega}\right)\right] \\
Q_n H_{n\omega} &= (1-\alpha) \left[w_i \left(1 - s_{\omega}\right)\right]
\end{aligned}$$

which gives

$$\begin{aligned}
C_{n\omega} &= \frac{\alpha w_i \left(1 - s_{\omega}\right)}{p} \\
H_{n\omega} &= \frac{(1-\alpha) w_i \left(1 - s_{\omega}\right)}{Q_n}
\end{aligned}$$

Plugging these results back into the worker's problem gives us the new objective to maximize:

$$\begin{aligned}
&\max_{s_{\omega}} \underbrace{\frac{b_{ni\omega}}{\kappa_{ni}} w_i P_n^{-\alpha} Q_n^{-(1-\alpha)}}_{\text{constant wrt } s_{\omega}} \times \left(1 - s_{\omega}\right) s_{\omega}^{\gamma} \\
&\text{s.t. } s_{\omega} \in [0,1]
\end{aligned}$$

which will yield the same s_{ω} as simply solving:

$$\begin{aligned}
&\max_{s_{\omega}} \left(1 - s_{\omega}\right) s_{\omega}^{\gamma} \\
&\text{s.t. } s_{\omega} \in [0,1]
\end{aligned}$$

If we ignore the constraint for the moment and then later return to verify that it's satisfied, we get the FOC and subsequent equalities:

$$
\begin{aligned}
&\frac{\partial}{\partial s_\omega}\left(1-s_\omega\right)s_\omega^\gamma = 0\\
\Rightarrow&\gamma s_\omega^{\gamma-1} - (\gamma+1)s_\omega^\gamma = 0\\
\Rightarrow&\gamma - (\gamma+1)s_\omega = 0\\
\Rightarrow& s_\omega = \frac{\gamma}{\gamma+1}
\end{aligned}
$$

which also gives that time spent working is

$$1 - s_\omega = \frac{1}{\gamma+1}$$

$s_\omega \in [0,1]$ as long as $\gamma \geq 0$.

3.2 Derivation of $c_{ni}(j)$ Expression

3.2.1 Write the Problem

This is the derivation of (5).

When making consumption decisions surrounding $c_{ni}(j)$, an individual is solving the following problem:

$$
\begin{aligned}
&\max_{\{c_{ni}(j)\}} C_n\\
\text{s.t. } &\sum_i \int_0^{M_i} p_{ni}(j)c_{ni}(j)dj = \alpha X_n
\end{aligned}
$$

or expanded out:

$$
\begin{aligned}
&\max_{\{c_{ni}(j)\}} \left[\sum_{i\in N}\int_0^{M_i} c_{ni}(j)^\rho dj\right]^{\frac{1}{\rho}}\\
\text{s.t. } &\sum_i \int_0^{M_i} p_{ni}(j)c_{ni}(j)dj = \alpha X_n
\end{aligned}
$$

where the αX_n term comes from the fact that people spend α of their total expenditures, X_n, on goods.

3.2.2 Lagrangian and FOCs

The Langragian for this problem is then:

$$\mathcal{L} = \left[\sum_{i\in N}\int_0^{M_i} c_{ni}(j)^\rho dj\right]^{\frac{1}{\rho}} + \lambda\left(\alpha X_n - \sum_i \int_0^{M_i} p_{ni}(j)c_{ni}(j)dj\right)$$

which gives the relevant FOC:

$$\begin{aligned}
\{c_{ni}(j)\} \qquad & \frac{\partial}{\partial c_{ni}(j)}\left[\left(\sum_{i\in N}\int_0^{M_i} c_{ni}(j)^\rho dj\right)^{\frac{1}{\rho}}\right] - \lambda p_{ni}(j) = 0 \\
& \Leftrightarrow \frac{1}{\rho}\left[\sum_{i\in N}\int_0^{M_i} \rho c_{ni}(j)^\rho dj\right]^{\frac{1}{\rho}-1} c_{ni}(j)^{\rho-1} - \lambda p_{ni}(j) = 0 \\
& \Leftrightarrow C_n^{1-\rho} c_{ni}(j)^{\rho-1} = \lambda p_{ni}(j) \\
& \Leftrightarrow c_{ni}(j) = \lambda^{\frac{1}{\rho-1}} p_{ni}(j)^{\frac{1}{\rho-1}} C_n \\
& \Leftrightarrow c_{ni}(j) = \lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n \qquad \text{since } \sigma = \frac{1}{1-\rho}
\end{aligned} \tag{19}$$

3.2.3 Solve for λ

From there, we can define the dual price index

$$P_n \equiv \left(\sum_{i\in N}\int_0^{M_i} p_{ni}(j)^{1-\sigma} d_j\right)^{\frac{1}{1-\sigma}} \tag{20}$$

and revisit to our budget constraint:

$$\begin{aligned}
\alpha X_n &= \sum_i \int_0^{M_i} p_{ni}(j)c_{ni}(j)dj \\
&= \sum_i \int_0^{M_i} p_{ni}(j)\lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n dj \quad \text{by (19)} \\
&= \lambda^{-\sigma} C_n \sum_i \int_0^{M_i} p_{ni}(j)^{1-\sigma} dj \\
&= \lambda^{-\sigma} C_n P_n^{1-\sigma} \\
\Rightarrow \lambda^{-\sigma} &= \frac{\alpha X_n}{C_n} P_n^{\sigma-1}
\end{aligned}$$

3.2.4 Plug in λ

Then, returning to (19), we can plug in our expression for $\lambda^{-\sigma}$:

$$
\begin{aligned}
c_{ni}(j) &= \lambda^{-\sigma} p_{ni}(j)^{-\sigma} C_n \\
&= \left(\frac{\alpha X_n}{C_n} P_n^{\sigma-1} \right) p_{ni}(j)^{-\sigma} C_n \\
&= \alpha X_n P_n^{\sigma-1} p_{ni}(j)^{-\sigma}
\end{aligned}
$$

which is what we wanted.

3.3 Derivation of $p_{ni}(j)$ Expression

This is the derivation of (7).

3.3.1 Marginal Cost

We will get the expression for $p_{ni}(j)$ by equating marginal cost and marginal revenue.

First, notice that since

$$
l_i(j) = F + \frac{x_i(j)}{A_i}
$$

cost is given by:

$$
\underbrace{\text{Cost}\,(x_i(j))}_{\text{total monetary cost}} = \underbrace{w_i}_{\text{wage}} \left[F + \frac{x_i(j)}{A_i} \right] \tag{21}
$$

However, to account for trade costs, we must consider the variable cost of delivering $q_{ni}(j)$ units of variety j to location n from location i:

$$
\underbrace{\text{Var Cost}\,(q_{ni}(j))}_{\text{variable monetary cost}} = \underbrace{w_i}_{\text{wage}} \underbrace{d_{ni}}_{\text{trade costs}} \left[\frac{q_{ni}(j)}{A_i} \right] \tag{22}
$$

This is fine for marginal cost, since the fixed cost will disappear when we take the derivative.

Also, notice that we've defined $q_{ni}(j)$ such that:

$$
x_i(j) = \sum_{n \in N} d_{ni} q_{ni}(j) \tag{23}
$$

where d_{ni} is the multiplier reflecting how many more units of variety j must be made to deliver $q_{ni}(j)$ units of variety j to location n from location i.

Then, marginal cost is given by:

$$\text{MC} = \frac{d}{dq_{ni}(j)} \text{Var Cost}\left(q_{ni}(j)\right) = \frac{w_i d_{ni}}{A_i}$$

3.3.2 Marginal Revenue

Under isoelastic (CES) demand, we can write the quantity demanded as

$$q_{ni}(j) = A_i p_{ni}(j)^{-\sigma}$$

Then, revenue is given by:

$$\text{Rev}(p_{ni}(j)) = p_{ni}(j) q_{ni}(j) = A_i p_{ni}(j)^{1-\sigma}$$

Then marginal revenue is given by:

$$\text{MR}_{ni}(j) = \frac{d\text{Rev}(p_{ni}(j))}{dp_{ni}(j)} \frac{dp_{ni}(j)}{dq_{ni}(j)} \tag{24}$$

by the chain rule.

Then notice that

$$\frac{d\text{Rev}(p_{ni}(j))}{dp_{ni}(j)} = A_i(1-\sigma) p_{ni}(j)^{-\sigma} \tag{25}$$

and

$$\frac{dq_{ni}(j)}{dp_{ni}(j)} = -\sigma A_i p_{ni}(j)^{-\sigma-1}$$

Since q and p are inversely related, we can write:

$$\frac{dp_{ni}(j)}{dq_{ni}(j)} = \left(\frac{dq_{ni}(j)}{dp_{ni}(j)}\right)^{-1} = -\frac{1}{\sigma A_i p_{ni}(j)^{\sigma+1}} \tag{26}$$

Then, plugging (25) and (26) into (24), we get:

$$\begin{aligned}
\text{MR}_{ni}(j) =& \frac{d\text{Rev}(p_{ni}(j))}{dp_{ni}(j)} \frac{dp_{ni}(j)}{dq_{ni}(j)} \\
=& A_i(1-\sigma) p_{ni}(j)^{-\sigma} \left(-\frac{1}{\sigma A_i p_{ni}(j)^{\sigma+1}}\right) \\
=& \left(\frac{\sigma-1}{\sigma}\right) p_{ni}(j)
\end{aligned}$$

3.3.3 Equating MR and MC

Then, equating marginal revenue and marginal cost, we get the desired expression:

$$
\begin{aligned}
\mathrm{MR}_{ni}(j) &= \mathrm{MC} \\
\Rightarrow \left(\frac{\sigma-1}{\sigma}\right) p_{ni}(j) &= \frac{d_{ni} w_i}{A_i} \\
\Rightarrow p_{ni}(j) &= \left(\frac{\sigma}{\sigma-1}\right) \frac{d_{ni} w_i}{A_i}
\end{aligned}
$$

3.4 Derivation of $x_i(j)$ Expression

This is the derivation of (8).

Zero profit implies that revenue minus variable cost equals fixed cost, which leads to the desired equation:

$$
\begin{aligned}
&\text{Revenue} - \text{Variable Cost} = \text{Fixed Cost} \\
\Rightarrow &\sum_n \left[p_{ni}(j) q_{ni}(j) - w_i d_{ni} \left(\frac{q_{ni}(j)}{A_i} \right) \right] = w_i F && \text{By (21) and (22)} \\
\Rightarrow &\sum_n \left[\frac{\sigma}{\sigma-1} \frac{d_{ni} w_i}{A_i} q_{ni}(j) - w_i d_{ni} \left(\frac{q_{ni}(j)}{A_i} \right) \right] = w_i F && \text{By (7)} \\
\Rightarrow &\sum_n \left[\frac{1}{\sigma-1} \frac{d_{ni}}{A_i} q_{ni}(j) \right] = F \\
\Rightarrow &\sum_n d_{ni} q_{ni}(j) = A_i F(\sigma-1) \\
\Rightarrow &x_i(j) = A_i F(\sigma-1) && \text{By (23)}
\end{aligned}
$$

3.5 Derivation of M_i Expression

This is a derivation of (9).

$$
\begin{aligned}
(1-s_\omega)L_i &= \int_0^{M_i} l_i(j) dj && \text{Labor market clearing} \\
&= \int_0^{M_i} \left[F + \frac{x_i(j)}{A_i} \right] dj && \text{By (6)} \\
&= \int_0^{M_i} F + \frac{A_i F(\sigma-1)}{A_i} dj && \text{By (8)} \\
&= \int_0^{M_i} \sigma F dj \\
&= \sigma F M_i
\end{aligned}
$$

By re-arranging, we get the expression:

$$
\begin{aligned}
M_i =& \frac{(1-s_\omega)L_i}{\sigma F} \\
=& \left(\frac{L_i}{\sigma F}\right)\left(\frac{1}{1+\gamma}\right) && \text{by (4)}
\end{aligned}
$$

which is what we wanted.

3.6 Derivation of π_{ni} Expression

This is a derivation of (10).

First, because it will be useful momentarily, note:

$$
\begin{aligned}
P_n =& \left[\sum_{k\in N}\int_0^{M_k} p_{nk}(j)^{1-\sigma}dj\right]^{\frac{1}{1-\sigma}} \\
\Rightarrow P_n^{1-\sigma} =& \sum_{k\in N}\int_0^{M_k} p_{nk}(j)^{1-\sigma}dj \\
\Rightarrow P_n^{1-\sigma} =& \sum_{k\in N}\int_0^{M_k} p_{nk}^{1-\sigma}dj && \text{By (7)} \\
\Rightarrow P_n^{1-\sigma} =& \sum_{k\in N} M_k p_{nk}^{1-\sigma} && (27)
\end{aligned}
$$

Now, begin with the baseline expression for share of expenditure:

$$
\begin{aligned}
\pi_{ni} =& \frac{\int_0^{M_i} p_{ni}(j)c_{ni}(j)dj}{\sum_{k\in N}\int_0^{M_k} p_{nk}(j)c_{nk}(j)dj} \\
=& \frac{\int_0^{M_i} p_{ni}(j)\alpha X_n P_n^{\sigma-1}p_{ni}(j)^{-\sigma}dj}{\sum_{k\in N}\int_0^{M_k} p_{nk}(j)\alpha X_n P_n^{\sigma-1}p_{nk}(j)^{-\sigma}dj} && \text{By (5)} \\
=& \frac{\int_0^{M_i} p_{ni}(j)^{1-\sigma}dj}{\sum_{k\in N}\int_0^{M_k} p_{nk}(j)^{1-\sigma}dj} && \text{Cancel terms} \\
=& \frac{\int_0^{M_i} p_{ni}(j)^{1-\sigma}dj}{P_n^{1-\sigma}} && \text{By (20)} \\
=& \frac{\int_0^{M_i} p_{ni}^{1-\sigma}dj}{\sum_{k\in N} M_k p_{nk}^{1-\sigma}} && \text{By (7) \& (27)} \\
=& \frac{M_i p_{ni}^{1-\sigma}}{\sum_{k\in N} M_k p_{nk}^{1-\sigma}}
\end{aligned}
$$

Thus, we now have the first desired equality:

$$\pi_{ni} = \frac{M_i p_{ni}^{1-\sigma}}{\sum_{k \in N} M_k p_{nk}^{1-\sigma}} \tag{28}$$

Now, notice that:

$$M_i p_{ni}^{1-\sigma} = \left(\frac{1}{1+\gamma}\right)\left(\frac{L_i}{\sigma F}\right)\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma} \qquad \text{By (7) and (9)}$$

Thus, we can re-write (28) to be:

$$\begin{aligned}
\pi_{ni} =& \frac{M_i p_{ni}^{1-\sigma}}{\sum_{k \in N} M_k p_{nk}^{1-\sigma}} \\
=& \frac{\left(\frac{1}{1+\gamma}\right)\left(\frac{L_i}{\sigma F}\right)\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{d_{ni}w_i}{A_i}^{1-\sigma}}{\sum_{k \in N}\left(\frac{1}{1+\gamma}\right)\left(\frac{L_k}{\sigma F}\right)\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{d_{nk}w_k}{A_k}^{1-\sigma}} \\
=& \frac{L_i\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} L_k \left(\frac{d_{nk}w_k}{A_k}\right)^{1-\sigma}}
\end{aligned}$$

which is what we wanted.

3.7 Derivation of $w_i L_i$ Expression

This is a derivation of (11).

There's not really much of a derivation here.

First, note that total labor income is expressed as:

$$\begin{aligned}
& w_i L_i (1 - s_\omega) \\
=& w_i L_i \left(\frac{1}{1+\gamma}\right) \qquad \text{By (4)}
\end{aligned}$$

Total revenue from goods produced in location i is given by:

$$\sum_{n \in N} \pi_{ni} \left(\frac{1}{1+\gamma}\right) \bar{v}_n R_n$$

Note that I'm using $\bar{v}_n$ to denote the average wage of workers in location n, rather than average income, with the distinction being that income would be wage scaled by share of time spent working.

There are no net profits, so total revenue minus total cost must be zero:

$$
\begin{aligned}
&\sum_{n\in N} \pi_{ni}\left(\frac{1}{1+\gamma}\right)\bar{v}_n R_n - w_i L_i \left(\frac{1}{1+\gamma}\right) = 0\\
\Rightarrow &\sum_{n\in N} \pi_{ni}\bar{v}_n R_n - w_i L_i = 0\\
\Rightarrow &\sum_{n\in N} \pi_{ni}\bar{v}_n R_n = w_i L_i
\end{aligned}
$$

3.8 Derivation of P_n Re-Expression

This is a derivation of (12).

$$
\begin{aligned}
P_n &= \left[\sum_{i\in N}\int_0^{M_i} p_{ni}(j)^{1-\sigma}dj\right]^{\frac{1}{1-\sigma}} && \text{By (20)}\\
&= \left[\sum_{i\in N}\int_0^{M_i} p_{ni}^{1-\sigma}dj\right]^{\frac{1}{1-\sigma}} && \text{By (7)}\\
&= \left[\sum_{i\in N} M_i p_{ni}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\\
&= \left[\sum_{i\in N}\left(\frac{1}{1+\gamma}\right)\frac{L_i}{\sigma F}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\frac{d_{ni}w_i}{A_i}^{1-\sigma}\right]^{\frac{1}{1-\sigma}} && \text{By (9) and (7)}\\
&= \left(\frac{\sigma}{\sigma-1}\right)\left(\frac{1}{1+\gamma}\right)^{\frac{1}{1-\sigma}}\left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}}\left[\sum_{i\in N} L_i\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}
\end{aligned}
$$

which is the first desired equality.

From there, we can note

$$
\pi_{nn} = \frac{L_n\left(\frac{d_{nn}w_n}{A_n}\right)^{1-\sigma}}{\sum_{i\in N} L_i\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma}} \qquad \text{By (10)} \tag{29}
$$

which let's us pick back up to write:

$$
\begin{aligned}
P_n &= \left(\frac{\sigma}{\sigma-1}\right)\left(\frac{1}{1+\gamma}\right)^{\frac{1}{1-\sigma}}\left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}}\left[\sum_{i\in N} L_i\left(\frac{d_{ni}w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\
&= \left(\frac{\sigma}{\sigma-1}\right)\left(\frac{1}{1+\gamma}\right)^{\frac{1}{1-\sigma}}\left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}}\left[\frac{L_n\left(\frac{d_{nn}w_n}{A_n}\right)^{1-\sigma}}{\pi_{nn}}\right]^{\frac{1}{1-\sigma}} && \text{By (29)} \\
&= \left(\frac{\sigma}{\sigma-1}\right)\left[\left(\frac{1}{1+\gamma}\right)\left(\frac{L_n}{\sigma F\pi_{nn}}\right)\right]^{\frac{1}{1-\sigma}}\frac{d_{nn}w_n}{A_n}
\end{aligned}
$$

which is the second desired expression.

3.9 Derivation of Indirect Utility

This is a derivation of (13).

Beginning with our utility function:

$$
U_{ni\omega} = \frac{b_{ni\omega}}{\kappa_{ni}}\left(\frac{C_{n\omega}}{\alpha}\right)^{\alpha}\left(\frac{H_{n\omega}}{1-\alpha}\right)^{1-\alpha} s_{\omega}^{\gamma}
$$

Optimization gives us that workers (living in region n and working in region i) spend α of their income, $w_i\left(\frac{1}{1+\gamma}\right)$, on consumption and $(1-\alpha)$ on housing. The price of C_n is P_n and the price of H_n is Q_n. Thus we have the expressions:

$$
P_nC_n = \alpha\left(\frac{1}{1+\gamma}\right)w_i \tag{30}
$$

$$
Q_nH_n = (1-\alpha)\left(\frac{1}{1+\gamma}\right)w_i \tag{31}
$$

Then we can re-write:

$$
\begin{aligned}
U_{ni\omega} =& \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{C_{n\omega}}{\alpha}\right)^{\alpha} \left(\frac{H_{n\omega}}{1-\alpha}\right)^{1-\alpha} s_{\omega}^{\gamma} && \text{by (2)} \\
=& \frac{b_{ni\omega}}{\kappa_{ni}} \left(\left(\frac{1}{1+\gamma}\right)\frac{\alpha w_i}{\alpha P_n}\right)^{\alpha} \left(\left(\frac{1}{1+\gamma}\right)\frac{(1-\alpha)w_i}{(1-\alpha)Q_n}\right)^{1-\alpha} s_{\omega}^{\gamma} && \text{by (30) and (31)} \\
=& \frac{b_{ni\omega}}{\kappa_{ni}} \left(\frac{1}{1+\gamma}\right) \left(\frac{w_i}{P_n}\right)^{\alpha} \left(\frac{w_i}{Q_n}\right)^{1-\alpha} s_{\omega}^{\gamma} \\
=& \frac{b_{ni\omega} w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}} \left(\frac{1}{1+\gamma}\right) s_{\omega}^{\gamma} \\
=& \frac{b_{ni\omega} w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}} \left(\frac{1}{1+\gamma}\right) \left(\frac{\gamma}{\gamma+1}\right)^{\gamma} && \text{by (3)} \\
=& \frac{b_{ni\omega} w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}} \left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]
\end{aligned}
$$

where the last line follows from

$$
\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma}\left[1-\frac{\gamma}{\gamma+1}\right] = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma} \cdot \frac{(\gamma+1)-\gamma}{\gamma+1} = \frac{1}{\gamma+1}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}
$$

This is the expression that we wanted.

3.10 Derivation of Fréchet Distribution for Indirect Utility

This is a derivation of (14).

Consider the following property of the Fréchet distribution:

Fréchet Distribution Monotonic Transformation Property

If $X \sim \text{Fréchet}(B, \epsilon)$, then $F_X(x) = \exp\left[-Bx^{-\epsilon}\right]$ for $x > 0$.

Then for a constant $c > 0$, $cX \sim \text{Fréchet}\left(c^{-\epsilon}B, \epsilon\right)$.

In other words,

$$
F_{cX}(u) = \Pr\{cX \le u\} = \Pr\{X \le u/c\} = \exp\left[-B(u/c)^{-\epsilon}\right] = \exp\left[-\left(c^{\epsilon}B\right)u^{-\epsilon}\right]
$$

Recall that

$$
b \sim \text{Fréchet}(B, \epsilon)
$$

and hence

$$G^b_{ni\omega}(b) = \exp\left[-B_{ni} b^{-\epsilon}\right]$$

Moreover, recall that

$$U_{ni\omega} = \underbrace{\frac{w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}} \left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]}_{\text{constant w.r.t. } \omega} \times b_{ni\omega}$$

Thus, employing out property of the Fréchet distribution, we have that

$$\begin{aligned} G_{ni}(U) &= \exp\left[-\left(\frac{w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}}\right)^{\epsilon} \left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]^{\epsilon} B_{ni} U^{-\epsilon}\right] \qquad \text{by the property above} \\ &= \exp\left[-B_{ni} (\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha})^{-\epsilon} \left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]^{\epsilon} w_i^{\epsilon} U^{-\epsilon}\right] \\ &= \exp\left[-\Psi_{ni} U^{-\epsilon}\right] \end{aligned}$$

if we define

$$\Psi_{ni} = B_{ni} \left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon} \left[\left(\frac{\gamma}{\gamma+1}\right)^{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]^{\epsilon}$$

which is the desired expression.

Also, notice that this is saying that

$$U_{ni\omega} \sim \text{Fréchet}\left(\Psi_{ni}, \epsilon\right) \tag{32}$$

3.11 Derivation of λ_{ni} Expression

This is a derivation of (16).

First, note that the pdf of $U_{ni\omega}$ is given by:

$$f_{ni}(u) = \frac{d}{du} G_{ni}(u) = \frac{d}{du}\left[\exp\left(-\Psi_{ni} u^{-\epsilon}\right)\right] = \epsilon \Psi_{ni} u^{-\epsilon-1} \exp\left[-\Psi_{ni} u^{-\epsilon}\right] \tag{33}$$

$$
\begin{aligned}
\lambda_{ni} =& \Pr\left(\underset{(mh)}{\text{argmax}}\, U_{mh\omega} = (ni)\right) \\
=& \Pr\left(U_{ni\omega} > U_{mh\omega} \text{ for all } (mh) \neq (ni)\right) \\
=& \int_0^\infty \underbrace{f_{ni}(u)}_{\text{pdf of } U_{ni\omega}} \Pr\left(U_{mh\omega} < u, \forall (m,h) \neq (n,i) \mid U_{ni\omega} = u\right) du \\
=& \int_0^\infty f_{ni}(u) \Pr\left(U_{mh\omega} < u, \forall (m,h) \neq (n,i)\right) du && \text{by independence} \\
=& \int_0^\infty f_{ni}(u) \prod_{(m,h)\neq(n,i)} \Pr\left(U_{mh,\omega} < u\right) du && \text{by independence} \\
=& \int_0^\infty f_{ni}(u) \prod_{(m,h)\neq(n,i)} G_{mh}(u) du && \text{by definition of CDF} \\
=& \int_0^\infty \epsilon \Psi_{ni} u^{-\epsilon-1} \exp\left[-\Psi_{ni} u^{-\epsilon}\right] \prod_{(m,h)\neq(n,i)} G_{mh}(u) du && \text{by (33)} \\
=& \int_0^\infty \epsilon \Psi_{ni} u^{-\epsilon-1} \exp\left[-\Psi_{ni} u^{-\epsilon}\right] \prod_{(m,h)\neq(n,i)} \exp\left[-\Psi_{mh} u^{-\epsilon}\right] du && \text{by (14)} \\
=& \int_0^\infty \epsilon \Psi_{ni} u^{-\epsilon-1} \exp[- \underbrace{\left(\Psi_{ni} + \sum_{(m,h)\neq(n,i)} \Psi_{mh}\right)}_{\sum_{r,s} \Psi_{rs}} u^{-\epsilon}] du \\
=& \int_0^\infty \epsilon \Psi_{ni} u^{-\epsilon-1} \exp\left[-\left(\sum_{r,s} \Psi_{rs}\right) u^{-\epsilon}\right] du \\
=& \Psi_{ni} \int_0^\infty \epsilon u^{-\epsilon-1} \exp\left[-\left(\sum_{r,s} \Psi_{rs}\right) u^{-\epsilon}\right] du
\end{aligned}
$$

Consider the following change of variables:

$$
\begin{aligned}
t &= u^{-\epsilon} \\
dt &= -\epsilon u^{-\epsilon-1} du \\
\Rightarrow & \epsilon u^{-\epsilon-1} du = -dt
\end{aligned}
$$

Notice that as u goes from 0 to ∞, t goes from ∞ to 0.

Thus, we can re-write:

$$
\begin{aligned}
\lambda_{ni} =& \Psi_{ni} \int_0^{\infty} \epsilon u^{-\epsilon-1} \exp\left[-\left(\sum_{r,s} \Psi_{rs}\right) u^{-\epsilon}\right] du \\
=& \Psi_{ni} \int_{t=\infty}^{t=0} \exp\left[-\left(\sum_{r,s} \Psi_{rs}\right) t\right] (-dt) \\
=& \Psi_{ni} \int_0^{\infty} \exp\left[-\left(\sum_{r,s} \Psi_{rs}\right) t\right] dt && \text{integral property} \\
=& \frac{\Psi_{ni}}{(-\sum_{r,s} \Psi_{rs})} \int_0^{\infty} \left(-\sum_{r,s} \Psi_{rs}\right) \left[-\left(\sum_{r,s} \Psi_{rs}\right) t\right] dt && \text{to make the inside of the integral equal } \tfrac{d}{dx}[e^{cx}] \\
=& \frac{\Psi_{ni}}{\sum_{r,s} \Psi_{rs}} \left[\lim_{t\to\infty} \exp\left(-\left(\sum_{r,s} \Psi_{rs}\right) t\right) - \exp(0)\right] && \text{by the fundamental theorem of calculus} \\
=& \frac{\Psi_{ni}}{-(\sum_{r,s} \Psi_{rs})} [0-1] \\
=& \frac{\Psi_{ni}}{\sum_{r,s} \Psi_{rs}}
\end{aligned}
$$

which is what we wanted.

3.12 Derivation of $\bar{U}$ Expression

This is a derivation of (18).

We are going to rely on the following properties of the Fréchet distribution:

Fréchet Distribution Max Stability Property

If[a] [b]

$$X_i \sim \text{Fréchet}(B_i, \epsilon)$$

and

$$Y = \max\{X_1, X_2, \ldots, X_n\}$$

Then

$$Y \sim \text{Fréchet}\left(\sum_{i=1}^{n} B_i, \epsilon\right)$$

[a]Note that this is under the parameterization where $X \sim \text{Fréchet}(B, \epsilon)$ corresponds to $F_X(x) = \exp\left[-Bx^{-\epsilon}\right]$. Sometimes, you will see people parameterize the Fréchet distribution as $X \sim \text{Fréchet}(B, \epsilon)$ corresponds to $F_X(x) = \exp\left[-\left(\frac{x}{B}\right)^{-\epsilon}\right]$.

[b]Find a citation for this.

Fréchet Distribution Property

If[a] $X \sim \text{Fréchet}(B, \epsilon)$ and $\epsilon > 1$, then

$$\mathbb{E}[X] = B^{\frac{1}{\epsilon}}\Gamma\left(1 - \frac{1}{\epsilon}\right)$$

[a]Note that this is under the parameterization where $X \sim \text{Fréchet}(B, \epsilon)$ corresponds to $F_X(x) = \exp\left[-Bx^{-\epsilon}\right]$. Sometimes, you will see people parameterize the Fréchet distribution as $X \sim \text{Fréchet}(B, \epsilon)$ corresponds to $F_X(x) = \exp\left[-\left(\frac{x}{B}\right)^{-\epsilon}\right]$.

Recall from (32) that:

$$U_{ni\omega} \sim \text{Fréchet}\left(\Psi_{ni}, \epsilon\right)$$

If workers are choosing the location that maximizes utility, then their selected utility follows:

$$\max_{n,i} U_{ni\omega} \sim \text{Fréchet}\left(\sum_{n,i} \Psi_{ni}, \epsilon\right)$$

by the D.2 property.

From which, as long as $\epsilon > 1$, we get

$$\mathbb{E}[\max_{n,i} U_{ni\omega}] = \left(\sum_{n,i} \Psi_{ni}\right)^{\frac{1}{\epsilon}} \Gamma\left(1 - \frac{1}{\epsilon}\right)$$

by the D.3 property.

Additionally, if any home-work pair gave strictly higher expected utility, workers would move to that pair until it's expected utility was no longer strictly higher; thus, it must be that the expected utility across all home-work pairs is the same. That is

$$\mathbb{E}[\max_{n,i} U_{ni\omega}] = \mathbb{E}[U_{ni\omega}] \text{ for all } n, i$$

Thus,

$$
\begin{aligned}
\bar{U} =& \mathbb{E}[U_{ni\omega}] \\
=& \mathbb{E}[\max_{n,i} U_{ni\omega}] \\
=& \left(\sum_{r,s} \Psi_{rs} \right)^{\frac{1}{\epsilon}} \Gamma \left(1 - \frac{1}{\epsilon} \right) \\
=& \Gamma \left(\frac{\epsilon - 1}{\epsilon} \right) \left[\sum_{r \in N} \sum_{s \in N} B_{ni} \left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha} \right)^{-\epsilon} w_i^{\epsilon} \left[\left(\frac{\gamma}{\gamma+1} \right)^{\gamma} - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \right]^{\epsilon} \right]^{\frac{1}{\epsilon}} \quad \text{by (15)} \\
=& \Gamma \left(\frac{\epsilon - 1}{\epsilon} \right) \left[\left(\frac{\gamma}{\gamma+1} \right)^{\gamma} - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \right] \left[\sum_{r \in N} \sum_{s \in N} B_{ni} \left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha} \right)^{-\epsilon} w_i^{\epsilon} \right]^{\frac{1}{\epsilon}}
\end{aligned}
$$

which is the desired expression.

Uncertainty

I'm not 100% sure if my argument here is written out correctly.