Caliendo Paper Notes

Dylan Baker

March 2025

Contents

1 Terms

1.1 Model Section Terms

- n or i: Location index
- N: Number of locations
- j or k: Sector index
- J: Number of sectors
- θ^j: Sector-specific productivity dispersion parameter
- t: Time index
- L_0^{nj}: The mass of households in location n and sector j at time $t = 0$
- w_t^{nj}: Wage in location n and sector j at time t
- $U\left(C_t^{nj}\right)$: Utility function over baskets of final local goods
- C_t^{nj}: Basket of final local goods

$$C_t^{nj} = \prod_{k=1}^{J} \left(c_t^{nj,k}\right)^{\alpha^k}$$

- $c_t^{nj,k}$: Consumption of sector k good in market nj at time t.
- α^k: Final consumption share of sector k goods
  - $\sum_{k=1}^J \alpha^k = 1$
- P_t^n: Ideal price index in location n at time t

$$P_t^n = \prod_{k=1}^J \left(P_t^{nk}/\alpha^k\right)^{\alpha^k}$$

- P_t^{nk}: Price index of sector k goods for final consumption in location n at time t
- $b^n > 0$: Consumption obtained by non-employed individuals through home production[1]
- $C_t^{n0} = b^n$: Consumption in sector zero in location n at time t, which represents non-employment.
- $\beta \geq 0$: Discount factor
- $\tau^{nj,ik} \geq 0$: Labor relocation costs from market nj to ik
- $\boldsymbol{\epsilon}_t^{ik}$: Household-specific idiosyncratic shock for each choice of market.
  - $F(\boldsymbol{\epsilon})$: CDF of the idiosyncratic shock

$$F(\boldsymbol{\epsilon}) = \exp(-\exp(-\boldsymbol{\epsilon} - \bar{\gamma}))$$

  - $\bar{\gamma}$:

$$\bar{\gamma} \equiv \int_{-\infty}^{\infty} x \exp(-x - \exp(-x)) dx$$

- v_t^{nj}: The lifetime utility of a household currently in location n and sector j at time t, with the expectation taken over future realizations of the idiosyncratic shock.

$$\begin{aligned} \mathrm{v}_t^{nj} =& U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu\epsilon_t^{ik}\right\} \\ \text{s.t. } C_t^{nj} \equiv& \begin{cases} b^n & \text{if } j = 0 \\ w_t^{nj}/P_t^n & \text{otherwise} \end{cases} \end{aligned}$$

- ν: Parameter that scales the variance of the idiosyncratic shock
- $V_t^{nj} \equiv \mathbb{E}\left[\mathrm{v}_t^{nj}\right]$: Expected lifetime utility of a representative agent in location n and sector j at time t
- $\mu_t^{nj,ik}$: The fraction of households that relocate from market nj to ik at time t
- A_t^{nj}: Time-varying region-sector (market) productivity, common across varieties
- z^{nj}: Variety-specific market productivity

[1] Not totally sure if I'm understanding this term.

- q_t^{nj}: Expression for the quantity produced of an intermediate good

$$q_t^{nj} = z^{nj}\left(A_t^{nj}\left(h_t^{nj}\right)^{\xi^n}\left(l_t^{nj}\right)^{1-\xi^n}\right)^{\gamma^{nj}}\prod_{k=1}^{J}\left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}$$

- l_t^{nj}: Labor inputs in location n and sector j at time t
- h_t^{nj}: Structure inputs in location n and sector j at time t
- $M_t^{nj,nk}$: Material inputs from sector k demanded by a firm in sector j and region n at time t
- $\gamma^{nj} \geq 0$: Share of value-added in the production of sector j and region n
  - This comes from $\gamma^{nj} + \sum_{k=1}^{J}\gamma^{nj,nk} = 1$ and the Cobb-Douglas form, so the value-add share is the share that comes from the non-material inputs.
- $\gamma^{n,nk} \geq 0$: The share of materials from sector k in the production of sector j and region n.
- ξ^n: The share of structures in value-add.
- r_t^{nj}: The rental price of structures in region n and sector j at time t.
- x_t^{nj}: Unit price of an input bundle

$$x_t^{nj} = B^{nj}\left(\left(r_t^{nj}\right)^{\xi^n}\left(w_t^{nj}\right)^{1-\xi^n}\right)^{\gamma^{nj}}\prod_{k=1}^{J}\left(P_t^{nk}\right)^{\gamma^{nj,nk}}$$

  - B^{nj}: A constant
- $\boldsymbol{\kappa}_t^{nj,ij}$: Iceberg trade costs between market location n and i in sector j.
  - Delivering one unit of good from market n to i requires producing $\boldsymbol{\kappa}_t^{nj,ij} \geq 1$ units of the good.
  - If a good is nontradable, then $\kappa = \infty$.
- z_j: Vector of productivity draws for sector j

$$z^j = \left(z^{1j}, z^{2j}, \ldots, z^{Nj}\right)$$

- Q_t^{nj}: The quantity produced of aggregate sectoral goods j in region n

$$Q_t^{nj} = \left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}} d\phi^j\left(z^j\right)\right)^{\eta^{nj}/\left(\eta^{nj}-1\right)}$$

- $\phi^j\left(z^j\right)$: The joint distribution over the vector z^j.

$$\phi^j\left(z^j\right) = \exp\left\{-\sum_{n=1}^{N}\left(z^{nj}\right)^{-\theta^j}\right\}$$

  - $\phi^{nj}\left(z^{nj}\right)$: The marginal distribution of z^{nj}.

$$\phi^{nj}\left(z^{nj}\right) = \exp\left\{-\left(z^{nj}\right)^{-\theta^j}\right\}$$

- $\tilde{q}_t^{nj}\left(z^j\right)$: The quantity demanded of an intermediate good of a given variety from the lowest-cost supplier.
- P_t^{nj}: The price index of the aggregate sectoral good.

$$P_t^{nj} = \Gamma^{nj}\left(\sum_{i=1}^{N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)^{-1/\theta^j}$$

  - Γ^{nj}: A constant.

- $\pi_t^{nj,ij}$: The share of total expenditure in market nj on goods j from market i

$$\pi_t^{nj,ij} = \frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{m=1}^{N}\left(x_t^{mj}\kappa_t^{nj,mj}\right)^{-\theta^j}\left(A_t^{mj}\right)^{\theta^j\gamma^{mj}}}$$

- ι^n: The share from the global portfolio received by region n.
- χ_t: The total revenue in the global portfolio.
- X_t^{nj}: Total expenditure on sector j good in region n

$$X_t^{nj} = \underbrace{\sum_{k=1}^{J}\gamma^{nk,nj}\sum_{i=1}^{N}\pi_t^{ik,nk}X_t^{ik}}_{\substack{\text{Value of total demand for } nj\\ \text{market goods used as}\\ \text{materials}}} + \alpha^j\underbrace{\left(\sum_{k=1}^{J}w_t^{nk}L_t^{nk} + \iota^n\chi_t\right)}_{\substack{\text{Value of final}\\ \text{demand in}\\ \text{region } n}}$$

- $\Theta_t \equiv (A_t, \kappa_t)$: Time-varying economy fundamentals
- $\bar{\Theta} \equiv (\Upsilon, H, b)$: Constant economy fundamentals

1.2 Dynamic Hat Algebra Section Terms

- $\dot{y}_{t+1}$: The proportional change in any scalar or vector between periods t and $t+1$.

$$\dot{y}_{t+1} \equiv \left(y_{t+1}^1/y_t^1, y_{t+1}^2/y_t^2, \ldots\right)$$

- $u_t^{nj} \equiv \exp\left(V_t^{nj}\right)$
- $\dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)$: (For all n and j) the equilibrium real wages in time differences as functions of the change in labor $\dot{L}_{t+1}$ and time-varying fundamentals $\dot{\Theta}_{t+1}$.

2 The Model

2.1 Initial Setup

"In each region-sector combination, there is a competitive labor market. In each market, there is a continuum of perfectly competitive firms producing intermediate goods."

Firms have a Cobb-Douglas Constant-Returns-to-Scale (CRS) production function, which utilizes labor, "composite local factor that we refer to as structures, and materials from all sectors."

We assume that productivities are "distributed Fréchet with a sector-specific productivity dispersion parameter θ^j."

Time is discrete, $t = 0, 1, 2, \ldots$.

"Households are forward looking, have perfect foresight, and optimally decide where to move given some initial distribution of labor across locations and sectors. Households face costs to move across markets and experience an idiosyncratic shock that affects their moving decision."

2.2 Household Problem

At $t = 0$, there is a mass of households in location n and sector j, denoted by L_0^{nj}. Households are either *employed* or *non-employed*.

If employed in location n and sector j at time t, workers inelastically supply a unit of labor and receive wage w_t^{nj}.

Given income, the household decides how to allocate their consumption over final goods across sectors with a Cobb-Douglas aggregator. Preferences, $U\left(C_t^{nj}\right)$, are over baskets of final local goods:

$$C_t^{nj} = \prod_{k=1}^{J} \left(c_t^{nj,k}\right)^{\alpha^k}$$

"Households are forward-looking and discount the future at rate $\beta \geq 0$. Migration decisions are subject to sectoral and spatial mobility costs."

Assumption 1

Labor relocation costs $\tau^{nj,ik} \geq 0$ depend on the origin (nj) and destination (ik) and are time invariant, additive, and measured in terms of utility.

Households have idiosyncratic shocks ϵ_t^{ik} for each choice of market.

The timing for the household's problem is as follows:

1. Households observe the economic conditions in each market, as well as their own idiosyncratic shocks.

2. Returns
   - If they begin the period in the labor market, they work and receive the market wage.
   - If they are non-employed in a region, they receive home production.
3. Households choose whether to relocate.

Formally:

$$
\begin{aligned}
\mathrm{v}_t^{nj} =& U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu \epsilon_t^{ik}\right\} \\
&\text{s.t. } C_t^{nj} \equiv \begin{cases} b^n & \text{if } j = 0 \\ w_t^{nj}/P_t^n & \text{otherwise} \end{cases}
\end{aligned}
$$

v_t^{nj} is the lifetime utility of a household currently in location n and sector j at time t, with the expectation taken over future realizations of the idiosyncratic shock. ν is a parameter that scales the variance of the idiosyncratic shock.

Households choose to move to the labor market with the highest utility net of costs.

Assumption 2

The idiosyncratic shock ϵ is i.i.d. over time and distributed Type-I Extreme Value with zero mean.

Note that Assumption 2 is common in the literature[2] and is useful to allow "for simple aggregation of idiosyncratic decisions made by households."

Let

$$
V_t^{nj} \equiv \mathbb{E}\left[\mathrm{v}_t^{nj}\right] \tag{1}
$$

Then

$$
V_t^{nj} = U\left(C_t^{nj}\right) + \nu \log\left(\sum_{i=1}^{N}\sum_{k=0}^{J} \exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}\right)
$$

The derivation is given in subsection 4.1.

"V_t^{nj} can be interpreted as the expected lifetime utility of a household before the realization of the household preference shocks or, alternatively, as the average utility of households in that market."

From there we can derive the fraction of households that relocate from market nj to ik:

[2] Caliendo et al. (2019) refers readers to
Aguirregabiria, V., & Mira, P. (2010). Dynamic discrete choice structural models: A survey. *Journal of Econometrics, 156*(1), 38-67.

$$\mu_t^{nj,ik} = \frac{\exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{1/\nu}} \tag{2}$$

To-do

Add derivation if desired.

All else equal (2) implies that markets with a higher lifetime utility (net of migration costs) will be the ones to attract more migrants. Moreover, it allows $\frac{1}{\nu}$ to be interpreted as a migration elasticity with respect to the expected utility of the destination market, i.e., $\beta V_{t+1}^{ik} - \tau^{nj,ik}$.

You can see the elasticity interpretation intuitively if you consider:

$$\begin{aligned}
\mu_t^{nj,ik} &= \frac{\exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{1/\nu}} \\
\Rightarrow \ln \mu_t^{nj,ik} &= \frac{1}{\nu}\left(\beta V_{t+1}^{ik} - \tau^{nj,ik} - \ln \sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)\right) \\
\Rightarrow \frac{\partial \ln \mu_t^{nj,ik}}{\partial(\beta V_{t+1}^{ik} - \tau^{nj,ik})} &\approx \frac{1}{\nu}
\end{aligned}$$

(2) "conveys all the information needed to determine how the distribution of labor across markets evolves over time. In particular,"

$$L_{t+1}^{nj} = \sum_{i=1}^{N}\sum_{k=0}^{J}\mu_t^{ik,nj}L_t^{ik} \tag{3}$$

(3) characterizes the distribution of employment and non-employment across markets.

Note that, under the given timing assumptions, labor in period t is "fully determined by forward-looking decisions at period $t-1$."

2.3 Production

"Firms in each sector and region are able to produce many varieties of intermediate goods. "

Production requires labor, structures, and materials from all sectors.

"TFP of an intermediate good is composed of two terms, a time-varying sectoral-regional component $\left(A_t^{nj}\right)$, which is common to all varieties in a region and sector, and a variety-specific component $\left(z^{nj}\right)$."

2.3.1 Intermediate Good Production

2.3.1.1 Production Function

We assume that the production function is CRS and Cobb-Douglas.

The output for a producer of an intermediate good is given by:

$$q_t^{nj} = z^{nj} \left(A_t^{nj} \left(h_t^{nj} \right)^{\xi^n} \left(l_t^{nj} \right)^{1-\xi^n} \right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(M_t^{nj,nk} \right)^{\gamma^{nj,nk}}$$

where

- l_t^{nj}: Labor inputs in location n and sector j at time t
- h_t^{nj}: Structure inputs in location n and sector j at time t
- $M_t^{nj,nk}$: Material inputs from sector k demanded by a firm in sector j and region n at time t
- $\gamma^{nj} \geq 0$: Share of value-added in the production of sector j and region n
    - This comes from $\gamma^{nj} + \sum_{k=1}^{J} \gamma^{nj,nk} = 1$ and the Cobb-Douglas form, so the value-add share is the share that comes from the non-material inputs.
- $\gamma^{n,nk} \geq 0$: The share of materials from sector k in the production of sector j and region n.
- ξ^n: The share of structures in value-add.

"Material inputs are goods from sector k produced in the same region n."

2.3.1.2 Unit Price of an Input Bundle

The unit price of an input bundle is given by:

$$x_t^{nj} = B^{nj} \left(\left(r_t^{nj} \right)^{\xi^n} \left(w_t^{nj} \right)^{1-\xi^n} \right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(P_t^{nk} \right)^{\gamma^{nj,nk}}$$

where B^{nj} is a constant.

Then the unit cost of a given intermediate good is given by:

$$\frac{x_t^{nj}}{z^{nj} \left(A_t^{nj} \right)^{\gamma^{nj}}}$$

since the cost for one unit of the inputs is x_t^{nj} and the productivity of the variety, $z^{nj} \left(A_t^{nj} \right)^{\gamma^{nj}}$, scales the amount you can produce with the relevant input.

2.3.1.3 Price of Good j in Region n

We take

- $\boldsymbol{\kappa}_t^{nj,ij}$: Iceberg trade costs between market location n and i in sector j.
  - Delivering one unit of good from market n to i requires producing $\boldsymbol{\kappa}_t^{nj,ij} \geq 1$ units of the good.
  - If a good is nontradable, then $\kappa = \infty$.

"Competition implies that the price paid for a particular variety of good j in region n is given by the minimum cost across regions, taking into account trade costs, and where the vector of productivity draws received by the different regions is "

$$z^j = \left(z^{1j}, z^{2j}, \ldots, z^{Nj}\right)$$

"That is, using z^j to index varieties:"

$$p_t^{nj}\left(z^j\right) = \min_i \left\{ \frac{\kappa_t^{nj,ij} x_t^{ij}}{z^{ij}\left(A_t^{ij}\right)^{\gamma^{ij}}} \right\}$$

2.3.2 Local Sectoral Aggregate Goods

2.3.2.1 Quantity of Aggregate Sectoral Goods

"Intermediate goods demanded from sector j and from all regions are aggregated into a local sectoral good denoted by Q and that can be thought of as a bundle of goods purchased from different regions."

Take

- Q_t^{nj}: The quantity produced of aggregate sectoral goods j in region n

$$Q_t^{nj} = \left(\int \left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}} d\phi^j\left(z^j\right)\right)^{\eta^{nj}/\left(\eta^{nj}-1\right)}$$

- $\phi^j\left(z^j\right)$: The joint distribution over the vector z^j.

$$\phi^j\left(z^j\right) = \exp\left\{-\sum_{n=1}^{N}\left(z^{nj}\right)^{-\theta^j}\right\}$$

  - $\phi^{nj}\left(z^{nj}\right)$: The marginal distribution of z^{nj}.

$$\phi^{nj}\left(z^{nj}\right) = \exp\left\{-\left(z^{nj}\right)^{-\theta^j}\right\}$$

- $\tilde{q}_t^{nj}\left(z^j\right)$: The quantity demanded of an intermediate good of a given variety from the lowest-cost supplier.

Note that "there are no fixed costs or barriers to entry and exit in the production of intermediate and sectoral goods. " Moreover, "[c]ompetitive behavior implies zero profit at all times."

2.3.2.2 Price of Aggregate Sectoral Goods

Given properties of the Fréchet distribution, the price of the aggregate sectoral good is given by:

$$P_t^{nj} = \Gamma^{nj}\left(\sum_{i=1}^{N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)^{-1/\theta^j}$$

where Γ^{nj} is a constant and we assume that $1+\theta^j > \eta^{nj}$.

2.3.2.3 Share of Total Expenditure in Market nj on Goods j from Market i

For the share of total expenditure in market nj on goods j from market i, we get the expression:

$$\pi_t^{nj,ij} = \frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{m=1}^{N}\left(x_t^{mj}\kappa_t^{nj,mj}\right)^{-\theta^j}\left(A_t^{mj}\right)^{\theta^j\gamma^{mj}}}$$

"This equilibrium condition resembles a gravity equation."

2.3.3 Market Clearing and Unbalanced Trade

2.3.3.1 Global Portfolio and Rent

We assume a mass 1 of rentiers in each region who own the local structures, rent them to firms, send their rents to a global portfolio, receive a constant share ι^n from the global portfolio, with $\sum_{n=1}^{N}\iota^n = 1$, and cannot re-locate.

The differences between the remittances and the income rentiers receive from the global portfolio generate imbalances, "which changes in magnitude as the rental prices change."

The imbalance is given by:

$$\sum_{k=1}^{J} r_t^{ik}H^{ik} - \iota^n\chi_t$$

where

$$\chi_t = \sum_{i=1}^{N}\sum_{k=1}^{J} r_t^{ik} H^{ik}$$

is the total revenue in the global portfolio.

2.3.3.2 Total Expenditure on Goods j in Region n

Let X_t^{nj} be the total expenditure on sector j good in region n. Then, goods market clearing implies

$$X_t^{nj} = \underbrace{\sum_{k=1}^{J} \gamma^{nk,nj} \sum_{i=1}^{N} \pi_t^{ik,nk} X_t^{ik}}_{\substack{\text{Value of total demand for } nj \\ \text{market goods used as} \\ \text{materials}}} + \underbrace{\alpha^j \left(\sum_{k=1}^{J} w_t^{nk} L_t^{nk} + \iota^n \chi_t \right)}_{\substack{\text{Value of final} \\ \text{demand in} \\ \text{region } n}}$$

2.3.3.3 Labor Expression

"Labor market clearing in region n and sector j is"

$$L_t^{nj} = \underbrace{\frac{1}{w_t^{nj}} \underbrace{\underbrace{\gamma^{nj}}_{\substack{\text{Value-} \\ \text{added} \\ \text{share}}} \underbrace{(1-\xi^n)}_{\substack{\text{Share of} \\ \text{value-add} \\ \text{paid to labor}}} \underbrace{\sum_{i=1}^{N} \pi_t^{ij,nj} X_t^{ij}}_{\substack{\text{Total} \\ \text{expenditure} \\ \text{from } ij \text{ market} \\ \text{on } nj \text{ goods}}}}_{\substack{\text{Total wage bill} \\ \text{in } nj}}}_{\substack{\text{Labor quantity} \\ \text{demanded in} \\ nj}}$$

"while the labor market clearing for structures in region n and sector j must satisfy"

$$H^{nj} = \frac{\gamma^{nj}\xi^n}{r_t^{nj}} \sum_{i=1}^{N} \pi_t^{ij,nj} X_t^{ij}$$

2.4 Equilibrium

2.4.1 Fundamentals and Parameters

"The endogenous state of the economy at any moment in time is given by the distribution of labor across all markets L_t."

The fundamentals of the economy are deterministic and include

- $A_t = \left\{A_t^{nj}\right\}_{n=1,j=1}^{N,J}$: Aggregate sectoral regional productivities
- $\boldsymbol{\kappa}_t = \left\{\boldsymbol{\kappa}_t^{nj,ij}\right\}_{n=1,i=1,j=1}^{N,N,J}$: Bilateral trade costs
- $Y = \left\{\tau^{nj,ik}\right\}_{n=1,j=0,i=1,k=0}^{N,J,J,N}$: Labor re-location costs
- $H = \left\{H^{nj}\right\}_{n=1,j=1}^{N,J}$: Stock of land and structures across markets
- $b = \{b^n\}_{n=1}^{N}$: Home production across regions

Time-varying fundamentals are denoted by

$$\Theta_t \equiv (A_t, \kappa_t)$$

Constant fundamentals are denoted by

$$\bar{\Theta} \equiv (\Upsilon, H, b)$$

The parameters in the economy, assumed constant throughout the paper, are given by:

- γ^{nj}: Value-added shares
- $(1 - \xi^n)$: Labor-share in value-added
- $\left(\gamma^{nk,nj}\right)$: Input-output coefficients
- ι^n: Portfolio shares
- α^j: Final consumption expenditure shares
- β: The discount factor
- θ: Trade elasticities
- ν: Migration elasticity

2.4.2 Equilibrium Definition

2.4.2.1 Temporary Equilibrium

Definition D.1: Temporary Equilibrium

Given $\left(L_t, \mathbf{\Theta}_t, \overline{\mathbf{\Theta}}\right)$, a temporary equilibrium is a vector of wages $w\left(L_t, \mathbf{\Theta}_t\,, \bar{\Theta}\right)$ that satisfies the equilibrium conditions of the static subproblem characterized by:

$$
\begin{aligned}
x_t^{nj} =& B^{nj}\left(\left(r_t^{nj}\right)^{\xi^n}\left(w_t^{nj}\right)^{1-\xi^n}\right)^{\gamma^{nj}} \prod_{k=1}^{J}\left(P_t^{nk}\right)^{\gamma^{nj,nk}} \\
P_t^{nj} =& \Gamma^{nj}\left(\sum_{i=1}^{N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)^{-1/\theta^j} \\
\pi_t^{nj,ij} =& \frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{m=1}^{N}\left(x_t^{mj}\kappa_t^{nj,mj}\right)^{-\theta^j}\left(A_t^{mj}\right)^{\theta^j\gamma^{mj}}} \\
X_t^{nj} =& \sum_{k=1}^{J}\gamma^{nk,nj}\sum_{i=1}^{N}\pi_t^{ik,nk}X_t^{ik} + \alpha^j\left(\sum_{k=1}^{J} w_t^{nk}L_t^{nk} + \iota^n\chi_t\right) \\
L_t^{nj} =& \frac{\gamma^{nj}\left(1-\xi^n\right)}{w_t^{nj}}\sum_{i=1}^{N}\pi_t^{ij,nj}X_t^{ij} \\
H^{nj} =& \frac{\gamma^{nj}\xi^n}{r_t^{nj}}\sum_{i=1}^{N}\pi_t^{ij,nj}X_t^{ij}
\end{aligned}
$$

"The temporary equilibrium of our model is the solution to a static multicountry interregional trade model."

We can then express real wages as:

$$
\omega^{nj}\left(L_t, \Theta_t, \bar{\Theta}\right) = w_t^{nj}/P_t^n
$$

2.4.2.2 Sequential Competitive Equilibrium

Consider

- $\mu_t = \left\{\mu_t^{nj,ik}\right\}_{n=1,j=0,i=1,k=0}^{N,J,N,J}$: The migration shares
- $V_t = \left\{V_t^{nj}\right\}_{n=1,j=0}^{N,J}$: Lifetime utilities

Definition D.2: Sequential Competitive Equilibrium

Given $\left(L_0, \{\Theta_t\}_{t=0}^{\infty}, \bar{\Theta}\right)$, a sequential competitive equilibrium of the model is a sequence of $\left\{L_t, \mu_t, V_t, w\left(L_t, \Theta_t, \bar{\Theta}\right)\right\}_{t=0}^{\infty}$ that solves equilibrium conditions

$$\begin{aligned} V_t^{nj} =& U\left(C_t^{nj}\right) + \nu \log\left(\sum_{i=1}^{N}\sum_{k=0}^{J} \exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}\right) \\ \mu_t^{nj,ik} =& \frac{\exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\sum_{m=1}^{N}\sum_{h=0}^{J} \exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{1/\nu}} \\ L_{t+1}^{nj} =& \sum_{i=1}^{N}\sum_{k=0}^{J} \mu_t^{ik,nj} L_t^{ik} \end{aligned}$$

and the temporary equilibrium at each t.

2.4.2.3 Stationary Equilibrium

Definition D.3: Stationary Equilibrium

A stationary equilibrium of the model is a sequential competitive equilibrium such that

$$\left\{L_t, \mu_t, V_t, w\left(L_t, \Theta_t, \bar{\Theta}\right)\right\}_{t-0}^{\infty}$$

are constant for all t.

3 Dynamic Hat Algebra

3.1 Solving the Model

We aim to study the general equilibrium implications of a change in fundamentals relative to a baseline economy.

Definition D.4: Baseline Economy

The baseline economy is the allocation

$$\{L_t, \mu_{t-1}, \pi_t, X_t\}_{t=0}^{\infty}$$

corresponding to the sequence of fundamentals $\{\Theta_t\}_{t=0}^{\infty}$ and to $\bar{\Theta}$.

We now consider how to solve for the baseline economy in time differences.

We use the notation:

- $\dot{y}_{t+1}$: The proportional change in any scalar or vector between periods t and $t+1$.
$$\dot{y}_{t+1} \equiv \left(y^1_{t+1}/y^1_t, y^2_{t+1}/y^2_t, \dots\right)$$

3.1.1 Temporary Equilibrium of the Baseline Economy After Changing Employment and Fundamentals

First, we'll consider how to solve for a temporary equilibrium of the baseline economy at $t+1$ after changing employment, $\dot{L}_{t+1}$, and fundamentals, $\dot{\Theta}_{t+1}$ – which we can do without estimates of Θ_t or $\bar{\Theta}$.

Proposition P.1

Given the allocation of the temporary equilibrium at $t, \{L_t, \pi_t, X_t\}$, the solution to the temporary equilibrium at $t+1$ for a given change in $\dot{L}_{t+1}$ and $\dot{\Theta}_{t+1}$ does not require information on the level of fundamentals at t, Θ_t, or $\bar{\Theta}$. In particular, it is obtained as the solution to the following system of nonlinear equations:

$$\begin{aligned}
\dot{x}^{nj}_{t+1} &= \left(\dot{L}^{nj}_{t+1}\right)^{\gamma^{nj}\xi^n} \left(\dot{w}^{nj}_{t+1}\right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(\dot{P}^{nk}_{t+1}\right)^{\gamma^{nj,nk}} \\
\dot{P}^{nj}_{t+1} &= \left(\sum_{i=1}^{N} \pi^{nj,ij}_t \left(\dot{x}^{ij}_{t+1}\dot{\kappa}^{nj,ij}_{t+1}\right)^{-\theta^j} \left(\dot{A}^{ij}_{t+1}\right)^{\theta^j\gamma^{ij}}\right)^{-1/\theta^j} \\
\pi^{nj,ij}_{t+1} &= \pi^{nj,ij}_t \left(\frac{\dot{x}^{ij}_{t+1}\dot{\kappa}^{nj,ij}_{t+1}}{\dot{P}^{nj}_{t+1}}\right)^{-\theta^j} \left(\dot{A}^{ij}_{t+1}\right)^{\theta^j\gamma^{ij}} \\
X^{nj}_{t+1} &= \sum_{k=1}^{J} \gamma^{nk,nj} \sum_{i=1}^{N} \pi^{ik,nk}_{t+1} X^{ik}_{t+1} + \alpha^j \left(\sum_{k=1}^{J} \dot{w}^{nk}_{t+1}\dot{L}^{nk}_{t+1} w^{nk}_t L^{nk}_t + \iota^n \chi_{t+1}\right) \\
\dot{w}^{nj}_{t+1}\dot{L}^{nj}_{t+1} w^{nj}_t L^{nj}_t &= \gamma^{nj}\left(1-\xi^n\right) \sum_{i=1}^{N} \pi^{ij,nj}_{t+1} X^{ij}_{t+1} \\
\text{where} \quad \chi_{t+1} &= \sum_{i=1}^{N}\sum_{k=1}^{J} \frac{\xi^i}{1-\xi^i} \dot{w}^{ik}_{t+1}\dot{L}^{ik}_{t+1} w^{ik}_t L^{ik}_t
\end{aligned}$$

This proposition highlights that we don't need to know the level of fundamentals to solve for the temporary equilibrium at $t+1$; we can solve with only the changes in employment and fundamentals.

3.1.2 Solving for the Sequential Equilibrium

Now, we'll show that "given an allocation at $t=0$, $\{L_0, \pi_0, X_0\}$, the matrix of gross migration flows at $t=-1$, μ_{-1}, and a sequence of changes in fundamentals, one can solve for the sequential equilibrium in time differences without needing to estimate the levels of fundamentals."

Definition D.5: Converging Sequence of Changes in Fundamentals

A converging sequence of changes in fundamentals is such that $\lim_{t\to\infty}\dot{\Theta}_t = 1$

Assumption

Agents have logarithmic preferences, $U\left(C_t^{nj}\right) \equiv \log\left(C_t^{nj}\right)$.

We denote

$$u_t^{nj} \equiv \exp\left(V_t^{nj}\right)$$

and

- $\dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)$: (For all n and j) the equilibrium real wages in time differences as functions of the change in labor $\dot{L}_{t+1}$ and time-varying fundamentals $\dot{\Theta}_{t+1}$.

"Namely, $\dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)$ is the solution to the system in Proposition 1."

Proposition P.2

Conditional on an initial allocation of the economy, $(L_0, \pi_0, X_0, \mu_{-1})$, given an anticipated convergent sequence of changes in fundamentals, $\left\{\dot{\Theta}_t\right\}_{t=1}^{\infty}$, the solution to the sequential equilibrium in time differences does not require information on the level of the fundamentals $\{\Theta_t\}_{t=0}^{\infty}$ or $\bar{\Theta}$ and solves the following system of nonlinear equations:

$$\begin{aligned}
\mu_{t+1}^{nj,ik} =& \frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{m=1}^{N}\sum_{h=0}^{J}\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}},\\
\dot{u}_{t+1}^{nj} =& \dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)\left(\sum_{i=1}^{N}\sum_{k=0}^{J}\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}\\
L_{t+1}^{nj} =& \sum_{i=1}^{N}\sum_{k=0}^{J}\mu_t^{ik,nj}L_t^{ik}
\end{aligned}$$

for all j, n, i, and k at each t, where $\left\{\dot{\omega}^{nj}\left(\dot{L}_t, \dot{\Theta}_t\right)\right\}_{n=1,j=0,t=1}^{N,J,\infty}$ is the solution to the temporary equilibrium given $\left\{\dot{L}_t, \dot{\Theta}_t\right\}_{t=1}^{\infty}$.

3.2 Solving for Counterfactuals

4 Derivations

4.1 Derivation of V_t^{nj}

This is a derivation of Equation 2.2.

As a quick starting point, note that we know the value of $U\left(C_t^{nj}\right)$, so the uncertainty is over subsequent periods:

$$\begin{aligned}
V_t^{nj} \equiv & \mathbb{E}\left[\mathrm{v}_t^{nj}\right] \\
= & \mathbb{E}\left[U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu\epsilon_t^{ik}\right\}\right] \\
= & U\left(C_t^{nj}\right) + \mathbb{E}\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu\epsilon_t^{ik}\right\}\right]
\end{aligned}$$

Denote the uncertain term as:

$$\Phi_t^{nj} \equiv \mathbb{E}\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu\epsilon_t^{ik}\right\}\right]$$

Recall that we assumed that the idiosyncratic shock ϵ is i.i.d. over time and distributed Type-I Extreme Value with zero mean. Then, the CDF of ϵ is:

$$F(\epsilon) = \exp(-\exp(-\epsilon - \bar{\gamma}))$$

where

$$\bar{\gamma} \equiv \int_{-\infty}^{\infty} x \exp(-x - \exp(-x))dx$$

is Euler's constant and

$$f(\epsilon) = \partial F / \partial \epsilon$$

Denote

$$\bar{\epsilon}_t^{ik,mh} \equiv \frac{\beta\left(V_{t+1}^{ik} - V_{t+1}^{mh}\right) - \left(\tau^{nj,ik} - \tau^{nj,mh}\right)}{\nu} \tag{4}$$

Then notice that:

$$
\begin{aligned}
&\Phi_t^{nj} \\
\equiv&\mathbb{E}\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}}\left\{\beta E\left[\mathrm{v}_{t+1}^{ik}\right]-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\right] \\
=&\mathbb{E}\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}}\left\{\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\right] && \text{by (1)} \\
=&\mathbb{E}\left[\sum_{i=1}^{N}\sum_{k=0}^{J}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\mathbf{1}\left\{\underset{\{m,h\}_{m=1,h=0}^{N,J}}{\operatorname{argmax}}\left\{\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}\right\}=ik\right\}\right] \\
=&\mathbb{E}\left[\sum_{i=1}^{N}\sum_{k=0}^{J}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}\mathbf{1}\left\{\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\right] \\
=&\sum_{i=1}^{N}\sum_{k=0}^{J}\mathbb{E}\left[\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}\mathbf{1}\left\{\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\right] \\
=&\sum_{i=1}^{N}\sum_{k=0}^{J}\mathbb{E}\left[\mathbb{E}\left[\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}\mathbf{1}\left\{\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\mid\epsilon_t^{ik}\right]\right] && \text{by LIE} \\
=&\sum_{i=1}^{N}\sum_{k=0}^{J}\mathbb{E}\left[\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}\mathbb{E}\left[\mathbf{1}\left\{\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right\}\mid\epsilon_t^{ik}\right]\right] && \text{by iid} \\
=&\sum_{i=1}^{N}\sum_{k=0}^{J}\mathbb{E}\left[\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}\Pr\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\mid\epsilon_t^{ik}\right)\right] && \text{binary variable property}
\end{aligned}
$$

Now, notice that

$$
\begin{aligned}
&\Pr\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}+\nu\epsilon_t^{mh}<\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\mid\epsilon_t^{ik}\right) \\
=&\Pr\left(\epsilon_t^{mh}<\epsilon_t^{ik}+\frac{\beta V_{t+1}^{ik}-\tau^{nj,ik}-\beta V_{t+1}^{mh}+\tau^{nj,mh}}{\nu}\mid\epsilon_t^{ik}\right) \\
=&\Pr\left(\epsilon_t^{mh}<\epsilon_t^{ik}+\bar{\epsilon}_t^{ik,mh}\mid\epsilon_t^{ik}\right) && \text{by (4)} \\
=&F\left(\epsilon_t^{ik}+\bar{\epsilon}_t^{ik,mh}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Phi_t^{nj} &= \sum_{i=1}^{N}\sum_{k=0}^{J}\mathbb{E}\left[\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\prod_{mh\neq ik}F\left(\epsilon_t^{ik}+\bar{\epsilon}_t^{ik,mh}\right)\right] \\
&= \sum_{i=1}^{N}\sum_{k=0}^{J}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)f\left(\epsilon_t^{ik}\right)\prod_{mh\neq ik}F\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}\right)d\epsilon_t^{ik}
\end{aligned}
$$

From there, we can note that

$$
\begin{aligned}
f\left(\epsilon_t^{ik}\right) &= \exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left(-\exp\left(-\epsilon_t^{ik}-\bar{\gamma}\right)\right) \\
F\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}\right) &= \exp\left[-\exp\left(-\left[\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}+\bar{\gamma}\right]\right)\right]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& f\left(\epsilon_t^{ik}\right)\prod_{mh\neq ik}F\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}\right) \\
&= \exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left(-\exp\left(-\epsilon_t^{ik}-\bar{\gamma}\right)\right)\exp\left[-\sum_{(m,h)\neq(i,k)}\exp\left(-\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}+\bar{\gamma}\right)\right)\right] \\
&= \exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left[-\exp\left(-\epsilon_t^{ik}-\bar{\gamma}\right)-\sum_{(m,h)\neq(i,k)}\exp\left(-\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}+\bar{\gamma}\right)\right)\right] \\
&= \exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left[-\sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(-\bar{\epsilon}_t^{ik,mh}-\epsilon_t^{ik}-\bar{\gamma}\right)\right] \qquad \text{since } \bar{\epsilon}_t^{ik,ik}=0 \\
&= \exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left[-\exp(-\epsilon_t^{ik}-\bar{\gamma})\sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(-\bar{\epsilon}_t^{ik,mh}\right)\right]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \Phi_t^{nj} \\
&= \sum_{i=1}^{N}\sum_{k=0}^{J}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)f\left(\epsilon_t^{ik}\right)\prod_{mh\neq ik}F\left(\bar{\epsilon}_t^{ik,mh}+\epsilon_t^{ik}\right)d\epsilon_t^{ik} \\
&= \sum_{i=1}^{N}\sum_{k=0}^{J}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\epsilon_t^{ik}\right)\exp(-\epsilon_t^{ik}-\bar{\gamma})\exp\left[-\exp(-\epsilon_t^{ik}-\bar{\gamma})\sum_{m=1}^{N}\sum_{h=0}^{J}\exp\left(-\bar{\epsilon}_t^{ik,mh}\right)\right]d\epsilon_t^{ik}
\end{aligned}
$$

Denote:

$$\lambda_t^{ik} \equiv \log \sum_{m=1}^{N} \sum_{h=0}^{J} \exp\left(-\bar{\epsilon}_t^{ik,mh}\right)$$

$$\zeta_t^{ik} \equiv \epsilon_t^{ik} + \bar{\gamma}$$

Break

Putting this derivation on pause here. (The below lines are directly from the paper.)

Then through a change of variables, we have:

$$\Phi_t^{nj} = \sum_{i=1}^{N} \sum_{k=0}^{J} \int_{-\infty}^{\infty} \left(\beta V_{t+1}^{ik} - \tau^{njik} + \nu\left(\zeta_t^{ik} - \bar{\gamma}\right)\right) \exp\left(-\zeta_t^{ik} - \exp\left(-\left(\zeta_t^{ik} - \lambda_t^{ik}\right)\right)\right) d\zeta_t^{ik}$$

Consider an additional change of variables using:

$$\tilde{y}_t^{ik} = \zeta_t^{ik} - \lambda_t^{ik}$$

Then we obtain:

$$\Phi_t^{nj} = \sum_{i=1}^{N} \sum_{k=0}^{J} \exp\left(-\lambda_t^{ik}\right) \left(\left(\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu\left(\lambda_t^{ik} - \bar{\gamma}\right)\right) + \nu \int_{-\infty}^{\infty} \tilde{y}_t^{ik} \exp\left(-\tilde{y}_t^{ik} - \exp\left(-\tilde{y}_t^{ik}\right)\right) d\tilde{y}_t^{ik}\right)$$