THE KLEIN-GORDON EQUATION ON ASYMPTOTICALLY MINKOWSKI SPACETIMES: CAUSAL PROPAGATORS

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Abstract. We construct the causal (forward/backward) propagators for the massive Klein-Gordon equation perturbed by a first order operator which decays in space but not necessarily in time. In particular, we obtain global estimates for forward/backward solutions to the inhomogeneous, perturbed Klein-Gordon equation, including in the presence of bound states of the limiting spatial Hamiltonians.

To this end, we prove propagation of singularities estimates in all regions of infinity (spatial, null, and causal) and use the estimates to prove that the Klein-Gordon operator is invertible mapping between adapted weighted Sobolev spaces. This paper builds off work of Vasy in which inverses of hyperbolic PDEs are obtained via construction of a Fredholm mapping problem using radial points propagation estimates. To deal with the presence of a perturbation which persists in time, we employ a class of pseudodifferential operators first explored in Vasy's many-body work.

Contents

1. Introduction

We consider the inhomogeneous Klein–Gordon equation on $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_z$ with coordinates (t, z):

$$\left[D_t^2 - (\Delta + m^2 + V)\right] u(t, z) = f(t, z) \tag{1.1}$$

where $D_t = -i\partial_t, D_{z_j} = -i\partial_{z_j}$, $\Delta = D_z \cdot D_z$ is the positive Laplacian, $m \in \mathbb{R}, m > 0$, and $V = V(t, z)$ is a smooth, real-valued potential function with spatial decay. In this paper we give detailed, quantitative estimates in phase space for the solution u to (1.1) in terms of the forcing f in all regions of spacetime infinity. We compile these phase space estimates to provide a novel construction of the causal (forward/backward) propagators $G_{+/-}$.

To illustrate a simplified version of our results, consider $V = V(z) \in S^{-2}(\mathbb{R}^n_z)$, meaning V decays like $1/|z|^2$ with corresponding derivative estimates. If the spatial Hamiltonian in equation (1.1) is positive, then, letting $\chi_{<0}(t)$ be a smooth cutoff to $t < 0$, we have, for any $\epsilon > 0$,

$$\|\left(\langle t,z\rangle^{-1/2-\epsilon} + \chi_{<0}(t)\langle t,z\rangle^{-1/2+\epsilon}\right) G_+ f(t,z)\|_{H^1(\mathbb{R}^{n+1})} \leq C\|\langle t,z\rangle^{1/2+\epsilon} f(t,z)\|_{L^2(\mathbb{R}^{n+1})}.$$

Such an estimate gives a global spacetime weighted $H^1(\mathbb{R}^{n+1})$ bound for the forward solution $u_+ = G_+ f$ in terms of a corresponding weighted $L^2(\mathbb{R}^{n+1})$ norm of the source f, with weight depending on the time direction; faster growth is allowed in the $t \to +\infty$ region, with slower growth as $t \to -\infty$, relative to a $\langle t,z\rangle^{1/2}$ threshold. This estimate is refined substantially in Theorem 1.1, below, in which we prove finer mapping properties for $G_{+/-}$ in a wider class of spaces with variable order spacetime weights and arbitrary differential orders. The theorem applies to a large class of Klein-Gordon operators which are asymptotically static and non-trapping.

By a "global" solution to (1.1), we mean one for which both the forcing $f(t,z)$ and the solution $u(t,z)$ are defined on the whole of the spacetime $\mathbb{R}^{n+1}_{t,z}$. In this work, we study primarily the two special solutions operators (i.e. propagators) to (1.1) which exhibit temporal causality, namely the forward and backward propagators $G_{+/-}$. These define solutions $u_{+/-} = G_{+/-} f$ with the property that they propagate solutions in the forward $(+)$ or backward $(-)$ time directions, meaning, for example for G_+, that

$$\operatorname{supp} f \subset \{t \geq T\} \implies \operatorname{supp} G_+ f \subset \{t \geq T\}\,. \tag{1.2}$$

Our main results on the causal propagators are Theorem 8.2, Theorem 8.3, and Theorem 8.4 below. We give a simplified version of these theorems now, assuming that

$$V = V_0(z) + V_1(t,z)$$

where $V_0 \in S^{-1}(\mathbb{R}^n)$ and V_1 a perturbation which is Schwartz in space and decaying in time:

$$\left|\partial_z^\alpha \partial_t^j V_1(t,z)\right| \leq C\langle t\rangle^{-1-j}\langle z\rangle^{-N}$$

for any $j \in \mathbb{N}_0$, multiindex $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{R}$ and some C depending on j and N. The assumptions in the theorem pertain to the spectral properties of the limiting spatial Hamiltonian

$$H_{V_0} = \Delta + m^2 + V_0(z).$$

Namely, if $V \equiv V_0(z)$ is static, we assume only that 0 is not an eigenvalue of H_{V_0} (see Theorem 8.4). Otherwise, i.e. if V_1 is non-zero, we assume that $H_{V_0} \geq c > 0$, i.e. the limiting Hamiltonian H_{V_0} is strictly positive. We always assume that the spectrum of H_{V_0} is strictly absolutely continuous at the threshold, i.e., that the spectrum of $\Delta + V_0$ is strictly absolutely continuous at 0, which for this V_0 is implied by the absence of resonances or eigenvalues for $\Delta + V_0$ at 0.

Theorem 1.1. *With V as above the forward propagator exists as a mapping between weighted L^2-based Sobolev spaces: for any $\epsilon > 0$, $s \in \mathbb{R}$,*

$$G_+ \colon \langle t,z\rangle^{-1/2-\epsilon} H^{s-1}(\mathbb{R}^{n+1}_{t,z}) \longrightarrow \langle t,z\rangle^{1/2+\epsilon} H^s(\mathbb{R}^{n+1}_{t,z}).$$

Moreover, if a spacetime function (see Section 2.2 for details)

$$\boldsymbol{\ell}_+(t,z) \in S^0_{\mathrm{cl}}(\mathbb{R}^{n+1}_{t,z})$$

satisfies

(1) *$\boldsymbol{\ell}_+ < -1/2$ in a neighborhood of future causal (timelike) infinity (i.e. in $t \gg 0$, $|z/t| < 1 + \epsilon$ for small $\epsilon > 0$) and*

(2) $\boldsymbol{\ell}_+ > -1/2$ *in a neighborhood of past causal (timelike) infinity (i.e. in* $t \ll 0, |z/t| < 1+\epsilon$ *for small* $\epsilon > 0$*) and*

(3) $\boldsymbol{\ell}_+$ *is monotone decreasing along future-directed null rays of* P_V*,*

then G_+ *extends to a bounded operator*

$$G_+ : \langle t,z\rangle^{\boldsymbol{\ell}_+(t,z)+1} H^{s-1}(\mathbb{R}^{n+1}_{t,z}) \longrightarrow \langle t,z\rangle^{\boldsymbol{\ell}_+(t,z)} H^{s}(\mathbb{R}^{n+1}_{t,z}) \,.$$

In particular, the weight $\boldsymbol{\ell}_+$ *can be taken arbitrarily high away from future causal infinity, yielding additional decay if* f *has additional decay.*

The analogous statements are true for the backward propagator G_-*, with* $\boldsymbol{\ell}_-(t,z) < -1/2$ *in a neighborhood of past causal infinity,* $t \ll 0, |z/t| < 1+\epsilon$*, and monotone decreasing on past-directed null rays, for example under the assumption of positivity of* $\Delta_z + m^2 + V_+$*.*

One of the main features of our approach is that it applies to perturbations of free Klein-Gordon within a large class of operators which limit to static potentials $V_\pm = V_\pm(z)$ as $t \to \pm\infty$. In particular, there is no need for the limiting potentials $V_\pm$ to be equal to each other as they are in Theorem 1.1. Though we allow more general V below, to fix the idea we begin by assuming that $V = V(t,z)$ is a smooth potential function with rapid spatial decay and smooth time dependence, approaching a static potential as $t \to \pm\infty$:

$$\forall t \in \mathbb{R},\ V(t,z) \in C^\infty(\mathbb{R}_t : \mathcal{S}(\mathbb{R}^n_z)),\ \text{and } V(t,z) - V_\pm(z) = O(|t|^{-1}) \text{ as } t \to \pm\infty,$$

and corresponding derivative estimates. The influence of the perturbation V on the analysis being the main object of study, we define the operator with explicit dependence on V:

$$P_V \coloneqq D_t^2 - H_V = D_t^2 - (\Delta + m^2 + V) \,. \tag{1.3}$$

In fact, we can allow V to be a differential operator of order 1 with symbolic behavior in z. Below (Section 3.4) we take V to be smooth on a compactification of spacetime, and this global smoothness condition is more general than the assumptions above. We can also allow $\Box_{g_{\text{mink}}} = D_t^2 - \Delta$ to be replaced by the d'Alembertian $\Box_g$ of a non-trapping, globally hyperbolic, asymptotically Minkowski metric g as described in prior work on the massless wave equation [2].

Thanks to a famous result of Klainerman [23], the forward solution $u_+ = G_+ f$ to $P_V u_+ = f$ with a sufficiently smooth and decaying source f is known to lie in $t^{-n/2}L^\infty(\mathbb{R}^{n+1}) \subset \langle t,z\rangle^{1/2+0}L^2(\mathbb{R}^{n+1})$ near future causal infinity. Our results below refine the L^2 statement, in that they allow use also to localize in frequency space in a neighborhood of the future radial set, i.e. the limit locus of bicharacteristic rays, a phase space subset over future null infinity defined below. One upshot is that the frequency-localization of the solution to the radial set effectively carries all of the asymptotic data of $G_+ f$ as $t \to +\infty$. In other words, there are frequency localizers Q_{rad} which cut off in phase space to the radial set, such that, for $f \in C^\infty_c(\mathbb{R}^{n+1})$,

$$(I - Q_{rad})G_+ f \in \mathcal{S}(\mathbb{R}^{n+1}), \quad \text{while } Q_{rad}G_+ f \in \langle t,z\rangle^{1/2+\epsilon}H^s(\mathbb{R}^{n+1}) \tag{1.4}$$

for any $s \in \mathbb{R}$ and any $\epsilon > 0$. We prove our result using propagation of singularities and radial points estimates, as we describe in detail shortly.

Propagator estimates of this type, i.e. global spacetime estimates using L^2-based, weighted Sobolev spaces, were established by Vasy for a large class of "scattering" operators, including the free Klein-Gordon operator [40]. They are a combination of principal type propagation of singularities estimates, radial points estimates, and microlocal elliptic estimates proven on non-compact regions of phase space. More accurately, they are proven up to and including infinity, meaning in phase space regions on a compactified spacetime. Propagation estimates in the scattering setting were first established by Melrose [32] in the case of the Helmholtz equation, generalizing Hörmander's propagation of singularities theorem [22] to scattering operators on non-compact scattering spaces. Melrose's paper [32] further introduced the radial point[1] propagation estimates (also in the scattering setting). For the free Klein-Gordon equation (and scattering perturbations thereof) one can use the phase space picture of Vasy/Melrose, based on the radial compactification $X = \overline{\mathbb{R}^{n+1}_{t,z}}$, a compact space, diffeomorphic to a closed ball, whose boundary points are the limits of geodesic rays. This compactification, illustrated in Figure 1, includes spatial and causal infinities as open subsets of the boundary (the shaded piece of the boundary in Figure 1 also represents causal infinity), while future null infinity and past null infinity are compressed to codimension one submanifolds of the boundary. The relevant phase space for the free Klein-Gordon, the scattering cotangent space ${}^{\mathrm{sc}}T^*X$, has X as the underlying spacetime and the standard τ, ζ (dual to t, z) momenta. This is described in detail in Section 2.

To understand the phase space nature of the estimates, consider the family of solutions to the free Klein-Gordon equation $P_0 u = 0$ obtained directly by Fourier transformation in the spatial variable, namely, for $g_\pm \in C_c^\infty(\mathbb{R}^n)$, with

$$\hat{u}(t,\zeta) = \sum_{\pm} e^{\pm it\sqrt{|\zeta|^2+m^2}} g_\pm(\zeta),$$

let $u(t,z) = \mathcal{F}^{-1}_{z\to\zeta}\hat{u}(t,\zeta)$. The infinite dimensional space of such u, in regions of the form $|z|/t \le c < 1$, behave asymptotically as

$$u = t^{-n/2}\left(a_+(z/t)e^{-im\sqrt{t^2-|z|^2}} + b_+(z/t)e^{im\sqrt{t^2-|z|^2}}\right)(1+O(1/t))$$

as $t \to +\infty$ with a_+, b_+ smooth functions easily computed in terms of the inverse Fourier transforms of the coefficients $g_\pm$. Here $y = z/t$ parametrizes future causal infinity. A similar expression holds as $t \to -\infty$. If one denotes the phase function $\phi_\pm(t,z) = \pm m\sqrt{t^2-|z|^2}$, then the radial set over future causal infinity is the subset of phase space defined by

$$\tau = D_t\phi_\pm = \frac{\pm mt}{\sqrt{t^2-|z|^2}}, \quad \zeta = D_z\phi_\pm = \frac{\mp mz}{\sqrt{t^2-|z|^2}}.$$

This future radial set is the union of two smooth manifolds in ${}^{\mathrm{sc}}T^*_{\iota^+}X$ which extend smoothly out to the compactified boundary of the fibers in ${}^{\mathrm{sc}}\overline{T}^*X$. Here ι^+ denotes future causal (timelike) infinity. The Q_{rad} in (1.4) can be taken to be the quantization of a cutoff function supported near this radial set, and, for these simple solutions, the conclusion of (1.4) follows from stationary phase.

The principal novelty of the current work lies in its treatment, via microlocal methods, of perturbations V that persist in time, in particular when there is asymptotic time dependence and separation of variables between time and space cannot be employed directly. If one thinks of the solutions of (1.1) in terms of their behavior along classical trajectories, then a geodesic

$$(t(s), z(s)) \text{ with } \zeta = dz/ds \neq 0$$

exits every compact spatial set, and thus as $s \to \pm\infty$, one expects solutions to behave asymptotically like those for free Klein-Gordon. However, on timelike trajectories with zero

[1]As far as the authors know, radial points as such were first studied in [11]

spatial momentum, the nature of the spatial Hamiltonians $\Delta + m^2 + V_\pm$ influences the asymptotic behavior substantially. The limits of such rays form two special points at causal infinity, one in the future and one in the past, which we call the "north pole", NP, and the "south pole", SP, and the lion's share of our work below is in proving estimates near NP and SP.

In particular, for nonzero $V = V(z)$, the operator P_V does not lie in the class of scattering operators $\mathrm{Diff}_{\mathrm{sc}}(\mathbb{R}^{n+1})$ defined by Melrose [32]. However, it does lie in a class of differential operators studied earlier by Vasy [36, 38] in his treatment of many-body Hamiltonians, called "3-body" or "many-body" or, as we write more frequently below, 3sc-operators. This family of operators on $\mathbb{R}^N$ includes not only many-body Hamiltonians, but also operators which decompose analogously near a family of collision planes, modeled as submanifolds C of infinity. In our case, the picture is comparatively simple; we have just one analogue of a collision plane and it is the straight line $z = 0$ in $\mathbb{R}^{n+1}$. For P_V, the relevant C that arises is simply the two poles,

$$C = \mathrm{NP} \cup \mathrm{SP},$$

Those two points lying in causal infinity are precisely the points where P_V fails to be a scattering operator. For a description of how these operators, which we follow Vasy in denoting by $\mathrm{Diff}_{3\mathrm{sc}}(\mathbb{R}^{n+1})$, arise in our setting, see Section 3. Correspondingly, we work on the resolved space $[X; C]$ obtained by blowing up the two poles. This blow-up introduces two new boundary hypersurfaces, each of which is essentially a copy of $\mathbb{R}^n_z$, and is in fact the limit locus of geodesics with zero spatial momentum, as depicted in Figure 2.

The result of the 3sc-analysis is a family of non-elliptic Fredholm problems for P_V, analogous to the Fredholm problems on asymptotically hyperbolic spaces established by Vasy [38]. Using this method, we prove global estimates on a-priori spaces of distributions which satisfy prescribed asymptotics (above threshold near past causal infinity and below threshold near future causal infinity). We let

$$H^{s,\boldsymbol{\ell}}_{\mathrm{sc}}(\mathbb{R}^{n+1}) \coloneqq \{u \in \mathcal{S}'(\mathbb{R}^{n+1}) : \langle t, z\rangle^{\boldsymbol{\ell}} u \in H^s_{\mathrm{sc}}(\mathbb{R}^{n+1})\}, \quad \|u\|_{H^{s,\boldsymbol{\ell}}} = \|u\|_{s,\boldsymbol{\ell}} \coloneqq \|\langle t, z\rangle^{\boldsymbol{\ell}} u\|_{H^s}.$$

denote the relevant weighted Sobolev spaces, where $\boldsymbol{\ell}(t, z)$ is a smooth spacetime weight and $H^s_{\mathrm{sc}}(\mathbb{R}^{n+1}) = H^s(\mathbb{R}^{n+1})$ is the standard Sobolev space of order s on $\mathbb{R}^{n+1}$. The a-priori spaces are defined as

$$\mathcal{X}^{s,\boldsymbol{\ell}} \coloneqq \{u \in H^{s,\boldsymbol{\ell}}_{\mathrm{sc}}(\mathbb{R}^{n+1}) \colon P_V u \in H^{s-1,\boldsymbol{\ell}+1}_{\mathrm{sc}}(\mathbb{R}^{n+1})\}, \quad \mathcal{Y}^{s,\boldsymbol{\ell}} \coloneqq H^{s,\boldsymbol{\ell}}_{\mathrm{sc}}(\mathbb{R}^{n+1}) \tag{1.5}$$

with

$$\|u\|^2_{\mathcal{X}^{s,\boldsymbol{\ell}}} = \|u\|^2_{s,\boldsymbol{\ell}} + \|P_V u\|^2_{s-1,\boldsymbol{\ell}+1}. \tag{1.6}$$

(In general, the weight $\boldsymbol{\ell}$ can have phase space dependence, although for time-persistent V it must be constant in a neighborhood of C. Genuine phase space dependence is permitted but not needed in the present paper, though it will play a role in future work.) Our construction of the propagators works by first establishing a Fredholm mapping property, and then proving invertibility of the mapping. The Fredholm property holds more generally than the invertibility, and in that case there are still causal propagators in the sense that there is a generalized inverse for any Fredholm map.

Given V, to obtain a Fredholm result (from which invertibility may or may not be concluded) we do *not* need to assume that the limiting Hamiltonians $H_{V_\pm}$ are positive. Instead, we assume only that

(1) $H_{V_\pm} = \Delta + m^2 + V_\pm$ has no resonance states or eigenvalues at m^2, or more generally that the absolutely continuous spectrum $[m^2, \infty)$ of $H_{V_\pm}$ is disjoint from the singular and point spectra, i.e. is purely absolutely continuous, and
(2) $H_{V_\pm}$ has no eigenvalue at 0.

The second assumption avoids the presence of linearly growing modes. In this case we have that P_V is a Fredholm operator between appropriate spaces.

Theorem 1.2. *Let V be any of the asymptotically static potentials described in Section 3.4 satisfying the assumptions just described. Let $\boldsymbol{\ell}_+$ or $\boldsymbol{\ell}_-$ be a spacetime weight as in Theorem 1.1, or more generally as described in Sections 2.8 and 8.1. Then*

$$P_V \colon \mathcal{X}^{s,\boldsymbol{\ell}_\pm} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_\pm+1}$$

is a Fredholm operator.

Analyzing P_V as a 3sc-operator allows us to prove estimates for $P_V u = f$ near C. In accordance with Vasy's results for many-body Hamiltonians, estimates at C are proven at each level of the temporal momentum component τ. This is accomplished in particular by analysis of the so-called *indicial operator* $\hat{N}_{\mathrm{ff}}(P_V)$. Near NP, the indicial operator is the Fourier transform in time of the limiting operator:

$$\hat{N}_{\mathrm{ff}}(P_V)(\tau) = \tau^2 - \Delta_z - m^2 - V_+ = \tau^2 - H_{V_+} ,$$

with an analogous indicial operator at SP. This operator's behavior has three distinct types:

(1) if $|\tau| > m$ then τ^2 lies in the continuous spectrum of H_{V_+},
(2) if $|\tau| < m$ then the operator $\hat{N}_{\mathrm{ff}}(P_V)(\tau)$ is elliptic on $\mathbb{R}^n$ in the scattering sense, and
(3) the borderline cases $\tau = \pm m$, which is where the radial set lies over NP.

In each of these three cases, we prove a different type of estimate. The three estimates are analogues of (1) principal type-propagation, (2) elliptic estimates, and (3) radial points estimates, which themselves come in above threshold and below threshold types.

In the case $|\tau_0| > m$, we prove a principal type propagation estimate of the form "if nothing goes in then nothing comes out" that appears in Vasy's original work, described there using broken geodesics. In other words, if $P_V u$ lies in some $H^{s-1,\ell+1}(\mathbb{R}^{n+1})$ when microlocalized near $\tau = \tau_0$, and u lies in $H^{s,\ell}(\mathbb{R}^{n+1})$ on all bicharacteristics which flow into NP at $\tau = \tau_0$, then u itself lies in $H^{s,\ell}(\mathbb{R}^{n+1})$ at $\tau = \tau_0$ over NP and satisfies a corresponding estimate. This estimate is the content of Proposition 6.1. The difference between ours and standard scattering propagation estimates is that over C, frequency localization is local in τ but global in the spatial momentum ζ. As a result, the propagation and radial points estimates always have assumptions on the flow on all bicharacteristics passing over C at the relevant τ-level.

The estimates at $\tau = \pm m$ are of the same flavor, in the sense that they are proven by making assumptions on the whole τ-level and its flow-out on the characteristic set. The difference is that they are radial points estimates, and thus exhibit the threshold phenomena typical for radial points, with different estimates depending on whether the distributions are above or below the threshold decay rate nearby.

While each of the estimates described above requires localizing in τ, frequency localization is a significant challenge over the poles C. One of the defining features of the scattering calculus ${}^{\mathrm{sc}}\Psi$ is that it provides a mechanism for frequency localization in the momentum variables τ, ζ – the symbols of the translation invariant vector fields D_t and the D_{z_j} – uniformly up to infinity, i.e. uniformly in dilation-invariant sets spacetime; near future causal infinity such conical neighborhoods can be described as open sets in $y = z/t$ with $x = 1/t < c$. Localization to the radial set is in particular accomplished by quantizing a cutoff function to the radial set localized over such a spacetime region. In the 3sc-setting, i.e. with non-zero V, frequency localization over C is complicated by the fact that the behavior of 3sc-operators is essentially "global" there. In Vasy's treatment, frequency localizers include functions of the original Hamiltonian, there of the form $\phi(H)$ for H a many-body Hamiltonian and $\phi \in C_c^\infty(\mathbb{R})$. In that case, because H is elliptic, $\phi(H)$ is a smoothing pseudodifferential operator. However, in our case, functions $\phi(P_V)$ are not well-behaved pseudodifferential operators, and we must first compose P_V with an invertible, globally elliptic 3sc-operator and then apply the functional calculus to obtain a localizer to the characteristic set. We use, for $E \geq 0$ sufficiently large,

$$G_\psi := \psi\left((D_t^2 + H_{V_0} + E)^{-1} P_{V_0}\right) .$$

which we show lies in ${}^{3\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^{n+1})$. The fact that G_ψ is not smoothing corresponds to the non-compactness of the characteristic set of P_0. We still have the commutation relation

$$[G_\psi, P_{V_0}] = 0$$

in our commutator construction. For more general V this commutation holds to leading order. Frequency localizers over C will be of the form QG_ψ where Q localizes to a τ-level over C. See Section 5.

Another novelty of our work is the treatment of large τ at C. For many-body Hamiltonians, non-compact regions of the non-interacting dual variable lie in the elliptic region, and are therefore not of direct interest. (It should be noted that a global Fredholm framework for many-body operators would still need to address these elliptic regions.) In contrast, for the 3sc formulation of Klein-Gordon, large τ lie in the non-radial part of the characteristic set and therefore exhibit principal type propagation. To establish propagation of singularities estimates for $\tau = \pm\infty$, we recognize the indicial family as a semiclassical scattering differential operator of the form introduced by Vasy–Zworski [41],

$$\hat{N}_{\mathrm{ff}}(P_V)(\tau) \in \Psi^{2,0,2}_{\mathrm{scl,sc}}(\mathbb{R}^n),$$

where the semiclassical parameter is $h = \pm 1/\tau$, say as $\tau \to \pm\infty$. An attractive picture emerges, in which the semiclassical principal symbol of $\hat{N}_{\mathrm{ff}}(P_V)(\tau)$ is exactly the restriction of the scattering principal symbol of P_V restricted to an appropriate hypersurface of a fiber compactified phase space.

The microlocal structure used in the underlying construction of the propagator for the free Klein-Gordon is the global sink-source radial point structure employed by Vasy [32, 40]. In particular, we exploit the global structure of the Hamilton flow in which all bicharacteristics flow from two components of the radial set to the other two; the former act as global sources for the flow and the latter act as sinks. On the characteristic set away from the radial set, we prove real principal type propagation estimates along bicharacteristic rays form the Hamiltonian flow. For the free Klein-Gordon operator, the Hamilton vector field $H_p = \tau\partial_t - \zeta \cdot \partial_z$ rescales to a smooth vector field on a the fiber compactification ${}^{\mathrm{sc}}\overline{T}^* X$ of the scattering cotangent bundle. There it induces a smooth extension of the bicharacteristic flow which preserves the characteristic set; the radial sets for families of sinks/sources for the flow. The propagator is the inverse of a Fredholm problem constructed using radial points estimates, as detailed below in Section 2.8 and 2.9.

As in Vasy [36], the flow we use to analyze P_V is the same flow as that for P_0. Away from C, this makes sense immediately since the assumptions on V make it so that P_V has the same scattering principal and subprincipal symbol as P_0. At C, as in Vasy [36], we use that the functional calculus localizers to the characteristic set can be approximated by those same functions of the corresponding localizers for the free operator. Specifically, we use that, under the given assumptions, with G_ψ the localizer for P_V and $G_{\psi,0}$ the free localizer, that

$$\hat{N}_{\mathrm{ff}}(G_\psi)(\tau) \text{ and } \hat{N}_{\mathrm{ff}}(G_{\psi,0})(\tau) \text{ have the same semiclassical principal symbol,}$$

which fits nicely below thanks to the semiclassical formulation of the indicial family. This is discussed in Section 5, together with the basic construction of the commutants that go into the positive commutator argument.

The mathematical study of the long-time behavior of solutions to massive wave equations is extensive and dates back at least to the pioneering work of Morawetz–Strauss [34], in which the authors found one of the first decay results for the Klein–Gordon equation. Quite a bit more work on the spectral and scattering theory of Klein–Gordon equations ensued, including the works of Lundberg [29] and Weder [42]. Klainerman's use of energy techniques [23], discussed above, is perhaps the result most directly related to the current work. More recent work has focused on energy decay (such as the works by Kopylova [27] or by Komech and Kopylova [24, 26] or those described in the survey article of Kopylova [25]), on Strichartz estimates (such as those by Kubo–Lucente [28]), or on asymptotics for related equations (such as the work of Bejenaru–Herr [4] for the Dirac equation).

The Fredholm approach to the construction of resolvents and propagators for non-elliptic operators using radial points estimates and anisotropic spaces is due to Vasy [12, 38, 39], while the radial points estimates used in his construction are due to Melrose [14, 32]. Adaptation of the method to more general non-elliptic scattering operators is due to Vasy [40]. Gérard–Wrochna [10] used the method to construct the Feynman propagator for the Klein–Gordon equation on asymptotically Minkowski spaces.

There is closely related work on the wave equation on Lorentzian scattering manifolds due to Baskin–Vasy–Wunsch [2, 3] and Hintz–Vasy [19]. These papers prove not only linear invertibility properties for the wave equation, but also semilinear results using weighted global spacetime estimates akin to those above, only there the relevant estimates are b-spaces. Subsequent work of Hintz and Vasy, which use in particular related microlocal methods including radial points estimates on non-compact spacetimes, establishes stability properties in mathematical GR [13, 20]. Use of the radial points estimates in semiclassical analysis to study resonances on asymptotically hyperbolic manifolds and Pollicot-Ruelle resonances, [5, 6]. Many of these approaches use anisotropic spaces, as they allow for threshold conditions to vary between components of the radial set [9].

Our is not the only recent work in microlocal analysis on hyperbolic PDE which uses a many-body approach; Hintz has used 3sc-operators, and a related class of 3b-pseudodifferential operators [15, 16, 17, 18, 21]. Also recently, Sussman has analyzed the asymptotics of solutions to Klein-Gordon near null infinity. This work includes a substantially more detailed study of solutions near null infinity, where we use only propagation of singularities estimates [35].

Outline of the paper. The paper is organized as follows. In Section 2, we analyze the free Klein-Gordon operator P_0, which serves as an instructive model model. In doing so, we recall the relevant features of the scattering calculus, including the global structure of the bicharacteristic flow of P_0 on the scattering cotangent bundle. We then prove Theorem 1.1 for P_0 and for scattering perturbations of P_0. This provides both an outline for our general approach to analysis of P_V and the actual estimates that will be used for P_V away from C. In Section 3 we discuss how P_V is not a scattering operator in general; it is instead a 3sc-differential operator, a class which we define and discuss the basic properties of, including the indicial operator of P_V. In Section 4 we display the basic properties of 3sc-pseudodifferential operators, including their symbol mappings and quantization, commutators, wavefront sets and elliptic sets. In Section 4.6 in particular we develop an elliptic theorem of 3sc-operators, including fiber infinity. Section 5 extends the work of Section 4 to prove functional calculus statements for non-elliptic 3sc-operators and we compute the symbols of some commutators which are used in the propagation proofs. In Sections 6 and 7 we prove the propagation estimates over the poles C (Section 6 proves the analogue of Hörmander's theorem and Section 7 establishes the radial point estimates). Finally, in Section 8, we put the foregoing work together to prove the existence of propagators using the Fredholm framework discussed above. In particular the main theorems are proven in this final numbered section. We finally include an index of notation after the main sections of the paper.

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2. The model case

In this section we prove Theorem 1.1 for the model operator

$$P_0 \coloneqq D_t^2 - (\Delta + m^2).$$

The majority of this work was described by Vasy [40]; we follow that development and then at the end of the section we describe related settings in which the same proof applies with minimal modifications.

In the case of P_0, the adapted Sobolev spaces are

$$\mathcal{X}^{s,\boldsymbol{\ell}} = \{u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}} : P_0 u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}\}, \quad \mathcal{Y}^{s,\boldsymbol{\ell}} = H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}, \tag{2.1}$$

for smooth function $\boldsymbol{\ell} = \boldsymbol{\ell}(t,z) \in S^0_{\mathrm{cl}}(\mathbb{R}^{n+1})$. We prove the invertibility of P_0 as a bounded operator between $\mathcal{X}^{s,\boldsymbol{\ell}}$ and $\mathcal{Y}^{s-1,\boldsymbol{\ell}+1}$ for suitable $\boldsymbol{\ell}$.

Theorem 2.1. *If $s \in \mathbb{R}$ and $\boldsymbol{\ell}_+$ a forward weight,*

$$P_0 : \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_++1} \tag{2.2}$$

is an isomorphism. Its inverse is the forward propagator. The same is true if $\boldsymbol{\ell}_+$ is replaced by a backward weight $\boldsymbol{\ell}_-$, in which case the inverse is the backward propagator.

2.1. **Outline.** As described in the introduction, the main step in the proof of Theorem 2.1 is to show that P_0 and P_0^* are Fredholm operators between $\mathcal{X}^{s,\boldsymbol{\ell}}$ and $\mathcal{Y}^{s-1,\boldsymbol{\ell}+1}$. In particular, $P_0 : \mathcal{X}^{s,\boldsymbol{\ell}} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}+1}$ is Fredholm provided we can establish the following two global estimates:

$$\|u\|_{s,\boldsymbol{\ell}} \leq C \left(\|P_0 u\|_{s-1,\boldsymbol{\ell}+1} + \|u\|_{-N,-M} \right),$$
$$\|u\|_{1-s,-1-\boldsymbol{\ell}} \leq C \left(\|P_0^* u\|_{-s,-\boldsymbol{\ell}} + \|u\|_{-N,-M} \right),$$

where M, N are sufficiently large that $H_{\mathrm{sc}}^{-N,-M} \hookrightarrow H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ and $H_{\mathrm{sc}}^{-N,-M} \hookrightarrow H_{\mathrm{sc}}^{1-s,-1-\boldsymbol{\ell}}$ are compact. These will hold only for suitable $\boldsymbol{\ell}$, and the outline of the estimates here motivates the choice of the forward and backward weights $\boldsymbol{\ell}_\pm$ in Section 2.8. Although for the free case we have $P_0^* = P_0$, it is useful to distinguish these as the second estimate above is used to analyze the cokernel of the Fredholm maps for P_0.

The global estimates for P_0 and P_0^* follow from several types of microlocal estimates. Indeed, via a microlocal partition of unity, we decompose phase space (the scattering cotangent bundle, described below) into open sets U where

(1) U localizes to the elliptic set of P_0.
(2) U is a neighborhood of a point in the characteristic set of P_0 at which the Hamilton vector field is non-radial. Here we bound u by $P_0 u$ and Eu, where E microlocalizes to a neighborhood that is in the past of the Hamiltonian flow, see Proposition 2.6. This is the typical microlocal propagation of singularities estimate.
(3) U is a neighborhood of a point in the characteristic set of P_0 at which the Hamilton vector field is radial.

We prove estimates in the corresponding regions.

(1) We have microlocal elliptic estimates in Propositions 2.2 and 2.3
(2) In a neighborhood of a point in the characteristic set and away from the radial set, we bound u by $P_0 u$ and Eu, where E microlocalizes to a neighborhood that is in the past of the Hamiltonian flow, see Proposition 2.6. This is the typical microlocal propagation of singularities estimate.
(3) Near the radial set, we have two different types of estimates: below the decay rate $\boldsymbol{\ell} < -1/2$, we have an estimate that is similar to the propagation of singularities estimate, see Proposition 2.13. Whereas for $\boldsymbol{\ell} > -1/2$, we have an estimate that mimics the elliptic estimate, but we need to assume that u is a-priori above the $-1/2$ threshold, see Proposition 2.11.

Given these four estimates – elliptic, principal type, and above and below threshold radial points estimates – we obtain the desired Fredholm estimate for P_0 (and, by working with adjoints, for P_0^*) in the space $H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ provided that $\boldsymbol{\ell} > -1/2$ (to apply the above threshold estimate) and $\boldsymbol{\ell} < -1/2$ (to apply the below threshold estimate). This motivates the use of a variable weight described below in Sections 2.6 and 2.8.

In microlocal regions near the radial set we always assume that $\boldsymbol{\ell}$ is constant, and emphasize this by denoting it by ℓ.

The rest of this section is devoted stating the estimates together with a sketch of the proofs and then the proof of Theorem 2.1.

2.2. **The scattering calculus.** As the operator P_0 is an element of Melrose's scattering calculus [32], which quantizes functions of the standard translation-invariant vector fields on

$\mathbb{R}^{n+1}$. Indeed, the *scattering differential operators* on a vector space $\mathbb{R}^N$ are given by

$$L \in \mathrm{Diff}^m_{\mathrm{sc}}(\mathbb{R}^N_w) \iff L = \sum_{|\alpha|\le m} a_\alpha(w) D_w^\alpha, \quad a_\alpha \in S^0_{\mathrm{cl}}(\mathbb{R}^N), \tag{2.3}$$

where $S^0_{\mathrm{cl}}(\mathbb{R}^N)$ is the space of classical symbols of order zero on $\mathbb{R}^N$. Observe that $P_0 = D_t^2 - (\Delta + m^2) \in \mathrm{Diff}^2_{\mathrm{sc}}(\mathbb{R}^{n+1})$. We now describe some relevant features of the scattering calculus.

Scattering operators are most easily understood as operators on a compactified space. Here we use the notation $X = \overline{\mathbb{R}^{n+1}}$ for this space, which is the *simultaneous radial compactification of spacetime.* In particular, points on the boundary of X (below simply called "infinity") represent the loci of endpoints of geodesics of arbitrary type (timelike/spacelike/null), where asymptotically parallel geodesics limit to the same point at infinity.

$$X := \overline{\mathbb{R}^{n+1}_{t,z}} = \mathbb{S}^{n+1}_+ . \tag{2.4}$$

Here X is simply a hemisphere of the unit sphere, $\mathbb{S}^{n+1}_+$; it is a compactification of $\mathbb{R}^{n+1}$ explicitly via the inclusion

$$(t,z) \mapsto \frac{(1,t,z)}{\langle t,z\rangle}, \qquad \langle t,z\rangle = (1+t^2+|z|^2)^{1/2}$$

The boundary of $\mathbb{S}^{n+1}_+$ is diffeomorphic to $\mathbb{S}^n$ and is given by

$$\partial \mathbb{S}^{n+1}_+ = \{(0,t,z) \colon t^2 + |z|^2 = 1\} . \tag{2.5}$$

In particular, a global boundary defining function on X is given by $(1+t^2+|z|^2)^{-1/2}$. In the region where $y = z/t$ is bounded, we may use $x = 1/t$ and y as local coordinates.

The smooth structure on X (as a manifold with boundary) distinguishes the classical symbols of order zero on $\mathbb{R}^{n+1}$ as smooth, i.e.,

$$C^\infty(X) = S^0_{\mathrm{cl}}(\mathbb{R}^{n+1}),$$

and thus one can rephrase the definition of $\mathrm{Diff}^m_{\mathrm{sc}}(\mathbb{R}^{n+1})$ by demanding coefficients a_α in (2.3) satisfy

$$a_\alpha \in C^\infty(X).$$

We also write

$$\mathrm{Diff}^m_{\mathrm{sc}}(X) = \mathrm{Diff}^m_{\mathrm{sc}}(\mathbb{R}^{n+1}),$$

and

$$\mathrm{Diff}^{m,r}_{\mathrm{sc}}(X) = \langle t,z\rangle^r\, \mathrm{Diff}^m_{\mathrm{sc}}(X).$$

The space $\mathrm{Diff}^m_{\mathrm{sc}}(X)$ is the universal enveloping algebra of the space of *scattering vector fields* $\mathcal{V}_{\mathrm{sc}}(X) := x\mathcal{V}_b(X)$, where x is a total boundary defining function for X and $\mathcal{V}_b(X)$ is the space of vector fields tangent to ∂X [31, 33]. The space of scattering vector fields is independent of the specific choice of boundary defining function and forms a Lie algebra. The scattering tangent bundle ${}^{\mathrm{sc}}TX$ is the vector bundle whose sections are scattering vector fields. If $(x,y) \in \mathbb{R}_+ \times \mathbb{R}^n$ are local coordinates on X with x a boundary defining function, then ${}^{\mathrm{sc}}TX$ is locally spanned over $C^\infty(X)$ by

$$\{x^2\partial_x, x\partial_y\} .$$

The dual bundle is the *scattering cotangent bundle* ${}^{\mathrm{sc}}T^*X$ and it is locally given by

$$\left\{\frac{dx}{x^2}, \frac{dy}{x}\right\}.$$

The (total) symbol of a scattering differential operator is best understood as a function on the doubly compactified phase space

$$ {}^{\mathrm{sc}}\overline{T}^*X = X \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}} \tag{2.6}$$

where τ, ζ are dual to t, z respectively, and $\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$ is the radial compactification of the momentum factor, or, as we will refer to it, the "fiber", as it is the fiber of the fibration ${}^{\mathrm{sc}}\overline{T}^*X \longrightarrow X$.

We define, for $m, r \in \mathbb{R}$,

$$ {}^{\mathrm{sc}}S^{m,r}(X) = \langle t, z\rangle^r \langle \tau, \zeta\rangle^m C^\infty(X \times \overline{\mathbb{R}^{n+1}}) \tag{2.7}$$

to be the space of classical scattering symbols of order m, r, and

$$ {}^{\mathrm{sc}}\Psi^{m,r}(\mathbb{R}^{n+1}) = \mathrm{Op}_L({}^{\mathrm{sc}}S^{m,r}(\mathbb{R}^{n+1}))$$

be the (classical) scattering pseudodifferential operators as defined by Melrose [32]. The principal symbol map, sending $\mathrm{Op}_L(a) = A$ to the equivalence class of a in

$$ j_{\mathrm{sc},m,r} \colon {}^{\mathrm{sc}}\Psi^{m,r} \longrightarrow {}^{\mathrm{sc}}S^{m,r}(X) / {}^{\mathrm{sc}}S^{m-1,r-1}(X), \tag{2.8}$$

is equivalent to the mapping taking a to its restriction to the boundary of $X \times \overline{\mathbb{R}^{n+1}}$:

$$ j_{\mathrm{sc},m,r}(A) = \langle t, z\rangle^{-r} \langle \tau, \zeta\rangle^{-m} a|_{\partial(X \times \overline{\mathbb{R}^{n+1}})}.$$

Here $\partial(X \times \overline{\mathbb{R}^{n+1}})$ is a union of two boundary hypersurfaces (bhs's), which we denote

$$ C_{\mathrm{sc}}(X) \coloneqq (\partial X \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}) \cup (X \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}) = \partial\, {}^{\mathrm{sc}}\overline{T}^*X. \tag{2.9}$$

The boundary hypersurface $\partial X \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$ is "spacetime infinity" while $X \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$ is "momentum" or "fiber infinity". The functions

$$ \rho_{\mathrm{base}} \coloneqq \langle t, z\rangle^{-1} \text{ and } \rho_{\mathrm{fib}} \coloneqq \langle \tau, \zeta\rangle^{-1}, \tag{2.10}$$

are boundary defining functions for spacetime and fiber infinity respectively.

We define

$$ \hat{N}_{\mathrm{sc},m,r}(A) \coloneqq \langle t, z\rangle^{-r} \langle \tau, \zeta\rangle^{-m} a|_{\partial X \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}}, \tag{2.11}$$

$$ \sigma_{\mathrm{sc},m,r}(A) \coloneqq \langle t, z\rangle^{-r} \langle \tau, \zeta\rangle^{-m} a|_{X \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}}. \tag{2.12}$$

We have that for all $p \in \partial X \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$,

$$ \hat{N}_{\mathrm{sc},m,r}(A)(p) = \sigma_{\mathrm{sc},m,r}(A)(p).$$

Note in particular that the symbol of L in equation (2.3), obtained by replacing $D_t^j D_z^\alpha$ by $\tau^j \zeta^\alpha$, is smooth on ${}^{\mathrm{sc}}\overline{T}^*X$ after it is multiplied by $\langle \tau, \zeta\rangle^{-m}$, and more generally the symbol of $L \in \mathrm{Diff}^{m,r}_{\mathrm{sc}}$ is smooth on ${}^{\mathrm{sc}}\overline{T}^*X$ after it is multiplied by $\langle t, z\rangle^{-r} \langle \tau, \zeta\rangle^{-m}$.

2.3. **Elliptic estimates.** We now recall the standard notions of operator wavefront set, elliptic set, and character set for operators on the scattering calculus, which are useful in this section as well as in comparison with the corresponding 3sc notions below.

The **(scattering) operator wavefront set** $\mathrm{WF}'(A) = \mathrm{WF}'_{\mathrm{sc}}(A) \subset C_{\mathrm{sc}}(X)$ is the essential support of the symbol a, where we recall that $\alpha \notin \text{ess-supp}(a)$ if an only if there is an open set $U \subset X \times \overline{\mathbb{R}^{n+1}}$ with $\alpha \in U$ such that a is Schwartz in U.

For $A \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$, the **(scattering) elliptic set** $\mathrm{Ell}(A)$ is defined by $\alpha \in \mathrm{Ell}(A)$ if and only if $\sigma_{\mathrm{sc},0,0}(A)(\alpha) \neq 0$. For $A \in {}^{\mathrm{sc}}\Psi^{m,\ell}$, $\alpha \in \mathrm{Ell}(A)$ if and only if $\sigma_{\mathrm{sc},m,\ell}(A) \neq 0$. The **(scattering) characteristic set** is the complement of the elliptic set:

$$\mathrm{Char}(A) = C_{\mathrm{sc}}(X) \setminus \mathrm{Ell}(A).$$

Note that this definition of ellipticity is equivalent to the standard one in which we demand that, in a neighborhood of α, the principal symbol $j_{\mathrm{sc},m,r}(A)$ is bounded below in a neighborhood of α in ${}^{\mathrm{sc}}\overline{T}^*X$ by

$$|j_{\mathrm{sc},m,r}(A)| \geq c > 0.$$

Given $\alpha \in \mathrm{Ell}(A)$, the standard elliptic parametrix construction holds in the scattering calculus, and there is $B \in {}^{\mathrm{sc}}\Psi^{-m,-r}(X)$ such that

$$\alpha \notin \mathrm{WF}'(I - BA) \cup \mathrm{WF}'(I - AB)$$

and thus we obtain the (scattering) elliptic estimates:

Proposition 2.2 (Corollary 5.5, [40])**.** *Let* $A \in {}^{\mathrm{sc}}\Psi^{m,r}(X)$*. Let* $B, G \in {}^{\mathrm{sc}}\Psi^{0,0}$*, and assume* $\mathrm{WF}'(G) \subset \mathrm{Ell}(A)$ *and* $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$*. Then for any* $M, N \in \mathbb{R}$*, there is a* $C > 0$ *such that*

$$\|Qu\|_{s,\ell} \leq C \left(\|GAu\|_{s-m,\ell-r} + \|u\|_{-N,-M} \right).$$

2.4. **Variable weight spaces.** As discussed briefly at the end of Section 2.1, our Fredholm estimates require Sobolev spaces with variable growth/decay order $\boldsymbol{\ell}$, which we review briefly, referring to Vasy [40, Sect. 3] for further properties and details. In our discussion of causal propagators, we require only that $\boldsymbol{\ell}$ depend on spacetime variables, but we discuss the general case for completeness.

Suppose $\boldsymbol{\ell} \in C^\infty({}^{\mathrm{sc}}\overline{T}^*X)$, and $0 < \delta < 1/2$. We define $a \in S^{m,\boldsymbol{\ell}}_\delta(\mathbb{R}^N_w)$ if $a \in C^\infty(\mathbb{R}^N_w \times \mathbb{R}^N_\theta)$ and

$$|D^\alpha_w D^\beta_\theta a| \leq C_{\alpha\beta} \langle w \rangle^{\boldsymbol{\ell} - |\alpha| + \delta|(\alpha,\beta)|} \langle \theta \rangle^{m - |\beta| + \delta|(\alpha,\beta)|}.$$

(Vasy uses two distinct δ, δ', but this is not needed here.) Then

$${}^{\mathrm{sc}}\Psi^{m,\boldsymbol{\ell}}_\delta(\mathbb{R}^N) = \mathrm{Op}_L(S^{m,\boldsymbol{\ell}}_\delta(\mathbb{R}^N)),$$

Standard symbolic constructions still work in the variable order setting, but one must work with equivalence classes of symbols rather than with restrictions. As an example, the principal symbol of $\mathrm{Op}_L(a) \in {}^{\mathrm{sc}}\Psi^{m,\boldsymbol{\ell}}(\mathbb{R}^N$ is the equivalence class $[a]$ of a in $S^{m,\boldsymbol{\ell}}_\delta / S^{m-1+2\delta,\boldsymbol{\ell}-1+2\delta}_\delta$.

A paradigmatic example of such a variable order symbol is the product

$$a_{m,\boldsymbol{\ell}}(w,\theta) := \langle w \rangle^{\boldsymbol{\ell}} \langle \theta \rangle^m, \tag{2.13}$$

with the δ loses incurred by differentiation because the exponent ℓ is a function. Note that $A_{m,\boldsymbol{\ell}} = \mathrm{Op}_L(a_{m,\boldsymbol{\ell}})$ is *not* a classical symbol but is still globally scattering elliptic in the sense that, for some $\epsilon > 0$,

$$|a_{m,\boldsymbol{\ell}}(t,z,\tau,\zeta)| \geq \epsilon \langle t, z \rangle^{\boldsymbol{\ell}} \langle \tau, \zeta \rangle^m.$$

We further note that $A_{m,\boldsymbol{\ell}}$ is invertible as a map of $\mathcal{S}(X) = \mathcal{S}(\mathbb{R}^{n+1})$ and hence as a map of $\mathcal{S}'(X) = \mathcal{S}'(\mathbb{R}^{n+1})$. We may therefore define the variable order Sobolev space $H_{\mathrm{sc}}^{m,\boldsymbol{\ell}}$ by

$$H_{\mathrm{sc}}^{m,\boldsymbol{\ell}} = \{u \in \mathcal{S}'(X) : A_{m,\boldsymbol{\ell}} u \in L^2\},$$

where $\ell \leq \inf \boldsymbol{\ell}$. If $\boldsymbol{\ell} = \boldsymbol{\ell}(w)$, then $\mathrm{Op}_L(a_{m,\boldsymbol{\ell}}) = \langle w \rangle^{\boldsymbol{\ell}} \mathrm{Op}_L(\langle \theta \rangle^m)$ and $H_{\mathrm{sc}}^{m,\boldsymbol{\ell}} = \langle w \rangle^{\boldsymbol{\ell}} H_{\mathrm{sc}}^m$ where $\langle w \rangle^{\boldsymbol{\ell}}$ simply acts as a spacetime-dependent weight. It follows essentially from Arzela-Ascoli that, for any $m > m'$ and $\boldsymbol{\ell} > \boldsymbol{\ell}'$ (the latter interpreted pointwise) then

$$H_{\mathrm{sc}}^{m,\boldsymbol{\ell}} \hookrightarrow H_{\mathrm{sc}}^{m',\boldsymbol{\ell}'}$$

is a compact inclusion.

Operators in ${}^{\mathrm{sc}}\Psi^{0,0}(X)$ are bounded on $H_{\mathrm{sc}}^{m,\boldsymbol{\ell}}$ and $A \in {}^{\mathrm{sc}}\Psi^{m,\boldsymbol{\ell}}$ maps $H_{\mathrm{sc}}^{m,\boldsymbol{\ell}} \to H_{\mathrm{sc}}^{0,0}$, so the standard elliptic estimates still apply. In particular, we have the following generalization of Proposition 2.2 for variable order spaces.

Proposition 2.3. *Suppose $A \in {}^{\mathrm{sc}}\Psi^{m,r}$ $G, B \in {}^{\mathrm{sc}}\Psi^{0,0}$ satisfy*

$$\mathrm{WF}'(B) \subset \mathrm{Ell}(G) \subset \mathrm{WF}'(G) \subset \mathrm{Ell}(A).$$

For each $s, M, N \in \mathbb{R}$ and $\boldsymbol{\ell} \in C^\infty({}^{\mathrm{sc}}\overline{T}^* X)$, there is a constant C so that

$$\|Bu\|_{s,\boldsymbol{\ell}} \leq C \left(\|GAu\|_{s-m,\boldsymbol{\ell}-r} + \|u\|_{-N,-M} \right).$$

2.5. **Hamiltonian flow and radial sets.** Having obtained estimates on the elliptic set, we now analyze the operator P_0 near its characteristic set. As the estimates obtained by Vasy [40] include both standard propagation estimates and estimates near the radial points (i.e., the submanifold of points where the rescaled Hamilton vector field vanishes on $\partial {}^{\mathrm{sc}}\overline{T}^* X$), we now turn our attention to the symbol of the free Klein-Gordon operator

$$P_0 := D_t^2 - (\Delta + m^2) \tag{2.14}$$

and the flow of its Hamilton vector field on the characteristic set.

The full symbol of P_0 is

$$p(t, z, \tau, \zeta) = \tau^2 - (|\zeta|^2 + m^2).$$

The standard coordinates on phase space we write as (t, z, τ, ζ), so that the characteristic set is given by

$$\mathrm{Char}(P_0) = \{(t, z, \tau, \zeta) : \tau^2 - |\zeta|^2 - m^2 = 0\}. \tag{2.15}$$

In open sets of the form $|z|/t < C$ (including for large C), we may use coordinates

$$x = 1/t, \quad y = z/t, \quad \xi, \quad \eta,$$

on ${}^{\mathrm{sc}}\overline{T}^* X$. We write the canonical one-form on ${}^{\mathrm{sc}}T^* X$ as

$$\tau dt + \zeta \cdot dz = \xi \frac{dx}{x^2} + \eta \cdot \frac{dy}{x}, \qquad \text{i.e. we write} \quad \zeta = \eta, \quad \tau = -\xi - \eta \cdot y. \tag{2.16}$$

The Hamilton vector field is defined by

$$H_p := \frac{\partial p}{\partial \tau} \partial_t + \frac{\partial p}{\partial \zeta} \partial_z - \frac{\partial p}{\partial t} \partial_\tau - \frac{\partial p}{\partial z} \partial_\zeta.$$

In the above coordinates we see that the Hamilton vector field is

$$\begin{aligned}\frac{1}{2}H_p &= \tau\partial_t - \zeta\cdot\partial_z \\ &= x\left(\tau(-x\partial_x - y\cdot\partial_y + (\eta\cdot y)\partial_\xi) - \zeta\cdot(\partial_y - \eta\partial_\xi)\right) \\ &= x\left((\xi+\eta\cdot y)x\partial_x - (\eta-(\xi+\eta\cdot y)y)\cdot\partial_y + \eta\cdot(\eta-(\xi+\eta\cdot y)y)\partial_\xi\right)\end{aligned} \tag{2.17}$$

It is a general fact that for $a \in {}^{\mathrm{sc}}S^{m,r}(X)$ with $A = \mathrm{Op}_L(a)$, a classical scattering symbol, one can rescale the Hamilton vector field to obtain a new vector field ${}^{\mathrm{sc}}H_a$ that is tangent to the boundary of ${}^{\mathrm{sc}}\overline{T}^*X$. One can take, for example,

$$ {}^{\mathrm{sc}}H_a = \langle t,z\rangle^{-l+1}\langle\tau,\zeta\rangle^{-m+1}H_a \in \mathcal{V}_b({}^{\mathrm{sc}}\overline{T}^*X), \tag{2.18}$$

where the latter containment means exactly that ${}^{\mathrm{sc}}H_a$ extends to a smooth vector field on ${}^{\mathrm{sc}}\overline{T}^*X$ and is tangent to the boundary; in particular the flow on ${}^{\mathrm{sc}}\overline{T}^*X$ restricts to the boundary $C_{\mathrm{sc}}(X)$ and defines a flow on there. The flow of ${}^{\mathrm{sc}}H_a$ on $\mathrm{Char}(A)$ is what we refer to as the bicharacteristic or Hamiltonian flow; over the spacetime interior this is the standard formulation of the bicharacteristic flow as a homogeneous degree zero restriction to the sphere bundle.

Remark 2.4. This tangency to the boundary in (2.18) is important in that it allows us to extend the Hamiltonian flow to the whole of ${}^{\mathrm{sc}}\overline{T}^*X$ and thus to extend the propagation of singularities estimates to infinite spacetime and/or large momenta regions. Since this tangency is unchanged under multiplication by a positive non-vanishing prefactor, there is some ambiguity in the definition of ${}^{\mathrm{sc}}H_a$. Moreover, multiplication by such a prefactor does not effect the propagation estimates. We use that in regions

$$0 \le x \le C, |y| < C, \tag{2.19}$$

that $x \sim \langle t,z\rangle^{-1}$, meaning

$$0 < c < x\cdot\langle t,z\rangle < 1/c,$$

and in such regions $\langle t,z\rangle^{-1}$ can be replaced by x in (2.18). A similar remark holds for the fiber variable (τ,ζ). That is, if we define $\rho = 1/\tau, \mu = \zeta/\tau$, then in regions

$$0 \le \rho \le C, |\mu| \le C, \tag{2.20}$$

we have

$$0 < c < \rho\cdot\langle\tau,\zeta\rangle < 1/c,$$

and in such regions $\langle\tau,\zeta\rangle^{-1}$ can be replaced by ρ in (2.18). It will be useful below that $|\rho|, |\mu| < C$ in a neighborhood of the characteristic of $\tau^2 - |\zeta|^2 - m^2 = 0$.

One can obtain a global definition of ${}^{\mathrm{sc}}H_a$ which uses x^s in place of $\langle t,z\rangle^{-s}$ and ρ^s in place of $\langle\tau,\zeta\rangle^{-s}$ in regions (2.19) and (2.20) by choosing global boundary defining functions which are equal to x and ρ respectively in those regions, but we do not make this process formal here; we simply use powers of x and ρ to rescale in regions where to do so is valid.

The relationship between the commutator $[A,B] \coloneqq AB - BA$ and the Hamiltonian flow underpins the propagation estimates in the next section. Indeed, if $A = \mathrm{Op}_L(a) \in {}^{\mathrm{sc}}\Psi^{m_1,r_1}(X)$ and $B = \mathrm{Op}_L(b) \in {}^{\mathrm{sc}}\Psi^{m_2,r_2}(X)$, then

$$\begin{gathered}[A,B] \in {}^{\mathrm{sc}}\Psi^{m_1+m_2-1,r_1+r_2-1}(X)\,, \\ j_{\mathrm{sc},m_1+m_2-1,r_1+r_2-1}\left(i[A,B]\right) = \langle t,z\rangle)^{-r_2}\langle\tau,\zeta\rangle^{-m_2}\cdot{}^{\mathrm{sc}}H_a(b)\,.\end{gathered} \tag{2.21}$$

It is of course possible to phrase this relationship in terms of the scattering principal symbols of A and B, our normalization of the principal symbol would then make the symbol's dependence on the orders explicit. The same relationship also holds if one of A or B lies in a variable order space; the only change in this case is that the principal symbol must be interpreted as an equivalence class of (variable order) symbols.

The radial set of H_p is given by

$$\mathcal{R} := \mathrm{Char}(P_0) \cap {}^{\mathrm{sc}}H_p^{-1}(0) \subset \partial^{\mathrm{sc}}\overline{T}^* X\,. \tag{2.22}$$

The rescaled vector field ${}^{\mathrm{sc}}H_p$ is non-vanishing on the portion of $C_{\mathrm{sc}}X$ lying over the interior of X; to locate the radial set we therefore consider it over $\partial X \times \overline{\mathbb{R}^{n+1}}$.

Despite the fact that ξ, η are dual (in the rescaled scattering sense) to x, y, it is sometimes useful to have different coordinates on the cotangent bundle. For example in the coordinates (x, y, τ, ζ) we have

$$\begin{aligned} H_p &= 2\tau(-x^2\partial_x - xy \cdot \partial_y) - x\zeta \cdot \partial_y \\ &= -2x\left(\tau x \partial_x + (\zeta + \tau y) \cdot \partial_y\right) \end{aligned} \tag{2.23}$$

Thus, in regions with $x \le C, |y| \le C$ and $0 \le \rho = 1/\tau \le C, |\mu| = |\zeta/\tau| < C$, by Remark 2.4, we have

$$(1/2){}^{\mathrm{sc}}H_p = -x\partial_x - (\mu + y) \cdot \partial_y, \tag{2.24}$$

This latter expression indicates how we can realize the zero locus of ${}^{\mathrm{sc}}H_p$ as a family of radial sinks and sources; namely, as long as $t, \tau > 0$, we can work in the following coordinates

$$x = \frac{1}{t}, \quad w = \frac{\zeta}{\tau} + y, \quad \rho = \frac{1}{\tau}, \quad \mu = \frac{\zeta}{\tau}, \tag{2.25}$$

to obtain

$$(1/2)H_p = (x/\rho)\left(-x\partial_x - w \cdot \partial_w\right), \tag{2.26}$$

This expression indeed shows that, in $\{t > 0\} \cap \{\tau > 0\}$, the radial set is $x = 0, w = 0$ in the characteristic set, that is, in these coordinates it is exactly

$$\mathcal{R}^f_+ = \{(x, w, \rho, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \colon x = 0, w = 0, 1 - |\mu|^2 - m^2\rho^2 = 0\}$$

which is a smooth submanifold of ${}^{\mathrm{sc}}\overline{T}^* X$ intersecting the boundary $\rho = 0$ normally *and is a radial sink of the flow.*

However, only the region $t > 0, \tau > 0$ is preserved by the flow, and in that region $x = 1/t$ and $\rho = 1/\tau$ are valid boundary defining functions for spacetime and fiber infinity respectively near the radial set; in the other three regions, i.e. the other combinations of signs in $\pm t > 0, \pm\tau > 0$, one can use analogous coordinates. For example, in $t < 0, \tau > 0$, using $-1/t$ and $1/\tau$ as bdf's, only the overall sign of the expression in (2.26) changes and in that region the radial set is a radial source. As for the set $\{t = 0\} \cup \{\tau = 0\}$ where no such decomposition holds, the latter set, $\{\tau = 0\}$, lying in the elliptic set of P_0, is irrelevant, whereas the former lies away from the radial sets but intersects the characteristic set; we analyze that part only in Proposition 2.5 when we look at the global properties of $\mathrm{Char}(P_0)$.

First, though, we require some basic definitions. We define future and past causal infinity as

$$\iota^+ := \overline{\{(t,z)\colon t \geq |z|\}} \cap \partial X\,,$$
$$\iota^- := \overline{\{(t,z)\colon -t \geq |z|\}} \cap \partial X\,,$$

where the closure is taken in X. Its boundary is a compressed null infinity,

$$S^+ := \partial\iota^+\,, \quad S^- := \partial\iota^-\,.$$

It is "compressed" in the sense that it is lower dimensional than null infinity $\mathscr{I}^\pm$, which is typically $n-$dimensional. Null infinity can be shown to be naturally identified with the faces introduced by blow up of $S^\pm$, a formalism not used here but very useful in the study of radiation fields [2, 3].

We then define the future and past radial sets as

$$\mathcal{R}^f := \mathcal{R} \cap {}^{\mathrm{sc}}\overline{T}^*_{\iota^+}X\,,$$
$$\mathcal{R}^p := \mathcal{R} \cap {}^{\mathrm{sc}}\overline{T}^*_{\iota^-}X\,.$$

and, moreover, using that $\tau = 0$ does not intersect the characteristic set, we have the further decomposition

$$\mathcal{R}^f = \mathcal{R}^f_+ \sqcup \mathcal{R}^f_-\,, \quad \mathcal{R}^p = \mathcal{R}^p_+ \sqcup \mathcal{R}^p_-\,,$$

where

$$\mathcal{R}^\bullet_+ := \mathcal{R}^\bullet \cap \{\tau \geq m\}\,,$$
$$\mathcal{R}^\bullet_- := \mathcal{R}^\bullet \cap \{\tau \leq -m\}\,.$$

From (2.26) we can already see that away from fiber infinity, the radial set has the form:

$$\mathcal{R}^f \cap \{|(\tau,\zeta)| < \infty\} = \{x = 0, \zeta = -\tau y, \tau^2 = |\zeta|^2 + m^2\}\,. \tag{2.27}$$

Proposition 2.5. *The characteristic set* $\mathrm{Char}(P_0)$ *consists of two connected components,* $\mathrm{Char}(P_0)_\pm = \mathrm{Char}(P_0) \cap \overline{\{\pm\tau > m - \epsilon\}}$. *On the component* $\mathrm{Char}(P_0)_+$*, the radial set* $\mathcal{R}^f_+$ *is a global sink, and* $\mathcal{R}^p_+$ *a global source, for the Hamiltonian flow, while on* $\mathrm{Char}(P_0)_-$, $\mathcal{R}^f_-$ *is a global source, and* $\mathcal{R}^p_-$ *a global sink. These sinks / sources are radial in the sense that the Hamilton vector field is of the form (2.26) in neighborhoods of each sink component, and satisfies the analogue of (2.26) with positive sign near the source components.*

The projection of the radial set to the base variables is causal infinity,

$$\pi(\mathcal{R}^f) = \iota^+, \pi(\mathcal{R}^p) = \iota^-\,,$$

and this projection is a diffeomorphism on each component of $\mathcal{R}$ *away from fiber infinity.*

Proof. We wish to use (2.26), and the three other expressions obtained by using the other combinations of $\rho = \pm 1/\tau, x = \pm 1/t$. First off, defining

$$\phi_t = t/\langle t,z\rangle, \quad \phi_z = z/\langle t,z\rangle, \tag{2.28}$$

we note that for $\delta_0 > 0$, on the region

$$\mathrm{Char}(P_0) \cap \{|\phi_t| \geq \delta_0\} \text{ we have } \langle\tau,\zeta\rangle \sim |\tau| \text{ and } \langle t,z\rangle \sim t,$$

where $\langle\tau,\zeta\rangle \sim |\tau|$ means there is a constant $c > 0$ so that $c \leq \frac{|\tau|}{\langle\tau,\zeta\rangle} \leq c^{-1}$. Thus in this region, if $t,\tau > 0$: (1) (2.26) holds and (2) ${}^{\mathrm{sc}}H_p = aH_p$ where $0 < c \leq a \leq C$ is a smooth, bounded,

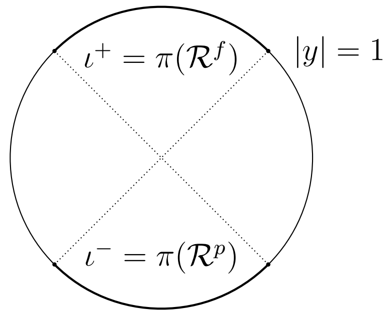

FIGURE 1. The projection of the radial set $\mathcal{R} \subset {}^{\mathrm{sc}}\overline{T}^* X$ to X. The dotted lines denote the light cone, $\{|t| = |z|\}$. The forward weights satisfy $\boldsymbol{\ell}_+ < -1/2$ in a neighborhood ι_+ and $\boldsymbol{\ell}_+ > -1/2$ in a neighborhood of ι_-.

positive function. All this is to say that (2.26) and the analogous expressions for the other sign choices $\rho = \pm 1/\tau, x = \pm 1/t$ completely describe the behavior of ${}^{\mathrm{sc}}H_p$ on $\mathrm{Char}(P_0)$ in regions where $|\phi_t| \geq \delta_0$. In particular, for example where $t > 0, \tau > 0$, $\mathcal{R}$ is a smooth radial sink given by $w = 0 = x$ and intersects fiber infinity (in this region where $\rho = 0$) normally. This region contains only $\mathcal{R}^f_+$, and the expressions in neighborhoods of the other components are easily derived.

It remains only to show that in $\mathrm{Char}(P_0) \cap \{|\phi_t| \leq \delta_0\}$, that all trajectories of the Hamiltonian flow leave this region. As it is useful below in our definition of spacetime weights for the causal propagators, we do this by showing that ϕ_t is a monotone quantity on the flow in regions $\phi_t \in (-(1/\sqrt{2}) + \delta, (1/\sqrt{2}) - \delta)$. (Note that on the spacetime boundary, $|\phi_t| = 1/\sqrt{2}$ is null infinity.) Indeed,

$$(1/2) H_p \phi_t = \frac{1}{\langle t, z \rangle} \left(\tau(1 - \phi_t^2) + \phi_t \zeta \cdot \phi_z \right)$$

Thus on $\mathrm{Char}(P_0)$, in the $\tau > 0$ region, where $\tau = \sqrt{|\zeta|^2 + m^2}$, using

$$\begin{aligned} (1/2) H_p \phi_t &\geq \sqrt{|\zeta|^2 + m^2}(1 - \phi_t^2) - |\phi_t||\zeta||\phi_z| \\ &\geq \sqrt{|\zeta|^2 + m^2}\sqrt{1 - \phi_t^2} \left(\sqrt{1 - \phi_t^2} - |\phi_t| \right). \end{aligned}$$

Using that $\sqrt{1 - \phi_t^2} - |\phi_t| \geq (1/\sqrt{2}) - |\phi_t|$ where $|\phi_t| \leq 1/\sqrt{2}$ we have, for some $c > 0$, that

$$c\,{}^{\mathrm{sc}}H_p \phi_t \geq (1/\sqrt{2}) - |\phi_t|,$$

when $|\phi_t| \leq 1/\sqrt{2}$, so ϕ_t is monotone increasing on the $\tau > 0$ component of $\mathrm{Char}(P_0)$, and a similar argument shows it is decreasing on the $\tau < 0$ component.

Thus every trajectory eventually leaves a neighborhood of $\phi_t = 0$, and the global structure of $\mathrm{Char}(P_0)$ is as stated. □

2.6. **Propagation of singularities in the scattering calculus.** We now discuss the propagation estimate away from the radial sets. Specifically, we have the standard real principal type propagation estimates for P_0, following Vasy [40, Theorem 4.4].

To make the statement cleaner, we define the following notion of control. Given sets $U_1, U_2, U_3 \subset {}^{\mathrm{sc}}\overline{T}^* X$, we say that U_1 is controlled along ${}^{\mathrm{sc}}H_p$ by U_2 through U_3 if for every $\alpha \in U_1 \cap \mathrm{Char}(P_0)$, there is a $\sigma < 0$ so that the bicharacteristic γ of ${}^{\mathrm{sc}}H_p$ with $\gamma(0) = \alpha$

has $\gamma(\sigma) \in U_2$ and $\gamma([\sigma, 0]) \subset U_3$. In other words, each point in U_1 lies on a bicharacteristic segment contained in U_3 that originates in U_2.

Proposition 2.6. *Suppose $B, E, G \in {}^{\mathrm{sc}}\Psi^{0,0}$ and that $\mathrm{WF}'(B)$ is controlled along ${}^{\mathrm{sc}}H_p$ by $\mathrm{Ell}(E)$ through $\mathrm{Ell}(G)$. Assume further that $\boldsymbol{\ell} \in C^\infty({}^{\mathrm{sc}}\overline{T}^* X)$ is decreasing monotonically along the Hamiltonian flow.*

For any M, N there is $C > 0$ such that if $Eu \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ and $GP_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$, then $Bu \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ and

$$\|Bu\|_{s,\boldsymbol{\ell}} \le C \left(\|Eu\|_{s,\boldsymbol{\ell}} + \|GP_0u\|_{s-1,\boldsymbol{\ell}+1} + \|u\|_{-N,-M}\right).$$

The same is true if $\mathrm{WF}'(B)$ is controlled along ${}^{\mathrm{sc}}H_{-p} = -{}^{\mathrm{sc}}H_p$ by $\mathrm{Ell}(E)$ through $\mathrm{Ell}(G)$ and $\boldsymbol{\ell}$ is monotone increasing along the ${}^{\mathrm{sc}}H_p$-flow.

Remark 2.7. The differential order s may also be taken variable in this proposition, monotone along the flow. This is irrelevant for our purposes and thus omitted. The operator P_0 may also be taken to be an element $P \in {}^{\mathrm{sc}}\Psi^{m,r}$ in which case the $s-1$ on the RHS is replaced by $s-m+1$ and the $\boldsymbol{\ell}+1$ replaced by $\boldsymbol{\ell}-r+1$; again this is irrelevant for our purposes.

As it informs the proof of the estimates in Section 6 below, we provide a sketch of the proof of Proposition 2.6. The proof proceeds in several steps. We first establish the following weaker estimate by a positive commutator argument:

Lemma 2.8. *Suppose $B, E, G \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$, and that $\mathrm{WF}'(B)$ is controlled along ${}^{\mathrm{sc}}H_p$ by $\mathrm{Ell}(E)$ through $\mathrm{Ell}(G)$, and $\boldsymbol{\ell} \in C^\infty({}^{\mathrm{sc}}\overline{T}^* X)$ is decreasing monotonically along the Hamiltonian flow. For any M, N, there is a $C > 0$ so that for all $u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ with $P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$,*

$$\|Bu\|_{s,\boldsymbol{\ell}} \le C \left(\|Eu\|_{s,\boldsymbol{\ell}} + \|GP_0u\|_{s-1,\boldsymbol{\ell}+1} + \|Gu\|_{s-1/2,\boldsymbol{\ell}-1/2} + \|u\|_{-N,-M}\right).$$

The same is true if $\mathrm{WF}'(B)$ is controlled along $-{}^{\mathrm{sc}}H_p$ by $\mathrm{Ell}(E)$ through $\mathrm{Ell}(G)$ and $\boldsymbol{\ell}$ is monotone increasing.

Proof of Lemma 2.8. We first prove the lemma for $u \in \mathcal{S}(X)$; a standard approximation argument extends the result to u as in the statement of the lemma.

Our aim is to exploit the relationship (2.21) between the commutator and the Hamiltonian flow. Indeed, given $\alpha \in \mathrm{WF}'(B) \cap \mathrm{Char}(P_0)$, we construct an operator

$$Q = A_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} Q_0 \in {}^{\mathrm{sc}}\Psi^{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}}(X), \quad Q_0 \in {}^{\mathrm{sc}}\Psi^{0,0}(X),$$

where $A_{s,\boldsymbol{\ell}} = \mathrm{Op}_L(a_{s,\boldsymbol{\ell}})$ with $a_{s,\boldsymbol{\ell}}$ the symbol defined in (2.13). We then consider

$$\frac{i}{2}\langle [P_0, Q^*Q]u, u\rangle = \mathrm{Im}\langle Qu, QP_0u\rangle.$$

On the one hand, for each $\epsilon > 0$, this quantity bounds the following from above:

$$\langle A_{\frac{1}{2},-\frac{1}{2}} A_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} Q_0 u, A_{-\frac{1}{2},+\frac{1}{2}} A_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} Q_0 P_0 u\rangle \ge -\frac{1}{4\epsilon}\|Q_0P_0u\|^2_{s-1,\boldsymbol{\ell}+1} - \epsilon\|Q_0u\|^2_{s,\boldsymbol{\ell}}.$$

On the other hand, we know the relationship between the principal symbol of $i[P_0, Q^*Q]$ and the Hamiltonian flow, which we exploit to construct Q symbolically. In principle, the construction requires three cases; when $\alpha \in \partial X \times \overline{\mathbb{R}^{n+1}}$, when $\alpha \in X \times \partial\overline{\mathbb{R}^{n+1}}$, and when $\alpha \in \partial X \times \partial\overline{\mathbb{R}^{n+1}}$. We treat the first two cases simultaneously; the third case is handled by including an additional boundary defining function in the definition of the symbol of Q.

Following Vasy [40, Section 4.3], we first construct $q_0 \in {}^{\mathrm{sc}}S^{0,0}(X)$ so that q_0 is elliptic at α and there are $c > 0$ and a compact set $K \subset \mathrm{Ell}(e)$ and

$$-\left(a^2_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} q_0 H_p q_0 + q_0^2 a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} H_p a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}}\right) - c a^2_{s,\boldsymbol{\ell}} q_0^2 \geq 0 \quad \text{off of } K.$$

In a small neighborhood of α we employ coordinates $q_1, \dots, q_{2(n+1)}$ on an open neighborhood W of α contained in $\mathrm{Ell}(G)$in ${}^{\mathrm{sc}}\overline{T}^*X$ so that ${}^{\mathrm{sc}}H_p = \partial_{q_1}$ and $q_{2(n+1)}$ is a boundary defining function. We further assume that $\alpha = (0, \dots, 0)$ in this system. This is possible because α is on the boundary and ${}^{\mathrm{sc}}H_p$ is non-vanishing and tangent to the boundary. We suppose that γ is the integral curve of ${}^{\mathrm{sc}}H_p$ passing through α with $\gamma(0) = \alpha$ and $\gamma(\sigma) = \beta$, $\sigma < 0$. Given open neighborhoods U_1 of $\gamma([\sigma, 0])$ and U_2 of β in W we take $\epsilon > 0$ so that the open rectangles

$$\{q : q_1 \in (\sigma - \epsilon, \epsilon), q_{2(n+1)} < \epsilon, |q_j| < \epsilon, j = 2, \dots, 2n+1\},$$
$$\{q : q_1 \in (\sigma - \epsilon, \sigma + \epsilon), q_{2(n+1)} < \epsilon, |q_j| < \epsilon, j = 2, \dots, 2n+1\}$$

around $\gamma([\sigma, 0])$ and β are contained in U_1 and U_2, respectively. Here q_1 acts as the flow parameter and $q_{2(n+1)}$ acts as the spacetime boundary defining function; powers $q^{2\boldsymbol{\ell}}_{2(n+1)}$ are spacetime weights.

We then set

$$\varphi(s) = \begin{cases} \exp(-\kappa s + (s-\epsilon)^{-1} - (s - \sigma + \epsilon)^{-1}) & \sigma - \epsilon < s < \epsilon, \\ 0 & \text{otherwise} \end{cases},$$

and let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth function that is identically one in a neighborhood of 0. We finally set

$$q_0 = \varphi(q_1)\chi^2\left(\frac{q_2}{\delta}\right) \dots \chi^2\left(\frac{q_{2n-1}}{\delta}\right) \coloneqq \varphi\chi^2.$$

For $\delta > 0$ sufficiently small, this function q_0 is supported in the rectangle above. Moreover, the explicit definition of φ allows us to bound φ in terms of $\varphi' = {}^{\mathrm{sc}}H_p(\varphi(q_1))$.

With $q = a_{s,\boldsymbol{\ell}} q_0$, the principal symbol of $(i/2)[P_0, Q^*Q]$ is then

$$q H_p q = a^2_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} \varphi' \varphi \chi^2 + \varphi^2 \chi^2 a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} H_p a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}}.$$

In the region of interest, the weight a is a non-vanishing smooth multiple of $q_{2(n+1)}^{-\boldsymbol{\ell}-\frac{1}{2}}$, i.e., $a_{s-1/2,\boldsymbol{\ell}+1/2} = b q_{2(n+1)}^{-\boldsymbol{\ell}-\frac{1}{2}}$, so

$$a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} H_p a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} = q_{2(n+1)}^{-2\boldsymbol{\ell}-1} b H_p b + b^2 q_{2(n+1)}^{-2\boldsymbol{\ell}-1} (\log q_{2(n+1)}) \left(-H_p \boldsymbol{\ell}\right).$$

The second term has a favorable sign as $\boldsymbol{\ell}$ is decreasing along the flow, while the first term will be controlled by the main term in the commutator.

Taking $\kappa > 0$ sufficiently large allows for the main term (arising from φ') to control the others.

We now set $e \in {}^{\mathrm{sc}}S^{0,0}(X)$ so that $E = \mathrm{Op}_L(e)$; as $K \subset \mathrm{Ell}(E)$, there is thus some $C > 0$ so that, with $q = a_{s-\frac{1}{2},\boldsymbol{\ell}+\frac{1}{2}} q_0$,

$$C a^2_{s,\boldsymbol{\ell}} e^2 - q H_p q - c a^2_{s,\boldsymbol{\ell}} q_0^2 \geq 0. \tag{2.29}$$

The construction of q_0 also guarantees that $\mathrm{WF}'(Q_0) \subset \mathrm{Ell}(G)$; explicit choice of the commutant Q_0 will also guarantee that this nonnegative quantity (2.29) above is a smooth sum of squares.[2] The Gårding inequality in Lemma 2.9 below then shows that

$$C\langle E^*Eu,u\rangle - \langle\frac{i}{2}[P_0,Q^*Q]u,u\rangle - c\langle Q_0^*A_{s,\boldsymbol{\ell}}^*A_{s,\boldsymbol{\ell}}Q_0u,u\rangle \gtrsim -\|Gu\|^2_{s-1/2,\boldsymbol{\ell}-1/2} - \|u\|^2_{-N,-M}.$$

In other words, we have the bound

$$\langle\frac{i}{2}[P_0,Q^*Q]u,u\rangle + c\|Q_0u\|^2_{s,\boldsymbol{\ell}} \lesssim \|Eu\|^2_{s,\boldsymbol{\ell}} + \|Gu\|^2_{s-1/2,\boldsymbol{\ell}-1/2} + \|u\|^2_{-N,-M}.$$

Putting the two bounds together and taking $\epsilon = c/2$ yields the estimate

$$\|Q_0u\|^2_{s,\boldsymbol{\ell}} \lesssim \|Gu\|^2_{s-1/2,\boldsymbol{\ell}-1/2} + \|Eu\|^2_{s,\boldsymbol{\ell}} + \|Q_0P_0u\|^2_{s-1,\boldsymbol{\ell}+1} + \|u\|^2_{-N,-M}.$$

□

For completeness, we include the "easy" version of the Gårding inequality used in the proof sketch above:

Lemma 2.9. *If $A \in {}^{\mathrm{sc}}\Psi^{m,\boldsymbol{\ell}}(X)$ has a nonnegative principal symbol that is a sum of squares, i.e., there are $B_1,\dots,B_k \in {}^{\mathrm{sc}}\Psi^{m/2,\boldsymbol{\ell}/2}(X)$ with*

$$\sigma_{\mathrm{sc},m,\boldsymbol{\ell}}(A) = \sum_{j=1}^{k} |\sigma_{\mathrm{sc},m/2,\boldsymbol{\ell}/2}(B_j)|^2,$$

and $G \in \Psi^{0,0}(X)$ satisfies $\mathrm{WF}'(A) \subset \mathrm{Ell}(G)$, then for every $M,N \in \mathbb{R}$, there is a constant C so that for all $u \in H_{\mathrm{sc}}^{m/2,\boldsymbol{\ell}/2}$,

$$\langle Au,u\rangle \geq -C\|Gu\|^2_{\frac{m-1}{2},\frac{\boldsymbol{\ell}-1}{2}} - C\|u\|^2_{-N,-M}.$$

Proof. This is an easy consequence of elliptic regularity, see [1] for details. □

An inductive application of Lemma 2.8 (combined with repeated adjustments to the supports of the symbols) then establishes Proposition 2.6 for $u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ with $P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$. Finally, a regularization argument finishes the proof. The regularization argument is technical and involves replacing the symbols in the proof of Lemma 2.8 with weaker approximating symbols. As the relevant symbol classes are not used in the rest of the paper, we refer the reader to Vasy [40, Section 4.4] and defer our discussion of regularization arguments to Section 6 below.

2.7. **Radial points estimates.** We also need estimates that hold near the radial set $\mathcal{R}$. In addition to illuminating the proof for the model setting, we also must employ estimates which hold microlocally near open subsets of $\mathcal{R}$ located away from C. In Section 7 below, we establish estimates that hold in a more general setting; these are then combined with estimates that are local on $\mathcal{R}$. For simplicity, we use the radial points estimates of [40] directly, and then show that these can easily be localized on $\mathcal{R}$.

We use the radial points estimates from Vasy [40] proven for a real principal type operator $P \in {}^{\mathrm{sc}}\Psi^{m,l}$ near a smooth submanifold of radial points (there denoted L). Specifically, we use the estimates in Proposition 4.12 of [40], which are the "first pass" estimates in which the norm $\|Bu\|_{s,r}$ is controlled by a norm $\|Gu\|_{s-1/2,r'}$ with $r' \in [r-1/2, r)$. An induction argument then removes this norm on the right hand side. We use these estimates because they clarify how easily the estimates can be localized along the radial set. The assumptions in Vasy's proposition about the Hamiltonian flow of P near the radial set are

[2]This allows us to avoid the use of the sharp Gårding inequality here.

given in equation (4.12) there; these assumptions are satisfied for radial vector fields as in equation (2.26). Vasy's quantity $\tilde{\beta}$ is 0 if $P - P^* \in \Psi^{m-2,l-2}$ (i.e. is lower than the expected order for real principal type operators). As P_0 is self-adjoint, we do not include that correction here.

As discussed by Vasy [40, Section 4.7], the regularity and decay orders s, r can be exchanged thanks to the symmetry in scattering analysis realized by the Fourier transform, which in particular maps $H^{s,r}_{\mathrm{sc}}$ isometrically to $H^{r,s}_{\mathrm{sc}}$. In particular, even though the radial set in Vasy's notes is at fiber infinity, it is a trivial change to adapt it to our setting. In contrast with the propagation estimates, the positivity near radial points is due entirely to the weight (in our case $\boldsymbol{\ell} + \frac{1}{2}$). This leads to two cases depending on the sign of the weight; we refer to these as "above threshold" and "below threshold" estimates. We first state the above threshold results, which bounds $H^{s,\boldsymbol{\ell}}_{\mathrm{sc}}$ norm of u near a component $\mathcal{R}'$ of the radial set of P_0.

Proposition 2.10 (Proposition 4.11 of [40], above threshold)**.** *Let $\mathcal{R}'$ denote any one of the four components of the radial set $\mathcal{R}$ of P_0. Suppose $\boldsymbol{\ell}$ is constant near $\mathcal{R}'$ and that $\boldsymbol{\ell} > -1/2$ there.*

Let $s, \ell' \in \mathbb{R}$, $\ell' > -1/2$, $\ell' \in [\boldsymbol{\ell} - 1/2, \boldsymbol{\ell})$. Then there are are $B, G \in {}^{\mathrm{sc}}\Psi^{0,0}$ with $\mathcal{R}' \subset \mathrm{Ell}(B)$ and $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$, such that: if $Gu \in H^{s-1/2,\ell'}_{\mathrm{sc}}$ and $GP_0u \in H^{s-1,\boldsymbol{\ell}+1}_{\mathrm{sc}}$, then $Bu \in H^{s,\boldsymbol{\ell}}_{\mathrm{sc}}$, and for all M, N there is $C > 0$ such that

$$\|Bu\|_{s,\boldsymbol{\ell}} \leq C \left(\|GP_0u\|_{s-1,\boldsymbol{\ell}+1} + \|Gu\|_{s-\frac{1}{2},\ell'} + \|u\|_{-N,-M} \right). \tag{2.30}$$

The operators B and G in the proposition are essentially microlocal cutoffs microsupported near $\mathcal{R}'$, with B microsupported in a compact subset of the elliptic set of G. Such operators can easily be constructed in the coordinates in (2.25) near $\mathcal{R}^f_+$ (and the analogous coordinates near the three other components of $\mathcal{R}$.) Indeed, for $\delta_1 > 2\delta_0 > 0$ sufficiently small and $c > 0$, we choose smooth bump functions $\phi_0, \phi_1, \chi_{>c}$ so that $\phi_i(s) = 1$ for $|s| < \delta_i$, $\phi_i(s) = 0$ for $|s| > 2\delta_i$, $i = 1, 2$, and $\chi_{>c}(s) = 1$ for $s \geq c$ and $\chi_{>c}(s) = 0$ for $s \leq c/2$. With these choices, we define:

$$B = \mathrm{Op}_L(b), \quad b = \chi_{>m-\delta_0}(1/\rho)\phi_0(x)\phi_0(|w|)\phi_0(1 - |\mu|^2 - m^2\rho^2),$$

with a similar construction for G:

$$G = \mathrm{Op}_L(g), \quad g = \chi_{>m-\delta_1}(1/\rho)\phi_1(x)\phi_1(|w|)\phi_1(1 - |\mu|^2 - m^2\rho^2).$$

Here, $\phi_i(1 - |\mu|^2 - m^2\rho^2)$ cuts off to the characteristic set, $\phi_i(x)\phi_i(|w|)$ to the radial set, and $\chi_{>m-\delta_i}(1/\rho) = \chi_{>m-\delta_i}(\tau)$ cuts of to the $\tau > 0$ portion of the radial set. It is not hard to check that, despite the presence of $1/\rho$, b and g are smooth scattering symbols microlocalized near the radial set. Note that

$$b, g \in S^{0,0}(\mathbb{R}^{n+1}),$$

as they are smooth functions on ${}^{\mathrm{sc}}\overline{T}^*X$. Indeed, they are supported in a set in which all the coordinates in (x, w, ρ, μ) are bounded, and thus they form smooth coordinates on ${}^{\mathrm{sc}}\overline{T}^*X$ in which x, ρ are boundary defining functions. In particular,

$$bg = b.$$

The proof of such an estimate follows from a commutator argument with a commutant Q^*Q with $Q = \mathrm{Op}_L(q)$,

$$q = \chi_{>m-\delta_1}\left(\frac{1}{\rho}\right)\phi_0(x)\phi_0(|w|^2)\phi_0(1-|\mu|^2-m^2\rho^2)x^{-\boldsymbol{\ell}-\frac{1}{2}}\rho^{-s+\frac{1}{2}}.$$

The commutator $\frac{i}{2}[P_0, Q^*Q]$ then has principal symbol

$$qH_pq = \frac{x}{\rho}\left((\boldsymbol{\ell}+\frac{1}{2}) - x\frac{\phi_0'(x)}{\phi_0(x)} - |w|^2\frac{\phi_0'(|w|^2)}{\phi_0(|w|^2)}\right)q^2. \tag{2.31}$$

As $\boldsymbol{\ell} > -\frac{1}{2}$, the first two terms have the same sign, which explains the absence of a term $\|Eu\|$ on the right side of the estimate. The other term can be absorbed into the first one if δ_0 is sufficiently small by observing that $\delta_0^2|w|^2 < 4\delta_0^2$ on the support of ϕ_0'. The remaining terms in the estimate arise as before.

We now localize the proposition to neighborhoods of particular closed subsets of $\mathcal{R}'$, allowing us later to combine them with more specialized estimates. In particular, we localize to $\mathcal{R}_+^f \cap \{|y| > 0\}$. Indeed, as $\mu = 0$ is the only point in $\mathcal{R}_+^f$ lying above $y = 0$, we use symbols as in the global radial point estimate together with an additional localizer in μ. In particular, we replace b with $b' = \chi_{>c_0}(|\mu|)b$ with a similar definition for $g' = \chi_{>c_1}(|\mu|)g$. Then, provided $c_i > 2\delta_i$, we find that

$$\text{ess-supp}(b) \subset \{|y| \geq c_0' - 2\delta_0\}, \quad \text{ess-supp}(g) \subset \{|y| \geq c_1' - 2\delta_1\}.$$

We assume moreover that $2c_1' < c_0'$ so that

$$b'g' = b' \quad \text{and} \quad \mathrm{WF}'(B') \subset \mathrm{Ell}(G').$$

We therefore have a family of cutoffs localized near the radial set and away from $\{y = 0\}$. For any $c > 0$, we can choose these operators so that

$$\mathcal{R}_+^f \cap \{|y| > c\} \subset \mathrm{Ell}(B') \quad \text{and} \quad \{y = 0\} \cap \mathrm{WF}'(G) = \varnothing.$$

From the proposition above and considerations using these microlocalizers on $\mathcal{R}$, we have the following:

Proposition 2.11 (Localized above threshold estimate)**.** *Let $\mathcal{R}'$ denote any one of the four components of the radial set $\mathcal{R}$ of P_0. Suppose $\boldsymbol{\ell}$ is constant near $\mathcal{R}'$ and that $\boldsymbol{\ell} > -1/2$ there.*

Let $s, \ell', M, N \in \mathbb{R}$, $\ell' > -1/2$ and $B', G' \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$ as above. If $G'u \in H_{\mathrm{sc}}^{-N,\ell'}(X)$ and $G'P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$, then $B'u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ Moreover, there is a constant $C > 0$ depending on $M, N, \boldsymbol{\ell}, \ell', s$ so that

$$\|B'u\|_{s,\boldsymbol{\ell}} \leq C\left(\|G'P_0u\|_{s-1,\boldsymbol{\ell}+1} + \|G'u\|_{-N,\ell'} + \|u\|_{-N,-M}\right).$$

Proof. We first show that the estimates in Proposition 2.10 hold with the B and G replaced by B' and G'.

Thus, we assume we are given $\boldsymbol{\ell}, \ell', s$ as Proposition 2.10 and, for G' as in the current proposition, that $G'u \in H_{\mathrm{sc}}^{s-1/2,\ell'}$ and $G'P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}-1}$. We wish to deduce that $B'u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$. Taking G as in Proposition 2.10, note that, if $\tilde{Q}$ a Fourier localizer to $|\mu > c|$ e.g.,

$$\tilde{Q} := \mathrm{Op}_L(\chi_{>c}(|\mu|)\phi(1-|\mu|^2-m^2\rho^2)) \in {}^{\mathrm{sc}}\Psi^{0,0},$$

for ϕ a bump function supported near 0 and $\chi_{>c}$ a localizer to $|\mu| \geq c$, then

$$G\tilde{Q}u \in H_{\mathrm{sc}}^{s-1/2,\ell'}.$$

Indeed, for $c > 0$ and $\delta_1 > 0$ sufficiently small, $\mathrm{WF}'(G\tilde{Q}) \subset \mathrm{Ell}\, G'$.

As for $GP_0\tilde{Q}u$, we have that $GP_0\tilde{Q}u = G\tilde{Q}P_0u + G[P_0, \tilde{Q}]u$, and $G[P_0, \tilde{Q}]u$ can be made lower order than expected because

$$\mathrm{WF}'([P_0, \tilde{Q}]) \cap \mathrm{WF}'(G) = \varnothing,$$

which follows directly from the form of the Hamilton vector field (2.26) and the definition of $\tilde{Q}$.[3] Hence,

$$G[P_0, \tilde{Q}] \in {}^{\mathrm{sc}}\Psi^{0,-2}.$$

Thus we have, again taking c, δ_1 sufficiently small,

$$\|GP_0\tilde{Q}u\|_{s-1,\boldsymbol{\ell}+1} \leq C\left(\|G'P_0u\|_{s-1,\boldsymbol{\ell}+1} + \|G'u\|_{s-1,\ell'} + \|u\|_{-N,-M}\right). \tag{2.32}$$

Thus, the hypotheses of Proposition 2.10 apply to $\tilde{Q}u$, and we obtain (2.30) for $u = \tilde{Q}u$ for B as in that estimate. But then there is B' with

$$\mathrm{WF}'(B') \subset \mathrm{Ell}(B\tilde{Q}),$$

obtained simply by taking $c > 0$ sufficiently large and δ_0 sufficiently small. We may thus use

$$\|B'u\|_{s,\boldsymbol{\ell}} \leq C\left(\|B\tilde{Q}u\|_{s,\boldsymbol{\ell}} + \|u\|_{-N,-M}\right)$$

in combination with (2.30) and (2.32) to give (2.30) for B' and G'. In other words, if $G'u \in H_{\mathrm{sc}}^{s-1/2,\ell'}$, $G'P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$, then $B'u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ together with the estimate

$$\|B'u\|_{s,\boldsymbol{\ell}} \leq C\left(\|G'P_0u\|_{s-1,\boldsymbol{\ell}+1} + \|G'u\|_{s-1/2\ell'} + \|u\|_{-N,-M}\right).$$

A standard argument using induction on s in half-integer steps then finishes the proposition. $\square$

We similarly have the below threshold and localized below threshold estimates. Note the presence of an additional term on the right side owing to the sign change in equation (2.31).

Proposition 2.12 (Proposition 4.11 of [40])**.** *Let $\mathcal{R}'$ denote any one of the four components of the radial set $\mathcal{R}$ of P_0. Suppose $\boldsymbol{\ell}$ is constant near $\mathcal{R}'$ and that $\boldsymbol{\ell} < -1/2$ there.*

Let $s \in \mathbb{R}$ and $B, E, G \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$ be such that $\mathrm{WF}'(B) \setminus \mathcal{R}'$ is controlled along ${}^{\mathrm{sc}}H_p$ by $\mathrm{Ell}(E)$ through $\mathrm{Ell}(G)$. If $Eu \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$, $GP_0u \in H_{\mathrm{sc}}^{s-,\boldsymbol{\ell}+1}$, and $Gu \in H_{\mathrm{sc}}^{s-\frac{1}{2},\boldsymbol{\ell}-\frac{1}{2}}$, then $Bu \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$ and, for any $M, N \in \mathbb{R}$, there is a constant C so that

$$\|Bu\|_{s,\boldsymbol{\ell}} \leq C\left(\|Eu\|_{s,\boldsymbol{\ell}} + \|GP_0u\|_{s-1,\boldsymbol{\ell}+1} + \|Gu\|_{s-1/2,\boldsymbol{\ell}-1/2} + \|u\|_{-N,-M}\right).$$

For the localized version of the below threshold estimate, we also introduce $e' = \chi_{>c_1}(|\mu|)e$ in addition to the definitions of b' and g' given for the localized above threshold estimate. The proof is essentially the same as the proof of Proposition 2.11.

Proposition 2.13 (Localized below threshold estimate)**.** *Let $\mathcal{R}'$ be any one of the four components of the radial set $\mathcal{R}$ of P_0. Suppose $\boldsymbol{\ell}$ is constant near $\mathcal{R}'$ and that $\boldsymbol{\ell} < -1/2$ there.*

Let $s, M, N \in \mathbb{R}$ and let $B', E', G' \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$ be as above. If $E'u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$, $G'P_0u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}+1}$, then $B'u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}$, and for any $M, N \in \mathbb{R}$, there is a constant C so that

$$\|B'u\|_{s,\boldsymbol{\ell}} \leq C\left(\|E'u\|_{s,\boldsymbol{\ell}} + \|G'P_0u\|_{s-1,\boldsymbol{\ell}+1} + \|u\|_{-N,-M}\right).$$

[3] In fact, in this case $[P_0, \tilde{Q}] \equiv 0$. In the case where P_0 is perturbed by a lower order scattering operator only the wavefront set containment holds.

2.8. **Weight functions, Fredholm estimates, and propagators.** We now describe in more detail how to combine the estimates of the previous sections to obtain Fredholm estimates for P_0. We pass to a microlocal partition of unity subordinate to a cover of ${}^{\mathrm{sc}}\overline{T}^*X$ by open neighborhoods of the components of the radial set $\mathcal{R}$, the characteristic set $\operatorname{Char} P_0$, and then the elliptic set. On each of these neighborhoods, we appeal to the estimates of previous sections. Away from the characteristic set, we appeal to elliptic estimates. Near the characteristic set, the above threshold estimate propagates regularity from a source component of the radial set to a small neighborhood of it; we then use the propagation estimates to conclude regularity in a neighborhood of a sink component, which is then propagated into the radial set by the below threshold estimate.

As noted earlier, it is impossible to satisfy simultaneously the conditions for the above threshold and below threshold estimates with a constant weight, so we appeal to variable weights. Because the radial estimates have no threshold conditions in the regularity order s, there is no need to allow variable order regularity. It is only the spacetime weight $\boldsymbol{\ell}$ that must vary, and the conditions it must satisfy are summarized in the following definition:

Definition 2.14. Let $\boldsymbol{\ell} \in C^\infty({}^{\mathrm{sc}}\overline{T}^*X)$. We call $\boldsymbol{\ell}$ admissible if $\boldsymbol{\ell}$ is monotone along the Hamiltonian flow within each component of the characteristic set and constant near the components of $\mathcal{R}$. Moreover, we say $\boldsymbol{\ell}$ is:

(1) *forward* if $\boldsymbol{\ell} > -1/2$ on $\mathcal{R}_+^p \cup \mathcal{R}_-^p$ and $\boldsymbol{\ell} < -1/2$ on $\mathcal{R}_+^f \cup \mathcal{R}_-^f$,
(2) *backward* if $\boldsymbol{\ell} < -1/2$ on $\mathcal{R}_+^p \cup \mathcal{R}_-^p$ and $\boldsymbol{\ell} > -1/2$ on $\mathcal{R}_+^f \cup \mathcal{R}_-^f$,
(3) *Feynman* if $\boldsymbol{\ell} < -1/2$ on $\mathcal{R}_+^f \cup \mathcal{R}_-^p$ and $\boldsymbol{\ell} > -1/2$ on $\mathcal{R}_+^p \cup \mathcal{R}_-^f$, and
(4) *anti-Feynman* if $\boldsymbol{\ell} > -1/2$ on $\mathcal{R}_+^f \cup \mathcal{R}_-^p$ and $\boldsymbol{\ell} < -1/2$ on $\mathcal{R}_+^p \cup \mathcal{R}_-^f$.

We write forward weights as $\boldsymbol{\ell}_+$ and backward weights as $\boldsymbol{\ell}_-$.

Note that the four types of weight functions described here correspond to the four distinguished parametrices of Duistermaat–Hörmander [8]. We encode which propagator we are considering by selecting an appropriate weight for the function spaces.

In particular, a forward (resp. backward) weight function decreases (resp. increases) as t increases, while a Feynman (resp. anti-Feynman) weight function decreases (resp. increases) along the global Hamiltonian flow. Forward and backward weight functions distinguish the causal (i.e., forward and backward) propagators.

For the causal propagators, the weights can be taken to be functions on spacetime (i.e., independent of τ, ζ), while the Feynman and anti-Feynman weights must be genuinely pseudodifferential. We focus now on the causal propagators, but the construction in the Feynman and anti-Feynman settings follows similar lines (though with a less explicit weight).

To construct the causal weights, we seek a function on spacetime that has the desired monotonicity and is equal to $-1/2 \pm \epsilon$ at $\mathcal{R}^{p/f}$. We therefore employ the function

$$\phi_t = \frac{t}{\langle t, z\rangle}$$

from the proof of Proposition 2.5. Indeed, we show there that ϕ_t is monotone increasing along the $\tau > 0$ component of the flow and decreasing on the $\tau < 0$ component. For any $\epsilon, \delta > 0$, we then let $f : [-1, 1] \to \mathbb{R}$ be any smooth, non-increasing function with $f(s) = -1/2 + \epsilon$ for $s < -1/\sqrt{2} + \delta$ and $f(s) = -1/2 - \epsilon$ for $s > 1/\sqrt{2} - \delta$ and set

$$\boldsymbol{\ell}(t, z) = f(\phi_t).$$

With this definition, $\boldsymbol{\ell}$ is a forward weight function, and $-1-\boldsymbol{\ell}$ is a backward weight function.

We now introduce some notation for the function spaces on which we expect P_0 to be Fredholm. To simplify matters, we focus on the the construction leading to the forward propagator, but the same argument shows that the other three choices of weights lead to Fredholm estimates. We let $\boldsymbol{\ell}_+$ denote a forward weight function, so that $-1-\boldsymbol{\ell}_+$ is a backward weight function. We recall that

$$\mathcal{X}^{s,\boldsymbol{\ell}_+} = \left\{ u \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}_+} \colon P_0 u \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}_++1} \right\}, \quad \mathcal{Y}^{s,\boldsymbol{\ell}_+} = H_{\mathrm{sc}}^{s,\boldsymbol{\ell}_+}$$

and $\mathcal{Y}^{s,\boldsymbol{\ell}_+}$ is equipped with the norm of $H_{\mathrm{sc}}^{s,\boldsymbol{\ell}_+}$, whereas

$$\|u\|_{\mathcal{X}^{s,\boldsymbol{\ell}_+}}^2 = \|u\|_{s,\boldsymbol{\ell}_+}^2 + \|P_0\|_{s-1,\boldsymbol{\ell}_++1}^2 .$$

Although we do not need this fact here, the space $\mathcal{X}^{s,\boldsymbol{\ell}_+}$ depends only on the principal symbol of P_0 and operators with the same principal symbol induce equivalent norms.

As $P_0 : \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_++1}$ is continuous, showing that it is Fredholm therefore reduces to the following two estimates:

$$\begin{aligned} \|u\|_{s,\boldsymbol{\ell}_+} &\leq C \left(\|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{-N,-M} \right), \\ \|u\|_{1-s,-1-\boldsymbol{\ell}_+} &\leq C \left(\|P_0 u\|_{-s,-\boldsymbol{\ell}_+} + \|u\|_{-N',-M'} \right), \end{aligned} \tag{2.33}$$

for some M, M', N, N' are such that the inclusions $H_{\mathrm{sc}}^{s,\boldsymbol{\ell}_+} \hookrightarrow H_{\mathrm{sc}}^{-N,-M}$ and $H_{\mathrm{sc}}^{1-s,-1-\boldsymbol{\ell}_+} \hookrightarrow H_{\mathrm{sc}}^{-N',-M'}$ are compact.

We take now an open cover O_1, O_2, O_3, O_4 of ${}^{\mathrm{sc}}\overline{T}^* X$ so that:

1. $\mathcal{R}_\pm^p \subset O_1 \subset \{\boldsymbol{\ell}_+ = -1/2 + \epsilon\}$,
2. $\mathcal{R}_\pm^f \subset O_2 \subset \{\boldsymbol{\ell}_+ = -1/2 - \epsilon\}$,
3. $\operatorname{Char} P_0 \subset O_1 \cup O_2 \cup O_3$,
4. O_3 is controlled along ${}^{\mathrm{sc}}H_p$ by O_1,
5. $O_2 \setminus \mathcal{R}$ is controlled along ${}^{\mathrm{sc}}H_p$ by O_3, and
6. $O_4 \subset \operatorname{Ell}(P_0)$.

Now we take a microlocal partition of unity $\mathrm{Id} = B_1 + B_2 + B_3 + B_4$, $B_i \in {}^{\mathrm{sc}}\Psi^{0,0}(X)$ with $\mathrm{WF}'(B_i) \subset O_i$.

If $u \in \mathcal{X}^{s,\boldsymbol{\ell}_+}$, then, by assumption, the above threshold estimate applies to u near $\mathcal{R}_\pm^p$ and so, by Proposition 2.10 (with $G = \mathrm{Id}$),

$$\|B_1 u\|_{s,\boldsymbol{\ell}_+} \leq C \left(\|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{s-1/2,\boldsymbol{\ell}'} \right), \tag{2.34}$$

where $\boldsymbol{\ell}' < \boldsymbol{\ell}_+$ and $-1/2 < \boldsymbol{\ell}' < \boldsymbol{\ell}_+$ on O_1. Now, as O_3 is controlled by O_1, Proposition 2.6 tells us

$$\|B_3 u\|_{s,\boldsymbol{\ell}_+} \leq C \left(\|B_1 u\|_{s,\boldsymbol{\ell}_+} + \|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{s-1/2,\boldsymbol{\ell}'} \right). \tag{2.35}$$

The hypotheses for Proposition 2.12 are now fulfilled and so we obtain

$$\|B_2 u\|_{s,\boldsymbol{\ell}_+} \leq C \left(\|B_3 u\|_{s,\boldsymbol{\ell}_+} + \|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{s-1/2,\boldsymbol{\ell}'} \right). \tag{2.36}$$

Because $\mathrm{WF}'(B_4) \subset \operatorname{Ell}(P_0)$, the elliptic estimates of Proposition 2.2 tell us

$$\|B_4 u\|_{s,\boldsymbol{\ell}_+} \leq C \left(\|P_0 u\|_{s-2,\boldsymbol{\ell}_+} + \|u\|_{s-1/2\boldsymbol{\ell}'} \right) \leq C \left(\|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{s-1/2,\boldsymbol{\ell}'} \right). \tag{2.37}$$

Because $\mathrm{Id} = B_1 + B_2 + B_3 + B_4$, we then have the estimate

$$\|u\|_{s,\boldsymbol{\ell}_+} \leq C \left(\|P_0 u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{s-1/2,\boldsymbol{\ell}'} \right),$$

and the inclusion $\mathcal{X}^{s,\boldsymbol{\ell}_+} \hookrightarrow H_{\mathrm{sc}}^{s-1/2,\boldsymbol{\ell}'}$ is compact.

To obtain the estimate

$$\|u\|_{1-s,-1-\boldsymbol{\ell}_+} \leq C\left(\|P_0^*\|_{-s,-\boldsymbol{\ell}_+} + \|u\|_{-s-1/2,\boldsymbol{\ell}'}\right),$$

we use the same chain of estimates, but with the roles of $\mathcal{R}^p$ and $\mathcal{R}^f$ exchanged. In other words, we propagate regularity from $\mathcal{R}^f$ to $\mathcal{R}^p$ along the Hamiltonian flow in $\mathrm{Char}(P_0)$. Here $\boldsymbol{\ell}'$ must be chosen analogously, i.e., $\boldsymbol{\ell}' < -1 - \boldsymbol{\ell}_+$ must also be greater than the threshold $-1/2$ near $\mathcal{R}^f$. By the standard argument of iterating by $1/2$ differential orders we replace the $-1/2$ on the right by an arbitrary differential order $-N$ and deduce (2.33). Note that the formulation of the estimate in (2.33) with the lower order error term on the right hand side follows from bounding the $\boldsymbol{\ell}'$ term by a small factor times that $\boldsymbol{\ell}$ term on the left; specifically, using that for any $M > 0$ and $\epsilon > 0$ there is $C(\epsilon) > 0$ such that $x^{-\boldsymbol{\ell}'} \leq C(\epsilon)x^M + \epsilon x^{\boldsymbol{\ell}}$, based off which we have, for any $N \in \mathbb{R}$, a $C > 0$ such that

$$\|u\|_{H_{\mathrm{sc}}^{-N,\boldsymbol{\ell}'}} \leq C\|u\|_{H_{\mathrm{sc}}^{-N,-M}} + \epsilon\|u\|_{H_{\mathrm{sc}}^{-N,-\boldsymbol{\ell}}}, \tag{2.38}$$

so for ϵ sufficiently small the last term can be absorbed onto the left hand side.

With these estimates, we have nearly proved Theorem 2.1:.

Proof of Theorem 2.1. The estimates above in equation (2.33) show that P_0 is Fredholm between the stated spaces. Indeed, the estimates directly imply that the operators have closed range and finite dimensional kernel. That they have finite dimensional cokernel follows from the identification of the cokernel with the kernel of the operator on the adjoint space, which for the forward weight $\boldsymbol{\ell}_+$ is the backward weight $-1-\boldsymbol{\ell}_+$.

The fact that P_0 is invertible on the stated space then follows if its kernel and cokernel are zero. This claim follows from the energy/Grönwall argument given below in the proof of Theorem 8.2. The cokernel of the forward problem is the kernel of a corresponding backward problem, and vice versa, so this completes the proof.

For the statement that the inverse is the forward propagator, let $f \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}_++1}$ for any forward weight $\boldsymbol{\ell}_+$, and assume that for some $T \in \mathbb{R}$,

$$\operatorname{supp} f \subset \{t \geq T\}.$$

Then the inverse mapping of (2.2) applied to f gives a solution u_+ to $P_0 u_+ = f$ with $u_+ \in H_{\mathrm{sc}}^{s,\boldsymbol{\ell}_+}$. Then u_+ satisfies the above threshold condition near the past radial sets, and just as with elements in the kernel, u_+ is Schwartz near past causal infinity. The same energy/Grönwall argument then shows that u_+ is identically zero in $t \leq T$, i.e. u_+ is the forward solution. □

2.9. **Scattering perturbations.** We finally observe that the estimates above are all symbolic in nature, so the same estimates hold for any operator with the same principal symbol and sub-principal symbol as P_0. In fact, we require only that these agree at $\partial X \times \overline{\mathbb{R}^n}$ as long as the underlying Lorentzian metric is non-trapping.

We therefore consider a "potential"

$$V \in \langle t,z\rangle^{-1}\operatorname{Diff}_{\mathrm{sc}}^1(X)$$

with $V - V^* \in \langle t,z\rangle^{-2}\operatorname{Diff}_{\mathrm{sc}}^0(X)$. We also allow the differential part

$$D_t^2 - D_z \cdot D_z =: \Box_{g_0}$$

of P_0 to be replaced by the wave operator for an asymptotically Minkowski metric g satisfying

$$g - g_0 \in S^{-2}\left(\mathbb{R}^{n+1}; \mathrm{Sym}^{0,2}\right),$$

where

$$g_0 = dt^2 - \sum_{j=1}^n dz_j^2.$$

We further demand that g be non-trapping, i.e., that all null geodesics of g approach $\mathscr{I}^\pm$ (or equivalently compressed null infinity $S^\pm$) in both directions along the flow. Then, with $\Box_g = -\sqrt{|g|}^{-1}\partial_i \sqrt{|g|} g^{ij} \partial_j$, we treat the operator

$$P_{g,V} := \Box_g - m^2 - V.$$

We observe that P_0 and $P_{g,V}$ have the same principal symbol at spatial infinity $\partial X \times \overline{\mathbb{R}^n}$ and the structure of the Hamiltonian flow is the same as that given in Proposition 2.5. Additionally, we note that $P_{g,V} - P_{g,V}^* \in \langle t, z\rangle^{-2} \operatorname{Diff}^0_{\mathrm{sc}}(X)$, so that the same propagation and radial point estimates over the boundary hold. The propagation estimates at fiber infinity hold with the same proof, and so we in fact have the following theorem:

Theorem 2.15. *If g is a non-trapping asymptotically Minkowski metric in the sense that*

$$g - g_0 \in S^{-2}\left(\mathbb{R}^{n+1}; \mathrm{Sym}^{0,2}\right),$$

and $V \in \langle t, z\rangle^{-1} \operatorname{Diff}^1_{\mathrm{sc}}(X)$ satisfies $V - V^* \in \langle t, z\rangle^{-2} \operatorname{Diff}^0(X)$, then Theorem 2.1 holds for the operator $P_{g,V}$. In other words, if $s \in \mathbb{R}$ and $\boldsymbol{\ell}_+$ is a forward weight, then

$$P_{g,V} : \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_+ + 1}$$

is an isomorphism and its inverse is the forward propagator. The same is true when $\boldsymbol{\ell}_+$ is replaced by a backward weight $\boldsymbol{\ell}_-$, in which case the inverse is the backward propagator.

3. Asymptotically static potentials

In this section we describe the spacetime geometry adaptations for the operator P_V. For static potentials $V = V(z)$, the operator P_V is not a scattering operator in the sense of Section 2; *V fails to be smooth on the north pole and south pole of X as described below.* We therefore pass to the minimal resolution of X on which V is smooth and thereby recognize our operator as an element of Vasy's many-body scattering calculus.

3.1. The resolution of X.

Recall that, for the spacetime compactification $X = \overline{\mathbb{R}^{n+1}_{t,z}}$, the coordinates $x = 1/t, y = z/t$ are valid up to the boundary ∂X in any region in which (x, y) are bounded. This clarifies the failure of smoothness of V at the point which $x = 0, y = 0$, which we refer to as the "north pole" and denote NP. Indeed, there, even for Schwartz potentials $V \in \mathcal{S}(\mathbb{R}^n)$, we see that $V(z) = V(y/x)$ fails even to be continuous at NP. The same goes for the "south pole", SP, where the coordinates $x = -1/t$, $y = z/t$ vanish. (We only write $x = -1/t$ when working near SP.) We set $C = \{\mathrm{NP}, \mathrm{SP}\} \subset \partial X$. Thus

$$C = \partial X \cap \overline{\{z = 0\}}. \tag{3.1}$$

and

$$\mathrm{NP} = \partial X \cap \overline{\{z = 0\}} \cap \overline{\{t > 0\}}, \quad \mathrm{SP} = \partial X \cap \overline{\{z = 0\}} \cap \overline{\{t > 0\}}.$$

On the other hand, $V \in \mathcal{S}(\mathbb{R}^n)$ is smooth on $X \setminus C$.

We are therefore led to consider the blow-up of C in X equipped with the blow-down map

$$\beta_C : [X; C] \longrightarrow X\,. \tag{3.2}$$

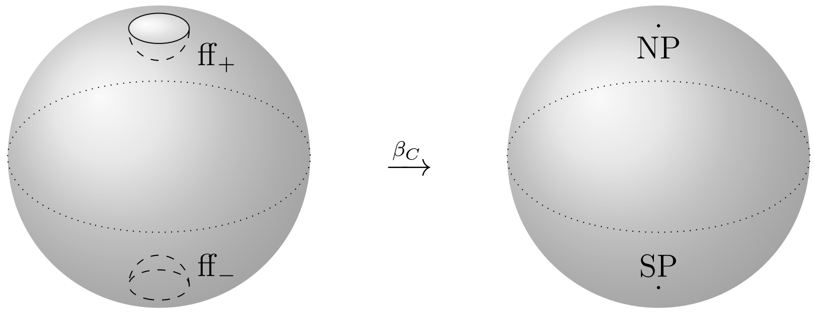

FIGURE 2. The blow-down map $\beta_C : [X;C] \to X$ of the 3sc-single space.

Static potentials that are Schwartz functions $V = V(z) \in \mathcal{S}(\mathbb{R}^n)$ are smooth on $[X;C]$.[4]

The space $[X;C]$ is the 3sc-*single space* and is a manifold with corners possessing three boundary hypersurfaces,

$$\begin{aligned} \mathrm{ff}_+ &:= \beta_C^*(\mathrm{NP})\,, \\ \mathrm{ff}_- &:= \beta_C^*(\mathrm{SP})\,, \\ \mathrm{mf} &:= \beta_C^*(\partial X)\,. \end{aligned}$$

Note that $\mathrm{ff}_\pm$ has interior which is isomorphic to $\mathbb{R}^n_z$. In what follows, we typically restrict our attention to ff_+ and write $\mathrm{ff} = \mathrm{ff}_+$.

Coordinates on $[X;C]$ can be understood in terms of those on X as follows. First off, coordinates near the boundary of X can be taken near any point $p \in \partial X$ near which $t \to \infty$, to be

$$x = 1/t, \quad y = z/t$$

where here x is a boundary defining function (bdf) of ∂X. Then in the region $|z| < C$, $t > 0$, we have the simple coordinates

$$x = 1/t, \quad z$$

with x being a bdf of ff in this region, while near the intersection of $\mathrm{ff} \cap \mathrm{mf}$, near any point there is at least one z_k for which $\hat{x} = 1/z_k$ is a bdf for ff and there one can use

$$\hat{x} = 1/z_k, \quad \hat{Y}_j = z_j/z_k (j \neq k), \quad y_k = z_k/t, \tag{3.3}$$

and here y_k is a bdf of mf.

3.2. 3sc**-differential operators.** Differential operators in the 3sc-calculus are given by

$$\mathrm{Diff}^m_{3\mathrm{sc}}(X) := \mathrm{Diff}^m_{\mathrm{sc}}(X) \otimes_{C^\infty(X)} C^\infty([X;C])\,. \tag{3.4}$$

More concretely, $L \in \mathrm{Diff}^m_{3\mathrm{sc}}$, if

$$L = \sum_{|\alpha|+k \leq m} a_{k,\alpha} D_t^k D_z^\alpha\,, \tag{3.5}$$

where the coefficients $a_{k,\alpha}$ are smooth on the blown up space $[X;C]$.

Using the x, z coordinates, it is easy to see that

$$P_V = D_t^2 - (\Delta + m^2 + V(z)) \in \mathrm{Diff}^2_{3\mathrm{sc}}(X). \tag{3.6}$$

[4]More general V must have a classical symbol expansion at infinity to be smooth on $[X;C]$.

On $[X; C]$, general differential operators in the 3sc-calculus are simply

$$\mathrm{Diff}^{m,l}_{3\mathrm{sc}} = \langle t, z\rangle^l \,\mathrm{Diff}^m_{3\mathrm{sc}},$$

which is to say we do not distinguish, in the notation, the rates of spatial decay or blow up of coefficients at the faces ff and mf. Thus, in particular

$$\mathrm{Diff}^m_{3\mathrm{sc}} = \mathrm{Diff}^{m,0}_{3\mathrm{sc}}\,.$$

The principal symbol of the operator P_V will have three components. Two of them are inherited directly from the scattering calculus; they are localized away from C, i.e. to the region of X where V is smooth, and are essentially the same as the scattering principal symbol. The other component of the principal symbol, defined only above C, is the "indicial operator", and is essentially the time Fourier transform of P_V restricted to C.

3.3. 3sc-**geometry.** We now aim to describe the domain of the principal symbol of a 3sc-differential operator.

The (radial compactification) of the three-body scattering cotangent bundle is, by definition, the pullback bundle

$$ {}^{3\mathrm{sc}}\overline{T}^*[X; C] \coloneqq \beta_C^* \,{}^{\mathrm{sc}}\overline{T}^* X. \tag{3.7}$$

Since we are working over $\mathbb{R}^{n+1}$, these bundles are trivial and thus there is a natural decomposition

$$ {}^{3\mathrm{sc}}\overline{T}^*[X; C] = [\overline{\mathbb{R}^{n+1}}; C] \times \overline{\mathbb{R}^{n+1}}.$$

The manifold with corners ${}^{3\mathrm{sc}}\overline{T}^*[X; C]$ has three boundary hypersurfaces, namely

$$ {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X; C], \quad {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X; C], \text{ and } {}^{3\mathrm{sc}}S^*[X; C]\,, \tag{3.8}$$

where the latter is the "fiber" boundary of ${}^{3\mathrm{sc}}\overline{T}^* X$, i.e. the ${}^{3\mathrm{sc}}S^*[X; C] = [\overline{\mathbb{R}^{n+1}}; C] \times \partial\overline{\mathbb{R}^{n+1}}$.

We denote the corresponding boundary defining functions as

$$\rho_{\mathrm{ff}}\,, \quad \rho_{\mathrm{mf}}\,, \quad \rho_{\mathrm{fib}}\,. \tag{3.9}$$

Moreover, we also define the total boundary defining function for the spacetime boundary $\rho_\infty \coloneqq \rho_{\mathrm{ff}}\rho_{\mathrm{mf}}$. As discussed in Remark 2.4, such boundary defining functions, which are used in particular in re-weighting of symbols and distributions below, can be multiplied by positive function without effecting the estimates in which they are used. One global choice of ρ_{ff} would be $\langle t, z\rangle^{-1}$, but (again see Remark 2.4), it is more convenient to assume that

$$\rho_\infty = x = 1/t$$

in regions $0 \le x \le C, |y| \le C$. Similarly, a global choice of ρ_{fib} would be $\langle \tau, \zeta\rangle^{-1}$, but it is more convenient for us to choose ρ_{fib} so that

$$\rho_{\mathrm{fib}} = \rho = 1/\tau$$

in regions $0 \le \rho \le C, |\mu| \le C$. We can take ρ_{mf} globally as

$$\rho_{\mathrm{mf}} = \langle z\rangle^{-1}.$$

In our analysis using the three-body calculus, near ff we will prove estimates which are global on the $\{\tau = \mathit{const.}\}$ subsets of phase space. Thus, following the notation of [36], we define the vector bundle

$$W^\perp \coloneqq \mathrm{span}_{\mathbb{R}}\Big(\frac{dx}{x^2}\Big) \subset {}^{\mathrm{sc}}T^*_C X. \tag{3.10}$$

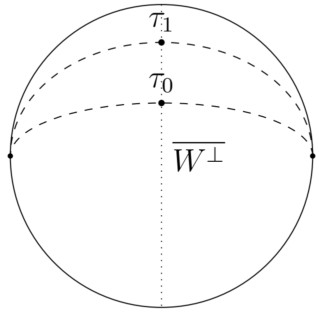

FIGURE 3. The dashed lines are the sets $\beta_C^{-1}\mathrm{cl}(\pi^{-1}(\tau))$ for $\tau = \tau_0, \tau_1$. The common boundary of these sets is the fiber equator, fibeq.

Thus $W^\perp$ is parametrized by $\tau \in \mathbb{R}$ corresponding to the form $-\tau(dx/x^2)$, both at NP and SP, recalling that near SP we write $x = -1/t$. Formally, $W^\perp \subset {}^{\mathrm{sc}}T^*_C X$ is defined as the annihilator of the subset $W \subset {}^{\mathrm{sc}}T_C X$ consisting of vectors arising from vector fields $U = xU'$ with $U' \in \mathcal{V}_{\mathrm{b}}(X)$ with U' tangent to C, where $\mathcal{V}_{\mathrm{b}}(X)$ are the vector fields tangent to the boundary. But the simple definition above suffices. In particular,

$$W^\perp = \mathbb{R} \sqcup \mathbb{R}, \tag{3.11}$$

where both copies of $\mathbb{R}$ are parametrized by τ, one over NP and the other over SP.

The orthogonal projection

$$\pi \colon {}^{3\mathrm{sc}}T^*_C X \longrightarrow W^\perp, \tag{3.12}$$

with action $(\tau, \zeta) \mapsto \tau$, does *not* extend smoothly to ${}^{\mathrm{sc}}\overline{T}^*_C X$, but the closures of the fibers $\pi^{-1}(\tau)$ are smooth submanifolds with common boundary, as depicted in Figure 3. Pulling back to the three-body space, we have

$$\beta_C^{-1}\mathrm{cl}(\pi^{-1}(\tau)) = \mathrm{ff} \times \mathrm{cl}\{(\tau, \zeta)\} \subset \mathrm{ff} \times \overline{\mathbb{R}^{n+1}} \subset {}^{3\mathrm{sc}}T^* X,$$

and thus

$$\beta_C^{-1}\mathrm{cl}(\pi^{-1}(\tau)) \simeq \mathrm{ff} \times \overline{\mathbb{R}^n} \simeq {}^{\mathrm{sc}}\overline{T}^* \mathbb{R}^n, \tag{3.13}$$

where the equivalence is induced simply by dropping the τ. This will arise below in the analysis of indicial operators of three-body operators, which will be τ-dependent families of scattering operators on $\mathbb{R}^n$ for which $\pm 1/\tau$ acts as semiclassical parameter.

The coincidence of the boundaries of the $\{\tau = \mathit{const.}\}$ sets in ${}^{\mathrm{sc}}\overline{T}^*_C X$ will be significant, below, as it will force the scattering symbols of indicial operators to be constant in τ. We will denote this common boundary, the fiber equator, by fibeq, so, for fixed $\tau_0 \in \mathbb{R}$,

$$\mathrm{fibeq} = \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}} \cap \overline{\{\tau = \tau_0\}} \subset \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}. \tag{3.14}$$

This set is independent of τ_0 and depicted in Figure 3. See Section 4.2 for further discussion of the role played by fibeq in quantization and as the locus of definition of the fiber symbol of the indicial operator.

The principal symbol of an element of $\mathrm{Diff}^m_{3\mathrm{sc}}(X)$ therefore has three pieces corresponding to the three boundary components of ${}^{3\mathrm{sc}}\overline{T}[X; C]$. The first two pieces are the two components of scattering principal symbol, (2.11) and (2.12), while the new piece is called the indicial operator. For differential operators in our context, it is the Fourier transform in time with the dual variable τ (dual to t) entering as a parameter. This piece of the principal symbol then yields a parametrized family of scattering differential operators on $\mathrm{ff} \simeq \overline{\mathbb{R}^n}$.

Indeed, the components $a_{k,\alpha}$ of a differential operator restrict to ff to smooth functions, and the indicial operator for L as in equation (3.5) is given by

$$\hat{N}_{\mathrm{ff}}(L)(\tau) = \widehat{L}_{\mathrm{ff}}(\tau) := \sum_{|\alpha|+k\leq m} (a_{k,\alpha}|_{\mathrm{ff}})\tau^k D_z^\alpha \,. \tag{3.15}$$

Note that if L were a scattering operator, then the coefficients $a_{k,\alpha}|_{\mathrm{ff}}$ would be constant at ff. Thus, when $L \in \mathrm{Diff}_{\mathrm{sc}}$, $\hat{N}_{\mathrm{ff}}(L)(\tau)$ is translation invariant in z, and can be identified via the Fourier tranform with its total symbol.

More generally, $\hat{N}_{\mathrm{ff}}(L)(\tau)$ must be regarded as operator-valued function of τ. In particular, as we will see in our discussion of ellipticity below, global ellipticity of a 3sc-operator is equivalent to invertibility of *all three components* of the principal symbol, in particular it requires invertibility of the indicial operator for each τ.

The 3sc-principal symbol of P_V is

$$j_{3\mathrm{sc}}(P_V) = \Big(\sigma_{3\mathrm{sc}}(P_V), \hat{N}_{\mathrm{mf}}(P_V), \hat{N}_{\mathrm{ff}}(P_V)(\tau)\Big)\,.$$

Here, $\sigma_{3\mathrm{sc}}(P_V)$ is the "standard" interior principal symbol, i.e. the scattering fiber principal symbol, $\hat{N}_{\mathrm{mf}}(P_V)$ is the spacetime boundary symbol, and $\hat{N}_{\mathrm{ff}}(P_V)(\tau)$ is the indicial operator. Concretely, $\sigma_{3\mathrm{sc}}(P_V)$ and $\hat{N}_{\mathrm{mf}}(P_V)$ are given by $(\tau^2 - |\zeta|^2 - m^2)\rho_{\mathrm{fib}}^2$ restricted to momentum infinity and mf, respectively. Convenient expressions for these are

$$\sigma_{3\mathrm{sc}}(P_V)(t, z, \tau, \zeta) = \tau^2 - |\zeta|^2,$$

while for $\hat{N}_{\mathrm{mf}}(P_V)$, a function defined at the spacetime boundary away from C,

$$\hat{N}_{\mathrm{mf}}(P_V)(y, \tau, \zeta) = \tau^2 - |\zeta|^2 - m^2.$$

These functions "match" in the sense that if you multiply by ρ_{fib}^2 they are equal as $\langle t, z\rangle \to \infty, \langle \tau, \zeta\rangle \to \infty$. For our static potential $V = V(z)$, the **indicial operator** is simply

$$\hat{N}_{\mathrm{ff}}(P_V)(\tau) = \tau^2 - (\Delta_z + m^2 + V(z)). \tag{3.16}$$

The absence of V in the second component of the symbol is a consequence of the assumption that V decays in z.

In fact, as described below in Section 4.2, the components of this principal symbol must satisfy matching conditions on the intersection of their domains. This is straightforward for the components $\sigma_{3\mathrm{sc}}(P_V)$ and $\hat{N}_{\mathrm{mf}}(P_V)$, which are restriction of *functions*, so matching simply means the values of their restrictions are equal. The component $\hat{N}_{\mathrm{ff}}(P_V)$, however, is a family of scattering operators, and the matching condition for P_V is that its scattering symbols are exactly the restrictions of $\sigma_{3\mathrm{sc}}(P_V)$ and $\hat{N}_{\mathrm{mf}}(P_V)$ to the boundary components of phase space over the front face. This is clarified for a general 3sc-operator below. This a generalization of the matching condition that holds for scattering symbols (2.11)-(2.12).

3.4. **Asymptotically static potentials V and generalizations.** Our results apply both to potential functions and to more general perturbations V. For potential functions, we treat smooth functions V which approach fixed spatial functions as $t \to \pm\infty$.

We assume in general that

$$V = V(t, z) \in \rho_{\mathrm{mf}} C^\infty([X; C]; \mathbb{R}). \tag{3.17}$$

Thus (recalling $\rho_{\mathrm{mf}} = \langle z \rangle^{-1}$) these are potential functions which are smooth on the whole of $[X; C]$, vanish at least to order one as $z \to \infty$ (i.e. at mf) and have, in general, non-vanishing limits at ff.

In particular, we have, in regions with $|z/t| < C$,

$$V = V_+(z) + V'(t,z), \quad V' = O(1/t) \text{ as } t \to +\infty$$

where V_+ is a symbol of order -1, meaning

$$|\partial_z^\alpha V_+(z)| \lesssim_\alpha \langle z \rangle^{-|\alpha|}$$

and V' satisfies the following estimates

$$|\partial_t^k \partial_z^\alpha V'(t,z)| \lesssim_{k,\alpha} \langle t, z \rangle^{-1-k} \langle z \rangle^{-|\alpha|} \,,$$

The estimates here are equivalent to the containment (3.17) if you assume in addition that the V_+ and V' has asymptotic expansions in $1/t$ and $\langle z \rangle^{-1}$. A simple example of such a potential is a $V = V(t,z) \in C^\infty(\mathbb{R}_t; \mathcal{S}(\mathbb{R}^n))$, with $V(t,z) \equiv V_\pm(z) \in \mathcal{S}(\mathbb{R}^n)$ for $\pm t \gg 0$.

More generally, for complex-valued V, we assume assume that $V \in \rho_{\mathrm{mf}} C^\infty([X; C]; \mathbb{C})$ with

$$2 \operatorname{Im} V = V - \overline{V} \in \langle t, z \rangle^{-2} C^\infty([X; C]; \mathbb{C}).$$

This ensures in particular that the "subprincipal symbol" does not influence the threshold weight of $-1/2$ that appears in Theorem 1.1. In terms of the asymptotic decomposition above, this means that

$$V = V_+(z) + V'(t,z) + i \operatorname{Im} V,$$

where V_+ and V' are as above and

$$|\partial_t^k \partial_z^\alpha \operatorname{Im} V(t,z)| \lesssim_{k,\alpha} \langle t, z \rangle^{-2-k} \langle z \rangle^{-|\alpha|} \,.$$

In fact, our results more generally, including to differential and pseudodifferential V with suitable regularity and decay hypotheses, and with an assumption on the subprincipal symbol $V - V^*$ which generalizes the assumption on $\operatorname{Im} V$ above.

4. The three-body scattering calculus for Klein-Gordon

In this section we will recall the key features of the 3sc-calculus adapted to our setting. As noted above, our P_V is not a scattering operator on the whole of $\mathbb{R}^{n+1}_{t,z}$ in the sense of Melrose, but it is a 3sc-operator in the sense of Vasy. We will now describe what that means in detail, what the 3sc-operators and their features look like in our setting, and the basic properties that inform our analysis.

Generally, the 3sc-calculus introduced by Vasy [36], is defined with respect to data which includes both the total space and collision planes. In contrast with the general case, in our setting, we have only the $\{z = z_0\}$ collision planes (really lines) for $z_0 \in \mathbb{R}^n$ fixed, corresponding to the points C on the boundary at infinity.

We can summarize the main features of this introductory section to the 3sc-calculus as follows.

- The 3sc-operators in our setting are the natural pseudodifferential generalization of the 3sc-differential operators defined above, exactly in the standard sense that they are quantizations of symbols whose behaviour is analogous to the behavior of the total symbols of elements of $\mathrm{Diff}^{m,l}_{3\mathrm{sc}}$. These pseudodifferential operators are denotes ${}^{3\mathrm{sc}}\Psi^{m,\ell}$ when they have differential order m and spacetime weight order ℓ.

- As with P_V (3.16), the principal symbols of these operators have three components. The first two, like the scattering symbols, are local, i.e. they are functions. The third is the **indicial operator**, which has both a $t = +\infty$ and a $t = -\infty$ component itself, is defined only from data that lives over C, and like the symbol of P_V for static V in (3.16), is a family of operators parametrized by τ, the dual variable to t. Given $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$ and focusing, as we do below, on NP, the indicial operator is denoted $\hat{N}_{\mathrm{ff},l}(A) = \hat{N}_{\mathrm{ff},l}(A)(\tau)$. *The principal symbol is multiplicative,* in the sense that the indicial operator of the composition of two 3sc-operator is the composition of the indicial operators. The behavior of $\hat{N}_{\mathrm{ff},l}(A)(\tau)$ in τ is semiclassical as $\tau \to \pm\infty$.
- Ellipticity is still appropriately construed as invertibility of the principal symbol. Namely, global ellipticity is exactly the assumption that the first two components of the principal symbol (which are functions) are non-zero, and that the indicial operator is semiclassically elliptic and invertible for all τ.
- There is also an appropriate notion of microlocal 3sc-ellipticity, and corresponding 3sc-microlocal elliptic estimates.
- As in Vasy's treatment, we use the standard L^2-based Sobolev spaces on $\mathbb{R}^{n+1}$. (In particular, we do not introduce spaces specifically adapted to the 3sc setting.) Thus, our Sobolev spaces are exactly those used in Section 2 above. All the estimates we state and prove are for distributions in the scattering (i.e. standard!) weighted Sobolev spaces $H^{m,l}_{\mathrm{sc}}(\mathbb{R}^{n+1}_{t,z})$.

Moreover, in this section we confront perhaps this most striking difference between 3sc and sc operators, namely that general commutators of 3sc operators do not have the standard loss of one order in comparison to composition. Namely, the analogue of (2.21) for 3sc-operators fails. Indeed, a static, potential function $V(z) \in \mathcal{S}(\mathbb{R}^n)$ lies in ${}^{3\mathrm{sc}}\Psi^{0,0}(X)$ (because it does not decay in time) and the partial derivative in a spatial coordinate $\partial_{z_j} \in \mathrm{Diff}^1_{\mathrm{sc}}(X) \subset {}^{3\mathrm{sc}}\Psi^{1,0}(X)$ while the commutator $[\partial_{z_j}, V] = \partial_{z_j} V$ has no additional time decay, and thus one concludes only that $[\partial_{z_j}, V] \in \mathrm{Diff}^{0,0}_{3\mathrm{sc}}(X) \subset {}^{3\mathrm{sc}}\Psi^{0,0}(X)$. Regarding commutators, we make the additional point.

- As in the examples just discussed, if $A \in {}^{3\mathrm{sc}}\Psi^{m_1,l_1}$ and $B \in {}^{3\mathrm{sc}}\Psi^{m_2,l_2}$, then in general $[A, B] \in {}^{3\mathrm{sc}}\Psi^{m_1+m_2-1,l_1+l_2}$. If either one of A or B satisfies a "centrality condition", which is essentially that $\hat{N}_{\mathrm{ff},l}(A)(\tau)$ is a function (as opposed to an operator) then in fact $[A, B] \in {}^{3\mathrm{sc}}\Psi^{m_1+m_2-1,l_1+l_2-1}$. In case this centrality condition is satisfied, a formula for the principal symbol of $[A, B]$ is given. See Section 4.3.

Our treatment is simplified in comparison to the general case of the 3sc-calculus, in which more complex arrangements of collision planes are treated. In particular, we present a simplified commutator formula for the indicial operator below. More important is the fact that in our setting, as we treat a hyperbolic operator, the characteristic set extends to fiber infinity; we must therefore discuss the behavior of the indicial operator for large $\pm\tau$, and we do so below in our treatment of the indicial operator as a semiclassical scattering operator.

4.1. 3sc-**pseudodifferential operators.** The space of (classical) three body scattering symbols is

$$ {}^{3\mathrm{sc}}S^{m,l}(X; C) \; = \rho_{\mathrm{fib}}^{-l} \rho_\infty^{-m} C^\infty({}^{\mathrm{sc}}\overline{T}^*[X; C]) \tag{4.1} $$

and the space of (classical) 3sc-pseudodifferential operators of order m, r is

$$ {}^{3\mathrm{sc}}\Psi^{m,r} = \mathrm{Op}_L(\S^{m,r}_{3\mathrm{sc}}) \tag{4.2} $$

by [36, Lemma 3.5], where, concretely, for $a \in S^{m,r}_{3\mathrm{sc}}$,

$$ A = \mathrm{Op}_L(a) = \int e^{i(t-t')\tau + i(z-z')\cdot\zeta} a(x, z, \tau, \zeta) d\tau d\zeta $$

(This is taken as the definition of ${}^{3\mathrm{sc}}\Psi^{m,r}$, whereas in the cited paper the space of PsiDO's is defined as an appropriate set of integral on the three-scattering double space; we do not use this latter definition directly in our work.) In the original (t, z) coordinates, a 3sc-symbol is a smooth function on $\mathbb{R}^{n+1}_{t,z} \times \mathbb{R}^{n+1}_{\vartheta}$ such that each seminorm

$$ \|a\|_{3\mathrm{sc},M} = \sum_{k+|\alpha|+|\beta| \le M} \sup_{t,z,\vartheta} \langle t, z\rangle^k \langle z\rangle^{-r+|\alpha|} \langle \vartheta\rangle^{-m+|\beta|} \, |\partial_t^k \partial_z^\alpha \partial_\vartheta^\beta a(t, z, \vartheta)| $$

is finite.

Remark 4.1. The spaces of classical scattering and 3sc-symbols can also be defined by reference to the compactified spaces. Namely, ${}^{\mathrm{sc}}S^{0,0}(\mathbb{R}^{n+1})$ are exactly the smooth functions on the compactified scattering cotangent bundle ${}^{\mathrm{sc}}\overline{T}^* X$ while ${}^{3\mathrm{sc}}S^{0,0}(\mathbb{R}^{n+1})$ is exactly the smooth functions on the compactified 3sc-cotangent bundle ${}^{3\mathrm{sc}}\overline{T}^* X$.

To define the indicial operator, we need a lemma to the effect that a 3sc-operator defines an operator on ff via extension and restriction to the boundary. From [36, Corollary 3.4] we have the mapping properties for smooth functions (the differential order plays no significant role here):

Lemma 4.2. *If $A \in {}^{3\mathrm{sc}}\Psi^{m,r}(X)$, then*

$$ A : \dot{C}^\infty(X) \longrightarrow \dot{C}^\infty(X) $$

and

$$ A : \rho^k_{\mathrm{mf}} \rho^{k'}_{\mathrm{ff}} C^\infty([X; C]) \longrightarrow \rho^{k+r}_{\mathrm{mf}} \rho^{k'+r}_{\mathrm{ff}} C^\infty([X; C]) \, . $$

We also note that sc-operators are 3sc-operators

$$ {}^{\mathrm{sc}}\Psi^{m,r}(X) \subset {}^{3\mathrm{sc}}\Psi^{m,r}(X) \, , $$

since the scattering symbol estimates imply in particular the $\|\bullet\|_{3\mathrm{sc},M}$ estimates above. (All our operators are assumed classical throughout.)

We recall the main boundedness property for 3sc-operations, which is proven using the standard square-root trick.

Proposition 4.3 ([36], Cor. 8.2)**.** *For $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{m,r}$ and $s, \ell \in \mathbb{R}$,*

$$ A \colon H^{m+s,r+\ell}_{\mathrm{sc}}(\mathbb{R}^{n+1}) \longrightarrow H^{s,\ell}_{\mathrm{sc}}(\mathbb{R}^{n+1}). \tag{4.3} $$

is bounded, with operator norm bounded by a seminorm $\|a\|_M$.

Having introduced the total symbols of 3sc-operators, we now consider the appropriate definition of their principal symbols. Indeed, recall from Section 2.2 that for a scattering operator, $\mathrm{Op}_L(a) = A \in {}^{\mathrm{sc}}\Psi^{m,l}(X)$, the two components of the principal symbol $j_{\mathrm{sc},m,r}(A) = (\sigma_{\mathrm{sc},m,r}(A), \hat{N}_{\mathrm{sc},m,r}(A))$ are the restriction sof the function $\langle \tau, \zeta\rangle^{-m} \langle t, z\rangle^{-r} a$ to the two components of $\partial^{3\mathrm{sc}}\overline{T}^* X$.

In principal, one could define the 3sc-principal symbol of $\mathrm{Op}_L(a) = A \in {}^{3\mathrm{sc}}\Psi^{m,r}$ to be the restriction of a to the four components of the boundary of ${}^{3\mathrm{sc}}\overline{T}^*X$. However, such a definition would have the limitation that it would not be multiplicative over ff, a limitation which is addressed by using, instead of the front face restriction, the family of indicial operators mentioned above and defined in Section 4.4 below.

When $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{0,0}$, the symbols over mf and at fiber infinity are simply the boundary component restrictions

$$\hat{N}_{\mathrm{mf}}(A) = a|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]} \in C^\infty({}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C])$$

and

$$\sigma_{3\mathrm{sc}}(A) = a|_{{}^{3\mathrm{sc}}S^*[X;C]} \in C^\infty({}^{3\mathrm{sc}}S^*[X;C]),$$

on in terms of the boundary defining functions in (3.9), they are the restrictions on $\rho_{\mathrm{mf}} = 0$ and $\rho_{\mathrm{fib}} = 0$, respectively. This is the 3sc generalization of the two scattering principal symbol components (2.11)-(2.12). When $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{m,r}$ and a needs to be re-weighted to have boundary restrictions, then we use the boundary defining functions most suitable to our analysis. We proceed to formalize this now.

4.2. **The principal symbol and the indicial operator.** Generalizing the indicial operator $\hat{N}_{\mathrm{ff}}(P_V)$ (3.16), we describe now how $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$ has an indicial operator $\hat{N}_{\mathrm{ff},l}(A) = \hat{N}_{\mathrm{ff},l}(A)(\tau)$, a smoothly-parametrized family of scattering operators on ff. The indicial family is one of the three components of the total 3sc-principal symbol of A, together with the (rescaled) restrictions of the symbol of A to mf and fiber infinity [36, Chap. 6]. The main goals of this subsection are three-fold:

- to define the indicial family $\hat{N}_{\mathrm{ff},l}(A)$ and show that it is the quantization of a boundary restriction of the symbol of A,
- to prove that $\hat{N}_{\mathrm{ff},l}(A)$ is in fact a semiclassical scattering operator, and to characterize those semiclassical scattering operators which arise as indicial operators of ${}^{3\mathrm{sc}}\Psi$ operators,
- to recall the 3sc-principal symbol $j_{3\mathrm{sc}}$, show that it is multiplicative, and define left-quantization on appropriate principal symbols. This is Proposition 4.8 below.

The properties of the indicial operator are simplified in our case, in comparison with the general 3sc-calculus, due C being zero-dimensional, and correspondingly $\overline{W^\perp}$ being one-dimensional. Thus, in particular $\hat{N}_{\mathrm{ff},l}(A)$ depends on a single parameter. What we develop here is the precise sense in the operator is semiclassical, and the precise sense in which a special class of semiclassical scattering operators over ff can be quantized into 3sc-operators.

Also note that, while we typically work near NP for brevity, there are in fact two components of the indicial operator corresponding to the two boundary hypersurfaces $\mathrm{ff} = \mathrm{ff}_+, \mathrm{ff}_-$, and the indicial operator in fact is two separate families, one over NP and one over SP. We often elide separate discussion of the SP component since the details are nearly identical to that of the NP component.

Beginning with $A \in {}^{3\mathrm{sc}}\Psi^{m,0}$, we assume here and throughout that $A = \mathrm{Op}_L(a)$ for a classical symbol a. By [36, Cor. 3.4], if $u \in C^\infty([X;C])$ then $Au \in C^\infty([X;C])$. If $f \in C^\infty([X;C])$ and $u|_{\mathrm{ff}} = f$, we define

$$A_\partial f := (Au)\,|_{\partial[X;C]},$$

which is well defined by Lemma 4.2. One can check that this definition is independent of the extension u of f. In particular, identifying ff with $\mathbb{R}^n$, if $f \in \mathcal{S}(\mathbb{R}^n)$, then the operator

$$A_{\mathrm{ff}} f \coloneqq A_\partial(f)|_{\mathrm{ff}}$$

is well defined.

Choosing $\tau_0 \in \mathbb{R}$, by [36, Lemma 6.1],

$$e^{i\tau_0/x} A e^{-i\tau_0/x} \in {}^{3\mathrm{sc}}\Psi^{0,0}$$

which gives the definition of the indicial operator at τ_0, namely:

$$\hat{N}_{\mathrm{ff}}(A)(\tau_0) \coloneqq \left(e^{i\tau_0/x} A e^{-i\tau_0/x}\right)_{\mathrm{ff}}. \tag{4.4}$$

(In particular, $A_{\mathrm{ff}} = \hat{A}_{\mathrm{ff}}(0)$.) We write

$$\widehat{A}_{\mathrm{ff}}(\tau_0) = \hat{N}_{\mathrm{ff}}(A)(\tau_0) \tag{4.5}$$

Given $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$ we have $x^l A \in {}^{3\mathrm{sc}}\Psi^{m,0}$ and define

$$\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0) = \widehat{x^l A}_{\mathrm{ff}}(\tau_0), \tag{4.6}$$

so in particular for $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$, $\widehat{A}_{\mathrm{ff},0} = \widehat{A}_{\mathrm{ff}}$. We also write for short

$$\hat{A}_{\mathrm{ff},\ell} \coloneqq \hat{N}_{\mathrm{ff},\ell}(A).$$

Given $A = \mathrm{Op}_L(a)$, we now derive a simple formula for $\hat{A}_{\mathrm{ff},\ell}$ in terms of a. Indeed in Lemma 4.4 we see that $\hat{A}_{\mathrm{ff},\ell}$ is the left quantization in z of an appropriate boundary restriction of a to ff.

In [36, p. 23] it is shown that A_{ff} can be obtained by restriction of the integral kernel of $A \in {}^{3\mathrm{sc}}\Psi^{m,0}$ to a boundary hypersurface sf_C of the scattering triple space $X^2_{3\mathrm{sc}}$; although we do not discuss this space in detail here, we note that, in particular, in the coordinates,

$$x = 1/t, \quad S = (x - x')/x^2, \quad z, \quad Y = (y - y')/x,$$

the kernel of A is conormal to the diagonal $S = 0, Y = 0$ smoothly down to $x = 0$.

$$\begin{aligned} Au &= \int K_A(t, z, t', z') u(t', z') dt' dz' \\ &= \int K_A(t, z, S/(1 - xS), \frac{z - Y}{1 - xS}) u\left(\frac{S}{1 - xS}, \frac{z - Y}{1 - xS}\right) \frac{1}{(1 - xS)^{n+2}} dS dY \end{aligned} \tag{4.7}$$

Then defining

$$A(x, S, z, Y) = (1 - xS)^{-(n+2)} K_A(t, z, \frac{S}{1 - xS}, \frac{z - Y}{1 - xS})$$

on $x = 0$, we obtain, as in [36, Eq. 4.15]

$$Au|_{x=0} = A_{\mathrm{ff}} u|_{x=0} = \int A(0, S, z, Y) u(0, z - Y) dS dY,$$

which is to say that, as a Schwartz kernel,

$$A_{\mathrm{ff}} = \int A(0, S, z, Y) dS$$

The functions

$$x, \quad \tilde{t} = t - t', \quad z, \quad \tilde{z} = z - z'$$

are also coordinates near sf_C. (These are the W_a, W^a functions from [37].) At $x = 0$ we have $S = \tilde{t}$ and $\tilde{z} = Y$. If $\mathrm{Op}_L(a) = A$, then, with $a_{\mathrm{ff}}(z, \tau, \zeta) = a|_{{}^{3\mathrm{sc}}\overline{T}_{\mathrm{ff}}X}$ and $a_0(x, z, \tau, \zeta) = a(1/x, z, \tau, \zeta)$

$$\begin{aligned} K_A &= \int e^{i\tilde{t}\tau + i\tilde{z}\cdot\zeta} a(t, z, \tau, \zeta) d\tau d\zeta \\ &= \int e^{i\tilde{t}\tau + i\tilde{z}\cdot\zeta} a_0(x, z, \tau, \zeta) d\tau d\zeta \\ \implies A_{\mathrm{ff}} &= \int e^{i\tilde{t}\tau + i\tilde{z}\cdot\zeta} a_{\mathrm{ff}}(z, \tau, \zeta) d\tau d\zeta dt = \int e^{i\tilde{z}\cdot\zeta} a_{\mathrm{ff}}(z, 0, \zeta) d\zeta. \end{aligned} \tag{4.8}$$

Conjugating by $e^{i\tau/x}$ we then obtain

Lemma 4.4. *Let $A = \mathrm{Op}_L(a)$ for $a \in S^{m,l}_{3\mathrm{sc}}$, and denote the (rescaled) restriction of a to ff by*

$$a_{\mathrm{ff}} \coloneqq (x^l a)|_{{}^{3\mathrm{sc}}\overline{T}_{\mathrm{ff}}X}, \tag{4.9}$$

Then $\hat{A}_{\mathrm{ff}}(\tau_0) \in {}^{\mathrm{sc}}\Psi^{m,0}(\mathbb{R}^n_z)$ and has kernel

$$\widehat{A}_{\mathrm{ff}}(\tau_0) = \int e^{i(z-z')\cdot\zeta} a_{\mathrm{ff}}(z, \tau_0, \zeta) d\zeta = \mathrm{Op}_{L,z}(a_{\mathrm{ff}}(z, \tau_0, \zeta)), \tag{4.10}$$

Proof. We simply refer to [36, Chap. 6] where it is shown the formal computations given above agree with the value of the operator. $\square$

We need to compute the next term in the expansion of $Au(x, z)$ as $x \to 0$. In terms of the distribution $A(x, S, z, Y)$ above, this is straightforward. Indeed, we have [36, Lemma 7.1]

$$Au = A_{\mathrm{ff}}u + x\left((\partial_x A)_{\mathrm{ff}}u + A_{\mathrm{ff}}\partial_x u - D_\tau \hat{A}_{\mathrm{ff}}(0)(z \cdot \partial_z u)\right) + O(x^2). \tag{4.11}$$

Here $\partial_x A$ denotes the derivative of $A(x, S, z, Y)$ (in these coordinates) restricted to $x = 0$, and (4.11) is derived by simply applying ∂_x to (4.7) and using the chain rule. The last two terms are computed from boundary values of A, and thus from the restricted symbol a_{ff}. The term $\partial_x A$, however, depends on the interior values of the symbol.

If $u \in x^{-r}C^\infty_{\mathrm{ff}}$, we set

$$\begin{aligned} u_{\mathrm{ff}} &\coloneqq (x^r u)|_{\mathrm{ff}}\,, \\ u'_{\mathrm{ff}} &\coloneqq \partial_x(x^r u)|_{\mathrm{ff}}\,, \end{aligned}$$

so that

$$u = x^{-r}\left(u_{\mathrm{ff}} + x u'_{\mathrm{ff}} + O(x^2)\right)$$

as $x \to 0$.

We will also give a more intuitive form in terms of symbols. If $A = \mathrm{Op}(a)$, where $a = a(x, z, x', z', \tau, \zeta)$

$$A'_{\mathrm{ff},\ell} \coloneqq \mathrm{Op}(x^\ell(\partial_x a + \partial_{x'} a)|_{x=x'=0})\,.$$

In particular, if $A = \mathrm{Op}_L(a)$, then $A'_{\mathrm{ff},\ell} = \mathrm{Op}_L(x^\ell \partial_x a|_{x=0})$.

Then, the expansion takes a more natural form.

Lemma 4.5. *Let $A = \operatorname{Op}(a)$ and $u \in x^{-r}C^\infty_{\mathrm{ff}}$, then*

$$x^{\ell+r}(Au)(x,z) = A_{\mathrm{ff}}u_{\mathrm{ff}} + x\left(A'_{\mathrm{ff}}u_{\mathrm{ff}} + (r-1)D_\tau \hat{A}_{\mathrm{ff}}(0)u_{\mathrm{ff}} + A_{\mathrm{ff}}u'_{\mathrm{ff}}\right) + O(x^2)\,.$$

Proof. We use different coordinates than Vasy, but one can check that the result is the same. Specifically, we take

$$x, z, S = \frac{1}{x} - \frac{1}{x'}\,, Y = z - \frac{z'}{1 - xS}\,,$$

and we calculate that

$$x' = \frac{x}{1-xS}\,, z - z' = Y + xS(z - Y)\,, z' = (1-xS)(z-Y)\,.$$

We can write $(Au)(x,z)$ in these coordinates as

$$x^{\ell+r}(Au)(x,z) = \int \tilde{A}(x,z,S,Y)\tilde{u}\left(\frac{x}{1-xS}, (1-xS)(z-Y)\right)\,dS\,dY\,,$$

where

$$\tilde{u}(x',z') = (x')^r u(x',z')\,,$$

and the integral kernel $\tilde{A}$ is given by

$$\tilde{A} = (2\pi)^{-(n+1)}\int e^{i\varphi}x^\ell(1-xS)^r a\left(x, z, \frac{x}{1-xS}, (1-xS)(z-Y), \tau, \zeta\right)\,d\tau\,d\zeta$$

with $\varphi = S\tau + Y\zeta + xS(z-Y)\zeta$.

We also set

$$\tilde{a}(x,z,x',z',\tau,\zeta) \coloneqq x^\ell a(x,z,x',z',\tau,\zeta)\,.$$

We have that

$$\begin{aligned}
\tilde{u}(0,z') &= u_{\mathrm{ff}}(z')\,,\\
\partial_{x'}\tilde{u}(0,z') &= u'_{\mathrm{ff}}(z')
\end{aligned}$$

and

$$\tilde{A}(0,z,S,Y) = (2\pi)^{-(n+1)}\int e^{iS\tau+iY\zeta}\tilde{a}(0,z,0,z-Y,\tau,\zeta)\,d\tau\,d\zeta\,.$$

We calculate the derivative of Au as

$$\begin{aligned}
\partial_x(x^{\ell+r}Au)(0,z) &= \int \partial_x\tilde{A}(0,z,S,Y)\tilde{u}(0,z-Y)\,dS\,dY\\
&\quad + \int \tilde{A}(0,z,S,Y)(\partial_x - S(z-Y)\partial_z)\tilde{u}(0,z-Y)\,dS\,dY\,.
\end{aligned}$$

By a straight-forward calculation using integration by parts, we obtain

$$\begin{aligned}\partial_x(x^{\ell+r}Au)(0,z) &= \int e^{i(S\tau+Y\zeta)}\left((\partial_x+\partial_{x'})\tilde{a}(0,z,0,z-Y,\tau,\zeta)\right)\tilde{u}(0,z-Y)\,d\tau\,d\zeta\,dS\,dY \\ &\quad + (r-1)\int e^{i(S\tau+Y\zeta)}D_\tau\tilde{a}(0,z,0,z-Y,\tau,\zeta)\tilde{u}(0,z-Y)\,d\tau\,d\zeta\,dS\,dY \\ &\quad + \int e^{i(S\tau+Y\zeta)}\tilde{a}(0,z,0,z-Y,\tau,\zeta)\partial_x\tilde{u}(0,z-Y)\,d\tau\,d\zeta\,dS\,dY \\ &= A'_{\mathrm{ff}}u_{\mathrm{ff}} + (r-1)D_\tau\hat{A}_{\mathrm{ff}}(0)u_{\mathrm{ff}} + A_{\mathrm{ff}}u'_{\mathrm{ff}}\,.\end{aligned}$$

□

Let $Q = \mathrm{Op}_L(x^{-\ell}q)$ for $q \in {}^{\mathrm{sc}}S^{0,0}$. The first term in the expansion is given by

$$Q_{\mathrm{ff}} = \mathrm{Op}_L(q(0,0,0,\zeta))\,.$$

We have that $x^\ell\partial_x x^{-\ell}q(x,xz) = \partial_x q(x,xz) + z\partial_y q(x,xz) + O(x)$ and consequently

$$Q'_{\mathrm{ff},\ell} = \mathrm{Op}_L(\partial_x q(0,0,0,\zeta) + z\partial_y q(0,0,0,\zeta))\,.$$

Importantly, $Q'_{\mathrm{ff},\ell}$ is *not* a Fourier multiplier.

In the case of the free Klein–Gordon operator, we have for $u \in C^\infty_{\mathrm{ff}}$ that

$$P_0 u = -(\Delta+m^2)u_{\mathrm{ff}} - x(\Delta+m^2)u'_{\mathrm{ff}} + O(x^2)\,.$$

Using this development of the indicial operator, we can now define the **principal symbol of a** 3sc **operator**, essentially by putting the indicial operator together with the mf and fiber infinity parts of the scattering principal symbol. Concretely, for $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$, we have the two restrictions

$$\hat{N}_{\mathrm{mf},m,\ell}(A) = \langle\tau,\zeta\rangle^{-m}x^\ell a|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]} \in C^\infty({}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C])$$

and

$$\sigma_{3\mathrm{sc},m,\ell}(A) = \langle\tau,\zeta\rangle^{-m}x^\ell a|_{{}^{3\mathrm{sc}}S^*[X;C]} \in C^\infty({}^{3\mathrm{sc}}S^*[X;C])$$

and the τ-dependent family of operators

$$\hat{N}_{\mathrm{ff},\ell}(A) \in C^\infty(\mathbb{R}_\tau;{}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff})).$$

These three objects form the 3sc**-principal symbol:**

$$\begin{aligned} j_{3\mathrm{sc},m,\ell} : {}^{3\mathrm{sc}}\Psi^{m,\ell} &\longrightarrow C^\infty({}^{3\mathrm{sc}}\overline{T}_{\mathrm{mf}}X) \times C^\infty({}^{3\mathrm{sc}}S^*[X;C]) \times C^\infty(\mathbb{R}_\tau;{}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff})) \\ A &\mapsto \left(\sigma_{3\mathrm{sc},m,\ell}(A), \hat{N}_{\mathrm{mf},m,l}(A), \hat{N}_{\mathrm{ff},m,\ell}(A)\right)\end{aligned} \tag{4.12}$$

Note that we have dropped the inclusion of the ff_- component of the indicial operator as otherwise the notation becomes too cumbersome.

We note that $\hat{N}_{\mathrm{ff},\ell}(A)$ is not an arbitrary element in $C^\infty(\mathbb{R}_\tau;{}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff}))$. Indeed, if we define the space of symbols

$$S^{m,\ell}(\mathbb{R}^n_z;\mathbb{R}^{n+1}_{\tau,\zeta}) = \{a \in C^\infty(\mathbb{R}^n;\mathbb{R}^{n+1}) : |\langle z\rangle^{-l+|\alpha|}\langle\tau,\zeta\rangle^{-m+j+|\beta|}D^\alpha_z D^j_\tau D^\beta_\zeta a| < \infty\}$$

Then $a_{\mathrm{ff}} = x^{-\ell}a|_{\mathrm{ff}}$ is a classical symbol, i.e. in $S^{m,0}_{cl}(\mathbb{R}^n_z;\mathbb{R}^{n+1}_{\tau,\zeta}) = \langle\tau,\zeta\rangle^m C^\infty(\overline{\mathbb{R}^n_z}\times\overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ and

$$\hat{N}_{\mathrm{ff},\ell}(A) \in \mathrm{Op}_z(S^{m,0}_{cl}(\mathbb{R}^n_z;\mathbb{R}^{n+1}_{\tau,\zeta})) \subset C^\infty(\mathbb{R}_\tau;{}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff})).$$

In Section 4.4 below, we will discuss the consequences of the fact that the indicial operator $\hat{N}_{\mathrm{ff},\ell}(A)$ is in fact a semiclassical-scattering operator in $h = \pm 1/\tau$ as $\tau \to \pm\infty$. These are exactly the symbols, defined for $m, l, r \in \mathbb{R}$, by

(4.13)
$$S^{m,l,r}_{\mathrm{scl,sc},\pm 1/\tau}(\mathbb{R}^n) = \left\{ a \in C^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_\mu \times \mathbb{R}_\tau) : |D^\alpha_z D^\beta_\mu D^j_h a| \lesssim \langle \mu \rangle^{m-|\beta|} \langle z \rangle^{l-|\alpha|} h^{-r} \ \forall \alpha, \beta, j \right\},$$

where $h = 1/\langle \tau \rangle$. In our case we often write $S^{m,l,r}_{\mathrm{scl,sc},\pm 1/\tau}(\mathrm{ff})$ as the operators we consider are more naturally viewed as functions on $\mathrm{ff} = \overline{\mathbb{R}^n_z}$. Then their quantizations are

$$\Psi^{m,l,r}_{\mathrm{scl,sc},\pm 1/\tau} = \mathrm{Op}_{L,\mathrm{scl}}(S^{m,l,r}_{\mathrm{scl,sc},\pm 1/\tau}). \tag{4.14}$$

This definition admits an obvious generalization to the case that $\mathbb{R}^n$ is an arbitrary scattering manifold is straightforward. In Proposition 4.7 below we show that for $A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$, $\hat{N}_{\mathrm{ff},\ell}(A) \in \Psi^{m,0,m}_{\mathrm{scl,sc},\pm 1/\tau}$.

We note that, for $\mathrm{Op}_L(a) = A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$, the indicial operator $\hat{N}_{\mathrm{ff},\ell}(A)$ is equivalent to the data $a_{\mathrm{ff}} = x^{-\ell} a|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}X}$ (since $a_{\mathrm{ff}}(z, \tau, \zeta)$ is the left reduction of $\hat{N}_{\mathrm{ff},\ell}(A)(\tau)$). Thus $j_{3\mathrm{sc},m,\ell}(A)$, like the scattering symbol in Section 2, is still given by the restriction data of the (appropriately weighted) symbol. Thus, it is automatic that the three components of $j_{3\mathrm{sc},m,\ell}(A)$ satisfy *matching conditions* at the intersections of the boundary hypersurfaces of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, i.e. $\langle \tau, \zeta \rangle^{-m} a_{\mathrm{ff}}(z, \tau, \zeta)$, $\hat{N}_{\mathrm{mf},m,\ell}(a)$ and $\sigma_{3\mathrm{sc},m,\ell}(a)$ are equal on the restriction to their common boundaries (see (4.15).)

Conversely, we have the following proposition, which tells us that we can quantize a 3sc-principal symbol provided the appropriate matching conditions of the three symbol components are satisfied, where again we do not include the ff_- component.

Proposition 4.6. *Let*

$$(a_1, a_2, A_\tau) \in C^\infty({}^{3\mathrm{sc}}S^*[X;C]) \times C^\infty({}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}X) \times \Psi^{m,0,m}_{\mathrm{scl,sc},\pm 1/\tau}$$

and let $\mathrm{Op}_{L,z}(a_0) = A_\tau$, *i.e. let* $a_0(z, \tau, \zeta)$ *be the* τ*-dependent family of symbols quantizing* $A_\tau \in {}^{\mathrm{sc}}\Psi^{m,0}(\mathbb{R}^n)$. *Then there is* $A \in {}^{3\mathrm{sc}}\Psi^{m,0}(\mathbb{R}^{n+1})$ *with* $j_{3\mathrm{sc},m,0}(A) = (a_1, a_2, A_\tau)$ *if and only if* $\langle \tau, \zeta \rangle^{-m} a_0 \in C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ *and*

$$\begin{gathered} a_1|_{{}^{3\mathrm{sc}}S^*_{\mathrm{mf}}[X;C]} = a_2|_{{}^{3\mathrm{sc}}S^*_{\mathrm{mf}}[X;C]}, \quad \langle \tau, \zeta \rangle^{-m} a_0|_{{}^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X;C]} = a_1|_{{}^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X;C]} \\ \langle \tau, \zeta \rangle^{-m} a_0|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}\cap\mathrm{mf}}[X;C]} = a_2|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}\cap\mathrm{mf}}[X;C]} \end{gathered} \tag{4.15}$$

This follows from the simple fact that the agreement of the functions a_1, a_2 and $\langle \tau, \zeta \rangle^{-m} a_0$ implies the existence of a function a extending all three functions; then $A = \mathrm{Op}_L(\langle \tau, \zeta \rangle^m a)$ is the desired operator. Note that in this case, with a_{ff} in (4.9) that

$$a_0 = a_{\mathrm{ff}}.$$

We now discuss multiplicativity of $j_{3\mathrm{sc}}$, using the composition theorem [36, Proposition 5.2]:

Proposition 4.7. *Let* $A \in {}^{3\mathrm{sc}}\Psi^{m_1,\ell_1}(X)$ *and* $B \in {}^{3\mathrm{sc}}\Psi^{m_2,l_2}(X)$. *The composition is well-defined as an operator*

$$A \circ B \in {}^{3\mathrm{sc}}\Psi^{m_1+m_2,l_1+l_2}(X)$$

and

$$(A \circ B)_\partial = A_\partial B_\partial \,.$$

Applying the latter equation to the conjugated operators in (4.4), we see that the indicial operator of a composition is the composition of the indicial operators. Summarizing these considerations, we obtain the basic features of the 3sc-principal symbol.

Proposition 4.8. *The kernel of the mapping of the principal symbol mapping (4.12) is exactly* ${}^{3\mathrm{sc}}\Psi^{m-1,\ell-1}$*, while the image is the set of those* $(q_1, q_2, \{Q_\tau\})$ *such that, with* q_0 *the left reduction of* Q_τ*, we have* $\langle \tau, \zeta\rangle^{-m} q_0 \in C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ *and the matching condition (4.15) hold. Moreover, the mapping* $j_{3\mathrm{sc}}$ *is multiplicative (but not commutative): for* $A \in {}^{3\mathrm{sc}}\Psi^{m_1,\ell_1}$*,* $B \in {}^{3\mathrm{sc}}\Psi^{m_2,\ell_2}$

$$j_{3\mathrm{sc},m_1+m_2,\ell_1+\ell_2}(AB) = j_{3\mathrm{sc},m_1,\ell_1}(A) j_{3\mathrm{sc},m_2,\ell_2}(B)\,,$$

where the product denotes component wise composition (i.e. multiplication in the first two components).

Proof. Because this is merely a statement about restriction of the symbol to boundary components of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, the only parts of the proposition that require attention are: (1) the multiplicativity of the symbol, which follows from Proposition 4.7 and (2) the statement that the indicial family lies in $\Psi^m_{\mathrm{scl},\mathrm{sc},\pm 1/\tau}$. The latter is precisely the content of Section 4.4. □

Remark 4.9. These propositions clarify that we cannot quantize any family of semiclassical scattering operators to be the indicial family of a 3sc-operator. In fact, the left reduction $a_{\mathrm{ff}}(z, \tau, \zeta)$ of the indicial operator $\hat{N}_{\mathrm{ff}}(A)(\tau)$ is smooth on $\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$, its fiber symbol, being the restriction to fiber infinity on each $\{\tau = \mathit{const.}\}$ slice, is independent of τ, i.e. with fibeq as in (3.14),

$$\sigma_{\mathrm{sc}}\left(\hat{N}_{\mathrm{ff}}(A)_{\mathrm{ff}}(\tau)\right) \equiv \sigma_{3\mathrm{sc}}(A)|_{\mathrm{ff}\times\mathrm{fibeq}} \in C^\infty(\mathrm{ff} \times \mathrm{fibeq}).$$

Here we use the τ dependent weight

$$\sigma_{\mathrm{sc}}\left(\hat{N}_{\mathrm{ff}}(A)_{\mathrm{ff}}(\tau)\right) = \langle \tau, \zeta\rangle^{-m} a_{\mathrm{ff}}(z, \tau, \zeta)|_{\mathrm{ff}\times\partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}} \tag{4.16}$$

Lemma 4.10. *Let* $A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$*, then* $A^* \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$ *with*

$$j_{3\mathrm{sc}}(A^*) = j_{3\mathrm{sc}}(A)^*\,,$$

meaning that

$$\sigma_{3\mathrm{sc}}(A^*) = \overline{\sigma_{3\mathrm{sc}}(A)}\,,$$
$$\hat{N}_{\mathrm{mf}}(A^*) = \overline{\hat{N}_{\mathrm{mf}}(A)}\,,$$
$$\hat{N}_{\mathrm{ff}}(A^*) = \hat{N}_{\mathrm{ff}}(A)^*\,.$$

Proof. The fact that the adjoint is again a 3sc-operator is a simple consequence of the definition via the 3sc-double space in Vasy [36, Eq. 3.13] and the principal symbol can be easily calculated using quantized symbols. □

Remark 4.11. Finally, we remark that in the above definitions of principal symbols, it is possible to replace $\langle \tau, \zeta\rangle^{-1}$ by any smooth boundary defining function ρ_{fib} of the fiber boundary. We used $\langle \tau, \zeta\rangle^{-1}$ to be explicit but all factors of using a ρ_{fib} which is equal to $1/\tau$ near the characteristic set is also useful.

4.3. **Commutators.** For two 3sc-operators $A \in {}^{3\mathrm{sc}}\Psi^{m_1,\ell_1}, B \in {}^{3\mathrm{sc}}\Psi^{m_2,\ell_2}$ the commutator in general does not decrease in order, i.e. in general $[A,B] \notin {}^{3\mathrm{sc}}\Psi^{m_1+m_2-1,\ell_1+\ell_2-1}$. For example, if $A = D_z$ and $B = f(z)$ for some function $f \in S^0(\mathbb{R}^n_z)$, then $[A,B]$ is the multiplication operator by $D_z f$, which is a 3sc-operator of order $(0,0)$, since there is no decay in t.

The commutator drops in order exactly if the indicial operators commute,

$$[\hat{N}_{\mathrm{ff}}(A), \hat{N}_{\mathrm{ff}}(B)] = 0,$$

which is not the case in the previous example.

Let $A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$. We say that A is in the 3sc-centralizer,

$$A \in Z^{3\mathrm{sc}}\Psi^{m,\ell} \text{ if and only if } [A,B] \in {}^{3\mathrm{sc}}\Psi^{m+m'-1,\ell+\ell'-1}$$

for all $B \in {}^{3\mathrm{sc}}\Psi^{m',\ell'}$. This is equivalent to the condition that

$$\hat{N}_{\mathrm{ff},\ell}(A)(\tau) = f(\tau)\,\mathrm{Id}\,, \tag{4.17}$$

for some $f \in C^\infty(\overline{W^\perp})$ (cf. Vasy [36, Lemma 6.5]).

Now, we will calculate the indicial operator of the commutator $[A,B]$ in the case that $[\hat{N}_{\mathrm{ff}}(A), \hat{N}_{\mathrm{ff}}(B)] = 0$. For this we calculate the Taylor expansion of $[A,B]u$ at $x = 0$.

Lemma 4.12. *Let $A \in {}^{3\mathrm{sc}}\Psi^{m_1,\ell_1}, B \in {}^{3\mathrm{sc}}\Psi^{m_2,\ell_2}$ and $u \in C^\infty_{\mathrm{ff}}$, then*

$$\begin{aligned} x^{\ell_1+\ell_2}[A,B]u(x,z) = [A_{\mathrm{ff}}, B_{\mathrm{ff}}] &+ x\big([A'_{\mathrm{ff}} - D_\tau \hat{A}_{\mathrm{ff}}(0), B_{\mathrm{ff}}] + [A_{\mathrm{ff}}, B'_{\mathrm{ff}} - D_\tau \hat{A}_{\mathrm{ff}}(0)]\big)u_{\mathrm{ff}} \\ &+ x[A_{\mathrm{ff}}, B_{\mathrm{ff}}](\partial_x u)_{\mathrm{ff}} + O(x^2)\,. \end{aligned}$$

Proof. We use Lemma 4.5 to see that

$$\begin{aligned} x^{\ell_1+\ell_2}ABu(x,z) &= A_{\mathrm{ff}}B_{\mathrm{ff}}u_{\mathrm{ff}} + x\left(A'_{\mathrm{ff}}B_{\mathrm{ff}} + (\ell_2-1)D_\tau\hat{A}_{\mathrm{ff}}(0)B_{\mathrm{ff}} + A_{\mathrm{ff}}B'_{\mathrm{ff}} - A_{\mathrm{ff}}D_\tau\hat{B}_{\mathrm{ff}}(0)\right)u_{\mathrm{ff}} \\ &\quad + xA_{\mathrm{ff}}B_{\mathrm{ff}}u'_{\mathrm{ff}} + O(x^2)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} [A,B]u(x,z) = [A_{\mathrm{ff}}, B_{\mathrm{ff}}] &+ x\big([A'_{\mathrm{ff}} - D_\tau \hat{A}_{\mathrm{ff}}(0), B_{\mathrm{ff}}] + [A_{\mathrm{ff}}, B'_{\mathrm{ff}} - D_\tau \hat{A}_{\mathrm{ff}}(0)]\big)u_{\mathrm{ff}} \\ &+ x\left(\ell_2 D_\tau \hat{A}_{\mathrm{ff}}(0)B_{\mathrm{ff}} - \ell_1 D_\tau \hat{B}_{\mathrm{ff}}(0)A_{\mathrm{ff}}\right)u_{\mathrm{ff}} \\ &+ x[A_{\mathrm{ff}}, B_{\mathrm{ff}}]u'_{\mathrm{ff}} + O(x^2)\,. \end{aligned}$$

□

Proposition 4.13. *If $A \in {}^{3\mathrm{sc}}\Psi^{m_1,\ell_1}, B \in {}^{3\mathrm{sc}}\Psi^{m_2,\ell_2}$ with $[\hat{N}_{\mathrm{ff},\ell_1}(A), \hat{N}_{\mathrm{ff},\ell_2}(B)] \equiv 0$, then*

$$\begin{aligned} \hat{N}_{\mathrm{ff},\ell_1+\ell_2-1}([A,B])(\tau) = [\hat{A}'_{\mathrm{ff}}(\tau) - D_\tau\hat{A}_{\mathrm{ff}}(\tau), \hat{B}_{\mathrm{ff}}(\tau)] &+ [\hat{A}_{\mathrm{ff}}(\tau), \hat{B}'_{\mathrm{ff}}(\tau) - D_\tau\hat{B}_{\mathrm{ff}}(\tau)] \\ &+ \ell_2 D_\tau\hat{A}_{\mathrm{ff}}(\tau)\hat{B}_{\mathrm{ff}}(\tau) - \ell_1 D_\tau\hat{B}_{\mathrm{ff}}(\tau)\hat{A}_{\mathrm{ff}}(\tau)\,. \end{aligned}$$

If $Q = \mathrm{Op}_L(x^{-\ell}q)$ for $q \in {}^{\mathrm{sc}}S^{0,0}$ supported in a neighborhood of NP and $P_0 = D_t^2 - (\Delta + m^2)$, then

$$\begin{aligned} \hat{N}_{\mathrm{sc}}(i[P_0,Q]) = x^{\ell-1}H_p(x^{-\ell}q)|_{x=0} &= -2x^\ell\left(\tau x\partial_x + (\zeta + \tau y)\partial_y\right)x^{-\ell}q|_{x=0} \\ &= 2\ell\tau q|_{x=0} - 2(\zeta+\tau y)\partial_y q|_{x=0}\,. \end{aligned}$$

At the north pole we have that

$$\hat{N}_{\mathrm{sc}}(i[P_0,Q])|_{\mathrm{NP}} = 2\ell\tau q(0,0,\tau,\zeta) - 2\zeta\partial_y q(0,0,\tau,\zeta)\,.$$

We can recover this from Lemma 4.12 as follows: we have that

$$\begin{aligned}\hat{Q}_{\mathrm{ff}}(\tau) &= \mathrm{Op}_L(q(0,0,\tau,\zeta))\,,\\ \hat{Q}'_{\mathrm{ff}}(\tau) &= \mathrm{Op}_L(\partial_x q(0,0,\tau,\zeta) + z\partial_y q(0,0,\tau,\zeta))\,,\\ (\widehat{P_0})_{\mathrm{ff}}(\tau) &= \tau^2 - (\Delta + m^2)\,,\\ (\widehat{P_0})'_{\mathrm{ff}}(\tau) &= 0\,.\end{aligned}$$

Since the indicial operator of a sc-operator is a Fourier multiplier, we have that

$$[(\widehat{P_0})_{\mathrm{ff}}(\tau_0), \hat{Q}_{\mathrm{ff}}(\tau_0)] = 0.$$

We have that

$$\begin{aligned}\hat{N}_{\mathrm{ff},\ell-1}(i[P_0,Q])(\tau) &= [\tau^2 - (\Delta+m^2), \hat{Q}'_{\mathrm{ff}}(\tau)] + 2\ell\tau\hat{Q}_{\mathrm{ff}}(\tau)\\ &= -2\,\mathrm{Op}_L(\zeta\partial_y q(0,0,\tau,\zeta)) + 2\ell\tau\,\mathrm{Op}_L(q(0,0,\tau,\zeta))\,.\end{aligned}$$

This is a special case of the general identity

$$\hat{N}_{\mathrm{ff},\ell_1+\ell_2-1}(i[A,B])(\tau) = \mathrm{Op}_L(\hat{N}_{\mathrm{sc}}(x^{\ell_1+\ell_2}H_a(b))|_{\mathrm{NP}})$$

for $A = \mathrm{Op}_L(a) \in {}^{\mathrm{sc}}\Psi^{m_1,\ell_1}$ and $B = \mathrm{Op}_L(b) \in {}^{\mathrm{sc}}\Psi^{m_2,\ell_2}$.

4.4. **The indicial operator as a semiclassical scattering operator.** Two features of semiclassical scattering operators will be crucial in our work below, namely, we will need, for $A \in {}^{3\mathrm{sc}}\Psi^{m,l}(\mathbb{R}^{n+1})$.

- to understand how the semiclassical scattering principal symbol of $\hat{N}_{\mathrm{ff},l}(A)$ can be seen as a function on certain parts of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, and
- to recall mapping properties of $\hat{N}_{\mathrm{ff},l}(A)$ on semiclassical Sobolev spaces.

The former is used in formulating ellipticity of A at $\pm\infty \in \overline{W^\perp}$, while the latter is used crucially in that part of the propagation estimates in Sections 6 and 7 below in which the indicial operator of the relevant commutator for the free Klein-Gordon operator P_0 is compared to that of P_V.

For background on semiclassical analysis we refer to [7, 43]. Our work below follows more closely the discussion of smooth semiclassical pseudodifferential operators found in [39].

We work with semiclassical-scattering PsiDO's of order m, l, r, specifically with (classical, smooth) elements in $\Psi^{m,l,r}_{\mathrm{scl,sc}}$, which are (by definition) semiclassical quantizations of symbols $\tilde{a} \in h^{-r}C^\infty([0,1)_h \times S^{m,l}_{\mathrm{sc}}(\mathbb{R}^n))$, i.e. operators

$$B(h) = \mathrm{Op}_{\mathrm{scl}}(\tilde{a}) = \frac{1}{h^{n/2}}\int e^{i\frac{z-z'}{h}\cdot\mu}\tilde{a}(z,\mu;h)d\mu$$

where $h^r a(z,\mu;h)$ is a family of scattering symbols of order m, l that is smooth in $h \in [0,1)$.

Given $A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$, recalling a_{ff} from (4.9), the fact that, for

$$h = \pm\tau,\ \mu = \zeta/\tau, \quad \text{as } \tau \to \pm\infty.$$

the symbol

$$\tilde{a}(z,\mu;h) = a_{\mathrm{ff}}(z, 1/h, \mu/h) \tag{4.18}$$

is semiclassical can be seen by relating the radial compactification $\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$ to the semiclassical symbol space by blowing up the $\tau = 0$ equator of $\partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$.

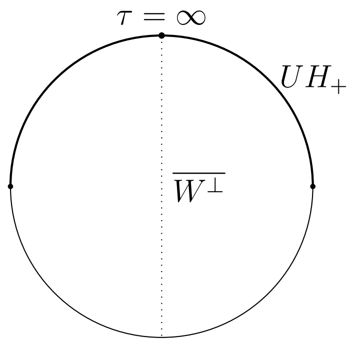

FIGURE 4. The upper half sphere UH_+.

Here, given $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$, the value of the principal symbol $\tau = +\infty$ will be the semiclassical principal symbol of $\mathrm{Op}_L(a_{\mathrm{ff}}) = \hat{A}_{\mathrm{ff},l}(\tau)$, i.e. the function of $z, \mu = \zeta/\tau$ given by the limit $\lim_{\tau \to +\infty, \mu = \zeta/\tau} \langle \tau, \zeta \rangle^{-m} a_{\mathrm{ff}}(z, \tau, \zeta)$. Provided the restriction makes sense (which it does for classical symbols) this is exactly

$$\langle \tau, \zeta \rangle^{-m} x^l a|_{h=1/\tau=0=x, \tau>0}(z, \mu)$$

This is just restriction fo the interior of the upper half-sphere in fiber infinity over ff, and thus, since $\langle \tau, \zeta \rangle^{-m} x^l a$ is a smooth function on the whole of ${}^{3\mathrm{sc}}\overline{T}^*[X; C]$, if we define

$$UH_+ = {}^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X; C] \cap \mathrm{cl}(\{\tau \geq 0\}). \tag{4.19}$$

Lemma 4.14. *Let $\mathrm{Op}_L(a) = A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$ and $a_{\mathrm{ff}} = (x^\ell a)|_{\mathrm{ff}}$. Then $\tilde{a}$ in (4.18) lies in the semiclassical symbol space $h^{-m} C^\infty([0,1)_h \times S^{m,0})$. Thus, $\hat{N}_{\mathrm{ff},\ell}(A)(\tau) \in \Psi^{m,0,m}_{\mathrm{scl,sc},\pm 1/\tau}$, the space defined in (4.14).*

For fixed $h = 1/\tau_0 = 0$, the scattering symbol $j_{\mathrm{sc},m,0}(\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0))$ satisfies

$$\sigma_{\mathrm{sc},m,0}(\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0)), = \langle \tau_0, \zeta \rangle^{-m} a_{\mathrm{ff}}|_{\mathrm{ff} \times \mathrm{fibeq}}$$
$$\hat{N}_{\mathrm{sc},m,0}(\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0)) = \langle \tau_0, \zeta \rangle^{-m} a_{\mathrm{ff}}|_{\partial\, \mathrm{ff} \times \{\tau = \tau_0\}}.$$

The semiclassical symbol at $h = 1/\tau = 0$ satisfies

$$\sigma_{\mathrm{scl},h=1/\tau}(\hat{N}_{\mathrm{ff},\ell}(A)(1/h)) = \langle \tau, \zeta \rangle^{-m} a_{\mathrm{ff}}|_{UH_+}. \tag{4.20}$$

and similarly for $h = -1/\tau$ and UH_-.

Proof. The function a_{ff} satisfies $\langle \tau, \zeta \rangle^{-m} a_{\mathrm{ff}} \in C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$. In regions of the form $-C < \tau < C$, the lemma asserts only that $\hat{N}_{\mathrm{ff},\ell}(A)(\tau) = \mathrm{Op}_{L,z}(a_{\mathrm{ff}}(z, \tau, \zeta))$ is a scattering operator of order m, which follows since on each slice $\tau = 0$ the restriction of a_0 is scattering of order m.

For regions of unbounded τ, we consider the blown up space

$$\beta_{scl} \colon [\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}; \mathrm{fibeq}] \longrightarrow \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}. \tag{4.21}$$

In this manifold with corners, the set $\mathrm{cl}(\beta_{scl}^{-1}\{\tau > 1\})$ is diffeomorphic to $[0,1)_{1/\tau} \times \overline{\mathbb{R}^n_{\mu = \zeta/\tau}}$ and thus smooth functions on the blown-up space define classical semiclassical scattering operators in $h = 1/\tau$ as $\tau \to \infty$. Now, $\beta_{\mathrm{sc}}^{-1}(\langle \tau, \zeta \rangle^{-m} a_0)$ is smooth on the whole of $[\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}; \mathrm{fibeq}]$

(because pullbacks of smooth functions via a blow down map are smooth) and thus the restriction to $\tau > 1$ is smooth. But in that region $\langle \tau, \zeta \rangle^{-1} \sim \langle \mu \rangle^{-1}(1/\tau)$, so in face $\tau^{-m}\langle \mu \rangle^{-m} a_0$ is smooth, which is what we wanted.

The same goes for $h = -1/\tau$ as $\tau \to -\infty$, i.e. $\mathrm{cl}(\beta_{scl}^{-1}\{\tau < -1\}) = [0,1)_{-1/\tau} \times \overline{\mathbb{R}^n_{\mu=\zeta/\tau}}$. □

Given the lemma, it makes sense to extend $\hat{N}_{\mathrm{ff},\ell}(A)(\tau)$ to $h = 1/\tau = 0$ as the semiclassical symbol of $\hat{N}_{\mathrm{ff},\ell}(A)$ at $h = 0$. Indeed, we define $\overline{W^\perp}$ to be the radial compactification of $W^\perp$. Thus, over NP, it is

$$\overline{\mathbb{R}_\tau} = \mathbb{R}_\tau \cup \{\pm\infty\}. \tag{4.22}$$

We then extend $\hat{N}_{\mathrm{ff},\ell}(A)$ to $\overline{W^\perp}$ be defining

$$\hat{N}_{\mathrm{ff},\ell}(A)(\pm\infty) := \langle \tau, \zeta \rangle^{-m} a_{\mathrm{ff}}|_{UH_\pm} \tag{4.23}$$

This leads to a natural notion of ellipticity for A at $+\infty \in \overline{W^\perp}$, namely that $\hat{N}_{\mathrm{ff},\ell}(A)$ be semiclassically elliptic at $h = 1/\tau = 0$, i.e. its semiclassical symbol is nowhere zero. See Definition 4.18 and after.

We recall that the semiclassical Sobolev space is defined as (cf. Zworski [43, Section 8.3])

$$H^{s,\ell}_{\mathrm{scl}}(\mathbb{R}^n) := \left\{u \in \mathcal{S}'(\mathbb{R}^n) : \langle \zeta \rangle^s \mathcal{F}_h(\langle z \rangle^\ell u) \in L^2(\mathbb{R}^n)\right\}, \tag{4.24}$$

$$\|u\|^2_{H^{s,\ell}_{\mathrm{scl}}} := (2\pi h)^{-n} \int \langle \zeta \rangle^{2s} |\mathcal{F}_h(\langle z \rangle^\ell u)(\zeta)|^2 \, d\zeta\,. \tag{4.25}$$

Here,

$$(\mathcal{F}_h u)(\zeta) := \int e^{-\frac{i}{h}\langle z, \zeta \rangle} u(z)\, dz$$

denotes the semiclassical Fourier transform.

Semiclassical scattering operators are bounded on semiclassical Sobolev spaces:

Proposition 4.15. *If $A \in \Psi^{m,r,k}_{\mathrm{scl,sc}}$, then extends as a bounded operator*

$$A : h^k H^{m+s,r+\ell}_{\mathrm{scl}} \longrightarrow H^{s,\ell}_{\mathrm{scl}}\,.$$

We also recall that for semiclassical operators ellipticity implies invertibility since we can bound the error term in the parametrix construction by the semiclassical parameter and then use a Neumann series to show invertibility (see Zworski [43, Theorem 4.29]).

Proposition 4.16. *Let $A = A(h) \in \Psi^{m,r,k}_{\mathrm{scl,sc}}$ be elliptic, then there exists $h_0 > 0$ such that for all $h \in (0, h)$, $A(h)^{-1}$ exists as a bounded operator*

$$A(h)^{-1} : H^{s,\ell}_{\mathrm{scl}} \longrightarrow h^k H^{m+s,r+\ell}_{\mathrm{scl}}\,.$$

4.5. **Wavefront sets and elliptic sets.** We now develop an appropriate notion of operator wavefront set in this context. We note that there are distinct notions of operator wavefront set even in the original works on the 3sc-calculus. Given $A \in {}^{3\mathrm{sc}}\Psi^{m,l}(X)$, there is ${}^{3\mathrm{sc}}\mathrm{WF}(A)$ and ${}^{\mathrm{sc}}\mathrm{WF}(A)$, the former from Definition 9.1 of [36] and the latter from Definition 5.1 of [37]. Both of these do not treat fiber infinity over ∂X, so our notion of wavefront set must generalize these. Rather than recalling both in detail, we give one definition of operator wavefront set, which we will denote by ${}^{3\mathrm{sc}}\mathrm{WF}(A)$.

First we point out the source of subtlety in the definition of operator wavefront set in this context. We wish in particular to have a notion of wavefront set for which

$$\forall A, B \in {}^{3\mathrm{sc}}\Psi^{*,*}, \quad {}^{3\mathrm{sc}}\mathrm{WF}'(AB) \subset {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{3\mathrm{sc}}\mathrm{WF}'(B). \tag{4.26}$$

The issue arises at points $\alpha \in {}^{3\mathrm{sc}}\overline{T}[X;C]$ lying over the intersection of mf and ff, that is, $\alpha \in {}^{3\mathrm{sc}}T^*_{\mathrm{mf}\cap\mathrm{ff}}[X;C]$. A natural notion of wavefront may assert that α is *not* in the wavefront set of $\mathrm{Op}_L(a) = A$ if a vanishes to infinite order in a neighborhood of α. But any neighborhood U of α in ${}^{3\mathrm{sc}}\overline{T}$ has that $U \cap {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]$ is non-empty and open. But the global nature of $\hat{N}_{\mathrm{ff},\ell}(A)$ means that the symbol of A^2 is not necessarily trivial in $U \cap {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]$ if a is.

To avoid this issue we essentially do not include points in $\mathrm{mf} \cap \mathrm{ff}$ in the operator wavefront set. Rather, as we describe in detail now, away from C we use the scattering definition of ${}^{3\mathrm{sc}}\mathrm{WF}'$ and over ff only look at τ levels. We define

$${}^{\mathrm{sc}}\dot{\overline{T}}X = \left({}^{\mathrm{sc}}\overline{T}^*X \setminus {}^{\mathrm{sc}}\overline{T}^*_C X\right) \cup \overline{W^\perp}, \tag{4.27}$$

the compactification of the compressed scattering cotangent bundle from Section 5 of [37]. In that paper, interpreted in our context, defines the compressed scattering cotangent bundle to be

$${}^{\mathrm{sc}}\dot{T}X = ({}^{\mathrm{sc}}T^*X \setminus {}^{\mathrm{sc}}T^*_C X) \cup W^\perp,$$

i.e. not compactified. This space has a mapping

$$\pi^\perp \colon {}^{\mathrm{sc}}T^*X \longrightarrow {}^{\mathrm{sc}}\dot{T}^*X \tag{4.28}$$

which is identity everywhere except on $T^*_C X$ where it is projection onto $W^\perp$ (which is just the mapping $(\tau, \zeta) \mapsto \tau$ on the fiber.)

We now define the operator wavefront set ${}^{3\mathrm{sc}}\mathrm{WF}'(A)$ of a 3sc-PsiDO A. We follow [37] here as opposed to [36] here as there are two different definitions, and we indicate the difference. Given $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$, $j_{3\mathrm{sc},m,\ell}(A)$ is defined as a function on the set from [36, Eq. 9.1]

$$C_{3\mathrm{sc}}[X;C] := {}^{3\mathrm{sc}}S^*[X;C] \cup {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C] \cup \overline{W^\perp}. \tag{4.29}$$

Its value on these three components are exactly the three components of $j_{3\mathrm{sc},m,\ell}(A)$. However, as described above, points at the boundary of ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}X$ are problematic when considered as part of the operator wavefront set. Thus we instead define what is effectively the locus in ${}^{\mathrm{sc}}\dot{\overline{T}}^*X$ which avoids those points in $C_{3\mathrm{sc}}[X;C]$, namely

$$\dot{C}_{3\mathrm{sc}}[X;C] := {}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \cup \overline{W^\perp} \tag{4.30}$$

and use this to define both ${}^{3\mathrm{sc}}\mathrm{WF}'$ and ${}^{3\mathrm{sc}}\mathrm{Ell}$ below.

Since ${}^{3\mathrm{sc}}\mathrm{WF}'(A)$ will specify points in ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$ near which the left symbol a is rapidly decreasing, we need a mechanism for relating subsets of $\dot{C}_{3\mathrm{sc}}[X;C]$ to subsets of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$. This will work by associating ${}^{\mathrm{sc}}\overline{T}^*X \setminus {}^{\mathrm{sc}}\overline{T}^*_C X$ naturally to its image via the blow-down β_C, while points $\overline{W^\perp}$ will correspond to the natural τ slices over ff.

Thus, we define

$$\gamma_{3\mathrm{sc}} : \dot{C}_{3\mathrm{sc}}[X;C] \longrightarrow \mathcal{P}(\partial {}^{3\mathrm{sc}}\overline{T}^*[X;C]) \tag{4.31}$$

as

$$\begin{aligned}
\gamma_{3\mathrm{sc}}(p) &= \{p\} && \text{for } p \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X\,,\\
\gamma_{3\mathrm{sc}}(\tau) &= \beta_C^{-1}(\pi^\perp)^{-1}\{\tau\} && \text{for } \tau \in W^\perp\,,\\
\gamma_{3\mathrm{sc}}(\pm\infty) &= UH_\pm && \text{for } \pm\infty \in \partial\overline{W^\perp}\,.
\end{aligned}$$

By abuse of notation, for a set $S \subset \dot{C}_{3\mathrm{sc}}[X;C]$ we write

$$\gamma_{3\mathrm{sc}}(S) \coloneqq \bigcup_{p\in S} \gamma_{3\mathrm{sc}}(p)\,. \tag{4.32}$$

We used above that ${}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X$ is naturally identified with ${}^{3\mathrm{sc}}S^*_{[X;C]\setminus\mathrm{ff}}[X;C] \cup {}^{3\mathrm{sc}}\overline{T}^*_{\partial[X;C]\setminus\mathrm{ff}}[X;C]$ via the blow down map.

The topology on ${}^{\mathrm{sc}}\dot{T}X$ is that in which a neighborhood basis near $\alpha \in W^\perp$ is induced by the open neighborhoods of $\gamma_{3\mathrm{sc}}(\alpha)$ (4.32), while in ${}^{\mathrm{sc}}\dot{\overline{T}}X$, open sets around boundary points are defined as usual for radial compactifications.

For a symbol $a \in {}^{3\mathrm{sc}}S^{m,\ell}$, we define the essential support ${}^{3\mathrm{sc}}\operatorname{ess-supp}(a) \subset \partial^{3\mathrm{sc}}\overline{T}^*[X;C]$ by declaring $p \in {}^{3\mathrm{sc}}\operatorname{ess-supp}(a)^c$ if and only if there exists $U \subset {}^{3\mathrm{sc}}\overline{T}^*[X;C]$ open and $\chi \in C_c^\infty({}^{3\mathrm{sc}}\overline{T}^*[X;C])$ such that $p \in U$, $\chi|_U \equiv 1$ and $\chi a \in {}^{3\mathrm{sc}}S^{-\infty,-\infty}$.

Definition 4.17. Let $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{m,l}(X)$. The operator wavefront set

$${}^{3\mathrm{sc}}\mathrm{WF}'(A) \subset \dot{C}_{3\mathrm{sc}}[X;C]$$

is defined as follows: a point $p \in \dot{C}_{3\mathrm{sc}}[X;C]$ is *not* in the wavefront set,

$$p \in {}^{3\mathrm{sc}}\mathrm{WF}'(A)^c \text{ if and only if } \gamma_{3\mathrm{sc}}(p) \cap {}^{3\mathrm{sc}}\operatorname{ess-supp}(a) = \varnothing\,.$$

Moreover, we define

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{fib}}(A) &\coloneqq {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{\mathrm{sc}}S^*_{X\setminus C}X\,,\\
\mathrm{WF}'_{\mathrm{mf}}(A) &\coloneqq {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X\,,\\
\mathrm{WF}'_{\mathrm{ff}}(A) &\coloneqq {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap \overline{W^\perp}\,.
\end{aligned}$$

We can write the complements of each of the components as

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{fib}}(A)^c = \{&\alpha \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \colon \exists U \subset {}^{\mathrm{sc}}S^*_{X\setminus C}X \text{ open such that } \alpha \in U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}\,,\\
\mathrm{WF}'_{\mathrm{mf}}(A)^c = \{&\alpha \in {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \colon \exists U \subset {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \text{ open such that } \alpha \in U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}\,,\\
\mathrm{WF}'_{\mathrm{ff}}(A)^c = \{&\tau \in W^\perp \colon \exists\ \epsilon > 0 \text{ such that } a(A) \text{ vanishes to}\\
&\text{infinite order on } \beta^{-1}(\pi^\perp)^{-1}[\tau-\epsilon, \tau+\epsilon]\}\\
&\cup \{\pm\infty : \exists \text{ open } U \subset \partial^{3\mathrm{sc}}\overline{T}^*[X;C] \text{ such that } UH_\pm \subset U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}.
\end{aligned}$$

Now we define the elliptic sets. Over mf and fiber infinity, the definition of ellipticity is exactly as in the standard scattering case, i.e. non-vanishing (or, for operators acting on sections of vector bundles, invertibility) of the principal symbol. Over ff in $W^\perp$, the correct notation of ellipticity is invertibility between appropriate scattering Sobolev spaces.

To define the elliptic set, we note that the two components of the symbol $\sigma_{3\mathrm{sc},m,\ell}(A)$ and $\hat{N}_{\mathrm{mf},m,\ell}(A)$ define, by restriction, functions on ${}^{\mathrm{sc}}S^*_{X\setminus C}X$ and ${}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X$, respectively.

Definition 4.18. Let $A \in {}^{3\mathrm{sc}}\Psi^{m,\ell}$. The 3sc-elliptic set ${}^{3\mathrm{sc}}\mathrm{Ell}(A)$ (whose m, ℓ dependence is suppressed from the notation) is

$$
{}^{3\mathrm{sc}}\mathrm{Ell}(A) = \mathrm{Ell}_{\mathrm{fib}}(A) \cup \mathrm{Ell}_{\mathrm{mf}}(A) \cup \mathrm{Ell}_{\mathrm{ff}}(A) \subset \dot{C}_{3\mathrm{sc}}[X; C]\,,
$$

with

$$
\begin{aligned}
\mathrm{Ell}_{\mathrm{fib}}(A) &= \{\alpha \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \colon \sigma_{3\mathrm{sc},m,\ell}(A)(\alpha) \neq 0\}\,,\\
\mathrm{Ell}_{\mathrm{mf}}(A) &= \{\alpha \in {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \colon \hat{N}_{\mathrm{mf},m,\ell}(A)(\alpha) \neq 0\}\,,
\end{aligned}
$$

while

$$
\begin{aligned}
\mathrm{Ell}_{\mathrm{ff}}(A) = \{\tau \in W^\perp \colon &\hat{N}_{\mathrm{ff},\ell}(A)(\tau) \text{ is scattering elliptic and invertible}\}\\
&\cup \{\pm\infty \in \partial\overline{W^\perp} : \sigma_{3\mathrm{sc},m,\ell}(A) \text{ is nowhere vanishing on } UH_\pm\}.
\end{aligned}
$$

Note that it makes sense to speak of invertibility of scattering elliptic operators. Indeed, if $B \in {}^{\mathrm{sc}}\Psi^{m,l}$ is (scattering) elliptic then $B \colon H^{s,r}_{\mathrm{sc}} \longrightarrow H^{s-m,r-l}_{\mathrm{sc}}$ is Fredholm. By scattering ellipticity, its kernel and cokernel consist of Schwartz functions, and therefore the invertibility of this Fredholm mapping is independent of s, r, i.e. if it is invertible for any s, r then it is invertible for all s, r. Thus, we say a (globally) elliptic, scattering operator $B \in {}^{\mathrm{sc}}\Psi^{m,l}$ is "invertible" if any (and thus all) of these Fredholm operators is invertible.

We now describe in more detail the significance of $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(A)$ and $\pm\infty \in \mathrm{Ell}_{\mathrm{ff}}(A)$. For the former, we first consider the assumption that $\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0)$ is scattering elliptic. Recall that if $\mathrm{Op}_L(a) = A$, then $\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0) = \mathrm{Op}_{L,z}(a_{\mathrm{ff}})$ (4.10). Our discussion of symbols in Section 4.2 allows us to identify the scattering principal symbol of $\hat{A}_{\mathrm{ff},l}(\tau)$ with the appropriate boundary restrictions of a; following Remark 4.9, choosing $\tau-$dependent defining function we have

$$
j_{\mathrm{sc},m,0}(\hat{A}_{\mathrm{ff},l}(\tau_0)) = (\langle \tau, \zeta\rangle^{-m} a_{\mathrm{ff}}|_{\mathrm{ff}\,\times\,\mathrm{fibeq}}, \langle \tau, \zeta\rangle^{-m} a_{\mathrm{ff}}|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}\,\cap\,\mathrm{mf}}\cap\{\tau=\tau_0\}}).
$$

Thus if $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(A)$ then these two components are nowhere vanishing, and the scattering operator $\hat{N}_{\mathrm{ff},\ell}(A)(\tau_0)$ is invertible.

As for the condition $+\infty \in \mathrm{Ell}_{\mathrm{ff}}(A)$ (the $-$ case is similar) this is equivalent to $\hat{N}_{\mathrm{ff},\ell}(A)$ being semiclassically elliptic as $\tau = 1/h \to +\infty$. Indeed, we have $\langle \tau, \zeta\rangle^{-m} x^l a$ is invertible on the whole of UH_+. For $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$, the indicial operator $\hat{N}_{\mathrm{ff},\ell}(A)(\tau)$ is a semiclassical scattering PsiDO with semiclassical principal symbols at $\tau = \pm\infty$, so ellipticity at $+\infty \in \overline{W^\perp}$ (over ff_+) is exactly the condition that

$$
\sigma_{\mathrm{scl},m}(\hat{N}_{\mathrm{ff},\ell}(A)) = x^l \langle \tau, \zeta\rangle^{-m} a|_{UH_+}
$$

is nowhere vanishing.

4.6. **Elliptic regularity.** We have the following microlocal parametrix theorem, which extends Lemma 9.3 and Remark 9.4 ([36]) to include fiber infinity.

Proposition 4.19. *Let $A \in {}^{3\mathrm{sc}}\Psi^{m,l}(X)$ and $K \subset {}^{3\mathrm{sc}}\mathrm{Ell}(A)$ be compact. Then there exists $G \in {}^{3\mathrm{sc}}\Psi^{-m,-l}(X)$ such that*

$$K \cap {}^{3\mathrm{sc}}\mathrm{WF}'(AG - \mathrm{Id}) = K \cap {}^{3\mathrm{sc}}\mathrm{WF}'(GA - \mathrm{Id}) = \varnothing\,.$$

Proof. We include only the construction for $+\infty \in \overline{W^\perp}$.

Let $+\infty \in \mathrm{Ell}_{\mathrm{ff}}(A)$. Let $a_{\mathrm{ff}} = x^l a|_{\mathrm{ff}}$, so $a_{\mathrm{ff}} \in \langle \tau, \zeta\rangle^m C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$. The ellipticity condition at $+\infty$ is exactly that $\langle \tau, \zeta\rangle^m a_{\mathrm{ff}}|_{UH_+}$ is invertible. This implies that $\hat{A}_{\mathrm{ff},l}(\tau)$ is invertible for sufficiently large $\tau > \tau_0$ for $\tau_0 \gg 0$ fixed by Proposition 4.16. We will show that one can quantize $\hat{A}_{\mathrm{ff}}(\tau)^{-1}$ to a 3sc operator, and use this to perform a parametrix construction for A near $+\infty \in \overline{W^\perp}$.

Let $UH_+ \subset U \subset U'$ where U, U' are open sets such that: (1) $\langle \tau, \zeta\rangle^m a_{\mathrm{ff}}$ is invertible on U' and (2) there exists $\chi \in C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ with $\chi|_U \equiv 1$ and $\operatorname{supp}\chi \subset U'$. (The second condition is easily achievable since UH_+ is an embedded p-submanifold of $\mathrm{ff} \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$.) One can, in particular, arrange that $\chi = \chi(\tau, \zeta)$ (since $UH_+ = \mathrm{ff} \times \{\tau \geq 0\} \cap \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$)) and that $\chi(\tau) \equiv 1$ for $\tau > \tau_0$ for some fixed τ_0. Then

$$\chi a_{\mathrm{ff}}^{-1} \in \langle \tau, \zeta\rangle^{-m} C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}),$$

and thus $\mathrm{Op}_{L,z}(\chi a_{\mathrm{ff}}^{-1}) \in \Psi^{-m,0,-m}_{\mathrm{scl,sc},\pm 1/\tau}$. Moreover, if $q \in C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ has $\operatorname{supp} q \subset U$, then $q(\chi a_{\mathrm{ff}}^{-1} a_{\mathrm{ff}} - 1) \equiv 0$ and thus

$$\mathrm{Op}_{L,z}(q)(\mathrm{Op}_{L,z}(\chi a_{\mathrm{ff}}^{-1})\hat{A}_{\mathrm{ff},l} - 1) \in \Psi^{-1,0,-1}_{\mathrm{scl,sc},\pm 1/\tau}$$

The preceding is the first step in a semiclassical-scattering parametrix construction which yields $\tilde{g} \in \langle \tau, \zeta\rangle^{-m} C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$, such that

$$\mathrm{Op}_{L,z}(q)(\mathrm{Op}_{L,z}(\tilde{g})\hat{A}_{\mathrm{ff},l} - 1) \in \Psi^{-\infty,0,-\infty}_{\mathrm{scl,sc},\pm 1/\tau}.$$

Indeed, following the standard elliptic parametrix construction, if g_0 has

$$E = \mathrm{Op}_{L,z}(q)(\mathrm{Op}_{L,z}(g_N)\hat{A}_{\mathrm{ff},l} - 1) \in \Psi^{-N,0,-N}_{\mathrm{scl,sc},\pm 1/\tau},$$

then one can solve $-b_N a_{\mathrm{ff}} = \sigma_{\mathrm{scl,sc},-N,0,-N}(E)$ on the support of χ and then take a Borel sum to obtain $\tilde{g}$.

Letting $\chi_{\tau_0} \in C^\infty(\mathbb{R})$ have $\chi_{\tau_0}(\tau) = 1$ for $\tau > \tau_0 + 1$ and $\operatorname{supp}\chi_{\tau_0} \subset [\tau_0, \infty)$, then

$$\hat{K}(\tau) \coloneqq \chi_{\tau_0}(\tau)\left(\hat{A}_{\mathrm{ff},l}^{-1} - \mathrm{Op}_{L,z}(\tilde{g})\right) \in \Psi^{-\infty,0,-\infty}_{\mathrm{scl,sc},\pm 1/\tau},$$

(This follows from the argument [30] used in the proof of Corollary 4.21, where one compares the parametrix and actual operator and shows they differ by a residual operator.) But this implies that

$$(1 - \chi_{\tau_0})\tilde{g} + \chi_{\tau_0}(\tau)a_{\mathrm{ff}}^{-1} \in \langle \tau, \zeta\rangle^{-m} C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}),$$

Indeed, this follows since $\tilde{g} \in \langle \tau, \zeta\rangle^{-m} C^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ and $\chi_{\tau_0}(\tau)a_{\mathrm{ff}}^{-1} - \tilde{g}$ is order $-\infty, 0, -\infty$.

Thus

$$\hat{G} \coloneqq (1 - \chi_{\tau_0})\,\mathrm{Op}_{L,z}(\tilde{g}) + \chi_{\tau_0}(\tau)A_{\mathrm{ff},l}^{-1} \in \Psi^{-m,0,-m}_{\mathrm{scl,sc},\pm 1/\tau},$$

and

$$\hat{G} - \mathrm{Op}_{L,z}(\tilde{g}) \in \Psi^{-\infty,0,\infty}_{\mathrm{scl,sc},\pm 1/\tau}.$$

To quantize this $\hat{G}$ to a full 3sc-PsiDO, we need to prescribe matching values on the other faces ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]$ and mf of ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]$, but this is easily done. Indeed, the symbols of G, $\mathrm{Op}_{L,z}(g)$, and a_{ff}^{-1} agree at fiber UH_+ (meaning concretely that when multiplied by $\langle\tau,\zeta\rangle^m$ they restrict to the same function there) and thus in a sufficiently small neighborhoods $V \subset V'$ of UH_+ in ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C] \cup \mathrm{mf}$ one can choose another cutoff χ' which is 1 on V and supported in V', and let $g = \chi a_{\mathrm{ff}}^{-1}$ on V. Then this g matches the principal symbol of $\hat{G}$ and thus there is a smooth function which we also call g on the whole of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$ such that $\mathrm{Op}_L(x^{-l}g) = G \in {}^{3\mathrm{sc}}\Psi^{-m,l,m}$ and: (1) $g = a_{\mathrm{ff}}^{-1}$ on V, $\hat{G}_{\mathrm{ff}}(\tau) = \hat{G}(\tau) = \hat{A}_{\mathrm{ff}}^{-1}(\tau)$ for $\tau > \tau_0$.

Thus for $Q \in {}^{3\mathrm{sc}}\Psi^{0,0,0}$ close to UH_+,

$$E = Q(GA - I) \in {}^{3\mathrm{sc}}\Psi^{-1,-1,-1}.$$

This is the first step in a parametrix construction which produces a residual error. Indeed, to find $G_1 \in {}^{3\mathrm{sc}}\Psi^{-m-1,-l-1}$ such that $G_1 A + E \in {}^{3\mathrm{sc}}\Psi^{-2,-2,-2}$ near UH_+, one solves $(\hat{G}_1)_{\mathrm{ff},-l-1}\hat{A}_{\mathrm{ff},l} = \hat{E}_{\mathrm{ff},-1}$, i.e. $(\hat{G}_1)_{\mathrm{ff},-l-1} = \hat{E}_{\mathrm{ff},-1}\hat{A}^{-1}_{\mathrm{ff},l}$, for $\tau > \tau_0$. We have already seen that $\hat{A}^{-1}_{\mathrm{ff},l}$ can be approximated in $\Psi^{-\infty,0,-\infty}_{\mathrm{scl,sc},\pm 1/\tau}$ by an operator in $\Psi^{-m,0,-m}_{\mathrm{scl,sc},\pm 1/\tau}$, so again using the ellipticity on near UH_+ on the other bhs's of ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, we have G_1. The inductive construction of the higher order approximations is standard. □

From the elliptic parametrix construction we get elliptic estimates. In this section we record these estimates and use them to discuss an important application to globally elliptic 3sc-operator.

Proposition 4.20. *Let $u \in \mathcal{S}'$ and $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$. Let $B, Q' \in {}^{3\mathrm{sc}}\Psi^{0,0}$. If ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(A) \cap {}^{3\mathrm{sc}}\mathrm{Ell}(Q')$, then $Q'Au \in H^{s,r}_{\mathrm{sc}}$ implies $Bu \in H^{s+m,r+l}_{\mathrm{sc}}$, and for any $M, N \in \mathbb{R}$ there is $C > 0$ such that*

$$\|Bu\|_{s,r} \leq C\left(\|Q'Au\|_{s-m,r-l} + \|u\|_{-N,-M}\right).$$

Proof. This follows from the standard argument using the mapping property in Proposition 4.3. Namely, Let G be as in Proposition 4.20 with respect to $K = {}^{3\mathrm{sc}}\mathrm{WF}'(B)$. Then

$$\begin{aligned} n\|Bu\|_{s,r} &\leq \|(GA - I)Bu\|_{s,r} + \|GABu\|_{s,r} \\ &\leq \|(GA - I)Bu\|_{s,r} + \|G(I - Q')ABu\|_{s,r} + \|GQ'ABu\|_{s,r} \\ &\leq C\left(\|Q'Au\|_{s-m,r-l} + \|u\|_{-N,-M}\right). \end{aligned}$$

where we used Proposition 4.3 and (4.26). □

Thus an operator $A \in {}^{3\mathrm{sc}}\Psi^{m,l}$ that is globally elliptic satisfies the (global) elliptic estimate, namely that for any $M, N \in \mathbb{R}$, there is $C > 0$ such that,

$$\|u\|_{s,r} \leq C(\|Au\|_{s-m,r-l} + \|u\|_{-N,-M}). \tag{4.33}$$

In particular, as have now established, there are global parametrices $G, G' \in {}^{3\mathrm{sc}}\Psi^{-m,-l}$ such that

$$GA - I,\ AG' - I \in {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}. \tag{4.34}$$

We use these parametrices in the corollary below.

In order to define microlocalizers over ff, we will need an operator which is globally elliptic and commutes with P_{V_0}. To accomplish this, we switch the signs in P_{V_0} to make it elliptic; that is, we consider, for $E \geq 0$,

$$D_t^2 + H_{V_0} + E \tag{4.35}$$

We will use the inverse of this operator below to define such microlocalizers in Section 5.1. Specifically, we will need:

Corollary 4.21. *The operator $D_t^2 + H_{V_0} + E$ is invertible for some $E \geq 0$, and*

$$(D_t^2 + H_{V_0} + E)^{-1} \in {}^{3\mathrm{sc}}\Psi^{-2,0}(\mathbb{R}^{n+1}).$$

Proof. The operator $D_t^2 + H_{V_0} + E$ is elliptic on mf and fiber infinity for any $E > 0$. The indicial operator is

$$\tau^2 + H_{V_0} + E$$

and this operator is invertible for all τ for E sufficiently large. Indeed, all negative elements in the spectrum of $\Delta_z + V_0(z)$ are eigenvalues. This means that for $-E < 0$ in the spectrum of $\Delta_z + m^2 + V_0$, there exists $w \in L^2(\mathbb{R}^n_z)$ such that

$$(H_{V_0} + E)w(z) = 0\,.$$

But then by scattering ellipticity of $H_{V_0} + E$, $w \in \mathcal{S}$. On the other hand, for all Schwartz w,we can integrate by parts to obtain

$$\langle (\Delta_z + m^2 + V_0)w, w\rangle = \int |\nabla_z w|^2 + \langle V_0 w, w\rangle + (m^2 + E)|w(z)|^2 dz.$$

Since $\langle V_0 w, w\rangle \geq -\epsilon/2|V_0 w|^2 - (1/2\epsilon)|w|^2$ and we can bound $|V_0 w| \leq C(|\nabla_z w|^2 + |w|^2)$, we obtain a lower bound, for $\epsilon > 0$ sufficiently small and E sufficiently large,

$$\langle (\Delta_z + m^2 + V_0 + E)w, w\rangle \geq \frac{1}{2}\int |\nabla_z w|^2 + (m^2 + E)|w(z)|^2 dz > 0$$

for all Schwartz w. Thus for E sufficiently large, $\hat{N}_{\mathrm{ff}}(D_t^2 + H_{V_0} + E)(\tau)$ is invertible for all τ, hence $D_t^2 + H_{V_0} + E$ is globally 3sc-elliptic.

On the other hand, by the same integration by parts argument, $D_t^2 + H_{V_0} + E$ is positive for E sufficiently large, hence is invertible.

To show that $(D_t^2 + H_{V_0} + E)^{-1} \in {}^{3\mathrm{sc}}\Psi^{-2,0}$, we employ an argument from [30]. Namely, by global ellipticity, we have parametrices G, G' for $(D_t^2 + H_{V_0} + E)^{-1}$ as in (4.34), and writing $G(D_t^2 + H_{V_0} + E) - I = K$ and $(D_t^2 + H_{V_0} + E)G' - I = K'$ we obtain

$$\begin{aligned}(D_t^2 + H_{V_0} + E)^{-1} &= (G(D_t^2 + H_{V_0} + E) + K)(D_t^2 + H_{V_0} + E)^{-1}((D_t^2 + H_{V_0} + E)G' + K') \\ &= G((D_t^2 + H_{V_0} + E)G' + K') + KG'K' + K(D_t^2 + H_{V_0} + E)^{-1}K'.\end{aligned}$$

Since for any operator $L\colon \mathcal{S} \longrightarrow \mathcal{S}'$ we have ${}^{3\mathrm{sc}}\Psi^{-\infty,-\infty} \circ L \circ {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty} \subset {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}$, this and the composition properties for 3sc-PsiDOs show that the right hand side above lies in ${}^{3\mathrm{sc}}\Psi^{-2,0}$, which is what we wanted. $\square$

5. Functional calculus, commutators, and special symbol classes

As discussed in the introduction, our localizers near ff will include functions of the limiting static operator P_{V_+} which act as microlocalizers to the characteristic set. Indeed, from Section 4.3, we recall that ζ-dependence in the indicial operator can lead to failure of the basic commutator order relations. Thus, again following Vasy, our comutator will be local a composition of an operator whose symbol is purely τ-dependent over C composed with a function of the operator $P_{V_\pm}$. To prepare for this, in this section, for static V_0, we wish to consider functions of P_{V_0}.

However, we must take special care in defining these functions, as it is not automatic that a function of a differential operator of positive order is pseudodifferential. As a cautionary example, consider the free Hamiltonian function $p = \tau^2 - |\zeta|^2 - m^2$. This is a classical symbol, but, for $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(0) = 1$, the composition $\psi(p)$ is not a symbol, in fact $D_\tau^k(\psi(p))$ grows to order k as τ goes to infinity along the characteristic set. Correspondingly, $\psi(P_0)$ is not a pseudodifferential operator of any order.

To handle this issue we consider instead the normalization of p in which we divide essentially by $\langle \tau, \zeta\rangle^2$ to make an order 0 classical symbol. Consider, for $E \geq 0$, the operator

$$G_{\psi,0} \coloneqq \psi\left((D_t^2 + D_z \cdot D_z + m^2 + E)^{-1} P_0\right), \tag{5.1}$$

which is the Fourier multiplier for the function

$$\psi\left(\frac{\tau^2 - |\zeta|^2 - m^2}{\tau^2 + |\zeta|^2 + m^2 + E}\right) = \psi\left(\frac{p}{\tau^2 + |\zeta|^2 + m^2 + E}\right). \tag{5.2}$$

This $G_{\psi,0}$ depends on E; the value of which is fixed such that the operator in (4.35) is invertible. Since the ratio $p/(\tau^2 + |\zeta|^2 + m^2 + E)$ is a smooth function on the whole of ${}^{\mathrm{sc}}\overline{T}^* X$, its composition with ψ is also. Thus we have

$$G_{\psi,0} \in {}^{\mathrm{sc}}\Psi^{0,0}(X)\,.$$

5.1. Functional calculus.

In this section, we construct the operator G_ψ as an element in the 3-scattering calculus. We will construct the functional calculus for general self-adjoint ${}^{3\mathrm{sc}}\Psi^{0,0}$-operators.

Our main result is a generalization of Proposition 10.2 [36], which relates to functions of a many-body Hamiltonian H. For such H, one obtains that for $\psi \in C_c^\infty(\mathbb{R})$,

$$\psi(H) \in {}^{3\mathrm{sc}}\Psi^{-\infty,0},$$

while in contrast, we see from the example of $G_{\psi,0}$ that we do not expect smoothing operator when taking functions of the Klein-Gordon operator. We will apply the following proposition to functions of $(D_t^2 + H_{V_0} + E)^{-1} P_{V_0}$ in Definition 5.6 below.

Proposition 5.1. *Let $\psi \in C_c^\infty(\mathbb{R})$ and $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$ be self-adjoint. Then*

$$\psi(A) \in {}^{3\mathrm{sc}}\Psi^{0,0}$$

and its principal symbol satisfies $j_{3\mathrm{sc}}(\psi(A)) = \psi(j_{3\mathrm{sc}}(A))$, meaning

$$\sigma_{3\mathrm{sc}}(\psi(A)) = \psi(\sigma_{3\mathrm{sc}}(A)), \quad \hat{N}_{\mathrm{mf}}(\psi(A)) = \psi(\hat{N}_{\mathrm{mf}}(A)), \quad \hat{N}_{\mathrm{ff}}(\psi(A)) = \psi(\hat{N}_{\mathrm{ff}}(A)) \tag{5.3}$$

Remark 5.2. Since operators in ${}^{3\mathrm{sc}}\Psi^{0,0}$ are bounded on L^2, the assumption that A is self-adjoint is equivalent to A being symmetric.

We start by recalling the results for the scattering calculus.

Lemma 5.3. *Let $A \in {}^{\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^n)$ be self-adjoint. For each $k \in \mathbb{N}$ there exists a family of order k parametrices $B_k(z) \in {}^{\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^n)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ such that*

$$(A - z)B_k(z) - \mathrm{Id} = R_{1,z} \in {}^{\mathrm{sc}}\Psi^{-k,-k}(\mathbb{R}^n)\,,$$
$$B_k(z)(A - z) - \mathrm{Id} = R_{2,z} \in {}^{\mathrm{sc}}\Psi^{-k,-k}(\mathbb{R}^n)$$

and the seminorms of order k of $B_z, R_{1,z}, R_{2,z}$ are bounded by $C_k|\mathrm{Im}\, z|^{-c(k)}$.

Proof. The proof is the same as [36, Lemma 10.1]. The crucial point is that the principal symbol of $A-z$ is $j_{\mathrm{sc}}(A)-z$ and since A is self-adjoint, $j_{\mathrm{sc}}(A)$ is real and therefore $j_{\mathrm{sc}}(A)-z$ is invertible for $z \in \mathbb{C} \setminus \mathbb{R}$. ☐

Proposition 5.4. *Let $A \in {}^{\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^n)$ be self-adjoint and $\psi \in C_c^\infty(\mathbb{R})$. Then*

$$\psi(A) \in {}^{\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^n)$$

with principal symbol

$$j_{\mathrm{sc}}(\psi(A)) = \psi(j_{\mathrm{sc}}(A))\,.$$

Proof. The proof is the same as Proposition 10.2 in Vasy [36], but using the above lemma for the parametrix. Namely, we use the Helffer–Sjöstrand formula to define

$$\psi(A) := -\frac{1}{2\pi i}\int \bar{\partial}_z\tilde{\psi}(z)(A - z)^{-1}\, dz \wedge d\bar{z}\,,$$

where $\tilde{\psi}$ is a compactly supported almost analytic extension of ψ. Then we define

$$A_{\psi,k} := -\frac{1}{2\pi i}\int \bar{\partial}_z\tilde{\psi}(z)B_k(z)\, dz \wedge d\bar{z}\,,$$

where $B_k(z)$ is the k-th order parametrix of $A - z$ as in the previous lemma. We can use asymptotic summation to obtain a limit $\tilde{A}_\psi$ of $A_{\psi,k}$ and we conclude that

$$\psi(A) - \tilde{A}_\psi : C^{-\infty} \longrightarrow \dot{C}^\infty$$

and the formula for the principal symbol follows from the explicit form of $\psi(A)$. ☐

Now, we turn to the case of $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$. We will an analogous argument and therefore we start by constructing a parametrix for $A - z$.

Lemma 5.5. *Let $A \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ be self-adjoint. For each $k \in \mathbb{N}$ there exists a family of order k parametrices $B_z \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ such that*

$$(A - z)B_k(z) - \mathrm{Id} = R_{1,k}(z) \in {}^{3\mathrm{sc}}\Psi^{-k,-k}(X)\,,$$
$$B_k(z)(A - z) - \mathrm{Id} = R_{2,k}(z) \in {}^{3\mathrm{sc}}\Psi^{-k,-k}(X)$$

and the seminorms of order k of $B_k(z), R_{1,k}(z), R_{2,k}(z)$ are bounded by $C_k|\mathrm{Im}\, z|^{-c(k)}$.

Proof. To begin with, we claim that, for $z \in \mathbb{C}\setminus\mathbb{R}$, that $A-z$ is globally 3sc-elliptic. Indeed, by self-adjointness, $\sigma_{3\mathrm{sc}}(A)$ and $\hat{N}_{\mathrm{mf}}(A)$ are real valued, so $\sigma_{3\mathrm{sc}}(A) - z$ and $\hat{N}_{\mathrm{mf}}(A) - z$ are non-zero. Moreover, $\hat{N}_{\mathrm{ff}}(A)(\tau) \in {}^{\mathrm{sc}}\Psi^{0,0}(\mathrm{ff})$ is self-adjoint by Lemma 4.10, and thus $\hat{N}_{\mathrm{ff}}(A - z)(\tau) = \hat{N}_{\mathrm{ff}}(A)(\tau) - z$ is invertible for $z \in \mathbb{C} \setminus \mathbb{R}$.

The triple

$$\Big((\sigma_{3\mathrm{sc}}(A) - z)^{-1}, (\hat{N}_{\mathrm{mf}}(A) - z)^{-1}, (\hat{N}_{\mathrm{ff}}(A)(\tau) - z)^{-1}\Big)$$

satisfies the conditions of Proposition 4.6 and we find a symbol $b_1(z) \in C^\infty({}^{3\mathrm{sc}}\overline{T}^* X)$ such that

$$j_{3\mathrm{sc}}(b_1(z)) = \left((\sigma_{3\mathrm{sc}}(A) - z)^{-1}, (\hat{N}_{\mathrm{mf}}(A) - z)^{-1}, (\hat{N}_{\mathrm{ff}}(A)(\tau) - z)^{-1}\right) .$$

Therefore, we can quantize $b_k(z)$ to an operator $\mathrm{Op}_L(b_1(z)) = B_1(z) \in {}^{3\mathrm{sc}}\Psi^{0,0}$ satisfying

$$\begin{aligned} (A - z)B_1(z) - \mathrm{Id} &= R_{1,1}(z) , \\ B_1(z)(A - z) - \mathrm{Id} &= R_{2,1}(z) \end{aligned}$$

with $R_{1,1}(z), R_{2,1}(z) \in {}^{3\mathrm{sc}}\Psi^{-1,-1}(X)$ and by the chain rule $B_1(z), R_{1,1}(z), R_{2,1}(z)$ satisfy the seminorm bounds.

Define

$$\begin{aligned} B_k(z) &= \left(\mathrm{Id} + \sum_{j=1}^{k-1} R_{1,1}^j\right) B_1(z) , \\ B'_k(z) &= B_1(z) \left(\mathrm{Id} + \sum_{j=1}^{k-1} R_{2,1}^j\right) . \end{aligned}$$

□

Proof of Proposition 5.1. Let $\tilde{\psi} \in C_c^\infty(\mathbb{C})$ be an almost analytic extension of ψ. By the Helffer–Sjöstrand formula, we have that

$$\psi(A) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\psi}(z)(A - z)^{-1}\, dz \wedge d\bar{z} .$$

Denote by $B_k(z)$ a family of order k parametrices as in the previous lemma and define the operator $A_{\psi,k}$ by

$$A_{\psi,k} = -\frac{1}{2\pi i} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\psi}(z) B_k(z)\, dz \wedge d\bar{z} .$$

Since $\tilde{\psi}$ is compactly supported, we have that $A_{\psi,k} \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ for all k. Denote the error term by $F_k(z) = (A - z)^{-1} - B_k(z)$. We have that $|\mathrm{Im}\, z|^{c'(k)} F_k(z)$ is uniformly bounded on $\mathcal{B}(H_{\mathrm{sc}}^{r,s}(X), H_{\mathrm{sc}}^{r+k,s+k}(X))$ for some $c'(k)$ by the previous lemma and Proposition 4.3. Hence,

$$\psi(A) - A_{\psi,k} \in \mathcal{B}(H_{\mathrm{sc}}^{r,s}(X), H_{\mathrm{sc}}^{r+k,s+k}(X)) . \tag{5.4}$$

Also we have that

$$A_{\psi,k+1} - A_{\psi,k} \in {}^{3\mathrm{sc}}\Psi^{-(k+1),-(k+1)}(X) .$$

By a standard asymptotic summation argument, we obtain $\tilde{A}_\psi \in {}^{3\mathrm{sc}}\Psi^{0,0}$ such that

$$\tilde{A}_\psi \sim A_{\psi,1} + \sum_{k=1}^{\infty} A_{\psi,k+1} - A_{\psi,k} .$$

By (5.4), we have that

$$\psi(A) - \tilde{A}_\psi : C^{-\infty}(X) \longrightarrow \dot{C}^\infty(X)$$

is continuous, hence an element in ${}^{\mathrm{sc}}\Psi^{-\infty,-\infty}(X) = {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}(X)$ and therefore

$$\psi(A) \in {}^{3\mathrm{sc}}\Psi^{0,0}(X) .$$

We calculate the principal symbol as

$$\begin{aligned} j_{3\mathrm{sc}}(\psi(A)) &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \bar\partial_z\tilde\psi(z) j_{3\mathrm{sc}}((A-z)^{-1})\, dz\wedge d\bar z \\ &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \bar\partial_z\tilde\psi(z) (j_{3\mathrm{sc}}(A)-z)^{-1}\, dz\wedge d\bar z \\ &= \psi(j_{3\mathrm{sc}}(A))\,. \end{aligned}$$

Note that for the $\hat N_{\mathrm{ff}}$-component, we have used Proposition 5.4. □

Now, we are able to define the operator G_ψ. We fix $E \geq 0$ such that $D_t^2 + H_{V_0} + E$ is invertible (see Corollary 4.21).

Definition 5.6. For $\psi \in C_c^\infty(\mathbb{R})$ and $V_0 \in S^{-1}(\mathbb{R}^n_z)$, we set

$$G_\psi := \psi\left((D_t^2 + H_{V_0} + E)^{-1} P_{V_0}\right)\,. \tag{5.5}$$

Remark 5.7. G_ψ depends on V_0 explicitly, but we drop this from the notation as it causes no confusion; when we work near NP, G_ψ is defined with $V_0 = V_+$ and when we work near SP, G_ψ is defined with $V_0 = V_-$. We do not use G_ψ away from C.

From Proposition 5.1 it follows that $G_\psi \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ with indicial operator

$$\hat N_{\mathrm{ff}}(G_\psi)(\tau) = \psi\left((\tau^2 + H_{V_0} + E)^{-1}(\tau^2 - H_{V_0})\right)\,. \tag{5.6}$$

Moreover, both the fiber and mf components of the symbol of G_ψ are independent of V. Indeed, by the assumptions in Section 3.4, we have $V|_{\mathrm{mf}} = 0$, so since $V \in {}^{3\mathrm{sc}}\Psi^{0,0}$, we have

$$\sigma_{3\mathrm{sc}}(G_\psi) = \psi\left(\frac{p}{\tau^2+|\zeta|^2+m^2+E}\right)\bigg|_{{}^{3\mathrm{sc}}S[X;C]} = \psi\left(\frac{\tau^2-|\zeta|^2}{\tau^2+|\zeta|^2}\right) \tag{5.7}$$

$$\hat N_{\mathrm{mf}}(G_\psi) = \psi\left(\frac{p}{\tau^2+|\zeta|^2+m^2+E}\right)\bigg|_{{}^{\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]} \tag{5.8}$$

It will be crucial below that, although G_ψ is not smoothing, its indicial operator is smoothing for each τ. This makes it possible to compose G_ψ with quantizations of functions of τ alone; an arbitrary composition of even a compactly supported function of τ and a symbol in τ, ζ is not necessarily a PsiDO. On the other hand, we see from $G_{\psi,0}$ above that we expect $\hat N_{\mathrm{ff}}(G_\psi)$ to have symbol rapidly decaying as $|\zeta| \to \infty$, and indeed it does.

Proposition 5.8. *We have that*

$$\hat N_{\mathrm{ff}}(G_\psi) \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc}}\,.$$

Proof. We can use the functional calculus for positive self-adjoint scattering operators of order $(m,0)$, [36, Proposition 10.2] applied to the function, for $E \geq 0$, $\tilde\psi(t) := \psi((\tau^2+t+E)^{-1}(\tau^2-t))$.[5] Then, for every fixed τ, $\hat N_{\mathrm{ff}}(G_\psi)(\tau) \in {}^{\mathrm{sc}}\Psi^{-\infty,0}(\mathbb{R}^n)$. Since $\tilde\psi$ is smooth in $h = 1/\tau$ up to $h = 0$, the assertion follows. □

[5]Strictly speaking this is not a smooth function on $\mathbb{R}$, but since the spectrum of H_{V_0} is bounded from below by E we can change $\tilde\psi$ to a smooth function without changing $\tilde\psi(H_{V_0})$

Since V_0 is static, $D_t^2 + H_{V_0}$ and $D_t^2 - H_{V_0}$ commute and therefore

$$[G_\psi, P_{V_0}] = 0\,. \tag{5.9}$$

Consequently,

$$[G_\psi, P_V] = -[G_\psi, V'] \in {}^{3\mathrm{sc}}\Psi^{1,-1}\,,$$

which means that G_ψ and P_V commute to leading order.

The indicial operators of G_ψ and $G_{\psi,0}$, which the Fourier multiplier defined by (5.1), differ by a lower order operator:

Lemma 5.9. *We have that*

$$(\widehat{G_\psi})_{\mathrm{ff}} - (\widehat{G_{\psi,0}})_{\mathrm{ff}} \in \Psi^{-\infty,-1,-1}_{\mathrm{scl,sc}}\,.$$

Proof. We have that

$$\widehat{G_\psi}_{\mathrm{ff}}(\tau) = \psi\left((\tau^2 + H_{V_0} + E)^{-1}(\tau^2 - H_{V_0})\right)\,.$$

By [36, Proposition 10.3] and the fact that $j_{\mathrm{sc},2,0}(H_{V_0}) = j_{\mathrm{sc},2,0}(H_0)$, we have that

$$j_{\mathrm{sc}}\left((\widehat{G_\psi})_{\mathrm{ff}}(\tau)\right) = j_{\mathrm{sc}}\left((\widehat{G_{\psi,0}})_{\mathrm{ff}}(\tau)\right)$$

The semiclassical principal symbol is given by

$$\sigma_{\mathrm{scl},h=1/\tau}\left((\widehat{G_\psi})_{\mathrm{ff}}\right) = \psi\left((1+h^2\Delta)^{-1}(1-h^2\Delta)\right)\,,$$

which is independent of V and therefore agrees with the semiclassical principal symbol of $G_{\psi,0}$. □

Next, we prove the "shrinking window" lemma that states that if we choose ψC_c^∞ supported in a sufficiently small neighborhood of 0, then the operator norm of $\hat{N}_{\mathrm{ff}}(G_\psi)(\tau)$ is arbitrary small considered when considered as a map $H^{1,1}_{\mathrm{scl}} \to L^2$. This lemma is crucial in proving the positive commutator estimate, because we can control the error terms coming from commutators with the potential.

Lemma 5.10. *Assuming the spectrum of H_{V_0} is purely absolutely continuous at m^2, then there is $\delta > 0$, such that, for every $\varepsilon > 0$ there exists $\sigma > 0$ such that if $\psi \in C_c^\infty(\mathbb{R})$ has* $\operatorname{supp}\psi \subset (-\sigma,\sigma)$ *and $0 \le \psi \le 1$, then*

$$\|\mathbb{1}_{[m^2-\delta,+\infty)}(\tau)\hat{N}_{\mathrm{ff}}(G_\psi)(\tau)\circ\iota\|_{\mathcal{B}(H^{1,1}_{\mathrm{scl}},L^2)} < \varepsilon$$

where $\iota\colon H^{1,1}_{\mathrm{scl}} \hookrightarrow L^2$ is the natural inclusion map.

Remark 5.11. The same statement holds true for $(-\infty, -m^2+\delta]$.

Proof. First, let $\kappa > m^2$ be fixed; we claim that there ψ as in the lemma such that for $\tau \in [m^2-\delta, \kappa]$, that

$$\hat{N}_{\mathrm{ff}}(G_\psi)(\tau)\circ\iota\colon H^{1,1}_{\mathrm{scl}} \longrightarrow L^2$$

has mapping norm less than ϵ. To see this, recall that from Proposition 4.15 we have $\hat{N}_{\mathrm{ff}}(G_\psi)$ bounded on L^2 uniformly in τ. Since $\iota\colon H^{1,1}_{\mathrm{scl}}(\mathrm{ff}) \hookrightarrow L^2(\mathrm{ff})$ is compact, it suffices to show that for any sequence of $\psi_n \in C_c^\infty(\mathbb{R})$ with $\psi_n(s) = 1$ for $|s| \le 1/n$ and $0 \le \psi_n \le 1$, that $\hat{N}_{\mathrm{ff}}(G_{\psi_n}) \to 0$ in the strong operator topology on L^2, uniformly for $\tau \in [m^2-\delta,\kappa]$. This in turn follows from continuous functional calculus, specifically from (5.6), which identifies $\hat{N}_{\mathrm{ff}}(G_{\psi_n})$ with ψ_n of

$$(\tau^2 + H_{V_0} + E)^{-1}(\tau^2 - H_{V_0}) = (1 + \rho^2 H_{V_0} + \rho^2 E)^{-1}(1 - \rho^2 H_{V_0}),$$

which has purely absolutely continuous spectrum in a neighborhood of 0 by assumption.

On the other hand, the difference with the free operator satisfies

$$(\tau^2 + H_{V_0} + E)^{-1}(\tau^2 - H_{V_0}) - (\tau^2 + H_0 + E)^{-1}(\tau^2 - H_0) \in \Psi^{-1,-1,-1}_{\mathrm{scl,sc}}, \tag{5.10}$$

and for any $\epsilon' > 0$ there is $\kappa > 0$ such that this difference has norm $< \epsilon'$ for $\tau \geq \kappa$. Again functional calculus gives that $G_\psi(\tau) - G_{\psi_0}(\tau)$ has norm less then ϵ'. But $G_{\psi_0}(\tau)$ can be seen to satisfy the conclusion of the lemma explicitly, so the lemma follows for G_ψ.

If $\tilde{\psi}$ satisfies $\tilde{\psi}\psi = \tilde{\psi}$, then also

$$G_{\tilde{\psi}} G_\psi = G_{\tilde{\psi}}, \tag{5.11}$$

and thus the bound holds for all such $\hat{N}_{\mathrm{ff}}(G_{\tilde{\psi}})$ and the lemma is proven. □

5.2. **Further elliptic regularity near C.** If $P_V u \in H^{s,r}_{\mathrm{sc}}$, then away from C, then on ${}^{3\mathrm{sc}}\mathrm{Ell}(P_V)$ we have the estimates in Section 4.6. At C, frequency localization becomes global in the interaction variables, and, as discussed in above, we use G_ψ to localize near the characteristic set. Thus, for $\psi \in C^\infty_c$ with $\psi(s) = 1$ for $|s| < \delta$, $\delta > 0$, the operator

$$\mathrm{Id} - G_\psi$$

is morally speaking a localizer to an elliptic region over NP (or SP). Indeed, for P_V with V non-static, if Q is a localizer to a neighborhood of ${}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}}X$ and G_ψ is defined using V_+, then $Q(I - G_\psi)$ is a reasonable generalization of localization to the elliptic set from the sc setting.

Thus we expect to have an elliptic regularity statement using $I - G_\psi$, and such a statement will be crucial below as all the propagation estimates are proven first using commutators which have factors of G_ψ to localize *to* the characteristic set. Specifically, if $P_V u \in H^{s,r}_{\mathrm{sc}}$ then we should have $(I - G_\psi)u \in H^{s+2,r}_{\mathrm{sc}}$ as well, at least locally near *poles*. That is what we prove in this section.

We do so with a parametrix construction Note first that for static $V = V_0$, we have

$$E_\psi P_{V_0} = \mathrm{Id} - G_\psi \tag{5.12}$$

on the nose, since, if

$$E_\psi = F_\psi \circ (D_t^2 + H_{V_0} + E)^{-1} \tag{5.13}$$

we can set

$$F_\psi = f_\psi((D_t^2 + H_{V_0} + E)^{-1} P_{V_0}) \quad \text{with} \quad f_\psi(s) = \frac{1 - \psi(s)}{s}.$$

This definition of F_ψ makes sense as a bounded operator on L^2 by continuous functional calculus since $(D_t^2 + H_{V_0} + E)^{-1} P_{V_0} \in {}^{3\mathrm{sc}}\Psi^{0,0}$ is a bounded symmetric operator. Note that f_ψ is not compactly supported, so the results of Section 5 do not apply directly.

If we knew that $E_\psi \in {}^{3\mathrm{sc}}\Psi^{-2,0}$ then we would immediately have estimates. That is the purpose of the following lemma.

Lemma 5.12. *Let $\psi \in C^\infty_c(\mathbb{R})$ with $\psi(s) = 1$ for $|s| < \delta$. Then there is $\chi_{\mathrm{NP}} \in C^\infty(X)$ supported sufficiently close to* NP *and*

$$E_\psi \in {}^{3\mathrm{sc}}\Psi^{-2,0}$$

such that

$$\chi_{\mathrm{NP}} \cdot (E_\psi P_V - (I - G_\psi)) \in {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}.$$

The analogous statement holds near SP*, see Remark 5.7.*

The lemma gives estimates:

Corollary 5.13. *Let $s, r \in \mathbb{R}$, $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(s) = 1$ for $|s| < \delta$. Let $Q, G \in {}^{3\mathrm{sc}}\Psi^{0,0}$. If ${}^{3\mathrm{sc}}\mathrm{WF}'(Q) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(\chi_{\mathrm{NP}}) \cap {}^{3\mathrm{sc}}\mathrm{Ell}(G)$, then*

$$\|Q(I - G_\psi)u\|_{s+2,r} \lesssim \|GP_V u\|_{s,r} + \|u\|_{-N,-M},$$

with the left hand side being finite if the right hand side is finite.

The analogous statement holds near SP*, see Remark 5.7.*

Proof of corollary from lemma. This follows exactly as in elliptic estimates, namely from Lemma 5.12, we have

$$\begin{aligned}\|Q(I - G_\psi)u\|_{s+2,r} &\lesssim \|Q\chi_{\mathrm{NP}}(I - G_\psi)u\|_{s+2,r} + \|u\|_{-N,-M}\\ &\lesssim \|Q\chi_{\mathrm{NP}}E_\psi P_V u\|_{s+2,r} + \|u\|_{-N,-M}\\ &\lesssim \|GP_V u\|_{s,r} + \|u\|_{-N,-M}\end{aligned}$$

where the last line uses the boundedness properties of $Q\chi_{\mathrm{NP}}E_\psi \in {}^{3\mathrm{sc}}\Psi^{-2,0}$. □

Now we prove the lemma.

Proof. It is more convenient to work with F_ψ as its symbols will be defined using functional calculus of bounded operators. Thus we seek $F_\psi \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with

$$\chi_{\mathrm{NP}} \cdot \big(F_\psi \circ (D_t^2 + H_{V_+} + E)^{-1} P_V - (I - G_\psi)\big) \in {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}.$$

The lemma follows immediately by defining E_ψ as in (5.13).

Now we construct F_ψ. We do so iteratively at the symbolic level. We will use repeatedly that if $\psi, \tilde\psi \in C_c^\infty(\mathbb{R})$,

$$\psi\tilde\psi = \tilde\psi \implies (I - G_\psi)(I - G_{\tilde\psi}) = I - G_\psi.$$

At each stage of the construction we shrink the region in which the symbol conditions are satisfied. Fixing $K \subset \{s : \psi(s) = 1\}$, we take $\psi_1 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_1 \subset K$. We take $F_1 \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with

$$E_1 := F_1(D_t^2 + H_{V_+} + E)^{-1} P_V - (I - G_{\psi_1}) \in {}^{3\mathrm{sc}}\Psi^{-1,-1}.$$

The front face symbol condition is then exactly given by the function f_{ψ_1} (5.13),

$$\hat N_{\mathrm{ff}}(F_1)(\tau) = f_{\psi_1}((\tau^2 + H_{V_+} + E)^{-1}(\tau^2 - H_{V_+})). \tag{5.14}$$

(That such an indicial family exists follows from an analogous construction in the scl, sc calculus.) The two other symbol conditions are easy to arrange as they can be solved on off the characteristic set of P_V. For the inductive step, assume that for some $\psi_n \in C_c^\infty(\mathbb{R})$ which is 1 near 0, with $\operatorname{supp}\psi_n \subset K$, and $F_1, \ldots, F_n$ with $F_j \in {}^{3\mathrm{sc}}\Psi 1 - j, 1 - j$ such that

$$E_n := (I - G_{\psi_n})(\sum_{j=1}^n F_j)(D_t^2 + H_{V_+} + E)^{-1}(D_t^2 - H_{V_+}) - (I - G_{\psi_n}) \in \Psi^{-n,-n}.$$

Then we solve for F_{n+1} such that

$$F_{n+1}(D_t^2 + H_{V_+} + E)^{-1}(D_t^2 - H_{V_+}) + (I - G_{\psi_n})K_n \in {}^{3\mathrm{sc}}\Psi^{-n-1,-n-1}.$$

which is to say $\hat{N}_{\mathrm{ff},-n}(F_{n+1})(\tau) = F_\psi((\tau^2 + H_{V_+} + E)^{-1}(-\tau^2 + H_{V_+}))\hat{N}_{\mathrm{ff},-n}(E_n)$. Then for any $\psi_{n+1} \in C_c^\infty(\mathbb{R})$ with $\psi_{n+1}\psi_n = \psi_n$,

$$(I - G_{\psi_{n+1}})(\sum_{j=1}^{n+1} F_j)(D_t^2 + H_{V_+} + E)^{-1}(D_t^2 - H_{V_+}) - (I - G_{\psi_{n+1}}) \in \Psi^{-n-1,-n-1}.$$

Thus by induction there are $F_1, \dots,$ with $F_j \in {}^{3\mathrm{sc}}\Psi^{1-j,1-j}$ such that

$$(I - G_\psi)(\sum_{j=1}^{n} F_j)(D_t^2 + H_{V_+} + E)^{-1}(D_t^2 - H_{V_+}) - (I - G_\psi) \in \Psi^{-n,-n},$$

for any n, and taking the Borel sum gives

$$F_\psi = (I - G_\psi)(\sum_{j=1}^{\infty} F_j)$$

It remains only to know that $f_\psi((\tau^2 + H_{V_+} + E)^{-1}(\tau^2 - H_{V_+})) \in \Psi^{0,0,0}_{\mathrm{scl},\mathrm{sc}}$, but this follows from an identical parametrix argument. □

5.3. **Localization near the characteristic set.** We now construct the operators which we use as microlocalizers and commutators near C. There will be of the form

$$\mathrm{Op}_L(q)G_\psi\,, \tag{5.15}$$

where $q \in C^\infty({}^{\mathrm{sc}}T^*X)$. We are mainly interested in the case that

$$q|_{T_{\mathrm{NP}}}(\tau, \zeta) = q(0, 0, \tau, \mu) = f(\tau) \tag{5.16}$$

for some $f \in C^\infty(\overline{W^\perp})$, for this mimics the centrality condition in (4.17), with the crucial difference that in general these q do not lie in ${}^{3\mathrm{sc}}S^{m,l}(X; C)$. Indeed, even for $f(\tau) \in C_c^\infty(\mathbb{R}_\tau)$, any function q satisfying (5.16) is not a symbol since it does not exhibit additional vanishing in ζ under application of ∂_τ.

We will see that, despite the fact that such q are not 3sc-symbols, that the operator in (5.15) is a 3sc-operator provided

$$q \in C^\infty([{}^{\mathrm{sc}}\overline{T}^*X; \mathrm{fibeq}]) \tag{5.17}$$

This is the same blow up that appears in the proof of Lemma 4.14. Rather than describe this blow up in detail, we simply say that q satisfies (5.17) if: (1) it is a classical symbol in ζ of order zero and smooth in τ in regions $|\tau| < C$, (2) in regions with $|\zeta| < C$, $\pm\tau > C > 0$ it is smooth in $\rho = \pm 1/\tau$ down to $\rho = 0$, and (3) in regions where $\mu = \zeta/\rho$ is bounded, it is a classical symbol in μ, smooth in ρ down to $\rho = 0$.

While, (5.17) will ensure that (5.15) lies in ${}^{3\mathrm{sc}}\Psi^{0,0}$, the condition (5.16) will ensure that commutators with P_V have the correct order (i.e. lose one order compared with composition.) This is established in the following proposition, in which we use $\rho = 1/\tau$ to modify the differential order so that the centrality condition is preserved; this leads to the appearance of factors of $\langle \tau, \zeta \rangle / \tau$ in the local components of the symbol. Also (see Remark 2.4) we use powers of x to rescale the symbol. Here we also define all symbols using a fiber defining function ρ_{fib} which is equal to ρ near the characteristic set.

This proof follows from the same arguments as in Vasy [36, Proposition 13.1]. The main difference is that since G_ψ is of differential order 0 and vanishes to infinite order at fibeq (by Proposition 5.8), the product is a 3sc-operator if q is smooth on $[{}^{\mathrm{sc}}T^*X; \mathrm{fibeq}]$.

Proposition 5.14. *Let $q \in C^\infty([{}^{\mathrm{sc}}T^*X; \mathrm{fibeq}])$ satisfying (5.16), and assume that there is $c > 0$ such that $\operatorname{supp} q \cap [-c, c] = \varnothing$. Then for $s, \ell \in \mathbb{R}$,*

$$\mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi \in {}^{3\mathrm{sc}}\Psi^{s,\ell}(X)\,.$$

The components of the principal symbol are

$$\hat{N}_{\mathrm{ff}}(\mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi)(\tau) = \rho^{-s}f(\tau)\cdot\hat{N}_{\mathrm{ff}}(G_\psi)(\tau)\,,$$

and

$$\begin{aligned}\sigma_{3\mathrm{sc}}(\mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi) &= q|_{{}^{\mathrm{sc}}S^*X}\cdot\sigma_{3\mathrm{sc}}(G_\psi)\,,\\ \hat{N}_{\mathrm{mf}}(\mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi) &= q|_{{}^{\mathrm{sc}}T^*_{\partial X}X}\cdot\hat{N}_{\mathrm{mf}}(G_\psi)\,,\end{aligned}$$

where we have used implicitly $\rho_\infty = x$, $\rho_{\mathrm{fib}} = \rho$ near NP *on* $\operatorname{supp}(\sigma_{3\mathrm{sc}}(G_\psi))$ *and* $\operatorname{supp}(\hat{N}_{\mathrm{mf}}(G_\psi))$.

We note that the commutator with P_V decreases the order as expected:

Proposition 5.15. *The commutator of P_V and QG_ψ satisfies $[P_V, QG_\psi] \in {}^{3\mathrm{sc}}\Psi^{s+1,\ell-1}(X)$ and*

$$\begin{aligned}\tau^{-s}\hat{N}_{\mathrm{ff},\ell-1}([P_V, QG_\psi]) = -f(\tau)[\partial_x V'|_{x=0}, (\widehat{G_\psi})_{\mathrm{ff}}]&\\ -[H_{V_0}, \mathrm{Op}_L(\partial_x q(0,0,\tau,\zeta) + z\partial_y q(0,0,\tau,\zeta))](\widehat{G_\psi})_{\mathrm{ff}}&\\ -2i\ell\tau f(\tau)(\widehat{G_\psi})_{\mathrm{ff}}\,.&\end{aligned}$$

Proof. Without loss of generality, we may assume that $s = 0$. By Proposition 5.14, we have that the front face symbols of QG_ψ and P_V commute. To calculate the indicial operator of the commutator, we use Proposition 4.13 with $A = P_V$ and $B = QG_\psi$. We have that

$$\begin{aligned}\hat{A}_{\mathrm{ff}} &= \tau^2 - H_{V_0}\,,\\ D_\tau\hat{A}_{\mathrm{ff}} &= -2i\tau\,,\\ \hat{A}'_{\mathrm{ff}} &= -\partial_x V'|_{x=0}\,,\\ \hat{B}_{\mathrm{ff}} &= f(\tau)(\widehat{G_\psi})_{\mathrm{ff}}\,,\\ D_\tau\hat{B}_{\mathrm{ff}} &= -if'(\tau)(\widehat{G_\psi})_{\mathrm{ff}} + f(\tau)D_\tau(\widehat{G_\psi})_{\mathrm{ff}}\,,\\ \hat{B}'_{\mathrm{ff}} &= \mathrm{Op}_L(\partial_x q(0,0,\tau,\zeta) + z\partial_y q(0,0,\tau,\zeta))(\widehat{G_\psi})_{\mathrm{ff}} + f(\tau)(\widehat{G_\psi})'_{\mathrm{ff}}\,.\end{aligned}$$

Using that $(\widehat{G_\psi})_{\mathrm{ff}}$ and H_{V_0} commutate we calculate

$$\begin{aligned}[\hat{A}'_{\mathrm{ff}} - D_\tau\hat{A}_{\mathrm{ff}}, \hat{B}_{\mathrm{ff}}] &= -f(\tau)[\partial_x V'|_{x=0}, (\widehat{G_\psi})_{\mathrm{ff}}]\,,\\ [\hat{A}_{\mathrm{ff}}, \hat{B}'_{\mathrm{ff}} - D_\tau\hat{B}_{\mathrm{ff}}] &= -[H_{V_0}, \hat{B}'_{\mathrm{ff}}]\,,\\ &= -[H_{V_0}, \mathrm{Op}_L(\partial_x q(0,0,\tau,\zeta) + z\partial_y q(0,0,\tau,\zeta))](\widehat{G_\psi})_{\mathrm{ff}} - f(\tau)[H_{V_0}, (\widehat{G_\psi})'_{\mathrm{ff}}]\,.\end{aligned}$$

The second summand vanishes because $(H_{V_0})' = 0$ implies that $(\widehat{G_\psi})'_{\mathrm{ff}} = 0$. This proves the claim. ∎

The positive commutator argument will evaluate the commutator of the unperturbed operator in the scattering calculus and therefore we need a to compare the commutator in the perturbed and unperturbed setting (cf. [36, Corollary 13.4]):

Corollary 5.16. *Let*

$$R(\tau) \coloneqq \tau^{-s} \left(\hat{N}_{\mathrm{ff}}([P_V, QG_\psi])(\tau) - \hat{N}_{\mathrm{ff}}([P_0, QG_{\psi,0}])(\tau) \right) .$$

Then,

$$R \in \Psi^{-\infty,-1,0}_{\mathrm{scl,sc}}$$

and

$$\|R(\tau)\|_{\mathcal{B}(L^2, H^{1,1}_{\mathrm{scl}})} \lesssim \sup\{|D^\alpha_{x,y,\tau} D^\beta_\zeta q(0,0,\tau,\zeta)| \colon |\alpha| \le 1, |\beta| \le cn\} ,$$

where $c > 0$ is a universal constant and the implied constant is independent of τ and q.

Proof. For brevity, we set $q_x(\tau,\zeta) \coloneqq \partial_x q(0,0,\tau,\zeta)$ and $q_y(\tau,\zeta) = \partial_y q(0,0,\tau,\zeta)$. From the previous proposition, we calculate that

$$\begin{aligned} R(\tau) = &-f(\tau)[\partial_x V'|_{x=0}, (\widehat{G_\psi})_{\mathrm{ff}}] + [V_0, \mathrm{Op}_L(q_x + zq_y)](\widehat{G_\psi})_{\mathrm{ff}} \\ &- i\left(-2\,\mathrm{Op}_L(\zeta q_y) + 2\ell\tau f(\tau)\right)\left((\widehat{G_\psi})_{\mathrm{ff}} - (\widehat{G_{\psi,0}})_{\mathrm{ff}}\right) . \end{aligned}$$

We have that $(\widehat{G_\psi})_{\mathrm{ff}} \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc}}$ and therefore the first term is in $\Psi^{-\infty,-1,0}_{\mathrm{scl,sc}}$ and bounded independently of q and τ. The commutator $[V_0, \mathrm{Op}_L(q_x + zq_y)]$ is in $\Psi^{0,-1,0}_{\mathrm{scl,sc}}$ and we can estimate its operator norm as a map $H^{1,0}_{\mathrm{sc}} \to H^{1,1}_{\mathrm{sc}}$ by

$$C \sup\{|D^\alpha_{x,y} D^\beta_\zeta q(0,0,\tau,\zeta)| \colon |\alpha| = 1, |\beta| \le cn\} ,$$

and composition with $(\widehat{G_\psi})_{\mathrm{ff}}$ is a bounded map $L^2 \to H^{1,1}_{\mathrm{sc}}$ with the same norm.

From Lemma 5.9, we have that $(\widehat{G_\psi})_{\mathrm{ff}} - (\widehat{G_{\psi,0}})_{\mathrm{ff}} \in \Psi^{-\infty,-1,-1}_{\mathrm{scl,sc}}$ and

$$\mathrm{Op}_L(\zeta q_y) - \ell\tau f(\tau) \in \Psi^{0,0,1}_{\mathrm{scl,sc}}$$

and the operator norm is bounded by

$$C \sup\{|D^\alpha_y D^\beta_\zeta q(0,0,\tau,\zeta)| \colon |\alpha| \le 1, |\beta| \le cn\} ,$$

which completes the proof. □

5.4. **Gårding type theorems.** In this section, we state and prove a sharp Gårding type theorem for 3sc-operators. In contrast to Vasy [36], we use a localization operator G_ψ that is in ${}^{3\mathrm{sc}}\Psi^{0,0}$ and therefore we have to take the fiber principal symbol in account. We will consider the general situation of a localizer $\psi(A)$, where $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$ is self-adjoint. In the case of the Klein-Gordon equation, we take $A = (D_t^2 + H_{V_0} + E^2)^{-1} P_{V_0}$ in which case $\psi(A) = G_\psi$.

Our proof again follows [36], in particular using a method for construction of square roots of operators, which we recall now. In [36] this appears as Lemma C.1, but as we do not use it directly here, we merely state the result and recall the method of proof.

Assume that we are given self-adjoint operators $A, Q \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$, $c > 0$ and $\psi \in C^\infty_c(\mathbb{R})$ real-valued with $\psi(x) = 1$ for $|x| < \delta$ for some $\delta > 0$. Then, if we have a bound from below of the form If

$$\psi(A)Q\psi(A) \ge c\psi(A)^2 ,$$

then for any $c' \in (0,c)$ and $\phi \in C^\infty_c(\mathbb{R})$ with $\phi\psi = \phi$, we can find a square root $B \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ in the sense that

$$\phi(A)(Q - c')\phi(A) = \phi(A)B^*B\phi(A) . \tag{5.18}$$

We recall the proof almost verbatim from [36], the main difference being that our $\psi(A)$ is in ${}^{3\mathrm{sc}}\Psi^{0,0}$ as opposed to ${}^{3\mathrm{sc}}\Psi^{-\infty,0}$. Define

$$P = \psi(A)Q\psi(A) + c(\mathrm{Id} - \psi(A)^2) \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)\,.$$

Since $P \geq c$ we have that $P - c' \geq c - c' > 0$. Then we can apply Proposition 5.1 to take the square root of $P - c'$, i.e. we take $f(P - c')$ with a function $f \in C_c^\infty(\mathbb{R})$ such that $f(t) = \sqrt{t}$ on the spectrum of $P - c'$. The function exists because $\sigma(P - c') \subset [c - c', C]$ for some $C > 0$. We then have that

$$\tilde{P} \coloneqq (P - c')^{1/2} \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)\,.$$

We choose a $\psi_1 \in C_c^\infty(\mathbb{R})$ with $\psi_1 \equiv 1$ on $\operatorname{supp}\phi$ and $\psi_1 \equiv 0$ on $\operatorname{supp}(1 - \psi)$. We calculate that

$$\begin{aligned}\psi_1(A)\tilde{P}^2\psi_1(A) &= \psi_1(A)(P - c')\psi_1(A)\\ &= \psi_1(A)(A - c')\psi_1(A)\,.\end{aligned}$$

Let $B = \tilde{P}\psi_1(A)$, then multiplying the previous equation yields the equation (5.18).

We now have the non-sharp Gårding inequality which operates under the assumption that the principal symbol is strictly positive.

Proposition 5.17. *Let $A, Q, C \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ be self-adjoint and assume $\hat{N}_{\mathrm{ff}}(A) \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc},\pm 1/\tau}$. Suppose that the principal symbol of C satisfies*

$$\begin{aligned}\sigma_{3\mathrm{sc}}(C) &= c_{\mathrm{fib}} \cdot \psi_0(\sigma_{3\mathrm{sc}}(A))^2\,,\\ \hat{N}_{\mathrm{mf}}(C) &= c_{\mathrm{mf}} \cdot \psi_0(\hat{N}_{\mathrm{mf}}(A))^2\,,\\ \hat{N}_{\mathrm{ff}}(C) &= c_{\mathrm{ff}} \cdot \psi_0(\hat{N}_{\mathrm{ff}}(A))^2\,,\end{aligned}$$

where $\psi_0 \in C_c^\infty(\mathbb{R})$, $c_{\mathrm{fib}} \in C^\infty({}^{\mathrm{sc}}S^*_{X\setminus C}X)$, $c_{\mathrm{mf}} \in C^\infty({}^{\mathrm{sc}}T^*_{\partial X\setminus C}X)$ and $c_{\mathrm{ff}} \in C^\infty(\overline{W^\perp})$. We assume that

(1) $c_0 \leq c_\bullet \leq c_0'$ for $\bullet \in \{\mathrm{fib}, \mathrm{mf}, \mathrm{ff}\}$ and some $c_0', c_0 > 0$,
(2) $\psi_0(x) = 1$ for $|x| < \delta_0$,
(3) there exists $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(x) = 1$ for $|x| \leq \delta_1$ and $\operatorname{supp}\psi \cap \operatorname{supp}(1 - \psi_0) = \varnothing$ such that

$$\begin{aligned}\psi(\sigma_{3\mathrm{sc}}(A))\sigma_{3\mathrm{sc}}(Q)\psi(\sigma_{3\mathrm{sc}}(A)) &\geq c_{\mathrm{fib}}\psi(\sigma_{3\mathrm{sc}}(A))^2\,,\\ \psi(\hat{N}_{\mathrm{mf}}(A))\hat{N}_{\mathrm{mf}}(Q)\psi(\hat{N}_{\mathrm{mf}}(A)) &\geq c_{\mathrm{mf}}\psi(\hat{N}_{\mathrm{mf}}(A))^2\,,\\ \psi(\hat{N}_{\mathrm{ff}}(A))\hat{N}_{\mathrm{ff}}(Q)\psi(\hat{N}_{\mathrm{ff}}(A)) &\geq c_{\mathrm{ff}}\psi(\hat{N}_{\mathrm{ff}}(A))^2\,.\end{aligned} \tag{5.19}$$

Then for any $\varepsilon \in (0, 1)$ and $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \cap \operatorname{supp}(1 - \psi) = \varnothing$, there exists $R \in {}^{3\mathrm{sc}}\Psi^{-1,-1}(X)$ such that

$$\phi(A)Q\phi(A) \geq (1 - \varepsilon)\phi(A)C\phi(A) + R\,.$$

Proof. The idea is to construct a square-root of $Q - (1 - \varepsilon)C$ modulo lower order terms. We follow the methodology described before the proof to take square roots first of each of the symbols individually. That is, we write, for $\bullet \in \{\mathrm{mf}, \mathrm{ff}\}$,

$$P_\bullet(\tau) = \hat{N}_\bullet(\psi(A)Q\psi(A)) + c_\bullet(\mathrm{Id} - \hat{N}_\bullet(\psi(A))^2)$$

and

$$P_{\mathrm{fib}}(\tau) = \sigma_{3\mathrm{sc}}(\psi(A)Q\psi(A)) + c_{\mathrm{fib}}(\mathrm{Id} - \sigma_{3\mathrm{sc}}(\psi(A))^2).$$

Then, with $f \in C_c^\infty(\mathbb{R})$ with $f(x) = \sqrt{x}$ for $\epsilon \leq x \leq c$ for c sufficiently large, we let

$$B_\bullet = f(P_\bullet - (1-\epsilon)c_\bullet)\psi_1(A), \quad \bullet \in \{\mathrm{fib}, \mathrm{mf}, \mathrm{ff}\}\,.$$

The symbols $B_{\mathrm{fib}}, B_{\mathrm{ff}}, B_{\mathrm{mf}}$ satisfy the conditions of Proposition 4.6; the matching conditions follow easily, and the fact that B_{ff} lies $\Psi^{0,0,0}_{\mathrm{scl,sc},\pm 1/\tau}$ and satisfies the appropriate smoothness condition follows from the fact that P_{ff} is semiclassically scattering elliptic and $\hat{N}_{\mathrm{ff}}(\psi_1(A)) \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc},\pm 1/\tau}$.

Therefore we find a $B \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ with $\sigma_{3\mathrm{sc}}(B) = B_{\mathrm{fib}}$, $\hat{N}_{\mathrm{ff}}(B) = B_{\mathrm{ff}}$ and $\hat{N}_{\mathrm{mf}}(B) = B_{\mathrm{mf}}$. Hence there is a $R \in {}^{3\mathrm{sc}}\Psi^{-1,-1}(X)$ such that

$$\phi(A)\,(Q - (1-\varepsilon)C)\,\phi(A) = \phi(A)B^*B\phi(A) + R\,.$$

Since $\phi(A)B^*B\phi(A) \geq 0$ this proves the proposition. □

Lastly, we have the sharp Gårding inequality that only assumes that the principal symbol is non-negative.

Proposition 5.18. *Let $A, Q, C \in {}^{3\mathrm{sc}}\Psi^{0,0}(X)$ be self-adjoint and assume $\hat{N}_{\mathrm{ff}}(A) \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc},\pm 1/\tau}$.*

Suppose that the principal symbol of C satisfies

$$\begin{aligned} \sigma_{3\mathrm{sc}}(C) &= c_{\mathrm{fib}} \cdot \psi_0(\sigma_{3\mathrm{sc}}(A))^2\,, \\ \hat{N}_{\mathrm{mf}}(C) &= c_{\mathrm{mf}} \cdot \psi_0(\hat{N}_{\mathrm{mf}}(A))^2\,, \\ \hat{N}_{\mathrm{ff}}(C) &= c_{\mathrm{ff}} \cdot \psi_0(\hat{N}_{\mathrm{ff}}(A))^2\,, \end{aligned}$$

where $\psi_0 \in C_c^\infty(\mathbb{R})$, $c_{\mathrm{fib}} \in C^\infty({}^{\mathrm{sc}}S^*_{X\setminus C}X)$, $c_{\mathrm{mf}} \in C^\infty({}^{\mathrm{sc}}T^*_{\partial X\setminus C}X)$ and $c_{\mathrm{ff}} \in C^\infty(\overline{W^\perp})$.

Assume that

(1) $c_\bullet \geq 0$ for $\bullet \in \{\mathrm{fib}, \mathrm{mf}, \mathrm{ff}\}$,
(2) either c_{ff} vanishes in a neighborhood of $\pm\infty$ or $c_{\mathrm{ff}}(\pm\infty) > 0$,
(3) if $c_{\mathrm{fib}}(\xi) = 0$ for $\xi \in {}^{\mathrm{sc}}S^*_{X\setminus C}X$, then $\sigma_{3\mathrm{sc}}(Q)(\xi) = 0$ and the analogous condition for c_{mf} and c_{ff},
(4) $\sqrt{c_\bullet} \in C^\infty$ and vanishes to infinite order at points ξ where $c_\bullet(\xi) = 0$,
(5) the symbols

$$c_{\mathrm{ff}}^{-1}\hat{N}_{\mathrm{ff}}(Q), \quad c_{\mathrm{mf}}^{-1}\hat{N}_{\mathrm{mf}}(Q), \quad c_{\mathrm{fib}}^{-1}\sigma_{3\mathrm{sc}}(Q)$$

are bounded together with all their derivatives on $\overline{W^\perp}$, ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]$, ${}^{3\mathrm{sc}}S^*[X;C]$, respectively,
(6) $\psi_0(x) = 1$ for $|x| < \delta_0$,
(7) there exists $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(x) = 1$ for $|x| \leq \delta_1$ and $\operatorname{supp}\psi \cap \operatorname{supp}(1-\psi_0) = \varnothing$ such that

$$\begin{aligned} \psi(\sigma_{3\mathrm{sc}}(A))\sigma_{3\mathrm{sc}}(Q)\psi(\sigma_{3\mathrm{sc}}(A)) &\geq c_{\mathrm{fib}}\psi(\sigma_{3\mathrm{sc}}(A))^2\,, \\ \psi(\hat{N}_{\mathrm{mf}}(A))\hat{N}_{\mathrm{mf}}(Q)\psi(\hat{N}_{\mathrm{mf}}(A)) &\geq c_{\mathrm{mf}}\psi(\hat{N}_{\mathrm{mf}}(A))^2\,, \\ \psi(\hat{N}_{\mathrm{ff}}(A))\hat{N}_{\mathrm{ff}}(Q)\psi(\hat{N}_{\mathrm{ff}}(A)) &\geq c_{\mathrm{ff}}\psi(\hat{N}_{\mathrm{ff}}(A))^2\,. \end{aligned} \tag{5.20}$$

Then for any $\varepsilon \in (0,1)$ *and* $\phi \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}\phi \cap \operatorname{supp}(1-\psi) = \varnothing$*, there exists* $R \in {}^{3\mathrm{sc}}\Psi^{-1,-1}(X)$ *such that*

$$\phi(A)Q\phi(A) \geq (1-\varepsilon)\phi(A)C\phi(A) + R\,.$$

Proof. In the case that $c_{\mathrm{ff}}(\tau) = 0$ for $\tau \gg 0$, the argument is identical to that in [36, Proposition C.3]. In the case $c_{\mathrm{ff}}(+\infty) \neq 0$, we put the previous lemma microlocally near $+\infty \in \overline{W^\perp}$ and then the same argument again from [36]. □

6. Propagation estimates over C

We now prove propagation of singularities estimates over C. As elsewhere, since the arguments are identical at the two points in C, we focus on NP. Our commutant construction follows [36, Chap. 14] closely, and as such we attempt to be faithful to the notation there for ease of comparison, although we make some changes in order to decrease the overall amount of notation, which is substantial.

Note that τ is preserved along the flow at NP; as described above, the global nature of the operator above ff leads to propagation phenomenon analogous to diffraction, namely that singularities entering at a given τ level in the characteristic set at NP may emerge, still at level τ, in any direction. Thus, if we wish to control a distribution u at a specific τ_0 in $\overline{W^\perp}$ over NP, we must assume a priori control of u along all bicharacteristics which enter NP at that τ_0 level.

To formulate this rigorously, recall $\gamma_{3\mathrm{sc}} : \mathcal{P}(C_{3\mathrm{sc}}[X;C]) \longrightarrow \mathcal{P}(\partial^{3\mathrm{sc}}\overline{T}^*[X;C])$ defined in (4.31), which in particular associates to each $\tau_0 \in \overline{W^\perp}$ the full $\{\tau = \tau_0\}$ slice above ff. We also define the projection onto τ levels as follows. Recalling that $\mathrm{Char}(P_0) \subset {}^{\mathrm{sc}}\overline{T}^*X$, i.e. that we include fiber infinity in the characteristic set of P_0, we define the map that records both the spacetime location and the τ level of a point in the characteristic set of P_0, (possibly $\pm\infty$),

$$\pi_{X,\tau} : \mathrm{Char}(P_0) \longrightarrow X \times \overline{\mathbb{R}}\,, \tag{6.1}$$

$$(x,y,\tau,\zeta) \mapsto (x,y,\tau)\,. \tag{6.2}$$

This map is well-defined up to the fiber boundary since $\mathrm{Char}(P_0)$ has empty intersection with the fiber equator.

Proposition 6.1. *Let* $B_0 \in {}^{3\mathrm{sc}}\Psi^{s,\ell}, E_0 \in {}^{3\mathrm{sc}}\Psi^{s,\ell}, G \in {}^{3\mathrm{sc}}\Psi^{s-1,\ell+1}$ *for some* $s,\ell \in \mathbb{R}$*. Assume that*

(1) $\mathrm{WF}'_{\mathrm{ff}}(E_0) = \varnothing$*,*
(2) ${}^{3\mathrm{sc}}\mathrm{WF}'(B_0) \cup {}^{3\mathrm{sc}}\mathrm{Ell}(E_0) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$
(3) For all $\alpha \in \gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathrm{Ell}(G))$*, we have that* ${}^{\mathrm{sc}}H_p(\alpha) \neq 0$*.*
(4) For all $\alpha \in \mathrm{Char}(P_0)$ *that are incoming to* ${}^{3\mathrm{sc}}\mathrm{WF}'(B_0)$*, in the sense that*

$$\pi_{X,\tau}(\alpha) \in \pi_{X,\tau}(\gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathrm{WF}'(B_0)) \cap \mathrm{Char}(P_0))\,,$$

there exists $s_\alpha \in (-\varepsilon, 0)$ *such that*

$$\exp(s_\alpha\, {}^{\mathrm{sc}}H_p)(\alpha) \in {}^{3\mathrm{sc}}\mathrm{Ell}(E_0)$$

and for all $s \in [s_\alpha, 0]$*,*

$$\exp(s\, {}^{\mathrm{sc}}H_p)(\alpha) \in \gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathrm{Ell}(G))\,.$$

For each $u \in H_{\mathrm{sc}}^{-N,-M}$ *with* $E_0 u \in L^2$, $GP_V u \in L^2$ *it follows that* $B_0 u \in L^2$ *and*

$$\|B_0 u\| \lesssim_{M,N} \|E_0 u\| + \|GP_V u\| + \|u\|_{-N,-M}.$$

The proof of this proposition comes at the end of this section.

Remark 6.2. In words, the proposition states that, to obtain estimates at a given τ level over NP, we must control the backward flow out of the entire sphere $|\zeta|^2 = \tau^2 + m^2$ in a neighborhood of NP. The elliptic set of E_0 must control this set in the sense that it must contain a transversal of the sphere earlier along the flow. In particular, the elliptic set of G over ff must contain the whole of $\mathrm{WF}'_{\mathrm{ff}}(B_0)$.

Remark 6.3. The same proposition holds if the wavefront set of B_0 is controlled by the for backward flow of the elliptic set of E_0, in which case $s_\alpha \in (0, \epsilon)$.

We have two types of estimates in this section, one microlocalized near $\tau \in W^\perp$ with $m < |\tau| < \infty$, for which the arguments follow most closely those in [36], and the other microlocalized at $\tau = \pm\infty \in \overline{W^\perp}$, which requires more substantial modifications.

The proofs in these two settings are similar, not only in their overall structure, but in the specific functions which define the commutators and the proofs of the various properties of the attendant operators. We thus focus on the case $\pm\infty \in \overline{W^\perp}$, and in fact to $\tau = +\infty$, and the argument for finite τ (as for $\tau = -\infty$) is a straightforward adaptation.

Both cases involve the consideration of the set Σ (not the characteristic set!) which is the closure in ${}^{\mathrm{sc}}\overline{T}^* X$ of the set $\zeta \cdot y = 0$ in a region $|y| < c$ of NP. In the coordinates (x, y, ρ, μ) above in equation (2.25), Σ is

$$\Sigma = \{(x, y, \rho, \mu) : \mu \cdot y = 0, \mu \neq 0, |y| < c\}, \tag{6.3}$$

and the value of c is irrelevant below as we will localize our estimates in small neighborhoods of NP. In these coordinates it is clear that Σ is smooth up to fiber infinity. (This same Σ is used in Vasy, but there only its finite ξ, ζ points are relevant; we use it out to infinity.)

We also assume for simplicity for most of this section that

$$V - V^* = 0.$$

Indeed, without this assumption the commutators which arise below involve $P_{(V+V^*)/2}$ and the $V - V^*$ appears as an error, but it is clearer to make this realness assumption and use commutators with P_V, and then discuss the generalization of V later under the assumption in (6.30).

We first work with the free Klein-Gordon operator P_0 and then relate its commutators to those of P_V. The commutators of P_0 can be analyzed directly using its Hamilton vector field, and we need in particular to analyze this vector field's behavior at Σ, both near and away from fiber infinity over NP.

Lemma 6.4. *The set* Σ *is a smooth submanifold of* ${}^{\mathrm{sc}}\overline{T}^* X$. *Moreover, the Hamilton vector field* ${}^{\mathrm{sc}}H_p$ *is transversal to* $\Sigma \cap {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}} X$ *in a neighborhood of* $\mathrm{Char}(P_0)$.

Proof. At $y = 0$, away from fiber infinity, $H_p = -2x(\zeta \cdot \partial_y)$, so the rescaled Hamilton vector field ${}^{\mathrm{sc}}H_p$ is ${}^{\mathrm{sc}}H_p = -\mu \cdot \partial_y$ at NP (2.24). Since the condition defining Σ is $\mu \cdot y = 0$, thus

$${}^{\mathrm{sc}}H_p(\mu \cdot \eta) = -|\mu|^2 \tag{6.4}$$

on NP and is thus non-zero near the characteristic set near NP, which is what we wanted. $\square$

Remark 6.5. Consequently, there exists a neighborhood $U' \subset {}^{\mathrm{sc}}\overline{T}^*X$ of $\mathrm{Char}(P_0) \cap {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}}X$ on which we can solve the Cauchy problem

$$H_p f = 0\,, \quad f|_\Sigma = f_0\,.$$

The proposition will follow from estimates localized near points in $\overline{W^\perp}$, and the positive commutator argument we use to establish such estimates is accomplished using commutators localized near NP and near τ. Formally speaking, we will use a positive commutator argument analogous to that used in the scattering setting discussed in Section 2. We use the commutators constructed for this purpose in Section 5.3 of the form

$$i[P_V, G_\psi Q^* Q G_\psi] \tag{6.5}$$

where $\psi \in C_c^\infty$ and Q is constructed analogously to the corresponding commutant in [36]. In particular, we will take $q \in C^\infty([{}^{\mathrm{sc}}T^*X; \mathrm{fibeq}])$ satisfying the centrality condition $q|_{\mathrm{NP}} = f(\rho)$ where $f \in C^\infty(\overline{W^\perp})$. Concretely,

$$Q = \mathrm{Op}_L(x^{-(\ell+1/2)}\rho^{-(s-1/2)}q)\,, \quad q = \chi_\partial(x)\tilde{q}\,, \tag{6.6}$$

for a cutoff function χ_∂ supported near 0 and $\tilde{q}$ a function on ${}^{\mathrm{sc}}T^*_{\partial X}X$, and $\tilde{q}$ essentially given by [36, Eq. 14.20], with modification that we clarify below. Furthermore, $\tilde{q}$ itself is defined first near the characteristic set, this is the function $\tilde{q}_0$ in [36, Eq. 14.11], and then on a complement of a neighborhood of the characteristic set using a partition of unity. We describe this in detail below.

First, we make the following remarks about this commutator and its important features, both of which appear in the proof in Vasy [36].

- As discussed in Section 4.3, $QG_\psi \in {}^{3\mathrm{sc}}\Psi^{s-1/2,\ell+1/2}$ satisfies
$$[P_V, G_\psi Q^* Q G_\psi] \in {}^{3\mathrm{sc}}\Psi^{2s,2\ell},$$
and, by choosing the support of $f = q|_{\mathrm{NP}}$ localized around a given $\tau_0 \in \overline{W^\perp}$, we will obtain estimates localized near that τ_0. (Note we want more than simply localization as we need a positive commutator.)
- We do not use (or more accurately we do not attempt to define) the Hamilton vector field of P_V directly, and thus we do not directly compute the principal symbol of $i[P_V, G_\psi Q^* Q G_\psi]$ in terms of some action on q. Instead, we *compare* $i[P_V, G_\psi Q^* Q G_\psi]$ to an operator whose principal symbol we know explicitly. (See just below these remarks for an elaboration on this comparison.)

It instructive to consider first the case $V = 0$ and the commutator $[P_0, G_{\psi,0} Q^* Q G_{\psi,0}]$, where $G_{\psi,0}$ is the corresponding function of the free Klein-Gordon operator P_0 in equation (5.1). In this case compute $\hat{N}_{\mathrm{ff}}(i[P_0, G_{\psi,0} Q^* Q G_{\psi,0}])$ in terms of

$$f(\rho) \coloneqq \tilde{q}|_{\mathrm{NP}},$$

and ${}^{\mathrm{sc}}H_p\tilde{q}$. Recalling, from (2.23)-(2.24), that away from the characteristic set we use the rescaling ${}^{\mathrm{sc}}H_p = (\rho/x)H_p$, we have

$$\hat{N}_{\mathrm{ff},2\ell}(i[P_0, G_{\psi,0} Q^* Q G_{\psi,0}])(\tau) = \rho^{-2s}\widehat{(G_{\psi,0})}_{\mathrm{ff}}\left(2\left({}^{\mathrm{sc}}H_p\tilde{q}\right)f(\tau) + 2(2\ell+1)f(\tau)^2\right)\widehat{(G_{\psi,0})}_{\mathrm{ff}}. \tag{6.7}$$

(This is an easy consequence of $\hat{N}_{\mathrm{ff},\ell}(A)(\tau) = \hat{N}_{\mathrm{sc}}(A)(0,0,\tau,D_z)$ for $A \in {}^{\mathrm{sc}}\Psi^{m,\ell}$.) Thus, we seek a $\tilde{q}$ which gives positivity when differentiated by the Hamilton vector field, and we proceed to the construction of $\tilde{q}$ now.

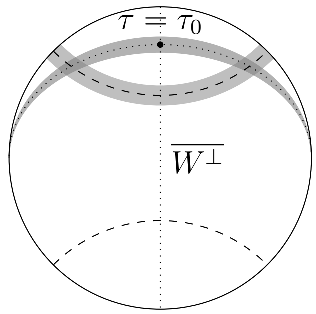

FIGURE 5. Localization near the characteristic set and at $\tau_0 \in \overline{W^\perp}$. The two gray areas are the support of cutoffs around Σ_+ and $(\pi^\perp)^{-1}(\tau)$, respectively. The intersection is a neighborhood around $\Sigma(\tau)$.

Let $\chi_0 \in C^\infty(\mathbb{R})$ given by

$$\chi_0(t) = \begin{cases} e^{-1/t} & t \geq 0\,, \\ 0 & t \leq 0\,. \end{cases} \tag{6.8}$$

A key feature of χ_0 is that

$$\chi_0(t) = t^2 \chi_0'(t),$$

Choose $\chi_1 \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\begin{aligned} \chi_1(t) &= 0 \quad \text{for } t \leq 0\,, \\ \chi_1(t) &= 1 \quad \text{for } t \geq 1\,, \\ \chi_1'(t) &\geq 0\,. \end{aligned}$$

Vasy's construction of $\tilde{q}_0$ uses functions N and ω, and we retain this notation with appropriate modifications of their definitions. We choose a neighborhood

$$U' \subset \mathrm{Char}(P_0) \cap {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}} X$$

as in Remark 6.5 and we define a function $N \in C^\infty(U')$, which will act as our flow parameter from Σ, by

$${}^{\mathrm{sc}}H_p N = 1\,, \quad N|_\Sigma = 0\,,$$

and by the transversality of ${}^{\mathrm{sc}}H_p$ to Σ in (6.4), we see that away from $\mu = 0$ and near Σ, i.e. on sets of the form $|\mu| \geq c > 0$, $|\mu \cdot y| \leq c$, $|y| < c$, we have

$$c_1(\mu \cdot y) \leq N \leq c_2(\mu \cdot y)\,.$$

for some $c_1 < c_2$. Thus N has the dual features that it is commensurable with $\mu \cdot y$ (near Σ and away from $\mu = 0$) and parallel along the flow. This N is *identical* to that in [36], we merely use that it is smooth up to the fiber boundary $\rho = 0$ near the characteristic set.

In proving estimates at $+\infty \in \overline{W^\perp}$, we define an ω which differs from the one in Vasy, namely we let $\omega \in C^\infty(U')$ be given by the solution of

$${}^{\mathrm{sc}}H_p \omega = 0\,, \quad \omega|_\Sigma = |y|^2 + \rho^2. \tag{6.9}$$

This ω is used to localize near $\omega = 0$, which here is the set $\rho = 0, y = 0$, i.e. fiber infinity over NP. Note that at finite τ_0 levels one can use, exactly as in Vasy, $\omega|_\Sigma = |y|^2 + |\tau - \tau_0|^2$.

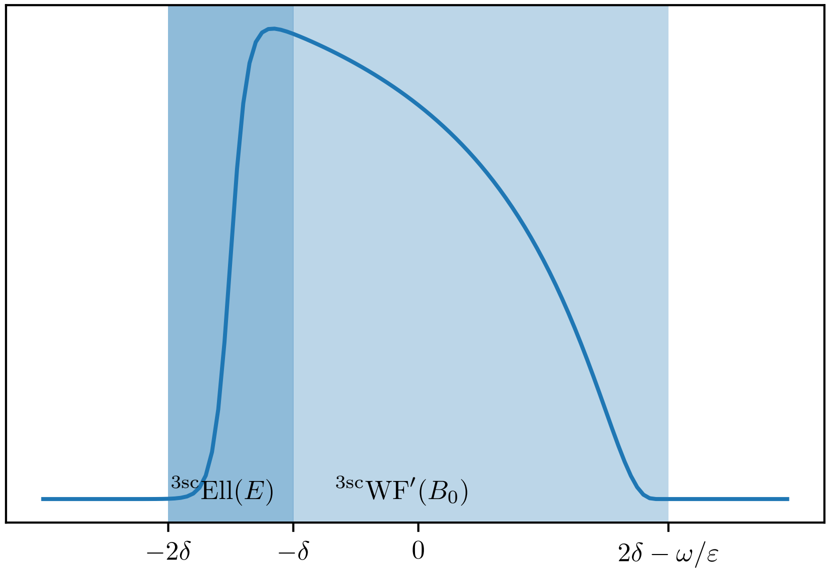

FIGURE 6. The function $\tilde{q}_0$ in the direction of the flow measured by N. The north pole is at $N = 0$.

Thus exactly as in [36, Eq. 14.10-14.11], for $\epsilon, \delta, \beta > 0$, we set

$$\phi = N + \omega/\epsilon,$$

and define the function

$$\tilde{q}_0(x, y, \rho, \theta) \coloneqq \chi_0(\beta^{-1}(2 - \phi/\delta))\chi_1(N/\delta + 2)\,,$$

depicted in Figure 6. In particular the support of $\tilde{q}_0$ is contained in $\phi \leq 2\delta$ and $N \geq -2\delta$. Thus, we have the bounds

$$|N| \leq 2\delta\,, \quad |\omega| \leq 4\varepsilon\delta\,, \quad |\phi| \leq 6\delta \tag{6.10}$$

on the support of $\tilde{q}_0$. In particular, the choice of δ determines how far from NP we have to control u with E and ϵ determines the interval around $+\infty \in \overline{W^\perp}$ we want to control.

We now wish to extend $\tilde{q}_0$ to a function $\tilde{q}$ defined on the whole of ${}^{\mathrm{sc}}\overline{T}^*X$, i.e. also away from the characteristic set; since $\tilde{q}_0$ is defined on the characteristic set we can do this easily enough with a bump function that is 1 near $\mathrm{Char}(P_0)$ and in fact we will choose such a function that is smooth on ${}^{\mathrm{sc}}\overline{T}^*X$.

Such cutoffs to the characteristic set were discussed in Section 5, and we recall that smooth functions of $p/(\tau^2 + |\zeta|^2 + 1)$ are smooth on the whole of ${}^{\mathrm{sc}}\overline{T}^*X$. Thus we define, for $\chi \in C_c^\infty(\mathbb{R})$ with χ identically 1 near zero,

$$\chi_{\mathrm{Char}} = \chi\left(\frac{\tau^2 - |\zeta|^2 - m^2}{\tau^2 + |\zeta|^2 + 1}\right) = \chi\left(\frac{p}{\tau^2 + |\zeta|^2 + 1}\right). \tag{6.11}$$

We choose χ such that $\operatorname{supp}\chi_{\mathrm{Char}} \subset U'$ in order that $\chi_{\mathrm{Char}}\tilde{q}_0$ is well-defined. The choice of a function χ as opposed to the ψ used in 5 is deliberate, as we hope to avoid confusion in the use of χ_{Char} as a cutoff to the characteristic set and ψ which is used to take functions of an operator. For the same reason the denominator has 1 instead of m^2+E, as the symbol of χ_{Char} is irrelevant; it is just a cutoff to the characteristic set. We note that χ_{Char} differs from Vasy's cutoff $\rho(g)$ [36, Eq. 14.20]; in the setting of that paper, the characteristic set is compactly contained in the interior of ${}^{\mathrm{sc}}T^*X$ so the behavior near fiber infinity is irrelevant.

Define $\tilde{q}$ by

$$\tilde{q} = \chi_{\mathrm{Char}}\tilde{q}_0 + (1-\chi_{\mathrm{Char}})\chi_0(\beta^{-1}(2-\omega_0/(\tilde{\epsilon}\delta))), \tag{6.12}$$

Note that $\tilde{q}$ is constant in ζ on NP. We have that

$$N|_{\mathrm{NP}} = 0\,, \quad \omega|_{\mathrm{NP}} = \rho^2\,,$$

therefore we may define $f \in C^\infty(\overline{W^\perp})$ by

$$f(\rho) = \tilde{q}|_{\mathrm{NP}} = \chi_0(\varrho(\rho))\,, \tag{6.13}$$

where

$$\varrho(\rho) \coloneqq \beta^{-1}\left(2 - \frac{\rho^2}{\epsilon\delta}\right)$$

and thus $\tilde{q}$ satisfies the centrality condition (4.17).

We now compute the Hamilton vector field applied to $\tilde{q}_0$:

$${}^{\mathrm{sc}}H_p(\tilde{q}_0) = -\beta^{-1}\delta^{-1}\chi_0'(\beta^{-1}(2-\phi/\delta))\chi_1(N/\delta+2) + \delta^{-1}\chi_0(\beta^{-1}(2-\phi/\delta))\chi_1'(N/\delta+2)\,.$$

We have used here that ${}^{\mathrm{sc}}H_pN = 1$ and ${}^{\mathrm{sc}}H_p\omega = 0$. Note that $\chi_1(N/\delta+2)$ is constantly 1 in a neighborhood of ff. Thus we now draw the important conclusion that over NP this expression simplifies to

$$f^\flat(\rho) \coloneqq -{}^{\mathrm{sc}}H_p(\tilde{q}_0)|_{\mathrm{NP}} = \beta^{-1}\delta^{-1}\chi_0'(\varrho) \in C^\infty(\overline{W^\perp}), \tag{6.14}$$

the minus sign being included as $f^\flat \geq 0$ will be used as an upper bound below. We will use below that we can bound f in terms of $f^\flat$; for some $C>0$

$$f(\rho) \leq Cf^\flat(\rho). \tag{6.15}$$

We may define g_0 and $\tilde{e}_0$ by

$$g_0^2 \coloneqq 2\beta^{-1}\delta^{-1}\left(\chi_0(\beta^{-1}(2-\phi/\delta))\chi_0'(\beta^{-1}(2-\phi/\delta))\right)\chi_1(N/\delta+2)^2,$$

and

$$\tilde{e}_0^2 \coloneqq 2\delta^{-1}\chi_0^2\chi_1\chi_1'. \tag{6.16}$$

Setting

$$r \coloneqq \frac{M\delta}{2\beta}\left(2-\phi/\delta\right)^2\,,$$

we have the following.

Lemma 6.6. *For any $\varepsilon, \delta, M > 0$ there exist $\beta > 0$ large such that $r \in [0,1)$ and*

$${}^{\mathrm{sc}}H_p(\tilde{q}_0^2) + M\tilde{q}_0^2 = -(1-r)g_0^2 + \tilde{e}_0^2 \tag{6.17}$$

on U'.

Proof. We start by observing that from the bound of N and ω in (6.10), we have that

$$|2-\phi/\delta|\leq 8.$$

Therefore, if we choose $\beta>0$ such that

$$\beta>8^2\delta M\,,$$

then $r<1$.

To establish (6.17), we first observe that

$${}^{\mathrm{sc}}H_p(\tilde{q}_0^2)=-g_0^2+\tilde{e}_0^2\,.$$

Moreover, we calculate

$$\begin{aligned}
M\tilde{q}_0^2&=M\chi_0^2\left(\beta^{-1}(2-\phi/\delta)\right)\chi_1^2\\
&=M\frac{(2-\phi/\delta)^2}{\beta^2}\left(\chi_0\chi_0'\right)\left(\beta^{-1}(2-\phi/\delta)\right)\chi_1^2\\
&=rg_0^2\,,
\end{aligned}$$

where we used the explicit relationship between $\chi_0(t)=t^2\chi_0'(t)$. □

Taking $\beta>0$ as in the previous lemma, we define

$$\tilde{b}_0\coloneqq(1-r)^{1/2}g_0\,.$$

Noting that $\tilde{b}_0$ is defined only on U', we can extend to a function b as with $\tilde{q}$ and $\tilde{q}_0$ above, namely by writing

$$b\coloneqq\chi_\partial(x)\tilde{b}$$

with

$$\tilde{b}\coloneqq\chi_{\mathrm{Char}}\tilde{b}_0+(1-\chi_{\mathrm{Char}})\left((1-M\delta\varrho)\frac{2}{\beta\delta}\chi_0(\varrho)\chi_0'(\varrho)\right)^{1/2}.$$

Here the parenthetical term on the right is equal to $\tilde{b}_0$ over NP, and thus $\tilde{b}$ gives a globally defined function which restricts to

$$\tilde{b}|_{\mathrm{NP}}(\rho)=\left(2(1-M\delta\varrho)f(\rho)f^\flat(\rho)\right)^{1/2}. \tag{6.18}$$

Similarly, we set

$$\tilde{e}=\chi_{\mathrm{Char}}\tilde{e}_0,\quad e=\chi_\partial(x)\tilde{e}_0,$$

We note that

$$(\operatorname{supp}e)\cap{}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}}X=\varnothing,$$

and thus e is in fact a standard scattering symbol. From (6.16), we see that

$$\operatorname{supp}e\subset\{N+\omega/\epsilon\leq 2\delta\}\cap\{-2\delta\leq N\leq-\delta\}, \tag{6.19}$$

and in addition is supported near the characteristic set thanks to the χ_{Char}. Thus e *is supported near the bicharacteristic rays flowing into $\rho=0$ over* NP *as N increases, i.e. along the direction of the Hamiltonian flow.*

To utilize q,b and e in an expression related to commutator $i[P_V,G_\psi Q^*QG_\psi]$, we will choose a $\psi\in C_c^\infty(\mathbb{R})$ with sufficiently small support that, away from the front face, only the function $\tilde{q}_0$ appears in the principal symbol of QG_ψ. Indeed, recall that for arbitrary $\psi \in C_c^\infty(\mathbb{R})$, by Proposition 5.14 we have

$$\sigma_{3\mathrm{sc},s-1,\ell+1}(QG_\psi) = (q|_{{}^{\mathrm{sc}}S^*X}) \cdot \sigma_{3\mathrm{sc}}(G_\psi)$$
$$\hat{N}_{\mathrm{mf},s-1,\ell+1}(QG_\psi) = (q|_{{}^{\mathrm{sc}}\overline{T}^*_{\partial X}X}) \cdot \hat{N}_{\mathrm{mf}}(G_\psi)$$
$$\hat{N}_{\mathrm{ff},\ell+1}(QG_\psi) = \rho^{-(s-1/2)} f \cdot \hat{N}_{\mathrm{ff}}(G_\psi).$$

Thus, if ψ is chosen with sufficiently small support such that

$$\chi_{\mathrm{Char}} \cdot \psi\left((\tau^2 + |\zeta|^2 + m^2 + E)^{-1} p\right) = \psi\left((\tau^2 + |\zeta|^2 + m^2 + E)^{-1} p\right),$$

We have that

$$\left(q\psi\left((\tau^2 + |\zeta|^2 + m^2 + E)^{-1} p\right)\right|_{\partial X} = \tilde{q}_0\, \psi\left((\tau^2 + |\zeta|^2 + m^2 + E)^{-1} p\right)\Big|_{\partial X},$$

so over the boundary away from ff only $\tilde{q}_0$ appears in the symbol of QG_ψ.

We thus get our desired positivity at the level of the principal symbol in the first two components, i.e. at fiber infinity and over mf. Indeed, directly from Lemma 6.6, we have

$$\sigma_{3\mathrm{sc},2s,2\ell}(i[P_V, G_\psi Q^* Q G_\psi]) = \left(-\tilde{b}_0^2 + \tilde{e}_0^2 + (2(2\ell+1) - M)\tilde{q}_0^2\right) \sigma_{3\mathrm{sc}}(G_\psi)^2 \tag{6.20}$$

$$\hat{N}_{\mathrm{mf},2s,2\ell}(i[P_V, G_\psi Q^* Q G_\psi]) = \left(-\tilde{b}_0^2 + \tilde{e}_0^2 + (2(2\ell+1) - M)\tilde{q}_0^2\right) \hat{N}_{\mathrm{mf}}(G_\psi)^2. \tag{6.21}$$

In order to use the sharp Gårding type theorem in Proposition 5.18, we must have a similar inequality over ff. This can be done by again possibly reducing the size of the support of ψ, and using operator norm bounds to compare the indicial operator of $[P_V, G_\psi Q^* Q G_\psi]$ to that of $[P_0, G_{\psi,0} Q^* Q G_{\psi,0}]$. We note that we have a formula for the free indicial operator in terms of the functions defined above, namely, from (6.7),

$$\hat{N}_{\mathrm{ff},2\ell}\left(i[P_0, G_{\psi,0} Q^* Q G_{\psi,0}]\right) = \left(-2(1 - M\delta\varrho) f f^\flat + (2(2\ell+1) - M) f^2\right) \rho^{-2s} \hat{N}_{\mathrm{ff}}(G_{\psi,0})^2, \tag{6.22}$$

Thus, for P_0, the indicial operator has similar structure to the other symbol components, and f can be bounded in terms of $f^\flat$. However, we cannot easily calculate the indicial operator of the perturbed commutator. Following Vasy, we will use a "window shrinking" argument to prove that the difference of the free and perturbed commutators are small provided that the cutoff function ψ in the localizer G_ψ is supported sufficiently close to 0.

Lemma 6.7. *For every $\varepsilon' > 0$ there exists $\psi \in C_c^\infty$ with $\psi = 1$ on $(-\delta, \delta)$ where $\delta = \delta(\varepsilon')$ such that*

$$\rho^{2s} \hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\psi Q^* Q G_\psi]) - (2(2\ell+1) - M)\, f^2 \cdot \hat{N}_{\mathrm{ff}}(G_\psi)^2$$
$$\leq -(2 - \varepsilon')(1 - M\delta\varrho) f^\flat f \cdot \hat{N}_{\mathrm{ff}}(G_\psi)^2.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(s) = 1$ for $|s| \leq 1$ to be chosen later. We consider the operator defined on ff which is the difference of the commutator and the local part of the free commutator in (6.22) but with the G_φ localizer:

$$R_\varphi := \rho^{2s} \hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\varphi Q^* Q G_\varphi]) - \left(-2(1 - M\delta\varrho) f f^\flat + (2(2\ell+1) - M)\, f^2\right) \hat{N}_{\mathrm{ff}}(G_\varphi)^2.$$

Note that, if $\psi \in C_c^\infty(\mathbb{R})$ has $\psi\phi = \psi$ then

$$R_\psi = G_\psi R_\varphi G_\psi.$$

We have that

$$\begin{aligned}R_\varphi = \rho^{2s}\left(\hat{N}_{\mathrm{ff},2\ell}\{i[P_V, G_\varphi Q^* Q G_\varphi] - i[P_0, G_{\varphi,0} Q^* Q G_{\varphi,0}]\}\right)\\ + \rho^{2s}\hat{N}_{\mathrm{ff},2\ell}\left(i[P_0, G_{\varphi,0} Q^* Q G_{\varphi,0}]\right) - \left(-2(1-M\delta\varrho) f f^\flat + (2(2\ell+1)-M) f^2\right)\hat{N}_{\mathrm{ff}}(G_{\varphi,0})^2\\ + \left(-2(1-M\delta\varrho) f f^\flat + (2(2\ell+1)-M) f^2\right)\left(\hat{N}_{\mathrm{ff}}(G_{\varphi,0})^2 - \hat{N}_{\mathrm{ff}}(G_\varphi)^2\right).\end{aligned}$$

We claim that

$$R_\varphi \in \Psi^{-\infty,-1,0}_{\mathrm{scl,sc}}$$

and that there is a $C > 0$ independent of $\rho = 1/\tau$ such that

$$\|R_\varphi(\rho)\|_{\mathcal{B}(L^2, H^{1,1}_{\mathrm{scl}})} \le C f^\flat(\rho) f(\rho)\,. \tag{6.23}$$

To see this, note that in the expression for $R_\varphi(\rho)$, line two vanishes by (6.22). For the third line in that expression, one gets (6.23) from Lemma 5.9 (and Proposition 5.8) and estimating f by $f^\flat$ as in (6.15). Finally, for the first line, we use Corollary 5.16 (and Lemma 5.9 and Proposition 5.8 again).

Taking $\varepsilon = \varepsilon'/C$ and ψ as in Lemma 5.10, we obtain

$$\begin{aligned}\rho^{2s}\hat{N}_{\mathrm{ff},2s,2\ell}(i[P_V, G_\psi Q^* Q G_\psi]) - (2(2\ell+1)-M) f^2 \hat{N}_{\mathrm{ff}}(G_\psi)^2\\ = 2(1-M\delta\varrho) f^\flat f \cdot \hat{N}_{\mathrm{ff}}(G_\psi)^2 + G_\psi R_\varphi G_\psi\\ \le 2(1-M\delta\varrho) f^\flat f \cdot \hat{N}_{\mathrm{ff}}(G_\psi)^2 - \varepsilon' \cdot f^\flat f\,.\end{aligned} \tag{6.24}$$

This is nearly what the lemma claims, only missing the $\hat{N}_{\mathrm{ff}}(G_\psi)^2$ on the ϵ' term, but is obtained by multiplying both sides by yet another $\hat{N}_{\mathrm{ff}}(G_{\tilde\psi})$ for yet another $\tilde\psi$ with $\tilde\psi\psi = \tilde\psi$. □

Now we can fix $\varepsilon' \in (0, 1/4)$ and $\psi \in C_c^\infty$ as in the lemma above. Choose $\phi \in C_c^\infty$ with $\phi(s) = 1$ for $|s| \le \delta(\varepsilon')/2$ and $\operatorname{supp}\phi \subset (-\delta(\varepsilon'), \delta(\varepsilon'))$. We define

$$B = \mathrm{Op}_L(x^{-\ell}\rho^{-s} b) G_\phi\,, \tag{6.25}$$

$$E = \mathrm{Op}_L(x^{-\ell}\rho^{-s} e) G_\phi\,, \tag{6.26}$$

$$Q_0 = \mathrm{Op}_L(x^{-\ell}\rho^{-s} q) G_\phi\,. \tag{6.27}$$

Lemma 6.8. *There exists $F \in {}^{3\mathrm{sc}}\Psi^{2s-1,2\ell-1}$ and $\varepsilon \in (0,1)$ such that ${}^{3\mathrm{sc}}\mathrm{WF}'(F) \subset {}^{3\mathrm{sc}}\mathrm{WF}'(B)$ and*

$$i[P_V, G_\phi Q^* Q G_\phi] + (M - 2(2\ell+1)) Q_0^* Q_0 \le -(1-\varepsilon) B^* B + E^* E + F\,.$$

Proof. By the definition of ϕ, there exists $\psi_1 \in C_c^\infty$ such that

$$\begin{aligned}\operatorname{supp}\phi \cap \operatorname{supp}(1-\psi_1) = \varnothing\,,\\ \operatorname{supp}\psi_1 \cap \operatorname{supp}(1-\psi) = \varnothing\,.\end{aligned}$$

We set

$$\begin{aligned}A &:= i[P_V, G_\psi Q^* Q G_\psi] + (M - 2(2\ell+1)) G_\psi \mathrm{Op}_L(x^{-\ell}\rho^{-s} q)^* \mathrm{Op}_L(x^{-\ell}\rho^{-s} q) G_\psi\\ &\quad - G_\psi \mathrm{Op}_L(x^{-\ell}\rho^{-s} e)^* \mathrm{Op}_L(x^{-\ell}\rho^{-s} e) G_\psi\,,\\ C &:= -G_\psi \mathrm{Op}_L(x^{-\ell}\rho^{-s} b)^* \mathrm{Op}_L(x^{-\ell}\rho^{-s} b) G_\psi\,.\end{aligned}$$

We have that

$$G_\phi A G_\phi = i[P_V, G_\phi Q^* Q G_\phi] + (M - 2(2\ell+1)) Q_0^* Q_0 - E^* E\,,$$
$$G_\phi C G_\phi = -B^* B\,.$$

Hence, to prove the claim, we can apply Proposition 5.18 with ψ_1. It remains to verify the inequality of the principal symbols. For the main face and fiber symbol, this is (6.20) and (6.21). For $\hat{N}_{\mathrm{ff}}$, we observe that

$$\rho^{2s}\hat{N}_{\mathrm{ff},2\ell}(A) = \rho^{2s}\hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\psi Q^* Q G_\psi]) + (M - 2(2\ell+1)) f^2 \hat{N}_{\mathrm{ff}}(G_\psi)^2\,,$$
$$\rho^{2s}\hat{N}_{\mathrm{ff},2\ell}(C) = -2(1 - M\delta\varrho) f(\rho) f^\flat(\rho) \hat{N}_{\mathrm{ff}}(G_\psi)^2$$

and by Lemma 6.7, we have that

$$\hat{N}_{\mathrm{ff},2\ell}(A) \le (1 - \varepsilon'/2)\hat{N}_{\mathrm{ff},2\ell}(C)\,.$$

□

Remark 6.9. As opposed to the case of scattering operators in Section 2, we cannot use a simplified version of the Gårding inequality, because we truly only have an inequality of principal symbols.

For $r \in (0,1)$ and $\delta_1, \delta_2 \in (0,\infty)$, we define

$$J_{r,\delta_1,\delta_2} := (1 + r/x)^{-\delta_2}(1 + r/\rho)^{-\delta_1}\,.$$

We have that

$$\mathrm{Op}_L(J_{r,\delta_1,\delta_2}) \to \mathrm{Id} \ \text{ as } r \to 0$$

strongly in $\mathcal{B}(H_{\mathrm{sc}}^{s',\ell'})$ for every $s' > 0, \ell' > 0$ (cf. Vasy [38, p. 408]).

Using (2.23), we calculate

$$H_p J_{r,\delta_1,\delta_2} = -2\tau x \delta_2 \frac{r}{x + r} J_{r,\delta_1,\delta_2}\,.$$

Hence, we can choose $\tilde{M} > 0$ independent of r, such that for all τ, x, we have

$$\tilde{M} J_{r,\delta_1,\delta_2} \ge \frac{1}{\tau x} H_p J_{r,\delta_1,\delta_2}\,. \tag{6.28}$$

Taking $M > \tilde{M} + 2(2\ell + 1)$ and using Lemma 6.6, we have that

$$H_p(x^{-(2\ell+1)}\rho^{-(2s-1)} J_{r,\delta_1,\delta_2}^2 q^2) \le x^{-2\ell}\rho^{-2s} J_{r,\delta_1,\delta_2}^2(-b^2 + e^2 + O(\rho \cdot x)) \tag{6.29}$$

on U'.

$$Q_r := \mathrm{Op}_L(x^{-(\ell+1/2)}\rho^{-(s-1/2)} J_{r,\delta_1,\delta_2} q) G_\phi\,,$$
$$B_r := \mathrm{Op}_L(x^{-\ell}\rho^{-s} J_{r,\delta_1,\delta_2} b) G_\phi\,,$$
$$E_r := \mathrm{Op}_L(x^{-\ell}\rho^{-s} J_{r,\delta_1,\delta_2} e) G_\phi\,.$$

We have that

$$Q_r \in {}^{3\mathrm{sc}}\Psi^{s-1/2-\delta_1,\ell+1/2-\delta_2} \text{ and } B_r, E_r \in {}^{3\mathrm{sc}}\Psi^{s-\delta_1,\ell-\delta_2}\,.$$

Since

$$J_{r,\delta_1,\delta_2} = \rho^{\delta_1} x^{\delta_2}(\rho + r)^{-\delta_1}(x + r)^{-\delta_2}\,,$$

we have that

$$\hat{N}_{\mathrm{ff},\ell+1/2-\delta_2}(Q_r) = \rho^{\delta_1}\hat{N}_{\mathrm{ff},\ell+1/2}(Q)\,,$$
$$\hat{N}_{\mathrm{mf}}(Q_r) = \rho^{\delta_1}x^{\delta_2}\hat{N}_{\mathrm{mf}}(Q)\,,$$
$$\sigma_{3\mathrm{sc}}(Q_r) = \rho^{\delta_1}x^{\delta_2}\sigma_{3\mathrm{sc}}(Q)$$

and similarly for B_r and E_r.

Lemma 6.10. *There exists $F_r' \in {}^{3\mathrm{sc}}\Psi^{2s-1-\delta_1,2\ell-1-\delta_2}$ such that ${}^{3\mathrm{sc}}\mathrm{WF}'(F_r') \subset {}^{3\mathrm{sc}}\mathrm{WF}'(B)$ and*

$$i[P_V, Q_r^*Q_r] \leq -(1-\varepsilon)B_r^*B_r + E_r^*E_r + F_r'\,,$$

and $F_r' \in \mathcal{B}(H_{\mathrm{sc}}^{2s-1,2\ell-1})$ is uniformly bounded as $r \to 0$.

Proof. The argument is essentially the same as the proof of Lemma 6.8, but using (6.29) to obtain the symbol inequalities. □

We first prove a variant of the propagation estimate with the specific B and E.

We now relax the condition on the imaginary part of P_V. Namely, we use the assumption that

$$V'' = (V - V^*)/2 \in {}^{3\mathrm{sc}}\Psi^{0,-2}\,. \tag{6.30}$$

Lemma 6.11. *Let $u \in H_{\mathrm{sc}}^{-N,-M}$, B, E as in (6.25)-(6.26), and $G \in {}^{3\mathrm{sc}}\Psi^{s-1,\ell+1}$ with*

$${}^{3\mathrm{sc}}\mathrm{WF}'(B) \cup {}^{3\mathrm{sc}}\mathrm{Ell}(E) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G).$$

If $Eu, GP_Vu \in L^2$, then $Bu \in L^2$ and we have the estimate

$$\|Bu\| \lesssim_{N,M} \|Eu\| + \|GP_Vu\| + \|u\|_{-N,-M}\,.$$

Remark 6.12. Note that here we make no explicit assumption about the relationship between the Hamiltonian flow and the elliptic set of G as in Proposition 6.1. This is because the operators here are given explicitly in (6.25)-(6.26). Here ${}^{3\mathrm{sc}}\mathrm{WF}'(B)$ by construction is an open neighborhood of $+\infty \in \overline{W^\perp} \subset \dot{C}_{3\mathrm{sc}}[X;C]$.

Proof. We take $\delta_1 = s + N$ and $\delta_2 = \ell + M$. Then for $r \in (0,1)$, we have that

$$Q_r \in {}^{3\mathrm{sc}}\Psi^{-N-1/2,-M+1/2}\,.$$

Therefore, $Q_ru \in H_{\mathrm{sc}}^{-1/2,1/2}$ and $Q_rP_Vu \in H_{\mathrm{sc}}^{1/2,-1/2}$ and the pairing of Q_ru and Q_rP_Vu is well-defined. We write $P_V = P_{\tilde{V}} - iV''$, where $P_{\tilde{V}}$ and V'' are both formally self-adjoint and we have that

$$2\,\mathrm{Im}\langle Q_rP_Vu, Q_ru\rangle = \langle i[P_{\tilde{V}}, Q_r^*Q_r]u, u\rangle - \langle (Q_r^*Q_rV'' + V''Q_r^*Q_r)u, u\rangle\,. \tag{6.31}$$

Since $V'' \in {}^{3\mathrm{sc}}\Psi^{0,-2}$, we have that

$$F_r'' \coloneqq F_r' - (Q_r^*Q_rV'' + V''Q_r^*Q_r) \in {}^{3\mathrm{sc}}\Psi^{2N-1,2M-1}$$

and using Lemma 6.10 with $P_{\tilde{V}}$, we obtain that

$$\|B_ru\|^2 \lesssim \|E_ru\|^2 - 2\,\mathrm{Im}\langle Q_rP_Vu, Q_ru\rangle + \langle F_r''u, u\rangle\,.$$

We have

$$|\mathrm{Im}\langle Q_rP_Vu, Q_ru\rangle| \geq -\frac{1}{4\mu}\|{}^{\mathrm{sc}}\Lambda_{-1/2,1/2}Q_rP_Vu\|^2 - \mu\|{}^{\mathrm{sc}}\Lambda_{1/2,-1/2}Q_ru\|^2\,.$$

Since ${}^{\mathrm{sc}}\Lambda_{1/2,-1/2}Q_r$ is a 0th order multiple of B_r and ${}^{\mathrm{sc}}\Lambda_{-1/2,1/2}$ is a 0th order multiple of $Q_{1,r}$, we can absorb $\mu\|{}^{\mathrm{sc}}\Lambda_{1/2,-1/2}Q_r u\|^2$ into $\frac{1}{2}\|B_r u\|^2$ if we choose $\mu \in (0,1)$ small enough modulo lower order terms. Then, we conclude that

$$\|B_r u\|^2 \lesssim \|{}^{\mathrm{sc}}\Lambda_{-1/2,1/2}Q_r P_V u\|^2 + \|E_r u\|^2 + |\langle F_r'' u, u\rangle| \,.$$

The right-hand side is bounded as $r \to 0$ and since $B_r \to B$ as $r \to 0$, we conclude that $Bu \in L^2$ and

$$\|Bu\|^2 \lesssim \|{}^{\mathrm{sc}}\Lambda_{-1/2,1/2}QP_V u\|^2 + \|Eu\|^2 + |\langle F'' u, u\rangle| \,.$$

Set

$$B_1 = \mathrm{Op}_L(x^{-\ell+1/2}\rho^{-s+1/2}b)G_\phi \,.$$

By elliptic regularity, we have that

$$\langle F'' u, u\rangle \lesssim \|B_1 u\|^2 + \|u\|_{-N,-M}^2 \,,$$
$$\|{}^{\mathrm{sc}}\Lambda_{-1/2,1/2}QP_V u\| \lesssim \|GP_V u\| + \|u\|_{-N,-M} \,.$$

Hence, we have proved the estimate

$$\|Bu\| \lesssim_{N,M} \|Eu\| + \|GP_V u\| + \|u\|_{-N,-M} \,.$$

To remove the B_1 term we inductively apply the previous estimate to the error term starting with the estimate

$$\|B_1 u\| \lesssim_{N,M} \|B_2 u\| + \|Eu\| + \|GP_V u\| + \|u\|_{-N,-M} \,,$$

where

$$B_2 = \mathrm{Op}_L(x^{-\ell+1}\rho^{-s+1}b)G_\phi \,.$$

After a finite amount of steps we have that $s - k/2 < -N$ and $\ell - k/2 < -M$ and therefore we have that $\|B_k u\| \lesssim \|u\|_{-N,-M}$. □

Now we can prove Proposition 6.1.

Proof of Proposition 6.1. Let B_0, E_0 and G be as in the proposition. First, we claim that there are B, E as in Lemma 6.11 such that ${}^{3\mathrm{sc}}\mathrm{WF}'(E)$ is controlled by ${}^{3\mathrm{sc}}\mathrm{Ell}(E_0)$. Indeed, this is accomplished by taking $\delta > 0$ small in the definition of q (and hence of B, E) and thus ${}^{3\mathrm{sc}}\mathrm{WF}'(E)$ is small neighborhood of the backward flow out of $+\infty$, see (6.19) and below. Thus, by standard scattering propagation in Section 2, we have that

$$\|Eu\| \lesssim_{N,M} \|E_0 u\| + \|GP_V u\| + \|u\|_{-N,-M}$$

and thus by Lemma 6.11, we control B in terms of E_0

$$\|Bu\| \lesssim_{N,M} \|E_0 u\| + \|GP_V u\| + \|u\|_{-N,-M} \,.$$

On the other hand, by the elliptic regularity of Corollary 5.13, for $\tilde{Q} \in {}^{\mathrm{sc}}\Psi^{s,\ell}$ with $+\infty \in \mathrm{Ell}_{\mathrm{ff}}(\tilde{Q})$ with ${}^{3\mathrm{sc}}\mathrm{WF}'(\tilde{Q})$ sufficiently small, we have

$$\|\tilde{Q}(I - G_\psi)u\| \lesssim \|GP_V u\| + \|u\|_{-N,-M},$$

where we willingly relinquish the additional differential order of regularity that comes in Corollary 5.13. In fact, we choose $\tilde{Q}$ so that $\hat{N}_{\mathrm{ff},\ell}(\tilde{Q})(+\infty) = f(0)$ with f given by (6.13) and set

$$B' = \tilde{Q}(I - G_\psi) + B \,.$$

We then have:

(1) $B'u \in L^2$ and satisfies the same estimate as Bu,
(2) $+\infty \in \mathrm{Ell}_{\mathrm{ff}}(B')$.

The second claim follows directly from (4.23) and

$$\begin{aligned}\hat{N}_{\mathrm{ff},\ell}(B')(+\infty) &= f(0)(1-\hat{N}_{\mathrm{ff}}(G_\psi)(+\infty)) + \hat{N}_{\mathrm{ff},\ell}(B)(+\infty)\\ &= f(0)(1-\hat{N}_{\mathrm{ff}}(G_\psi)(+\infty)) + f(0)\hat{N}_{\mathrm{ff}}(G_\psi)(+\infty)\\ &= f(0) > 0\,.\end{aligned}$$

The first claim follows from putting the bounds for B and $\tilde{Q}(I-G_\psi)$ together. More precisely,

$$\|B'u\| \lesssim_{N,M} \|E_0u\| + \|GP_Vu\| + \|u\|_{-N,-M}\,. \tag{6.32}$$

Finally, we control B_0u in terms of $B'u$ and E_0u as follows. Let $Q \in {}^{\mathrm{sc}}\Psi^{0,0}$ be such that

$$\begin{gathered}{}^{3\mathrm{sc}}\mathrm{WF}'(Q) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B')\\ \mathrm{WF}'_{\mathrm{ff}}(\mathrm{Id}-Q) = \varnothing.\end{gathered}$$

Then

$$\|QB_0u\| \lesssim_{N,M} \|B'u\| + \|u\|_{-N,-M}\,. \tag{6.33}$$

Moreover, ${}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id}-Q)B_0$ is controlled by ${}^{3\mathrm{sc}}\mathrm{Ell}(B') \cup {}^{3\mathrm{sc}}\mathrm{Ell}(E_0)$ through flow lines which avoid NP, and thus by the scattering propagation estimates

$$\|(\mathrm{Id}-Q)B_0u\| \lesssim_{N,M} \|E_0u\| + \|B'u\| + \|GP_Vu\| + \|u\|_{-N,-M}\,. \tag{6.34}$$

Consequently, we obtain

$$\begin{aligned}\|B_0u\| &\le \|QB_0u\| + \|(\mathrm{Id}-Q)B_0u\|\\ &\lesssim_{N,M} \|E_0u\| + \|B'u\| + \|GP_Vu\| + \|u\|_{-N,-M}\\ &\lesssim_{N,M} \|E_0u\| + \|GP_Vu\| + \|u\|_{-N,-M}\,.\end{aligned}$$

where in the last line we used (6.32). This proves the proposition. □

7. Radial point estimates over C

In the previous section, we had to exclude the points $\tau_0 \in \{\pm m\} \subset \overline{W^\perp}$ because the Hamilton vector field vanishes there. Indeed, by (2.27), we have that ${}^{\mathrm{sc}}\overline{T}^*_{\mathrm{NP}}X \cap \mathcal{R}^f = \{x = 0, y = 0, \zeta = 0, \tau = \pm m\}$. In Section 2.7, we proved localized radial point estimates away from the poles. Hence it remains to prove radial point estimates near C and $\tau_0 \in \{\pm m\}$. These estimates take a similar form as the scattering radial point estimates. As usual, we work only near NP.

The propositions in this section are conceptually a combination of the radial points estimates reviewed in Section 2.7 and the commutator construction with operators of the form QG_ψ used in Section 6. As in the scattering case, the form of the symbol q used in the radial points commutator is simpler than that of the standard propagation, in that it is essentially a bump function localizing to the radial set multiplied by a weight; for us it is a bump function localizing to $\tau = \pm m \in W^\perp$, see (7.1). Since the commutators QG_ψ are localized in bounded τ, they are smoothing, and thus the only relevant weight is the spacetime weight.

As in Section 6, the theorem is microlocal only in the 3sc-sense near $\pm m \in W^\perp$, meaning only in the inverse image of $\pm m$ under the $\pi^\perp$ map in (4.28). For example, the above threshold estimates imply that if a distribution u lies in the above threshold space $H^{s,-1/2+\epsilon}_{\mathrm{sc}}$

near $(\pi^\perp)^{-1}(m)$, and $P_V u$ lies in $H_{\mathrm{sc}}^{s-1,1/2+\epsilon'}$ there, then u lies in $H_{\mathrm{sc}}^{s,-1/2+\epsilon'}$ in a smaller neighborhood of $(\pi^\perp)^{-1}(m)$.

Proposition 7.1 (Above threshold radial point estimate). *Let $\ell, \ell' \in \mathbb{R}$ with $\ell \geq \ell' > -1/2$ and*

$$\tau_0 = \pm m \in W^\perp.$$

Let $B_1 \in {}^{3\mathrm{sc}}\Psi^{s,\ell'}$, $G \in {}^{3\mathrm{sc}}\Psi^{s-1,\ell+1}$ and assume that ${}^{3\mathrm{sc}}\mathrm{Ell}(B_1) \subset {}^{3\mathrm{sc}}\mathrm{WF}'(G)$. Then, there exists $B_0 \in {}^{3\mathrm{sc}}\Psi^{s,\ell}$ with

$$\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(B_0)$$

and ${}^{3\mathrm{sc}}\mathrm{WF}'(B_0) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B_1)$ such that for any $M, N \in \mathbb{R}$, we have

$$\|B_0 u\| \lesssim \|G P_V u\| + \|B_1 u\| + \|u\|_{-N,-M}\,,$$

provided the right hand side is finite.

Proposition 7.2 (Below threshold radial point estimate). *Let $\ell \in \mathbb{R}$ with $\ell < -1/2$, and let $B_0, E \in {}^{3\mathrm{sc}}\Psi^{s,\ell}$, $G \in {}^{3\mathrm{sc}}\Psi^{s-1,\ell+1}$, such that*

(1) ${}^{3\mathrm{sc}}\mathrm{WF}'_{\mathrm{ff}}(E) = \varnothing$,
(2) ${}^{3\mathrm{sc}}\mathrm{WF}'(B_0) \cup {}^{3\mathrm{sc}}\mathrm{Ell}(E) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$,
(3) $\gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathrm{Ell}\, G) \cap (\mathcal{R}^f_- \cup \mathcal{R}^p_+) = \varnothing$,
(4) for every $\alpha \in \mathrm{Char}(P_0)$ such that

$$\pi_{X,\tau}(\alpha) \in \pi_{X,\tau}\left(\gamma_{3\mathrm{sc}}\left({}^{3\mathrm{sc}}\mathrm{WF}'(B_0)\right) \cap \mathrm{Char}(P_0) \setminus (\mathcal{R}^f_+ \cup \mathcal{R}^p_-)\right),$$

there exists $s_\alpha \in (-\varepsilon, 0)$ such that

$$\exp(s_\alpha {}^{\mathrm{sc}}H_p)(\alpha) \in {}^{3\mathrm{sc}}\mathrm{Ell}(E)\,,$$

and for all $s \in [s_\alpha, 0]$,

$$\exp(s_\alpha {}^{\mathrm{sc}}H_p)(\alpha) \in \pi_{3\mathrm{sc}}^{-1}({}^{3\mathrm{sc}}\mathrm{Ell}(G))\,.$$

For any $M, N \in \mathbb{R}$, the estimate

$$\|B_0 u\| \lesssim \|G P_V u\| + \|E u\| + \|u\|_{-N,-M}$$

holds.

The same statement holds if $\mathcal{R}^f_+ \cup \mathcal{R}^p_-$ is replaced by $\mathcal{R}^f_- \cup \mathcal{R}^p_+$ with forward control instead of backward control.

Again for the construction of the commutator, we assume that $V - V^* = 0$ and then later we can run the argument under the assumption (6.30).

We proceed to the commutator construction. We will work near $\tau_0 = m$, as the result near $\tau_0 = -m$ follows via identical arguments. To avoid cumbersome distinction between the above and below threshold cases, we introduce a constant $\kappa \in \{\pm 1\}$ in the arguments below. The case that $\kappa = 1$ is used in the above threshold estimate and the case that $\kappa = -1$ is used in the below threshold estimate.

Let $\delta_1, \delta_2 > 0$ to be chosen later. Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ be non-negative cutoff functions with $\operatorname{supp} \chi_i \subset [-\delta_i, \delta_i]$ and $\chi_i(s) = 1$ for $s \in [-\delta_i/2, \delta_i/2]$, $i = 1, 2$. Below, δ_1 will be taken sufficiently small with respect to δ_2 so that an error term arising on mf has a definite sign.

Consider the function

$$q(x, y, \tau) = \chi_2(|y|^2)\chi_1(x)\chi_1(\tau - m), \tag{7.1}$$

which is identically 1 on a small neighborhood of the set $\tau = m$ over C.

As opposed to the previous section, we directly calculate the commutator with the regularized symbols. Since q is compactly supported in τ, we only need a regularization for the decay order. Hence, for $r \in (0,1)$, we set

$$J_{r,\delta} \coloneqq (1 + r/x)^{-\delta} \,. \tag{7.2}$$

We have that $J_{r,\delta} = x^{\delta}(x+r)^{-\delta}$ and therefore

$$\hat{N}_{\mathrm{ff},-\delta}(J_{r,\delta}) = r^{-\delta} \,.$$

As before, by (2.23), we have

$$H_p J_{r,\delta} = -2\tau x \delta \frac{r}{x+r} J_{r,\delta}$$

Setting

$$q_r^{\flat}(x,y,\tau,\zeta) = \chi_1(x)\chi_1(\tau - m)\left(\left(2\ell + 1 - 2\delta\frac{r}{x+r}\right)\tau\chi_2(|y|^2) - 4(\zeta + \tau y)\cdot y\, \chi_2'(|y|^2)\right)$$

we have that

$$H_p(x^{-(\ell+1/2)}J_{r,\delta}q) = x^{-(\ell-1/2)}J_{r,\delta}\cdot\left(q_r^{\flat} + O(x)\right) \,.$$

Since $x \geq 0$, we have that

$$2\ell + 1 - 2\delta\frac{r}{x+r} \geq 2(\ell + 1/2 - \delta)$$

and thus for $\delta > \ell - 1/2$, we have $2\ell + 1 - 2r/(x+r) > 0$.

Let $\ell \in \mathbb{R}$, $\kappa(\ell + 1/2) > 0$, and define

$$\begin{aligned} b_r(x,y,\tau) &= \sqrt{2\tau\kappa\left(2\ell + 1 - 2\delta\frac{r}{x+r}\right)\chi_2(|y|^2)\chi_1(x)\chi_1(\tau - m)} \\ &= \sqrt{2\tau\kappa\left(2\ell + 1 - 2\delta\frac{r}{x+r}\right)}\cdot q(x,y,\tau)\,, \\ e(x,y,\tau,\zeta) &= \sqrt{8(\zeta + \tau y)\cdot y\chi_2(|y|^2)(-\chi_2'(|y|^2))\chi_1(x)\chi_1(\tau - m)} \end{aligned} \tag{7.3}$$

and we can write

$$2q_r^{\flat}q = \kappa b_r^2 + e^2$$

and consequently,

$$H_p(x^{-(2\ell+1)}J_{r,\delta}^2 q^2) = x^{-2\ell}J_{r,\delta}^2\left(\kappa b_r^2 + e^2 + O(x)\right) \,. \tag{7.4}$$

We then define

$$\begin{aligned} f(\tau) &\coloneqq q|_C = \chi_1(\tau - m)\,, \\ f^{\flat}(\tau) &\coloneqq q_r^{\flat}|_C = (2\ell + 1)\tau\chi_1(\tau - m) \end{aligned}$$

and observe that

$$2ff^{\flat} = \kappa\, b_r|_C^2 \,.$$

We define the microlocal commutant as

$$\tilde{Q}_r \coloneqq \mathrm{Op}_L(x^{-(\ell+1/2)}J_{r,\delta}q) \,.$$

Lemma 7.3. *For every $\varepsilon' > 0$ there exist $\delta = \delta(\varepsilon')$ and $\psi \in C_c^\infty$ with $\psi = 1$ on $(-\delta, \delta)$ such that*

$$\kappa \hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\psi \tilde{Q}_r^* \tilde{Q}_r G_\psi]) \geq (2 - \varepsilon') r^{-2\delta} f f^\flat \hat{N}_{\mathrm{ff}}(G_\psi)^2 \,.$$

Proof. The proof is almost identical to the proof of Lemma 6.7. We write

$$\begin{aligned} R_1 &:= \kappa \hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\varphi \tilde{Q}_r^* \tilde{Q}_r G_\varphi]) - 2r^{-2\delta} f f^\flat \hat{N}_{\mathrm{ff}}(G_\varphi)^2 \\ &= \hat{N}_{\mathrm{ff},2\ell}(i[P_V, G_\varphi \tilde{Q}_r^* \tilde{Q}_r G_\varphi] - i[P_0, G_{\varphi,0} \tilde{Q}_r^* \tilde{Q}_r G_{\varphi,0}]) \\ &\quad + \hat{N}_{\mathrm{ff},2\ell}\left(i[P_0, G_{\varphi,0} \tilde{Q}_r^* \tilde{Q}_r G_{\varphi,0}]\right) - 2r^{-2\delta} f f^\flat \hat{N}_{\mathrm{ff}}(G_{\varphi,0})^2 \\ &\quad + 2r^{-2\delta} f f^\flat \left(\hat{N}_{\mathrm{ff}}(G_{\varphi,0})^2 - \hat{N}_{\mathrm{ff}}(G_\varphi)^2\right) . \end{aligned}$$

We use Corollary 5.16, equation (7.4), and Lemma 5.9 to obtain that $R_1 \in \Psi^{-\infty,-1,0}_{\mathrm{scl,sc}}$ and

$$\|R_1(\tau)\|_{\mathcal{B}(L^2, H^{1,1}_{\mathrm{scl}})} \leq C f(\tau) f^\flat(\tau) \,.$$

Since we assumed that H_{V_0} has purely absolutely continuous spectrum in a small neighborhood of m^2, we can apply Lemma 5.10 to obtain the claimed inequality. □

Again we choose $\varepsilon' \in (0, 1/4)$ and ψ in the lemma above and we choose $\phi \in C_c^\infty$ with $\phi(s) = 1$ for $|s| \leq \delta(\varepsilon')/2$ and $\operatorname{supp} \phi \subset (-\delta(\varepsilon'), \delta(\varepsilon'))$. Define

$$\begin{aligned} Q_r &:= \mathrm{Op}_L(x^{-(\ell+1/2)} J_{r,\delta} q) G_\phi \,, \\ B_r &:= \mathrm{Op}_L(x^{-\ell} J_{r,\delta} b_r) G_\phi \,, \\ E_r &:= \mathrm{Op}_L(x^{-\ell} J_{r,\delta} e) G_\phi \,. \end{aligned}$$

We have that $Q_r \in {}^{3\mathrm{sc}}\Psi^{-\infty,\ell+1/2-\delta}(X)$ and $B_r, E_r \in {}^{3\mathrm{sc}}\Psi^{-\infty,\ell-\delta}(X)$. The indicial operators are given by

$$\begin{aligned} &\hat{N}_{\mathrm{ff},\ell+1/2-\delta}(Q_r)(\tau) = r^{-\delta} \chi_1(\tau - m) \widehat{(G_\phi)}_{\mathrm{ff}}(\tau) \,, \\ &\hat{N}_{\mathrm{ff},\ell-\delta}(B_r)(\tau) = 2r^{-\delta} \sqrt{\tau \kappa(\ell + 1/2)} \chi_1(\tau - m) \widehat{(G_\phi)}_{\mathrm{ff}}(\tau) \,, \\ &\hat{N}_{\mathrm{ff},\ell-\delta}(E_r)(\tau) \equiv 0 \,. \end{aligned} \tag{7.5}$$

Lemma 7.4. *There exists $F_r \in {}^{3\mathrm{sc}}\Psi^{-\infty,2(\ell-\delta)-1}$ such that $\mathrm{WF}'_{3\mathrm{sc}}(F_r) \subset \mathrm{WF}'_{3\mathrm{sc}}(B)$ and*

$$(1 - \varepsilon) B_r^* B_r \leq \kappa(i[P_V, Q_r^* Q_r] - E_r^* E_r) + F_r$$

and

$$F_r \in \mathcal{B}(H^{s,2\ell-1}_{\mathrm{sc}}, L^2)$$

is uniformly bounded as $r \to 0$.

Proof. Using the sharp Gårding type theorem, Proposition 5.18, we only have to prove that

$$\begin{aligned} \hat{N}_{\mathrm{mf},2\ell}(B_r^* B_r) &= \kappa \hat{N}_{\mathrm{mf},2\ell}(i[P_V, Q_r^* Q_r] - E_r^* E_r) \,, \\ (1 - \varepsilon') \hat{N}_{\mathrm{ff},2\ell}(B_r^* B_r) &\leq \kappa \hat{N}_{\mathrm{ff},2\ell}(i[P_V, Q_r^* Q_r] - E_r^* E_r) \,. \end{aligned}$$

The first claim follows from (7.4) and the fact that

$$\hat{N}_{\mathrm{mf},2\ell}(i[P_V, Q_r^* Q_r]) = H_p(x^{-(2\ell+1)} q^2) \hat{N}_{\mathrm{mf}}(G_\phi^2) \,.$$

For the second equality, we observe that

$$\hat{N}_{\mathrm{ff},2\ell}(E_r^* E_r) = 0$$

together with Lemma 7.3 implies the claimed inequality. $\square$

We define

$$\begin{aligned} Q &:= \mathrm{Op}_L(x^{-(\ell+1/2)} q) G_\phi \,, \\ B &:= \mathrm{Op}_L(x^{-\ell} b_0) G_\phi \,, \\ E &:= \mathrm{Op}_L(x^{-\ell} e) G_\phi \,. \end{aligned}$$

With these operators, we can now state the above and below threshold radial point estimates.

We now only assume that

$$V'' = -\operatorname{Im} P_V \in {}^{3\mathrm{sc}}\Psi^{0,-2} \,.$$

Lemma 7.5 (Above threshold estimate)**.** *Let $\ell, \ell' \in \mathbb{R}$ with $\ell \geq \ell' > -1/2$ and let $u \in H_{\mathrm{sc}}^{s,\ell'}$ and assume that $x^{-1/2} Bu, x^{-1/2} Q P_V u \in L^2$. Then $Bu \in L^2$ and we have the estimate*

$$\|Bu\| \lesssim \|x^{-1/2} Bu\| + \|x^{-1/2} Q P_V u\| + \|u\|_{-N,-M} \,. \tag{7.6}$$

Lemma 7.6 (Below threshold estimate)**.** *Let $\ell \in \mathbb{R}$ with $\ell < -1/2$ and let $u \in H_{\mathrm{sc}}^{-N,-M}$ and assume that $x^{-1/2} Bu, x^{-1/2} Q P_V u, Eu \in L^2$. Then $Bu \in L^2$ and we have the estimate*

$$\|Bu\| \lesssim \|x^{-1/2} Bu\| + \|x^{-1/2} Q P_V u\| + \|Eu\| + \|u\|_{-N,-M} \,.$$

Proof of Lemma 7.5. We take $\delta = \ell - \ell'$, so that $Q_r \in {}^{3\mathrm{sc}}\Psi^{-\infty,\ell'+1/2}$. As in (6.31), we have that

$$2 \operatorname{Im} \langle Q_r P_V u, Q_r u \rangle = \langle i[P_{\tilde{V}}, Q_r^* Q_r] u, u \rangle - \langle (Q_r^* Q_r V'' + V'' Q_r^* Q_r) u, u \rangle \,.$$

Since $V'' \in {}^{3\mathrm{sc}}\Psi^{0,-2}$, we have that

$$F_r' := F_r + Q_r^* Q_r V'' + V'' Q_r^* Q_r \in {}^{3\mathrm{sc}}\Psi^{-\infty, 2\ell'-1}$$

and

$$\begin{aligned} \|B_r u\|^2 &\lesssim \langle i[P_{\widetilde{V}}, Q_r^* Q_r] u, u \rangle - \|E_r u\|^2 + \langle F_r u, u \rangle \\ &= 2 \operatorname{Im} \langle Q_r P_V u, Q_r u \rangle - \|E_r u\|^2 + \langle F_r' u, u \rangle \\ &\leq \mu \|x^{1/2} Q_r u\|^2 + \frac{1}{4\mu} \|x^{-1/2} \phi P_V u\|^2 + |\langle F_r' u, u \rangle| \end{aligned}$$

Taking $\mu \in (0,1)$ small enough we can absorb $x^{1/2} Q_r u$ into $B_r u$ since $x^{1/2} Q_r$ is a 0th order multiple of B_r. Hence, we have that

$$\|B_r u\|^2 \lesssim \|x^{-1/2} Q_r P_V u\|^2 + |\langle F_r' u, u \rangle| \,.$$

By assumption the right-hand side is bounded uniformly as $r \to 0$, and since $B_r \to B$, we have that $Bu \in L^2$. The claimed estimate follows from a standard elliptic estimate for F_r':

$$|\langle F_r' u, u \rangle| \lesssim \|x^{-1/2} B_r u\|^2 + \|u\|_{-N,-M} \,.$$

$\square$

Proof of Lemma 7.6. We use the same argument, but start with $u \in H_{\mathrm{sc}}^{-N,-M}$ and use $\delta = \ell - M$. Moreover, since $\kappa = -1$ we have to keep the term $\|E_r u\|^2$. $\square$

Proof of Proposition 7.1. We proceed in two steps, first we show that we can estimate the right hand side of (7.6) by the right hand side claimed in Proposition 7.1 and then we construct B_0 from B such that B_0 is elliptic at $\tau = m$.

Step 1: We have to show that if $B_1 u \in L^2$, then

$$\|Bu\| \lesssim \|GP_V u\| + \|u\|_{-N,-M}\,.$$

By assumption G is elliptic on ${}^{3\mathrm{sc}}\mathrm{WF}'(Q)$ and $G, x^{-1/2}Q$ are both of spatial order $\ell + 1$. Therefore

$$\|x^{-1/2}QP_V u\| \lesssim \|GP_V u\| + \|u\|_{-N,-M}\,.$$

By the previous lemma and a standard induction argument to replace $x^{-1}B$ by B_1, we obtain

$$\|Bu\| \lesssim \|GP_V u\| + \|B_1 u\| + \|u\|_{-N,-M}\,.$$

Step 2: Now we have to show that we can estimate

$$\|B_0 u\| \lesssim \|GP_V u\| + \|B_1 u\| + \|u\|_{-N,-M}$$

for some $B_0 \in {}^{3\mathrm{sc}}\Psi^{s,\ell}$ with $m \in \mathrm{Ell}_{\mathrm{ff}}(B_0)$ and ${}^{3\mathrm{sc}}\mathrm{WF}'(B_0) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$. If B was elliptic at $m \in \overline{W^\perp}$, then we could just take $B_0 = B$.

Let $Q' = \mathrm{Op}_L(q') \in {}^{\mathrm{sc}}\Psi^{0,0}$ with

$$\hat{N}_{\mathrm{ff}}(Q')(m) = 2\sqrt{m(\ell + 1/2)} \cdot \mathrm{Id}$$

and ${}^{3\mathrm{sc}}\mathrm{WF}'(Q')$ sufficiently small and set

$$B' \coloneqq B + x^{-\ell} Q'(\mathrm{Id} - G_\psi)\,.$$

Then

$$\hat{N}_{\mathrm{ff},\ell}(B') = 2\sqrt{\tau(\ell + 1/2)}\chi(\tau - m)\hat{N}_{\mathrm{ff}}(G_\psi) + 2\sqrt{m(\ell + 1/2)}\hat{N}_{\mathrm{ff}}(\mathrm{Id} - G_\psi)$$

and

$$\hat{N}_{\mathrm{ff},\ell}(B')(m) = 2\sqrt{m(\ell + 1/2)} \cdot \mathrm{Id}\,.$$

Therefore, $m \in \mathrm{Ell}_{\mathrm{ff}}(B')$. By Corollary 5.13, we have that

$$\begin{aligned}\|B'u\| &\leq \|Bu\| + \|x^{-\ell} Q'(\mathrm{Id} - G_\psi)\| \\ &\lesssim \|GP_V u\| + \|B_1 u\| + \|u\|_{-N,-M}\,.\end{aligned}$$

□

Proof of Proposition 7.2. The argument is basically the same as in the above threshold case, but we can absorb $\|B_1 u\|$ into $\|Eu\|$. More precisely, we can estimate

$$\|B_1 u\| \lesssim \|GP_V u\| + \|Eu\| + \|u\|_{-N,-M}$$

by propagation of singularities. □

8. Construction of the causal propagators

Using the estimates above, we can now complete the construction of the causal propagators. The argument is similar in structure to the construction done for scattering perturbations V in Sections 2.8 and 2.9. We will now state the precise assumptions for a 3sc perturbation V.

Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ such that

(1) there exist $V_\pm \in S^{-1}(\mathbb{R}^n_z)$ and $C > 0$ such that for all $t > C$ and $|z/t| < C$,
$$V(t,z) - V_+(z) \in {}^{3\mathrm{sc}}\Psi^{1,-1}\,,$$
and for all $t < -C$ and $|z/t| < C$,
$$V(t,z) - V_-(z) \in {}^{3\mathrm{sc}}\Psi^{1,-1}\,.$$

(2) The imaginary part $(V - V^*)/2$ satisfies
$$(V - V^*)/2 \in {}^{3\mathrm{sc}}\Psi^{0,-2}\,.$$

(3) The Hamiltonians $H_{V_\pm} = \Delta + m^2 + V_\pm$ have purely absolutely continuous spectrum near $[m^2, \infty)$ and finitely many eigenvalues in $(-\infty, m^2)$.

If the Hamiltonians $H_{V_\pm}$ have bound states we need more decay for $V - V_\pm$. In that case we additionally have to assume positivity of the Hamiltonians to conclude invertibility of P_V.

We construct the causal propagators, working in $\mathcal{X}$ spaces based on exactly the same scattering Sobolev spaces used in the scattering setting in Section 2.8. Recall the spacetime-dependent forward and backward weight functions $\boldsymbol{\ell}_\pm$ from Definition 2.14. In addition to being monotone along the flow, we must have that the weight functions $\boldsymbol{\ell}_\pm$ are constant in an open neighborhood of ${}^{\mathrm{sc}}\overline{T}^*_C X$. Indeed, all of our estimates over the poles are proven with constant weights near C, and in fact we do not even define the indicial operator in the presence of variable weights. Note that we can take $\boldsymbol{\ell}_\pm \in C^\infty(X)$ i.e. a function on spacetime, which is constant in neighborhoods of both past and future causal infinity, as discussed in the proof of Proposition 2.5. Note that this cannot be achieved for the Feynman weights, since they have to satisfy $\boldsymbol{\ell} > -1/2$ at $\mathrm{NP} \cap \{\tau = -m\}$ and $\boldsymbol{\ell} < -1/2$ on $\mathrm{NP} \cap \{\tau = m\}$.

We fix $s \in \mathbb{R}$ and admissible forward and backward weights $\boldsymbol{\ell}_+$, $\boldsymbol{\ell}_-$. Define
$$\begin{aligned}\mathcal{X}^{s,\boldsymbol{\ell}_\pm} &:= \{u \in H^{s,\boldsymbol{\ell}_\pm}_{\mathrm{sc}} \colon P_V u \in H^{s-1,\boldsymbol{\ell}_\pm+1}_{\mathrm{sc}}\}\,,\\ \mathcal{Y}^{s,\boldsymbol{\ell}_\pm} &:= H^{s,\boldsymbol{\ell}_\pm}_{\mathrm{sc}}\,.\end{aligned}$$

Remark 8.1. Note that these $\mathcal{X}^{s,\boldsymbol{\ell}_\pm}$ spaces depend on V, but we do not include this in the notation. However, if $W \in {}^{3\mathrm{sc}}\Psi^{1,-1}$, then
$$W : H^{s,\boldsymbol{\ell}_\pm}_{\mathrm{sc}} \longrightarrow H^{s-1,\boldsymbol{\ell}_\pm+1}_{\mathrm{sc}}\,,$$
hence $\mathcal{X}^{s,\boldsymbol{\ell}_\pm}$ only depends on $V_\pm$.

The first theorem is for the case that $H_{V_\pm}$ have no eigenvalues.

Theorem 8.2. *Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ satisfying (1), (2), and (3) above. If the Hamiltonians $H_{V_\pm}$ have no discrete spectrum, then the mapping*
$$P_V \colon \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_++1}$$
is invertible and its inverse is the forward propagator in the sense of (1.2). The same is true for the backward propagator with $\boldsymbol{\ell}_+$ replaced by $\boldsymbol{\ell}_-$.

In the case that the Hamiltonians $H_{V_\pm}$ have discrete spectrum we have to strengthen the assumptions on the decay of $V - V_\pm$: To simplify our exposition, we make the further assumption that, as $t \to \pm\infty$, for $V_\pm$ the asymptotic values of V in Section 3.4, we have an additional order of spacetime decay,

Theorem 8.3. *Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ satisfying (1), (2), and (3). Assume that as $t \to \pm\infty$ we have*

$$V - V_\pm(z) \in {}^{3\mathrm{sc}}\Psi^{1,-2}(\mathbb{R}^{n+1}). \tag{8.1}$$

Then the mapping

$$P_V \colon \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_+ +1} \tag{8.2}$$

is Fredholm.

Moreover,

1. *if H_{V_-} is positive, then (8.2) is injective and*
2. *if H_{V_+} is positive, then (8.2) is surjective.*

If $H_{V_\pm}$ are both positive, then P_V is invertible and its inverse is the forward propagator in the sense of (1.2). The same is true for the backward propagator with $\boldsymbol{\ell}_+$ replaced by $\boldsymbol{\ell}_-$.

Finally, if V is static we can drop the assumption of no decaying modes.

Theorem 8.4. *Let $V = V(z) \in S^{-2}(\mathbb{R}^n_z)$ with (2) and (3), then*

$$P_V \colon \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_+ +1}$$

is invertible.

Remark 8.5. The conclusions of the three theorems remain true if $D_t^2 - \Delta_z$ is replaced by $\Box_g$ for a non-trapping, asymptotically Minkowski perturbation g.

8.1. **Assuming no bound states.** We start by proving Theorem 8.2. The assumption that there are no eigenvalues in $(-\infty, m^2)$ implies that

$$(-m, m) \subset \mathrm{Ell}_{\mathrm{ff}}(P_V). \tag{8.3}$$

Indeed, recalling $(\widehat{P_V})_{\mathrm{ff}}(\tau) = \tau^2 - (\Delta_z + m^2 + V_+(z))$, we see that for $|\tau| < m$, $(\widehat{P_V})_{\mathrm{ff}}(\tau)$ is scattering elliptic. Thus (see Section 4.6), for $|\tau| < m$, we have that $\tau \in \mathrm{Ell}_{\mathrm{ff}}(P_V)$ if and only if $\tau^2 - (\Delta_z + m^2 + V_+)$ is in fact *invertible*. Thus, ellipticity of P_V at $\tau \in W^\perp \cap (-m, m)$ is equivalent to the non-existence of a bound state of $\Delta + V_+$ with energy $E = \tau^2 - m^2$. Thus in this section we will have (8.3) at both NP and SP.

For $u \in \mathcal{S}(\mathbb{R}^{n+1})$, define

$$E_u(t) \coloneqq \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t u|^2 + |\nabla_z u|^2 + (Vu)u + m^2 u^2 dz \geq c(t) \int_{\mathbb{R}^n} |u(t,z)|^2 dz, \tag{8.4}$$

where $c(t)$ is the minimum of $\sigma(\Delta + m^2 + V(t,z))$. If there are no bound states of V_-, then $\sigma(\Delta + m^2 + V_-) \geq c_0 > 0$, so by the Kato–Rellich theorem, there is t_0 such that for any $t < t_0$, $E_u(t) \geq (c - \delta)\|u\|^2$, which is to say in particular that $E_u(t) \geq 0$ for t sufficiently small. The same goes for $t \to +\infty$, so $E_u(t) \geq 0$ for any $|t| > t_0' > 0$.

We first prove that $P_V : \mathcal{X}^{s,\boldsymbol{\ell}_+} \to \mathcal{Y}^{s-1,\boldsymbol{\ell}_+ +1}$ is a Fredholm mapping. Again this reduces to showing the analogue of (2.33), namely that for any $N, M, s \in \mathbb{R}$, there is $C > 0$ such that, provided all quantities are finite,

$$\|u\|_{s,\boldsymbol{\ell}_+} \leq C\left(\|P_V u\|_{s-1,\boldsymbol{\ell}_++1} + \|u\|_{-N,-M}\right), \tag{8.5}$$
$$\|u\|_{1-s,-1-\boldsymbol{\ell}_+} \leq C\left(\|P_V u\|_{-s,-\boldsymbol{\ell}_+} + \|u\|_{-N,-M}\right),$$

Again, these estimates imply the Fredholm property by a standard argument.

To obtain them we again argue as in Section 2.8. Now we choose an open cover O_1, O_2, O_3, O_4 of the compressed cotangent bundle ${}^{\mathrm{sc}}\dot{\overline{T}}^* X$. Note here that, due to our assumption of no bound states, $\mathrm{Char}\, P_V = \mathrm{Char}\, P_0$.

1. $\mathcal{R}^p_\pm \subset O_1 \subset \{\boldsymbol{\ell}_+ = -1/2 + \epsilon\}$, in particular $\pm m \in O_1 \cap W^\perp$ over SP.
2. $\mathcal{R}^f_\pm \subset O_2 \subset \{\boldsymbol{\ell}_+ = -1/2 - \epsilon\}$, in particular $\pm m \in O_2 \cap W^\perp$ over NP.
3. $\mathrm{Char}\, P_V \subset O_1 \cup O_2 \cup O_3$.
4. $O_3 \cap \mathrm{Char}\, P_V$ is controlled along ${}^{\mathrm{sc}}H_p$ by O_1,
5. $O_2 \setminus \mathcal{R}^f$ is controlled along ${}^{\mathrm{sc}}H$ by O_3, and
6. $O_4 \subset {}^{3\mathrm{sc}}\mathrm{Ell}(P_V)$.

In this context, the meaning of item (3) is that, away from C, the standard characteristic set $\tau^2 - |\zeta|^2 - m^2$ lies in $O_1 \cup O_2 \cup O_3$, while over C,

$$[-\infty, -m] \cup [m, +\infty] \subset O_1 \cup O_2 \cup O_3 \cap \overline{W^\perp}$$

In fact, we will choose O_1, O_2, O_3 such that, for some $\epsilon > 0$,

$$(-m-\epsilon, -m+\epsilon) \cup (m-\epsilon, m+\epsilon) = O_1 \cap W^\perp \text{ over SP} \tag{8.6}$$

$$(-m-\epsilon, -m+\epsilon) \cup (m-\epsilon, m+\epsilon) = O_2 \cap W^\perp \text{ over NP} \tag{8.7}$$

$$[-\infty, -m-\epsilon/2) \cup (m+\epsilon/2, +\infty] = O_3 \cap \overline{W^\perp} \text{ over both SP and NP.} \tag{8.8}$$

In particular, O_1 and O_2 will be small neighborhoods around $\mathcal{R}$ over SP and NP respectively.

The meaning of (4) must now be understood in ${}^{\mathrm{sc}}\dot{\overline{T}}X$, in the sense that any point $\tau_0 \in \overline{W^\perp} \cap O_3$ over, say, SP, corresponds to all points $\tau_0^2 - |\zeta|^2 - m^2 = 0$ over SP, and the control assumption implies that for any such ζ, there exists some point $q \in {}^{\mathrm{sc}}\overline{T}^* X$ such that for some $s \in \mathbb{R}$: (1) $\exp_{{}^{\mathrm{sc}}H_p}(sq) = (\mathrm{SP}, \tau_0, \zeta)$ and (2) either $q \in O_1 \cap ({}^{\mathrm{sc}}\overline{T}^* X \setminus {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{SP}} X)$ or $q \in {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{SP}} X$ and $\tau(q) \in O_1$. Similarly for O_2 being controlled by O_3.

Such a collection O_1, O_2, O_3, O_4 of open sets can be constructed as follows. Our O_1 must be a set which contains the SP radial sets and $\pm m \in W^\perp$ over SP. For this we can take, near SP, a set of the form $(-m-\epsilon, -m+\epsilon) \cup (m-\epsilon, m+\epsilon) \subset W^\perp$ over SP, union with the set in ${}^{\mathrm{sc}}\overline{T}^* X \setminus {}^{\mathrm{sc}}\overline{T}^*_{\mathrm{SP}} X$:

$$\bigcup_\pm (\{|\tau \pm m| < \epsilon\}) \bigcup \left(\left\{ \frac{|\tau \pm m|}{\langle \tau, \zeta\rangle} < \epsilon \right\} \bigcap \{\langle \tau, \zeta \rangle > 1/\epsilon\} \right) \tag{8.9}$$
$$\bigcap \{t < -1/\epsilon, 0 < |y| < \epsilon\}.$$

This is a union of a basic neighborhood around $\tau = \pm m$ in the uncompactified $\dot{T}^* X$ union with an open set around the limit points of $\tau \pm m$ on the fiber boundary, all localized near SP by the intersection in the second line. We take the union of this with an open set containing $\mathcal{R}^p$ away from SP, for example with the coordinate function w which defines the radial set (see (2.25)), simply taking

$$\{|w| < \epsilon\} \bigcap \{t < -1/\epsilon, |y| > \epsilon/2\} \tag{8.10}$$

works. For O_2 we define the set in the exact same way but near NP.

For O_3 we will take an open neighborhood of $\mathrm{Char}(P_V)$ in the complement of $O_1 \cup O_2$. Again we can be explicit. We take O_3 to be as in (8.8) on $\overline{W^\perp}$ over both C, thus nearby O_3 we take as in (8.9), namely with identical to (8.9) except with $\{|\tau \pm m| > \epsilon/2\}$ in the first term. Doing the exact same over NP, we take the union of these with a small neighborhood of the $\mathrm{Char}(P_V)$ from the radial sets and the poles, for example, with

$$\{|w| > \epsilon/2\} \bigcap \{|y| > \epsilon/2\} \bigcap \{|\sigma_{\mathrm{sc},2,0}(P_V)| < \epsilon\}.$$

Then O_4 we take any open neighborhood of the closure of the complement $O_1 \cap O_2 \cap O_3$. Necessarily $O_4 \subset {}^{3\mathrm{sc}}\mathrm{Ell}(P_V)$.

Note that, due to the nature of ${}^{\mathrm{sc}}\dot{\overline{T}}^* X$ and, correspondingly, 3sc-ellipticity, the sets O_1 and O_3, for example, necessarily overlap at the intersection of the closures of $\{\tau = \pm m\}$ with the fiber boundary. This is simply because all closures of sets of the form $\{\tau = c\}$ intersect over the fiber boundary at the "fiber equator".

Keeping this in mind, we now choose a collection $B_1, B_2, B_3, B_4 \in {}^{3\mathrm{sc}}\Psi^{0,0}$ (in fact in ${}^{\mathrm{sc}}\Psi^{0,0}$) with

$$ {}^{3\mathrm{sc}}\mathrm{WF}'(B_i) \subset O_i \tag{8.11}$$

as in Section 2.8. Due to the overlap just described, the conditions ${}^{3\mathrm{sc}}\mathrm{WF}'(B_i) \subset O_i$ are ambiguous, in that they do not indicate the behavior of the B_i at the fiber equator above C. To clarify this, we take the approach that we require that

$$ \dot{C}_{3\mathrm{sc}}[X; C] \subset \bigcup_{i=1}^{4} {}^{3\mathrm{sc}}\mathrm{Ell}(B_i), \tag{8.12}$$

meaning each point in $\dot{C}_{3\mathrm{sc}}[X; C]X$ is is in the elliptic set of some B_i (4.30). Over SP or NP, since ellipticity at $\tau_0 \in W^\perp$ requires ellipticity of the symbol out to fiber infinity over $\tau = \tau_0$, the condition is incompatible with the B_i forming a partition of unity. Note however that any collection B_i satisfying (8.12) gives the "same control" as a partition of unity, namely, for any $N, M \in \mathbb{R}$, and any s, ℓ,

$$ \|u\|_{s,\ell} \lesssim \|B_1 u\|_{s,\ell} + \|B_2 u\|_{s,\ell} + \|B_3 u\|_{s,\ell} + \|B_4 u\|_{s,\ell} + \|u\|_{-N,-M} \tag{8.13}$$

(Indeed, in that case $\sum_{i=1}^4 B_i^* B_i$ is globally 3sc-elliptic and (8.13) follows from the Fredholm property for globally 3sc-elliptic operators and boundedness of B_i^*.)

To construct such B_i which satisfy both (8.11) and (8.12) we can use cutoff functions and the expressions for the O_i above. For example, we can take $B_1 = \mathrm{Op}_L(\chi_1)$ with $\chi_1 \colon {}^{\mathrm{sc}}\overline{T}^* X \longrightarrow \mathbb{R}$ identically 1 on sets of the forms (8.9) and (8.10) with the ϵ replace by a smaller $\epsilon' > 0$, and with support in the union of (8.11). Such B_1 has $\hat{N}_{\mathrm{ff}}(B_1)(\tau) = \mathrm{Id}$ for $\tau \in [-m - \epsilon', -m + \epsilon'] \cup [m - \epsilon', m + \epsilon']$ and indeed $\mathcal{R}^p \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B_1)$. We define B_2 and B_3 similarly.

Deducing the estimates in (8.5) for u using the B_i now follows exactly along the lines of the deduction in Section 2.8, by obtaining estimates exactly as in (2.34) - (2.37) with P_V replacing P_0. Thus, again, $u \in \mathcal{X}^{s,\boldsymbol{\ell}_+}$, then u is above threshold near $\mathcal{R}^p_\pm$, now in the 3sc-sense. That is:

(1) Putting together Proposition 2.11 and Proposition 7.1, we get (2.34) (with P_V replacing P_0) with the same $\boldsymbol{\ell}' < \boldsymbol{\ell}_+$ and $-1/2 < \boldsymbol{\ell}' < \boldsymbol{\ell}_+$.

(2) Again, since ${}^{3\mathrm{sc}}\mathrm{WF}'(B_3)$ is controlled by ${}^{3\mathrm{sc}}\mathrm{Ell}(B_1)$, by Proposition 6.1 we have (2.35) (with P_V).
(3) By Proposition 2.13 and Proposition 7.2 we have the below threshold estimate for B_2 in (2.36), and
(4) We have the 3sc-elliptic estimates so by Proposition 4.20 we obtain (2.37) (with P_V).

Since we control u by the $B_i u$ via (8.13) we again get the Fredholm estimates in (8.5), the second estimate being deduced by applying the above and below threshold estimates to the opposite radial sets. We again can absorb the $\boldsymbol{\ell}'$ term on the left by using (2.38). Thus the operator in (8.2) is Fredholm.

To see that P_V is invertible under the given assumptions, we need only check that its kernel and cokernel are zero. Distributions

$$u \in \ker\left(P_V \colon \mathcal{X}^{s,\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{s-1,\boldsymbol{\ell}_++1}\right)$$

must, thanks to the above threshold radial points estimates, in fact be rapidly decaying along with all their derivatives to the past. Using the energy $E_u(t)$ and a standard Grönwall argument gives that $u(t,z) \equiv 0$ for $t \gg 0$. Indeed, $dE_u(t)/dt \leq k(t)E_u(t)$ where $k(t) = O(1/|t|)^2$ as $t \to -\infty$. Since $E_u(t) \to 0$ as $t \to -\infty$ the differential Grönwall inequality shows that $E_u(t) \equiv 0$ for $t \ll 0$. By the positivity of the energy, we obtain that $u(t) = 0$ for $t \ll 0$. Then uniqueness of solutions to the Cauchy problem shows $u \equiv 0$ globally.

The cokernel of (8.2) can be identified with

$$\ker\left(P_V \colon \mathcal{X}^{1-s,1-\boldsymbol{\ell}_+} \longrightarrow \mathcal{Y}^{-s,-\boldsymbol{\ell}_+}\right)$$

and the same argument but using the energy estimates at $t \to +\infty$ shows the cokernel is zero.

We have shown that $P_V : \mathcal{X}^{s,\boldsymbol{\ell}_+} \to \mathcal{Y}^{s-1,\boldsymbol{\ell}_++1}$ for any $s \in \mathbb{R}$ and admissible forward weight $\boldsymbol{\ell}_+$ is invertible. Indeed the value of the inverse mapping is independent of the specific choice of s and $\boldsymbol{\ell}_+$. To see this, let $s' \in \mathbb{R}$ and $\boldsymbol{\ell}'_+$ an admissible weight and given $f \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}_++1} \cap H_{\mathrm{sc}}^{s'-1,\boldsymbol{\ell}'_++1}$, let $u \in \mathcal{X}^{s,\boldsymbol{\ell}_+}$ and $u' \in \mathcal{X}^{s',\boldsymbol{\ell}'_+}$ with $P_V u = f = P_V u'$. Then $u - u' \in \mathcal{X}^{\tilde{s},\tilde{\boldsymbol{\ell}_+}}$ for some $\tilde{s} \in \mathbb{R}$ and $\tilde{\boldsymbol{\ell}_+}$ admissible forward weight. Then $P_V(u-u') = 0$ and therefore $u = u'$.

Thus, we can unambiguously speak of the inverse of P_V, which we denote by $(P_V)^{-1}_{\mathrm{for}}$. The fact that $(P_V)^{-1}_{\mathrm{for}}$ satisfies the forward condition, (1.2), follows again from energy arguments. Indeed if $f \in H_{\mathrm{sc}}^{s-1,\boldsymbol{\ell}_++1}$ and $\operatorname{supp} f \subset \{t \geq T\}$, then $u = (P_V)^{-1}_{\mathrm{for}} f$ satisfies the above threshold estimates at $\mathcal{R}^p$ and thus is a Schwartz function as $t \to -\infty$ and the same energy argument shows that $\operatorname{supp} u \subset \{t \geq T\}$.

8.2. **With bound states.** In this section, we prove Theorem 8.3 and Theorem 8.4. We make appropriate adjustments to the above propagator construction to include the possibility that there are bound states of the Hamiltonian $\Delta + m^2 + V_\pm$ with (negative) energy bigger than 0. Such states appear as elements in the kernel of $(\widehat{P_V})_{\mathrm{ff}}(\tau)$ for $\tau \in (-m, m)$.

In discussing bound states, it is useful to distinguish the behavior of P_V at NP and SP, so for the remainder of this section we include the pole in the notation for the indicial operator:

$$(\widehat{P_V})_{\mathrm{ff},+}(\tau) = \text{ indicial operator of } P_V \text{ at NP},$$

while $(\widehat{P_V})_{\mathrm{ff},-}(\tau)$ is the indicial operator at SP. Similarly we write

$$W^\perp_+ = W^\perp \text{ over NP}$$

and $W_-^\perp = W^\perp$ over SP. Recall that, by scattering ellipticity, for $|\tau| < m$,

$$(\widehat{P_V})_{\mathrm{ff},\pm}(\tau)w = 0 \implies w \in \mathcal{S}(\mathbb{R}^n),$$

and thus by self-adjointness of $(\widehat{P_V})_{\mathrm{ff},\pm}(\tau)$,

$$\tau \in \mathrm{Ell}_{\mathrm{ff},\pm}(P_V) \iff \ker((\widehat{P_V})_{\mathrm{ff},\pm}(\tau)) = \{0\}. \tag{8.14}$$

We therefore have the elliptic estimate for P_V over ff.

Lemma 8.6. *Let $K \subset (-m, m) \subset W_+^\perp$ be a compact set such that*

$$\tau \in K \implies \ker((\widehat{P_V})_{\mathrm{ff},+}(\tau)) = \{0\}. \tag{8.15}$$

Then there is $Q \in {}^{3\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^{n+1})$ with $K \in \mathrm{Ell}_{\mathrm{ff}}(Q)$ such that for any $M, N \in \mathbb{R}$, $r, s \in \mathbb{R}$, and any $Q' \in {}^{3\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^{n+1})$ with ${}^{3\mathrm{sc}}\mathrm{WF}'(Q) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(Q') \cap {}^{3\mathrm{sc}}\mathrm{Ell}(P_V)$, there is $C > 0$ such that,

$$\|Qu\|_{s,r} \leq C\left(\|Q'P_V u\|_{s-2,r} + \|u\|_{-N,-M}\right).$$

The same goes near SP *with the relevant $+$'s replaced by $-$'s.*

Proof. This follows immediately from Proposition 4.20 and the fact that $K \subset \mathrm{Ell}_{\mathrm{ff}}(P_V)$. □

It is possible that there are finitely many points $\tau_0 \in (-m, m)$ such that $(\widehat{P_V})_{\mathrm{ff}}(\tau_0)$ is *not* invertible, and the remainder of this section proves estimates near such $\tau_0 \in W^\perp$.

First we establish some notation. Let $\lambda(\tau) := \sqrt{m^2 - \tau^2}$, define the eigenspace (set of bound states) of $\Delta_z + V_\pm(z)$ at frequency λ:

$$E_\pm(\lambda(\tau_0)) := \ker\left((\widehat{P_V})_{\mathrm{ff},\pm}(\tau_0)\colon H^{m,l}_{\mathrm{sc}}(\mathrm{ff}) \longrightarrow H^{m-2,l}_{\mathrm{sc}}(\mathrm{ff})\right).$$

Thus $E_+(\lambda(\tau_0))$ is the collection of bound states of $\Delta_z + V_+$ with frequency $\lambda(\tau)$. Again, $E_\pm(\lambda(\tau_0)) \subset \mathcal{S}(\mathbb{R}^n)$, so the eigenspace is independent of m, l. We define the set of τ values of bound states:

$$\mathcal{B}_{V,\pm} \subset \{\tau \in (-m, m)\colon E_\pm(\lambda(\tau)) \neq \{0\}\} \tag{8.16}$$

As discussed above, we assume that $\mathcal{B}_{V,\pm}$ is *finite*, and

$$0 \notin \mathcal{B}_{V,\pm}, \tag{8.17}$$

i.e. $\Delta_z + m^2 + V_+(z)$ has no eigenvalue at zero.

Our theorem for the causal propagators in the presence of bound states will follow the statement and proof of Theorem 8.2 closely; we show that *the same mapping* (8.2) is Fredholm, and then prove under additional assumptions on decaying modes that it is invertible.

The theorem is proven at the end of this section using the additional estimates proven near $\mathcal{B}_{V,\pm}$ in Proposition 8.7.

We now fix $\tau_0 \in \mathcal{B}_{V,+}$. There is therefore $w \in E(\lambda(\tau_0))$ with $w \not\equiv 0$ and we therefore have the oscillatory solution

$$P_{V_+}(e^{\pm i\tau_0 t}w) \equiv 0.$$

Since $w \subset \mathcal{S}(\mathbb{R}^n_z)$, this implies

$$e^{\pm i\tau_0 t}w(z) \in H^{\infty,-1/2-\epsilon}_{\mathrm{sc}}(\mathbb{R}^{n+1}).$$

for any $\epsilon > 0$ but not for $\epsilon = 0$. We therefore expect a spacetime weight threshold of $-1/2$ also for estimates localized near $\mathcal{B}_{V,+}$.

Near $\tau_0 \in W^\perp$, we will prove estimates with a spacetime weight loss identical to those in the radial points estimates above. We will see that this is a consequence of the fact that our approximate projection onto the solutions $e^{it\tau_0}w(z)$ intertwine P_V with $D_t^2 - \tau_0^2$ to leading order. Our main result will be the following.

Proposition 8.7. *Assume that* (8.1) *and* (8.17) *hold. Then for any* $\epsilon > 0$, *there exists* $Q \in {}^{3\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^{n+1})$ *such that*

$$[-m+\epsilon, m-\epsilon] \subset \mathrm{Ell}_{\mathrm{ff}}(Q), \tag{8.18}$$

and for any $G \in {}^{3\mathrm{sc}}\Psi^{0,0}$ *with* ${}^{3\mathrm{sc}}\mathrm{WF}'(Q) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$, *then for* $r, s, N, M \in \mathbb{R}$, *we have the following estimates.*

If $r < -1/2$, *there is* $C > 0$ *such that, if* $Qu \in H^{s,r}(\mathbb{R}^{n+1})$ *and* $GP_V u \in H^{s-2,r+1}(\mathbb{R}^{n+1})$, *then*

$$\|Qu\|_{s,r} \leq C(\|GP_V u\|_{s-2,r+1} + \|u\|_{-N,-M}). \tag{8.19}$$

If $r > -1/2$, *and* $r' \in \mathbb{R}$ *has* $r > r' > -1/2$, *then there is* $C > 0$ *such that, if* $Gu \in H^{-N,r'}(\mathbb{R}^{n+1})$ *and* $GP_V u \in H^{s-2,r+1}(\mathbb{R}^{n+1})$, *then* $Qu \in H^{s,r}$ *and*

$$\|Qu\|_{s,r} \leq C(\|GP_V u\|_{s-2,r+1} + \|Gu\|_{s-2,r'} + \|u\|_{-N,-M}). \tag{8.20}$$

The proposition follows from the elliptic regularity estimates in Lemma 8.6, and the following lemma, which is simply Proposition 8.7 microlocalized near a fixed $\tau_0 \in \mathcal{B}_{V,+}$.

Lemma 8.8. *Let* $\tau_0 \in \mathcal{B}_{V,+}$, *i.e. let* $\tau_0 \in (-m,m)$, $\tau_0 \neq 0$ *and* $E_+(\lambda(\tau_0)) \neq \{0\}$. *For any* $Q \in {}^{3\mathrm{sc}}\Psi^{0,0}(\mathbb{R}^{n+1})$ *with* $\tau_0 \subset \mathrm{Ell}_{\mathrm{ff}}(Q)$ *and* ${}^{3\mathrm{sc}}\mathrm{WF}'(Q)$ *sufficiently small, for any,* $G \in {}^{3\mathrm{sc}}\Psi$ *with* ${}^{3\mathrm{sc}}\mathrm{WF}'(Q) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$, *then for* $r, s, N, M \in \mathbb{R}$, *the estimates in Proposition* 8.7 *hold.*

The lemma is proven at the end of this section.

The proof proceeds by approximating projection onto the solutions $e^{i\tau_0 t}E_+(\lambda(\tau_0))$. If $\Pi_{\tau_0} = \Pi_{\tau_0}(z,z')$ is the integral kernel of orthogonal projection in $L^2(\mathbb{R}^n_z)$ onto $E_+(\lambda(\tau_0))$, i.e. for some orthonormal bases $\{w_j\}_{j=1}^N$ of $E_+(\lambda(\tau_0))$,

$$\Pi_{\tau_0}(z,z') \coloneqq \sum_{j=1}^N w_j(z) \cdot w_j(z') \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n),$$

then $e^{i(t-t')\tau_0}\Pi_{\tau_0}(z,z')$ is the integral kernel of this projection. However, as we will see below, $e^{i(t-t')\tau_0}\Pi_{\tau_0}(z,z')$ is not the integral kernel of a 3sc-PsiDO.

We proceed by smoothing in t in addition to projecting onto $E_+(\lambda(\tau_0))$; that is, we project onto a small τ-window around τ_0 on the t-Fourier transform side. Let $\chi_{\geq t_0} \in C^\infty(\mathbb{R})$ be a bump function supported near $+\infty$ with $\chi_{\geq t_0}(t) = 1$ for $t \geq t_0 - 1$ and $\chi_{\geq t_0}(t) = 0$ for $t \leq t_0 - 2$, and let $\chi_{\tau_0} \in C^\infty(\mathbb{R}_\tau)$ be a bump function supported near τ_0, so $\chi_{\tau_0}(\tau) = 1$ for $|\tau - \tau_0| < \delta$ and $\chi_{\tau_0}(\tau) = 0$ for $|\tau - \tau_0| \geq 2\delta$. Consider the operator defined by the integral kernel which: (1) cuts off to large time, (2) localizes in the t-momentum variable around τ_0, (3) projects onto $E_+(\lambda(\tau_0))$:

$$K_{\tau_0}(t,z,t',z') \coloneqq \chi_{\geq t_0}(t) \cdot \Pi_{\tau_0} \circ \mathcal{F}^{-1}_{\tau \to t} \circ \chi_{\tau_0}(\tau) \cdot \mathcal{F}_{t' \to \tau} \circ \chi_{\geq t_0}(t') \tag{8.21}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t-t')\tau} \chi_{\geq t_0}(t) \cdot \chi_{\tau_0}(\tau) \cdot \Pi_{\tau_0}(z,z') \cdot \chi_{\geq t_0}(t')\, d\tau \tag{8.22}$$

This K_{τ_0} will be used as a stand-in for projection onto $e^{i\tau_0 t}E_+(\lambda(\tau_0))$ near $t = +\infty$.

Lemma 8.9. *For $\delta > 0$ sufficiently small in the definition of χ_{τ_0},*

$$K_{\tau_0} \in {}^{3sc}\Psi^{-\infty,0}, \text{ and } (\widehat{K_{\tau_0}})_{\mathrm{ff}}(\tau) = \chi_{\tau_0}(\tau)\Pi_{\tau_0}, \tag{8.23}$$

and

$$^{3sc}\mathrm{WF}'[P_{V_+}, K_{\tau_0}] = \varnothing. \tag{8.24}$$

Proof. Proving that an operator lies in ${}^{3sc}\Psi$ can be done using the double space characterization in Section 3 of [36], but we argue directly using our work above. First we note that, for any $\chi \in C_c^\infty(\mathbb{R})$, the operator without the time cutoffs:

$$\tilde{K}_\chi := \int_{-\infty}^{\infty} e^{i(t-t')\tau} \cdot \chi(\tau) \cdot \Pi_{\tau_0}(z, z') \cdot d\tau \tag{8.25}$$

is a 3sc-operator since $\Pi_{\tau_0}(z, z') \in {}^{sc}\Psi^{-\infty,-\infty}(\mathbb{R}^n)$, in fact $\tilde{K}_\chi = \mathrm{Op}_L(k)$ for $k = \chi(\tau)a(z, \zeta)$ with $a \in \mathcal{S}(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta)$. This shows directly that $\tilde{K}_\chi$ is in ${}^{3sc}\Psi^{0,0}$ and has indicial operator as in (8.23). Then multiplying on the left by $\chi_{\geq t_0}(t)$ gives an operator $\mathrm{Op}_L(\chi_{\geq t_0}(t)k)$ which remains in ${}^{3sc}\Psi^{0,0}$ since k is rapidly decaying in z. The adjoint of that operator $\tilde{K}_\chi \chi_{\geq t_0}$ is thus also in ${}^{3sc}\Psi^{0,0}$. Then K_{τ_0} itself can be expressed as a composition $\chi_{\geq t_0}\tilde{K}_{\chi_1} \circ \tilde{K}_{\chi_2}\chi_{\geq t_0}$ where $\chi_1 = \chi_{\tau_0}$ and $\chi_2\chi_{\tau_0} = \chi_{\tau_0}$, and is thus in ${}^{3sc}\Psi^{0,0}$. The indicial operator statement follows from the composition and adjunction properties of indicial operators.

Finally, $[P_{V^+}, \tilde{K}_\chi] = 0$ for any χ. Since $[P_{V_+}, K_{\tau_0}]$ differs from $[P_{V^+}, \tilde{K}_{\chi_{\tau_0}}]$ only by terms with derivatives falling on $\chi_{\geq t_0}(t)$, the commutator $[P_{V_+}, K_{\tau_0}]$ is microsupported on

$$\chi'_{\geq t_0}(t) \subset \{t \in [t_0 - 2, t_0]\},$$

but then $\chi'_{\geq t_0}(t)\Pi_{\tau_0}$ has symbol which is rapidly decaying to the spacetime boundary, so (8.24) holds. □

We will use the approximate projection K_{τ_0} mainly by exploiting the following feature.

Corollary 8.10. $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(P_V + K_{\tau_0})$.

Proof. This follows directly from $(\widehat{P_V + K_{\tau_0}})_{\mathrm{ff}}(\tau_0) = (\widehat{P_V})_{\mathrm{ff}}(\tau_0) + \Pi_{\tau_0}$, since Π_{τ_0} is exactly projection onto the kernel of the self-adjoint operator $(\widehat{P_V})_{\mathrm{ff}}(\tau_0)$. □

To obtain estimates for $P_V u$ near $\tau_0 \in W^\perp$, we first use the elliptic estimates for $P_V + K_{\tau_0}$ near $\tau_0 \in W^\perp$, namely, that for any $Q, G \in {}^{3sc}\Psi^{0,0}$ with $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(Q)$ and ${}^{3sc}\mathrm{WF}'(Q) \subset {}^{3sc}\mathrm{Ell}(G) \cap {}^{3sc}\mathrm{Ell}(P_V + K_{\tau_0})$, for any $M, N \in \mathbb{R}$ there is $C > 0$ such that

$$\begin{aligned} \|Qu\|_{s+2,r} &\leq C\left(\|G(P_V + K_{\tau_0})u\|_{s,r} + \|u\|_{-N,-M}\right) \\ &\leq C(\|GP_V u\|_{s,r} + \|GK_{\tau_0}u\|_{s,r} + \|u\|_{-N,-M}). \end{aligned} \tag{8.26}$$

What we will show below is that, off the spacetime weight $r = -1/2$, for Q with sufficiently small support around $\tau_0 \in W^\perp$, that the $K_0 u$ term on the RHS can be bounded by $P_V u$. This will follow by applying ODE techniques to the easily verified formula

$$P_{V_+}K_{\tau_0}u = (D_t^2 - \tau_0^2)K_{\tau_0}u + Ru \tag{8.27}$$

where $R \in {}^{3sc}\Psi^{-\infty,-\infty}$, to obtain estimates for $K_{\tau_0}u$ in terms of $P_{V_+}K_{\tau_0}u$. Then using (8.24) we will remove the $K_{\tau_0}u$ from (8.26) entirely. Then applying ODE methods to the first term on the RHS, or using a positive commutator argument akin to that of Section 7, we obtain the following lemma.

Lemma 8.11. *Let $r \in \mathbb{R}$. Provided $r < -1/2$, for any $s_0, M, N \in \mathbb{R}$ there is C such that, if $K_{\tau_0}u \in H^{s,r}$ and $P_{V_+}K_{\tau_0}u \in H^{s,r+1}$, then for $Q, Q' \in {}^{3sc}\Psi^{0,0}$ with $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(Q)$ and ${}^{3sc}\mathrm{WF}'(Q) \subset {}^{3sc}\mathrm{Ell}(Q')$, we have $K_{\tau_0}u \in H^{s_0,r}$ and $P_{V_+}K_{\tau_0}u \in H^{s_0,r+1}$, and*

$$\|QK_{\tau_0}u\|_{s_0,r} \leq C(\|Q'P_{V_+}K_{\tau_0}u\|_{s,r+1} + \|u\|_{-N,-M}). \tag{8.28}$$

The same is true if, in the definition of K_{τ_0}, the projection Π_{τ_0} is replaced by orthogonal projection onto any subspace of $E_+(\lambda(\tau_0))$.

If instead $r > -1/2$, for any $s_0, M, N \in \mathbb{R}$ there is C such that, if $K_{\tau_0}u \in H^{s,r}$ and $P_{V_+}K_{\tau_0}u \in H^{s,r+1}$, possibly after taking χ_{τ_0} in K_{τ_0} with smaller support, we have $K_{\tau_0}u \in H^{s_0,r}$ and $P_{V_+}K_{\tau_0}u \in H^{s_0,r+1}$, and $r > r' > -1/2$,

$$\|QK_{\tau_0}u\|_{s_0,r} \leq C(\|Q'P_{V_+}K_{\tau_0}u\|_{s,r+1} + \|u\|_{-N,r'}). \tag{8.29}$$

The same is true if, in the definition of K_{τ_0}, the projection Π_{τ_0} is replaced by orthogonal projection onto any subspace of $E_+(\lambda(\tau_0))$.

With the lemma, we can now conclude the proof

Proof of Lemma 8.8. From (8.26) and Lemma 8.11, treating $r < -1/2$ first, we take Q, G and G' with ${}^{3sc}\mathrm{WF}'(G) \subset {}^{3sc}\mathrm{Ell}(G')$ to obtain

$$\begin{aligned}\|Qu\|_{s+2,r} &\lesssim \|GP_Vu\|_{s,r} + \|GK_{\tau_0}u\|_{s,r} + \|u\|_{-N,-M} \\ &\lesssim \|GP_Vu\|_{s,r} + \|G'P_{V_+}K_{\tau_0}u\|_{s,r+1} + \|u\|_{-N,-M} \\ &\lesssim \|GP_Vu\|_{s,r} + \|G'K_{\tau_0}P_{V_+}u\|_{s,r+1} + \|u\|_{-N,-M} \\ &\lesssim \|GP_Vu\|_{s,r} + \|G'K_{\tau_0}P_Vu\|_{s,r+1} + \|GK_{\tau_0}(P_V - P_{V_+})u\|_{s,r+1} + \|u\|_{-N,-M}\,,\end{aligned}$$

where we used $[P_{V_+}, K_{\tau_0}] \in {}^{3sc}\Psi^{-\infty,-\infty}$ in the fourth line.

The first two terms may both be bounded by $G''P_Vu$ for G'' with ${}^{3sc}\mathrm{WF}'(G) \cup {}^{3sc}\mathrm{WF}'(G') \subset {}^{3sc}\mathrm{Ell}(G'')$ by choosing ${}^{3sc}\mathrm{WF}'(K_{\tau_0})$ sufficiently small. Moreover, $P_V - P_{V_+} = -(V - V_+)$ and therefore by (8.1) the third term is controlled by $\|u\|_{-N,r-1}$ for any N, and we obtain overall that there is $C > 0$ such that, if all the terms below are finite, then we have an estimate

$$\|Qu\|_{s+2,r} \leq \|G''P_Vu\|_{s,r} + \|G''u\|_{-N,r-1} + \|u\|_{-N,-M}\,.$$

Iterating this estimate allows us to drop the $r-1$ term on the right.

The proof for $r > -1/2$ is similar. □

We can now use Proposition 8.7 and the arguments in Section 8.2 to prove Theorem 8.3.

Proof of Theorem 8.3. That P_V acting in (8.2) is Fredholm follows exactly as in the proof of Theorem 8.2 using exactly the same methodology, adding in $B_{5,\pm}$ elliptic on $(-m+\epsilon', m-\epsilon') \subset W_\pm^\perp$ and the estimates in Proposition 8.7.

The injectivity follows from exactly the same energy estimate argument, and the surjectivity is that same energy estimate argument applied to the adjoint.

The property (1.2) follows from the same argument as in the proof of Theorem 8.2. □

Proof of Theorem 8.4. For static $V = V(z)$, the Fredholm statement holds even in the presence bound states with energy less than $-m^2$. The fact that there are no elements in the kernel can be concluded directly from seperation of variables since on the finite family of eigenfunctions H_V the solutions are explicit and orthogonal to this family the energy argument holds. □

Index of Notation

- $\mathcal{S}$ is Schwartz functions, $\mathcal{S}'$ tempered distributions
- C_c^∞ is smooth and compactly supported
- $\dot{C}^\infty(M)$ for a manifold with corners M is the space of smooth functions which vanish to infinite order together with all their derivatives at the boundary
- $\sigma(A)$ the spectrum of an operator A
- $\lesssim$, used in an inequality when an unspecified positive constant is needed on the right hand side
- $H_V := \Delta + m^2 + V$, the Hamiltonian, for V possibly time-dependent, page 3
- $P_V := D_t^2 - H_V$, the Klein-Gordon operator, page 3
- $H_{\mathrm{sc}}^{s,\boldsymbol{\ell}}(\mathbb{R}^{n+1}), \mathcal{Y}^{s,\boldsymbol{\ell}}, \mathcal{X}^{s,\boldsymbol{\ell}}$ the weighted L^2-based Sobolev spaces and the a priori spaces, page 5
- $\mathrm{Diff}_{\mathrm{sc}}^m$ and $\mathrm{Diff}_{\mathrm{sc}}^{m,l}$ the scattering differential operators, page 11
- X, the radial compactification of $\mathbb{R}^{n+1}_{t,z}$, page 11
- ${}^{\mathrm{sc}}T^*X, {}^{\mathrm{sc}}\overline{T}^*X$ the scattering cotangent and its fiber compactification, page 12
- ρ_{base}, ρ_{fib}, boundary defining functions for ${}^{\mathrm{sc}}\overline{T}^*X$, page 12
- $j_{\mathrm{sc},m,l}(A)$ the scattering principal symbol, page 12
- $\sigma_{\mathrm{sc},m,l}(A), \hat{N}_{\mathrm{sc},m,l}(A)$ the fiber and normal components of the scattering principal symbol, respectively, page 12
- ${}^{\mathrm{sc}}S^{m,l}(X)$, scattering symbols, page 12,
- ${}^{\mathrm{sc}}\Psi^{m,\ell}$ of scattering operators, page 12,
- $C_{\mathrm{sc}}(X)$ the boundary of ${}^{\mathrm{sc}}\overline{T}^*X$, page 12
- $\mathrm{WF}'(A)$ the scattering operator wavefront set, Section 2.3
- $\mathrm{Ell}(A)$ the scattering elliptic set, Section 2.3
- $\mathrm{Char}(A) = C_{\mathrm{sc}}(X) \setminus \mathrm{Ell}(A)$, the characteristic set of A, Section 2.3
- H_p and ${}^{\mathrm{sc}}H_p$ the Hamilton vector field and its rescaling, equations (2.17) and (2.18), page 15
- $\mathcal{R}$ the radial set, and $\mathcal{R}^f_\pm, \mathcal{R}^p_\pm$ its four components, page 16 and below
- $\boldsymbol{\ell}$ variable order spacetime weight, Section 2.4
- $\boldsymbol{\ell}_\pm$ forward and backward weights, Section 2.8
- $V_+(z) = \lim_{t\to\infty} V(t)$, the limiting potential, Section 3.4, similarly for V_-
- $C \subset X$, the "poles", $C = \mathrm{NP} \cup \mathrm{SP}$, page 28
- $[X; C]$ the blow up of the poles in X, page 28
- β_C the blow down map, page 28
- $\mathrm{Diff}_{3\mathrm{sc}}^{m,l}$ the 3sc-differential operators, page 29
- ${}^{3\mathrm{sc}}T^*[X;C], {}^{3\mathrm{sc}}\overline{T}^*[X;C]$ the 3sc-cotangent bundle and compactification, page 30,
- $\rho_{\mathrm{ff}}, \rho_{\mathrm{mf}}, \rho_{\mathrm{fib}}$, boundary defining functions for ${}^{3\mathrm{sc}}\overline{T}^*X$, page 30
- $W^\perp$, the lines of τ over the poles, page 31
- $\pi\colon {}^{3\mathrm{sc}}T^*_C X \longrightarrow W^\perp$ the projection on $W^\perp$, page 31
- fibeq the fiber equator, page 31
- ${}^{3\mathrm{sc}}S^{m,l}(X)$, 3sc-symbols, page 34
- ${}^{3\mathrm{sc}}\Psi^{m,\ell}$ the 3sc-operators page 35
- $\hat{N}_{\mathrm{ff}}(A) = \hat{A}_{\mathrm{ff}}$ and $\hat{N}_{\mathrm{ff},\ell}(A)$ the indicial operators, page 37
- a_{ff} the weighted front face restriction, page 38
- $j_{3\mathrm{sc},m,\ell}(A)$ the principal symbol, equation (4.12), page 40

- $\sigma_{3\mathrm{sc},m,\ell}(A)$ the fiber symbol, $\hat{N}_{\mathrm{mf},m,\ell}(A)$ the "main face" symbol (restriction to the spacetime boundary), page 40
- $\Psi^{m,l,r}_{\mathrm{scl,sc},\pm 1/\tau}$ the two-sided semiclassical scattering operators, page 41
- UH_+ the upper half of fiber infinity over ff, page 45
- $H^{m,\ell}_{\mathrm{scl}}$ the semiclassical Sobolev spaces of order m, ℓ, page 46
- $\overline{W^\perp}$ the compactification of $W^\perp$, page 46
- ${}^{\mathrm{sc}}\dot{T}^*X$, ${}^{\mathrm{sc}}\dot{\overline{T}}^*X$ is the compressed cotangent bundle and its compactification, page 47
- $\pi^\perp$ the projection to ${}^{\mathrm{sc}}\dot{T}^*X$, page 47
- $C_{3\mathrm{sc}}[X;C]$, $\dot{C}_{3\mathrm{sc}}[X;C]$, equations (4.29) and (4.30), page 47
- $\gamma_{3\mathrm{sc}}$, page 47
- ${}^{3\mathrm{sc}}\mathrm{WF}'(A)$ the 3sc-operator wavefront set, Definition 4.17, page 48
- ${}^{3\mathrm{sc}}\mathrm{Ell}(A)$ the 3sc-elliptic set, Definition 4.18, page 49
- $G_{\psi,0}$, functional localizer, equation (5.1), page 53
- G_ψ, functional localizer, Definition (5.6), page 56
- $\pi_{X,\tau}$, page 65
- Σ, equation (6.3), page 66
- χ_0, χ_1 special cutoff functions, page 68

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