Math 608 - Spring 2025
Homework 3
5 February 2025

Read sections 4.5, 4.6, and 4.7.

1. If a result was proven in the text or in class you can just refer to it. Let X be a first countable topological space. Show that the following are equivalent:

i. X is compact.

ii. Every sequence in X has an accumulation point.

iii. Every sequence in X has a convergent subsequence.

2.

(a) If X is compact Hausdorff and $U \subset X$ is open, show that U is locally compact in the relative topology.

(b) Show that if E is locally compact Hausdorff, there is a Hausdorff space X so that E is homeomorphic to an open subset of X. (Hint: you already know this!)

3. Let $K \in C([0,1] \times [0,1])$. Define the integral operator

$$(Tf)(x) = \int_0^1 K(x,y)f(y)\,dy.$$

(a) Show that $T : C([0,1]) \to C([0,1])$.

(b) Show that in fact T is a *compact operator*, in the sense that the image $\{Tf : \|f\|_u \leq 1\}$ of the closed unit ball is precompact.

4. Consider $X = \mathbb{R}$ with the discrete topology and let $X^* = X \cup \{\infty\}$ be its one-point compactification.

(a) Describe the open sets in X^*. Hint: first determine the compact subsets of X.

(b) Describe $C(X^*)$.

5. Suppose X is a topological space for which there is a collection of continuous real-valued functions that separates points in X. Show X must be Hausdorff.

Quiz 3 Prove the exercise we need to finish the proof of Stone–Weierstrass. In other words, show there are numbers a_n so that the partial sums $\sum_{n=0}^{N} a_n t^n$ converge uniformly to $(1-t)^{1/2}$ for $0 \leq t \leq 1$. (Hints: Show that the Maclaurin series for $(1-t)^{1/2}$ is $a_0 = 1$ and $a_n = \frac{(2n-2)!}{2^n n((n-1)!)^2}$. Use Stirling's approximation to show that these are comparable to $1/n^{3/2}$ for large n and then use familiar results from calculus to show that the series converges uniformly. Show that the function it converges to satisfies $f(t) = -2(1-t)f'(t)$ and hence must be $(1-t)^{1/2}$.)

Additional practice problems Problems 4.52, 4.61, 4.64, 4.68.