

PRICE’S LAW ON MINKOWSKI SPACE IN THE PRESENCE OF AN INVERSE SQUARE POTENTIAL

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ABSTRACT. We consider the pointwise decay of solutions to wave-type equations in two model singular settings: the wave equation on Minkowski space with an inverse square potential as well as for the massless Dirac–Coulomb equation in $(3 + 1)$ -dimensions. In particular, we prove a form of Price’s Law for solutions to these equations and demonstrate the connection of the decay rate to radiation field asymptotics studied by the authors in other settings with singular structure. A novel feature of these models is that solutions exhibit different rates of decay as time increases than the standard Price’s Law on asymptotically flat space-times, which directly relates to long-range effects of the singularities.

1. INTRODUCTION

The pointwise decay of solutions to the linear wave equation on a space-time has many applications in terms of behavior of and lifespans of existence for small data solutions in nonlinear equations, far-field asymptotics, and stability of a given space-time in the general relativistic sense. Here we consider the long-time asymptotics of solutions of the wave equation on backgrounds with scale invariant singular potentials. Specifically, we analyze the pointwise decay rates in time for waves in the setting of diffractive and long range potentials. The models we consider here are those of the Dirac–Coulomb equation, which the first and second author considered in [BBG21], and the wave equation with an inverse square potential, which is related closely to work of the first and third authors in [BM16] and [BM19]. In three spatial dimensions, we prove pointwise decay estimates in time uniformly at any point in space that vary in an unexpected way from the standard t^{-3} decay rate given by Price’s Law for the wave equation in Minkowski space that has now been proven in a variety of settings.

As observed in the previous works of the first author with Wunsch–Vasy in [BVW15, BVW18], the pointwise decay rates of waves can be connected to the radiation field asymptotics of the operators considered. The radiation field for a wave equation on a given space-time is an asymptotic expansion whose exponents in the expansion along null infinity are the resonances of the spectral family of the singularly perturbed Laplacian on the hyperbolic space living at the “northern cap.” Resonances and radiation field asymptotics for the related problem of the wave equation on conic manifolds were computed in a previous papers by the first and third authors, [BM16] and [BM19] respectively. However, for pointwise decay estimates, we observe that the asymptotics of the radiation field do not tell the whole

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story. The locations of the resonances govern the time decay as t and r tend to infinity simultaneously, while the singular behavior of the resonant states lead to modified decay rates as t tends to infinity with r fixed.

In considering the wave equation on $\mathbb{R} \times \mathbb{R}^n$ with an inverse square potential, we assume u is a solution of the equation

$$(1) \quad \begin{aligned} \partial_t^2 u - \Delta u + \frac{F}{r^2} u &= 0, \\ (u, \partial_t u)|_{t=0} &= (\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \times C_c^\infty(\mathbb{R}^n \setminus \{0\}), \end{aligned}$$

where Δ is the negative definite Laplacian on \mathbb{R}^n and

$$F > -\frac{(n-2)^2}{4}.$$

Due to the Hardy inequality, the assumption on F implies that the Hamiltonian $-\Delta + \frac{F}{r^2}$ is essentially self-adjoint with form domain $H^1(\mathbb{R}^n)$ using the Friedrich's extension. See [BPST] for further discussions of this notion of self-adjoint extension for a range of F , but here we will simply always work with the Friedrich's extension.

For the massless Dirac–Coulomb equation, we consider $\psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$ a solution to

$$(2) \quad \begin{aligned} \left(\gamma^0 \left(\partial_t + \frac{iZ}{r} \right) + \sum_{j=1}^3 \gamma^j \partial_j \right) \psi &= 0, \\ \psi(0) = \psi_0 &\in (C_c^\infty(\mathbb{R}^3 \setminus \{0\}))^4 \end{aligned}$$

for $\gamma^{0,1,2,3}$ the gamma (or Dirac) matrices, meaning they satisfy the anticommutation relation

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2\eta^{\alpha\beta} Id_4,$$

where $\eta^{\alpha\beta}$ are the components of the Minkowski metric, i.e.,

$$\eta^{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 0 \\ 1 & \alpha = \beta \in \{1, 2, 3\} \\ 0 & \alpha \neq \beta \end{cases}.$$

Here Z is a real constant with $|Z| < \frac{1}{2}$, the bound again ensuring essential self-adjointness of the underlying Hamiltonian.

Our main results, given in Corollaries 3.2 and 3.4 below, can be stated in the following simplified form.

Theorem 1.1. *Let $\chi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. For $n = 3$ and u and ψ as defined in (1) and (2) respectively, we have*

$$\|u(t, \cdot) \chi(\cdot)\|_{W^{1,\infty}} \lesssim t^{-2-\alpha(F)}, \quad \|\psi(t, \cdot) \chi(\cdot)\|_{W^{1,\infty}} \lesssim t^{-2-\beta(Z)}.$$

where the functions

$$\alpha(F) = \sqrt{1 + 4F} - 1$$

and

$$\beta(Z) = 2\sqrt{1 - Z^2} - 1$$

arise from blowing up near the cone point as outlined below in Section 2. In addition, these decay rates are sharp.

In contrast, for $0 < \gamma < 1$, it follows from the works of [BM19, BBG21] that the solutions obey the estimates

$$|u(t, \gamma t, \theta)| \lesssim t^{-\frac{3}{2} + \sqrt{\frac{1}{4} + F}}, \quad |\psi(t, \gamma t, \theta)| \lesssim t^{-2 + \sqrt{1 - Z^2}}.$$

We emphasize that only the first part of Theorem 1.1 is new; the second estimate follows from the radiation field asymptotics in the prior work [BM19, BBG21]. The important aspect to note is that the two rates of decay as $t \rightarrow \infty$ *differ*. The singularity of the potential leads to slower time decay in compact spatial subsets.

Note that the results above are written to compare to a slower base time decay rate of t^{-2} rather than the familiar t^{-3} rate of Price's law as is expected to hold under generic sufficiently smooth perturbations $F = 0$ or $Z = 0$. In fact, our prior work [BBG21] yields the expected (t^{-3}) decay rate in the Dirac setting with $Z = 0$ due to a notable cancellation of terms that arise in the radiation field asymptotics, which we will discuss in more detail below. We also refer the reader to the works of Tataru [T13] and Morgan [M20] for an overview of the t^{-3} decay rate in the asymptotically Minkowski setting.

A model producing similar bimodal decay estimate in the case of asymptotically inverse square potentials on black hole backgrounds has been studied using vector field methods in the recent work of Gajic [Gajic]. Gajic's result builds upon the study of late time asymptotics for the wave equation beginning in the works [AAG1, AAG2, AAG3]. See also a forthcoming work of Van de Moortel and Gajic cited in [Gajic] addressing the Minkowski setting.

Remark 1.2. We have stated the results here in terms of inverse-square potentials for the wave equation and the Dirac–Coulomb equation, but a similar result holds on product cones in 3 spatial dimensions following the first and third author's work on [BM19]. Indeed, for a cone with a spherical cross-section of radius larger than $\sqrt{\frac{8}{3}}$, we observe a Price's Law with a time decay slower than the typical t^{-3} . By taking the radius sufficiently large this decay can be made slower than $t^{-2-\epsilon}$ for any $\epsilon > 0$. In this case, the index set from the radiation field alone would give an expected decay rate of t^{-3} , but the term arising from the first non-constant eigenfunction of the cross-section results in the slower decay rates. This observation provides some evidence of the long range nature of the interaction between diffraction and wave decay.

By modifying the approach taken to establish radiation field asymptotics on product cones by the first and third authors in [BM19], we can also compute the resonances and radiation fields for the Laplacian and the wave equation with inverse square potentials respectively. The time decay rates will depend explicitly on these resonances in the case of the Laplacian perturbed by singular potentials and is to that extent a manifestation of the long-range nature of such perturbations. As in the discussion below the main theorem, one interesting observation following from work of the first two authors with Booth [BBG21] is that there is an improvement in decay in time for solutions the Dirac equation in Minkowski space ($Z = 0$) in (2). The cancellation we referred to is due to the absence of a purely 0 spectral mode for the Dirac operator, but that gain disappears when you consider the Dirac–Coulomb equation because of the indicial roots associated with the lowest mode in that setting. Similar analysis shows that in the case $F < 0$, the decay rate for the wave equation worsens due precisely to singular behavior of the corresponding resonant states, which leads to the differing behavior on compact spatial subsets.

The main contributions of this article are to provide a canonical microlocal framework for proving pointwise wave decay even in the setting of singular potentials, as well as to extend Price's Law to an interesting physical case where the long-range effects of diffraction dramatically impact the behavior of waves. The starting point of our work is the recent work of the first author with Wunsch and Vasy [BVW15, BVW18]. Hintz [H20] took a similarly microlocal approach to Price's Law calculations by adapting functional analytic tools for wave equations on compactified space-times using work of Vasy and others [Vas13]. In addition, while the works of Hintz [H20] and Morgan [M20] considered potential perturbations of flat space, neither of those results could allow potentials with slow decay at infinity or singularities. See also the recent work of Looi [Looi21] and Morgan-Wunsch [MW21] for related studies. Here we present a modified Price's Law in exactly both of those cases for this family of relevant physical examples. In addition, since the methods come from a detailed analysis of the scattering matrix for these operators, one can with relative ease construct quasimodes that prove the sharpness of our decay estimates as stated in Theorem 1.1.

Remark 1.3. Here we work with the explicit operators of the form $\partial_t^2 - \Delta + \frac{F}{r^2}$, but it is natural to ask about equations of the form $\square_g + \frac{F}{r^2} + V$ for g an asymptotically flat metric and V a non-singular, lower order perturbation. The effects of these perturbations would most strongly manifest at low frequencies, and hence we conjecture that the decay rates we prove here would still be the expected behavior. Proving this would require adapting the second microlocalization tools of Vasy [Vas21-I, Vas21-II, Vas21-III] as implemented in the wave decay setting in [H20] to the conic setting, and will be an important topic for future work.

There is a rich history of recent results on Price's Law in the setting of stationary space times with static perturbations. There is a recent survey article by Schlag [Sch21] that gives a much more thorough overview of the subject than we can do here. The work initiated with harmonic by harmonic analysis of Donninger–Schlag–Soffer [DSS11] and Schlag–Soffer–Staubach [SSS10-I, SSS10-II], followed by the pioneering work of Tataru [T13] that relied upon Fourier localization techniques. The methods of Tataru have now been generalized to a large number of settings, [MTT12, MTT17], and in particular the role of regularity of the metric was recently studied in the work of Morgan [M20]. As mentioned above, as well as the work of Hintz [H20], in which Price's law on stationary space-times recently, including potentials using essentially the compactification machinery we are using, but all potential perturbations there were assumed to have r^{-3} decay and were smooth. A pointwise decay estimate for the wave equation with slow decay in one spatial dimension was proved by Donninger–Schlag in [DS10], though their result has regular potentials at $x = 0$ that decay slightly better than $|x|^{-2}$ at ∞ .

Dispersive estimates and nonlinear applications for wave and Schrödinger equations have been studied quite broadly. For instance, the wave equation with an inverse square potential has been studied in [BPST, PSS, MZZ]. The Schrödinger equation with an inverse square potential has been studied in [D, KMVZZ1, KMVZZ2] among many others. In particular, the distinction between $F > 0$ and $F < 0$ arises and there is actually a threshold $-1/4 + 1/25 < F^* < 0$ for which local well-posedness theory holds for critical semilinear equations with such potentials. In the long run, we hope that precise control on pointwise behavior of waves will give insight into further nonlinear applications, see [KMVZ, AM] for the most recent literature discussion on this problem to date.

In Section 2, we recall the compactification of our space-times and discuss the modifications to the case of these singular potentials. We then recall the nature of the radiation field expansions derived previously in [BM19, BBG21], and prove a microlocal lemma that allows us to lift the polyhomogeneous expansions we have in the radiation fields to prove error estimates in our time decay estimates. We then state and prove the main results in Section 3, which follow naturally from the results in Section 2 and the corresponding definitions of certain indexing sets that arise in each of our applications. Finally, in Appendix A, we provide the resonance calculations for the case of inverse square potentials that will give us the explicit decay rates we observe in our Price's Law statements.

2. COMPACTIFICATIONS AND PARTIAL POLYHOMOGENEITY

For both the Dirac–Coulomb equation and the wave equation on cones, we proceed by defining a compactification of the relevant space-time on which the introduced boundaries at “infinity” form the loci at which various asymptotic behaviors of the solutions take place. For both equations, the space-time decouples into the time factor \mathbb{R}_t times a spatial factor which is a cone on a closed manifold Z ; for the wave equation on cones, this Z is arbitrary, while for the Dirac–Coulomb equation the spatial slice is the polar coordinate interior $(0, \infty)_r \times \mathbb{S}^{n-1}$.

Concretely, suppose (Z, k) is a compact, connected $(n - 1)$ -dimensional Riemannian manifold without boundary. The metric cone on Z is a smooth Riemannian manifold on the interior of the open manifold with boundary $[0, \infty)_r \times Z$ given by the metric

$$g_c = dr^2 + r^2 k.$$

The metric g_c therefore extends to a smooth tensor on $[0, \infty) \times Z$ which is degenerate (i.e. not positive definite) at the boundary $\{0\} \times Z$, the locus of the conical singularity. We let $C(Z)$ denote the interior $(0, \infty)_r \times Z$. (This is nonstandard notation for the cone, which typically includes the cone point, but it is convenient as it accords with our usage below and in previous work.)

Thus, on the open manifold $M^\circ = \mathbb{R}_t \times C(Z)$, we consider the Lorentzian metric

$$g = -dt^2 + dr^2 + r^2 k.$$

Our compactification will allow us to regard M° as the interior of a compact manifold with corners, as follows.

We compactify $\mathbb{R}_t \times (0, \infty)_r$ (and thus $\mathbb{R}_t \times (0, \infty)_r \times Z$) by stereographic projection to a quarter-sphere \mathbb{S}_{++}^2 as depicted in Figure 1. In other words, the map $\mathbb{R}_t \times (0, \infty)_r \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ given by

$$(t, r) \mapsto \frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}$$

sends M° to the interior of the quarter-sphere given by

$$\mathbb{S}_{++}^2 = \{(z_1, z_2, z_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 \mid z_2 \geq 0, z_3 \geq 0\}.$$

The quarter-sphere \mathbb{S}_{++}^2 is a manifold with corners and has two boundary hypersurfaces defined by the boundary defining functions z_2 and z_3 . We let *cf* (or the *conic face*) be the hypersurface connecting NP to SP defined by the function

$$z_2 = \frac{r}{\sqrt{1 + t^2 + r^2}}$$

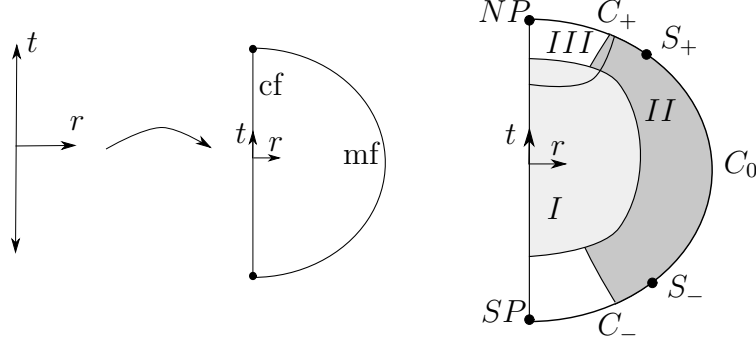


FIGURE 1. The compactification M of $\mathbb{R}_t \times (0, \infty)_r \times Z$ with the Z factor suppressed, mapping on the left with detail on the right. The image of $\mathbb{R}_t \times (0, \infty)_r$ can be thought of either as the right half disk D_+ or the quarter sphere \mathbb{S}_{++} . S_{\pm} are, respectively, collapsed future/past null infinity. C_{\pm} and C_0 are three subsets of mf

and we let mf (or the *main face*) be the face defined by

$$z_3 = \frac{1}{\sqrt{1 + t^2 + r^2}}.$$

Equivalently, we can compactify $\mathbb{R}_t \times (0, \infty)_r$ to a half ball $D_+ = \{w = (w_1, w_2) \in \mathbb{R}^2 : |w| \leq 1, w_2 \geq 0\}$ via the mapping

$$(t, r) \mapsto \frac{(t, r)}{1 + \sqrt{1 + t^2 + r^2}}.$$

We leave it to the interested reader to show that these compactifications are equivalent, in the sense that they induce diffeomorphisms of manifolds with corners. In particular, the boundary hypersurface (bhs) $w_2 = 0$ corresponds to $z_2 = 0$ and the bhs $|w| = 1$ corresponds to $z_3 = 0$.

There are several convenient coordinate systems valid in different regions of the compactification. For notational convenience, we will always use ρ to denote a defining function for mf and x to denote a defining function for cf . In region I (where we are bounded away from mf), it is convenient to take $x = r$, while in region II (where we are bounded away from cf), we can take $\rho = 1/r$. Finally, in region III (which is the source of most of the new technical work in this document), it is often convenient to take $\rho = 1/t$ and $x = r/t$ and write,

$$(3) \quad \text{region III} \simeq [0, 1]_{\rho} \times [0, c]_x \times Z,$$

where, again, for Dirac–Coulomb, $Z = \mathbb{S}^{n-1}$. Here $0 < c < 1$ is arbitrary and can be taken to be 1, though this makes the region more awkward to depict, and at any rate the interesting behavior in region III not covered by region II takes place near r/t small. *Because polyhomogeneity is independent of the choice of (equivalent) boundary defining function, we typically use whichever boundary defining functions are most convenient at the time.*

To understand the detailed asymptotic behavior of solutions we perform two blow ups of the compactification M . The first introduces future null infinity, which is the natural domain of definition of the radiation field, while the second blow up introduces the precise region needed to distinguish between the limits of timelike paths with zero and non-zero spatial momentum. The latter, as we have said, also distinguishes the two loci of differing asymptotic behavior of solutions at timelike infinity.

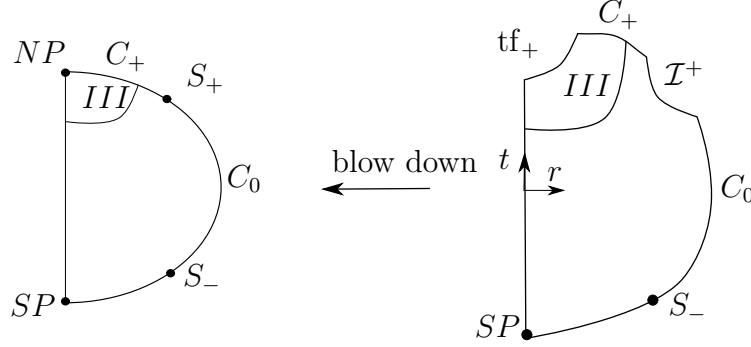


FIGURE 2. A schematic view of the space X (on the right) we use to study Price's Law. "Region III" is the neighborhood of timelike infinity in which the solutions under consideration exhibit distinct asymptotic behavior at the three boundary hypersurfaces.

We recall from previous work [BVW15, BVW18] the first blow up, which induces the construction of the manifold with corners on which the radiation field naturally lives.

Recall that we denote collapsed null infinity by S ; this is the intersection of the closure of maximally extended null geodesics with the introduced boundary mf. The submanifold is given by $S = \{v = \rho = 0\}$ where v is a smooth function on M which vanishes simply on the sets $x = \pm 1$. (Such v can be chosen so that v, ρ and coordinates on the link Z for local coordinate charts near S .) This submanifold naturally splits into two pieces according to whether $\pm t > 0$ near the component, which we denote by S_{\pm} . The complement of S in mf consists of three regions: the region C_0 being those points in mf where $v < 0$, while the region in mf where $v > 0$ has two components, denoted C_{\pm} according to whether $\pm t > 0$ nearby.

We now *blow up* S_+ in M by replacing it with its inward pointing spherical normal bundle.¹ In the product cone setting, this is equivalent to blowing up a pair of points in S^2_{++} and then taking the product with Y . This process replaces M with a new manifold $\overline{M} = [M; S_+]$ on which polar coordinates around the submanifold are smooth; the structure of this manifold with corners depends only on the submanifold S_+ (and not on the particular choice of defining functions v and ρ).

The space \overline{M} – which is not yet the final resolution we consider, as we have yet to blow up the north pole (zero momentum future timelike infinity) – is again a manifold with corners and has four boundary hypersurfaces: the lifts of C_+ , the union $C_0 \cup C_-$, the lift of cf, again denoted cf, and the new boundary hypersurface consisting of the pre-image of S_+ under the blow-down map. This new boundary hypersurface, which is essentially future null infinity, is denoted \mathcal{I}^+ . Moreover, \mathcal{I}^+ is naturally a fiber bundle over S_+ with fibers diffeomorphic to intervals. Indeed, the interior of the fibers is naturally an affine space (i.e., \mathbb{R} acts by translations, but there is no natural origin). Figure 2 depicts this blow-up construction. Given choices of functions v and ρ , the fibers of the interior of \mathcal{I}^+ in \overline{M} can be identified with $\mathbb{R} \times Z$ via the coordinate $s = v/\rho$.

Our focus on the $t > 0$ region of space-time is a matter of convenience only, it is also possible, indeed very useful, to blow up S_- and thereby obtain a boundary hypersurface corresponding to past null infinity. Indeed, the face obtained by the blow up of S_- is the

¹The reader may wish to consult Melrose's book [Mel93] for more details on the blow-up construction.

domain of the past radiation field. For Dirac-type equations, one can see distinct behavior when comparing \mathcal{I}^+ to the corresponding face \mathcal{I}^- over S_- , but these details are irrelevant for our discussion here which treats only the future timelike infinity behavior.

Finally, we blow up the “north pole” $\text{NP} = \{x = 0, \rho = 0\}$. We thereby obtain a manifold with corners $X = [\overline{M}; \text{NP}]$ with an additional boundary hypersurface tf_+ which separates cf from C_+ . Near the intersection of tf_+ with C_+ , we can use the boundary defining function $\rho_{\text{tf}_+} = x + \rho$, while a full set of coordinates is given by

$$\rho_{\text{tf}_+}, \quad y = \frac{x - \rho}{x + \rho}, \quad \theta,$$

where θ are any local coordinates on Z .

We now come to the main point of this blow up construction, which can be understood as follows. The solutions to the differential equations we consider here are tempered distributions which are conormal on \overline{M} and admit an asymptotic expansion at C_+ with polyhomogeneous coefficients. In this context, conormality² is equivalent to the existence of weights $\ell, \ell' \in \mathbb{R}$ such that for all $j, k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{\dim Z}$,

$$(x\partial_x)^j (\rho\partial_\rho)^k \partial_\theta^\alpha (x^{-\ell} \rho^{-\ell'} u) \in L^\infty,$$

while polyhomogeneity at C_+ is the condition that u admits an asymptotic expansion with terms of the form $\rho^s (\log \rho)^k a(x, \theta)$ with $s \in \mathbb{C}$, $k \in \mathbb{N}_0$ and k bounded for $\Re s < L$. The precise nature of the expansion here is the key to our arguments. Specifically, given an index set \mathcal{E} (meaning a subset $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}_0$ such that $\{(\sigma, k) \in \mathcal{E} : \text{Im } \sigma > -R\}$ is finite for each R) we assume that for each $L \in \mathbb{R}$, in sets $\{x < c\}$, i.e. away from \mathcal{I}_+ ,

$$(4) \quad u(x, \rho, \theta) = \sum_{(\sigma, k) \in \mathcal{E}, \text{Im } \sigma > -L} \rho^{i\sigma} (\log \rho)^k a_{\sigma k}(x, \theta) + \rho^L u'(x, \rho, \theta),$$

where the remainder $u'(x, \rho, \theta)$ is a bounded conormal function and the $a_{\sigma k}$ are themselves polyhomogeneous distributions on C^+ with a fixed index set \mathcal{F} at cf. (This means $a_{\sigma k} \sim \sum_{(\tau, l) \in \mathcal{F}} x^{i\tau} (\log x)^l c_{\tau, p}$ as $x \rightarrow 0$.) We say that such a distribution is **partially polyhomogeneous** at C^+ (near cf) with index set \mathcal{E} at C^+ and \mathcal{F} at cf. *Our main observation is that such a distribution u pulls back to the blown up space X to be fully polyhomogeneous at C^+ with index set \mathcal{E} and partially polyhomogeneous at tf with index set $\mathcal{F} + \mathcal{E}$ at tf and index set \mathcal{F} at cf.* This allows us to conclude the distinct asymptotic behavior described above. We formalize this pullback statement in Lemma 2.1.

In contrast with fully polyhomogeneous distributions, which admit expansions at every face, the partially polyhomogeneous distributions considered here admit only expansions at C^+ (in region III.)

This property of partial polyhomogeneity can be seen in the radiation field expansions for both (1) and (2) using the methods of proof in [BM19] and [BBG21] respectively. Indeed, this in the case of product cones, the result is proven directly in [BM19]. Indeed, there it can be seen that we have conormality relative to the conic singularity in x and polyhomogeneity in the ρ . We will discuss this in more detail below.

²Here we choose to use L^∞ -based conormal spaces as opposed to the H_b^s -based spaces used in [BM19] as the L^∞ -based spaces make the proof of Lemma 2.1 slightly clearer. The equivalence of L^∞ -based conormality and Sobolev space-based conormality follows from the Sobolev embedding theorem and is discussed in [Mel96] in Chapter 4.

Let

$$\mathbb{R}_{++}^2 = \{(z, w) \in \mathbb{R}^2 : z, w \geq 0\}$$

denote the closed upper right quadrant, let $H_1 = \{z = 0\}$, $H_2 = \{w = 0\}$. Let $v : \mathbb{R}_{++}^2 \rightarrow \mathbb{C}$ be a conormal distribution supported near 0, and assume that v is partially polyhomogeneous at H_1 with index sets \mathcal{E} at H_1 and \mathcal{F} at H_2 . Let ff denote the new face of the blow up $[\mathbb{R}_{++}^2 : (0, 0)]$.

Lemma 2.1. *Let $U \subset \mathbb{R}^N$ be an open set. Let v be a distribution on $\mathbb{R}_{++}^2 \times U$ which is partially polyhomogeneous at H_1 with index set \mathcal{E} at H_1 and \mathcal{F} at H_2 . Then the pullback of v to the blown-up space $[\mathbb{R}_{++}^2; \{(0, 0)\}] \times U$ is fully polyhomogeneous at H_1 with index set \mathcal{E} and partially polyhomogeneous at $\text{ff} \times U$ with index set $\mathcal{E} + \mathcal{F}$ at there and index \mathcal{F} at H_2 .*

Proof. We suppress the factor U in the domain as the smooth dependence on that parameter is irrelevant in the proof.

By assumption, near $(0, 0)$ we can write, for every $L, M \in \mathbb{R}$,

$$\begin{aligned} (5) \quad v &= \sum_{(\sigma, k) \in \mathcal{E}, \text{Im } \sigma > -L} z^{i\sigma} (\log z)^k a_{\sigma k} + z^L v' \\ &= \sum_{(\sigma, k) \in \mathcal{E}, \text{Im } \sigma > -L} z^{i\sigma} (\log z)^k \left(\sum_{(\tau, l) \in \mathcal{F}, \text{Im } \tau > -M} w^{i\tau} (\log w)^l c_{\sigma k \tau l} + w^M v'' \right) + z^L v' \end{aligned}$$

where v', v'' are conormal functions.

Blowing up, the functions $w, s = z/w$ define coordinates near the intersection of ff with H_1 . The first line of (5), with $z = sw$, continues to provide a polyhomogeneous expansions with index set \mathcal{E} but now with index set $\mathcal{E} + \mathcal{F}$ at ff since

$$v = \sum_{(\sigma, k) \in \mathcal{E}, \text{Im } \sigma > -L} s^{i\sigma} w^{i\sigma} (\log s + \log w)^k a_{\sigma k}(w) + (sw)^L v'.$$

It is easy to check that v' is conormal on $[\mathbb{R}_{++}^2; \{(0, 0)\}]$.

Near the intersection of ff with H_2 , $z, t = w/z$ define coordinates, with z a defining function of ff and t a defining function of H_2 , so using the second line of (5) we get

$$v = \sum_{\substack{(\sigma, k) \in \mathcal{E}, \text{Im } \sigma > -L \\ (\tau, l) \in \mathcal{F}, \text{Im } \tau > -M}} z^{i\sigma + i\tau} (\log z)^k t^{i\tau} (\log z + \log t)^l c_{\sigma k \tau l} + z^R v'''$$

where v''' is a conormal distribution and

$$R < \inf\{L, M - \text{Im } \sigma - \text{Im } \tau : (\sigma, 0) \in \mathcal{E}, (\tau, 0) \in \mathcal{F}\}.$$

This gives the asymptotic expansion with index set $\mathcal{E} + \mathcal{F}$ in z with terms having expansions in index set \mathcal{F} in t , and choosing M, L sufficiently negative gives the result. \square

3. ASYMPTOTIC EXPANSIONS FOR EQUATIONS WITH INVERSE SQUARE POTENTIALS

In this section we recall relevant versions of the main results of previous papers [BM19, BBG21]. We first turn our attention to the earlier paper [BM19] in the context of the wave equation on $\mathbb{R} \times \mathbb{R}^n$ with an inverse square potential. That paper describes the asymptotics of solutions of the wave equation on cones, but the techniques apply with no change to the wave equation on \mathbb{R}^n with an inverse square potential. In particular, the following theorem is an immediate consequence of those results:

Theorem 3.1. *Let u denote the solution of the wave equation with inverse square potential (1) with smooth, compactly supported initial data. If $\chi \in C_c^\infty([0, \infty))$ is supported in $[0, 1]$, then for all L , with the local coordinates coming from (3),*

$$\chi(x)u = \sum_{\operatorname{Im} \sigma_{j,k} > -L} \rho^{\frac{n-1}{2} + i\sigma_{j,k}} \chi(x) v_{j,k}(x, \theta) + \rho^{\frac{n-1}{2} + L} u',$$

where $u' \in I^1$. Here $\sigma_{j,k}$ are indexed by $j, k = 0, 1, 2, \dots$ and are given by

$$\sigma_{j,k} = -i \left(\frac{1}{2} + \sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F + k} \right),$$

when $\sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F} \notin \frac{1}{2} + \mathbb{Z}$, where λ_j are the eigenvalues of the spherical Laplacian. Similarly, $v_{j,k}$ are given explicitly in terms of hypergeometric functions and eigenfunctions ϕ_j of the spherical Laplacian:

$$v_{j,k} = x^{-\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F}} F(a, b, c; x) \phi_j(\theta),$$

and $F(a, b, c; x)$ is Gauss's hypergeometric function with parameters

$$\begin{aligned} a &= \frac{1}{2} + \sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F}, \\ b &= 1 + k + 2\sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F}, \\ c &= 1 + 2\sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F}. \end{aligned}$$

Of considerable importance is that the remainder term in Theorem 3.1 lies in a *fixed* conormal space independent of L .

Thus, to bring Lemma 2.1 to bear on solutions for the inverse square potential, we define the index sets

$$\mathcal{E} = \left\{ -i \left(\frac{n}{2} + k + \sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F} \right) : k = 0, 1, 2, \dots \right\}$$

when $\sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F} \notin \frac{1}{2} + \mathbb{Z}$ and

$$\mathcal{F} = \left\{ -i \left(\ell - \frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2} \right)^2 + \lambda_j + F} \right) : \ell = 0, 1, 2, \dots \right\}$$

using the resonance calculation in Appendix A. (Here there are no logarithms in the expansion so we simply drop the \mathbb{N}_0 factor from the index set definition.) Theorem 3.1 states that u on the blown \overline{M} is partially polyhomogeneous at C_+ with index set \mathcal{E} at C_+ and \mathcal{F} at cf. In particular, from Lemma 2.1, we conclude

Corollary 3.2. *Functions u as in the statement of Theorem 3.1 pulled back to the blown up space X from Section 2 are fully polyhomogeneous at C_+ with index set \mathcal{E} and partially polyhomogeneous at tf_+ with index set $\mathcal{E} + \mathcal{F}$ at tf_+ and \mathcal{F} at cf .*

Looking at the index set $\mathcal{E} + \mathcal{F}$, we have that the leading order decay rate is

$$1 + \sqrt{1 + 4F},$$

in particular the part of Theorem 1.1 regarding solutions to the wave equation with the inverse square potential holds.

Each term in the expansion given by the index set $\mathcal{E} + \mathcal{F}$ in both cases above is generically non-vanishing by arguments similar to those demonstrating sharpness in [H20], Section 5. The Price's Law decay rates in our theorem correspond to only the leading order term of the index set, hence the sharpness is determined by a single, linear condition. This term is non-zero precisely when the pairing of the data with the leading order co-resonant state associated to the leading order term in \mathcal{E} is non-vanishing, which is a generic condition.

Using the results of [BBG21], we obtain

Theorem 3.3. *If ψ is the solution to the massless Dirac–Coulomb system (2) with smooth, compactly supported initial data. If $\chi \in C_c^\infty([0, \infty))$ is supported in $[0, 1)$, then for all L ,*

$$\chi(x)\psi(\rho, x, \theta) = \sum_{\text{Im } \varsigma_{\kappa, \ell} > -L} \rho^{1+i\varsigma_{\kappa, \ell}} \chi(x) \varphi_{\kappa, \ell}(x, \theta) + \rho^{1+L} \psi',$$

where $\psi' \in I^1$. The numbers $\varsigma_{\kappa, \ell}$ are indexed by $\kappa \in \mathbb{Z} \setminus \{0\}$ and $\ell = 0, 1, 2, \dots$ and given by

$$\varsigma_{\kappa, \ell} = -i \left(1 + \ell + \sqrt{\kappa^2 - \mathbf{Z}^2} \right).$$

Similarly, the $\varphi_{\kappa, \ell}$ are given explicitly in terms of spinor spherical harmonics Ω_κ and hypergeometric functions:

$$\varphi_{\kappa, \ell} = x^{\sqrt{\kappa^2 - \mathbf{Z}^2} - 1} \left(F(a, b, c; x) \begin{pmatrix} \Omega_\kappa(\theta) \\ \Omega_{-\kappa}(\theta) \end{pmatrix} + F(a+1, b, c; x) \begin{pmatrix} \Omega_\kappa(\theta) \\ -\Omega_{-\kappa}(\theta) \end{pmatrix} \right)$$

with

$$\begin{aligned} a &= \sqrt{\kappa^2 - \mathbf{Z}^2} - i\mathbf{Z}, & b &= 1 + 2\sqrt{\kappa^2 - \mathbf{Z}^2} + \ell \\ c &= 1 + 2\sqrt{\kappa^2 - \mathbf{Z}^2}. \end{aligned}$$

Thus, for the case of Dirac–Coulomb we set

$$\mathcal{E}' = \left\{ -i \left(1 + \ell + \sqrt{\kappa^2 - \mathbf{Z}^2} \right) : \ell \in \{0, 1, 2, \dots\}, \kappa \in \mathbb{Z} \setminus \{0\} \right\}$$

and

$$\mathcal{F}' = \left\{ -i \left(j + \sqrt{\kappa^2 - \mathbf{Z}^2} - 1 \right) : j \in \{1, 2, \dots\}, \kappa \in \mathbb{Z} \setminus \{0\} \right\}.$$

Looking at the index set $\mathcal{E}' + \mathcal{F}'$, we have that the leading order decay rate is

$$1 + 2\sqrt{1 - \mathbf{Z}^2}.$$

Again from Lemma 2.1 we have the following corollary, which implies the part of Theorem 1.1 related to solutions to the Dirac–Coulomb equation.

Corollary 3.4. *Functions ψ as in the statement of Theorem 3.3 pulled back to the blown up space X from Section 2 are fully polyhomogeneous at C_+ with index set \mathcal{E}' and partially polyhomogeneous at tf_+ with index set $\mathcal{E}' + \mathcal{F}'$ at tf_+ and \mathcal{F}' at cf .*

APPENDIX A. RESONANCE CALCULATION

Let $\rho = \frac{1}{t+r}$ and $x = \frac{2r}{t+r}$ so that $r = \frac{x}{2\rho}$. If we set

$$\begin{aligned} L_0 &= \partial_t^2 - \Delta + \frac{F}{r^2} \\ &= \partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} + \frac{F}{r^2}, \end{aligned}$$

then L_0 lifts to

$$\begin{aligned} L_0 &= \rho^2(\rho\partial_\rho + x\partial_x)^2 + \rho^2(\rho\partial_\rho + x\partial_x) - \rho^2(2\partial_x - x\partial_x - \rho\partial_\rho)^2 \\ &\quad + \rho^2(2\partial_x - x\partial_x - \rho\partial_\rho) - 2\rho^2\frac{n-1}{x}(2\partial_x - x\partial_x - \rho\partial_\rho) - \frac{4\rho^2}{x^2}\Delta_{\mathbb{S}^{n-1}} + \frac{4F\rho^2}{x^2}. \end{aligned}$$

The paper [BM19] and its predecessors consider the reduced normal operator of

$$L = \rho^{-2}\rho^{-\frac{n-1}{2}}L_0\rho^{\frac{n-1}{2}}.$$

As this operator is homogeneous of degree 0 in ρ , the reduced normal operator is given by replacing $\rho\partial_\rho$ with $i\sigma$, yielding

$$\begin{aligned} P_\sigma = -\widehat{N}(L) &= 4(1-x)\partial_x^2 + \frac{n-1}{x}4\partial_x - \left(4 + (i\sigma + \frac{n-1}{2}) + 2(n-1)\right)\partial_x \\ &\quad + \frac{4}{x^2}\Delta_{\mathbb{S}^{n-1}} - \frac{4F}{x^2} - 2\left(\frac{n-1}{x}\right)(i\sigma + \frac{n-1}{2}). \end{aligned}$$

The poles of the inverse of this operator (on appropriate σ -dependent variable-order Sobolev spaces) yield the exponents seen above. The corresponding resonant states are solutions v of $P_\sigma v = 0$ lying in these same spaces.

For this particular operator, we can see the poles explicitly in terms of the failure of hypergeometric functions to remain linearly independent. Indeed, separating into angular modes ϕ_j with eigenvalues $-\lambda_j$, the radial coefficients v_j of a solution of $P_\sigma v_j = 0$ must satisfy

$$\begin{aligned} 0 &= xP_{\sigma,j}v_j \\ &= 4x(1-x)\partial_x^2 v_j + 4(n-1-x(n+i\sigma))\partial_x v_j - 4\left(\frac{F+\lambda_j}{x}\right)v_j - (n-1)(2i\sigma+n-1)v_j. \end{aligned}$$

Dividing by 4, setting $P = \frac{1}{4}xP_{\sigma,j}$ and letting $w = v_j$, w must satisfy

$$\begin{aligned} Pw &= x(1-x)\partial_x^2 w + (n-1-x(n+i\sigma))\partial_x w - \frac{F+\lambda_j}{x}w - \left(\frac{n-1}{2}\right)(i\sigma + \frac{n-1}{2})w \\ &= 0. \end{aligned}$$

Conjugating by x^α , where

$$\alpha = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + F + \lambda_j},$$

yields a hypergeometric equation for $y = x^{-\alpha}w$:

$$x(1-x)\partial_x^2 y + (n-1+2\alpha-x(n+i\sigma+2\alpha))\partial_x y - \left(\alpha + \frac{n-1}{2}\right)\left(\alpha + \frac{n-1}{2} + i\sigma\right)y = 0.$$

This is a hypergeometric differential equation with parameters (see, e.g., [DLMF] for notation)

$$\begin{aligned} a &= \frac{1}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + F}, \\ b &= \frac{1}{2} + i\sigma + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + F}, \\ c &= 1 + 2\sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + F}. \end{aligned}$$

The requirement that the solution v lies locally in H^1 near $x = 0$ implies that y must be a multiple of the hypergeometric function

$$y_1 = F(a, b, c; x) = F\left(\frac{1}{2} + s, \frac{1}{2} + i\sigma + s, 1 + 2s; x\right),$$

where $s = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + F}$. On the other hand, if a solution is to lie in the appropriate variable order Sobolev space globally, at C_+ it must be a multiple of

$$\begin{aligned} y_4 &= (1-x)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-x) \\ &= (1-x)^{-i\sigma} F\left(\frac{1}{2} + s, \frac{1}{2} - i\sigma + s, 1 - i\sigma; 1-x\right). \end{aligned}$$

The poles under consideration are therefore given by those σ for which y_4 is a multiple of y_1 . Kummer's connection formulae [DLMF, 15.10.18] allow us to write y_4 in terms of the basis of solutions y_1 and y_2 , where y_2 is given by

$$y_2 = x^{1-c} F(a-c+1, b-c+1, 2-c; x) = x^{-2s} F\left(\frac{1}{2} - s, \frac{1}{2} - s + i\sigma, 1 - 2s; x\right).$$

In this case, we have

$$y_4 = \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)} y_1 + \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} y_2$$

In particular, the coefficient of y_2 vanishes precisely when $\frac{1}{2} + s - i\sigma$ is a pole of the gamma function, i.e., when

$$\sigma = \sigma_{j,k} = -i\left(\frac{1}{2} + s + k\right), \quad k = 0, 1, \dots$$

Moreover, the resonant state v_j associated with $\sigma_{j,k}$ is a multiple of $x^\alpha y_1$ and therefore is polyhomogeneous with leading order behavior

$$x^{-\frac{n-2}{2}+s} \text{ as } x \rightarrow 0.$$

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