MATH 622: DIFFERENTIAL GEOMETRY I

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1. Introduction

This is the first semester of a two-semester graduate course providing an introduction to differential geometry.

We are mostly using Lee's Smooth Manifolds book but will supplement with others.

Topics roughly covered include:

- Examples of manifolds and submanifolds
- Differentiable manifolds and smooth maps between them
- Tangent and cotangent vectors and bundles, differentials, vector fields, Lie brackets
- Tensors
- Distributions and the Frobenius theorem
- Differential forms and integration; de Rham cohomology
- Lie groups
- Classical differential geometry of curves and surfaces
- Gauss–Bonnet

There are many paths to differential geometry. Common ones include algebraic geometry, algebraic topology, and general relativity. Another common path involves pizza (this is a joke about Gauss's Theorema Egregium).

2. Preliminaries

2.1. **Topology and topological manifolds.**

The main objects of study in this course are smooth manifolds. Before we define them, we'll first define topological manifolds. A naïve first definition is that a topological manifold is a topological space that "locally looks like $\mathbb{R}^n$".

Definition 1. A topological space is a set X equipped with a collection $\mathcal{T} \subset \mathcal{P}(X)$ so that

(1) $\mathcal{T}$ contains the empty set and the whole space: $\emptyset, X \in \mathcal{T}$;
(2) $\mathcal{T}$ is closed under finite intersections: if $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap \dots \cap U_n \in \mathcal{T}$.
(3) $\mathcal{T}$ is closed under arbitrary unions: if $\{U_i\}_{i \in I} \subset \mathcal{T}$, then $\cup_{i\in I} U_i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called open sets. A set $F \subset X$ is closed if $U = X \setminus F$ is open.

Suppose X and Y are topological spaces.

Definition 2. A function $f : X \to Y$ is continuous if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open.

Definition 3. A continuous function $f : X \to Y$ is a homeomorphism if it is a bijection and its inverse is continuous.

Definition 4. A topological space X is locally Euclidean if for every point $p \in X$ there is an open subset $U \subset X$ with $p \in U$ so that U is homeomorphic to an open subset of $\mathbb{R}^n$ for some n.

First working definition of a topological manifold: A topological manifold is a locally Euclidean topological space.

This definition seems to capture what we mean by "locally looks like $\mathbb{R}^n$", so why isn't this our definition? Well, bad things can happen here. Here are two examples that can be fleshed out.

- The line with two origins.
- The very long line.
- An uncountable set with the discrete topology.

One way to avoid these pitfalls is to add some conditions. Spivak uses metrizability, while Lee uses the following definition:

Definition 5. A manifold is a topological space M that is

(1) Hausdorff, i.e., for any $p, q \in M$, there are disjoint open neighborhoods of p and q;
(2) Second-countable, i.e., there is a countable base for its topology; and
(3) Locally Euclidean.

We won't prove the following theorem. The fastest proofs I know of go through homology.

Theorem 6 (Invariance of domain)**.** *If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is continuous and injective then $f(U)$ is open.*

One implication of this theorem is that the n in the definition is determined uniquely by the point and so is determined on each connected component of M. Throughout our course, we'll assume it's the same on each connected component and say $n = \dim M$.

Aside: connectivity.

Definition 7. A topological space X is connected if, given any open sets A, B with $A, B \neq \emptyset$, $A \cup B = X$, we must have $A \cap B \neq \emptyset$.

In other words, X is connected if the only sets that are both open and closed are $\emptyset$ and X. This splits X into connected components.

Many other things can go wrong. We'll ask that our manifolds be σ-compact (i.e., a countable union of compact sets).

2.2. **Calculus.** We now review some facts from multivariable calculus.

Definition 8. Suppose $U \subset \mathbb{R}^n$ is open. A function $f : U \to \mathbb{R}^m$ is differentiable at a point $a \in U$ if there is a linear map $L_a : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{x \to a} \frac{|f(x) - f(a) - L_a(x-a)|}{|x-a|} = 0.$$

We say that the linear map L_a is the (total) derivative of f at a.

If f is differentiable at all points in U, one can view the total derivative of f as a function $U \to \mathrm{End}(\mathbb{R}^n, \mathbb{R}^m)$, with the latter space being isomorphic to $\mathbb{R}^{n\times m}$. This total derivative is denoted by a variety of different notations, including f' and Df.

One immediate consequence of the definition of the derivative is the following version of "Taylor's theorem":

Theorem 9. *Suppose $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$ and W is any convex subset of U, then*

$$f(x) = f(a) + Df(a)(x-a) + R(x),$$

where

$$\lim_{x\to a}\frac{|R(x)|}{|x-a|}=0$$

The chain rule then reads: If $f:\mathbb{R}^n\to\mathbb{R}^m$ is differentiable at p and $g:\mathbb{R}^m\to\mathbb{R}^k$ is differentiable at $f(p)$, then $g\circ f:\mathbb{R}^n\to\mathbb{R}^k$ is differentiable at p with derivative $Dg\circ Df$ (with composition in the sense of linear maps).

If $f:U\to\mathbb{R}^m$ is continuous on U, differentiable on U, and its total derivative is a continuous function, we say that $f\in C^1(U;\mathbb{R}^m)$ or that f is C^1. When the codomain is $\mathbb{R}$ we typically drop the codomain from the notation. Defining higher derivatives via the chain rule is then straightforward; if f and its first k derivatives are all continuous on U then we say f is in C^k. If f and all of its derivatives are continuous, we say f is C^∞; if in addition f is an analytic function, we sometimes say f is C^ω. General functions in this course will be C^∞ unless otherwise specified.

The j-th partial derivative of a function $\phi:\mathbb{R}^n\to\mathbb{R}$ is given by

$$\partial_j\phi(x)=\frac{\partial\phi}{\partial x^j}(x)=\lim_{h\to 0}\frac{\phi(x+he_j)-\phi(x)}{h},$$

where e_j is the j-th standard basis vector in $\mathbb{R}^n$.

Picking bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ provides a matrix representation of the total derivative of a map $f:\mathbb{R}^n\to\mathbb{R}^m$. Indeed, a basis for $\mathbb{R}^m$ lets us write

$$f=\begin{pmatrix} f^1\\ f^2\\ \dots\\ f^m\end{pmatrix},$$

and then the total derivative Df has matrix

$$\begin{pmatrix}\partial_1 f^1 & \partial_2 f^1 & \dots & \partial_n f^1\\ \partial_1 f^2 & \partial_2 f^2 & \dots & \partial_n f^2\\ \vdots & \vdots & \ddots & \vdots\\ \partial_1 f^m & \partial_2 f^m & \dots & \partial_n f^m\end{pmatrix}.$$

Theorem 10 (Taylor's theorem)**.** *Let $U\subset\mathbb{R}^n$ be open and $a\in U$. Let $f\in C^{k+1}(U)$ for some $k\geq 0$. If V is any convex subset of U containing a then for all $x\in V$,*

$$f(x)=P_k(x)+R_k(x),$$

where P_k is the k-th order Taylor polynomial of f at a, i.e.,

$$P_k(x)=\sum_{m=0}^{k}\frac{1}{m!}\sum_{|\alpha|=m}\partial_\alpha f(a)(x-a)^\alpha,$$

and R_k is the remainder term given by

$$R_k(x)=\frac{1}{k!}\sum_{|\alpha|=k+1}(x-a)^\alpha\int_0^1(1-t)^k(\partial_\alpha f)(a+t(x-a))\,dt.$$

In particular, if all of the $(k+1)$-st partial derivatives of f are bounded by M then for all $x \in W$,

$$|f(x) - P_k(x)| \leq \frac{n^{k+1}M}{(k+1)!}|x-a|^{k+1}$$

Proof. Let's do it first for the one-dimensional version. Suppose $U \subset \mathbb{R}$, $g \in C^{k+1}(U)$, $W \subset U$ is an interval and $a \in W$. The Taylor polynomial of g is given by

$$P_k(x) = \sum_{m=0}^{k} \frac{1}{m!} g^{(m)}(a)(x-a)^m,$$

while the remainder term is given by

$$R_k(x) = \frac{1}{k!}(x-a)^{k+1} \int_0^1 (1-t)^k g^{(k+1)}(a + t(x-a))\, dt.$$

We'll prove it by induction. The statement for $k = 0$ follows immediately from the fundamental theorem of calculus and the chain rule applied to the function $g'(a + t(x-a))$. Now we assume it holds for some k and integrate the remainder $R_k(x)$ by parts:

$$\begin{aligned} R_k(x) &= \frac{1}{k!}(x-a)^{k+1} \left(-\frac{(1-t)^{k+1}}{k+1} g^{(k+1)}(a + t(x-a))|_0^1 + \frac{x-a}{k+1} \int_0^1 (1-t)^{k+1} g^{(k+2)}(a + t(x-a))\, dt \right) \\ &= \frac{1}{(k+1)!}(x-a)^{k+1} g^{(k+1)}(a) + R_{k+1}(x), \end{aligned}$$

so that indeed $f(x) = P_{k+1}(x) + R_{k+1}(x)$ for all $x \in W$. The final statement in one dimension follows by bounding the integrand by $M(1-t)^k$ and integrating.

For the more general statement we use the mutlivariable chain rule and the one-variable statement. (This is a homework assignment.) □

One way to think about Taylor's theorem for C^∞ functions is as a kind of divisibility theorem: If $f \in C^\infty$ vanishes as a point a, then f is in the C^∞-span of $(x^1 - a^1), (x^2 - a^2), \ldots, (x^n - a^n)$. In other words, the C^∞-ideal of functions vanishing at the point a is generated by these n functions.

Two of the most important results from multivariable calculus (though you probably did not learn them in a multivariable calculus class) are the inverse and implicit function theorems. We'll use the inverse function theorem repeatedly when talking about charts and the implicit function theorem will provide us with an essentially endless source of examples of smooth manifolds.

Theorem 11 (Inverse function theorem)**.** *Suppose $U \subset \mathbb{R}^n$ is open, $f \in C^1(U; \mathbb{R}^n)$, and $A = (Df)_p$ is the total derivative of f at p for some $p \in U$. If A is invertible (viewed as a linear map), then there is an open ball $B(p, r) \subset U$ with $r > 0$ so that*
(a) f is one-to-one on $B = B(p, r)$,
(b) $V = f(B)$ is open, and
(c) $g = (f|_B)^{-1}$ is C^1 on V.
(d) In addition, if f is C^k, so is g.

Proof. We start by writing

$$f(x) = f(p) + A(x-p) + R(x).$$

As f is C^1, we know that $R(x)$ is also C^1. Moreover, as we know that

$$\lim_{x\to p}\frac{|R(x)|}{|x-p|}=0,$$

given any $\epsilon>0$, we may find $r>0$ so that

$$|R(x)|\leq \epsilon\,|x-p|\,,\quad |DR(x)|\leq\epsilon$$

for all $x\in B(p,r)$. In particular, for all $x,y\in B(p,r)$, we have

$$\begin{aligned}|R(x)-R(y)|&=\left|\int_0^1 (DR)(y+t(x-y))(x-y)\,dt\right|\\&\leq \epsilon\,|x-y|\,.\end{aligned}$$

We now claim that by choosing $\epsilon>0$ we guarantee that f is one-to-one on $B=B(p,r)$. Because A is invertible, there is some $c>0$ (one can take c to be the size of the eigenvalue of A closest to 0) so that for all $x,y\in\mathbb{R}^n$,

$$|A(x-y)|\geq c\,|x-y|\,.$$

We now take $\epsilon=c/2$ and observe that for $x,y\in B(p,r)$,

$$\begin{aligned}|f(x)-f(y)|&=|A(x-y)+R(x)-R(y)|\\&\geq|A(x-y)|-|R(x)-R(y)|\geq c\,|x-y|-\frac{c}{2}\,|x-y|\geq\frac{c}{2}\,|x-y|\,,\end{aligned}$$

so that f is injective on $B=B(p,r)$. Moreover, as invertibility is an open condition and f is C^1, we can ensure (by possibly shrinking r) that $f'(x)$ is invertible on all $x\in B$.

Now, letting $V=f(B)$, we want to show that V is open. Pick $y_0\in V$, so there is some $x_0\in B$ with $f(x_0)=y_0$. Choose $t>0$ so that $\overline{B(x_0,t)}\subset B$, and take $\ell(x)=|f(x)-y_0|$ as a function on the sphere $|x-x_0|=t$. As f is one-to-one here and the sphere is compact, ℓ attains a positive minimum m here. Take $|y_1-y_0|<m/3$, so that

$$|f(x)-y_1|\geq|f(x)-y_0|-|y_0-y_1|\geq\frac{2m}{3}.$$

Now let $h(x)=|f(x)-y_1|^2$ on $\overline{B(x_0,t)}$. The previous calculation implies that $h(x)\geq 4m^2/9$ on the boundary of $\overline{B(x_0,t)}$, while $h(x_0)=|y_0-y_1|^2<m^2/9$, so the minimum of h is attained on the interior of $B(x_0,t)$. Let $x_1\in B(x_0,t)$ denote the point where this minimum is attained. As x_1 is therefore a critical point of h, we have

$$0=\nabla h(x_1)=2f'(x_1)^{\intercal}(f(x_1)-y_1).$$

We know that $f'(x_1)$ is invertible, so $f(x_1)=y_1$ and thus $B(y_0,m/3)\subset V$ and thus V is open.

We now show that $g=(f|_B)^{-1}$ is C^1 on V. (Its derivative should be $f'(g(y))^{-1}$.) Fix $y\in V$ and for $|k|$ small with $y+k\in V$, we find $x,x+h\in B$ so that $y=f(x)$ and $y+k=f(x+h)$. Setting $B=(f'(g(y))^{-1}$, we have

$$\frac{|g(y+k)-g(y)-Bk|}{|k|}=\frac{|x+h-x-B(f(x+h)-f(x))|}{|k|}=\frac{|-B\left(f(x+h)-f(x)-f'(x)h\right)|}{|k|}.$$

If we show that $|k| \geq c\,|h|$ for some $c > 0$ independent of h, then this quotient will go to 0 as $k \to 0$. Indeed, by using Taylor's theorem around p again we see that

$$\begin{aligned} |k| &= |f(x+h) - f(x)| = |Ah + R(x+h) - R(x)| \\ &\geq |Ah| - |R(x+h) - R(x)| \geq c\,|h| - \frac{c}{2}\,|x+h-x| = \frac{c}{2}\,|h|\,. \end{aligned}$$

We may therefore take a limit as $h \to 0$ to see that g is differentiable at y with derivative $f'(g(y))^{-1}$. As this function is continuous, the derivative of g is also continuous.

Finally, if f is C^k, then the chain rule applied to g shows that g is also C^k. $\square$

In discussing (and proving) the implicit function theorem it is helpful to think about splitting coordinates on the product space $\mathbb{R}^{n+m}$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we'll use (x, y) for a point in $\mathbb{R}^{n+m}$. Similarly, if $A : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is linear, we'll split it into (A_x, A_y), where $A_x : \mathbb{R}^n \to \mathbb{R}^m$ and $A_y : \mathbb{R}^m \to \mathbb{R}^m$ are given by

$$A_x(v) = A(v, 0), \quad A_y(w) = A(0, w).$$

Theorem 12 (Implicit function theorem)**.** *Suppose $f : E \to \mathbb{R}^m$ is C^1, where $E \subset \mathbb{R}^n \times \mathbb{R}^m$ is open. Suppose further that $f(p, q) = 0$ for some $(p, q) \in E$. Let $A = f'(p, q)$. If $A_y = (\partial_{y_j} f)_{j=1}^m$ is invertible, then there are open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ so that*

(a) for each $x \in W$ there is a unique y so that $(x, y) \in U$ and $f(x, y) = 0$, and

(b) letting $g(x)$ denote this y, then $g : W \to \mathbb{R}^m$ is C^1, $g(p) = q$, $f(x, g(x)) = 0$, and $g'(p) = -(A_y)^{-1}A_x$.

To get the formula for the derivative right, it is instructive to think about the (trivial) linear version of the theorem:

Theorem 13 (Implicit function theorem, linear version)**.** *If $A : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is linear and A_y is invertible, then for every $v \in \mathbb{R}^n$, there is a unique $w \in \mathbb{R}^m$ so that $A(v, w) = 0$.*

Proof. This is just the observation that

$$A(v, w) = A_x v + A_y w,$$

so that if this is to equal zero we must have

$$w = -A_y^{-1} A_x v.$$

$\square$

Proof of implicit function theorem. For $(x, y) \in E$, we define a new function $F : E \to \mathbb{R}^{n+m}$ by $F(x, y) = (x, f(x, y))$. Note that F is C^1 and DF has the following matrix at (p, q):

$$F'(p, q) = \begin{pmatrix} I_{n\times n} & 0 \\ \partial_{x_j} f(p, q) & \partial_{y_\ell} f(p, q) \end{pmatrix} = \begin{pmatrix} I_{n\times n} & 0 \\ A_x & A_y \end{pmatrix}.$$

As A_y is invertible, $F'(p, q)$ is also invertible, so by the inverse function we may find an open cube $U \subset \mathbb{R}^{n+m}$ so that F is one-to-one on U, $V = F(U)$ is open, and $(F|_U)^{-1}$ is C^1.

We now let $W = \{x \in \mathbb{R}^n \mid (x, 0) \in V\}$. Note that $p \in W$ because $f(p, q) = 0$ and that W is open because V is open. If $x \in W$, then $(x, 0) \in V$ and so there is some y with $(x, y) \in U$ with $f(x, y) = 0$.

This y is our candidate for $g(x)$; we must show it is unique. Indeed, if $(x, y') \in U$ also satisfies $f(x, y') = 0$, then

$$F(x, y') = (x, f(x, y')) = (x, f(x, y)) = F(x, y),$$

so $y' = y$ as F is injective on U. We now define $y = g(x)$ for $x \in W$ to be the unique y with $(x, y) \in U$ and $f(x, y) = 0$, i.e., so that $F(x, g(x)) = (x, 0)$ for all $y \in W$.

Letting $G = (F|_U)^{-1}$, we know that G is C^1 and $(x, g(x)) = G(x, 0)$, so g is also C^1. We finally calculate the derivative of g. Indeed, let $\Phi : \mathbb{R}^n \to \mathbb{R}^{n+m}$ be given by $\Phi(x) = (x, g(x))$, so then as a linear map, we have

$$\Phi'(x)v = (v, g'(x)v) \quad \text{for all } v \in \mathbb{R}^n.$$

Now, since $f(\Phi(x)) = 0$, we must have $f'(\Phi(x)) \circ \Phi'(x) = 0$, i.e., at (p, q) we have

$$0 = f'(p, q)(I, g'(p)) = A_x + A_y g'(p),$$

and so $g'(p) = -A_y^{-1} A_x$. □

The implicit function theorem will provide us with many examples of manifolds more or less "for free"; rather than taking the trouble of defining charts (to be discussed below), we instead can check the rank of a map to verify that the level set must be a manifold. Indeed, it can be viewed as giving you local homeomorphisms (as we'll see, you get something even better) putting a locally Euclidean structure on the level set.

As a quick example, consider the sphere $\mathbb{S}^n = \{\sum_{j=1}^{n+1} (x^j)^2 = 1\} \subset \mathbb{R}^{n+1}$. This set is the zero set of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$f(x^1, x^2, \dots, x^{n+1}) = \sum_{j=1}^{n+1} (x^j)^2 - 1.$$

Its total derivative at a point on the sphere is given by

$$Df_{(x^1, x^2, \dots, x^{n+1})} = \begin{pmatrix} 2x^1 & 2x^2 & \dots & 2x^{n+1} \end{pmatrix},$$

which has rank one as long as at least one of the x^j is non-vanishing (as it must be to lie on the sphere). The sphere is therefore a topological manifold.

3. Smooth manifolds and maps between them