Math 608 - Spring 2025
Homework 1
22 January 2025

If you do not remember much about metric spaces (or have not seen them before), read/review section 0.6.

Read sections 4.1, 4.2, and 4.3, with particular attention paid to Proposition 4.13, the examples on page 125, and the definition and properties of a subnet.

1. Suppose that (X, d) is a metric space.

(a) Suppose that $f : [0, \infty) \to [0, \infty)$ satisfies $f(0) = 0$, $f(x) > 0$ if $x > 0$, f is increasing, and f is subadditive, i.e., $f(x+y) \leq f(x) + f(y)$ for all x, y. Show that $f \circ d : X \times X \to [0, \infty)$ is a metric on X.

(b) Suppose that $f : [0, \infty) \to [0, \infty)$ is C^1 (continuously differentiable, i.e., continuous with continuous derivative), $f(0) = 0$, $f'(0) > 0$, $f'(x) \geq 0$ for all x, and f' is decreasing (i.e., $x \leq y$ implies $f'(x) \geq f'(y)$). Show that f is subadditive.

(c) Suppose d and d' are metrics on X. Show that the topology generated by d' is weaker than the topology generated by d (i.e., every open set in (X, d') is open in (X, d)) if and only if, given any $\epsilon > 0$ and $x \in X$, there is a $\delta > 0$ so that

$$d(x, y) < \delta \quad \implies \quad d'(x, y) < \epsilon.$$

(d) Conclude that if d is a metric on X, then so is $d' = \frac{d}{1+d}$, and these two metrics generate the same topology. (It is often convenient to be able to replace a metric d by d', which satisfies $d'(x, y) < 1$ for all $x, y \in X$.)

2. If $\{X_\alpha\}$ is a family of topological spaces, then $X = \prod_\alpha X_\alpha$ (with the product topology) is uniquely determined up to homeomorphism by the following universal property: There exist continuous maps $\pi_\alpha : X \to X_\alpha$ so that if Y is any topological space and $f_\alpha : Y \to X_\alpha$ are continuous functions for each α, then there is a unique continuous function $F : Y \to X$ so that $f_\alpha = \pi_\alpha \circ F$. (In other words, show that if you have two spaces X and X', together with corresponding maps π_α and π'_α, then X and X' must be homeomorphic. Hint: Use the uniqueness of F.)

3. If X is a set, $\mathcal{F}$ a collection of real-valued functions on X, and $\mathcal{T}$ the weak topology generated by $\mathcal{F}$, then $\mathcal{T}$ is Hausdorff if and only if for every $x, y \in X$ with $x \neq y$, there exists an $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

4. Prove Tietze's extension theorem, i.e., that in a normal space X, for a closed set $A \subset X$ and a continuous function $f : A \to \mathbb{R}$, there is a continuous function $g : X \to \mathbb{R}$ so that $g|_A = f$, by using the following steps:

1. Let $h = f/(1+|f|)$. Then $|h| < 1$.
2. Let $B = \{x \in A : h(x) \leq -\frac{1}{3}\}$ and $C = \{x \in A : h(x) > \frac{1}{3}\}$. Use Urysohn's lemma to show there is a continuous function h_1 on X so that $h_1|_B = -\frac{1}{3}$ and $h_1|_C = \frac{1}{3}$. Conclude that $|h(x) - h_1(x)| < \frac{2}{3}$ for $x \in A$.
3. Use induction to show that there is a continuous function h_n on X so that $|h_n(x)| \leq \frac{2^{n-1}}{3^n}$ for all $x \in X$ and so that $|h(x) - \sum_{i=1}^n h_i(x)| < \frac{2^n}{3^n}$ for all $x \in A$.
4. Show that the sequence (h_n) is uniformly summable to a continuous function k on X with $|k| \leq 1$ and $k|_A = h$.
5. Show that there is a continuous function $\phi : X \to \mathbb{R}$ which is equal to 1 on A and 0 on $\{x \in X : |k(x)| = 1\}$.
6. Set $g = \phi k/(1 - \phi\,|k|)$. Show this g works.

5. Prove that a topological space X is Hausdorff if and only if every net in X converges to at most one point. (See problem 4.32 in the text.)

Quiz 1

(a) Prove that if X_α is connected for each α, then $\prod X_\alpha$ (equipped with the product topology) is connected. See problem 4.27 for the precise statement; you can use problems 10 and 18 without proof.

(b) Show that the same does not necessarily hold for the box topology by showing that in the sequence space $\mathbb{R}^{\mathbb{N}}$, both the collection of bounded sequences and the collection of unbounded sequences are open.