Math 608 - Spring 2025
Homework 4
12 February 2025

Read Sections 5.1, 5.2. They are short but dense.

1. Suppose X and Y are compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x,y) = g(x)h(y)$ for $g \in C(X)$ and $h \in C(Y)$ is dense in $C(X \times Y)$.

2. Let X be a metric space. A function $f \in C(X)$ is called *Hölder continuous with exponent α* ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. If X is compact, show that $\{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

3. Let X be a compact metric space and $0 < \alpha \leq 1$. Show that the Hölder continuous functions on X with exponent α form a Banach space with norm

$$\|f\| = \|f\|_u + N_\alpha(f).$$

This space is often denoted $C^\alpha(X)$ (or $C^{0,\alpha}(X)$ if X is a manifold), though Folland uses $\Lambda_\alpha(X)$.

4. Suppose X is a Banach space.

(a) If $T \in \mathcal{L}(X,X)$ has $\|T\| < 1$, then $I - T$ is invertible and in fact the series $\sum_{n=0}^\infty T^n$ converges in $\mathcal{L}(X,X)$ to $I - T$.

(b) Show that if $A \in \mathcal{L}(X,X)$ is invertible and $\|A - B\| < \left\|A^{-1}\right\|^{-1}$, then B is invertible. (The set of invertible operators is therefore open in $\mathcal{L}(X,X)$.)

5. Suppose that X is a finite dimensional vector space. Let $e_1, \ldots, e_n$ be a basis for X and define

$$\left\|\sum_{j=1}^n a_j e_j\right\|_1 = \sum_{j=1}^n |a_j| .$$

(a) Show that $\|\cdot\|_1$ is a norm on X.

(b) Show that the map $(a_1, \ldots, a_n) \mapsto \sum_{j=1}^n a_j e_j$ is a continuous function from K^n (with the usual Euclidean topology) to X with the topology induced by $\|\cdot\|_1$.

(c) Show that the unit sphere in this norm is compact, i.e., $\{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.

(d) Show that any norm on X is equivalent to $\|\cdot\|_1$.

Quiz 4 Let X be a compact Hausdorff space. An *ideal* in $C(X;\mathbb{R})$ is a subalgebra $\mathcal{I}$ of $C(X;\mathbb{R})$ so that if $f \in \mathcal{I}$ and $g \in C(X;\mathbb{R})$, $fg \in \mathcal{I}$.

(a) If $\mathcal{I}$ is an ideal in $C(X;\mathbb{R})$, let $V(\mathcal{I}) = \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}\}$. Show that $V(\mathcal{I})$ is a closed subset of X. It is called the hull of $\mathcal{I}$.

(b) If $E \subset X$, let $J(E) = \{f \in C(X;\mathbb{R}) : f(x) = 0 \text{ for all } x \in E\}$. Show that $J(E)$ is a closed ideal in $C(X;\mathbb{R})$. It is called the kernel of E.

(c) Show that if $E \subset X$, then $V(J(E)) = \overline{E}$.

(d) If $\mathcal{I}$ is an ideal in $C(X;\mathbb{R})$, then $J(V(\mathcal{I})) = \overline{\mathcal{I}}$. (Hint: $J(V(\mathcal{I}))$ can be identified with a subalgebra of $C_0(U,\mathbb{R})$, where $U = X \setminus V(\mathcal{I})$.)

(e) Put the above together to show that the closed subsets of X are in one-to-one correspondence with the closed ideals of $C(X;\mathbb{R})$.

Additional practice problems Problems 5.8, 5.9, 5.10, 5.13.