MATH 622: DIFFERENTIAL GEOMETRY I

DEAN BASKIN

1. Introduction

This is the first semester of a two-semester graduate course providing an introduction to differential geometry.

We are mostly using Lee's Smooth Manifolds book but will supplement with others.

Topics roughly covered include:

- Examples of manifolds and submanifolds
- Differentiable manifolds and smooth maps between them
- Tangent and cotangent vectors and bundles, differentials, vector fields, Lie brackets
- Tensors
- Distributions and the Frobenius theorem
- Differential forms and integration; de Rham cohomology
- Lie groups
- Classical differential geometry of curves and surfaces
- Gauss–Bonnet

There are many paths to differential geometry. Common ones include algebraic geometry, algebraic topology, and general relativity. Another common path involves pizza (this is a joke about Gauss's Theorema Egregium).

2. Preliminaries

2.1. **Topology and topological manifolds.**

The main objects of study in this course are smooth manifolds. Before we define them, we'll first define topological manifolds. A naïve first definition is that a topological manifold is a topological space that "locally looks like $\mathbb{R}^n$".

Definition 1. A topological space is a set X equipped with a collection $\mathcal{T} \subset \mathcal{P}(X)$ so that

(1) $\mathcal{T}$ contains the empty set and the whole space: $\emptyset, X \in \mathcal{T}$;
(2) $\mathcal{T}$ is closed under finite intersections: if $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap \dots \cap U_n \in \mathcal{T}$.
(3) $\mathcal{T}$ is closed under arbitrary unions: if $\{U_i\}_{i\in I} \subset \mathcal{T}$, then $\cup_{i\in I} U_i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called open sets. A set $F \subset X$ is closed if $U = X \setminus F$ is open.

Suppose X and Y are topological spaces.

Definition 2. A function $f : X \to Y$ is continuous if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open.

Definition 3. A continuous function $f : X \to Y$ is a homeomorphism if it is a bijection and its inverse is continuous.

Definition 4. A topological space X is locally Euclidean if for every point $p \in X$ there is an open subset $U \subset X$ with $p \in U$ so that U is homeomorphic to an open subset of $\mathbb{R}^n$ for some n.

First working definition of a topological manifold: A topological manifold is a locally Euclidean topological space.

This definition seems to capture what we mean by "locally looks like $\mathbb{R}^n$", so why isn't this our definition? Well, bad things can happen here. Here are two examples that can be fleshed out.

- The line with two origins.
- The very long line.
- An uncountable set with the discrete topology.

One way to avoid these pitfalls is to add some conditions. Spivak uses metrizability, while Lee uses the following definition:

Definition 5. A manifold is a topological space M that is

(1) Hausdorff, i.e., for any $p, q \in M$, there are disjoint open neighborhoods of p and q;
(2) Second-countable, i.e., there is a countable base for its topology; and
(3) Locally Euclidean.

We won't prove the following theorem. The fastest proofs I know of go through homology.

Theorem 6 (Invariance of domain)**.** *If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is continuous and injective then $f(U)$ is open.*

One implication of this theorem is that the n in the definition is determined uniquely by the point and so is determined on each connected component of M. Throughout our course, we'll assume it's the same on each connected component and say $n = \dim M$.

Aside: connectivity.

Definition 7. A topological space X is connected if, given any open sets A, B with $A, B \neq \emptyset$, $A \cup B = X$, we must have $A \cap B \neq \emptyset$.

In other words, X is connected if the only sets that are both open and closed are $\emptyset$ and X. This splits X into connected components.

Many other things can go wrong. We'll ask that our manifolds be σ-compact (i.e., a countable union of compact sets).

2.2. **Calculus.** We now review some facts from multivariable calculus.

Definition 8. Suppose $U \subset \mathbb{R}^n$ is open. A function $f : U \to \mathbb{R}^m$ is differentiable at a point $a \in U$ if there is a linear map $L_a : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{x \to a} \frac{|f(x) - f(a) - L_a(x-a)|}{|x-a|} = 0.$$

We say that the linear map L_a is the (total) derivative of f at a.

If f is differentiable at all points in U, one can view the total derivative of f as a function $U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$, with the latter space being isomorphic to $\mathbb{R}^{n\times m}$. This total derivative is denoted by a variety of different notations, including f' and Df.

One immediate consequence of the definition of the derivative is the following version of "Taylor's theorem":

Theorem 9. *Suppose $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$ and W is any convex subset of U, then*

$$f(x) = f(a) + Df(a)(x-a) + R(x),$$

where

$$\lim_{x \to a} \frac{|R(x)|}{|x-a|} = 0$$

The chain rule then reads: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at p and $g : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $f(p)$, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at p with derivative $Dg \circ Df$ (with composition in the sense of linear maps).

If $f : U \to \mathbb{R}^m$ is continuous on U, differentiable on U, and its total derivative is a continuous function, we say that $f \in C^1(U; \mathbb{R}^m)$ or that f is C^1. When the codomain is $\mathbb{R}$ we typically drop the codomain from the notation. Defining higher derivatives via the chain rule is then straightforward; if f and its first k derivatives are all continuous on U then we say f is in C^k. If f and all of its derivatives are continuous, we say f is C^∞; if in addition f is an analytic function, we sometimes say f is C^ω. General functions in this course will be C^∞ unless otherwise specified.

The j-th partial derivative of a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\partial_j \phi(x) = \frac{\partial \phi}{\partial x^j}(x) = \lim_{h \to 0} \frac{\phi(x + he_j) - \phi(x)}{h},$$

where e_j is the j-th standard basis vector in $\mathbb{R}^n$.

Picking bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ provides a matrix representation of the total derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$. Indeed, a basis for $\mathbb{R}^m$ lets us write

$$f = \begin{pmatrix} f^1 \\ f^2 \\ \dots \\ f^m \end{pmatrix},$$

and then the total derivative Df has matrix

$$\begin{pmatrix} \partial_1 f^1 & \partial_2 f^1 & \dots & \partial_n f^1 \\ \partial_1 f^2 & \partial_2 f^2 & \dots & \partial_n f^2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f^m & \partial_2 f^m & \dots & \partial_n f^m \end{pmatrix}.$$

Theorem 10 (Taylor's theorem)**.** *Let $U \subset \mathbb{R}^n$ be open and $a \in U$. Let $f \in C^{k+1}(U)$ for some $k \geq 0$. If V is any convex subset of U containing a then for all $x \in V$,*

$$f(x) = P_k(x) + R_k(x),$$

where P_k is the k-th order Taylor polynomial of f at a, i.e.,

$$P_k(x) = \sum_{m=0}^{k} \frac{1}{m!} \sum_{|\alpha|=m} \partial_\alpha f(a)(x-a)^\alpha,$$

and R_k is the remainder term given by

$$R_k(x) = \frac{1}{k!} \sum_{|\alpha|=k+1} (x-a)^\alpha \int_0^1 (1-t)^k (\partial_\alpha f)(a + t(x-a))\, dt.$$

In particular, if all of the $(k+1)$-st partial derivatives of f are bounded by M then for all $x \in W$,

$$|f(x) - P_k(x)| \leq \frac{n^{k+1}M}{(k+1)!}|x-a|^{k+1}$$

Proof. Let's do it first for the one-dimensional version. Suppose $U \subset \mathbb{R}$, $g \in C^{k+1}(U)$, $W \subset U$ is an interval and $a \in W$. The Taylor polynomial of g is given by

$$P_k(x) = \sum_{m=0}^{k} \frac{1}{m!} g^{(m)}(a)(x-a)^m,$$

while the remainder term is given by

$$R_k(x) = \frac{1}{k!}(x-a)^{k+1} \int_0^1 (1-t)^k g^{(k+1)}(a + t(x-a))\, dt.$$

We'll prove it by induction. The statement for $k = 0$ follows immediately from the fundamental theorem of calculus and the chain rule applied to the function $g'(a + t(x-a))$. Now we assume it holds for some k and integrate the remainder $R_k(x)$ by parts:

$$\begin{aligned} R_k(x) &= \frac{1}{k!}(x-a)^{k+1}\left(-\frac{(1-t)^{k+1}}{k+1} g^{(k+1)}(a+t(x-a))|_0^1 + \frac{x-a}{k+1}\int_0^1 (1-t)^{k+1} g^{(k+2)}(a+t(x-a))\, dt \right) \\ &= \frac{1}{(k+1)!}(x-a)^{k+1} g^{(k+1)}(a) + R_{k+1}(x), \end{aligned}$$

so that indeed $f(x) = P_{k+1}(x) + R_{k+1}(x)$ for all $x \in W$. The final statement in one dimension follows by bounding the integrand by $M(1-t)^k$ and integrating.

For the more general statement we use the mutlivariable chain rule and the one-variable statement. (This is a homework assignment.) □

One way to think about Taylor's theorem for C^∞ functions is as a kind of divisibility theorem: If $f \in C^\infty$ vanishes as a point a, then f is in the C^∞-span of $(x^1 - a^1), (x^2 - a^2), \dots, (x^n - a^n)$. In other words, the C^∞-ideal of functions vanishing at the point a is generated by these n functions.

Two of the most important results from multivariable calculus (though you probably did not learn them in a multivariable calculus class) are the inverse and implicit function theorems. We'll use the inverse function theorem repeatedly when talking about charts and the implicit function theorem will provide us with an essentially endless source of examples of smooth manifolds.

Theorem 11 (Inverse function theorem). *Suppose $U \subset \mathbb{R}^n$ is open, $f \in C^1(U; \mathbb{R}^n)$, and $A = (Df)_p$ is the total derivative of f at p for some $p \in U$. If A is invertible (viewed as a linear map), then there is an open ball $B(p,r) \subset U$ with $r > 0$ so that*

(a) f is one-to-one on $B = B(p,r)$,
(b) $V = f(B)$ is open, and
(c) $g = (f|_B)^{-1}$ is C^1 on V.
(d) In addition, if f is C^k, so is g.

Proof. We start by writing

$$f(x) = f(p) + A(x-p) + R(x).$$

As f is C^1, we know that $R(x)$ is also C^1. Moreover, as we know that

$$\lim_{x \to p} \frac{|R(x)|}{|x-p|} = 0,$$

given any $\epsilon > 0$, we may find $r > 0$ so that

$$|R(x)| \leq \epsilon \, |x-p| \, , \quad |DR(x)| \leq \epsilon$$

for all $x \in B(p,r)$. In particular, for all $x, y \in B(p,r)$, we have

$$\begin{aligned} |R(x) - R(y)| &= \left| \int_0^1 (DR)(y + t(x-y))(x-y) \, dt \right| \\ &\leq \epsilon \, |x-y| \, . \end{aligned}$$

We now claim that by choosing $\epsilon > 0$ we guarantee that f is one-to-one on $B = B(p,r)$. Because A is invertible, there is some $c > 0$ (one can take c to be the size of the eigenvalue of A closest to 0) so that for all $x, y \in \mathbb{R}^n$,

$$|A(x-y)| \geq c \, |x-y| \, .$$

We now take $\epsilon = c/2$ and observe that for $x, y \in B(p,r)$,

$$\begin{aligned} |f(x) - f(y)| &= |A(x-y) + R(x) - R(y)| \\ &\geq |A(x-y)| - |R(x) - R(y)| \geq c \, |x-y| - \frac{c}{2} \, |x-y| \geq \frac{c}{2} \, |x-y| \, , \end{aligned}$$

so that f is injective on $B = B(p,r)$. Moreover, as invertibility is an open condition and f is C^1, we can ensure (by possibly shrinking r) that $f'(x)$ is invertible on all $x \in B$.

Now, letting $V = f(B)$, we want to show that V is open. Pick $y_0 \in V$, so there is some $x_0 \in B$ with $f(x_0) = y_0$. Choose $t > 0$ so that $\overline{B(x_0,t)} \subset B$, and take $\ell(x) = |f(x) - y_0|$ as a function on the sphere $|x - x_0| = t$. As f is one-to-one here and the sphere is compact, ℓ attains a positive minimum m here. Take $|y_1 - y_0| < m/3$, so that

$$|f(x) - y_1| \geq |f(x) - y_0| - |y_0 - y_1| \geq \frac{2m}{3}.$$

Now let $h(x) = |f(x) - y_1|^2$ on $\overline{B(x_0,t)}$. The previous calculation implies that $h(x) \geq 4m^2/9$ on the boundary of $\overline{B(x_0,t)}$, while $h(x_0) = |y_0 - y_1|^2 < m^2/9$, so the minimum of h is attained on the interior of $B(x_0,t)$. Let $x_1 \in B(x_0,t)$ denote the point where this minimum is attained. As x_1 is therefore a critical point of h, we have

$$0 = \nabla h(x_1) = 2f'(x_1)^{\intercal}(f(x_1) - y_1).$$

We know that $f'(x_1)$ is invertible, so $f(x_1) = y_1$ and thus $B(y_0, m/3) \subset V$ and thus V is open.

We now show that $g = (f|_B)^{-1}$ is C^1 on V. (Its derivative should be $f'(g(y))^{-1}$.) Fix $y \in V$ and for $|k|$ small with $y + k \in V$, we find $x, x + h \in B$ so that $y = f(x)$ and $y + k = f(x+h)$. Setting $B = (f'(g(y))^{-1}$, we have

$$\frac{|g(y+k) - g(y) - Bk|}{|k|} = \frac{|x + h - x - B(f(x+h) - f(x))|}{|k|} = \frac{|-B\left(f(x+h) - f(x) - f'(x)h\right)|}{|k|}.$$

If we show that $|k| \geq c\,|h|$ for some $c > 0$ independent of h, then this quotient will go to 0 as $k \to 0$. Indeed, by using Taylor's theorem around p again we see that

$$\begin{aligned}|k| &= |f(x+h) - f(x)| = |Ah + R(x+h) - R(x)| \\ &\geq |Ah| - |R(x+h) - R(x)| \geq c\,|h| - \frac{c}{2}\,|x+h-x| = \frac{c}{2}\,|h|\,.\end{aligned}$$

We may therefore take a limit as $h \to 0$ to see that g is differentiable at y with derivative $f'(g(y))^{-1}$. As this function is continuous, the derivative of g is also continuous.

Finally, if f is C^k, then the chain rule applied to g shows that g is also C^k. $\square$

In discussing (and proving) the implicit function theorem it is helpful to think about splitting coordinates on the product space $\mathbb{R}^{n+m}$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we'll use (x, y) for a point in $\mathbb{R}^{n+m}$. Similarly, if $A : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is linear, we'll split it into (A_x, A_y), where $A_x : \mathbb{R}^n \to \mathbb{R}^m$ and $A_y : \mathbb{R}^m \to \mathbb{R}^m$ are given by

$$A_x(v) = A(v, 0), \quad A_y(w) = A(0, w).$$

Theorem 12 (Implicit function theorem)**.** *Suppose $f : E \to \mathbb{R}^m$ is C^1, where $E \subset \mathbb{R}^n \times \mathbb{R}^m$ is open. Suppose further that $f(p, q) = 0$ for some $(p, q) \in E$. Let $A = f'(p, q)$. If $A_y = (\partial_{y_j} f)_{j=1}^m$ is invertible, then there are open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ so that*

(a) for each $x \in W$ there is a unique y so that $(x, y) \in U$ and $f(x, y) = 0$, and

(b) letting $g(x)$ denote this y, then $g : W \to \mathbb{R}^m$ is C^1, $g(p) = q$, $f(x, g(x)) = 0$, and $g'(p) = -(A_y)^{-1}A_x$.

To get the formula for the derivative right, it is instructive to think about the (trivial) linear version of the theorem:

Theorem 13 (Implicit function theorem, linear version)**.** *If $A : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is linear and A_y is invertible, then for every $v \in \mathbb{R}^n$, there is a unique $w \in \mathbb{R}^m$ so that $A(v, w) = 0$.*

Proof. This is just the observation that

$$A(v, w) = A_x v + A_y w,$$

so that if this is to equal zero we must have

$$w = -A_y^{-1} A_x v.$$

$\square$

Proof of implicit function theorem. For $(x, y) \in E$, we define a new function $F : E \to \mathbb{R}^{n+m}$ by $F(x, y) = (x, f(x, y))$. Note that F is C^1 and DF has the following matrix at (p, q):

$$F'(p, q) = \begin{pmatrix} I_{n\times n} & 0 \\ \partial_{x_j} f(p, q) & \partial_{y_\ell} f(p, q) \end{pmatrix} = \begin{pmatrix} I_{n\times n} & 0 \\ A_x & A_y \end{pmatrix}.$$

As A_y is invertible, $F'(p, q)$ is also invertible, so by the inverse function we may find an open cube $U \subset \mathbb{R}^{n+m}$ so that F is one-to-one on U, $V = F(U)$ is open, and $(F|_U)^{-1}$ is C^1.

We now let $W = \{x \in \mathbb{R}^n \mid (x, 0) \in V\}$. Note that $p \in W$ because $f(p, q) = 0$ and that W is open because V is open. If $x \in W$, then $(x, 0) \in V$ and so there is some y with $(x, y) \in U$ with $f(x, y) = 0$.

This y is our candidate for $g(x)$; we must show it is unique. Indeed, if $(x, y') \in U$ also satisfies $f(x, y') = 0$, then

$$F(x, y') = (x, f(x, y')) = (x, f(x, y)) = F(x, y),$$

so $y' = y$ as F is injective on U. We now define $y = g(x)$ for $x \in W$ to be the unique y with $(x, y) \in U$ and $f(x, y) = 0$, i.e., so that $F(x, g(x)) = (x, 0)$ for all $y \in W$.

Letting $G = (F|_U)^{-1}$, we know that G is C^1 and $(x, g(x)) = G(x, 0)$, so g is also C^1. We finally calculate the derivative of g. Indeed, let $\Phi : \mathbb{R}^n \to \mathbb{R}^{n+m}$ be given by $\Phi(x) = (x, g(x))$, so then as a linear map, we have

$$\Phi'(x)v = (v, g'(x)v) \quad \text{for all } v \in \mathbb{R}^n.$$

Now, since $f(\Phi(x)) = 0$, we must have $f'(\Phi(x)) \circ \Phi'(x) = 0$, i.e., at (p, q) we have

$$0 = f'(p, q)(I, g'(p)) = A_x + A_y g'(p),$$

and so $g'(p) = -A_y^{-1} A_x$. □

The implicit function theorem will provide us with many examples of manifolds more or less "for free"; rather than taking the trouble of defining charts (to be discussed below), we instead can check the rank of a map to verify that the level set must be a manifold. Indeed, it can be viewed as giving you local homeomorphisms (as we'll see, you get something even better) putting a locally Euclidean structure on the level set.

Corollary 14. *If $f : E \to \mathbb{R}^m$ is C^∞, where $E \subset \mathbb{R}^n \times \mathbb{R}^m$ is open, $f(p, q) = 0$, and $A = f'(p, q)$ has rank m, then the level set $f^{-1}(0)$ is locally Euclidean in a neighborhood of (p, q).*

As a quick example, consider the sphere $\mathbb{S}^n = \{\sum_{j=1}^{n+1} (x^j)^2 = 1\} \subset \mathbb{R}^{n+1}$. This set is the zero set of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$f(x^1, x^2, \ldots, x^{n+1}) = \sum_{j=1}^{n+1} (x^j)^2 - 1.$$

Its total derivative at a point on the sphere is given by

$$Df_{(x^1, x^2, \ldots, x^{n+1})} = \begin{pmatrix} 2x^1 & 2x^2 & \ldots & 2x^{n+1} \end{pmatrix},$$

which has rank one as long as at least one of the x^j is non-vanishing (as it must be to lie on the sphere). The sphere is therefore a topological manifold.

3. Smooth manifolds and maps between them

3.1. **Definitions.** Returning back to the main thread, we'd like to endow topological manifolds with smooth structures, i.e, with a rule for determining which functions are differentiable and how to differentiate them. Here's a naïve idea for what we'd like to do. We know that for every point in M there is a neighborhood U and a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$. We'd like to declare that $f : M \to \mathbb{R}$ is differentiable if $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is differentiable for every such pair (ϕ, U). This is roughly the right idea, but there's a significant problem: You typically can't differentiate homeomorphisms. (Indeed, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism it is "typically" differentiable *nowhere*.) If $\psi : V \to \psi(V)$ is another such homeomorphism and $U \cap V \neq \emptyset$,

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^1) : \mathbb{R}^n \to \mathbb{R}$$

need not be differentiable.

Here's our fix: we'll restrict our homeomorphisms to the class for which $\phi \circ \psi^{-1}$ is always differentiable.

We're typically interested in the C^∞ category, and I'll use "smooth" to mean this. Many of the results, on the other hand, apply in weaker or stronger settings and I encourage you to think about when they fail.

Notational conventions: now we'll typically use letters like x and y for these local homeomorphisms and we'll call them "coordinates". If U is an open subset of M for which $x : U \to x(U) \subset \mathbb{R}^n$ is a homeomorphism like this, we'll typically refer to (x, U) as a "chart". (The field is full of mapmaking analogies.)

Definition 15. If U, V are open subsets of M equipped with homeoemorphisms

$$\begin{aligned} x :&U \to x(U) \subset \mathbb{R}^n, \\ y :&V \to y(V) \subset \mathbb{R}^n, \end{aligned}$$

we say that (x, U) and (y, U) are C^∞-related if the "transition functions" are smooth, i.e., if

$$\begin{aligned} x \circ y^{-1} &: y(U \cap V) \to x(U \cap V), \\ y \circ x^{-1} &: x(U \cap V) \to y(U \cap V) \end{aligned}$$

are both smooth.

Definition 16. A family of C^∞-related homeomorphisms whose domains cover M is an *atlas* for M. A member (x, U) of an atlas is called a chart or a coordinate system for U.

Intuitive picture: it provides a way of assigning coordinates $(x^1(p), x^2(p), \ldots, x^n(p))$ to the points $p \in U$ so that you can treat it like $\mathbb{R}^n$.

If there is another homeomorphism that is C^∞-compatible with all of the charts in an atlas $\mathcal{A}$, you can create a new larger atlas $\mathcal{A}'$ by including it. Perhaps unsurprisingly, Zorn's lemma gives the following result:

Lemma 17. *If $\mathcal{A}$ is an atlas on M, $\mathcal{A}$ is contained in a unique maximal atlas $\mathcal{A}'$ for M.*

We can then define a smooth (or C^k, etc) manifold:

Definition 18. A C^∞ manifold (also called a smooth manifold) is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a maximal atlas for M.

Since each atlas is contained in a unique maximal atlas, specifying the manifold requires only specifying some atlas.

Examples

(1) $(\mathbb{R}^n, \mathcal{U})$, where $\mathcal{U}$ is the maximal atlas containing $(\mathrm{Id}, \mathbb{R}^n)$.
(2) $(\mathbb{R}, \mathcal{V})$, where $\mathcal{V}$ is the maximal atlas containing the chart $(x \mapsto x^3, \mathbb{R})$. Note that this is different from the standard structure! (Why?)
(3) $\mathbb{S}^n$ equipped with the stereographic projection maps $P_{j,\pm}$ for $j = 1, \ldots, n+1$:

$$\begin{aligned} P_{j,\pm} &: U_{j,\pm} \to \mathbb{R}^n, \\ U_{j,\pm} &= \{v \in \mathbb{R}^{n+1} \mid |v| = 1, v \neq \pm e_j\} \\ P_{j,\pm}(x) &= \Pi_j \left(\pm e_j + \frac{1}{\pm 1 - x^j}(x \mp e_j) \right), \end{aligned}$$

where e_j is the j-th standard basis vector in $\mathbb{R}^{n+1}$ and Π_j denotes the projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ given by dropping the j-th coordinate. (Draw a picture.)
(4) Products of manifolds have charts given by product charts.

(5) Any open subset of a manifold is again a manifold (with charts given by restriction).

Definition 19. $f : M \to N$ is smooth (or C^k, etc) if for every chart (x, U) $(x(U) \subset \mathbb{R}^n)$ on M and (y, V) $(y(V) \subset \mathbb{R}^m)$on N, the composition $y \circ f \circ x^{-1}$ is smooth.

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[d, "y"] \\ x(U) \arrow[u, "x^{-1}"] \arrow[r] & y(V) \end{tikzcd}$$

A function $f : M \to N$ is a diffeomorphism if it is a smooth homeomorphism so that f^{-1} is also smooth.

(We note here that all local notions that are invariant under diffeomorphism can be expressed on manifolds. A big example that we will occasionally use, sometimes without mention, is the property of a subset $A \subset M$ having measure zero.)

Properties of differentiable functions:

(1) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the sense of smooth manifold maps if and only if it is differentiable in the standard sense.
(2) $f : M \to \mathbb{R}^k$ is differentiable if and only if each component $f^j : M \to \mathbb{R}$ is differentiable.
(3) $f : M \to N$ is differentiable if and only if each coordinate function $y^i \circ f : M \to \mathbb{R}$ is differentiable for coordinates on N.
(4) If (x, U) is a chart, then $x : U \to x(U)$ is a diffeomorphism.

For $f : \mathbb{R}^n \to \mathbb{R}$, let's use the following notation (at least for now) for the partial derivative at a in the j-th coordinate direction:

$$D_j f(a) = \lim_{h \to 0} \frac{f(a^1, \ldots, a^{j-1}, a^j + h, a^{j+1}, \ldots, a^n) - f(a)}{h}.$$

Now suppose $f : M \to \mathbb{R}$ and (x, U) is a chart on M. For $p \in U$, we define

$$\frac{\partial f}{\partial x^j}(p) = \frac{\partial f}{\partial x^j}|_p = D_j \left(f \circ x^{-1} \right) (x(p)).$$

Proposition 20. *If (x, U) and (y, V) are charts on M and $f : M \to \mathbb{R}$ is differentiable, then*

$$\frac{\partial f}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i}.$$

Proof. At a point p in V, we have

$$\frac{\partial f}{\partial y^i}(p) = D_i(f \circ y^{-1})(y(p)) = D_i \left((f \circ x^{-1}) \circ (x \circ y^{-1}) \right) (y(p)).$$

Using the chain rule and unwinding the meanings of the expressions finishes the proof. $\square$

We'll fill this space soon with some words about manifolds with boundary.

3.2. **Partitions of unity.** In your first homework, you constructed compactly supported smooth functions on $\mathbb{R}$ that were identically one on a neighborhood of the origin. By taking products, rescaling, and translating, given any two open cubes C_1 and C_2 with $C_1 \subset \overline{C_1} \subset C_2$, you can construct smooth functions on $\mathbb{R}^n$ that are identically one on C_1 and supported in C_2. Similarly, given any $0 < r_1 < r_2$, by using your one-variable function, you can use $|x - p|$ as an argument and construct a smooth function that is identically one on $B(p, r_1)$, bounded between 0 and 1, and supported in $B(p, r_2)$. We'll call such functions smooth bump functions.

Definition 21 (From Lee's book)**.** Suppose $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an arbitrary open cover of a topological space M. A *partition of unity subordinate to* $\mathcal{U}$ is an indexed family $\{\psi_\alpha\}_{\alpha\in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ so that

(a) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and $x \in M$,
(b) $\operatorname{supp} \psi_\alpha \subset U_\alpha$ for all $\alpha \in A$,
(c) The family of supports $\{\operatorname{supp} \psi_\alpha\}_{\alpha\in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many α.
(d) $\displaystyle\sum_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

On a smooth manifold M, a smooth partition of unity is one for which each function is smooth.

The existence of partitions of unity is essentially why we wanted the second-countability hypothesis in our definition of a manifold. To construct them, we first need to know that we can fill out open covers with sets that look like coordinate balls in $\mathbb{R}^n$.

There are various relaxations of the hypotheses of this useful lemma available, but we'll state it right now in the form that we'll use.

Lemma 22 (Adapted from Lee's Theorem 1.15)**.** *Given a smooth manifold M and an open cover $\mathcal{U}$ of M, there is a countable, locally finite open refinement of $\mathcal{U}$ consisting of open sets B_i diffeomorphic to coordinate balls in $\mathbb{R}^n$. Moreover, the (closed) cover $\{\overline{B_i}\}$ is also locally finite.*

Proof. We first observe that the collection $\mathcal{B}$ of all open sets in M diffeomorphic to coordinate balls in $\mathbb{R}^n$ is a basis[1] for the topology of M.

We let K_j, $j = 1, 2, \ldots$ be compact sets so that $K_i \subset K_{i+1}$ and $\cup_i K_i = M$. (The existence of such a family follows from the fact that M is Hausdorff and second-countable.) For each j, let $C_j = K_{j+1} \setminus \operatorname{Int} K_j$ and $V_j = \operatorname{Int} K_{j+2} \setminus K_{j-1}$ (where $K_j = \emptyset$ if $j < 1$), so that $C_j \subset V_j$, C_J is compact, and V_j is open. For each $x \in C_j$, there is some $U_x \in \mathcal{U}$ containing x, and, since $\mathcal{B}$ is a basis, some $B_x \in \mathcal{B}$ with $x \in B_x \subset U_x \cap V_j$.

The collection of all such sets as x ranges over C_j is an open cover of C_j and so has a finite subcover. The union of all of these finite subcovers as $j = 1, 2, \ldots$ is a countable open subcover of M that refines $\mathcal{U}$. Because the subcover of C_j consists of sets contained in V_j, and $V_j \cap V_i = \emptyset$ if $|j - j'| > 2$, the resulting cover is locally finite. Because the closures are also in V_j, the cover by the closures remains locally finite. ∎

[1]Recall that this means that, for any point p and open set U containing p, there is some $B \in \mathcal{B}$ with $p \in B \subset U$.

Theorem 23 (Lee, Theorem 2.23)**.** *Suppose M is a smooth manifold with or without boundary and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of M. There exists a smooth partition of unity subordinate to $\mathcal{U}$.*

Proof. We'll just prove the boundary-less case now; the case with boundary is similar. (EXERCISE!) By Lemma 22, $\mathcal{U}$ has a countable, locally finite refinement $\{B_i\}$ consisting of neighborhoods B_i diffeomorphic to coordinate balls in $\mathbb{R}^n$ so that $\{\overline{B_i}\}$ is also locally finite.

For each i, we can find some slightly larger B_i' diffeomorphic to a coordinate ball in $\mathbb{R}^n$ so that $\overline{B_i} \subset B_i'$ as well as a coordinate map $x_i : B_i' \to \mathbb{R}^n$ so that $x_i(B_i) = B(0, r_i)$ and $x_i(B_i') = B(0, r_i')$ for some $0 < r_i < r_i'$. Let $H_i : \mathbb{R}^n \to \mathbb{R}$ denote a smooth function that is positive in $B(0, r_i)$ and zero elsewhere, and then define

$$f_i = \begin{cases} H_i \circ x_i & \text{on } B_i' \\ 0 & \text{on } M \setminus \overline{B_i} \end{cases}.$$

In the overlap $B_i' \setminus \overline{B_i}$, the two definitions both yield the zero function, so f_i is a well-defined smooth function on M.

Define $f : M \to \mathbb{R}$ by $f(x) = \sum_i f_i(x)$. As the cover $\overline{B_i}$ is locally fintie, this sum has only finitely many nonzero terms in a neighborhood of each point and therefore defines a smooth function. Each f_i is nonnegative everywhere and positive on B_i, and every point of M is in some B_i, so $f(x) > 0$ everywhere on M. The functions $g_i = f_i(x)/f(x)$ are thus smooth and so $0 \leq g_i(x) \leq 1$ and $\sum_i g_i(x) = 1$. Note that each g_i has support $\overline{B_i}$, which is contained in U_α for some α.

After grouping together terms and re-indexing (EXERCISE), we finish the proof. $\square$

4. Tangent and cotangent bundles

4.1. **The cotangent bundle of $\mathbb{R}^n$.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth (or, indeed, C^1 suffices), then