Chapter I
Background Material

I.1 Basic Notation

Let us collect some elementary notation. First of all, there are some basic sets which we will refer to:

$$\begin{aligned}
\mathbb{N} &:= \{\text{ positive integers, i.e., excluding } 0 \} \\
\mathbb{N}_0 &:= \{\text{ nonnegative integers, i.e., including } 0 \} \\
\mathbb{Z} &:= \{\text{ the integers } \} \\
\mathbb{R} &:= \{\text{ the real numbers } \} \\
\mathbb{C} &:= \{\text{ the complex numbers } \}
\end{aligned}$$

The symbols ":=" and "=:" always denote equalities which define the object on the side with the colon.

Given a mapping $f : A \longrightarrow B$ between any two sets A and B, we call A the *domain* and B the *range* of f. The set $f(A)$ is called the *image*. If $B' \subset B$, we write

$$f^{-1}(B') := \{\, a \in A \,|\, f(a) \in B' \,\}$$

and call this the *preimage* of B' under f. If $B' = \{b\}$ contains a single element, we will usually write $f^{-1}(b) := f^{-1}(\{b\})$. If $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ for some $n, m \in \mathbb{N}$, the preimage under f of a single point is also called a *level set* of f.

I.2 Differentiability of Maps

We briefly give the basic definitions for differentiability of maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and state (without proof) central results which will be of importance throughout the course.

Differentiability

Definition I.1 (Differentiability). *Let $U \in \mathbb{R}^n$ be open, $x \in U$ and $f : U \longrightarrow \mathbb{R}^m$. f is said to be* differentiable *at x if there is a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{h \to o} \frac{f(x+h) - f(x) - L(h)}{|h|} = 0\ .$$

The linear map L is called the differential *of f at x and denoted by $Df\big|_x$.*

Letting $n = m = 1$, this gives the usual definition of the derivative f' of a function $\mathbb{R} \longrightarrow \mathbb{R}$. Please note that in this case we have $Df\big|_x(h) = f'(x) \cdot h$, where $h \in \mathbb{R}$.

Remark I.2. A vertical bar with a subscript x will always mean "(evaluated) at the point x", for instance $Df\big|_x$ is the differential of f *at* x. The reason for this is that we need to distinguish between the point x where the differential is taken and any vector h to which the differential is applied. Whenever we deal with objects where this distinction is important, we will use this notation.

For the moment, consider a real-valued function $\phi : U \longrightarrow \mathbb{R}$. A concept closely related to the differential, yet slightly simpler, is that of directional derivatives:

Definition I.3 (Directional Derivative). *Let $U \subset \mathbb{R}^n$ be open, $x \in U$ and $\phi : U \longrightarrow \mathbb{R}$. If, for $h \in \mathbb{R}^n$, the limit*

$$\lim_{t\to 0} \frac{\phi(x+th) - \phi(x)}{t}$$

exists, we call it the directional derivative *of ϕ at x in direction h and denote it by $\partial_h\phi(x)$.*

The important special case of *partial derivatives* is obtained by choosing as direction one of the elements of the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. These are denoted by

$$\frac{\partial\phi}{\partial x_j}(x) := \partial_{e_j}\phi(x) \ . \tag{I.1}$$

(Here, and in the following, we write $x = (x_1, \ldots, x_n)$ with respect to this standard basis.) Collecting the partial derivatives in a vector, we define the *gradient* of ϕ at x by

$$\nabla\phi(x) := \left(\frac{\partial\phi}{\partial x_1}(x), \ldots, \frac{\partial\phi}{\partial x_n}(x)\right) \ . \tag{I.2}$$

Now let us return to maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ again. Since any such map $f : U \longrightarrow \mathbb{R}^m$ can be written as a vector of *component functions*

$$f = (f_1, \ldots, f_m) \ , \tag{I.3}$$

where $f_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ gives the j-th component of $f(x)$ (with respect to the standard basis of $\mathbb{R}^m$), we may collect all partial derivatives of all component functions in an $m \times n$–matrix:

$$\mathcal{J}f\big|_x := \begin{pmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix} \tag{I.4}$$

This is the *Jacobian*[1] of f at x. More generally, we denote the matrix consisting of the partial derivatives of the component functions $f_{j_1}, \ldots, f_{j_l}$ with respect to directions $e_{i_1}, \ldots, e_{i_k}$ by

$$\frac{\partial(f_{j_1}, \ldots, f_{j_l})}{\partial(x_{i_1}, \ldots, x_{i_l})}\Big|_x \ . \tag{I.5}$$

[1] Carl Gustav Jacob Jacobi, ∗ 1804 in Potsdam, † 1851 in Berlin

(I.5) is the submatrix of $\mathcal{J}f\big|_x$ obtained by crossing out the rows and columns corresponding to component functions and coordinates we did not use.

The partial derivatives (and their collection in terms of the Jacobian) are an important tool to check whether a given map is differentiable and, if so, to compute the differential.

Theorem I.4. *Let $U \subset \mathbb{R}^n$ be open, $x \in U$ and $f : U \longrightarrow \mathbb{R}^m$.*

i) If, for all $1 \le i \le n$ and $1 \le j \le m$, the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist and are continuous on an open neighbourhood of x, then f is differentiable at x.

ii) If f is differentiable at x, then

$$Df\big|_x(h) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x_0) \cdot h_i \right) e_j = \sum_{j=1}^{m} \langle \nabla f_j , h \rangle e_j \,,$$

holds for all $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{R}^m$.

The sums in the second item of Theorem I.4 can of course be interpreted as a matrix multiplication and the corresponding matrix is exactly the Jacobian,

$$Df\big|_x(h) = \mathcal{J}f\big|_x \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} ,$$

where, as before, $h = (h_1, \ldots, h_n)$ refers to the standard basis. Hence the Jacobian is the representation of the differential with respect to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Remark I.5 (The Differential vs. the Jacobian). It is important to distinguish between the linear map $Df\big|_x$ and the matrix $\mathcal{J}f\big|_x$, the latter is just one representative for the former. For maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, we might have a canonical representative (the one with respect to the standard bases), but when studying differentials of functions between surfaces or submanifolds, there is no single distinguished representative.

Directional derivatives of a function $\phi : U \longrightarrow \mathbb{R}$ have the same domain and range as ϕ, $\partial_h \phi : U \longrightarrow \mathbb{R}$. This allows us to iterate taking directional derivatives and obtain higher order (directional or partial) derivatives:

Definition I.6 (Higher Order Derivatives). *Let $U \subset \mathbb{R}^n$ be open and $\phi : U \longrightarrow \mathbb{R}$. If, for $h_1, \ldots, h_l \in \mathbb{R}^n$ and $i_1, \ldots, i_l \in \mathbb{N}_0$, the iterated directional derivatives*

$$\left(\partial_{h_1}\right)^{i_1} \left(\cdots \left(\partial_{h_k}\right)^{i_l} \phi \right) ,$$

exist, this is called a directional derivative of ϕ of order $k = i_1 + \cdots + i_l$. *If the h_j are taken to be elements of the standard basis of $\mathbb{R}^n$, it is called a* partial derivative of order k.

We will also use the usual shorthand for higher order partial derivatives, i.e.,

$$\frac{\partial^k \phi}{\partial x_{j_1}^{i_1} \cdots \partial x_{j_l}^{i_l}} := \left(\partial_{e_{j_1}}\right)^{i_1} \left(\cdots \left(\partial_{e_{j_k}}\right)^{i_l} \phi \right).$$

Definition I.7 (The Classes C^k). *Let $U \subset \mathbb{R}^n$ be open and $f = (f_1, \dots, f_m) : U \longrightarrow \mathbb{R}^m$. If all partial derivatives of orders up to k of all component functions f_j exist and are continuous on U, we say that f is k-*times continuously differentiable *on U and write $f \in C^k(U; \mathbb{R}^m)$. For $k = \infty$, we call f* smooth *and write $f \in C^\infty(U; \mathbb{R}^m)$. (For $m = 1$, we shall simply write $f \in C^k(U)$.)*

We end this collection of definitions with some more vocabulary for specific types of maps. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and $f : U \longrightarrow V$. We say that f is

i) an *immersion*, if f is differentiable on U and $Df\big|_x$ is injective for all $x \in U$,

ii) a *submersion*, if f is differentiable on U and $Df\big|_x$ is surjective for all $x \in U$,

iii) an *embedding*, if f is a homeomorphism onto $f(U)$ and an immersion,

iv) a (*smooth*) *diffeomorphism*, if f is a homeomorphism onto V and both f and f^{-1} are differentiable (smooth) on U and V, respectively.

Please observe that the dimensions n and m may pose obstructions to the existence of such maps: If $n > m$, there cannot be any immersions (and consequently embeddings), if $m > n$, there cannot be any submersions, and diffeomorphisms can only exist for $n = m$.

Remark I.8 (On Smoothness). It is often a matter of taste what exactly the word *smooth* refers to. In this course, by a smooth function we understand a function which is arbitrarily often continuously differentiable; in other references, smoothness refers to a function being continuously differentiable *as often as one might need*.

More importantly, when doing differential geometry, one can choose to consider smooth objects only: smooth submanifolds and surfaces, smooth functions between them, smooth vector fields etc. Or, one might be interested in studying the same questions for objects which are only of class C^k for some finite k. Since this is an introductory course, we restrict ourselves to the study of smooth things even though most definitions and results carry over (almost) unchanged to the more general setting. Then again, we will need the smooth versions of the Inverse and Implicit Function Theorems, compare Theorems I.10 and I.12.

Important Theorems for Differentiable Maps

We close this review by stating (without proof) some important results: The first two concern the differentials of compositions and inverses hence can be seen as algebraic properties of the differential (as a map on spaces of differentiable functions).

Theorem I.9 (Chain Rule). *Let $U \subset \mathbb{R}^{n_1}$, $V \subset \mathbb{R}^{n_2}$ be open subsets and $f : U \longrightarrow \mathbb{R}^{n_2}$, $g : V \longrightarrow \mathbb{R}^{n_3}$. Let $x \in U$ and suppose that $f(U) \subset V$.*

Then, if f is differentiable at x and g is differentiable at $f(x)$, $g \circ f$ is differentiable at x and

$$D(g \circ f)\big|_x = Dg\big|_{f(x)} \circ Df\big|_x \quad \text{and} \quad \mathcal{J}(g \circ f)\big|_x = \mathcal{J}g\big|_{f(x)} \cdot \mathcal{J}f\big|_x \,.$$

Theorem I.10 (Inverse Function Theorem). *Let $U \subset \mathbb{R}^n$ be open, $x \in U$ and $f : U \longrightarrow \mathbb{R}^n$ be continuously differentiable (smooth).*

If $Df\big|_x$ is invertible, i.e., the Jacobian is nonsingular, then f is a local (smooth) diffeomorphism, i.e., there are neighbourhoods U' of x and V' of $f(x)$ such that $f : U' \longrightarrow V'$ is a (smooth) diffeomorphism. In particular, $f^{-1} : V' \longrightarrow U'$ is differentiable (smooth) and its differential at $y = f(x)$ given by

$$D(f^{-1})\big|_{f(x)} = Df\big|_x^{-1} \,.$$

The next result is of a more computational value, though it is important for quite a few proofs as well. Depending on the source, it is named after Schwarz[2] or Young[3].

Theorem I.11 (Theorem of Schwarz-Young). *Let $U \subset \mathbb{R}^n$ be open, $x \in U$, $1 \le i, j \le n$ and $f : U \longrightarrow \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial x_j}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous on an open neighbourhood of x, then the same holds for $\frac{\partial^2 f}{\partial x_j \partial x_i}$ and we have*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \,.$$

The last result shows in which circumstances a set of equations can be solved by use of an implicit function. Here, an implicit function is an auxiliary function having the set of solutions to said equation as a level set, compare Section I.1.

Theorem I.12 (Implicit Function Theorem). *Let $U \subset \mathbb{R}^n$ be open and $f : U \longrightarrow \mathbb{R}^m$ be continuously differentiable (smooth). Suppose $x_0 \in \mathbb{R}^{n-m}$, $y_0 \in \mathbb{R}^m$ so that $(x_0, y_0) \in U$. Let $c = f(x_0, y_0)$.*

If the matrix $\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_m)}\big|_{(x_0,y_0)}$ is invertible, then there are open neighbourhoods $U' \subset \mathbb{R}^{n-m}$ of x_0 and $U'' \subset \mathbb{R}^m$ of y_0 and a continuously differentiable (smooth) map $g : U' \longrightarrow U''$ so that: For all $x \in U'$, $y \in U''$ we have

$$f(x,y) = c \qquad \Longleftrightarrow \qquad y = g(x) \,.$$

[2] Herrman Amandus Schwarz, ∗ 1843 in Hermsdorf, Silesia, † 1921 in Berlin
[3] Grace Chisholm Young, ∗ 1868 in Haslemere near London, † 1944 in England

In this case, the differential of g is given by

$$\frac{\partial(g_1,\dots,g_m)}{\partial(x'_1,\dots,x'_{n-m})}\Big|_{x_0} = -\left(\frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_m)}\Big|_{(x_0,y_0)}\right)^{-1}\frac{\partial(f_1,\dots,f_m)}{\partial(x'_1,\dots,x'_{n-m})}\Big|_{(x_0,y_0)} .$$

Example I.13 (For the Implicit Function Theorem). Let $n = 2$, $m = 1$ and consider the function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \quad , \qquad f(x,y) = x^2 + y^2 .$$

For $c = 1$, the set

$$f^{-1}(1) = \{\,(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\,\} = \mathbb{S}^1$$

is the unit circle. Letting x_0, $y_0 \in \mathbb{R}$, we see that

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = 2x_0 \quad , \qquad \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = 2y_0 .$$

These "matrices" are invertible if and only if $x_0 \neq 0$ or $y_0 \neq 0$, respectively. If $y_0 \neq 0$, for instance, the Implicit Function Theorem tells us that we may implicitly solve the equation $f(x,y) = 1$ near (x_0,y_0) for y, i.e., find a function $g : U' \longrightarrow U''$ satisfying

$$f(x,y) = 1 \quad \Longleftrightarrow \quad g(x) = y ,$$

near (x_0,y_0). Moreover, we obtain

$$\frac{\partial g}{\partial x}\Big|_{x_0} = -\Big(\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\Big)^{-1} \cdot \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = -(2y_0)^{-1} \cdot (2x_0) = -\frac{x_0}{y_0} .$$

It should be clear that, if for instance $y > 0$, g is given by $g(x) = \sqrt{1-x^2}$. This is also in line with the formula for the differential:

$$\frac{\partial}{\partial x}(1-x^2)^{\frac{1}{2}} = -\frac{x}{(1-x^2)^{\frac{1}{2}}} = -\frac{x}{g(x)} = -\frac{x}{y} .$$

But please be aware that the Implicit Function Theorem does not give a construction for g, but only ensures its existence!

I.3 Multivariate Integration

Let us now briefly sketch the theory of Lebesgue[4] integrable functions. Again, we start with the basic definitions and then give some important results.

Lebesgue integration

A *cuboid* in $\mathbb{R}^n$ is a cartesian product $Q = I_1 \times \cdots \times I_n$ of nonempty, bounded intervals $I_i \subset \mathbb{R}$. Its *n-dimensional volume* is defined as

$$\mathrm{vol}_n(Q) := |I_1| \cdots |I_n| ,$$

[4] Henri Lebesgue, ∗ 1875 in Beauvais, Oise, † 1941 in Paris

where $|I_i|$ denotes the length of I_i. A *step function* is a finite linear combination of characteristic functions of cuboids,

$$\varphi(x) := \sum_{i=1}^{k} c_i \, \chi_{Q_i}(x) \ ,$$

and it is easy to define a sensible notion of integral for step functions:

$$\int_{\mathbb{R}^n} \varphi(x)\, dx := \sum_{i=1}^{k} c_i \operatorname{vol}_n(Q_i) \tag{I.6}$$

Now, given any function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we say that

$$\Phi = \sum_{i=1}^{\infty} c_i \, \chi_{Q_i}$$

is an *envelope* for f, if the Q_i are open cuboids, the coefficients are nonnegative, $c_i \geq 0$, and if $|f(x)| \leq \Phi(x)$ for all $x \in \mathbb{R}^n$. The *content* $I(\Phi)$ of an envelope is defined as the (infinite!) sum of the volumes of its cuboids, weighted by the appropriate coefficients:

$$I(\Phi) := \sum_{i=1}^{\infty} c_i \operatorname{vol}_n(Q_i)$$

The notion of content allows us to define the L^1*–seminorm*, which in turn leads to the space of (Lebesgue) integrable functions:

Definition I.14 (Lebesgue integral). *Let* $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$.

i) The L^1–seminorm *of* f *is defined as*

$$\|f\|_1 := \inf \left\{ \, I(\Phi) \;\middle|\; \Phi \textit{ is an envelope for } f \, \right\} .$$

ii) We say that f *is* (Lebesque) integrable, *if there is a sequence* $\varphi_1, \varphi_2, \ldots$ *of step functions such that* $\|f - \varphi_k\|_1 \longrightarrow 0$ *as* $k \to \infty$. *In this case, the* (Lebesgue) integral *of* f *is defined as*

$$\int_{\mathbb{R}^n} f(x)\, dx := \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi_k(x)\, dx \ . \tag{I.7}$$

iii) If $U \subset \mathbb{R}^n$ *is open and* $f : U \longrightarrow \overline{\mathbb{R}}$, *we say that* f *is* integrable on U *if its extension* $\widetilde{f}$ *by* 0 *to* $\mathbb{R}^n$ *is integrable and in this case write*

$$\int_{U} f(x)\, dx := \int_{\mathbb{R}^n} \widetilde{f}(x)\, dx \ .$$

We denote the space of integrable functions by $\mathcal{L}^1(U)$.

The space $\mathcal{L}^1(U)$ is an infinite dimensional real vector space and the integral, as a mapping

$$\int \, : \mathcal{L}^1 \longrightarrow \overline{\mathbb{R}} \ , \tag{I.8}$$

is well-defined, linear and continuous with $\left|\int f\right| \leq \|f\|_1$. (Please note that it is not a *bounded* functional since we do not require the L^1–seminorm to be finite.) Moreover, we have the following two important properties:

$$f \leq g \implies \int f \leq \int g$$
$$f \geq 0 \implies \int f = \|f\|_1$$

But, what is probably most important: If a function $\mathbb{R} \longrightarrow \mathbb{R}$ is Riemann integrable, then it is Lebesgue integrable as well and the two integrals are the same. Thus, the actual computation of Lebesgue integrals of functions of one real variable is done in the usual way.

Important Theorems on Integrable Functions

With respect to the computation of integrals of multivariate functions, there are three important results: The Theorem of Fubini[5] and the closely related Theorem of Tonelli[6], and the theorem on integration by substitution.

Theorem I.15 (Fubini). *Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, $x = (u,v) \in \mathbb{R}^k \times \mathbb{R}^l$ where $k + l = n$. Then:*

i) The function $f(\,\cdot\,, v) : \mathbb{R}^k \longrightarrow \overline{\mathbb{R}}$, $u \longmapsto f(u,v)$ is integrable on $\mathbb{R}^k$ for almost every choice of $v \in \mathbb{R}^l$.

ii) The function $F : \mathbb{R}^l \longrightarrow \overline{\mathbb{R}}$, $v \longmapsto \int_{\mathbb{R}^k} f(u,v)\,du$ is integrable on $\mathbb{R}^l$.

iii) We have

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^l} \Big(\int_{\mathbb{R}^k} f(u,v)\,du \Big)\,dv\,.$$

The Theorem of Fubini can be used to reduce a multivariate integral to multiple one dimensional integrals, with the sole prerequisite that f be integrable on $\mathbb{R}^n$. What is often easier to decide is that f is *locally integrable*, i.e., for each point $x \in \mathbb{R}^n$, there is a neighbourhood $U \subset \mathbb{R}^n$ of x so that $f\big|_U \in \mathcal{L}^1(U)$. (In particular, any continuous function is locally integrable!) Then, the Theorem of Tonelli tells us, that we can use a *Fubini-type calculation* to decide whether f is integrable on all of $\mathbb{R}^n$:

Theorem I.16 (Tonelli). *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be locally integrable. Then, f is integrable over $\mathbb{R}^n$ if and only if there is a permutation π of $\{1, \ldots, n\}$ so that the iterated integral*

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \big|f(x_1, \ldots, x_n)\big|\,dx_{\pi(1)} \cdots dx_{\pi(n)}$$

[5] Guido Fubini, ∗ 1879 in Venezia, † 1943 in New York City
[6] Leonida Tonelli, ∗ 1885 in Gallipolli, † 1946 in Pisa

exists. In this case, we have

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1,\ldots,x_n)\,dx_{\widetilde{\pi}(1)} \cdots dx_{\widetilde{\pi}(n)}\ ,$$

for any permutation $\widetilde{\pi}$ *of* $\{1,\ldots,n\}$.

The important ingredient to the calculation of integrals is the well-known theorem on integration by substitution. This is a straight forward generalisation of the one dimensional case.

Theorem I.17 (Integration by Substitution). *Let* $U,V \subset \mathbb{R}^n$ *be open,* $T : U \longrightarrow V$ *be a diffeomorphism and* $f : V \longrightarrow \overline{\mathbb{R}}$. *Then,* f *is integrable over* V *if and only if* $(f\circ T)|\det DT|$ *is integrable over* U *and, if so, we have*

$$\int_{T(U)} f(y)\,dy = \int_U (f\circ T)(x)\big|\det DT\big|_x\big|\,dx\ .$$

Example I.18 (Volume of Three-Balls). Let us, as an example, compute the volume of a ball

$$B_R = \{\, x\in\mathbb{R}^3 \,|\, |x| \le R \,\}$$

in $\mathbb{R}^3$. This is, by definition, given by integrating the function 1 over B_R:

$$\mathrm{vol}_3(B_R) := \int_{B_R} 1\,dx$$

Extending the function 1 to the cuboid $[-R,R]^3$, and then using Theorem I.16, we see that 1 is integrable on $[-R,R]^3$, but then it is integrable on $B_R \subset [-R,R]^3$ as well. Without going into the details of null sets (i.e. sets of measure 0), we note that

$$\int_{B_R} f(x)\,dx = \int_{\mathring{B}_R\setminus P} f(x)\,dx\ ,$$

where P is the half-plane given by $P = \{\, x\in\mathbb{R}^3 \,|\, x_1 = 0,\ x_2 \ge 0 \,\}$. Now polar coordinates give a diffeomorphism of $\mathring{B}_R \setminus P$ with a simple, open cuboid:

$$T : Q = (0,R)\times(0,\pi)\times(0,2\pi) \longrightarrow \mathring{B}_R\setminus P \quad , \qquad (r,\vartheta,\varphi) \longmapsto \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

Noting that $|\det DT| = |r^2\sin\vartheta| = r^2\sin\vartheta$, substitution yields

$$\mathrm{vol}_3(B) = \int_B 1\,dx = \int_{T(Q)} 1\,dx = \int_Q 1\cdot r^2\sin\vartheta\, d(r,\vartheta,\varphi)\ .$$

Then, applying Fubini's Theorem twice, we get

$$\mathrm{vol}_3(B_R) = \int_{(0,R)} \int_{(0,\pi)} \int_{(0,2\pi)} r^2\sin\vartheta\, d\varphi d\vartheta dr\ ,$$

and this evaluates to the well-known formula

$$\mathrm{vol}_3(B_R) = \frac{4}{3}\pi R^3 .$$

I.4 Ordinary Differential Equations

Let us not delve to deeply into the rich theory of ordinary differential equations but only state a differentiable version of the Theorem of Picard[7]-Lindelöf[8] (which is sometimes also referred to as the Cauchy[9]-Lipschitz[10] Theorem) and a result on the dependence on initial data.

An (*ordinary*) *differential equation* of order k is an equation of the form

$$x^{(k)}(t) = F\big(t; x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\big) , \tag{I.9}$$

where $x^{(j)}$ denotes the j-th derivative of x with respect to t and $F : I \times \Omega \longrightarrow \mathbb{R}^n$ is continuous. Here, $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^k$ are supposed to be open and connected sets. A (vector-valued) C^k–function $x : I \longrightarrow \mathbb{R}^n$ is a (*global*) *solution* to (I.9), if we have

$$\big(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\big) \in I \times \Omega \tag{I.10}$$

for all $t \in I$, which is to say that x may be used as an input for the equation and

$$x^{(k)}(t) = F\big(t; x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\big) , \tag{I.11}$$

for all $t \in I$, which is to say that the vector $\big(t, x(t), \ldots, x^{(k)}(t)\big)$ indeed solves the equation.

An *initial value problem* for (I.9) consists of finding a solution to (I.9) which additionally satisfies

$$\big(x, x^{(1)}, \ldots, x^{(k-1)}\big)(t_0) := \big(x(t_0), x^{(1)}(t_0), \ldots, x^{(k-1)}(t_0)\big) = y_0 , \tag{I.12}$$

where $(t_0, y_0) \in I \times \Omega$ is given.

Important Theorems for ODEs

There are two results which are of interest to us: Most importantly the existence and uniqueness of solutions to initial value problems and then a result which shows that if we perturb the initial value problem slightly (in a differentiable manner), then the solution is perturbed only slightly (and differentiably) as well:

[7] Émile Picard, ∗ 1856 in Paris, † 1941 ibidem
[8] Ernst Leonard Lindelöf, ∗ 1870 in Helsinki, † 1946 ibidem
[9] Augustin-Louis Cauchy, ∗ 1789 in Paris, † 1857 in Sceaux
[10] Rudolf Lipschitz, ∗ 1832 in Königsberg (Kaliningrad), † 1903 in Bonn

Theorem I.19 (of Picard-Lindelöf). *Let $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^k$ be open and connected sets and suppose that $F : I \times \Omega \longrightarrow \mathbb{R}^n$ is continuous and uniformly bounded for all $(t, y) \in I \times \Omega$ and differentiable in $y \in \Omega$. Let $(t_0, y_0) \in I \times \Omega$. Then, there is a solution $x : I \longrightarrow \mathbb{R}^n$ to the initial value problem*

$$x^{(k)}(t) = F\big(t; x(t), \dots, x(k-1)(t)\big) \quad , \qquad \big(x, \dots, x^{(k-1)}\big)(t_0) = y_0$$

and this solution is uniquely determined by F and (t_0, y_0).

Proposition I.20 (Dependence on Parameters). *In the situation of Theorem I.19, assume that the data F and (t_0, y_0) depend differentiably on a parameter λ. Then, the solution given by Theorem I.19 depends differentiably on λ as well.*

References and Further Reading

The following list just aims at giving a first place to look for further details, additional or different explanations or more references. There are far too many good text books on analysis, general topology and differential geometry to name them all. The book being closest to what we are doing here is [dC1], although the notation might differ to some extent, followed by [Pr]. Despite the age of its first edition, [Sp1] is still a very good textbook on differential geometry, though the amount of five volumes is a bit intimidating.

On Differential and Riemannian Geometry:

[dC1] Manfredo P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1976.

[dC2] Manfredo P. do Carmo, *Riemannian Geometry*, 2nd. pr., Birkhäuser, Basel, 1992.

[Pr] Andrew Pressley, *Elementary Differential Geometry*, Springer, London, 2001.

[Sp1] Michael Spivak, *A Comprehensive Introduction to Differential Geometry : Volumes I–V*, 3rd ed., Publish or Perish, Houston, TX, 1999.

On Analysis and General Topology:

[Ar] V. I. Arnold, *Ordinary Differential Equations*, 2nd pr., Springer, Berlin – Heidelberg, 2006.

[PM] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, Springer, London, 1991.

[Ru] Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, NY, 1976.

[Sp2] Michael Spivak, *Calculus on Manifolds*, Perseus Books Publishing, New York, NY, 1965.