Frequency, Severity, and Aggregate Losses with Coverage Modifications

Chapter 8

Stat 346 - Short-term Actuarial Math

Introduction

- In the previous weeks, we have assumed that the loss amount X is also the claim amount paid.
- However, there are policy modifications for which the insurer may only be liable for a portion of this loss amount.
- For example:
  - deductibles
  - policy limits
  - coinsurance
- For purposes of notation, we shall denote the modified loss amount to be Y and will be referred to as the claim amount paid by the insurer. More precisely, Y^L denotes the per-loss variable while Y^P denotes the per-payment variable.
- We shall refer to X as the loss amount random variable.

Relationships between deductibles and limits

- For any loss random variable X, it can be shown that

$$X = (X - d)_+ + (X \wedge d).$$

- The interpretation to this is intuitively clear: for a policy with a deductible d, losses below d are not covered and therefore can be covered by another policy with a limit of d.
- The expectations are therefore equal:

$$\mathsf{E}(X) = \mathsf{E}\left[(X - d)_+\right] + \mathsf{E}(X \wedge d).$$

- In the case where there is a deductible d, the insurer's savings can therefore be thought of as $S = (X \wedge d)$.
- The expected savings (due to the deductible) expressed as a percentage of the loss (no deductible at all) is called the Loss Elimination Ratio:

$$\mathsf{LER} = \frac{\mathsf{E}(X \wedge d)}{\mathsf{E}(X)}.$$

Review Example

An insurance company offers two types of policies: Type Q and Type R. You are given:

- Type Q has no deductible, but has a policy limit of \$3,000.
- Type R has no policy limit, but has a deductible of d.
- Losses follow a Pareto(α, θ) distribution with $\alpha = 3$ and $\theta = 2,000$.

Calculate d so that both policies have the same expected claim amount per loss.

Coinsurance

- For policies with coinsurance, claim amount is proportional to the loss amount by a coinsurance factor.
- Coinsurance factor is denoted by α, where $0 < \alpha < 1$ so that the claim payment random variable is

$$Y = \alpha X.$$

- Its density can be expressed as

$$f_Y(y) = \frac{1}{\alpha} f_X\left(\frac{y}{\alpha}\right).$$

- Its expected value is clearly $\mathsf{E}(Y) = \alpha \mathsf{E}(X)$.

Combining policy modifications

- It is possible to combine deductibles and/or policy limits together with coinsurance factors.
- We shall adopt the convention that the coinsurance factor is applied after the application of any deductible or limit.
- So for example, consider the case where we have a policy deductible d, a policy limit u and a coinsurance factor α. The amount paid per loss random variable is given by

$$Y^L = \alpha \left[(X \wedge u) - (X \wedge d) \right].$$

An illustration

A Health Maintenance Organization (HMO) currently pays full cost of any emergency room care to its clients.

You are given that the cost of an emergency room care has an Exponential distribution with mean 1,000.

The company is evaluating the possible savings of imposing a deductible of $200 per emergency room visit, to be paid by the client.

1. Calculate the resulting loss elimination ratio due to a deductible of $200. Interpret this ratio.
2. Suppose the HMO decides to impose a per loss deductible of $200 per emergency room visit, along with a policy limit of $5,000 and a coinsurance factor of 80%. For every visit to the emergency room, calculate the expected claim amount per loss event and the expected claim amount per payment event made by the HMO.

The effect of inflation

- Assume that there is a gap between the time of loss and the time the payments are made, and the insurer is obligated to cover inflation losses.
- For modeling purposes, this means that instead of a loss of X, now the loss to consider is $(1+r)X$, assuming an inflation rate of r during the period.
- Note that for policies with deductibles, the deductible is subtracted after the inflation has been taken into account. Therefore, the effect on the payment amount is greater than $100r\%$ for two reasons:
  1. There are now more claims exceeding the deductible.
  2. The deductible amount is usually not increased for inflation, so that those claims exceeding the deductible will increase by more than the rate of inflation, even before the inflation.

Calculating the expected claim per loss

- Assume a deductible of d, a coinsurance of α, a policy limit of u, and the inflation rate of r.
- The expected claim per loss will be

$$\mathsf{E}(Y^L) = \mathsf{E}[\alpha((1+r)X \wedge u)] - \mathsf{E}[\alpha((1+r)X \wedge d)].$$

- This can be re-expressed as

$$\mathsf{E}(Y^L) = \alpha(1+r)\left[\mathsf{E}\left(X \wedge \frac{u}{1+r}\right) - \mathsf{E}\left(X \wedge \frac{d}{1+r}\right)\right].$$

Calculating the expected claim per payment

- We can express the probability of the loss (after inflation) exceeding the deductible, and therefore producing a claim, as follows:

$$\Pr((1+r)X > d) = \Pr\left(X > \frac{d}{1+r}\right) = 1 - F_X\left(\frac{d}{1+r}\right).$$

- The expected claim amount per payment is therefore

$$\mathsf{E}(Y^P) = \frac{\mathsf{E}(Y^L)}{1 - F_X\left(\frac{d}{1+r}\right)}.$$

Illustration

To illustrate, consider the case of the HMO in the previous example with a \$200 deductible, \$5,000 policy limit and an 80% coinsurance factor.

Now, assume a 5% uniform inflation. Calculate the new expected claim amounts per loss and per payment.

Individual vs Aggregate Deductibles

- In the context of insurance, especially when dealing with aggregate losses, understanding the distinction between individual and aggregate deductibles is crucial.
- **Individual Deductibles (Per Loss Deductible):**
  - Applied to each individual loss.
  - The insurer pays only the part of each claim that exceeds the deductible.
  - Common in standard insurance policies.
- **Aggregate Deductibles (Stop Loss Deductible):**
  - Applied to the total amount of losses over a specified period.
  - The insurer starts paying for losses only when the total aggregate losses exceed the deductible amount.
  - Used in policies with high frequency of small losses.
- These deductibles serve different purposes:
  - *Individual Deductibles* manage the frequency of small claims.
  - *Aggregate Deductibles* manage the total cost of claims over a period.

Aggregate Loss with Poisson Claims and Exponential Severity

- Number of claims (N) follows a Poisson distribution.
- Severity of each claim (X_i) follows an Exponential distribution with $\theta = 750$.
- Individual deductible: $d = 250$ per claim.

Calculate the expected aggregate loss $E(S)$ both with and without the deductible.

Net Stop-Loss Premium

- The net stop-loss premium is an important concept in actuarial science, used to determine the premium for reinsurance contracts.
- It is defined as the expected payment under a stop-loss reinsurance with a deductible d, and can be calculated as:

$$E(S-d)_+ = \int_d^\infty [1 - F_S(x)]dx = \int_d^\infty (x-d) f_S(x)dx,$$

where S is the aggregate loss, $F_S(x)$ is the cumulative distribution function of S, and $f_S(x)$ is the probability density function of S.

- For discrete distributions, this becomes:

$$E(S-d)_+ = \sum_{x=d+1}^{\infty} (x-d) f_S(x).$$

Example: Setup

- Consider a portfolio where the number of claims N and the amount of each claim X are both discrete random variables.
- Let N have the following distribution:
  - $\Pr(N = 0) = 0.6$
  - $\Pr(N = 1) = 0.4$
- Let X have the following distribution:
  - $\Pr(X = 100) = 0.5$
  - $\Pr(X = 200) = 0.5$
- Our goal is to calculate $\Pr(S = x)$ for all possible values of x and determine the net stop-loss premium for a deductible d.

Probabilities for Aggregate Loss S

- Based on the distributions of N and X, the aggregate loss S can take a few discrete values.
- The table below shows the probabilities $\Pr(S = x)$:

Aggregate Loss S	Probability $\Pr(S = x)$
0	0.6
100	0.2
200	0.2

- These probabilities are calculated based on the combined probabilities of N and X.

Net Stop-Loss Premium Calculation

- Given a deductible $d = 50$, we calculate the net stop-loss premium.
- The net stop-loss premium formula is:

$$E(S-d)_+ = \sum_{x=d}^{\infty} (x-d) f_S(x).$$

- Using the probabilities from the previous table:

$$\begin{aligned} E(S-50)_+ &= (100-50) \times 0.2 + (200-50) \times 0.2 \\ &= 50 \times 0.2 + 150 \times 0.2 \\ &= 10 + 30 \\ &= 40. \end{aligned}$$

- The net stop-loss premium with a deductible of 50 is 40.

Rule 1: Conditional Expectation Formula

- When the probability of the aggregate loss S falling between two values a and b is zero (i.e., $\Pr(a < S < b) = 0$), the net stop-loss premium can be calculated using a linear interpolation:

$$E[(S-d)_+] = \frac{b-d}{b-a}E[(S-a)_+] + \frac{d-a}{b-a}E[(S-b)_+]$$

- This relationship is particularly useful for discrete loss distributions where S can only take specific values.

Example for Rule 1

- Consider an aggregate loss S that can only take values 100, 500, or 1000.
- Let's calculate the net stop-loss premium for a deductible $d = 300$, using $a = 100$ and $b = 500$.
- Assume $E[(S - 100)_+] = 400$ and $E[(S - 500)_+] = 750$.
- Applying the formula:

$$E[(S - 300)_+] = \frac{500 - 300}{500 - 100} \times 400 + \frac{300 - 100}{500 - 100} \times 750$$

- This yields $E[(S - 300)_+] = 575$.

Rule 2: Discrete Loss Increments

- This recursive formula applies when the aggregate loss S takes values in multiples of a fixed amount h.
- The formula is given by:

$$E[(S-(j+1)h)_+] = E[(S-jh)_+] - h(1-F_S(jh))$$

- It's used to calculate the expected excess loss over each successive deductible level jh.

Example for Rule 2

- Suppose S takes values in multiples of $h = 100$ and $E[(S-100)_+] = 200$.
- Let's calculate $E[(S-200)_+]$ and $E[(S-300)_+]$.
- Assume $\Pr(S \geq 100) = 0.8$ and $\Pr(S \geq 200) = 0.6$.
- Applying the formula for $E[(S-200)_+]$:

$$E[(S-200)_+] = 200 - 100 \times 0.8 = 120$$

- Similarly, for $E[(S-300)_+]$:

$$E[(S-300)_+] = E[(S-200)_+] - 100 \times 0.6 = 60$$

Practice Problem Using Recursive Formulas

- You are given an insurance portfolio where the aggregate loss S can only take values in multiples of 100. The probabilities for S are as follows:
  - $\Pr(S = 100) = 0.1$
  - $\Pr(S = 200) = 0.15$
  - $\Pr(S = 300) = 0.25$

    $\vdots$
- You know that $E[(S - 100)_+] = 460$. Using the second recursive formula, calculate the net stop-loss premium for a deductible of \$300.