Option Pricing Models

Binomial Tree

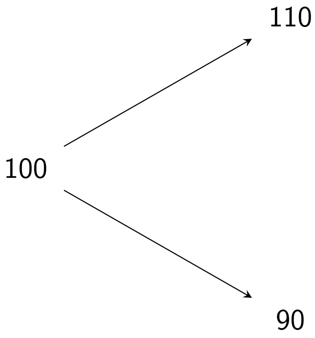
What if we make a simple assumption about the future price of a stock:

- it can only possibly be one of two points at some point in the future, one that is greater than the current value and one that is less than the current value.

Suppose the stock at time 0 is 100. Then at time 1 it can only be either 90 or 110.

Binomial Tree

This is drawn in a tree diagram going from left to right



Binomial Option Pricing

We will use the following notation:

S	initial value of the stock
u	Proportional increase of the stock if it increases
d	Proportional decrease of the stock if it decreases
Su	Value of the stock at t if it increases
Sd	Value of the stock at t if it decreases
C_u or P_u	Payoff of the option at t if the stock increases
C_d or P_d	Payoff of the option at t if the stock decreases
p	Probability of a movement up
$1 - p$	Probability of a movement down

Binomial Option Pricing

For example, suppose $S = 100$ and it can increase to 110 or decrease to 90 in one year and we are pricing a call option with a strike price of 105.

- $Su = 110$ and $u = 1.10$,
- $Sd = 90$ and $d = 0.90$
- The payoff when the stock increases is $C_u = 5$,
- The payoff when the stock decreases is $C_d = 0$.

Practice

Suppose a stock worth 40 can increase to 45 or decrease to 38 in 4 months. A certain call option has a strike price of 41. Determine the values of u, d, C_u, and C_d.

Practice

Suppose a stock worth 40 can increase to 45 or decrease to 38 in 4 months. A certain call option has a strike price of 41. Determine the values of u, d, C_u, and C_d.

- $u = 45/40 = 1.125$
- $d = 38/40 = 0.95$
- $C_u = 4$
- $C_d = 0$

Risk Neutral Pricing

Let p be the probability that the stock increases to Su, where $0 < p < 1$. Then $1 - p$ is the probability that the stock decreases to Sd. Then how much is a call option worth?

- With probability p the call option is worth C_u.
- With probability $1 - p$ the call option is worth C_d.

The actuarially fair payoff of the call option is then the expected value of the payoff, $pC_u + (1 - p)C_d$. The price of the call option is the present value of this

$$C = e^{-rt}[pC_u + (1 - p)C_d]$$

Risk Neutral Pricing

This method of pricing options is called risk neutral pricing and turns out to be very valuable. If p is not given it can be calculated using

$$p = \frac{e^{rt} - d}{u - d}$$

There is some theory behind this that this is the only p where you cannot mathematically make money with no risk (arbitrage).

Risk Neutral Pricing

Going back to our example, we had a stock worth 40 with $u = 1.125$ and $d = 0.95$, $r = 0.03$ and $\delta = 0.06$. A 41 strike call had possible payoffs of $C_u = 4$ and $C_d = 0$.

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.03)(1/3)} - .95}{1.125 - .95} = 0.343$$

The the call price is

$$C = e^{-rt}[pC_u + (1-p)C_d] = e^{-.03(1/3)}[0.343(4) + 0.657(0)] = 1.36$$

This is the same (with some rounding error) as the previous approach.

Risk Neutral Pricing

Again, the same formula hold for puts, but using possible payouts for puts:

$$P = e^{-rt}[pP_u + (1-p)P_d]$$

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.
- Now we find the risk neutral probability

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.
- Now we find the risk neutral probability

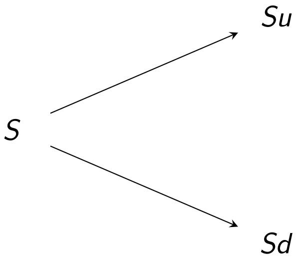
$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

- Now we calculate the put premium:

$$P = e^{-rt}[pP_u + (1-p)P_d] = e^{-.05(1/2)}[.486(0) + .514(4.80)] = 1.98$$

Binomial Tree

Returning to binomial trees. The following is written in terms of stock prices.



Binomial Tree

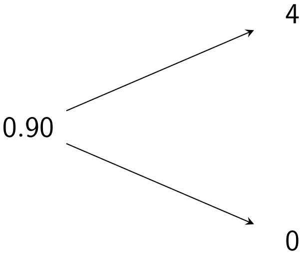
But trees can also be written in terms of the value of the call option.

$$\begin{tikzcd} & C_u \\ C \arrow[ur] \arrow[dr] & \\ & C_d \end{tikzcd}$$

C is the call premium and C_u and C_d are the possible call payoffs.

Binomial Tree

In the example where the stock price is 40 and it could increase to 45 or decrease to 38, the 41 strike option has an option tree of



Two Period Binomial Tree

Consider a **two period binomial tree**.

$$
\begin{tikzcd}
 & & Su^2 \\
 & Su \arrow[ur] \arrow[dr] & \\
\$S \arrow[ur] \arrow[dr] & & Sud \\
 & Sd \arrow[ur] \arrow[dr] & \\
 & & Sd^2
\end{tikzcd}
$$

Two Period Binomial Tree

The idea here is that there is a step size, h, which in a two period binomial tree is always $h = t/2$. Then in one step size, the new stock value can be either Su or Sd.

- If the stock at time h is Su, then at time $2h = t$, the stock price can move up to Su^2 or down to Sud
- If the stock at time h is Sd, then at time $2h = t$, the stock price can move up to Sdu or down to Sd^2

Because of the way we set this up, $Sud = Sdu$, so the tree is a **recombining** tree because at time h the nodes combine when the paths meet up.

Two Period Binomial Tree

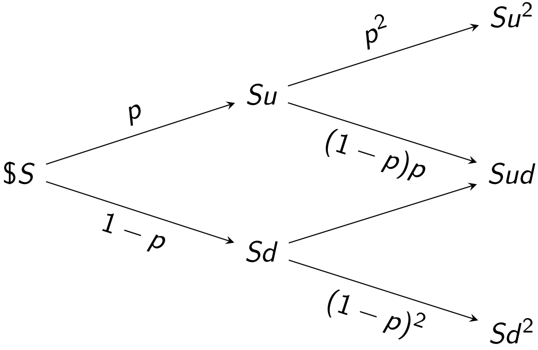
We will start with the easy way of pricing a two period binomial tree first. Recall the formula for p was $p = \frac{e^{rt}-d}{u-d}$. For a multi-period tree, the risk neutral probability is

$$p = \frac{e^{rh} - d}{u - d}$$

The only difference is that the the t is replaced with an h.

Two Period Binomial Tree

Using this p we can find the probability of ending up in certain places.



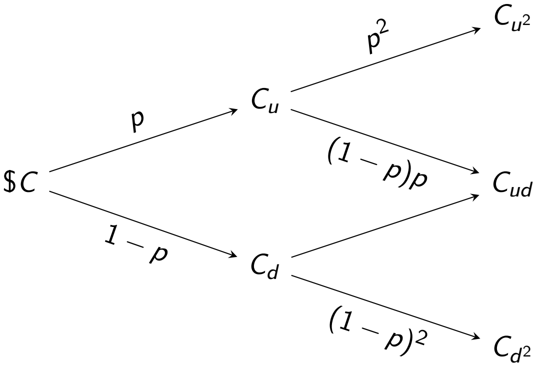
Two Period Binomial Tree

We know that

- There is a p^2 probability of ending up at Su^2
- a $2p(1-p)$ probability of ending up at Sud
- a $(1-p)^2$ probability of ending up at Sd^2

Two Period Binomial Tree

Now consider the option-based binomial tree with two periods



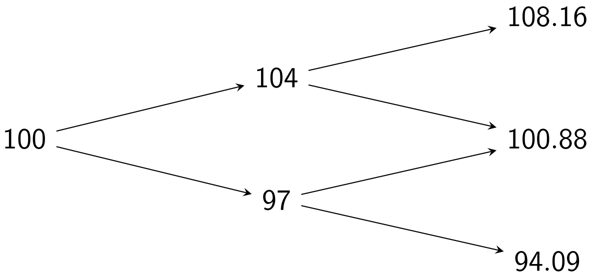
Two Period Binomial Tree

We know that

- C_{u^2} is the payoff of the option if the stock price at $2h$ is Su^2 $(max(0, Su^2 - K))$
- C_{ud} is the payoff of the option if the stock price at $2h$ is Sud $(max(0, Sud - K))$
- C_{d^2} is the payoff of the option if the stock price at $2h$ is Sd^2 $(max(0, Sd^2 - K))$
- C_u is the value of the call option position at time h is the stock price is S_u.
- C_d is the value of the call option position at time h is the stock price is S_d.
- C is the option premium

Two Period Binomial Tree

C_{u^2}, C_{ud} and C_{d^2} are easily determine using the payoff of the call options at the particular stock prices. For example, suppose a stock costs \$100 amd $u = 1.04$ and $d = 0.97$. The stock tree would be



Two Period Binomial Tree

Suppose a certain call option had a strike price of 98.

- If the stock price is 108.16, then the payoff is $108.16 - 98 = 10.16$
- If the stock price is 100.88, then the payoff is $100.88 - 98 = 2.88$
- If the stock price is 94.09, then the payoff is 0, because the option is not exercised.

Two Period Binomial Tree

- That means there is a p^2 probability of a payoff of 10.16,
- a $2p(1-p)$ probability of a payoff of 2.88
- and a $(1-p)^2$ probability of a payoff of 0.

The risk neutral price is then
$C = e^{-rt}[p^2(10.18) + 2p(1-p)(2.88) + (1-p)^2(0)]$

Two Period Binomial Tree

In general, the premium for a call option using a two period binomial tree is

$$C = e^{-rt}[p^2 C_{u^2} + 2p(1-p)C_{ud} + (1-p)^2 C_{d^2}]$$

The premium for a put option using a two period binomial tree is

$$P = e^{-rt}[p^2 P_{u^2} + 2p(1-p)P_{ud} + (1-p)^2 P_{d^2}]$$

Example

The put option payoffs are $P_{u^2} = 0$, $P_{ud} = 0.417$, and $P_{d^2} = 2.953$.

We must now find p.

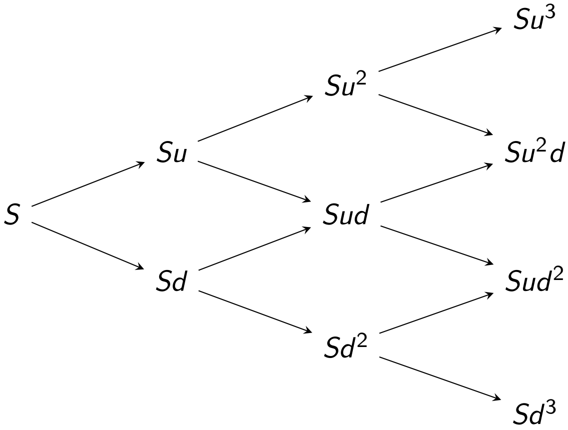
$$p = \frac{e^{rh} - d}{u - d} = \frac{e^{(.04)(1/12)} - .975}{1.027 - 0.975} = 0.545$$

The put option premium is

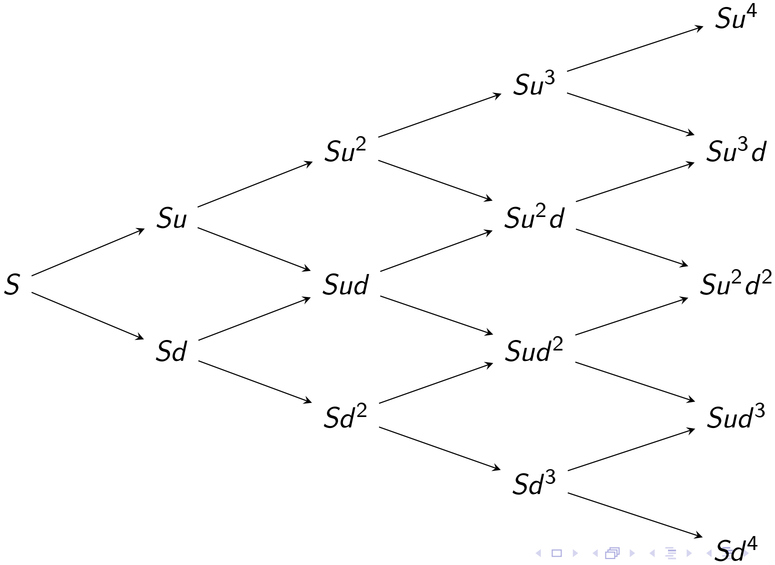
$$P = e^{-.04(1/6)}[(0.545)^2 0 + 2(.545)(1-.545)(0.417) + (1-0.545)^2 2.953]$$

$$= 0.81$$

Multi-Period Binomial Tree



Multi-Period Binomial Tree



Multi-Period Binomial Tree

The probability of getting to a stock price $Su^j d^k$, which is the probability of j up movements and k down movements, is

$$Pr(S_t = Su^j d^k) = \left(\begin{array}{c} j+k \\ k \end{array} \right) p^j (1-p)^k$$

Multi-Period Binomial Tree

This means that the price of a call option using a m-period binomial tree is

$$C = e^{-rt} \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p^{m-k} (1-p)^k C_{u^{m-k} d^k}$$

Example

Suppose a stock costs \$50 and the binomial tree has $h = .25$, $t = 1$, $u = 1.05$ and $d = 0.95$, and $p = 0.60$. What is the price of the 1 year at the money call option. Assume $r = 0$.

- $Su^4 = 60.775$ so $C_{u^4} = 10.775$
- $Su^3d = 54.987$ so $C_{u^3d} = 4.987$
- $Su^2d^2 = 49.75$ so $C_{u^2d^2} = 0$ as do all the other payoffs, (C_ud^3 and C_d^4)

So the call premium is

$$C = (.6^4)(10.775) + 4(.6^3)(.4)4.987 + 0 = 3.12$$

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.
The binomial tree for the stock price is

$$
\begin{tikzcd}
 & & & 169.54 \\
 & & 159.06 \arrow[ur] \arrow[dr] & \\
 & 149.23 \arrow[ur] \arrow[dr] & & 151.23 \\
140 \arrow[ur] \arrow[dr] & & 141.88 \arrow[ur] \arrow[dr] & \\
 & 133.11 \arrow[ur] \arrow[dr] & & 134.89 \\
 & & 126.55 \arrow[ur] \arrow[dr] & \\
 & & & 120.32
\end{tikzcd}
$$

Practice

$C_{u^3} = 24.54,\ C_{u^2d} = 6.23,\ C_{ud^2} = 0.$

The call premium is

$$C = e^{-.06/2}[(0.4857)^3(24.54) + 3(.4857)^2(.5143)(6.23)] = 4.93$$

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

$Sd^4 = 45.285$ and $Sud^3 = 47.43$, and all the rest are higher than 46, so the payoff is 0 except for Sd^4.

$$P = e^{-.03/3}[(1 - .494)^4(.715)] = .046$$

Binomial Tree Limit

Consider a 100-strike option with $S = 100$, $r = .06, \delta = .06, \sigma = .1$ and $t = 1$. What happens as the number of binomial periods increases?

Num of Periods	Premium
1	4.70
2	3.32
3	4.07
4	3.53
5	3.94
6	3.60

Binomial Tree Limit

Continued ...

Num of Periods	Premium
7	3.89
8	3.64
9	3.86
10	3.66
11	3.84
12	3.67
⋮	⋮
50	3.73
100	3.75
1000	3.755

Binomial Tree Limit

Eventually it settles near a specific value. As fun as it would be to do a 1000 period binomial tree, there is an easier way to find out what that limit is.

A set of equations gives us the limit. They are called the **Black-Scholes** equations.

Black-Scholes

The Black-Scholes equations:

$$
\begin{aligned}
d_1 &= \frac{\log(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T} \\
C &= SN(d_1) - Ke^{-rT}N(d_2) \\
P &= Ke^{-rT}N(-d_2) - SN(-d_1)
\end{aligned}
$$

where $N(\cdot)$ is the normal CDF function, found from a table.

Black-Scholes

Consider a 100-strike option with $S = 100$, $r = .06$, $\sigma = .1$ and $T = 1$. Assume dividend rate $\delta = 0$. What is the Black-Scholes price?

Black-Scholes

Consider a 100-strike option with $S = 100$, $r = .06$, $\sigma = .1$ and $T = 1$. Assume dividend rate $\delta = 0$. What is the Black-Scholes price?

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \\ &= \frac{\log(100/100) + (.06 + .5 \times .1^2) \times 1}{.1 \times \sqrt{1}} \\ &= 0.65 \\ d_2 &= d_1 - \sigma\sqrt{T} \\ &= 0.65 - 0.1 \times \sqrt{1} \\ &= 0.55 \end{aligned}$$

Black-Scholes

Consider a 100-strike option with $S = 100$, $r = .06$, $\sigma = .1$ and $T = 1$. What is the Black-Scholes price?

$$\begin{aligned} d_1 &= 0.65 \\ d_2 &= 0.55 \end{aligned}$$

The call option price is:

$$\begin{aligned} C &= 100N(d_1) - 100e^{-0.06}N(d_2) \\ &= 100N(0.65) - 100e^{-0.06}N(0.55) \\ &= 7.459322 \end{aligned}$$

Black-Scholes Practice

The current price of a stock is \$40, the risk free rate is $r = .03$, and the volatility of the stock is $\sigma = .1$. Using Black-Scholes, what is the price of a call option that expires in 9 months to purchase the stock at a strike price of 39?

$$
\begin{aligned}
d_1 &= 0.595 \\
d_2 &= 0.509
\end{aligned}
$$

Black-Scholes Practice

The current price of a stock is \$40, the risk free rate is $r = .03$, and the volatility of the stock is $\sigma = .1$. Using Black-Scholes, what is the price of a call option that expires in 9 months to purchase the stock at a strike price of 39?

$$\begin{aligned} d_1 &= 0.595 \\ d_2 &= 0.509 \end{aligned}$$

The call option price is:

$$\begin{aligned} C &= 40N(d_1) - 39e^{-0.03 \cdot 0.75}N(d_2) \\ &= 2.483579 \end{aligned}$$

Black-Scholes Practice for Put Option

Using the same parameters, what is the price of a *put* option?

Black-Scholes Practice for Put Option

Using the same parameters, what is the price of a *put* option? The put option price is:

$$\begin{aligned} P &= 39e^{-0.03 \cdot 0.75} N(-d_2) - 40N(-d_1) \\ &= 0.6158774 \end{aligned}$$

Put-Call Parity

Definition: Put-Call Parity establishes a relationship between the prices of a European call option and a European put option with the same strike price and expiration date.

Formula:

$$C - P = S - Ke^{-rT}$$

where:

- C = price of the call option
- P = price of the put option
- S = current stock price
- K = strike price
- r = risk-free interest rate
- T = time to maturity

Put-Call Parity Example

Given:

- Stock price $(S) = \$100$
- Strike price $(K) = \$100$
- Risk-free rate $(r) = 5\%$
- Time to maturity $(T) = 1$ year
- Call option price $(C) = \$10$

Find: Price of the put option (P)
Calculation:

$$P = C - S + Ke^{-rT} = 10 - 100 + 100e^{-0.05\times 1}$$

$$P = 10 - 100 + 95.12 = \$5.12$$

Put-Call Parity Practice

Practice Problem: Calculate the price of the put option given:

- Stock price $(S) = \$150$
- Strike price $(K) = \$155$
- Risk-free rate $(r) = 3\%$
- Time to maturity $(T) = 6$ months
- Call option price $(C) = \$8$

Delta Hedging

Definition: Delta hedging is an options strategy that aims to reduce, or hedge, the directional risk associated with price movements in the underlying asset by adjusting the position in the underlying asset and its options.

Delta (Δ):

$$\Delta = \frac{\partial C}{\partial S}$$

- For calls, Δ ranges from 0 to 1.
- For puts, Δ ranges from -1 to 0.

Delta Hedging for Calls
Strategy and Calculations

Black-Scholes Formula for Δ (Call Option):

$$\Delta_C = N(d_1)$$

where $d_1 = \frac{\log(S/K)+(r+0.5\sigma^2)T}{\sigma\sqrt{T}}$

Strategy:

- Short $\Delta_C \times$ Number of Options shares of the underlying stock.
- Shorting shares means selling shares you do not currently own, expecting to buy them back at a lower price.

Purpose: This strategy ensures that gains in the option's value due to increases in the underlying stock's price are offset by losses in the shorted stock position, leading to a less volatile overall investment position.

Delta Hedging for Puts

Strategy and Calculations

Black-Scholes Formula for Δ (Put Option):

$$\Delta_P = -N(-d_1) = N(d_1) - 1$$

where $d_1 = \frac{\log(S/K)+(r+0.5\sigma^2)T}{\sigma\sqrt{T}}$

Strategy:

- Buy $|\Delta_P| \times$ Number of Options shares of the underlying stock.
- Buying shares (longing) involves purchasing shares with the expectation that their value will increase.

Purpose: Longing shares in the context of put options hedging ensures that losses due to a decrease in the stock price (which increases the value of the put option) are offset by gains in the longed stock position, stabilizing the overall investment value.

Understanding Shorting and Longing in Hedging

Shorting Shares:

- Investors sell shares they do not own by borrowing them.
- The goal is to buy back the shares at a lower price and return them to the lender, pocketing the difference as profit.

Longing Shares:

- Investors purchase shares outright with the belief that the share price will increase.
- Profits are made when the shares are sold at a higher price than they were bought.

Role in Hedging: Both strategies are used to counterbalance the directional risk associated with holding options, aiming to neutralize the financial impact of significant price swings in the underlying asset.

Delta Hedging Example: Call

Given:

- Stock price $(S) = \$100$
- Delta of call option $(\Delta_C) = 0.6$
- Number of options $= 100$

Objective: Construct a delta-neutral portfolio
Action: Short 60 shares of the stock (since $100 \times 0.6 = 60$)

Delta Hedging Example: Put

Given:

- Stock price $(S) = \$100$
- Delta of put option $(\Delta_P) = -0.4$
- Number of options $= 100$

Objective: Construct a delta-neutral portfolio
Action: Buy 40 shares of the stock (since $100 \times -0.4 = -40$, and we negate the negative sign by buying)

Delta Hedging Practice

Practice Problems:

1. With a delta of 0.5 for a call option, how many shares should be shorted for a delta-neutral position if you own 150 options?
2. With a delta of -0.3 for a put option, how many shares should be bought for a delta-neutral position if you own 200 options?