Binomial Option Pricing Models

Options

An option is a contract that gives the owner of the option the right but not the obligation to buy or sell an asset if they wish. This can be thought of as a contract where the transaction takes place only if it is profitable for the owner of the contract.
Ownership of an option costs money upfront, called a premium.

Options

Option terminology:

- Premium - The price to buy an option (not the asset)
- Strike Price - The agreed upon price that the asset is bought or sold for.
- Exercise - If it is profitable for the owner of the option, they will make the transaction happen. This is called exercising the option.
- Expiration date - the date after which the the option may no longer be exercised.
- European option - This type of option can only be exercised at the expiration date
- American Option - This type of option can be exercised at any point prior to the expiration date
- Spot price - The price of the underlying asset at any given time

We will mainly deal with European options. If I don't specify, I mean European option.

Options

A call option is the right to buy an asset at expiration if the price is right
A put option is the right to sell an asset if the price is right

Options

Suppose the car company purchased an option to buy steel at \$400/ton at expiration.

- If the spot price of steel at expiration was more than \$400/ton, the car company would exercise the option and pay for the steel
- If the spot price of steel at expiration was less than \$400/ton, the car company would NOT exercise the option and make a payoff of 0

The payoff of a call option is $\max(0, \text{Spot Price} - \text{Strike Price})$

Options

"'r,fig.width = 4.5, fig.height = 3.5, echo=FALSE payoff ¡-
function(x) 0 + (x - 40000)(x ¿ 40000)
curve(payoff,from=30000,to=50000,main='Payoff Function for a
Call Option',xlab='Spot
Price',ylab='Payoff',col='blue',cex.main=.75,cex.axis=.75,cex.lab=.75,lw
= 2) abline(h = 0) points(40000,0,pch=4,col='red',cex=2) "'

Options

Suppose that same car company purchased a put option to sell a fleet of cars in 1 year for \$800,000.
- If the price of the fleet of cars at expiration is more than 800,000, the car company will NOT exercise the put option and have a payoff of 0. - If the price of the fleet of cars at expiration is less than 800,000, the car company will exercise the put option sell the fleet of cars.
The payoff of a put option is $\max(0, \text{Strike Price} - \text{Spot Price})$

Options

‘‘‘r,fig.width = 4.5, fig.height = 3.5, echo=FALSE payoff ¡-
function(x) 0 + (800000-x)(x ¡ 800000)
curve(payoff,from=700000,to=900000,main='Payoff Function for
a Put Option',xlab='Spot
Price',ylab='Payoff',col='blue',cex.main=.75,cex.axis=.75,cex.lab=.75,lw
= 2) abline(h = 0) points(800000,0,pch=4,col='red',cex=2) ‘‘‘

Options

Options will cost a premium at time 0. While payoff doesn't consider the premium, the profit will consider the premium. For both call and put options
(Profit = Payoff − Future Value of Premium)
The logic here is that the money spent on the premium is either borrowed and accrues interest or can't be invested and the loss must be accumulated forward.

Options

For this class, the interest rate is almost always continuous. So if P is the premium and the expiration time is t and the continuous interest rate is r, then the future value of the premium is (Pe^{rt} The time, t, is often measured in years. Having a premium doesn't change whether or not you exercise.

Options

Plotting profit as a function of spot price at expiration for an option must account for the premium.
‘‘‘r,fig.width = 4.5, fig.height = 3.5, echo=FALSE payoff ¡- function(x) 0 + (x - 40000)(x ¿ 40000) - 1000 curve(payoff,from=30000,to=50000,main='Profit Function for a Call Option',xlab='Spot Price',ylab='Profit',col='blue',cex.main=.75,cex.axis=.75,cex.lab=.75,lwd = 2) abline(h = 0) points(40000,0,pch=4,col='red',cex=2) ‘‘‘

Options

"'r,fig.width = 4.5, fig.height = 3.5, echo=FALSE payoff ¡- function(x) 0 + (800000-x)(x ¡ 800000) - 10000 curve(payoff,from=700000,to=900000,main='Profit Function for a Put Option',xlab='Spot Price',ylab='Profit',col='blue',cex.main=.75,cex.axis=.75,cex.lab=.75,lwd = 2) abline(h = 0) points(800000,0,pch=4,col='red',cex=2) "'

Practice

A call option is purchased for \$12 to buy a stock with a strike price of 1000, an expiration date of 6 months. The risk free continuous interest rate is $r = 4\%$. Determine the payoff and profit at expiration for the following spot prices at expiration:
- 950 - 1000 - 1010 - 1050

Practice

A put option is purchased for \$3 to sell a stock with a strike price of 40, with an expiration date of 4 months. The risk free continuous interest rate is $r = 3\%$. Determine the spot price at expiration that would make this position have 0 profit.

Moneyness

An option can be classified by its moneyness:
- In-the-money means that if the option was exercised immediately, it would make money - Out-of-the-money means that if the option was exercised immediately, it would lose money - At-the-money means that the option would break even if exercised immediately The current spot price (not spot price at expiration) is compared to the strike price to determine the moneyness

Binomial Tree

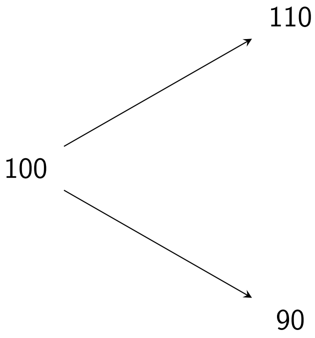
What if we make a simple assumption about the future price of a stock:

- it can only possibly be one of two points at some point in the future, one that is greater than the current value and one that is less than the current value.

Suppose the stock at time 0 is 100. Then at time 1 it can only be either 90 or 110.

Binomial Tree

This is drawn in a tree diagram going from left to right



Binomial Option Pricing

We will use the following notation:

S	initial value of the stock
u	Proportional increase of the stock if it increases
d	Proportional decrease of the stock if it decreases
Su	Value of the stock at t if it increases
Sd	Value of the stock at t if it decreases
C_u or P_u	Payoff of the option at t if the stock increases
C_d or P_d	Payoff of the option at t if the stock decreases
p	Probability of a movement up
$1 - p$	Probability of a movement down

Binomial Option Pricing

For example, suppose $S = 100$ and it can increase to 110 or decrease to 90 in one year and we are pricing a call option with a strike price of 105.

- $Su = 110$ and $u = 1.10$,
- $Sd = 90$ and $d = 0.90$
- The payoff when the stock increases is $C_u = 5$,
- The payoff when the stock decreases is $C_d = 0$.

Practice

Suppose a stock worth 40 can increase to 45 or decrease to 38 in 4 months. A certain call option has a strike price of 41. Determine the values of u, d, C_u, and C_d.

Practice

Suppose a stock worth 40 can increase to 45 or decrease to 38 in 4 months. A certain call option has a strike price of 41. Determine the values of u, d, C_u, and C_d.

- $u = 45/40 = 1.125$
- $d = 38/40 = 0.95$
- $C_u = 4$
- $C_d = 0$

Risk Neutral Pricing

Let p be the probability that the stock increases to Su, where $0 < p < 1$. Then $1 - p$ is the probability that the stock decreases to Sd. Then how much is a call option worth?

- With probability p the call option is worth C_u.
- With probability $1 - p$ the call option is worth C_d.

The actuarially fair payoff of the call option is then the expected value of the payoff, $pC_u + (1 - p)C_d$. The price of the call option is the present value of this

$$C = e^{-rt}[pC_u + (1 - p)C_d]$$

Risk Neutral Pricing

This method of pricing options is called risk neutral pricing and turns out to be very valuable. If p is not given it can be calculated using

$$p = \frac{e^{rt} - d}{u - d}$$

There is some theory behind this that this is the only p where you cannot mathematically make money with no risk (arbitrage).

Risk Neutral Pricing

Going back to our example, we had a stock worth 40 with $u = 1.125$ and $d = 0.95$, $r = 0.03$ and $\delta = 0.06$. A 41 strike call had possible payoffs of $C_u = 4$ and $C_d = 0$.

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.03)(1/3)} - .95}{1.125 - .95} = 0.343$$

The the call price is

$$C = e^{-rt}[pC_u + (1-p)C_d] = e^{-.03(1/3)}[0.343(4) + 0.657(0)] = 1.36$$

This is the same (with some rounding error) as the previous approach.

Risk Neutral Pricing

Again, the same formula hold for puts, but using possible payouts for puts:

$$P = e^{-rt}[pP_u + (1-p)P_d]$$

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

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- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.

Practice

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- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.
- Now we find the risk neutral probability

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

Practice

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$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

- Now we calculate the put premium:

$$P = e^{-rt}[pP_u + (1-p)P_d] = e^{-.05(1/2)}[.486(0) + .514(4.80)] = 1.98$$

Binomial Tree

Returning to binomial trees. The following is written in terms of stock prices.

$$\begin{array}{ccc} & & Su \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & Sd \end{array}$$

Binomial Tree

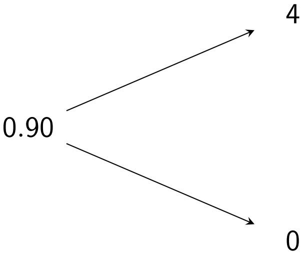
But trees can also be written in terms of the value of the call option.

$$\begin{tikzcd} & C_u \\ C \arrow[ur] \arrow[dr] & \\ & C_d \end{tikzcd}$$

C is the call premium and C_u and C_d are the possible call payoffs.

Binomial Tree

In the example where the stock price is 40 and it could increase to 45 or decrease to 38, the 41 strike option has an option tree of



Two Period Binomial Tree

Consider a **two period binomial tree**.

$$
\begin{tikzcd}
 & & Su^2 \\
 & Su \arrow[ur] \arrow[dr] & \\
\$S \arrow[ur] \arrow[dr] & & Sud \\
 & Sd \arrow[ur] \arrow[dr] & \\
 & & Sd^2
\end{tikzcd}
$$

Two Period Binomial Tree

The idea here is that there is a step size, h, which in a two period binomial tree is always $h = t/2$. Then in one step size, the new stock value can be either Su or Sd.

- If the stock at time h is Su, then at time $2h = t$, the stock price can move up to Su^2 or down to Sud
- If the stock at time h is Sd, then at time $2h = t$, the stock price can move up to Sdu or down to Sd^2

Because of the way we set this up, $Sud = Sdu$, so the tree is a **recombining** tree because at time h the nodes combine when the paths meet up.

Two Period Binomial Tree

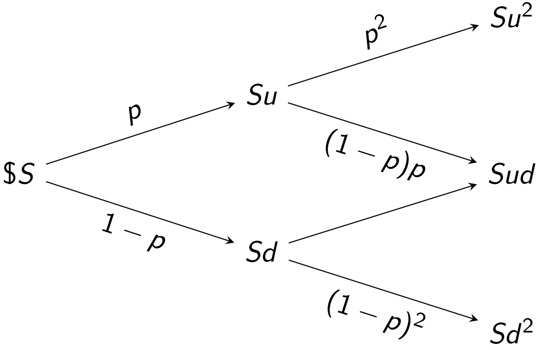
We will start with the easy way of pricing a two period binomial tree first. Recall the formula for p was $p = \frac{e^{rt}-d}{u-d}$. For a multi-period tree, the risk neutral probability is

$$p = \frac{e^{rh} - d}{u - d}$$

The only difference is that the the t is replaced with an h.

Two Period Binomial Tree

Using this p we can find the probability of ending up in certain places.



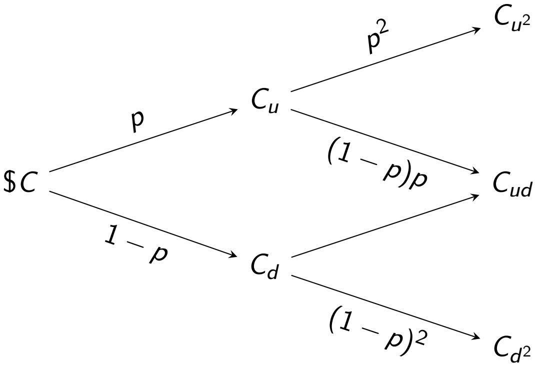
Two Period Binomial Tree

We know that

- There is a p^2 probability of ending up at Su^2
- a $2p(1-p)$ probability of ending up at Sud
- a $(1-p)^2$ probability of ending up at Sd^2

Two Period Binomial Tree

Now consider the option-based binomial tree with two periods



Two Period Binomial Tree

We know that

- C_{u^2} is the payoff of the option if the stock price at $2h$ is Su^2 $(max(0, Su^2 - K))$
- C_{ud} is the payoff of the option if the stock price at $2h$ is Sud $(max(0, Sud - K))$
- C_{d^2} is the payoff of the option if the stock price at $2h$ is Sd^2 $(max(0, Sd^2 - K))$
- C_u is the value of the call option position at time h is the stock price is S_u.
- C_d is the value of the call option position at time h is the stock price is S_d.
- C is the option premium

Two Period Binomial Tree

C_{u^2}, C_{ud} and C_{d^2} are easily determine using the payoff of the call options at the particular stock prices. For example, suppose a stock costs \$100 amd $u = 1.04$ and $d = 0.97$. The stock tree would be

$$\begin{tikzcd}
 & & 108.16 \\
 & 104 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 100.88 \\
 & 97 \arrow[ur] \arrow[dr] & \\
 & & 94.09
\end{tikzcd}$$

Two Period Binomial Tree

Suppose a certain call option had a strike price of 98.

- If the stock price is 108.16, then the payoff is $108.16 - 98 = 10.16$
- If the stock price is 100.88, then the payoff is $100.88 - 98 = 2.88$
- If the stock price is 94.09, then the payoff is 0, because the option is not exercised.

Two Period Binomial Tree

- That means there is a p^2 probability of a payoff of 10.16,
- a $2p(1-p)$ probability of a payoff of 2.88
- and a $(1-p)^2$ probability of a payoff of 0.

The risk neutral price is then
$C = e^{-rt}[p^2(10.18) + 2p(1-p)(2.88) + (1-p)^2(0)]$

Two Period Binomial Tree

In general, the premium for a call option using a two period binomial tree is

$$C = e^{-rt}[p^2 C_{u^2} + 2p(1-p)C_{ud} + (1-p)^2 C_{d^2}]$$

The premium for a put option using a two period binomial tree is

$$P = e^{-rt}[p^2 P_{u^2} + 2p(1-p)P_{ud} + (1-p)^2 P_{d^2}]$$

Example

The put option payoffs are $P_{u^2} = 0$, $P_{ud} = 0.417$, and $P_{d^2} = 2.953$.

We must now find p.

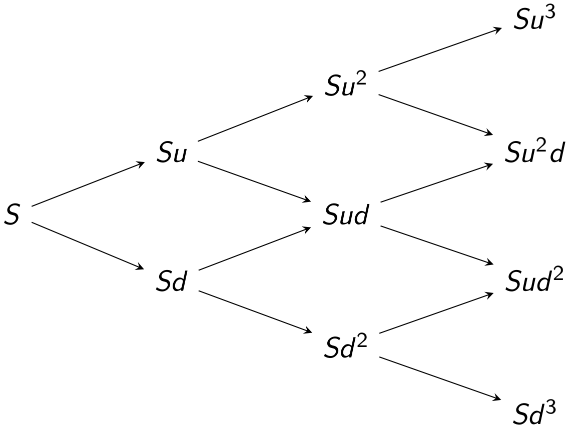
$$p = \frac{e^{rh} - d}{u - d} = \frac{e^{(.04)(1/12)} - .975}{1.027 - 0.975} = 0.545$$

The put option premium is

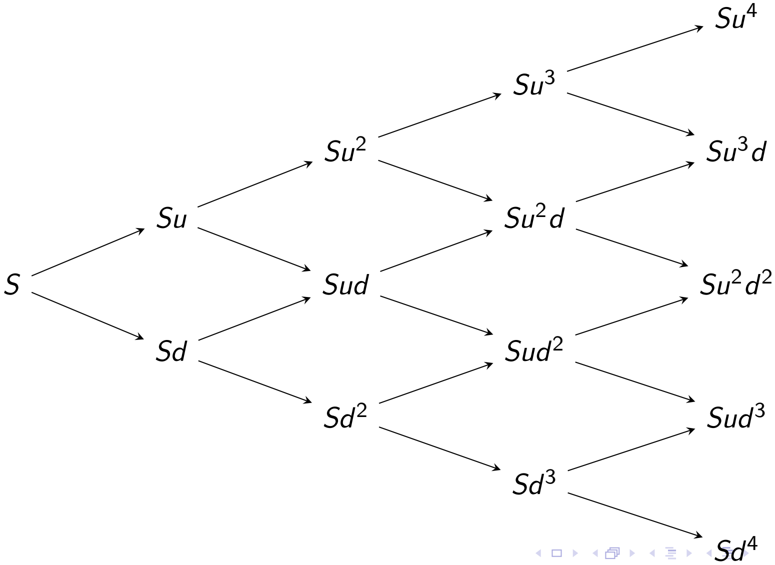
$$P = e^{-.04(1/6)}[(0.545)^2 0 + 2(.545)(1-.545)(0.417) + (1-0.545)^2 2.953]$$

$$= 0.81$$

Multi-Period Binomial Tree



Multi-Period Binomial Tree



Multi-Period Binomial Tree

The probability of getting to a stock price $Su^j d^k$, which is the probability of j up movements and k down movements, is

$$Pr(S_t = Su^j d^k) = \begin{pmatrix} j+k \\ k \end{pmatrix} p^j (1-p)^k$$

Multi-Period Binomial Tree

This means that the price of a call option using a *m*-period binomial tree is

$$C = e^{-rt} \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p^{m-k} (1-p)^k C_{u^{m-k} d^k}$$

Example

Suppose a stock costs \$50 and the binomial tree has $h = .25$, $t = 1$, $u = 1.05$ and $d = 0.95$, and $p = 0.60$. What is the price of the 1 year at the money call option. Assume $r = 0$.

- $Su^4 = 60.775$ so $C_{u^4} = 10.775$
- $Su^3d = 54.987$ so $C_{u^3d} = 4.987$
- $Su^2d^2 = 49.75$ so $C_{u^2d^2} = 0$ as do all the other payoffs, ($C_u d^3$ and C_d^4)

So the call premium is

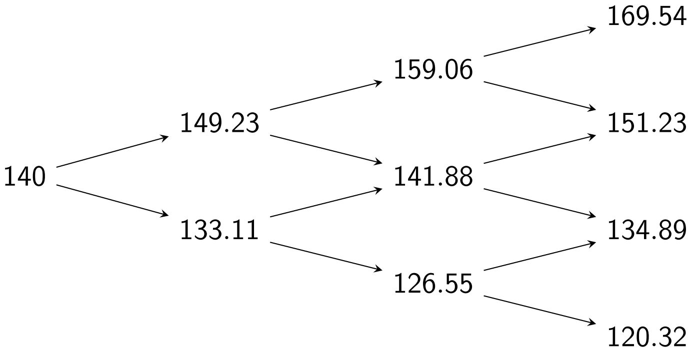
$$C = (.6^4)(10.775) + 4(.6^3)(.4)4.987 + 0 = 3.12$$

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.
The binomial tree for the stock price is



Practice

$C_{u^3} = 24.54, \; C_{u^2d} = 6.23, \; C_{ud^2} = 0.$

The call premium is

$$C = e^{-.06/2}[(0.4857)^3(24.54) + 3(.4857)^2(.5143)(6.23)] = 4.93$$

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

$Sd^4 = 45.285$ and $Sud^3 = 47.43$, and all the rest are higher than 46, so the payoff is 0 except for Sd^4.

$$P = e^{-.03/3}[(1 - .494)^4(.715)] = .046$$