Review of Mathematical Statistics

Chapter 10

Stat 477 - Loss Models

Concepts in mathematical statistics

- This is assumed knowledge (please review on your own!).
- Model estimation: parametric versus nonparametric
- Types of estimates: point estimates and interval estimates
- Properties of estimators: unbiasedness, consistency and efficiency
- Hypothesis testing: read section 12.4
- Chapter 12

Parametric versus nonparametric estimation

- In model estimation, we are interested in estimating the distribution of a random variable X.
- For the paramteric approach, the estimation involves determining usually a finite number of parameters:
  - Suppose X has CDF $F(x;\theta)$ and PDF $f(x;\theta)$ where θ is usually referred to as parameter (which could be a vector of parameters).
  - Parameter θ is usually unknown and must be estimated using observable data. The estimator is typically denoted by $\hat{\theta}$ and is a function of the observable sample.
  - Once θ is estimated, the distribution of X is completely specified.
- In the nonparametric approach, the distribution (either through $F(x)$ or $f(x)$) of X is estimated directly for all values of x without specifying a parametric form. The result is a nonparametric estimate of either of these functions.

Point and interval estimates

- Suppose $X_1, \ldots, X_n$ denote a random sample from the distribution of X, with observed values often denoted by $x_1, \ldots, x_n$. Any function $h(X_1, \ldots, X_n)$ is often referred to as a statistic.
- A statistic $h(X_1, \ldots, X_n)$ that is used to estimate the parameter θ is called a point estimator and is often denoted by $\hat{\theta}$. Replacing the random sample with the corresponding observed values $x_1, \ldots, x_n$ is often referred to as the point estimate.
- In contrast, an interval estimator of an unknown parameter is a random interval constructed from the sample data, which covers the true value of θ at a certain probability level.
- Suppose $\hat{\theta}_L$ and $\hat{\theta}_U$ be two statistics such that $\hat{\theta}_L < \hat{\theta}_U$. The interval $(\hat{\theta}_L, \hat{\theta}_U)$ is said to be a $100(1-\alpha)\%$ confidence interval of θ if

$$\Pr(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) \geq 1 - \alpha.$$

Normal distribution

Suppose $X_1, \ldots, X_n$ is a random sample from a N(μ, σ^2) distribution where the variance σ^2 is unknown. Then:

- The sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a point estimate of the population mean μ.
- A $100(1-\alpha)\%$ confidence interval of μ is given by the expression

$$\left(\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right),$$

  where $t_{n-1,\alpha/2}$ is the $100(1-\alpha/2)$-th quantile of a t-distribution with $n-1$ degrees of freedom and s is the sample standard deviation.

Properties of an estimator

- **Unbiasedness** An estimator $\hat{\theta}$ is said to be unbiased if and only if $\mathsf{E}(\hat{\theta}) = \theta$ for all θ. The bias is defined to be $\text{Bias}_{\hat{\theta}}(\theta) = \mathsf{E}(\hat{\theta}) - \theta$.
- **Asymptotically unbiased** Let $\hat{\theta}_n$ be an estimator of θ based on a sample size of n. Then, the estimator is said to asymptotically unbiased if
$$\lim_{n\to\infty} \mathsf{E}(\hat{\theta}) = \theta.$$
- **Consistency** An estimator $\hat{\theta}_n$ is said to be (weakly) consistent if for all $\varepsilon > 0$ and θ, we have
$$\lim_{n\to\infty} \Pr(|\hat{\theta}_n - \theta| > \varepsilon) = 0.$$
  - A sufficient (but not necessary) condition for weak consistency is that the estimator is asymptotically unbiased and $\lim_{n\to\infty} \text{Var}(\hat{\theta}_n) = 0$.

Illustrative example 1a

Suppose X has the Uniform distribution on $(0, \theta)$ and consider the estimator of θ

$$\hat{\theta}_n = \max(X_1, \ldots, X_n).$$

- Show that this estimator is asymptotically unbiased.
- Prove that it is also a consistent estimator of θ.

Mean-squared error

- **Mean-squared error** The mean-squared error (MSE) of an estimator is defined to be
$$\text{MSE}_{\hat{\theta}}(\theta) = \text{E}[(\hat{\theta} - \theta)^2].$$
It can be shown that
$$\text{MSE}_{\hat{\theta}}(\theta) = \text{Var}(\hat{\theta}) + [\text{Bias}_{\hat{\theta}}(\theta)]^2.$$
- **Efficient estimator** For any two unbiased estimators $\hat{\theta}_a$ and $\hat{\theta}_b$ of θ, $\hat{\theta}_a$ is said to be more efficient than $\hat{\theta}_b$ if $\text{Var}(\hat{\theta}_a) < \text{Var}(\hat{\theta}_b)$.
- **UMVUE** An estimator $\hat{\theta}$ is said to be uniformly minimum variance unbiased estimator (UMVUE) if
  - it is unbiased; and
  - for any true value of θ, there is no other unbiased estimator that has a smaller variance.

Illustrative example 1b

Suppose X has the Uniform distribution on $(0, \theta)$ and consider the estimator of θ

$$\hat{\theta}_n = \max(X_1, \ldots, X_n).$$

- Calculate the mean and variance of this estimator.
- Evaluate the MSE of this estimator.
- Now consider the two estimators: $\hat{\theta}_a = 2\bar{X}$ and $\hat{\theta}_b = \frac{n+1}{n} \max(X_1, \ldots, X_n)$. Compare the MSE of these two estimators.