Binomial Option Pricing Models

Binomial Tree

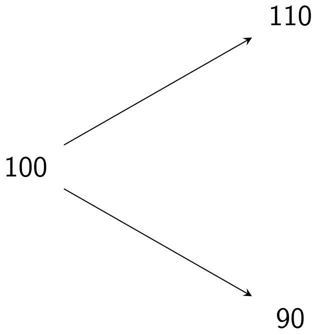
What if we make a simple assumption about the future price of a stock:

- it can only possibly be one of two points at some point in the future, one that is greater than the current value and one that is less than the current value.

Suppose the stock at time 0 is 100. Then at time 1 it can only be either 90 or 110.

Binomial Tree

This is drawn in a tree diagram going from left to right



Binomial Option Pricing

We will use the following notation:

S	initial value of the stock
u	Proportional increase of the stock if it increases
d	Proportional decrease of the stock if it decreases
Su	Value of the stock at t if it increases
Sd	Value of the stock at t if it decreases
C_u or P_u	Payoff of the option at t if the stock increases
C_d or P_d	Payoff of the option at t if the stock decreases
p	Probability of a movement up
$1 - p$	Probability of a movement down

Binomial Option Pricing

For example, suppose $S = 100$ and it can increase to 110 or decrease to 90 in one year and we are pricing a call option with a strike price of 105.

- $Su = 110$ and $u = 1.10$,
- $Sd = 90$ and $d = 0.90$
- The payoff when the stock increases is $C_u = 5$,
- The payoff when the stock decreases is $C_d = 0$.

Practice

Suppose a stock worth 40 can increase to 45 or decrease to 38 in 4 months. A certain call option has a strike price of 41. Determine the values of u, d, C_u, and C_d.

Practice

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- $u = 45/40 = 1.125$
- $d = 38/40 = 0.95$
- $C_u = 4$
- $C_d = 0$

Risk Neutral Pricing

Let p be the probability that the stock increases to Su, where $0 < p < 1$. Then $1 - p$ is the probability that the stock decreases to Sd. Then how much is a call option worth?

- With probability p the call option is worth C_u.
- With probability $1 - p$ the call option is worth C_d.

The actuarially fair payoff of the call option is then the expected value of the payoff, $pC_u + (1 - p)C_d$. The price of the call option is the present value of this

$$C = e^{-rt}[pC_u + (1 - p)C_d]$$

Risk Neutral Pricing

This method of pricing options is called risk neutral pricing and turns out to be very valuable. If p is not given it can be calculated using

$$p = \frac{e^{rt} - d}{u - d}$$

There is some theory behind this that this is the only p where you cannot mathematically make money with no risk (arbitrage).

Risk Neutral Pricing

Going back to our example, we had a stock worth 40 with $u = 1.125$ and $d = 0.95$, $r = 0.03$ and $\delta = 0.06$. A 41 strike call had possible payoffs of $C_u = 4$ and $C_d = 0$.

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.03)(1/3)} - .95}{1.125 - .95} = 0.343$$

The the call price is

$$C = e^{-rt}[pC_u + (1 - p)C_d] = e^{-.03(1/3)}[0.343(4) + 0.657(0)] = 1.36$$

This is the same (with some rounding error) as the previous approach.

Risk Neutral Pricing

Again, the same formula hold for puts, but using possible payouts for puts:

$$P = e^{-rt}[pP_u + (1-p)P_d]$$

Practice

A stock currently costs \$75. In 6 months it can grow to \$80.50 or fall to \$72. The interest rate is $r = 0.05$. A put option has a strike price of $K = 76.80$. Use risk neutral pricing to determine the price of the option.

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- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.

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- First we calculate all relevant values, $u = 1.073$, $d = 0.96$, $P_u = 0$ and $P_d = 4.80$.
- Now we find the risk neutral probability

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

Practice

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$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{(.05)(1/2)} - .96}{1.073 - .96} = 0.578$$

- Now we calculate the put premium:

$$P = e^{-rt}[pP_u + (1-p)P_d] = e^{-.05(1/2)}[.486(0) + .514(4.80)] = 1.98$$

Binomial Tree

Returning to binomial trees. The following is written in terms of stock prices.

$$\begin{tikzcd} & Su \\ S \arrow[ur] \arrow[dr] & \\ & Sd \end{tikzcd}$$

Binomial Tree

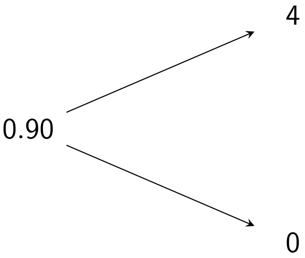
But trees can also be written in terms of the value of the call option.

$$
\begin{tikzcd}
 & C_u \\
C \arrow[ur] \arrow[dr] & \\
 & C_d
\end{tikzcd}
$$

C is the call premium and C_u and C_d are the possible call payoffs.

Binomial Tree

In the example where the stock price is 40 and it could increase to 45 or decrease to 38, the 41 strike option has an option tree of



Two Period Binomial Tree

Consider a **two period binomial tree**.

$$
\begin{tikzcd}
 & & Su^2 \\
 & Su \arrow[ur] \arrow[dr] & \\
\$S \arrow[ur] \arrow[dr] & & Sud \\
 & Sd \arrow[ur] \arrow[dr] & \\
 & & Sd^2
\end{tikzcd}
$$

Two Period Binomial Tree

The idea here is that there is a step size, h, which in a two period binomial tree is always $h = t/2$. Then in one step size, the new stock value can be either Su or Sd.

- If the stock at time h is Su, then at time $2h = t$, the stock price can move up to Su^2 or down to Sud
- If the stock at time h is Sd, then at time $2h = t$, the stock price can move up to Sdu or down to Sd^2

Because of the way we set this up, $Sud = Sdu$, so the tree is a **recombining** tree because at time h the nodes combine when the paths meet up.

Two Period Binomial Tree

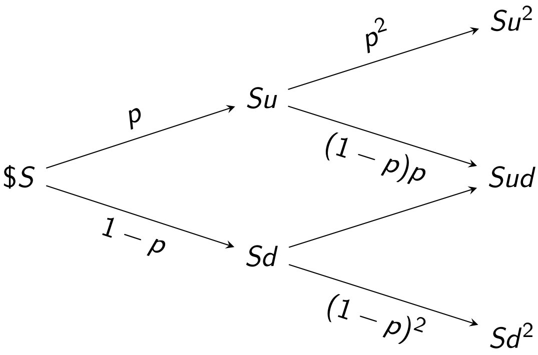
We will start with the easy way of pricing a two period binomial tree first. Recall the formula for p was $p = \frac{e^{rt}-d}{u-d}$. For a multi-period tree, the risk neutral probability is

$$p = \frac{e^{rh} - d}{u - d}$$

The only difference is that the the t is replaced with an h.

Two Period Binomial Tree

Using this p we can find the probability of ending up in certain places.



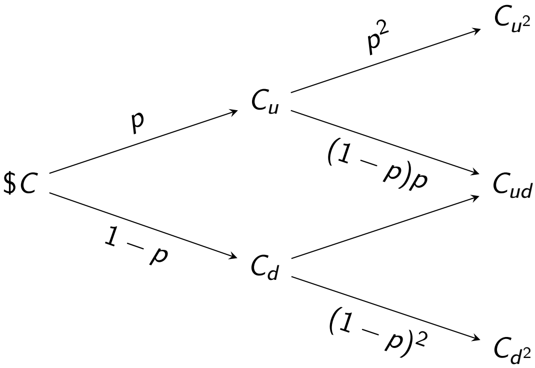
Two Period Binomial Tree

We know that

- There is a p^2 probability of ending up at Su^2
- a $2p(1-p)$ probability of ending up at Sud
- a $(1-p)^2$ probability of ending up at Sd^2

Two Period Binomial Tree

Now consider the option-based binomial tree with two periods



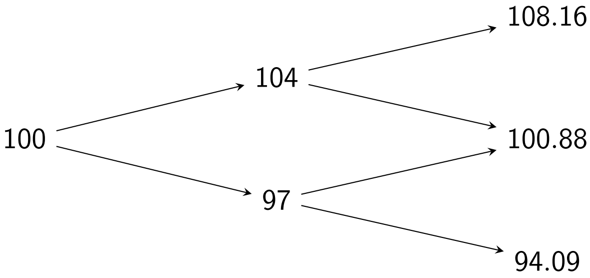
Two Period Binomial Tree

We know that

- C_{u^2} is the payoff of the option if the stock price at $2h$ is Su^2 $(max(0, Su^2 - K))$
- C_{ud} is the payoff of the option if the stock price at $2h$ is Sud $(max(0, Sud - K))$
- C_{d^2} is the payoff of the option if the stock price at $2h$ is Sd^2 $(max(0, Sd^2 - K))$
- C_u is the value of the call option position at time h is the stock price is S_u.
- C_d is the value of the call option position at time h is the stock price is S_d.
- C is the option premium

Two Period Binomial Tree

C_{u^2}, C_{ud} and C_{d^2} are easily determine using the payoff of the call options at the particular stock prices. For example, suppose a stock costs \$100 amd $u = 1.04$ and $d = 0.97$. The stock tree would be



Two Period Binomial Tree

Suppose a certain call option had a strike price of 98.

- If the stock price is 108.16, then the payoff is $108.16 - 98 = 10.16$
- If the stock price is 100.88, then the payoff is $100.88 - 98 = 2.88$
- If the stock price is 94.09, then the payoff is 0, because the option is not exercised.

Two Period Binomial Tree

- That means there is a p^2 probability of a payoff of 10.16,
- a $2p(1-p)$ probability of a payoff of 2.88
- and a $(1-p)^2$ probability of a payoff of 0.

The risk neutral price is then
$C = e^{-rt}[p^2(10.18) + 2p(1-p)(2.88) + (1-p)^2(0)]$

Two Period Binomial Tree

In general, the premium for a call option using a two period binomial tree is

$$C = e^{-rt}[p^2 C_{u^2} + 2p(1-p)C_{ud} + (1-p)^2 C_{d^2}]$$

The premium for a put option using a two period binomial tree is

$$P = e^{-rt}[p^2 P_{u^2} + 2p(1-p)P_{ud} + (1-p)^2 P_{d^2}]$$

Example

The put option payoffs are $P_{u^2} = 0$, $P_{ud} = 0.417$, and $P_{d^2} = 2.953$.

We must now find p.

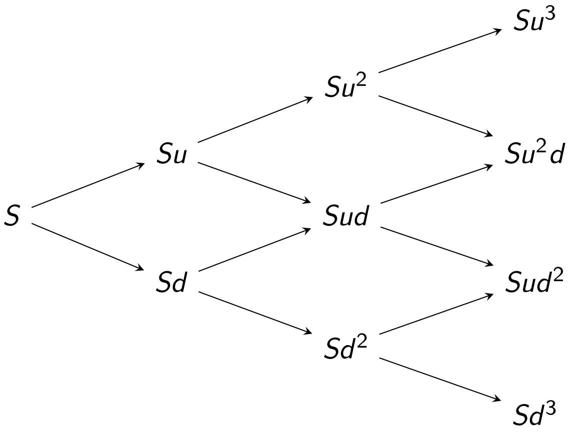
$$p = \frac{e^{rh} - d}{u - d} = \frac{e^{(.04)(1/12)} - .975}{1.027 - 0.975} = 0.545$$

The put option premium is

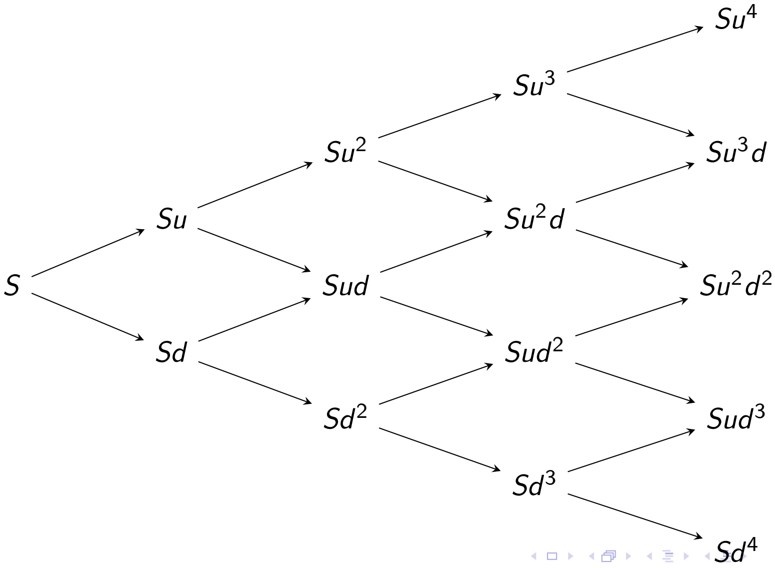
$$P = e^{-.04(1/6)}[(0.545)^2 0 + 2(.545)(1-.545)(0.417) + (1-0.545)^2 2.953]$$

$$= 0.81$$

Multi-Period Binomial Tree



Multi-Period Binomial Tree



Multi-Period Binomial Tree

The probability of getting to a stock price $Su^j d^k$, which is the probability of j up movements and k down movements, is

$$Pr(S_t = Su^j d^k) = \left(\begin{array}{c} j+k \\ k \end{array} \right) p^j (1-p)^k$$

Multi-Period Binomial Tree

This means that the price of a call option using a *m*-period binomial tree is

$$C = e^{-rt} \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p^{m-k} (1-p)^k C_{u^{m-k} d^k}$$

Example

Suppose a stock costs \$50 and the binomial tree has $h = .25$, $t = 1$, $u = 1.05$ and $d = 0.95$, and $p = 0.60$. What is the price of the 1 year at the money call option. Assume $r = 0$.

- $Su^4 = 60.775$ so $C_{u^4} = 10.775$
- $Su^3d = 54.987$ so $C_{u^3d} = 4.987$
- $Su^2d^2 = 49.75$ so $C_{u^2d^2} = 0$ as do all the other payoffs, (C_ud^3 and C_d^4)

So the call premium is

$$C = (.6^4)(10.775) + 4(.6^3)(.4)4.987 + 0 = 3.12$$

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.

Practice

Determine the 6 month 145-strike call option premium using a 3 period binomial tree with $S = 140$, $r = 0.06$, $u = 1.0659$, $d = 0.9508$, and $p = 0.4857$.
The binomial tree for the stock price is

$$
\begin{tikzcd}
 & & & 169.54 \\
 & & 159.06 \arrow[ur] \arrow[dr] & \\
 & 149.23 \arrow[ur] \arrow[dr] & & 151.23 \\
140 \arrow[ur] \arrow[dr] & & 141.88 \arrow[ur] \arrow[dr] & \\
 & 133.11 \arrow[ur] \arrow[dr] & & 134.89 \\
 & & 126.55 \arrow[ur] \arrow[dr] & \\
 & & & 120.32
\end{tikzcd}
$$

Practice

$C_{u^3} = 24.54,\ C_{u^2 d} = 6.23,\ C_{ud^2} = 0.$

The call premium is

$$C = e^{-.06/2}[(0.4857)^3(24.54) + 3(.4857)^2(.5143)(6.23)] = 4.93$$

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

Practice

Determine the price of a 46 strike 4 month put option using a 4 period binomial tree with $S = 50$, $r = 0.03$, $u = 1.022$, $d = 0.976$, and $p = 0.494$.

$Sd^4 = 45.285$ and $Sud^3 = 47.43$, and all the rest are higher than 46, so the payoff is 0 except for Sd^4.

$$P = e^{-.03/3}[(1 - .494)^4(.715)] = .046$$