Severity Models - Special Families of Distributions

Sections 5.3-5.4

Stat 346 - Short-term Actuarial Math

Introduction

- Given that a claim occurs, the (individual) claim size X is typically referred to as claim severity.
- While typically this may be of continuous random variables, sometimes claim sizes can be considered discrete.
- When modeling claims severity, insurers are usually concerned with the tails of the distribution. There are certain types of insurance contracts with what are called long tails.

Parametric distributions

A parametric distribution consists of a set of distribution functions with each member determined by specifying one or more values called "parameters".

The parameter is fixed and finite, and it could be a one value or several in which case we call it a vector of parameters. Parameter vector could be denoted by θ.

Parametric distributions

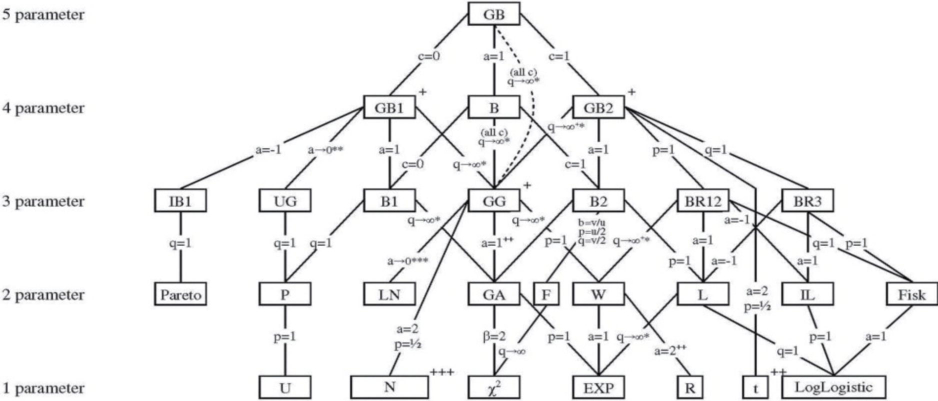
Some parameters have special names:

- Location parameters: controls the shift of the distribution. If $f(x-\mu)$ belongs to the same family of distribution as f then f is called a location family and μ is a location parameter.
- Scale parameter: controls the spread of the distribution. If $\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$ belongs to the same family as f, then f is a scale distribution and σ is a scale parameter.
- Note: a distribution where $\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$ belongs to the same family as f is called a location scale family.
- Rate parameter: For some distributions you can have a rate parameter which is the inverse of the scale parameter $1/\sigma$
- Shape parameter: All parameters that are not location or scale parameters and are not functions of location and scale parameter is called a shape parameter.

Some parametric claim size distributions

- Normal - easy to work with, but careful with getting negative claims. Insurance claims usually are never negative.
- Gamma/Exponential - use this if the tail of distribution is considered 'light'; applicable for example with damage to automobiles.
- Lognormal - somewhat heavier tails, applicable for example with fire insurance.
- Burr/Pareto - used for heavy-tailed business, such as liability insurance.
- Inverse Gaussian - not very popular because complicated mathematically.

Most distributions are connected



* $q \to \infty$ with $b = \beta q^{1/a}$
** $a \to 0$ with $p = d/a$
*** $a \to 0$ with $b = (\sigma^2 a^2)^{1/a}$, $p = (a\mu + 1)/\sigma^2 a^2$

\+ The distribution of the inverse is obtained if the sign of a is changed
++ The 1/2 t corresponds to a=2, p=1/2
+++ The 1/2 Normal corresponds to a=2, p=1/2

The Normal distribution

The Normal distribution is a clear example:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter vector $\theta = (\mu, \sigma)$. It is well known that μ is the mean and σ^2 is the variance.

Some important properties:

- Standard Normal when $\mu = 0$ and $\sigma = 1$.
- If $X \sim \mathsf{N}(\mu, \sigma)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathsf{N}(0, 1)$.
- Sums of Normal random variables is again Normal.
- If $X \sim \mathsf{N}(\mu, \sigma)$, then $cX \sim \mathsf{N}(c\mu, c\sigma)$.

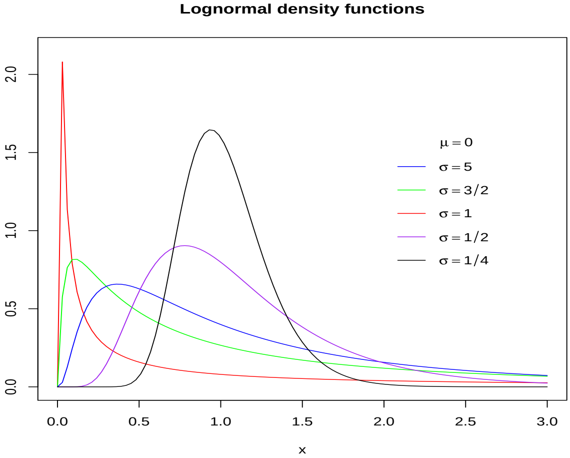
The Lognormal distribution

- If $Y \sim$ Normal(μ, σ), then $X = \exp(Y)$ is lognormal and we write $X \sim$ Lognormal(μ, σ). Its density can be expressed as

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\log x - \mu)^2/2\sigma^2}, \quad \text{for } x > 0.$$

- Thus, if X is lognormal, then $\log(X)$ is Normal.
- Moments: $\mathsf{E}(X^k) = \exp(k\mu + k^2\sigma^2/2)$
- Mean: $\mathsf{E}(X) = e^{\mu + \sigma^2/2}$ Variance: $\mathsf{Var}(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$
- Derive the mode.

Lognormal densities for various σ's



The Gamma distribution

- We shall write $X \sim$ Gamma(α, θ) if density has the form

$$f_X(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, \quad \text{for } x > 0;\ \alpha, \theta > 0,$$

with α, the shape parameter, and θ, the scale parameter.

- Higher moments: $\mathsf{E}(X^k) = \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\theta^k$
- Mean: $\mathsf{E}(X) = \alpha\theta$ Variance: $\mathsf{Var}(X) = \alpha\theta^2$
- Special cases:
  - Exponential: When $\alpha = 1$, we have $X \sim$ Exp (θ).
  - Chi-square: When $\alpha = n/2$ and $\theta = 2$, we have a chi-squared distribution with n degrees of freedom.

The Pareto distribution

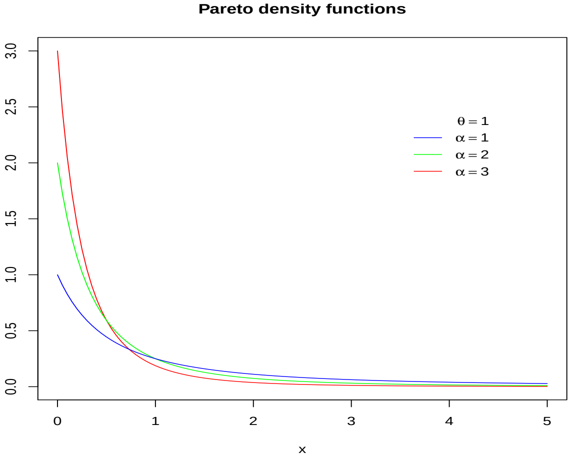
- We shall write $X \sim$ Pareto(α, θ) if density has the form

$$f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad \text{for } x > 0,$$

  where $\alpha > 0$ and $\theta > 0$.

- α, the shape parameter; θ, the scale parameter.
- CDF: $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$.
- Mean: $\mathrm{E}(X) = \frac{\theta}{\alpha - 1}$
- Higher moments: $\mathrm{E}(X^k) = \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)}\theta^k\Gamma(k+1), \quad \text{for } -1 < k < \alpha.$
- Variance: (derive it!)

Pareto densities for various α's



Some important properties

- A positive scalar multiple of a Pareto is again a Pareto.
    - If $X \sim \text{Pareto}(\alpha, \theta)$ and c is a positive constant, then $cX \sim \text{Pareto}(\alpha, c\theta)$.
    - This also explains why θ is called the scale parameter.
- The Pareto distribution is a continuous mixture of exponentials with Gamma mixing weights.
    - If $(X|\Lambda = \lambda) \sim \text{Exp}(1/\lambda)$ and $\Lambda \sim \text{Gamma}(\alpha, 1/\theta)$, then unconditionally, $X \sim \text{Pareto}(\alpha, \theta)$.
    - To be derived in lecture - also part of generating new distributions.

The Burr distribution

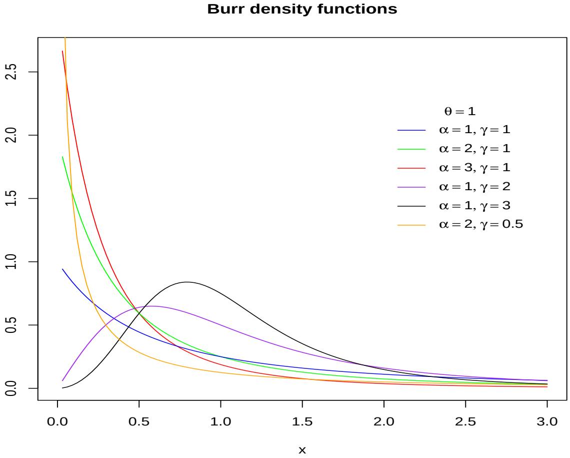
- We shall write $X \sim$ Burr(α, θ, γ) if density has the form

$$f_X(x) = \frac{\alpha\gamma(x/\theta)^\gamma}{x[1 + (x/\theta)^\gamma]^{\alpha+1}}, \quad \text{for } x > 0,$$

  where $\alpha > 0$, $\theta > 0$ and $\gamma > 0$.

- α, the shape parameter; θ, the scale parameter.
- Sometimes more precisely called Burr Type XII distribution and the Pareto is a special case when $\gamma = 1$.
- Higher moments: $\mathsf{E}(X^k) = \dfrac{\Gamma(1 + k/\gamma)\Gamma(\alpha - k/\gamma)}{\Gamma(\alpha)}\theta^k$, provided $-\gamma < k < \alpha\gamma$.

Burr densities for various parameter values

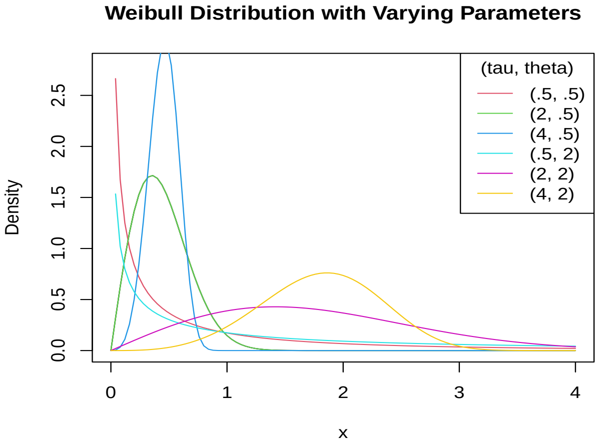


Weibull Distribution

- The Weibull distribution is commonly used in reliability analysis and extreme value modeling.
- It is characterized by two parameters:
  - Shape parameter (τ) controls the shape of the distribution.
  - Scale parameter (θ) controls the scale or rate at which events occur.
- The probability density function (PDF) of the Weibull distribution is given by:

$$f_X(x) = \frac{\tau \left(\frac{x}{\theta}\right)^{\tau} e^{-(x/\theta)^{\tau}}}{x}, \quad x \geq 0$$

Weibull densities for various parameter values



Inverse Distributions

- Inverse distributions are a class of probability distributions that are less common but offer valuable modeling options in severity models.
- They are known for being more challenging mathematically but can provide flexibility in capturing certain characteristics of data.
- Here are some commonly used inverse distributions:

- Inverse Gamma Distribution
- Inverse Weibull Distribution
- Inverse Pareto Distribution
- Inverse Burr Distribution

The relationship between a distribution and it's inverse is if $X \sim \text{Gamma}(\alpha, \theta)$ then $Y = 1/X \sim \text{InvGamma}(\alpha, 1/\theta)$

What we need to know

There are a few things you should be able to. do with these distributions

- Recognize the distributions based on the main part of the distribution
- Classify the distributions based on the existence of moments
- Understand what happens when you multiply by a constant

Recognizing Distributions

Understanding how to identify a distribution based on a proportional function can be helpful. For example, the Gamma distribution density function could be written as

$$f(x) \propto x^{\alpha-1}e^{-x/\theta}$$

A density function with the constant removed is sometimes called a kernel.

Recognizing Distributions

Let X be a random variable with density function $f(x) = k(x+3)^{-4}, \ x > 0$. What is $E(X)$.

Strategy 1:

- Solve for k by setting $\int_0^\infty k(x+3)^{-4}dx = 1$ and solving for k.
- Find $E(X) = \int_0^\infty kx(x+3)^{-4}dx$

Strategy 2:

- Recognize that this is a Pareto distribution with $\theta = 3$ and $\alpha = 3$.
- Find $E(X)$ using the tables without any integrals

Recognizing Distributions

Let X be a random variable with density function $f(x) = k\sqrt{x}e^{-8x^{3/2}},\ x > 0$. What is $VaR_{.95}(X)$?

Existence of Moments

The moments of a distribution, $E(X^k)$, can help identify a distribution. Let's look at a few specific examples

- Exponential distribution: $E(X^k) = \theta^k k!$. Moments exist even for large values of k.
- Pareto distribution: $\mathsf{E}(X^k) = \dfrac{\Gamma(\alpha - k)}{\Gamma(\alpha)}\theta^k\Gamma(k+1)$, for $-1 < k < \alpha$. When $k > \alpha$, the moments are infinite. A Pareto with $\alpha < 2$ has an infinite variance. With $\alpha < 1$ it has an infinite mean.
- Inverse Gamma: $\mathsf{E}(X^k) = \dfrac{\theta^k\Gamma(\alpha - k)}{\Gamma(\alpha)}$, for $k < \alpha$. Similar to the Pareto, the value of α will dictate the existence of moments.

Some applications make sense with tails so large the moments are infinite. Other will not.

Existence of Moments

You believe that X follows a Burr distribution with parameters $alpha = 4$, $\theta = 1300$ and $\gamma = .4$. How many (integer) moments exist for this distribution?

Multiplying by a Constant

- When you multiply a random variable X by a constant c, you get a new random variable $Y = cX$.
- This operation has a significant impact on probability distributions.

Example: Multiplying a Gamma Distribution

If $X \sim \text{Gamma}(\alpha, \theta)$, then what is the distribution of $Y = cX$?

- Property: For a constant $c > 0$, if $Y = cX$, then the probability density function of Y is given by:

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$$

Analytical Solution: Multiplying a Gamma Distribution

Substituting the Gamma PDF:

$$f_Y(y) = \frac{1}{c} \cdot \frac{1}{\theta^\alpha \Gamma(\alpha)} \left(\frac{y}{c}\right)^{\alpha-1} e^{-\frac{y/c}{\theta}}$$

Simplifying:

$$f_Y(y) = \frac{1}{(c\theta)^\alpha \Gamma(\alpha)} \cdot y^{\alpha-1} e^{-\frac{y}{c\theta}}$$

This is the probability density function of $Y = cX$. It follows a Gamma distribution with parameters:

Shape Parameter: α

Scale Parameter: $c\theta$

Multiplying Distributions by a Constant

- When you multiply a random variable by a constant, certain distributions maintain their form with adjusted parameters.

Distribution Properties After Multiplying by a Constant

- **Normal Distribution:**
  - If $X \sim \mathrm{N}(\mu, \sigma)$, then $cX \sim \mathrm{N}(c\mu, c\sigma)$.
- **Lognormal Distribution:**
  - If $X \sim \mathrm{Lognormal}(\mu, \sigma)$, then $cX \sim \mathrm{Lognormal}(\mu + \log(c), \sigma)$.
- **Gamma Distribution:**
  - If $X \sim \mathrm{Gamma}(\alpha, \theta)$, then $cX \sim \mathrm{Gamma}(\alpha, c\theta)$.
- **Pareto Distribution:**
  - If $X \sim \mathrm{Pareto}(\alpha, \theta)$, then $cX \sim \mathrm{Pareto}(\alpha, c\theta)$.
- **Burr Distribution:**
  - If $X \sim \mathrm{Burr}(\alpha, \theta, \gamma)$, then $cX \sim \mathrm{Burr}(\alpha, c\theta, \gamma)$.
- **Weibull Distribution:**
  - If $X \sim \mathrm{Weibull}(\tau, \theta)$, then $cX \sim \mathrm{Weibull}(\tau, c\theta)$.

Impact of Economic Factors on Distributions

Distributional shapes often remain unchanged, but they may experience a constant shift. To illustrate this concept, consider the events of 2020 when the supply chain was disrupted due to factors like Covid-19, leading to shortages of replacement parts for American mechanics. This supply shortage resulted in rising prices, leading to inflation.

- Inflation shifted the cost landscape for auto repairs.
- Despite rising prices, 2020 also witnessed a decrease in claim counts, attributed to reduced car usage during the pandemic.

Example

Suppose losses for an auto claim followed a Weibull distribution with $\alpha = 3$ and $\theta = 600$. In 2020, inflation caused auto claim losses to increase by 50%.

- What is the new distribution of losses?
- How much did the mean of losses change?
- How much did the variance of losses change?

The linear exponential family

- A random variable X has a distribution from the linear exponential family if its density has the form

$$f_X(x;\theta) = \frac{1}{q(\theta)} p(x) e^{r(\theta)x},$$

where $p(x)$ depends only on x and $q(\theta)$ is a normalizing constant.

- The support of X must not depend on θ.
- Mean:

$$\mathsf{E}(X) = \mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$$

- Variance:

$$\mathsf{Var}(X) = v(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$$

Some members of the linear exponential family

- Continuous:
    - Normal
    - Gamma (and all special cases e.g. Exponential, Chi-squared)
- Discrete:
    - Poisson
    - Binomial
    - Negative Binomial