Introductory Probability Principles

Definitions and Examples

Random Variable: A random variable is a quantity that take on different values resulting from random chance.

Examples:

- How much precipitation we get in a given day
- Whether or not a basketball player makes their next shot
- The number of mistakes Dr. Richardson makes in a particular set of slides

Typically we will denote a random variable by a capital letter. Y is a random variable.

Definitions and Examples

Probability Distribution: A probability distribution is a mathematical representation of what values may appear as a realization of a random variable and how often they may occur.

Examples:

- When you flip a coin, there is a 50% chance it will land on heads and 50% chance it will land on tails
- Rolling a dice has a 1/6 chance of each number 1 through 6

Typically we say that a random variable follows or has a probability distribution. Random variable Y follows a certain probability distribution.

Definitions and Examples

Two types of probability distributions:

- A **discrete** random variable will have a **discrete** probability distribution. This means that there are a countable number of possible values the random variable can be. Flipping a coin, rolling a dice, counting cars, etc..
- A **continuous** random variable will have a **continuous** probability distribution. In this case, regardless of how the data is recorded, there are an uncountable number of possible values. This basically means that there is some range somewhere that the random value can take on any possible values in that range. Height, speed, and percentages are examples. Some large countable sets are typically treated as continuous, like money or population.

Definitions and Examples

When we describe a random variable in the following terms:

- **Outcome**: It is a possible result of a random experiment. For instance, when you roll a dice, getting a 3 is an outcome.
- **Sample Space**: It is the set of all possible outcomes of an experiment. In the case of a dice roll, the sample space is {1, 2, 3, 4, 5, 6}.
- **Event**: It is a subset of the sample space. An event could be a single outcome or a combination of outcomes. For instance, getting an odd number when rolling a dice is an event, which can be represented as the set {1, 3, 5}.
- **Probability**: It is a measure of the likelihood that an event occurs. The probability of an event A, denoted as $P(A)$, is a number between 0 and 1, inclusive. The higher the probability, the more likely the event is to occur.

Definitions and Examples

Suppose we conduct a random experiment: Ask a random passerby to pick a favorite color.

- A possible outcome of this experiment is someone saying "blue."
- The sample space is the list of all possible colors.
- An event would be that they say one of the main 7 colors of the rainbow (ROYGBIV).
- The probability of an event could be that we determine the likelihood someone says one of the 7 main colors is about 70%, or 0.7.

Definitions and Examples

If all outcomes are equally likely, then
$P(A) = \frac{\text{number of outcomes in A}}{\text{total number of outcomes}}$.

Suppose that Y is the random variable for a dice roll. Let A be the probability of rolling an even number. There are 3 ways to roll an even number and 6 ways in total to roll a dice. So the probability is $P(A) = \frac{3}{6} = \frac{1}{2}$.

Also of note is that $P(\text{not } A) = 1 - P(A)$.

Definitions and Examples

For two events A and B, we can look at $P(A \text{ and } B)$ and $P(A \text{ or } B)$. These values are connected through

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

There are two special cases to consider:

- **Mutually Exclusive**: Events A and B are mutually exclusive if they cannot both occur at the same time. If A and B are mutually exclusive, then $P(A \text{ and } B) = 0$ and

$$P(A \text{ or } B) = P(A) + P(B)$$

.

- **Independent: Events** A and B are independent if A occurring has no effect on whether or not B occurs. If A and B are independent, then

$$P(A \text{ and } B) = P(A) \times P(B)$$

Definitions and Examples

Suppose the event A is the event of rain or snow in a particular day and the event B is the event that my kids plays in a soccer game that same day. Would these two events be mutually exclusive? Would they be independent?

Definitions and Examples

Suppose the event A is the event of the next car coming down the road being red and the event B is the event that the next car is blue and we live in a world where cars only have one color. Would these two events be mutually exclusive? Would they be independent? If 20% of the cars are red and 30% are blue, what is

- $Pr(A \text{ or } B)$
- $Pr(A \text{ and } B)$

Definitions and Examples

Suppose the event A is the event of the next car coming down the road being red and the event B is the event that the next car is a Tesla. Would these two events be mutually exclusive? Would they be independent? Assume that the two events are indeed independent and that 20% of cars are red and 5% off cars are

- $Pr(A \text{ or } B)$
- $Pr(A \text{ and } B)$

Definitions and Examples

Beyond these basic probabilities, you will be responsible for understanding the following discrete probabilities:

- How many times will event A occur in n independent realizations of X.
- Probability of event A never happening in n independent realizations.
- Probability of event A occurring given event B also occurred.
- How many independent realizations can occur before event A occurs.

Binomial Distribution

Let $p = Pr(A)$ be the probability an event occurs. Suppose we observe n independent realizations of a random variable X . Then we can determine how many times A occurs using n and p. These are called Binomial probabilities.

In fact we can define a new random variable, Y , that is equal to the number of times A occurs in n independent trials. Y is called a binomial random variable and follows a binomial distribution.

Binomial Distribution

The binomial distribution is a probability distribution that describes the number of successes in a fixed number of independent experiments of the same probability of success.

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where:

- $P(Y = k)$: Probability of k successes in n trials
- $\binom{n}{k}$: Combination, representing the number of ways to choose k successes
- p: Probability of success in a single trial
- $1 - p$: Probability of failure in a single trial

Binomial Random Variable

A binomial random variable is the number of successes in n independent trials with the same probability of success p.

$$Y \sim B(n, p)$$

where:

- Y: Binomial random variable
- n: Number of trials
- p: Probability of success in each trial

Example

Consider a binomial experiment with $n = 5$ trials and probability of success $p = 0.3$.

$$Y \sim B(5, 0.3)$$

Let's calculate the probability distribution of Y:

Probability Distribution Table

k (Number of successes)	$P(Y=k)$	Probability
0	$\binom{5}{0}\cdot(0.3)^0\cdot(0.7)^5$	0.16807
1	$\binom{5}{1}\cdot(0.3)^1\cdot(0.7)^4$	0.36015
2	$\binom{5}{2}\cdot(0.3)^2\cdot(0.7)^3$	0.30870
3	$\binom{5}{3}\cdot(0.3)^3\cdot(0.7)^2$	0.13230
4	$\binom{5}{4}\cdot(0.3)^4\cdot(0.7)^1$	0.02835
5	$\binom{5}{5}\cdot(0.3)^5\cdot(0.7)^0$	0.00243

Probability of No Success

The probability that a success never happens, i.e., $k = 0$, can be calculated directly from the binomial formula:

$$P(Y = 0) = \binom{n}{0} \cdot p^0 \cdot (1 - p)^n$$

For our example where $n = 5$ and $p = 0.3$, this probability is:

$$P(Y = 0) = \binom{5}{0} \cdot (0.3)^0 \cdot (0.7)^5 \approx 0.16807$$

Probability of At Least One Success

The probability that a success occurs at least once, i.e., $Y \geq 1$, can be calculated as one minus the probability of no success:

$$P(Y \geq 1) = 1 - P(Y = 0)$$

Using the calculated value from the previous slide:

$$P(Y \geq 1) = 1 - 0.16807 \approx 0.83193$$

Scenario: Binomial Probability in Practice

Imagine you are a data scientist at a tech company. You are conducting A/B testing on the user interface of a product. Based on initial tests, the probability that a user prefers the new interface over the old one is 0.45.

You decide to gather more data and survey 10 random users.

Questions:

1. What is the probability that exactly 4 out of the 10 users prefer the new interface?
2. What is the probability that none of the users prefer the new interface?
3. What is the probability that at least one user prefer the new interface?

Try solving this problem using the binomial probability formula:

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k}$$

Introduction to Geometric Distribution

The geometric distribution is a probability distribution that describes the number of trials needed before the first success in a sequence of independent trials, each with the same probability p of success.

The probability mass function of the geometric distribution is given by:

$$P(Z = k) = (1 - p)^{k-1} \cdot p$$

where:

- $P(Z = k)$: Probability that the first success occurs on the k^{th} trial
- p: Probability of success in each trial
- $1 - p$: Probability of failure in each trial

Example

Let's consider a scenario where the probability of success in a single trial is $p = 0.3$.
Questions:

1. What is the probability that the first success occurs on the 3rd trial?
2. What is the probability that the first success occurs within the first 5 trials?

We can use the geometric distribution formula to find the answers:

$$P(Z = k) = (1 - p)^{k-1} \cdot p$$