What Are Conditional Probabilities?

Definition

Conditional probability is the probability of an event occurring given that another event has already occurred.

- Notation: $Pr(A|B)$ represents the conditional probability of event A given that event B has occurred.
- It quantifies how the probability of A changes when we have information about B.

Conditional Probability Formula

The conditional probability of event A given event B is calculated using the formula:

$$Pr(A|B) = \frac{Pr(A \text{ and } B)}{Pr(B)}$$

- $Pr(A \text{ and } B)$ is the probability that both A and B occur.
- $Pr(B)$ is the probability of event B occurring.

Interpretation

Conditional probability measures the proportion of times event A occurs when event B is known to have occurred.

Example

Problem

You roll a fair six-sided die. What is the probability of getting a 4 given that you rolled an even number?

- Sample Space: $\{1, 2, 3, 4, 5, 6\}$
- Event A: Rolling a 4
- Event B: Rolling an even number
- Calculate: $Pr(A|B)$

Example=

- $Pr(A)$: Probability of rolling a 4 $= \frac{1}{6}$
- $Pr(B)$: Probability of rolling an even number $= \frac{3}{6}$ (3 even numbers out of 6)
- $Pr(A \text{ and } B)$: Probability of rolling a 4 and an even number $= \frac{1}{6}$
- Using the formula: $Pr(A|B) = \frac{Pr(A \text{ and } B)}{Pr(B)}$
- $Pr(A|B) = \frac{1/6}{3/6} = \frac{1}{3}$

So, the probability of rolling a 4 given that you rolled an even number is $\frac{1}{3}$.

Joint Probability and Conditional Probability

Joint Probability

Joint probability $Pr(A \text{ and } B)$ is the probability of both events A and B occurring simultaneously.

Relationship

We can express joint probability using conditional probability:

$$Pr(A \text{ and } B) = Pr(A|B) \cdot Pr(B)$$

Example

Problem

In a golf game, let A be the event of hitting a hole-in-one on a randomly chosen hole, and B be the event of the hole being a par 3. What is the joint probability of both a randomly chosen hole being a par 3 and hitting a hole in 1?

- $Pr(A|B)$: Probability of hitting a hole-in-one on a par 3 hole $= \frac{1}{1000}$
- $Pr(B)$: There are 18 holes in a course and 4 are par 3 $= \frac{4}{18}$
- Calculate: $Pr(A \text{ and } B)$

Example

- We want to find $Pr(A \text{ and } B)$, which is the joint probability of both hitting a hole-in-one and hitting the fairway, given that we hit the fairway off the tee.
- Using the relationship between joint and conditional probability:

$$Pr(A \text{ and } B) = Pr(A|B) \cdot Pr(B)$$

- We know $Pr(A|B)$ is the probability of hitting a hole-in-one on a par 3 hole. This is still $\frac{1}{1000}$.
- We know $Pr(B)$ is the probability of randomly choosing a par 3 hole, which is $\frac{4}{218}$.
- So, we can calculate:

$$Pr(A \text{ and } B) = \frac{1}{1000} \cdot \frac{4}{18} = \frac{2}{900} = 0.00222$$

Introduction

Bayes' Rule

Bayes' Rule, also known as Bayes' Theorem, is a fundamental concept in probability theory. It allows us to update the probability of an event based on new evidence or information.

Definition of Bayes' Rule

Bayes' Rule

Bayes' Rule relates the conditional probability of event A given event B to the conditional probability of event B given event A as follows:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

- $P(A|B)$: Probability of event A given event B
- $P(B|A)$: Probability of event B given event A
- $P(A)$: Prior probability of event A
- $P(B)$: Prior probability of event B

Bayes' Rule Interpretation

Interpretation

Bayes' Rule allows us to update our belief in the probability of event A based on new information provided by event B.

- $P(A|B)$: Updated probability of A after considering B
- $P(B|A)$: Probability of B given A
- $P(A)$: Prior belief in the probability of A
- $P(B)$: Prior belief in the probability of B

Law of Total Probability in Bayes' Rule

Understanding the Denominator

The denominator in Bayes' Rule, $P(B)$, can be broken up into pieces using the law of total probability. This gives us a comprehensive view of all the ways event B can occur.

- Using the law of total probability, the denominator in Bayes' Rule can be expressed as:

$$P(B) = P(B|A) \cdot P(A) + P(B|\text{not } A) \cdot P(\text{not } A)$$

- Where:
  - $P(B|A)$: Probability of event B given A has occurred.
  - $P(B|\text{not } A)$: Probability of event B given A has not occurred.
  - $P(A)$: Prior probability of event A.
  - $P(\text{not } A)$: Prior probability of the complement of event A.

Illustration of Bayes' Rule

Example

Let's illustrate Bayes' Rule with a medical diagnosis scenario.

- Suppose a rare disease occurs in 1% of the population ($P(Disease) = 0.01$).
- A diagnostic test for the disease has a 95% true positive rate ($P(PositiveTest|Disease) = 0.95$) and a 10% false positive rate ($P(PositiveTest|NoDisease) = 0.10$).
- Calculate the probability of having the disease given a positive test result: $P(Disease|PositiveTest)$.

Illustration of Bayes' Rule (cont.)

- We have (using some shorthand notation to make it fit):

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|NoD) \cdot P(NoD)}$$

- Plugging in the values:

$$P(Disease|PositiveTest) = \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.10 \cdot 0.99} \approx 0.087$$

- So, the probability of having the disease given a positive test result is approximately 8.7%.

Scenario

You are a data analyst at a music streaming service. The service offers two types of subscriptions: Premium and Free.

- Premium subscribers enjoy an ad-free experience and higher audio quality. Free subscribers hear ads between songs and have lower audio quality.
- 80% of users are Free subscribers ($P(\text{Free}) = 0.8$), while the rest are Premium subscribers ($P(\text{Premium}) = 0.2$).
- You want to determine the probability that a user is a Premium subscriber if they have rated a song 5 stars: $P(\text{Premium}|\text{5 Stars})$.
- Premium subscribers are more likely to rate songs 5 stars, so $P(\text{5 Stars}|\text{Premium}) = 0.7$, while for Free subscribers, it's lower at $P(\text{5 Stars}|\text{Free}) = 0.3$.

Calculate the probability that a user is a Premium subscriber if they have rated a song 5 stars.

Solution

We can use Bayes' Rule to calculate the probability of a user being a Premium subscriber given that they have rated a song 5 stars:

$$P(\text{Premium}|\text{5 Stars}) =$$

$$\frac{P(\text{5 Stars}|\text{Premium}) \cdot P(\text{Premium})}{P(\text{5 Stars}|\text{Premium}) \cdot P(\text{Premium}) + P(\text{5 Stars}|\text{Free}) \cdot P(\text{Free})}$$

- $P(\text{5 Stars}|\text{Premium}) = 0.7$
- $P(\text{Premium}) = 0.2$
- $P(\text{5 Stars}|\text{Free}) = 0.3$
- $P(\text{Free}) = 0.8$

Solution (cont.)

- Plugging in the values into Bayes' Rule:

$$P(\text{Premium}|\text{5 Stars}) = \frac{0.7 \cdot 0.2}{0.7 \cdot 0.2 + 0.3 \cdot 0.8}$$

- After calculations:

$$P(\text{Premium}|\text{5 Stars}) \approx 0.368$$

So, the probability that a user is a Premium subscriber if they have rated a song 5 stars is approximately 36.8%.