7.1 # 1, 3, 5, 8, 10, 13, 17, 19, 25, 29

1.) Determine if the matrix is symmetric.

$\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix} = A \qquad A^T = \begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$ Since $A^T = A$, this matrix is symmetric

3.) $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = A \qquad A^T = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ Not symmetric $A \neq A^T$

5.) $\begin{bmatrix} -6 & 2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & -6 \end{bmatrix} = A \qquad A^T = \begin{bmatrix} -6 & 0 & 0 \\ 2 & -6 & 0 \\ 0 & 2 & -6 \end{bmatrix}$ Not symmetric $A \neq A^T$

8.) Determine if the matrix is orthogonal. If so, find the inverse.

$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = P$ P is an orthogonal matrix if $P^T = P^{-1}$.
ie. if $P^TP = I$ where P is square

$$P^TP = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

10.) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = P$

$$P^TP = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \neq I$$

P is not an orthogonal matrix

13.) Orthogonally diagonalize the matrix.

$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A \qquad \det(A-\lambda I) = (3-\lambda)(3-\lambda)-1 = \lambda^2-6\lambda+8 = (\lambda-4)(\lambda-2) \Rightarrow \lambda = 4, 2$

$\lambda=4 \quad A-\lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x_1 - x_2 = 0, \ \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$

Basis for eigenspace $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Normalize this basis vector $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\lambda=2 \quad A-\lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x_1 + x_2 = 0, \ \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$

Basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Normalized: $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

17.) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ has eigenvalues 5, 2, -2   A is symmetric

$\lambda=5 \quad A-\lambda I = \begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &\text{ free} \end{aligned} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3$ Normalize: $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\lambda=2 \quad A-\lambda I = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \\ x_3 &\text{ free} \end{aligned} \quad \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$ Normalize: $\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$\lambda=-2 \quad A-\lambda I = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= 0 \\ x_3 &\text{ free} \end{aligned} \quad \vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$ Normalize: $\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

7.1 continued

19.) $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ has eigenvalues $7, -2$. A is symmetric

$\lambda = 7$ $\quad A - \lambda I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -\frac{1}{2}x_2 + x_3$, x_2, x_3 free $\quad \vec{x} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$

$\lambda = -2$ $\quad A - \lambda I = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -x_3$, $x_2 = -\frac{1}{2}x_3$, x_3 free $\quad \vec{x} = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} x_3$

The vectors $\vec{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are not orthogonal, so we find an orthonormal basis for the eigen space. The orthogonal projection of $\vec{v}_2$ onto $\vec{v}_1$ is

$\hat{v}_2 = \frac{-1/2}{\frac{1}{4}+1} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} = \frac{-2}{5} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ 0 \end{bmatrix}$. So $\vec{v}_1$ and $\vec{v}_2 - \hat{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/5 \\ -2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$

are orthogonal. We normalize them and get that

$\left\{ \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{55} \\ 2/\sqrt{55} \\ 5/\sqrt{55} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace corresponding to $\lambda = 7$. The set $\left\{ \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace corresponding to $\lambda = -2$.

So

$$P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{55} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{55} & -1/3 \\ 0 & 5/\sqrt{55} & 2/3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

25.) True/False.

a.) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

b.) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$ then $\vec{u} \cdot \vec{v} = 0$

c.) An $n \times n$ symmetric matrix has n distinct real eigenvalues,

d.) For a nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.

a.) True b.) True c.) False d.) False

29.) Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

Since A is orthogonally diagonalizable, $A = PDP^{-1}$ where P is orthogonal and D is diagonal.

$A^{-1} = (PDP^{-1})^{-1} = P^{-1}D^{-1}P$ Since D^{-1} is still diagonal, A^{-1} is orthogonally diagonalizable.