5.1 # 2, 6, 7, 11, 13, 15, 19, 21, 23, 24, 25, 27, 31

2.) Is $\lambda = -3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?

$\lambda = -3$ is an eigenvalue $\Leftrightarrow A\vec{x} = -3\vec{x}$ has a non trivial soln

$\Leftrightarrow (A+3I)\vec{x} = \vec{0}$ has a nontrivial soln

$$A+3I = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 6 & 12 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has free variable $\Rightarrow$ has nontrivial soln

So, $\boxed{\text{Yes}}$

6.) Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigen vector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.

Is $A\vec{x}$ a multiple of $\vec{x}$? $A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix}$ This is not a multiple of $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ so $\boxed{\text{No}}$

7.) Is $\lambda = 4$ an eigen value of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

$$A-4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable $\Rightarrow$ has a nontrivial soln $\Rightarrow \lambda = 4$ is an eigenvalue.

Any nonzero solution to $(A-4I)\vec{x} = \vec{0}$ is an eigenvector.

$x_1 = -x_3$
$x_2 = -x_2$
x_3 free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector. So is any multiple of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

11.) Find a basis for the eigenspace corresponding to the eigenvalues

$A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$, $\lambda = -1, 7$

For $\lambda = -1$: $A + I = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 2 & -3 & 0 \\ -4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = \tfrac{3}{2}x_2$, x_2 free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = -1$ is $\left\{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right\}$

11.) continued

For $\lambda=7$: $A-7I = \begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix}$ $\left[\begin{array}{cc|c} -6 & -3 & 0 \\ -4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -\frac{1}{2}x_2$
x_2 free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

A basis for the eigenspace corresponding to $\lambda=7$ is $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$.

13.) Same directions as in #11.

$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\lambda = 1, 2, 3$

For $\lambda=1$: $A-I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solutions to $(A-I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda=2$: $A-2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solutions to $(A-2I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda=3$: $A-3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solns to $(A-3I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

5.1 Continued

15.) $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$, $\lambda = -5$ $\quad A + 5I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 + x_2 + x_3 = 0$, x_2, x_3 free

So Solutions to $(A+5I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

A basis for the eigenspace corresponding to $\lambda = -5$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

19.) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

The columns of A are linearly dependent, so A is not invertible by IMT. Therefore zero is an eigenvalue of A.

21.) True/False. Let A be an $n \times n$ matrix.

a.) If $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x}$, then λ is an eigen value of A.

b.) A matrix A is not invertible iff zero is an eigenvalue of A.

c.) A number c is an eigenvalue of A iff the equation $(A - cI)\vec{x} = \vec{0}$ has a nontrivial soln.

d.) Finding an eigen vector of A may be difficult, but checking whether a given vector is an eigenvector is easy.

e.) To find the eigenvalues of A, reduce A to echelon form.

a.) False b.) True c.) True d.) True e.) False

23.) Explain why a 2×2 matrix can have at most two distinct eigen values. Explain why an $n \times n$ matrix can have at most n distinct eigen values.

The set of eigenvectors corresponding to distinct eigenvalues is linearly independent by Thm 2. For an $n \times n$ matrix, the eigen vectors have n entries. Having more than n distinct eigen values means you have more than n eigen vectors. A set with more vectors than entries in each vector is linearly dependent.

24.) Construct an example of a 2x2 matrix with only one distinct eigenvalue. The eigenvalues of a triangular matrix are the entries on the main diagonal. So choose any 2x2 triangular matrix with the same number on its diagonal.

$$\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}, \quad \text{etc...}$$

$\lambda = 3$ $\qquad$ $\lambda = -4$

25.) Let λ be an eigenvalue of an invertible matrix A. Show λ^{-1} is an eigenvalue of A^{-1}. If λ is an eigenvalue of A then by defn there is a nonzero vector $\vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$. Since A is invertible, we can apply A^{-1} to both sides of the equation, to get $\underbrace{A^{-1}A}_{\text{identity}}\vec{x} = A^{-1}\lambda\vec{x} \Rightarrow \vec{x} = A^{-1}\lambda\vec{x}$

(where $\vec{x}$ is this nonzero vector)

Since λ is a scalar, we can rearrange, $\vec{x} = \lambda A^{-1}\vec{x}$ and then multiply both sides by λ^{-1}. $\lambda^{-1}\vec{x} = \underbrace{\lambda^{-1}\lambda}_{1} A^{-1}\vec{x} \Rightarrow \lambda^{-1}\vec{x} = A^{-1}\vec{x}$

Since $\vec{x}$ is non zero, λ^{-1} is an eigenvalue of A^{-1}.

27.) Show that λ is an eigen value of A if and only if λ is an eigenvalue of A^T.

λ is an eigen value of $A \Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial soln.

$(A-\lambda I)\vec{x} = \vec{0}$ has a non trivial soln $\Leftrightarrow A - \lambda I$ is not invertible (IMT)

$A - \lambda I$ is not invertible $\Leftrightarrow (A-\lambda I)^T$ is not invertible. $\Leftrightarrow A^T - \lambda I$ is not invertible

$(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$ $\qquad \Leftrightarrow \lambda$ is an eignenvalue of A^T

31.) A is the standard matrix of T. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin. Without writing A, find an eigenvalue of A and describle the eigen space. A line through the origin is the set of all multiples of some nonzero vector $\vec{v}$. T reflects points across this line, so points on the line stay put, meaning $T(\vec{v}) = \vec{v}$ or in terms of the matrix $A\vec{v} = \vec{v}$. $\vec{v}$ is nonzero, so its an eigenvector corresponding to $\lambda = 1$.

The eigenspace corresponding to $\lambda = 1$ is Span$\{\vec{v}\}$.