4.4 # 2, 3, 5, 7, 10, 11, 13, 15, 17, 21, 23, 32

2.) Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\quad \vec{x} = -2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -26 \\ 1 \end{bmatrix}$

3.) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$

5.) Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to the given basis $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$

$\vec{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & 3 & -1 \\ -2 & -5 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

7.) $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right] \begin{array}{l} \\ R_1+R_2 \\ 3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right] \begin{array}{l} 3R_2+R_1 \\ \\ R_3/10 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

10.) Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$ $\quad P_{\mathcal{B}} = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3] = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$

11.) Use an inverse matrix to find $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$$

4.4

13.) The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\vec{p}(t) = 1+4t+7t^2$ relative to $\mathcal{B}$.

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2) = 1+4t+7t^2$$

$$c_1 + c_3 + (c_2+2c_3)t + (c_1+c_2+c_3)t^2 = 1+4t+7t^2$$

$c_1+c_3=1$

$c_2+2c_3=4$

$c_1+c_2+c_3=7$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{array}\right] -R_1+R_3 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 6 \end{array}\right] -R_2+R_3 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right] \begin{array}{l} R_3+R_2 \\ R_3/(-2) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] -R_3+R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad [\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

15.) True/False. $\mathcal{B}$ is a basis for a vector space V.

a.) If $\vec{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$ coordinate vector of $\vec{x}$ is in $\mathbb{R}^n$.

b.) If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}\vec{x}$ for $\vec{x}$ in V.

c.) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

a.) True b.) False c.) False

17.) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{array}\right] 3R_1+R_2 \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} R_2+R_1 \\ R_2/(-2) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$x_1 = 5+5x_3$

$x_2 = -2-x_3$

x_3 free

infinitely many answers to the Problem

one answer: $\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} 10 \\ -3 \\ 1 \end{bmatrix}$

4.4 continued

21.) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. (Hint: Multiplication by A should transform a vector $\vec{x}$ into $[\vec{x}]_{\mathcal{B}}$.)

Since $P_{\mathcal{B}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$, $P_{\mathcal{B}}^{-1}$ is the matrix we are looking for.

$P_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}$ so $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$.

23.) V is a vector space, $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis and $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is the coordinate mapping. Show that the coordinate mapping is one-to-one. (Hint: Suppose $[\vec{u}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}}$ for some $\vec{u}, \vec{w} \in V$ and show that $\vec{u} = \vec{w}$.)

$[\vec{u}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ So $\vec{u} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ and $\vec{w} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ therefore $\vec{u} = \vec{w}$. Thus the coordinate mapping is one-to-one.

32.) Let $\vec{p}_1(t) = 1 + t^2$, $\vec{p}_2(t) = t - 3t^2$, $\vec{p}_3(t) = 1 + t - 3t^2$

a.) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}^2$.

b.) Consider the basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ for $\mathbb{P}^2$. Find $\vec{q} \in \mathbb{P}^2$ s.t. $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

a.) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ This is row equivalent to I_3, so its invertible, and so the columns are linearly independent and span $\mathbb{R}^3$. Since $\mathbb{R}^3$ is isomorphic to $\mathbb{P}^2$,

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ forms a basis for $\mathbb{P}^2$.

b.) $\vec{q} = -1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$ $\vec{q}(t) = 1 + 3t - 10t^2$