Example 16.3.7 Find the derivative of

$$f(x)\left(\frac{x^4 - 2x^2 + 1}{x^2 - 1}\right)^5.$$

(Note: The notes I gave you last class has $x^4 + 2x^2 + 1$ in the numerator. Can you still simplify?)

$$x^4 - 2x^2 + 1 = (x^2 - 1)(x^2 - 1)$$

$$\left(\frac{x^2 - 2x + 1}{x^2 - 1}\right)^5 = (x^2 - 1)^5$$

$$\frac{d}{dx} f(x) = 5(x^2 - 1)^4 \, \underbrace{2x}_{g'(x)}$$

Groups 16.3.8 Does xe^{x^2} have a maximum?

$$\frac{d}{dx}\left(xe^{x^2}\right)$$

$$x \quad * \quad e^{x^2}$$

$$\frac{d}{dx}\left(xe^{x^2}\right) = (1)e^{x^2} + x\left(e^{x^2}\right)'$$

$$u = x^2 \qquad f(u) = e^u$$

$$u' = 2x \qquad f' = e^u$$

$$\left(e^{x^2}\right)' = 2xe^{x^2}$$

$$\frac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + x\left(2xe^{x^2}\right)$$

$$= e^{x^2}\left(1 + 2x^2\right) = 0$$

$$2x^2 = -1$$

$$x^2 = -\frac{1}{2}$$

Quiz ex

$$f(x) = xe^{2x}$$

$$f'(x) = e^{2x} + x\left(e^{2x}\right)'$$

$$= e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$$

$$f'(x) = 0 \iff e^{2x}(1+2x) = 0$$

$$\iff x = -\frac{1}{2}$$

Chapter 17

Implicit Differentiation and its Applications

17.1 The derivative of x^x

17.1. Goals

- finding the derivative of x^x
- logarithmic differentiation

Example 17.1.1

(a) What are the derivatives of x^n and b^x? Do either of these rules work for x^x when $x > 0$?

(b) Can we make it so that x is not a power? (hint: do you remember your log rules?)

$$(-0.5)^{-0.5} = \frac{1}{\sqrt{-0.5}}$$

a) is $\frac{df}{dx} = x x^{x-1}$? $\qquad x x^{x-1}$

is $\frac{df}{dx} = x^x \ln x$?

b) $y = x^x$

if we had $\ln(x^x) = x \ln x$

$$\boxed{\ln y = \ln(x^x) = x \ln x}$$

want $\frac{dy}{dx}$

Example 17.1.2 Find the derivative of $\overset{f(x)}{\cancel{y}} = x^x$.

$$\ln(y) = x \ln x$$

check

$$(\ln(y))' = (x \ln x)' = 1 + \ln x$$

$$y = y(x) \qquad \ln(f(x))$$

chain rule

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)} = \frac{y'}{y}$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

17.2 Logarithmic differentiation

17.2. Goals

- using logarithmic differentiation

Example 17.2.1 Find the tangent to the curve $f(x) = (x^2+1)^x$ at $x = 0$.

Reason #1 to use log differentiation

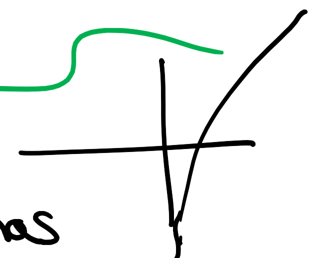
function$^{\text{function}}$

$f(x)^{g(x)}$

domain: all real #s

$y = (x^2+1)^x$

$\ln y = \ln\left((x^2+1)^x\right)$



log x has domain x>0

$\ln y = x \ln(x^2+1)$

$\frac{y'}{y} = x\left(\ln(x^2+1)\right)' + \ln(x^2+1)$

$\frac{y'}{y} = x\frac{2x}{x^2+1} + \ln(x^2+1)$

CR check

$y' = \left(x\frac{2x}{x^2+1} + \ln(x^2+1)\right)y$

$\boxed{y' = \left(x\frac{2x}{x^2+1} + \ln(x^2+1)\right)(x^2+1)^x}$

$y'(0) = (0+0)\cancel{(\sim)} = 0$

$y = mx+b \quad m = 0$

$y(0) = 1$

$\boxed{y = 1 \text{ is eqn for tangent}}$

$\frac{d}{dx}\left(\ln(f(x))\right) = f'(x)\frac{1}{f(x)}$

$= \frac{f'(x)}{f(x)}$

Example 17.2.2

(a) what is the domain of $(x-1)^{1-x^2}$? don't worry

$x \geq 1$

(b) on this domain, find $f'(2)$.

chain rule

$$\ln(y) = (1-x^2)\ln(x-1)$$ product

$$\frac{y'}{y} = -2x\ln(x-1) + (1-x^2)\frac{1}{x-1}$$

$$y' = (\text{mess})y$$

$$y' = \left(-2x\ln(x-1) + \frac{1-x^2}{x-1}\right)(x-1)^{1-x^2}$$

$$y'(2)\left(0 + \frac{-3}{1}\right) * 1 = -3$$

Question 17.2.3 What are the properties of logarithms that we know?

$$\ln(AB) = \ln(A) + \ln(B)$$
$$\ln(A/B) = \ln(A) - \ln(B)$$
$$\ln(A^n) = n\ln(A)$$
$$\ln(e^x) = x \Leftarrow \log_b(b^x) = x$$
$$\ln(e) = 1 \qquad e^{\ln x} = x$$

Example 17.2.4 Find the derivative of $y = 2x^{e^x}$

$$\ln\left(\sqrt{\frac{(x+1)x^5}{e^{x^2}(x-9)^{12}}}\right)$$

Example 17.2.5 Find the derivative of $y = \dfrac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$

$$\ln y = \ln\left(\frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}\right)$$

$$\ln y = \ln\left((x+3)^5(x^2+7x)^8\right) - \ln\left(x(x^2+5)^3\right)$$

$$= \ln\left((x+3)^5\right) + \ln\left((x^2+7x)^8\right) - \ln x - \ln\left((x^2+5)^3\right)$$

$$\ln y = 5\ln(x+3) + 8\ln(x^2+7x) - \ln x - 3\ln(x^2+5)$$

$$\boxed{\boxed{\frac{y'}{y}} = 5\frac{1}{x+3} + 8\frac{2x+7}{x^2+7x} - \frac{1}{x} - 3\frac{2x}{x^2+5}}$$

$$\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}$$

$$y' = (\text{mess}) * \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$$

17.2.1 Extra Examples

Example 17.2.6 Find the derivatives of

(a) $\dfrac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$

(b) $e^{2x}(x^2+3)^5(2x^2+1)^3$

(c) $\left(e^{x-1}\right)^{x+1}$

17.3 Implicit differentiation

17.3. Goals

- using the ideas of the previous section to find $\frac{dy}{dx}$ of implicitly defined functions

Spot the mistake 17.3.1 Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.2 Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.3 What kinds of information can we use to sketch the graph of a curve?

Example 17.3.4 Sketch a graph of the curve $y^2 = x^3 - x$ (don't worry about concavity)

Example 17.3.5 Find all points where the tangent to

$$x^3 + y^3 = 1$$

is horizontal or vertical

Example 17.3.7 Find the slope of the tangent to

$$x^3 + y^3 = 6xy$$

at the point $(3, 3)$.

Example 17.3.8 Find the absolute maximum and minimum y-values of the ellipse

$$2x^2 + 4xy + 3y^2 = 6.$$