1.4 #1, 4, 7, 9, 11, 13, 17, 19, 22, 23, 25, 31

1.) Compute the product using (a) the definition, (b) the row-vector rule. If a product is undefined, explain why.

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

This product is undefined because the number of columns in the matrix is not the same as the number of entries in the vector.

4.) $\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

a.) $1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

b.) $\begin{bmatrix} 1\cdot 1 + 3\cdot 2 + (-4)\cdot 1 \\ 3\cdot 1 + 2\cdot 2 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

7.) Write the vector equation $x_1\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$ as a matrix equation.

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

9.) Write the system as a vector equation and matrix equation.

$5x_1 + x_2 - 3x_3 = 8$

$2x_2 + 4x_3 = 0$

Vector equation:

$x_1\begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

matrix equation:

$\begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

11.) $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$ Write the augmented matrix for the linear system corresponding to $A\vec{x} = \vec{b}$. Then solve the system and write the solution as a vector.

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{array}\right] \begin{array}{l} \\ -R_1+R_2 \\ 3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{array}\right] \begin{array}{l} \\ R_2/2 \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -12 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3/(-12) \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} -3R_3+R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] -3R_2+R_1 \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] 4R_3+R_1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}}$$

1.4

13.) $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A? Why or why not.

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{array}\right] \begin{matrix} \\ 2R_1+R_2 \\ -3R_1+R_3 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{array}\right] R_2+R_3 \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

The linear system associated to this augmented matrix is consistent.

Therefore $\vec{u}$ is in Span $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix} \right\}$, meaning $\vec{u}$ is in the plane spanned by the columns of A.

17.) $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ How many rows of A contain a pivot position? Does the equation $A\vec{x} = \vec{b}$ have a solution for each $\vec{b}$ in $\mathbb{R}^4$

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \begin{matrix} \\ R_1+R_2 \\ \\ -2R_1+R_3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & 8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \begin{matrix} \\ \\ 2R_2+R_3 \\ 3R_2+R_4 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3/16 \\ R_4/5 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} -R_3+R_4 \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only three of the rows have a pivot position, not every row. Therefore $A\vec{x} = \vec{b}$ doesn't have a solution for every $\vec{b}$ in $\mathbb{R}^4$.

19.) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A above? Do the columns of A span $\mathbb{R}^4$?

Asking if each vector in $\mathbb{R}^4$ can be written as a linear combination of the columns of A is the same as asking if $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^4$. The answer is still no. Therefore the columns of A do not span $\mathbb{R}^4$.

22.) $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$ Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \quad \begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

There is a pivot position in every row, so yes $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$.

1.4 continued

23.) True / False

a.) The equation $A\vec{x} = \vec{b}$ is referred to as a vector equation.

b.) A vector $\vec{b}$ is a linear combination of the columns of a matrix A iff the equation $A\vec{x} = \vec{b}$ has at least one solution.

c.) The equation $A\vec{x} = \vec{b}$ is consistent if the augmented matrix $[A \ \vec{b}]$ has a pivot in every row.

d.) The first entry in the product $A\vec{x}$ is a sum of products.

e.) If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^m$.

f.) If A is an $m \times n$ matrix and if the equation $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ in $\mathbb{R}^m$, then A cannot have a pivot position in every row.

a.) False b.) True c.) False d.) True e.) True f.) True

25.) Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

$c_1 = -3, \ c_2 = -1, \ c_3 = 2$

31.) Let A be a 3×2 matrix. Explain why the equation $A\vec{x} = \vec{b}$ cannot be consistent for all $\vec{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$ A will not have a pivot position in every row, so $A\vec{x} = \vec{b}$ cannot be consistent for all $\vec{b}$ in $\mathbb{R}^3$. This is true whenever A has more rows than columns.