6.3 # 1, 6, 7, 9, 11, 13, 17, 21, 23, 24

1.) Assume $\{\vec{u}_1, \ldots, \vec{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$ Write $\vec{x}$ as a sum of two vectors, one in Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, the other in Span$\{\vec{u}_4\}$.

The vector in Span$\{\vec{u}_4\}$ is $\frac{\vec{x} \cdot \vec{u}_4}{\vec{u}_4 \cdot \vec{u}_4} u_4 = \frac{50+24-2}{25+9+2} \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$

Since $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$, we can solve for $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$.

$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix}$ is in Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

So $\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$ where $\begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} \in$ Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} \in$ Span$\{\vec{u}_4\}$.

6.) Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\vec{y}$ onto Span$\{\vec{u}_1, \vec{u}_2\}$.

$\vec{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u}_1 \cdot \vec{u}_2 = -4(0) + -1(1) + 1(1) = 0$ So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{y \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{-24-4+1}{16+1+1} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{0+4+1}{0+1+1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$

$$\boxed{\hat{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}}$$

7.) Let W be the subspace spanned by the $\vec{u}$'s and write $\vec{y}$ as a sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \begin{bmatrix}1\\3\\5\end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = 5+3-8=0$ So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set. Since W is also spanned by the $\vec{u}$'s, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis of W.

We then use the Orthogonal Decomposition Theorem. $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$ is in W and $\vec{z} = \vec{y} - \hat{y}$ is in $W^\perp$.

$$\hat{y} = \frac{1+9-10}{1+9+4}\begin{bmatrix}1\\3\\-2\end{bmatrix} + \frac{5+3+20}{25+1+16}\begin{bmatrix}5\\1\\4\end{bmatrix} = 0 + \frac{2}{3}\begin{bmatrix}5\\1\\4\end{bmatrix} = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} \qquad \vec{z} = \begin{bmatrix}1\\3\\5\end{bmatrix} - \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} = \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}.$$

$$\boxed{\vec{y} = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} + \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}}$$

9.) $$\vec{y} = \begin{bmatrix}4\\3\\3\\-1\end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}-1\\3\\1\\-2\end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = -1+3+0-2=0$

$\vec{u}_1 \cdot \vec{u}_3 = -1+0+0+1=0$

$\vec{u}_2 \cdot \vec{u}_3 = 1+0+1-2=0$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set that spans W, so it is an orthogonal basis of W. We again use the Orthogonal Decomposition Theorem.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3 = \frac{4+3-1}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{-4+9+3+2}{1+9+1+4}\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} + \frac{-4+3-1}{3}\begin{bmatrix}-1\\0\\1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}2\\2\\0\\2\end{bmatrix} + \begin{bmatrix}-2/3\\2\\2/3\\-4/3\end{bmatrix} + \begin{bmatrix}2/3\\0\\-2/3\\-2/3\end{bmatrix} = \begin{bmatrix}2\\4\\0\\0\end{bmatrix}, \quad \vec{z} = \begin{bmatrix}4\\3\\3\\-1\end{bmatrix} - \begin{bmatrix}2\\4\\0\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\3\\-1\end{bmatrix} \qquad \boxed{\vec{y} = \begin{bmatrix}2\\4\\0\\0\end{bmatrix} + \begin{bmatrix}2\\-1\\3\\-1\end{bmatrix}}$$

11.) Find the closest point to $\vec{y}$ in the subspace W spanned by $\vec{v}_1, \vec{v}_2$.

$$\vec{y} = \begin{bmatrix}3\\1\\5\\1\end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_2 = 3-1-1-1=0$, So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set that spans W, so it is an orthogonal basis.

By the Best approximation theorem, $\hat{y}$ is the closest point in W to $\vec{y}$.

$$\hat{y} = \frac{9+1-5+1}{9+3}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{3-1+5-1}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$$

$\boxed{\hat{y} = \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix} \text{ is the closest point in } W \text{ to } \vec{y}}$

6.3 continued

13.) Find the best approximation to $\vec{z}$ by vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$.

$$\vec{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_2 = 2-1-1=0$ So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set. The set of vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$ is the Span$\{\vec{v}_1, \vec{v}_2\}$. $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for Span$\{\vec{v}_1, \vec{v}_2\}$ and the best approximation for $\vec{z}$ in this space is

$$\hat{z} = \frac{\vec{z}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{z}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \frac{6+7-6+3}{4+2+9}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} + \frac{3-7-3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} - \frac{7}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

17.) Let $\vec{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

a.) Let $U = [\vec{u}_1 \ \vec{u}_2]$. Compute U^TU and UU^T.

b.) Compute $\text{proj}_W\vec{y}$ and $(UU^T)\vec{y}$.

a.)
$$U^TU = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/9+1/9+4/9 & -4/9+2/9+4/9 \\ -4/9+2/9+2/9 & 4/9+4/9+1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$UU^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/9+4/9 & 2/9-4/9 & 4/9-2/9 \\ 2/9-4/9 & 1/9+4/9 & 2/9+2/9 \\ 4/9-2/9 & 2/9+2/9 & 4/9+1/9 \end{bmatrix} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$

b.)
$\vec{u}_1 \cdot \vec{u}_2 = -4/9 + 2/9 + 2/9 = 0$

$\vec{u}_1 \cdot \vec{u}_1 = 4/9 + 1/9 + 4/9 = 1$

$\vec{u}_2 \cdot \vec{u}_2 = 4/9 + 4/9 + 1/9 = 1$

$\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for W, so by thm 10 (pg 351 Lay) $\text{proj}_W\vec{y} = (UU^T)\vec{y}$

$$\text{Proj}_W\vec{y} = (UU^T)\vec{y} = \frac{1}{9}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 32-16+2 \\ -8+40+4 \\ 8+32+5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

21.) True / False. All vectors and subspaces are in $\mathbb{R}^n$.

a.) If $\vec{z}$ is orthogonal to $\vec{u}_1$ and to $\vec{u}_2$ and if $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ then $\vec{z}$ must be in $W^{\perp}$.

b.) For each $\vec{y}$ and each subspace W, the vector $\vec{y} - \text{proj}_W \vec{y}$ is orthogonal to W.

c.) The orthogonal projection $\hat{y}$ of $\vec{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{y}$.

d.) If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

e.) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space U.

a.) TRUE b.) TRUE c.) FALSE d.) TRUE e.) TRUE

23.) Let A be an $m \times n$ matrix. Prove that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written in the form $\vec{x} = \vec{p} + \vec{u}$, where $\vec{p}$ is in Row A and $\vec{u}$ is in Nul A. Also, show that if the equation $A\vec{x} = \vec{b}$ is consistent, then there is a unique $\vec{p}$ in Row A s.t. $A\vec{p} = \vec{b}$.

By the orthogonal decomposition thm any $\vec{x} \in \mathbb{R}^n$ can be written as $\vec{x} = \vec{p} + \vec{u}$ where $\vec{p} \in \text{Row } A$ and $\vec{u} \in (\text{Row } A)^T$. By thm 3 in Lay 6.1, $(\text{Row } A)^T = \text{Nul } A$.

Next, if $A\vec{x} = \vec{b}$ is consistent, then let $\vec{x}$ be a solution and $\vec{x} = \vec{p} + \vec{u}$ where $\vec{p} \in \text{Row} A$

Then $A\vec{p} = A(\vec{x} - \vec{u}) = A\vec{x} - A\vec{u} = \vec{b} - \vec{0} = \vec{b}$. (Since $\vec{u} \in \text{Nul } A$, $A\vec{u} = \vec{0}\}$ Thus $A\vec{p} = \vec{b}$ has at least one soln. Suppose there is more than one soln, that is, for $\vec{p} \neq \vec{p}'$ $A\vec{p} = \vec{b}$ and $A\vec{p}' = \vec{b}$. Then $A\vec{p} = A\vec{p}'$ implies $A\vec{p} - A\vec{p}' = A(\vec{p} - \vec{p}') = \vec{0}$ then $\vec{p} - \vec{p}' \in \text{Nul } A = (\text{Row } A)^T$.

$\vec{p} = \vec{p}' + (\vec{p} - \vec{p}')$ satisfies the orthogonal decomposition theorem, but so does $\vec{p} = \vec{p} + \vec{0}$. Since the orthogonal decomposition theorem gives a unique vector, $\vec{p}' = \vec{p}$. Therefore $\vec{p}$ is unique.

6.3 Continued

24.) Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\vec{w}_1, \dots \vec{w}_p\}$ and let $\{\vec{v}_1, \dots \vec{v}_q\}$ be an orthogonal basis for $W^\perp$.

a.) Explain why $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$ is an orthogonal set.

Since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is an orthogonal basis, by definition these vectors are pairwise orthogonal. Similarly for $\{\vec{v}_1, \dots \vec{v}_q\}$. Then for any $\vec{v}_j \in \{\vec{v}_1, \dots \vec{v}_q\}$ and any $\vec{w}_i$ in $\{\vec{w}_1, \dots, \vec{w}_p\}$, $\vec{w}_i \in W$ and $\vec{v}_j \in W^\perp$, so $\vec{w}_i \cdot \vec{v}_j = \vec{0}$. Therefore $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots, \vec{v}_q\}$ is orthogonal.

b.) Explain why the set in part (a) spans $\mathbb{R}^n$.

For any $\vec{y} \in \mathbb{R}^n$, $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} \in W$ and $\vec{z} \in W^\perp$ by the orthogonal decomposistion theorem. Since $\hat{y} \in W$, $\hat{y} = a_1\vec{w}_1 + \dots + a_p\vec{w}_p$ for some scalars a_i since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis for W. Similarly $\vec{z} = b_1\vec{v}_1 + \dots + b_q\vec{v}_q$ for some scalars b_j. Thus $\vec{y} = a_1\vec{w}_1 + \dots + a_p\vec{w}_p + b_1\vec{v}_1 + \dots b_q\vec{v}_q$. Thus any $\vec{y} \in \mathbb{R}^n$ can be written as a linear combination of $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$. So this set spans $\mathbb{R}^n$.

c.) Show that $\dim W + \dim W^\perp = n$.

Part (a) tells us $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$ is linearly independent, and part (b) tells us it spans $\mathbb{R}^n$. Therefore this set is a basis for $\mathbb{R}^n$.

$\dim \mathbb{R}^n = p + q$, but we know $\dim \mathbb{R}^n = n$ and $\dim W = p$ since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis and $\dim W^\perp = q$ since $\{\vec{v}_1, \dots, \vec{v}_q\}$ is a basis.

Hence $n = \dim W + \dim W^\perp$