1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 4 & 2 & 3 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in $(\mathrm{Col}(A))^\perp$?

$x_1 = 2 - 1.5x_4$
$x_2 = -8 + 3.5x_4 \Rightarrow$ yes
$x_3 = 3$
x_4 free

2. Let $W = \mathrm{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find a vector $\mathbf{u}$ in $W^\perp$.

(b) Does $\{\mathbf{u}, \mathbf{w}_1, \mathbf{w}_2\}$ form a basis for $\mathbb{R}^3$?

a) I suppose I never said "nonzero"

solve $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

so $x_1 = 0$, $x_2 = x_3$, x_3 free $\quad \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$ for any t

b) yes

3. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ (that is, find the projection of $\mathbf{u}$ onto $\mathbf{v}$).

(b) Use (a) to find a vector orthogonal to $\mathbf{v}$.

a) $\mathrm{proj}_v u = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \dfrac{-1+2}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \\ 1/3 \end{bmatrix}$

b) $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ 1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -7/6 \\ -1/6 \\ 2/3 \end{bmatrix}$

4. Consider the point $P = (1, 2, 3)$ and the plane $W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the closest point in W to $\mathbf{u}$.

(b) What is the distance between $\mathbf{u}$ and W?

(c) Check your answer using my "projection calculator" (link at end of problem set).

1) Solve $A^T A \hat{x} = A^T \vec{b}$ to find weights

2)

a) $A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad A^T \vec{b} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

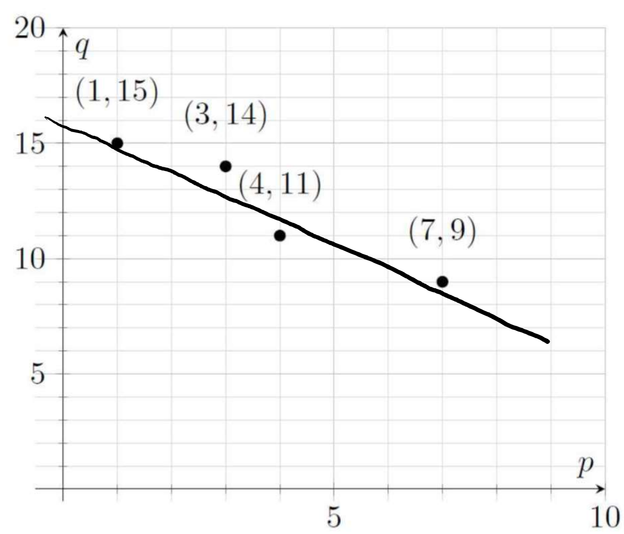
$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \end{array}\right] \quad \hat{x}_1 = 0, \ \hat{x}_2 = 1/2$

$\Rightarrow$ closest point in W is $\frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$

b) $\mathrm{dist}(u, W) = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2}$

Least Squares

5. A coffee company would like to better understand how the demand of coffee q (in thousands of lbs) is related to the unit price of coffee p (in \$/lb). So far, all they have is some data (illustrated below).



For economic reasons, they expect p and q to satisfy a linear relationship, i.e.

$$q = cp + d;$$

for some constants c and d. They'd like to find the line that is "closest" to their data to try to better understand this relationship.

(a) Use the data to generate a system of linear equations in c and d (there will be one equation for each data point).

(b) Is the system of equations in (a) consistent?

(c) Find the least-squares solution to the system of equations from (a).

(d) Add the graph of the line corresponding to the least-squares solution to the illustration above. In what sense is this the "closest" line to the data?

(e) Based on your model, what will be the demand if the unit price of coffee is \$10/lb? What should the company set the price of coffee to be if they want to maximize revenue?

$(1,15), (3,14), (4,11), (7,9)$

a) $\begin{cases} 15 = c + d \\ 14 = 3c + d \\ 11 = 4c + d \\ 9 = 7c + d \end{cases}$

b) NO

c) $A = \begin{bmatrix} 1 & 1 & 15 \\ 3 & 1 & 14 \\ 4 & 1 & 11 \\ 7 & 1 & 9 \end{bmatrix}$

$A^TA = \begin{bmatrix} 75 & 15 \\ 15 & 4 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 164 \\ 49 \end{bmatrix}$

$\hat{x} \approx \begin{bmatrix} -1.05 \\ 16.2 \end{bmatrix}$

$y = -1.05x + 16.2$

d) $q \approx -10.53 + 16.2 = 5.67$

6. Suppose you run an experiment and collect the following data.

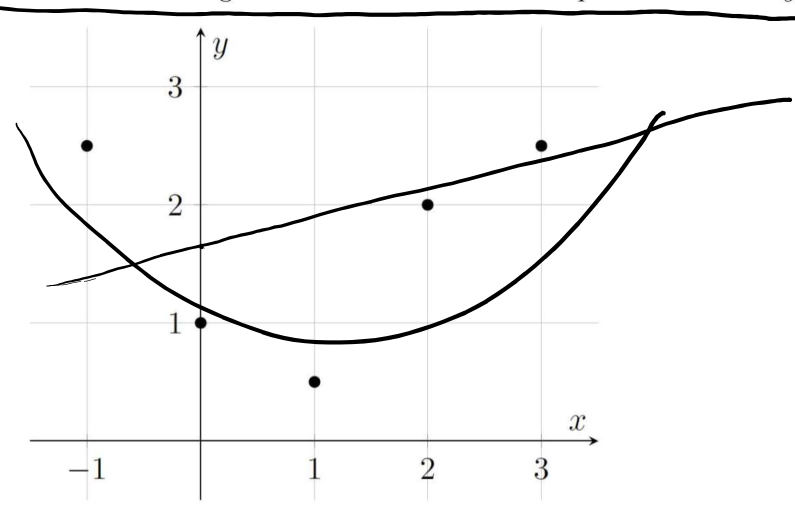
x	y
-1	2.5
0	1
1	0.5
2	2
3	2.5

$$\left[\begin{array}{cc|c} -1 & 1 & 2.5 \\ 0 & 1 & 1 \\ 1 & 1 & 0.5 \\ 2 & 1 & 2 \\ 3 & 1 & 2.5 \end{array}\right]$$

(a) You expect x and y to satisfy a linear relationship, $y = cx + d$. Use the data to generate a system of equations in c and d, and then find the least-squares solution to this system. $y = 0.1x + 1.6$

(b) ~~Add the graph of the line corresponding to the least-squares solution to the data below.~~ Do you think this linear model is a good model for the relationship between x and y?

any answer fine



7. After reassessing the situation in (2), you decide that it may make more sense for y to be a quadratic function of x, i.e.

$$y = a + bx + cx^2$$

for some numbers a, b, and c.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2.5 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0.5 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 2.5 \end{array}\right]$$

(a) Use the data to generate a system of equations in a, b, and c. Note that this system is *linear* in a, b, and c.

(b) Find the least-squares solution to the system from (a). $y = 1.17 - 0.76x + 0.43x^2$

(c) Add the graph of your least-squares quadratic to the illustration in 2(b). Does this look like a better fit than the least-squares line?

(d) no grade, just comment For the system in 2(a) and the system in 3(a), find each squared distance between your approximation $A\mathbf{x}$ and b, (i.e. find $\|A\mathbf{x} - \mathbf{b}\|$ in both cases). Which is smaller? Does this agree with your answer to 3(c)?

(e) Use the better of the two models to predict the value of y when $x = 6$.

12.09

$1.17 - 4.56 + 15.48$

8. You've been given a bunch of data relating the demand of a good q (in thousands of units) to its unit price p (in dollars per unit) shown below

p	1.0	1.5	4.5	4.8	5.3	6.5	7.7	8.0	9.5	10.5
q	29	28.5	23.5	23.0	22.1	19.7	17.1	16.9	14.5	12.8

You expect that p and q will satisfy a linear relationship, i.e.

$$q = cp + d$$

for some numbers c and d. You want to build a linear model for the relationship between p and q in order to make predictions for the demand associated with prices outside of your data set. In order to assess a model's ability to make predictions, we first *randomly* divide the data into two sub-collections, the "training set" and the "test set":

training

p	1.5	4.8	5.3	6.5	10.5
q	28.5	23.0	22.1	19.7	12.8

test

p	1.0	4.5	7.7	8.0	9.5
q	29	23.5	17.1	16.9	14.5

We then build a model using only the data from the training set, and use the unused data from the test set to assess the quality of the model's predictions.

(a) Use the data from the training set to generate a system of equations in c and d. Write this system as a matrix equation $A_1\mathbf{x} = \mathbf{b}_1$, where $\mathbf{x} = (c, d)$. Then, find the least squares solution $\widehat{\mathbf{x}}$ to this system of equations. $y = -1.75x + 31.25$

(b) Use the data from the test set to generate a system of equations in c and d. Write this system as a matrix equation $A_2\mathbf{x} = \mathbf{b}_2$.

(c) On average, how well does the model from (a) predict the demands associated with the prices in the test set? That is, what is the mean squared error $\frac{1}{5}\|A_2\widehat{\mathbf{x}} - \mathbf{b}_2\|$, where $\widehat{\mathbf{x}}$ is the least squares solution you found in (a)?

$$\left[\begin{array}{cc|c} 1 & 1 & 29 \\ 1 & 4.5 & 23.5 \\ 1 & 7.7 & 17.1 \\ 1 & 8.0 & 16.8 \\ 1 & 9.5 & 14.5 \end{array}\right]$$

$$\frac{1}{5}\left\| \begin{bmatrix} 1 & 1 \\ 1 & 4.5 \\ 1 & 7.5 \\ 1 & 8.0 \\ 1 & 9.5 \end{bmatrix} * \begin{bmatrix} 31.25 \\ -1.75 \end{bmatrix} - \begin{bmatrix} 29 \\ 23.5 \\ 17.1 \\ 16.9 \\ 14.5 \end{bmatrix} \right\| \approx \frac{1}{5} 5 = 1$$