MATH 118, Spring 2020, Multivariable Calculus
Lecture Notes

Taken in part from
Calculus, 6e,
McCallum et al.
and *Multivariable Calculus, 8e,*
James Stewart

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Yale University, 2020

NOTE: I will update these notes as often as I can with the topics and examples (which will be worked out by hand in a separate document) we cover in class.

Contents

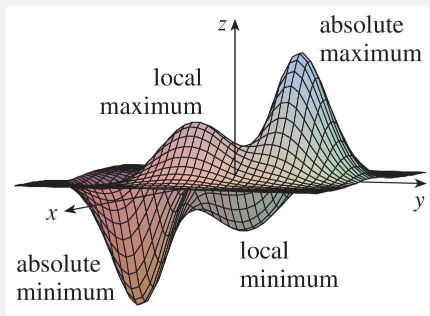
Lecture 8 Local extrema

Stewart 14.1, McCallum 12.3, 12.5

- second derivative test
- extrema from contour maps

Question 8.1. In single variable calculus: how did we look for local (relative) extrema?

Definition 8.2. $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points near (a, b). $f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all points near (a, b).

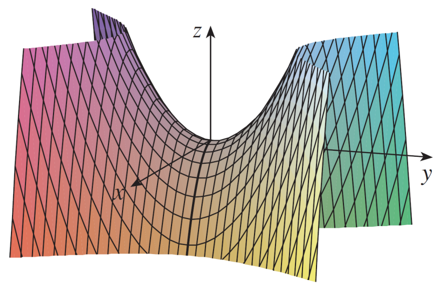


Theorem 8.3. If f has a local extremum at (a, b) and f_x and f_y exist there, then $f_x(a, b) = f_y(a, b) = 0$.

Definition 8.4. The point $P(a, b)$ is called a **critical point of** f if $f_x(a, b) = f_y(a, b) = 0$.

Example 8.5. Find the critical point(s) of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Example 8.6. Find the critical point of $f(x, y) = y^2 - x^2$. Is this a relative extremum?

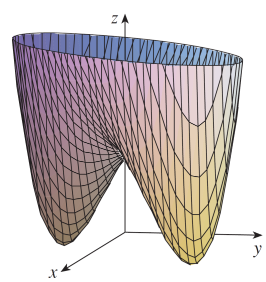


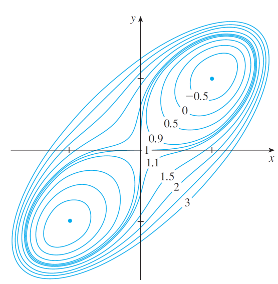
Theorem 8.7. If (a, b) is a critical point of $f(x, y)$ (i.e. if $\nabla f(a, b) = 0$ or is undefined, then let

$$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - \big(f_{xy}(a,b)\big)^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local minimum
(c) If $D < 0$, then $f(a, b)$ is a saddle point

Example 8.8. Find the local extrema and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

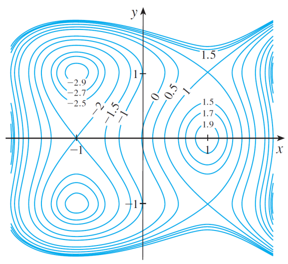




Poll 8.9. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

How many critical points does f have?

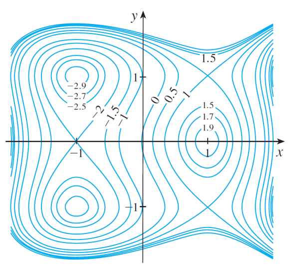


(a) 0
(b) 3
(c) 6
(d) I don't know.

Poll 8.10. The contour map below is for the surface

$$f(x,y) = 3x - x^3 - 2y^2 + y^4.$$

Classify the extrema of f.

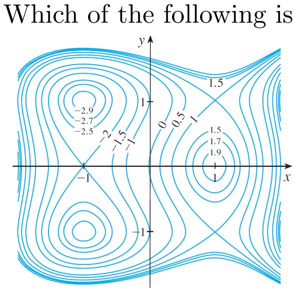


(a) f has three local maximums and no local minimums
(b) f has two local maximums and one local minimum
(c) f has three local minimums and no local maximum
(d) f has two local minimums and one local maximum

Poll 8.11. The contour map below is for the surface

$$f(x,y) = 3x - x^3 - 2y^2 + y^4.$$

Which of the following is ***not*** a saddle point of f?

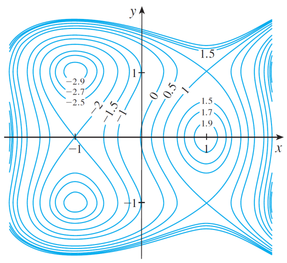


(a) $(-1,0)$
(b) $(1,1)$
(c) $(-1,1)$
(d) $(1,-1)$

Poll 8.12. The contour map below is for the surface

$$f(x,y) = 3x - x^3 - 2y^2 + y^4.$$

Which is a reasonable guess for the absolute minimum of f in this window?

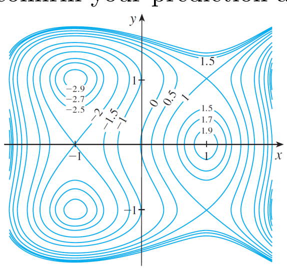


(a) $f(x,y) = -3$
(b) $f(x,y) = 1$
(c) $f(x,y) = -3.4$
(d) $f(x,y) = -1.6$

Example 8.13. The contour map below is for the surface

$$f(x,y) = 3x - x^3 - 2y^2 + y^4.$$

Use the map to classify the relative extrema and find saddle points of f, then confirm your prediction analytically.



Lecture 9 Global extrema

- open/closed/bounded/unbounded
- parameterizing the boundary of a surface
- finding global extrema

Definition 9.1. For a function f on a set D, $f(a,b)$ is an absolute (or global) maximum if $f(a,b) \geq f(x,y)$ for all (x,y) in D, and an absolute (or global) minimum if $f(a,b) \leq f(x,y)$ for all (x,y) in D.

Theorem 9.2. If f is continuous on a closed and bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum and absolute minimum value on D.

Theorem 9.3. To find the extreme values of a continuous function $f(x,y)$ on a closed and bounded set D

(a) Find the values of f at its critical points in D

(b) Find the extreme values of f on the boundary of D

(c) The biggest value from (1) and (2) is the absolute maximum, the smallest is the absolute minimum

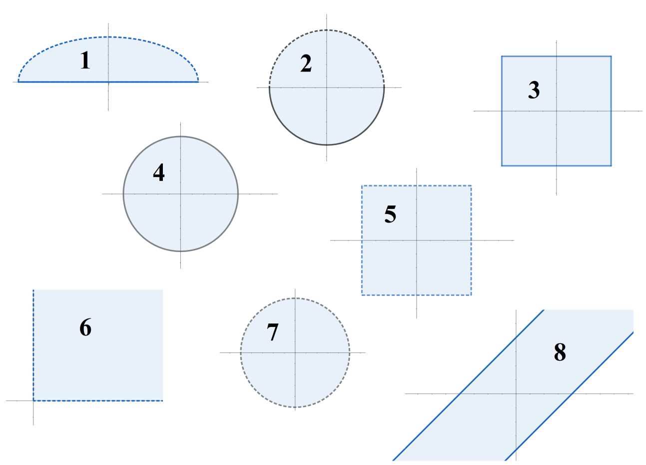
Poll 9.4. Decide if each of the regions below is

(a) closed but not bounded

(b) bounded but not closed

(c) closed but not bounded

(d) bounded but not closed



Example 9.5. Find the global extrema of

$$f(x,y) = x^3 - 3x - y^3 + 12y$$

on the trapezoid with vertices

$$(-2,-2), \quad (2,2), \quad (2,3), \quad \text{and } (-2,3).$$

Find a graph at `https://ggbm.at/ebkugczt`!

Groups 9.6. Find the global extrema of

$$f(x,y) = x^2 - 2xy + 2y$$

on the set

$$D = \{(x,y)|0 \leq x \leq 3, 0 \leq y \leq 2\}.$$

Find a graph at `https://ggbm.at/jpexbayj`!

Think, Pair, Share 9.7. How would you determine the function values of

$$f(x,y) = 2x^3 + y^4$$

on the boundary of the set

$$D = \{(x,y)|x^2 + y^2 \leq 1\}?$$

Find a graph at `https://ggbm.at/ttaxru3v`!