# 5.3 # 1, 4, 5, 9, 11, 15, 17, 21, 24, 26

**1.)** Let $A = PDP^{-1}$ and compute $A^4$. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^4 = (PDP^{-1})^4 = PD\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}DP^{-1} = PD^4P^{-1}, \quad D^4 = \begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix}$$ since D is diagonal

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

**4.)** $A = PDP^{-1}$ is given. Compute $A^k$ where $k$ is an arbitrary positive integer.

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3(-3)^k + 4(-2)^k & 6(-3)^k - 6(-2)^k \\ -2(-3)^k + 2(-2)^k & 4(-3)^k - 3(-2)^k \end{bmatrix}$$

**5.)** $A = PDP^{-1}$ is given. Use the diagonalization thm to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

The diagonal entries of D are eigenvalues corresponding to the eigenvectors given by the columns of P respectively. So $\lambda = 2$ corresponds to $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ and $\lambda = 3$ corresponds to both $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. So $\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for the eigenspace for $\lambda = 2$ and $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace for $\lambda = 3$.

**9.)** Diagonalize the matrix, if possible.

$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ $\det(A - \lambda I) = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$ So $\lambda = 3$

$A - 3I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ $(A - 3I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} +1 & +1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, $x_2$ free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ So $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace, but A is not diagonalizable because A has only 1 linearly independent eigenvector. It needs exactly 2 because A is 2x2.

11.) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$ For $\lambda = -1$ $(A+I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2 - x_3$, $x_2, x_3$ free

$\lambda = -1, 5$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.

For $\lambda = 5$ $(A-5I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} -5 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$

So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for the eigenspace

I took a multiple of $\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$ to get rid of the fractions

Since $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ are linearly independent, (check this if not obvious)

we can let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

15.) $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ For $\lambda = 0$ $(A+0I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = x_3$, $x_2 = -x_3$, $x_3$ free $\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 0, 1$

For $\lambda = 1$ $(A-I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2 - x_3$, $x_2, x_3$ free $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17.) $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$ This matrix is triangular, $\lambda = 2$.

$(A-2I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$, $x_3$ free $\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the 1-dimensional eigenspace. Since we can find at most 1 lin. indep eigenvector (and not exactly 3), A is not diagonalizable.

# 5.3 continued

**21.)** True/False. $A, B, P, D$ are all $n \times n$ matrices.

a.) $A$ is Diagonalizable if $A = PDP^{-1}$ for some matrix $D$ and some invertible matrix $P$.

b.) If $\mathbb{R}^n$ has a basis of eigenvectors of $A$, then $A$ is diagonalizable.

c.) $A$ is diagonalizable iff $A$ has $n$ eigenvalues, counting multiplicities.

d.) If $A$ is diagonalizable, then $A$ is invertible.

a.) False (only if $D$ is diagonal) b.) True c.) False d.) False.

**24.)** $A$ is a $3 \times 3$ matrix with two eigen values. Each eigenspace is one-dimensional. Is $A$ diagonalizable? Why?

No, if each eigenspace is one-dimensional, we can have at most 2 Linearly independent eigenvectors. A $3 \times 3$ matrix needs exactly 3 linearly independent eigenvectors to be diagonalizable.

**26.)** $A$ is a $7 \times 7$ matrix with three eigenvalues. One eigenspace is two dimensional, and one of the other eigen spaces is three (3) dimensional. is it possible that $A$ is not diagonalizable? Justify.

Yes, if the third eigenspace is only 1-dimensional then we could have at most $2+3+1=6$ linearly indep. eigenvectors and $A$ would not be diagonalizable.