6.2 # 3, 6, 8, 9, 11, 14, 20, 21, 23, 26, 27, 28, 29

3.) Determine if the set of vectors is orthogonal.

$\left\{ \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} = -12+21-9=0$ $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = 6-7+1=0$

$\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -18-3-9=-30 \neq 0$ The set is not orthogonal.

6.) $\left\{ \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\}$ $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} = -20-4+0+24=0$ $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = 15-12+0-3=0$

$\begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = -12+3-15-8=-32 \neq 0$ The set is not orthogonal

8.) Show that $\{u_1, u_2\}$ or $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\vec{x}$ as a linear combination of the $\vec{u}$'s.

$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ $\vec{u}_1 \cdot \vec{u}_2 = -6+6=0$, So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

Since $\vec{u}_1, \vec{u}_2$ are non-zero, the set is linearly independent by Thm 4. Since $\dim \mathbb{R}^2 = 2$ and our set is lin. indep consisting of exactly 2 vectors, it is a basis of $\mathbb{R}^2$. Thus we have an orthogonal basis.

$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2$ where $c_1 = \dfrac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = -\dfrac{3}{2}$ and $c_2 = \dfrac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{3}{4}$

$\vec{x} = \frac{-3}{2}\vec{u}_1 + \frac{3}{4}\vec{u}_2$

9.) Same directions as #8.

$u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix}$, $u_3 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$, $\vec{x} = \begin{bmatrix}8\\-4\\3\end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = -1+1=0$

$\vec{u}_1 \cdot \vec{u}_3 = 2-2=0$

$\vec{u}_2 \cdot \vec{u}_3 = -2+4-2=0$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

Since the set is orthogonal and doesn't contain $\vec{0}$, it is lin. indep.

Since $\dim \mathbb{R}^3 = 3$, the set is a basis for $\mathbb{R}^3$.

$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$ where $c_1 = \dfrac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \dfrac{8+3}{1+1} = \dfrac{11}{2}$

$$\boxed{\vec{x} = \tfrac{11}{2}\vec{u}_1 - \tfrac{7}{6}\vec{u}_2 + \tfrac{2}{3}\vec{u}_3}$$

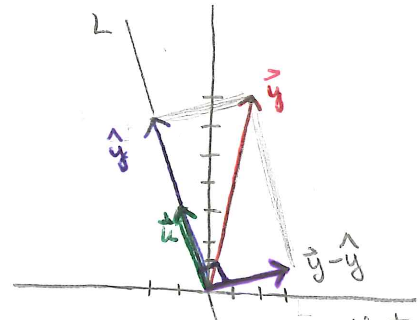
$c_2 = \dfrac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{-8-16+3}{1+16+1} = \dfrac{-21}{18} = \dfrac{-7}{6}$

$c_3 = \dfrac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \dfrac{16-4-6}{4+1+4} = \dfrac{6}{9} = 2/3$

11.) Compute the orthogonal projection of $\vec{y} = \begin{bmatrix}1\\7\end{bmatrix}$ onto the line through $\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.

L is the line through $\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.

$\hat{y} = \text{Proj}_L \vec{y} = \left(\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\dfrac{-1+21}{1+9}\right)\begin{bmatrix}-1\\3\end{bmatrix} = 2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$



Notice that $\vec{y}-\hat{y}$ is orthogonal to $\vec{u}$

* I copied the problem wrong, so this isn't exactly # 11, but the solution is correct for the problem I wrote.

14.) Let $\vec{y} = \begin{bmatrix}2\\6\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}7\\1\end{bmatrix}$. Write $\vec{y}$ as the sum of a vector in Span$\{\vec{u}\}$ and a vector orthogonal to $\vec{u}$.

$\hat{y} = \text{Proj}_L \vec{y} = \left(\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\dfrac{14+6}{49+1}\right)\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}14/5\\2/5\end{bmatrix}$ $\quad \hat{y}$ is in Span$\{\vec{u}\}$ ($\hat{y}$ is a scalar multiple of $\vec{u}$)

$\vec{y}-\hat{y} = \begin{bmatrix}2\\6\end{bmatrix} - \begin{bmatrix}14/5\\2/5\end{bmatrix} = \begin{bmatrix}-4/5\\28/5\end{bmatrix}$ This is orthogonal to $\vec{u}$ because

$\begin{bmatrix}-4/5\\28/5\end{bmatrix}\cdot\begin{bmatrix}7\\1\end{bmatrix} = \dfrac{-28}{5} + \dfrac{28}{5} = 0$

$$\boxed{\vec{y} = \begin{bmatrix}14/5\\2/5\end{bmatrix} + \begin{bmatrix}-4/5\\28/5\end{bmatrix}}$$

6.2 Continued

20.) Determine if the set is orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$\left\{ \underset{\vec{v}}{\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}}, \underset{\vec{u}}{\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}} \right\}$$

$\vec{v} \cdot \vec{u} = -\frac{2}{9} + \frac{2}{9} + 0 = 0$ So, these vectors are orthogonal.

$\|\vec{v}\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$ So, $\vec{v}$ is a unit vector

$\|\vec{u}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + 0} = \sqrt{5}/3$ $\vec{u}$ is not a unit vector

The set given is not orthonormal, but we can normalize $\vec{u}$.

$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$ is orthonormal.

21.) $$\left\{ \underset{\vec{v}}{\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}}, \underset{\vec{u}}{\begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}}, \underset{\vec{x}}{\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \right\}$$

$\vec{v} \cdot \vec{u} = \frac{3}{10} + \frac{-3}{20} + \frac{-3}{20} = 0$

$\vec{v} \cdot \vec{x} = 0 + \frac{-3}{\sqrt{2}\sqrt{20}} + \frac{3}{\sqrt{2}\sqrt{20}} = 0$

$\vec{u} \cdot \vec{x} = 0 + \frac{1}{\sqrt{2}\sqrt{20}} + \frac{-1}{\sqrt{2}\sqrt{20}} = 0$

The set is orthogonal.

$\|\vec{v}\| = \sqrt{\frac{1}{10} + \frac{9}{20} + \frac{9}{20}} = 1$ $\|\vec{u}\| = \sqrt{\frac{9}{10} + \frac{1}{20} + \frac{1}{20}} = 1$ $\|\vec{x}\| = \sqrt{0 + \frac{1}{2} + \frac{1}{2}} = 1$

The set is orthonormal.

23.) True/False. All vectors are in $\mathbb{R}^n$.

a) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b.) If $\vec{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c.) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d.) A matrix with orthonormal columns is an orthogonal matrix.

e.) If L is a line through $\vec{0}$ and if $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto L, then $\|\hat{y}\|$ gives the distance from $\hat{y}$ to L.

a.) True b.) True c.) False d.) False e.) False

26.) Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

A set of nonzero orthogonal vectors in $\mathbb{R}^n$ is a linearly indep. set by Thm 4. Since W is spanned by these vectors, we have that the Set is a basis for W. Thus $\dim W = n$. The only n-dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, so $W = \mathbb{R}^n$.

27.) Let U be a square matrix with orthonormal columns. Explain why U is invertible.

If the columns of U are orthonormal, they are orthogonal and any orthogonal subset of $\mathbb{R}^n$ is linearly independent (Thm 4). Thus the columns of U are linearly independent, so U is invertible by the IMT.

28.) Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

An orthogonal matrix is a square, invertible matrix U such that $U^{-1} = U^T$. Since U is invertible, $I = UU^{-1} = UU^T = (U^T)^T U^T$. Since $(U^T)^T U^T = I$, by thm 6, U^T has orthonormal columns, which means U has orthonormal rows.

Since U^T is invertible, by the IMT, its columns span $\mathbb{R}^n$. Thus the rows of U span $\mathbb{R}^n$. Hence the rows of U are an orthonormal basis for $\mathbb{R}^n$.

29.) Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix.

U and V are invertible, so UV is invertible and $(UV)^{-1} = V^{-1}U^{-1}$.

Also $U^{-1} = U^T$ and $V^{-1} = V^T$ so $(UV)^{-1} = V^{-1}U^{-1} = V^T U^T = (UV)^T$.

Since UV is invertible and $(UV)^{-1} = (UV)^T$, UV is an orthogonal matrix.