Example 3.5.7. Determine if the following sets of vectors are linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ (b) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$v_2 = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\Longrightarrow$ dependent

linearly <u>indep</u> b/c not multiples

Proposition 3.5.8 (Sets of two vectors)**.** *A set of two vectors* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is linearly independent if and only if neither of the vectors is a multiple of the other.*

$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, & \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, & \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \end{array}$

$v_1 + v_2 = v_3$
so dependent

Question 3.5.9. What can you say about linear dependence/independence of a set of p vectors in $\mathbb{R}^n$ if $p > n$? (i.e. more vectors than entries in each vector)

From HW

$\Rightarrow$ free variable (more colm than rows)

$\Rightarrow$ dependent

e.g.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \right\}$$

dependent!

converse not true!

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$$

dependent

Theorem 3.5.10 (Too many vectors)**.** *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.*

e.g. $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

$x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$

Definition 3.5.12. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis for** V if

(a) $\mathcal{B}$ is a linearly independent set, and

(b) $\mathcal{B}$ spans all of V; that is,

$$V = \text{Span}(\mathcal{B}) = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

The **dimension** of a vector space is the number of vectors in any basis for the space.

Example 3.5.13. Let $\mathbf{v}_1 = \begin{bmatrix}3\\0\\-6\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}-4\\1\\7\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}-2\\1\\5\end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$?

Check lin indep by reducing

$$[v_1\ v_2\ v_3] = \begin{bmatrix}3 & -4 & -2\\0 & 1 & 1\\-6 & 7 & 5\end{bmatrix} \sim \begin{bmatrix}3 & -4 & -2\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}$$

pivot in every column $\Rightarrow$ linearly indep ✓

if $A = [v_1\ v_2\ v_3]$ has a pivot in every row ~~reduces to identity~~

then yes $A\vec{x} = \vec{b}$ has soln for all $\vec{b}$ (prev HW)

true, just relax

Columns of A span $\mathbb{R}^3$ if there's a pivot in every row

Question 3.5.14. What is the dimension of $\mathbb{R}^n$?

suppose $\{b_1, \dots, b_p\} = B$ is a basis

- i.e. B spans $\mathbb{R}^n$
- B is linearly indep

- $A = [b_1\ b_2\ \cdots\ b_p]$
  has n rows
  linearly indep $\Rightarrow$ less columns or equal than rows
  $p \leq n$

- $b_1, b_2, \dots b_p$ spans $\mathbb{R}^n$ only if A has pivots in each row
  $\Rightarrow p \geq n$

$p = n$

so dim of $\mathbb{R}^n$ is n

Remark: for the columns of A to be a basis for $\mathbb{R}^n$, A has to be $n \times n$ invertible mtx (pivot in every row)

still indep
doesn't span

no pivot in last col
$\Rightarrow$ linearly dep

Example 3.5.15. Which of the following is a basis for $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

still spans but
isn't indep

<u>not</u> a basis
for $\mathbb{R}^3$
not a pivot in
last row

<u>need</u> 3 vectors

pivot in every row
$\Rightarrow$ spans
pivot in every col
$\Rightarrow$ lin indep

$\Rightarrow$ basis!

Remark 3.5.16. In one sense, a basis for V is a spanning set of V that is as small as possible. In another sense, a basis for V is a linearly independent set that is as large as possible.

Example 3.5.17. Find a basis for Col U, where $U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$$\operatorname{Col} U = \operatorname{Span}\left\{ \underset{u_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \underset{u_2}{\begin{bmatrix}4\\0\\0\\0\end{bmatrix}}, \underset{u_3}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}, \underset{u_4}{\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}}, \underset{u_5}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \right\}$$

$u_2 = 2u_1$

$\boxed{\text{Defn basis: - spans, - lin indep}}$

$\{u_1, u_2, u_3, u_4, u_5\}$ - spans but dependent

$$\operatorname{Span}\{u_1, u_2, u_3, u_4, u_5\} = \operatorname{Span}\{u_1, 4u_1, u_3, u_4, u_5\} = \operatorname{Span}\{u_1, u_3, u_4, u_5\}$$

also $u_4 = 2u_1 - u_3$

so

$$\operatorname{Span}\{u_1, u_3, u_4, u_5\} = \operatorname{Span}\{u_1, u_3, u_5\}$$

$$\operatorname{Col}(U) = \operatorname{Span}\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}$$

This is lin indep

$\Longrightarrow$ basis!

Example 3.5.18. Below, A is row equivalent to U from the last example. Find a basis for Col A.

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the dimension of Col(A)?

a

we did row ops on A to get U

is it just $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}8\\1\\0\\0\end{bmatrix}$?

lin indep ✓

does not span Col A (there's no last entry!)

BUT! row operations preserve linear independence!

$a_2 = 4a_1$ (b/c $u_2 = 4u_1$)

$a_4 = 2a_1 - a_3$ (b/c $u_4 = 2u_1 - u_3$)

So $\left\{ \begin{bmatrix}1\\3\\2\\5\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\5\\2\\8\end{bmatrix} \right\}$ spans Col A and linearly indep!

Theorem 3.5.19. *The pivot columns $\boxed{\text{of a matrix } A}$ form basis for* Col A. *The dimension of* Col(A) *is the number of pivots.*

if $A \sim U$ this does not mean Col A = Col U

Example 3.5.20. Find a basis for Nul A, where A is the same as the previous example:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the dimension of Nul(A)?

1) start with a spanning set
2) throw out any dependent vectors!

$$\text{Nul}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$$

solve $[A \mid \vec{0}]$

if you do that

$$\boxed{\begin{cases} x_1 = -4x_2 - 2x_4 \\ x_2 \text{ free} \\ x_3 = x_4 \\ x_4 \text{ free} \\ x_5 = 0 \end{cases}}$$

plug in $x_2 = 1$, $x_4 = 0$ $\sim \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

CHECK

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$x_2 = 1$, $x_4 = 0$ $\qquad$ $x_4 = 1$, $x_2 = 0$

so it's a basis

Theorem 3.5.21. *The dimension of* Nul(A) *is the number of free variables of A. In other words, the dimension of* Nul(A) *is the number of columns minus the number of pivots.*

$$\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$