ex

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$

$A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

$A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$

NOTE: These slides contain *both* Sections 3.1 and 3.2.

3.1 Introduction to Determinants

McDonald Fall 2018, MATH 2210Q, 3.1&3.2 Slides

3.1 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 4, 8, 13, 20, 21, 37, 39.
- Recommended: 11, 31, 32.

Definition 3.1.1. For $n \geq 2$, let $A = [a_{ij}]$ be a $n \times n$ matrix. We define $A_{k\ell}$ to be the $(n-1) \times (n-1)$ matrix obtained by deleting the kth row and ℓth column of A. We also set $\det(a) = a$ for any real number a. The **determinant** of A is the alternating sum

$$|A| = \det A = \cancel{a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - a_{14}A_{14} + \cdots + (-1)^{n+1}\det A_{1n}.}$$

ugh

Remark 3.1.2. This is a *recursive* definition. That is, we need to know how to compute the determinants of the $A_{k\ell}$ first, before we can compute the determinant of A.

Example 3.1.3. Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{bmatrix}$

lines, not brackets

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$= 1 \cdot \begin{vmatrix} 3 & -2 \\ -3 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & -2 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 \\ 0 & -3 \end{vmatrix}$

$= 1\left(3 \cdot 0 - (-2)(-3)\right) - 2\left(2 \cdot 0 - (-2) \cdot 0\right)$

$= -6$

Definition 3.1.4. Given $A = [a_{ij}]$, the (i,j)**-cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Theorem 3.1.5. *The determinant of an $n \times n$ matrix A can be computed by a* **cofactor expansion** *across any row or down any column. The expansion of across the ith row is*

$$|A| = \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$|A| = \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example 3.1.6. Use a cofactor expansion across the third row to compute $\det A$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{bmatrix}$$

across 3^{rd} row

$$\det A = a_{31} \det A_{31} - a_{32} \det A_{32} + a_{33} \det A_{33}$$

$$= \cancel{0 \cdot \begin{vmatrix} 2 & 0 \\ 3 & -2 \end{vmatrix}} - (-3) \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} + \cancel{0 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}}$$

$$= 3(1 \cdot (-2) - 0 \cdot 2) = -6$$

across 3^{rd} column

$$\det A = a_{13} \det A_{13} - a_{23} \det A_{23} + a_{33} \det A_{33}$$

$$= \cancel{0 \begin{vmatrix} 2 & 3 \\ 0 & -3 \end{vmatrix}} - (-2) \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} + \cancel{0 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}}$$

$$= 2(1 \cdot (-3) - 2 \cdot 0) = -6$$

Example 3.1.7. Compute the determinant of $A = \begin{bmatrix} 3 & 1 & -2 & 6 & 1 \\ 0 & 2 & 5 & -2 & 3 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$

look for column or row with most zeros.

$$\det A = 3 \cdot \begin{vmatrix} 2 & 5 & -2 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & -3 & 0 \end{vmatrix} - \cancel{0 \cdot |A_{21}| + 0 \cdot |A_{31}| - 0 \cdot |A_{41}| + 0 \cdot |A_{51}|}$$

$$= 3 \cdot \left(2 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{vmatrix} - \cancel{0 \cdot \begin{vmatrix} 5 & -2 & 3 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{vmatrix}} + \cancel{0 \cdot |*|} - \cancel{0 \cdot |*|} \right)$$

$$= 6 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{vmatrix} = 6(-6) = -36$$

from previous example

matrix was "easy" to compute b/c it was almost Δ

> **Theorem 3.1.8.** *If A is an $n \times n$ triangular matrix, then*
> $$\det A = a_{11}a_{22}a_{33} \cdots a_{nn}.$$

Remark 3.1.9. This suggests a nice strategy. Turn A into a triangular matrix! We could try to reduce A to echelon form, U. How are determinants affected by row operations?

3.2 Properties of Determinants

3.2 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 8, 10, 16, 17, 20, 27, 34.
- Recommended: 2, 3, 26, 32, 40.

Theorem 3.2.1 (Row Operations)**.** *Let A be a square matrix.*

(a) *If a multiple of one row of A is added to another to produce B, then* $\det B = \det A$.

(b) *If two rows of A are interchanged to produce B, then* $\det B = -\det A$.

(c) *If one row of A is multiplied by k to produce B, then* $\det B = k \det A$.

Example 3.2.2. Compute $\det A$ where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

note, det

lets reduce before cofactor exp.

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & -5 \\ 3 & 2 \end{vmatrix} - 0 \begin{vmatrix} -4 & 2 \\ 3 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} -4 & 2 \\ 0 & -5 \end{vmatrix}$$

$$= 1 \cdot (0 \cdot 2 - (-5) \cdot 3) = 15$$

~ or ~

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= -1 \cdot 3 \cdot (-5)$$

$$= 15$$

Example 3.2.3. Compute $\det A$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$. factor 2

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & -2 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2 \cdot 1 \cdot 3 \cdot (-6) \cdot 1 = -36$$

Suppose an $n \times n$ matrix A can be reduced to echelon form U using only row replacements and row interchanges. Since U is in echelon form, it is triangular, so $\det U = u_{11}u_{22}u_{33}\cdots u_{nn}$.

Proposition 3.2.4. *If an $n \times n$ matrix A can be reduced to echelon form U using only row replacements and k row interchanges, then*

$$\det A = (-1)^k u_{11}u_{22}u_{33}\cdots u_{nn}.$$

general strat: bring to echelon form, mult along diagonal

computers take $2n^3/3$ ops 25x25 takes ~10,000 here it is!

> **Theorem 3.2.5.** *A square matrix A is invertible if and only if* $\det A \neq 0$.

Example 3.2.6. Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

two identical rows $\Longrightarrow$ not invertible $\Longrightarrow \det A = 0$

note $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 3.2.7. Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

cancel this

$$= -\left(-2 \cdot \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix}\right)$$

swap

$$= 2 \cdot 1 \cdot (-3) \cdot 5 = -30$$

Theorem 3.2.8. *If A and B are $n \times n$ matrices, then* $\det AB = (\det A)(\det B)$.

Example 3.2.9. Verify Theorem 3.2.8 for $A = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\det A = 5 \cdot 1 - 0 \cdot 2 = 5$$
$$\det B = 4 \cdot 1 - 2 \cdot 3 = -2$$

$$(\det A)(\det B) = 5 \cdot (-2) = -10$$

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2+15 & 4+20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 17 & 24 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 1 & 2 \\ 17 & 24 \end{vmatrix} = 24 - 34 = -10$$

Example 3.2.10. Let A and P be square matrices with P invertible, and show that $\det(PAP^{-1}) = \det A$.

$$\begin{aligned} \det(PAP^{-1}) &= (\det P) \cdot (\det A) \cdot (\det P^{-1}) \\ &= (\det P)(\det P^{-1})(\det A) \\ &= (\det PP^{-1}) \det A \\ &= \det I \quad \det A \\ &= 1 \cdot \det A \\ &= \det A \end{aligned}$$

finally a use for the transpose!

Theorem 3.2.11. *If A is an $n \times n$ matrix, then* $\det A^T = \det A$.

Remark 3.2.12. This means we can perform operations on the *columns* of a matrix in the same way that we perform row operations, and expect the same effect on the determinant.

Example 3.2.13. Compute $\det A$, where $A = \begin{bmatrix} -5 & 2 & 2 & 2 \\ 3 & 0 & 3 & 5 \\ -4 & 0 & 4 & 0 \\ -2 & 0 & 2 & -2 \end{bmatrix}$.

$$\det A = \begin{vmatrix} -5 & 2 & 2 & 2 \\ 3 & 0 & 3 & 5 \\ -4 & 0 & 4 & 0 \\ -2 & 0 & 2 & -2 \end{vmatrix} \overset{C_1 = C_3 + C_1}{=} \begin{vmatrix} -5 & 2 & 2 & 2 \\ 6 & 0 & 3 & 5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} \overset{C_3 = C_4 + C_3}{=} \begin{vmatrix} -5 & 2 & 4 & 2 \\ 6 & 0 & 8 & 5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$\overset{\text{swap } C_1 \,\&\, C_2}{=} - \begin{vmatrix} 2 & -5 & 4 & 2 \\ 0 & 6 & 8 & 5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -2 \cdot 6 \cdot 4 \cdot (-2)$$

$$= 96$$

Theorem 3.2.14 ("Column" Operations)**.** *Let A be a square matrix.*

(a) *If a multiple of one column of A is added to another to produce B, then* $\det B = \det A$.

(b) *If two columns of A are interchanged to produce B, then* $\det B = -\det A$.

(c) *If one column of A is multiplied by k to produce B, then* $\det B = k \det A$.