Example 16.3.7 Find the derivative of

$$f(x) \left(\frac{x^4 - 2x^2 + 1}{x^2 - 1}\right)^5.$$

(Note: The notes I gave you last class has $x^4 + 2x^2 + 1$ in the numerator. Can you still simplify?)

$$x^4 - 2x^2 + 1 = (x^2 - 1)(x^2 - 1)$$

$$\left(\frac{x^2 - 2x + 1}{x^2 - 1}\right)^5 = (x^2 - 1)^5$$

$g'(x)$

$$\frac{d}{dx} f(x) = 5(x^2 - 1)^4 \, 2x$$

Groups 16.3.8 Does xe^{x^2} have a maximum?

$$\frac{d}{dx}\left(xe^{x^2}\right)$$

$$x \quad * \quad e^{x^2}$$

$$\frac{d}{dx}\left(xe^{x^2}\right) = (1)e^{x^2} + x\left(e^{x^2}\right)'$$

$$u = x^2 \qquad f(u) = e^u$$

$$u' = 2x \qquad f' = e^u$$

$$\left(e^{x^2}\right)' = 2xe^{x^2}$$

$$\frac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + x\left(2xe^{x^2}\right)$$

$$= e^{x^2}(1 + 2x^2) = 0$$

$$2x^2 = -1$$

$$x^2 = -\frac{1}{2}$$

$$*$$

Quiz ex

$$f(x) = xe^{2x}$$

$$f'(x) = e^{2x} + x\left(e^{2x}\right)'$$

$$= e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$$

$$f'(x) = 0 \iff e^{2x}(1+2x) = 0$$

$$\iff x = -\frac{1}{2}$$

Example 17.1.1

(a) What are the derivatives of x^n and b^x? Do either of these rules work for x^x when $x > 0$?

(b) Can we make it so that x is not a power? (hint: do you remember your log rules?)

$$(-0.5)^{-0.5} = \frac{1}{\sqrt{-0.5}}$$

a) is $\frac{df}{dx} = x x^{x-1}$? $x x^{x-1}$

is $\frac{df}{dx} = x^x \ln x$?

b) $y = x^x$

if we had $\ln(x^x) = x \ln x$

$$\boxed{\ln y = \ln(x^x) = x \ln x}$$

want $\frac{dy}{dx}$

Example 17.1.2 Find the derivative of $\overset{f(x)}{\cancel{y}} = x^x$.

$$\ln(y) = x \ln x$$

$$(\ln(y))' = (x\ln x)' = 1 + \ln x \quad \text{check}$$

$$y = y(x) \qquad \ln(f(x))$$

chain rule

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)} = \frac{y'}{y}$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = y(1+\ln x) = x^x(1+\ln x)$$

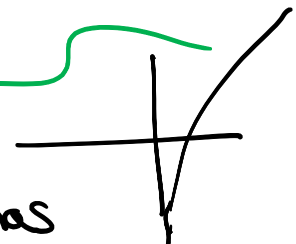
Example 17.2.1 Find the tangent to the curve $f(x) = (x^2+1)^x$ at $x = 0$.

Reason #1 to use log differentiation

function^function

$f(x)^{g(x)}$

domain: all real #s



log x has domain x>0

$$y = (x^2+1)^x$$

$$\ln y = \ln\left((x^2+1)^x\right)$$

$$\ln y = x \ln(x^2+1)$$

$$\frac{y'}{y} = x\left(\ln(x^2+1)\right)' + \ln(x^2+1)$$

$$\frac{y'}{y} = x\,\frac{2x}{x^2+1} + \ln(x^2+1)$$

CR check

$$y' = \left(x\,\frac{2x}{x^2+1} + \ln(x^2+1)\right)y$$

$$\boxed{y' = \left(x\,\frac{2x}{x^2+1} + \ln(x^2+1)\right)(x^2+1)^x}$$

$$y'(0) = (0+0)\,\cancel{(\sim)} = 0$$

$y = mx + b$ $\quad m = 0$

$$y(0) = 1$$

$\boxed{y = 1 \text{ is eqn for tangent}}$

$$\frac{d}{dx}\left(\ln(f(x))\right) = f'(x)\frac{1}{f(x)} = \frac{f'(x)}{f(x)}$$

Example 17.2.2

(a) what is the domain of $(x-1)^{1-x^2}$?

(b) on this domain, find $f'(2)$.

don't worry

$x \geq 1$

chain rule

$$\ln(y) = (1-x^2)\ln(x-1)$$

product

$$\frac{y'}{y} = -2x\ln(x-1) + (1-x^2)\frac{1}{x-1}$$

$$y' = (\text{mess})y$$

$$y' = \left(-2x\ln(x-1) + \frac{1-x^2}{x-1}\right)(x-1)^{1-x^2}$$

$$y'(2)\left(0 + \frac{-3}{1}\right) * 1 = -3$$

Question 17.2.3 What are the properties of logarithms that we know?

$$\ln(AB) = \ln(A) + \ln(B)$$
$$\ln(A/B) = \ln(A) - \ln(B)$$
$$\ln(A^n) = n\ln(A)$$
$$\ln(e^x) = x \Longleftarrow \log_b(b^x) = x$$
$$\ln(e) = 1 \qquad e^{\ln x} = x$$

Example 17.2.4 Find the derivative of $y = 2x^{e^x}$

done outside class

$$\ln y = \ln\left(2x^{e^x}\right) = e^x \ln(2x)$$

$$\frac{y'}{y} = e^x * \frac{2}{2x} + e^x \ln(2x) \quad \text{(prod rule)}$$

$$y' = \left(e^x \frac{1}{x} + e^x \ln(2x)\right) 2x^{e^x}$$

y

Quiz example:

$$y = f(x) = \left(x^4 + x^2 + 3\right)^{2x+1}$$

$$\ln(y) = \ln\left(\left(x^4 + x^2 + 3\right)\right)(2x+1)$$

$$\frac{y'}{y} = 2\ln\left(x^4 + x^2 + 3\right) + (2x+1)\frac{4x^3 + 2x}{x^4 + x^2 + 3}$$

product rule

$$\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}$$

$$\ln\left(\sqrt{\frac{(x+1)x^5}{e^{x^2}(x-9)^{12}}}\right)$$

Example 17.2.5 Find the derivative of $y = \dfrac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$

$$\ln y = \ln\left(\frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}\right)$$

$$\ln y = \ln\left((x+3)^5(x^2+7x)^8\right) - \ln\left(x(x^2+5)^3\right)$$

$$= \ln\left((x+3)^5\right) + \ln\left((x^2+7x)^8\right) - \ln x - \ln\left((x^2+5)^3\right)$$

$$\ln y = 5\ln(x+3) + 8\ln(x^2+7x) - \ln x - 3\ln(x^2+5)$$

$$\boxed{\boxed{\frac{y'}{y}} = 5\frac{1}{x+3} + 8\frac{2x+7}{x^2+7x} - \frac{1}{x} - 3\frac{2x}{x^2+5}}$$

$$\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}$$

$$y' = (\text{mess}) * \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$$

Example 17.2.6 Find the derivatives of

(a) $\dfrac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$

$$\ln y = \ln\left(\frac{xe^{5x}}{(x+1)^2\sqrt{x-2}}\right) = \ln(xe^{5x}) - \ln((x+1)^2\sqrt{x-2})$$

$$= \ln(x) + \ln(e^{5x}) - \ln((x+1)^2) - \ln(\sqrt{x-2})$$

$$y = \ln x + 5x - 2\ln(x+1) - \tfrac{1}{2}\ln(x-2)$$

$$\frac{y'}{y} = \frac{1}{x} + 5 - \frac{2}{x+1} - \frac{1}{2}\frac{1}{x-2}$$ check

$$y' = (\text{mess}) * \frac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$$

(b) $e^{2x}(x^2+3)^5(2x^2+1)^3$

$$y' = \left(\frac{1}{x} + 5 - \cdots - \frac{1}{2}\frac{1}{x-2}\right) * y$$

$$\ln(y) = \ln\left(e^{2x}(x^2+3)^5(2x^2+1)^3\right)$$

$$= 2x + 5\ln(x^2+3) + 3\ln(2x^2+1)$$

$$\frac{y'}{y} = 2 + 5\frac{2x}{x^2+3} + 3\frac{4x}{2x^2+1}$$

(c) $\left(e^{x-1}\right)^{x+1}$

done outside of class:

with logs

$$\ln(y) = \ln(e^{x-1})(x+1)$$

$$= (x-1)(x+1)$$

$$= x^2 - 1$$

$$\Longleftrightarrow \frac{y'}{y} = 2x$$

$$y' = 2x(e^{x-1})^{x+1}$$

without logs

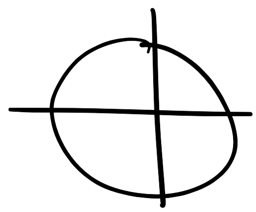
$$y = (e^{x-1})^{x+1} = e^{(x-1)(x+1)}$$

$$= e^{x^2-1}$$

$$y' = 2xe^{x^2-1}$$ chain rule

Spot the mistake 17.3.1 Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$



$$y^2 = 1 - x^2$$

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = -2x(1-x^2)^{-1/2}$$

not really a function

If I ask for $\frac{dy}{dx}$ at $x=0$, how do you know which one I'm asking for? It's really a function of x and y

$\ln y = f(x)$

$\ln(u) = f(x)$

$\frac{1}{u} u' = f'(x)$

$\frac{y'}{y} = f'(x)$

Example 17.3.2 Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

we can suppose
$y = y(x)$ (function of x)
and so

$$\frac{d}{dx}(y(x))^2 = 2y(x) \cdot y'(x)$$

$$\frac{d}{dx}(x^2+1)^2 = 2(x^2+1)2x$$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0$$

$$\boxed{2x + 2y\frac{dy}{dx} = 0}$$

$$\Longleftrightarrow 2y\frac{dy}{dx} = -2x$$

$$\Longleftrightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$\frac{dy}{dx} = -\frac{x}{y}$

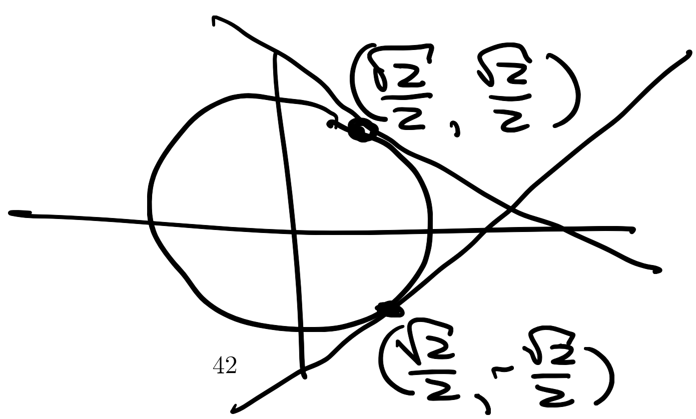
what does this mean?

$\Longrightarrow$ $\frac{dy}{dx}$ depends on both x, y

e.g.

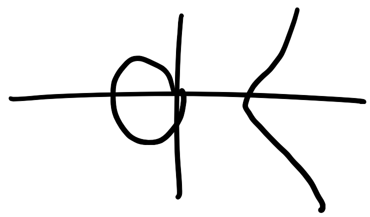
$$\frac{dy}{dx}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -1$$

$$\frac{dy}{dx}\left(\frac{\sqrt{2}}{\sqrt{2}}, -\frac{\sqrt{2}}{2}\right) = 1$$



Find vert and horiz tangents to

Example 17.3.4 ~~Sketch a graph of the~~ curve $y^2 = x^3 - x$ (don't worry about concavity)



want to find $\frac{dy}{dx}$

$(y^2)' = (x^3 - x)'$

$y^2 = x^3 - x$

$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - x)$

$f(u) = u^2$ $\quad$ $\underline{2yy' = 3x^2 - 1}$

$u = y$

$y' = \frac{3x^2-1}{2y}$ $\qquad$ $\frac{dy}{dx} = \frac{3x^2-1}{2y}$

horizontal tangent $\Leftrightarrow$ $y' = 0 = \frac{dy}{dx}$

$\frac{3x^2-1}{2y} = 0 \Leftrightarrow 3x^2 - 1 = 0$

$x = \pm \frac{\sqrt{3}}{3}$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}}$

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vertical tangent means $\frac{dy}{dx}$ approaches ∞

as $y \to 0$ denominator $\to 0$

$\frac{5}{1/10}$, $\frac{5}{1/100}$, $\frac{5}{1/1000}$ $\frac{5}{1/1000000}$

$= 50$ $\quad = 500$ $\quad = 5000$ $\quad = 5{,}000{,}000$

$\Rightarrow$ vert tangent when denominator $= 0$

$\frac{dy}{dx} = \frac{3x^2-1}{2y}$ $\qquad 2y = 0 \Leftrightarrow y = 0$

Example 17.3.8 Find the ~~absolute maximum and minimum y values~~ horiz and vert tangents of the ellipse

$$\cancel{2x^2 + 4xy + 3y^2 =} 6.$$

$$(x+2y)^2 = 2xy + 3$$

redone outside of class:

$$\frac{d}{dx}(x+2y)^2 = \frac{d}{dx}(2xy+3)$$

$$\frac{d}{dx}(x+2y(x))^2 = \frac{d}{dx}(2xy(x)+3)$$

$$2(x+2y)\left(1+2\frac{dy}{dx}\right) = 2x\frac{dy}{dx} + 2y$$

prod rule ch rule

$$2(x+2y) + 2(x+2y)\left(2\frac{dy}{dx}\right) = 2x\frac{dy}{dx} + 2y$$

$$4(x+2y)\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2(x+2y)$$

$$\frac{dy}{dx}\left(4(x+2y) - 2x\right) = 2y - 2(x+2y)$$

$$\frac{dy}{dx} = \frac{2y - 2(x+2y)}{4(x+2y) - 2x}$$

$$= -\frac{2x+2y}{2x+8y}$$

$$= -\frac{x+y}{x+4y}$$

Strategy (in class)