4.7 #1, 3, 5, 7, 9, 11, 13, 15

1.) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for a vector space V, and suppose $\vec{b}_1 = 6\vec{c}_1 - 2\vec{c}_2$ and $\vec{b}_2 = 9\vec{c}_1 - 4\vec{c}_2$.

a.) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b.) Find $[\vec{x}]_{\mathcal{C}}$ for $\vec{x} = -3\vec{b}_1 + 2\vec{b}_2$. Use part (a.)

a.) $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$ So $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$

b.) $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

3.) Let $\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$ and $\mathcal{W} = \{\vec{w}_1, \vec{w}_2\}$ be bases for V, and let P be a matrix whose columns are $[\vec{u}_1]_{\mathcal{W}}$ and $[\vec{u}_2]_{\mathcal{W}}$. Which of the following is satisfied by P for all $\vec{x} \in V$?

i) $[\vec{x}]_{\mathcal{U}} = P[\vec{x}]_{\mathcal{W}}$ ii) $[\vec{x}]_{\mathcal{W}} = P[\vec{x}]_{\mathcal{U}}$

5.) Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be bases for a vector space V and suppose $\vec{a}_1 = 4\vec{b}_1 - \vec{b}_2$, $a_2 = -\vec{b}_1 + \vec{b}_2 + \vec{b}_3$ and $\vec{a}_3 = \vec{b}_2 - 2\vec{b}_3$.

a.) Find the change of coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

b.) Find $[\vec{x}]_{\mathcal{B}}$ for $x = 3\vec{a}_1 + 4\vec{a}_2 + \vec{a}_3$.

a.) $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

b.) $[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{A}}{P} [\vec{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

7.) $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ are bases for $\mathbb{R}^2$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$\vec{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

To find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ we row reduce $[\vec{c}_1\ \vec{c}_2\ \vec{b}_1\ \vec{b}_2]$ to RREF.

$$\begin{bmatrix} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{bmatrix} \xrightarrow{5R_1+R_2} \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{bmatrix} \xrightarrow{R_2/(-8)} \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}. \quad \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}} = \frac{1}{-1}\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

9.) $\vec{b}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\vec{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 & 4 & 8 \\ 2 & 2 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad \text{So } \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}.$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}} = \frac{1}{-2}\begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}$$

11.) $\mathcal{B}$ and $\mathcal{C}$ are bases of a vector Space V. Mark True/False.

a.) The columns of the change-of-coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are $\mathcal{B}$ coordinate vectors of the vectors in $\mathcal{C}$.

b.) If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the standard basis for V, then $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is the same as the change of coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4.

a.) FALSE

b.) TRUE.

4.7 continued

13.) In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2,\ 3 - 5t + 4t^2,\ 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1+2t$.

$$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad [\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} \quad [\vec{b}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad \text{So} \quad \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

$\vec{x} = -1 + 2t$ $\quad [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ $\quad$ We are asked to find $[\vec{x}]_{\mathcal{B}}$.

$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$, So we solve

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \text{So } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

15.) Fill in a justification to complete the proof of thm 15.

Given $\vec{v}$ in V, there exist scalars $x_1, \ldots, x_n$ such that $\vec{v} = x_1\vec{b}_1 + \ldots + x_n\vec{b}_n$ because (a). Apply the coordinate mapping determined by the basis $\mathcal{C}$, and obtain $[\vec{v}]_{\mathcal{C}} = x_1[\vec{b}_1]_{\mathcal{C}} + \ldots + x_n[\vec{b}_n]_{\mathcal{C}}$ because (b). This eqn may be written in the form $[\vec{v}]_{\mathcal{C}} = \left[[b_1]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $*$ by the defn of (c). This shows that the matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ satisfies $[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{v}]_{\mathcal{B}}$ for each $\vec{v} \in V$, because the vector on the right side of $*$ is (d).

(a) $\mathcal{B}$ is a basis for V

(b) the coordinate mapping is a linear transformation

(c) the product of a matrix and a vector

(d) the coordinate vector of $\vec{v}$ relative to $\mathcal{B}$.