MATH 111, Intro to Functions and Calc II, S2020, Lecture Notes

Taken in part from
An Integrated Approach to Functions and their Rates of Change
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NOTE: I will update these notes as often as I can with the topics and examples (which will be worked out by hand in a separate document) we cover in class.

Contents

Chapter 16

Taking the Derivative of Composite Functions

16.1 The chain rule

16.1. Goals

- review derivatives
- express derivatives of composite functions $f(g(x))$

Groups 16.1.1. What are the derivatives of the following functions:

- x^n
- e^x
- b^x
- $ln(x)$
- $\log_b(x)$
- $f(x) \pm g(x)$
- $f(x)g(x)$
- $f(x)/g(x)$

Think, Pair, Share 16.1.2. What does the derivative represent?

Question 16.1.3. Suppose we're selling ice cream. The price that we set depends on the demand, that is, $p = f(u)$ where u is measured in people who want ice cream, and P is measured in dollars. Of course, the demand of our ice cream depends on the temperature, so we also have $u(t)$ where t is degrees Fahrenheit. So we guess, our revenue *really* depends on the temperature outside. How could we find the rate of change of our revenue with respect to temperature?

Theorem 16.1.4. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Question 16.1.5. How does this look with prime notation? What's nice about Leibnitz notation?

Groups 16.1.6. Suppose the price of our ice cream in dollars is

$$P(x) = 3x + 1,$$

where x is the demand in people. and the demand is $x = 1 - t^2$ where t is temperature measured in degrees Fahrenheit.

(a) write p as a function of t

(b) find $p'(t)$ (called the *marginal profit*), and its units

Example 16.1.7. What are the derivatives of $f(kx)$ and $f(x+k)$? Why does this make sense geometrically?

Groups 16.1.8. Write the functions as a composition

(a) $(x^2+1)^{10}$

(b) $\ln(x^2+2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.9. Find the derivatives of

(a) $(x^2+1)^{10}$

(b) $\ln(x^2+2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.10. Suppose the population of frogs in a pond is e^{g}, where g is the temperature of the pond in Celcius and the average temperature in the month of February is $0.25t + 14$ where t is in days. What is the rate of change of frogs with respect to time?

Groups 16.1.11. Suppose a rectangle is inscribed inside the ellipse

$$\frac{x^2}{9} + 4y = 1.$$

What's the largest possible area of such a rectangle?

16.2 The derivative of x^n for any real number n

16.2. Goals

- prove the power rule a different way
- find the derivative of b^x a different way
- find derivatives of quotients a different way

Example 16.2.1. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Question 16.2.2. Can we recover the derivative of b^x for $b > 0$ using only the chain rule?

Groups 16.2.3. Now, let's consider x^n.

(a) rewrite x^n using $\ln x$ and e^x.

(b) find the derivative of the function you found in (a)

(c) what did you prove?

Question 16.2.4. Can we find the derivative of $f(x)/g(x)$ without using the quotient rule?

16.3 Using the chain rule

16.3. Goals

- do more with the chain rule
- derive a formula for functions of multiple compositions
- understand the importance of simplification

Theorem 16.3.1. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Example 16.3.2. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Groups 16.3.3. Differentiate $e^{\sqrt{x^3+1}}$.

Think, Pair, Share 16.3.4. Can we find a general rule for the derivative of $f(g(h(x)))$?

Example 16.3.5. Differentiate

$$\ln\left(\sqrt{\frac{1+x}{(1-x)^3}}\right).$$

Example 16.3.6. Differentiate

$$\frac{8^{x^2+1}}{\left(2^x\right)^x}$$

Example 16.3.7. Find the derivative of

$$\left(\frac{x^4-2x^2+1}{x^2-1}\right)^5.$$

(Note: The notes I gave you last class has x^4+2x^2+1 in the numerator. Can you still simplify?)

Groups 16.3.8. Does xe^{x^2} have a maximum?

Chapter 17

Implicit Differentiation and its Applications

17.1 The derivative of x^x

17.1. Goals
- finding the derivative of x^x
- logarithmic differentiation

Example 17.1.1.

(a) What are the derivatives of x^n and b^x? Do either of these rules work for x^x when $x > 0$?

(b) Can we make it so that x is not a power? (hint: do you remember your log rules?)

Example 17.1.2. Find the derivative of $y = x^x$.

Definition 17.1.3. The method we just used, by taking logs of both sides and using the chain rule, is called **logarithmic differentiation**.

17.2 Logarithmic differentiation

17.2. Goals

- using logarithmic differentiation

Example 17.2.1. Find the tangent to the curve $f(x) = (x^2 + 1)^x$ at $x = 0$.

Example 17.2.2.

(a) what is the domain of $(x - 1)^{1-x^2}$?

(b) on this domain, find $f'(2)$.

Question 17.2.3. What are the properties of logarithms that we know?

Example 17.2.4. Find the derivative of $y = 2x^{e^x}$

Example 17.2.5. Find the derivative of $y = \dfrac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$

17.2.1 Extra Examples

Example 17.2.6. Find the derivatives of

(a) $\dfrac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$

(b) $e^{2x}(x^2+3)^5(2x^2+1)^3$

(c) $\left(e^{x-1}\right)^{x+1}$

17.3 Implicit differentiation

17.3. Goals

- using the ideas of the previous section to find $\frac{dy}{dx}$ of implicitly defined functions

Spot the mistake 17.3.1. Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.2. Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.3. What kinds of information can we use to sketch the graph of a curve?

Example 17.3.4. Sketch a graph of the curve $y^2 = x^3 - x$ (don't worry about concavity)

Example 17.3.5. Find all points where the tangent to

$$x^3 + y^3 = 1$$

is horizontal or vertical

Procedure 17.3.6 (using implicit differentiation)**.**

Example 17.3.7. Find the slope of the tangent to

$$x^3 + y^3 = 6xy$$

at the point $(3, 3)$.

Example 17.3.8. Find the absolute maximum and minimum y-values of the ellipse

$$2x^2 + 4xy + 3y^2 = 6.$$