MATH 118

3D Geometry, cross products and lines

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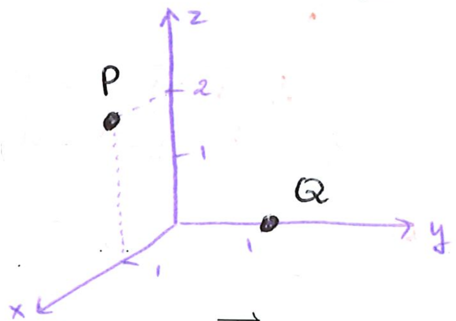
1. Consider the points $P(1,0,2)$ and $Q(0,1,0)$.

   (a) Draw a set of 3D axes and plot P and Q.



   (b) Find the vector $\overrightarrow{PQ}$, and use it to find the distance between P and Q.

$$\overrightarrow{PQ} = \begin{bmatrix} 0-1 \\ 1-0 \\ 0-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

so the distance between P and Q is $\|\overrightarrow{PQ}\| = \sqrt{(-1)^2+(1)^2+(-2)^2} = \sqrt{6}.$

2. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Calculate each of the following.

   (a) $\mathbf{u} \times \mathbf{v}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \hat{i}(0-3) + \hat{j}(6-1) + \hat{k}(1-0)$$

$$= \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

   (b) $\mathbf{v} \times \mathbf{u}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \hat{i}(3-0) + \hat{j}(1-6) + \hat{k}(0-1)$$

$$= \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix}$$

(c) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$.

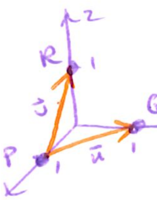
$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} = -3 + 0 + 3 = 0$$

$\therefore$ $\vec{u} \times \vec{v}$ is orthogonal to <u>both</u> $\vec{u}$ and $\vec{v}$.

(d) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$.

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} = -6 + 5 + 1 = 0$$

3. Consider the triangle with vertices $\overset{P}{(1,0,0)}$, $\overset{Q}{(0,1,0)}$ and $\overset{R}{(0,0,1)}$.



(a) Find the area of the triangle.

Let $\vec{u} = \vec{PQ} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\underset{\overset{\shortparallel}{\vec{PR}}}{\vec{v}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Then $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} \hat{i}(1-0) \\ +\hat{j}(0+1) \\ +\hat{k}(0+1) \end{matrix} =$

$\therefore$ area of triangle $= \frac{1}{2} \|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{1^2+1^2+1^2} = \frac{\sqrt{3}}{2}$.

(b) Based on your answer to (a), are the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ colinear (*i.e.* do they all lie on the same line)?

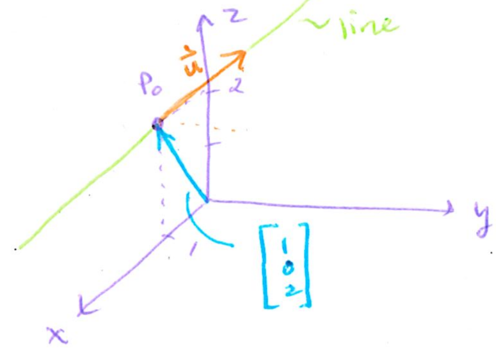
No, if they were all on the same line, the resulting triangle wouldn't have any area (area $= 0$).

(c) Find a nonzero vector that is perpendicular to the plane containing the triangle.

Let $\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ($\leftarrow$ is perpendicular to $\vec{u}$ and $\vec{v}$ and therefore the plane containing $\vec{u}$ and $\vec{v}$)

4. Write down the vector equation of the line passing through $P(1, 0, 2)$ in the direction of $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$$\vec{r}(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$



5. A line L passes through the points $P(1, -1, 1)$ and $Q(-1, 2, -1)$. Find the point at which L intersects the xy-plane, if it exists.

STRATEGY: ① Find the (vector) equation of the line.

② Set the z-component equal to 0 (xy-plane is where $z=0$)

③ Use ② to identify coordinates of point of intersection.

① Let $\vec{u} = \vec{PQ} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$. The vector equation of the line through P and Q is $\vec{r}(t) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1-2t \\ -1+3t \\ 1-2t \end{bmatrix}$

② $z = 1 - 2t = 0 \Rightarrow t = 1/2$.

③ The coordinates of the point of intersection correspond to $t = 1/2$.

$$\left.\begin{aligned} x &= 1-2t \\ y &= -1+3t \\ z &= 1-2t \end{aligned}\right\} \text{ at } t = 1/2 \rightarrow \begin{aligned} x &= 0 \\ y &= 1/2 \\ z &= 0 \end{aligned}$$

$\therefore$ The line intersects the xy-plane at $(0, \frac{1}{2}, 0)$.