MATH 118
Local extrema

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1. Rederive the equation of the plane tangent to the surface $z = f(x, y)$ at (x_0, y_0) by completing the following steps.

   (i) Recognize the surface as the level surface of a function of three variables $F(x, y, z)$.

$\underbrace{z = f(x,y)}$ is the level surface $F(x,y,z) = 0$ of

$\Rightarrow f(x,y) - z = 0$ $\boxed{F(x,y,z) = f(x,y) - z}$

   (ii) Find a vector perpendicular to the surface, and therefore perpendicular to the tangent plane, at (x_0, y_0).

$\vec{\nabla} F(x_0, y_0, \overset{f(x_0,y_0)}{\overset{\parallel}{z_0}})$ is perpendicular to the level surface

$\vec{\nabla} F = \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix}$ $\therefore$ $\boxed{\begin{bmatrix} f_x(x_0,y_0) \\ f_y(x_0,y_0) \\ -1 \end{bmatrix}}$ is perpendicular to the surface at $x = x_0$, $y = y_0$

   (iii) Find the z-coordinate of the point on the surface corresponding to $x = x_0$ and $y = y_0$. Then write down the equation of the tangent plane.

$z_0 = f(x_0, y_0)$.

$\therefore$ Equation of tangent plane is :

$$\begin{bmatrix} f_x(x_0,y_0) \\ f_y(x_0,y_0) \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - f(x_0,y_0) \end{bmatrix} = 0 \iff f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0$$

$$\iff \boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

2. Find all of the critical points of the function $f(x, y) = x^2 + 2y^2 + x^2y + 4$.

① $\frac{\partial f}{\partial x} = 2x + 2xy = 0 \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0$ or $y = -1$.

② $\frac{\partial f}{\partial y} = 4y + x^2 = 0$

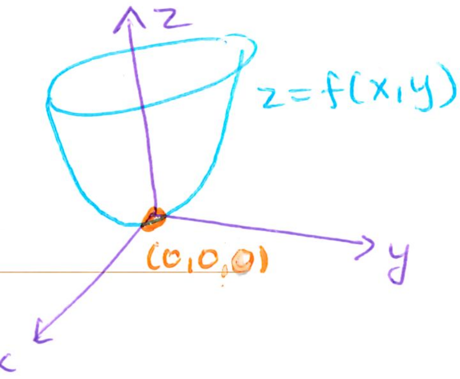
$\underline{x = 0}$: ② $\Rightarrow 4y = 0 \Rightarrow y = 0$ $\therefore$ CP is $(0,0)$.

$\underline{y = -1}$: ② $\Rightarrow -4 + x^2 = 0 \Rightarrow x^2 = 4$
$\Rightarrow x = \pm 2$ $\therefore$ CPs are $(2,-1), (-2,-1)$.

$\boxed{\therefore \text{ CPs are } (0,0), (2,-1), (-2,-1).}$

3. To better understand the Second Derivative Test, let's look at three simple cases. For each of the following functions, find and classify all of the critical points. Then make a rough sketch of the graph of the function.

(I) $f(x, y) = x^2 + y^2$



$f_x = 2x = 0 \Rightarrow x = 0$

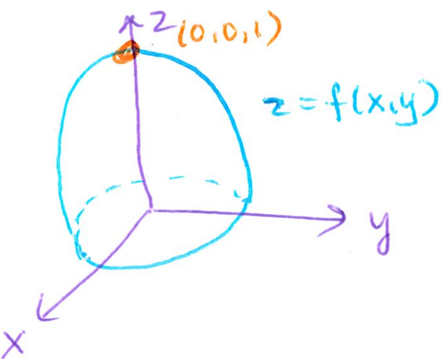
$f_y = 2y = 0 \Rightarrow y = 0$

$\therefore$ (0,0) is the only CP.

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0.$

$\Rightarrow D(0,0) = (2)(2) - (0)^2 = 4 > 0$
$f_{xx}(0,0) = 2 > 0$ $\Big\}$ $\therefore$ $f(0,0)$ is a local minimum.

(II) $f(x, y) = 1 - x^2 - y^2$



$f_x = -2x = 0 \Rightarrow x = 0$

$f_y = -2y = 0 \Rightarrow y = 0$

$\therefore$ (0,0) is the only CP.

$f_{xx} = -2,\ f_{yy} = -2,\ f_{xy} = 0.$

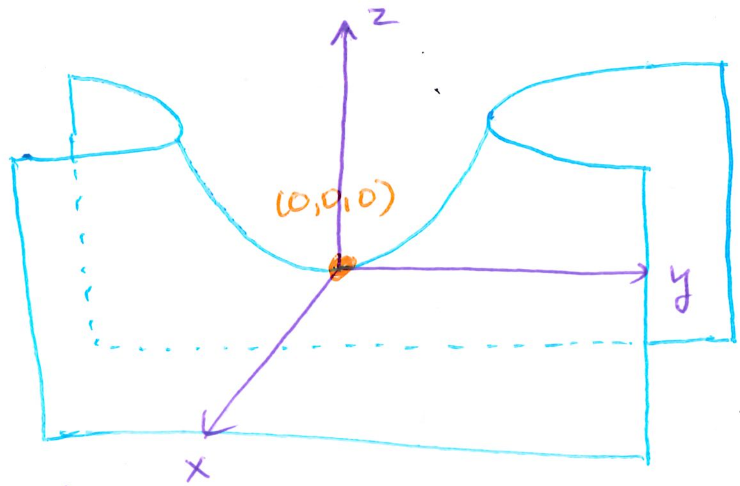
$\Rightarrow D(0,0) = (-2)(-2) - (0)^2 = 4 > 0$
$f_{xx}(0,0) = -2 < 0$ $\Big\}$ $\therefore$ $f(0,0)$ is a local max.

(III) $f(x, y) = y^2 - x^2$

$f_x = -2x = 0 \Rightarrow x = 0$

$f_y = 2y = 0 \Rightarrow y = 0$

$\therefore$ (0,0) is the only CP.



$f_{xx} = -2,\ f_{yy} = 2,\ f_{xy} = 0$

$\Rightarrow D(0,0) = (-2)(2) - (0)^2 = -4 < 0$

$\therefore$ $f(0,0)$ is a saddle point.

4. For each of the critical points found in question 2, classify it as either a local minimum, local maximum or a saddle point.

$f_{xx} = 2+2y$

$f_{yy} = 4$

$f_{xy} = 2x$

CP (x_0, y_0)	$D(x_0,y_0)$	classification.
$(0,0)$	$(2)(4)-0^2 = 8>0$	$f_{xx}(0,0)=2>0$ $\Rightarrow f(0,0)$ is a local min.
$(2,-1)$	$(0)(4)-(4)^2=-16<0$	$D<0 \Rightarrow f(2,-1)$ is a saddle point.
$(-2,-1)$	$(0)(4)-(-4)^2=-16<0$	$D>0 \Rightarrow f(-2,-1)$ is a saddle point.

5. Find and classify all of the critical points of $f(x, y) = (x - 2y)(4 - xy)$.

$f_x = 1\cdot(4-xy)+(x-2y)(-y) = 4-xy-xy+2y^2 = 4-2xy+2y^2=0$.

$f_y = (-2)(4-xy)+(x-2y)(-x) = -8+2xy-x^2+2xy = -8+4xy-x^2=0$.

$$\Rightarrow \begin{cases} 4-2xy+2y^2=0 & ① \\ -8+4xy-x^2=0 & ② \end{cases} \xRightarrow{\text{multipiy by 2}} \begin{aligned} 8-4xy+4y^2&=0 \\ -8+4xy-x^2&=0 \end{aligned}$$

add $0+0+4y^2-x^2=0$

$\Rightarrow x^2=4y^2$.

$\Rightarrow x=\pm 2y$

$\underline{x=2y}$: $① \Rightarrow 4-2(2y)y+2y^2=0 \Rightarrow 4-2y^2=0$

$\Rightarrow y^2=2 \Rightarrow y=\pm\sqrt{2}$

$\Rightarrow x=\pm 2\sqrt{2}$

$\therefore$ CPs are $(2\sqrt{2},\sqrt{2}), (-2\sqrt{2},-\sqrt{2})$.

$\underline{x=-2y}$: $① \Rightarrow 4-2(-2y)y+2y^2=0 \Rightarrow 4+6y^2=0$ NO SOLUTION.

$f_{xx} = -2y$

$f_{yy} = 4x$

$f_{xy} = -2x+4y$

CP (x_0,y_0)	$D(x_0,y_0)$	classification
$(2\sqrt{2},\sqrt{2})$	$(-2\sqrt{2})(8\sqrt{2})-(\cancel{-4\sqrt{2}+4\sqrt{2}})^2$ $=-32<0$	$D<0 \Rightarrow$ saddle point
$(-2\sqrt{2},-\sqrt{2})$	$(2\sqrt{2})(-8\sqrt{2})-(0)^2$ $=-32<0$	$D<0 \Rightarrow$ saddle point