4.1 Vector Spaces and Subspaces

McDonald Fall 2018, MATH 2210Q, 4.1 Slides

4.1 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 1, 3, 8, 13, 23, 31.
- Recommended: 12, 15, 17, 22, 32.

A lot of the theory in Chapters 1 and 2 used simple and obvious algebraic properties of $\mathbb{R}^n$, which we discussed in Section 1.3. Many other mathematical systems have the same properties. The properties we are interested in are listed in the following definition.

Definition 4.1.1. A **vector space** is a nonempty set V of objects, called *vectors*, on which two operations are defined: *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V, and all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector, $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Example 4.1.2. Show the set of all $m\times n$ matrices with entries in $\mathbb{R}$, denoted $M_{m\times n}$, is a vector space.

The following properties of vector spaces are also useful.

Proposition 4.1.3. *For each* $\mathbf{u}$ *in* V *and scalar* c*,*

$$0\mathbf{u} = \mathbf{0}$$
$$c\mathbf{0} = \mathbf{0}$$
$$-\mathbf{u} = (-1)\mathbf{u}$$

Example 4.1.4. Show that the set $\mathbb{P}_n$ of all polynomials of degree at most n is a vector space.

Example 4.1.5. Show that the set W of all real-valued functions on $\mathbb{R}$ (or $[a, b]$) is a vector space.

Definition 4.1.6. A **subspace** of a vector space is a subset H of V that has the properties:

(a) The zero vector of V is in H.

(b) H is closed under vector addition: for every $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

(c) H is closed under scalar multiplication: for all $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H.

Example 4.1.7. The set $\{\mathbf{0}\}$ is a subspace of any vector space, called the **zero subspace**.

Example 4.1.8. Let $H = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$. Show that H is a subspace of $M_{2\times 2}$.

Example 4.1.9. Show that $H = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$ is a subspace of W, from 4.1.5.

Example 4.1.10. Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V. Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of V.

Theorem 4.1.11. *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.*

Definition 4.1.12. We call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the **subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. For any subspace H of V, a **spanning set** (or **generating set**) for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 4.1.13. Let

$$H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}.$$

Show that H is a subspace of $\mathbb{R}^4$.

Example 4.1.14. Show $H = \{at^2 + at + a : a \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{P}_n$.

Example 4.1.15. Show that $\mathbb{D} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 < 1 \right\}$ is not a subspace of $\mathbb{R}^2$.

4.1.1 Additional Notes and Problems