6.1 # 3, 5, 10, 16, 18, 19, 23, 25, 27, 29

3.) Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ Compute:

$$\frac{1}{\vec{w}\cdot\vec{w}}\vec{w}$$

$$\vec{w}\cdot\vec{w} = \vec{w}^T\vec{w} = \begin{bmatrix} 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 3(3) - 1(-1) - 5(-5) = 9 + 1 + 25 = 35$$

$$\frac{1}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{1}{35}\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$$

5.) $\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$ $\quad \vec{u}\cdot\vec{v} = \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = -4 + 12 = 8$, $\vec{v}\cdot\vec{v} = \begin{bmatrix} 4 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 16 + 36 = 52$

$$\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{8}{52}\right)\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \left(\frac{2}{13}\right)\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$$

10.) Find a unit vector in the direction of $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

$$\vec{u} = \frac{1}{\sqrt{(-6)^2 + 4^2 + (-3)^2}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{61}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$$

16.) Determine if the vectors are orthogonal.

$\vec{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ $\quad \vec{u}\cdot\vec{v} = \begin{bmatrix} 12 & 3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 24 - 9 - 15 = 0$

Since $\vec{u}\cdot\vec{v} = 0$, $\vec{u}$ and $\vec{v}$ are orthogonal.

18.) $\vec{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$ $\quad \vec{y}\cdot\vec{z} = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = -3 - 56 + 60 + 0 = 1$

Since $\vec{y}\cdot\vec{z} \neq 0$, $\vec{y}$ and $\vec{z}$ are <u>not</u> orthogonal.

19.) True/False. All vectors are in $\mathbb{R}^n$.

a.) $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

b.) For any scalar c, $\vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$.

c.) If the distance from $\vec{u}$ to $\vec{v}$ equals the distance from $\vec{u}$ to $-\vec{v}$, then $\vec{u}$ and $\vec{v}$ are orthogonal

d.) For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

e.) If vectors $\vec{v}_1, \ldots \vec{v}_p$ span a subspace W and if $\vec{x}$ is orthogonal to each $\vec{v}_j$ for $j=1,\ldots,p$, then $\vec{x}$ is in $W^\perp$.

a.) True   b.) True   c.) True   d.) False   e.) True

23.) Let $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. Compute and compare $\vec{u}\cdot\vec{v}$, $\|\vec{u}\|^2$, $\|\vec{v}\|^2$ and $\|\vec{u}+\vec{v}\|^2$. Do not use the pythagorean thm.

$\vec{u}\cdot\vec{v} = -14+20-6=0$

$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = 2^2+(-5)^2+(-1)^2 = 30$

$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = (-7)^2+(-4)^2+6^2 = 101$

$\|\vec{u}+\vec{v}\|^2 = (-5)^2+(-9)^2+(5)^2 = 131$

25.) Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\vec{v}$.

Hint: Consider $\vec{v}=\vec{0}$ and $\vec{v}\neq 0$.

$\begin{bmatrix} x \\ y \end{bmatrix}$ is orthogonal to $\vec{v}$ if the dot product equals zero, that is, if $ax+by=0$. If $\vec{v}=\vec{0}$ then $0x+0y=0$ for any values of x, y. In this case, $H=\mathbb{R}^2$. If $\vec{v}\neq 0$, then either $a\neq 0$ or $b\neq 0$. Suppose $a\neq 0$. Then $ax+by=0 \iff ax=-by \iff x=\left(\frac{-b}{a}\right)y$. Then $H = \left\{ \begin{bmatrix} \frac{-b}{a}y \\ y \end{bmatrix} : y\in\mathbb{R} \right\}$. H has basis $\left\{ \begin{bmatrix} -b/a \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$.

6.1 continued

27.) Suppose a vector $\vec{y}$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$. Show that $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$.

We Know that $\vec{y}\cdot\vec{u}=0$ and $\vec{y}\cdot\vec{v}=0$. We want to show that $\vec{y}\cdot(\vec{u}+\vec{v})=0$.

$$\begin{aligned}\vec{y}\cdot(\vec{u}+\vec{v}) &= \vec{y}\cdot\vec{u}+\vec{y}\cdot\vec{v}\\ &= 0+0\\ &= 0\end{aligned}$$

Thus $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$

29.) Let $W=\text{Span}\{\vec{v}_1,\ldots,\vec{v}_p\}$. Show that if $\vec{x}$ is orthogonal to each $\vec{v}_j$, for $1\le j\le p$, then $\vec{x}$ is orthogonal to every vector in W.

Suppose $\vec{x}$ is orthogonal to each $\vec{v}_j$. Then $\vec{x}\cdot\vec{v}_j=0$ for each $\vec{v}_j$.

Any vector $\vec{w}$ in W is of the form $\vec{w}=c_1\vec{v}_1+\ldots+c_p\vec{v}_p$

then
$$\begin{aligned}\vec{x}\cdot\vec{w} &= \vec{x}\cdot(c_1\vec{v}_1+\ldots+c_p\vec{v}_p)\\ &= \vec{x}\cdot(c_1\vec{v}_1)+\ldots+\vec{x}\cdot(c_p\vec{v}_p)\\ &= c_1(\vec{x}\cdot\vec{v}_1)+\ldots+c_p(\vec{x}\cdot\vec{v}_p)\\ &= c_1(0)+\ldots+c_p(0)\\ &= 0\end{aligned}$$

Since $\vec{x}\cdot\vec{w}=0$, $\vec{x}$ is orthogonal to $\vec{w}$ for any $\vec{w}$ in W.