22.1 Net Change

Goals

- relating net change to geometry

Example 22.1.1. You're driving a car with a broken odometer. *Why do calculus students always find themselves in such bizarre hypotheticals?* you wonder. "Not to worry!" your passenger says, "I know calculus!" At two minute intervals, you check your speedometer, and your friend quickly converts to miles per minute. You collect the following data:

time (minutes)	speed (miles per minute)
0	0.75
2	0.7
4	0.65
6	0.5
8	0.6
10	1

In this problem, you'll use the information you've been provided to estimate the total distance traveled.

time (minutes)	speed (miles per minute)
0	0.75
2	0.7
4	0.65
6	0.5
8	0.6
10	1

speed = $\frac{\text{distance}}{\text{time}}$

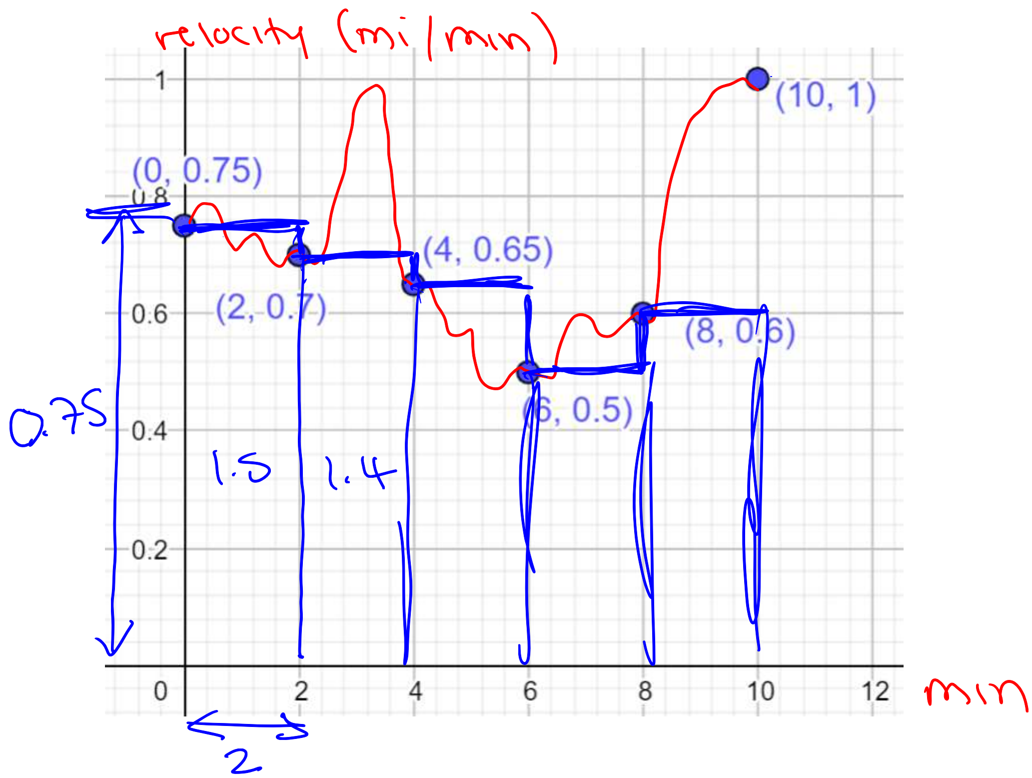
distance = speed * time

(a) Estimate the distance traveled during each interval in the following table, by assuming the speed on each interval is constant and given by the speed at the start of the interval.

with speeds determined by left endpoint

time interval (minutes)	estimated distance (miles)
$(0, 2)$	0.75 mi/min * 2 min = 1.5 mi
$(2, 4)$	1.4
$(4, 6)$	1.3
$(6, 8)$	1
$(8, 10)$	1.2
total distance traveled in ten minutes	total estimate 6.4 mi

(b) How can this method be represented graphically? What are we really adding up?



position ≈ area of rectangles.

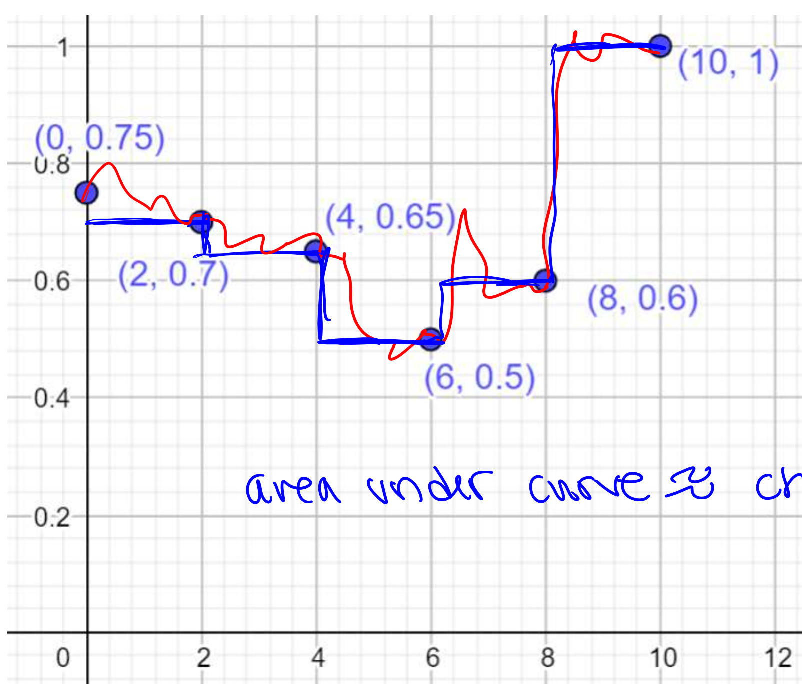
time (minutes)	speed (miles per minute)
0	~~0.75~~
2	0.7
4	0.65
6	0.5
8	0.6
10	1

(c) Estimate the distance traveled during each interval in the following table, by assuming the speed on each interval is constant and given by the speed at the end of the interval.

with speeds determined by left endpoint

time interval (minutes)	estimated distance (miles)
$(0, 2)$	$0.7 * 2 = 1.4$
$(2, 4)$	1.3
$(4, 6)$	1
$(6, 8)$	1.2
$(8, 10)$	2
total distance traveled in ten minutes	$\approx$ 6.9 miles

(d) How can this method be represented graphically? What are we really adding up?



area under curve $\approx$ change in pos.

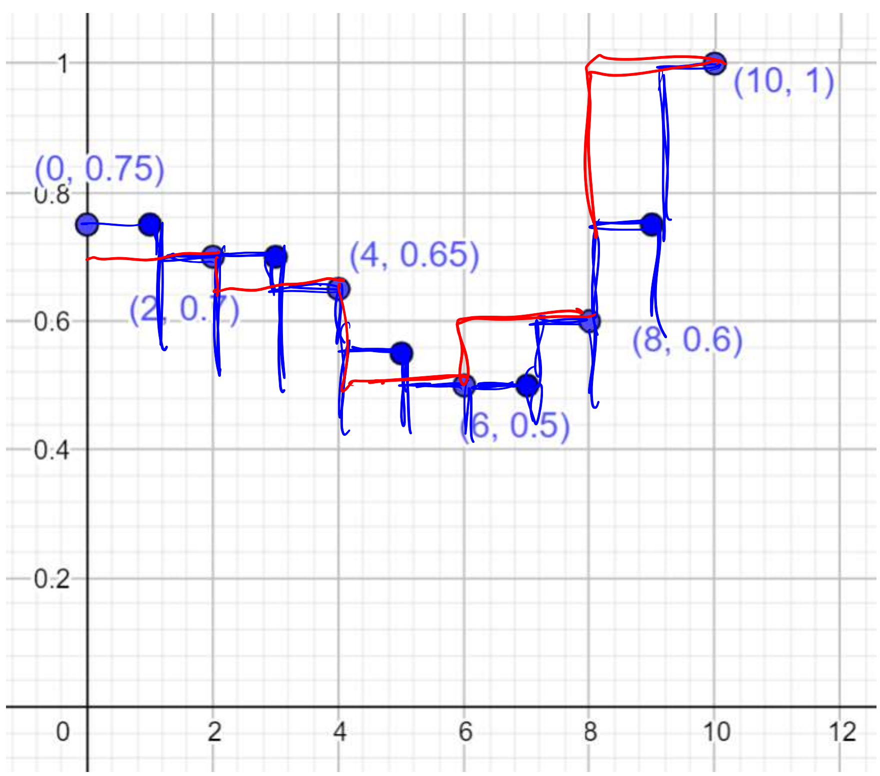
time (minutes)	speed (miles per minute)
0	0.75
1	0.75
2	0.7
3	0.7
4	0.65
5	0.55
6	0.5
7	0.5
8	0.6
9	0.75
10	1

(e) Estimate the total distance traveled during each interval in the following table, by assuming the speed on each interval is constant and given by the speed at the end of the interval.

on $(0,1)$ travel 0.75 mi/min $*$ 1 min $= 0.75$

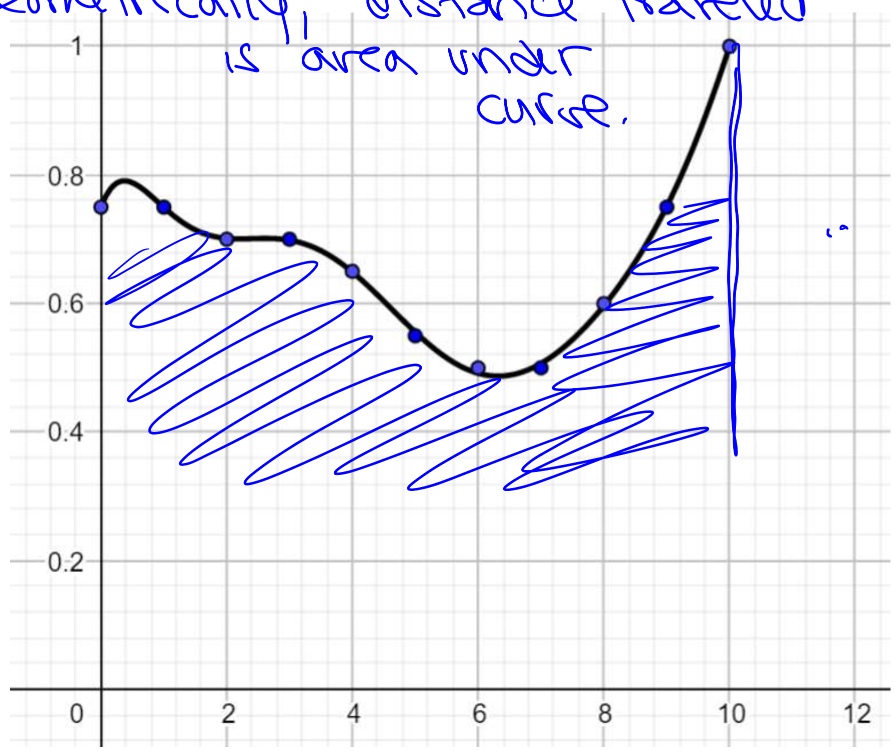
$0.75 + 0.7 + 0.7 + \cdots + 0.75 + 1 = 6.7$

(f) How can this method be represented graphically? What are we really adding up?



(g) If we were able to continuously record our speed, it might look something like the following curve.

geometrically, distance traveled is area under curve.



How is the total distance traveled represented geometrically on this graph? How could you estimate it?

Observation 22.1.2. When $v(t)$ represents speed of an object, the total distance traveled from time $t = a$ to $t = b$ is the area under the graph of $v(t)$ on the interval (a, b).

Observation 22.1.3. If $f(x)$ represents the rate of change of a quantity, then the net change in that quantity from $x = a$ to $x = b$ is the signed area under f from a to b.

22.2 The Definite Integral

Goals

- introduction to definite integrals
- integral as signed area

Definition 22.2.1. If we're adding a bunch of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n$$

then it's convenient to use the notation (called "Sigma Notation")

$$\boxed{\sum_{i=1}^{n} a_i.}$$

Σ greek "S" stands for sum

Here, i is called the *index*, and the numbers below and above the sigma symbol say to add all the a_i from $i = 1$ and stop at $i = n$.

Example 22.2.2. Some examples of this notation...

terms

stop

$$\sum_{n=1}^{5} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

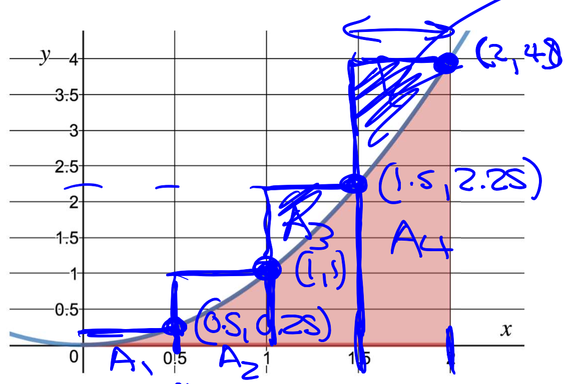
start

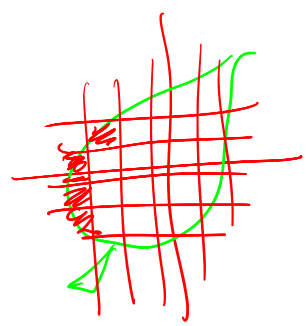
$$\sum_{k=2}^{4} 2^k = 2^2 + 2^3 + 2^4$$

Example 22.2.3. Estimate the area under the curve $y = x^2$ from $x = 0$ to $x = 2$.

over estimate

all the points are on curve $(x, f(x))$





try splitting into 4 rectangles

width of rectangles = 0.5

$A_1 = hw = 0.25 * 0.5$

$A_2 = hw = 1 * 0.5$

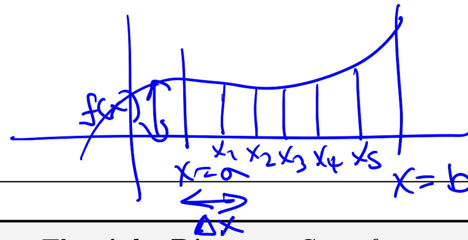
$A_3 = hw = 2.25 * 0.5$

$A_4 = hw = 4 * 0.5$

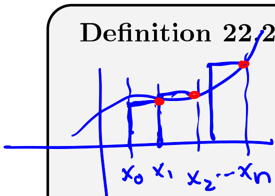
Area under x^2

$\approx A_1 + A_2 + A_3 + A_4$

$= 3.75$



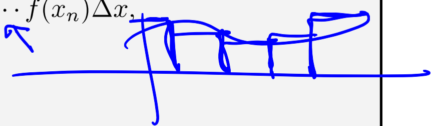
Definition 22.2.4. The **right Riemann Sum** from $x = a$ to $x = b$ of $f(x)$ is



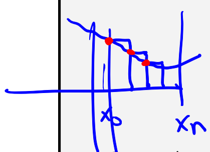
$$R_n = \sum_{i=1}^{n} \boxed{f(x_i)\Delta x} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x,$$

areas of rectangles

right endpts



and the **left Riemann Sum** from $x = a$ to $x = b$ of $f(x)$ is

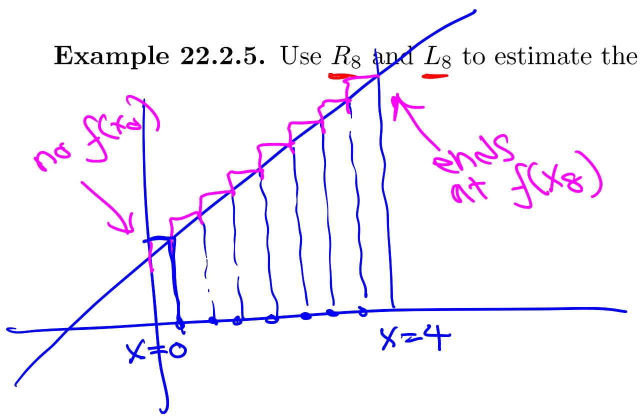


$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots f(x_{n-1})\Delta x,$$

areas of rectangles

where $\Delta x = \frac{b-a}{n}$ is the width of each rectangle, and the endpoints of each rectangle are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = x_1 + \Delta x$, and in general $x_i = x_{i-1} + i\Delta x$.

Example 22.2.5. Use R_8 and L_8 to estimate the area under $f(x) = 2x + 1$ from 0 to 4.



R_8 means 8 rectangles right endpts

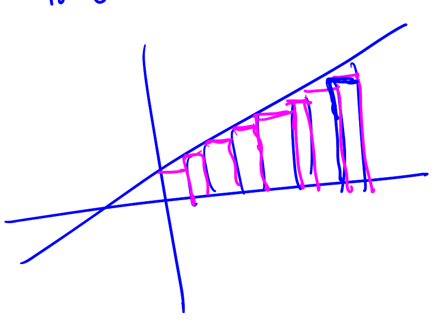
$$\Delta x = \frac{b-a}{n} = \frac{4-0}{8} = 0.5$$

$\Delta x = 0.5$

$x_0 = a = 0$
$x_1 = 0.5$
$x_2 = 1$
$x_3 = 1.5$
$x_4 = 2$
$x_5 = 2.5$
$x_6 = 3$
$x_7 = 3.5$
$x_8 = 4 = b$

$$R_8 = \sum_{n=1}^{8} f(x_n)\Delta x = 0.5 * f(0.5) + 0.5\, f(1) + 0.5\, f(1.5) + \cdots + 0.5\, f(4)$$
$$= 0.5\,(2) + 0.5(3) + 0.5(4) + \cdots + 0.5(9)$$
$$= 22$$

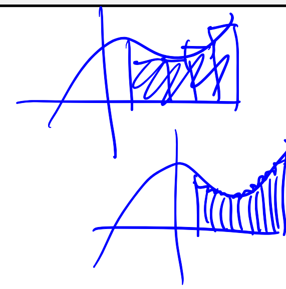
$$L_8 = \sum_{n=0}^{7} f(x_n)\Delta x = 0.5\, f(0) + 0.5\, f(0.5) + 0.5\, f(1) + \cdots + 0.5\, f(3.5)$$
$$= 0.5\,(1 + \cdots + 8)$$
$$= 18$$



Definition 22.2.6. If $f \geq 0$, then the **area** of the region under f (and above the axis) is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} [f(x_1)\Delta x + \cdots f(x_n)\Delta x].$$

For continuous f, this limit always exists, and is equal to $\lim_{n\to\infty} L_n$.



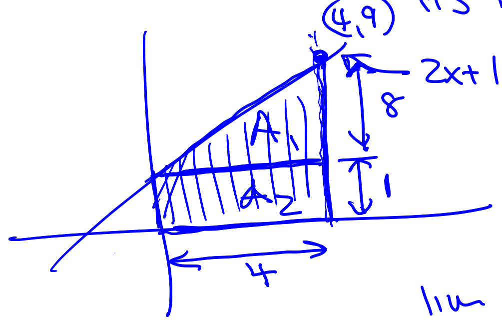
Example 22.2.7. Evaluate the limit of the right sum

definition of area

$$A = \boxed{\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

where $\boxed{f(x) = 2x + 1,\ x_0 = 0, \text{ and } x_n = 4}$

what is $\lim_{n\to\infty} R_n$? it's the area



$A_1 = \frac{8*4}{2} = 16$

$A_2 = 4*1 = 4$

$\lim_{n\to\infty} R_n = 16+4 = 20$