MATH 118
Local and global extrema

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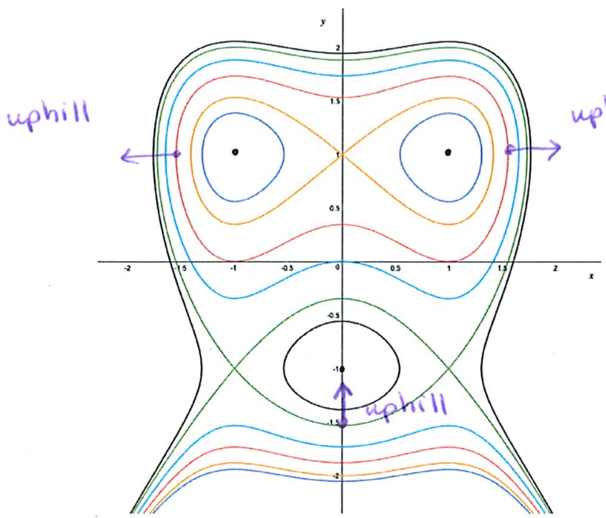
1. (a) The contour plot for $z = f(x, y)$ is shown below. Suppose $f_x(1.5, 1) > 0$, $f_x(-1.5, 1) < 0$ and $f_y(0, -1.5) > 0$, and that $\{(-1, 1), (0, 1), (1, 1), (-1, -1), (0, -1), (1, -1)\}$ is the set of all of the critical points of $f(x, y)$. Guess the classification of the critical points of $f(x, y)$.



$\Rightarrow$ (1, 1) and (1, -1) are local minima

(0, -1) is a local maxima

(-1, -1), (1, -1) and (0, 1) are saddle points.

(b) The function in part (a) is $f(x, y) = x^4 - 2x^2 + y^3 - 3y$. Confirm your answer in (a) by calculating and classifying the critical points of $f(x, y)$.

$\frac{\partial f}{\partial x} = 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0 \Rightarrow x = 0, 1, -1.$

$\frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \Rightarrow 3(y^2-1) = 0 \Rightarrow y = 1, -1.$

$\therefore$ critical points are (0, 1), (0, -1), (1, 1), (1, -1), (-1, 1), (-1, -1).

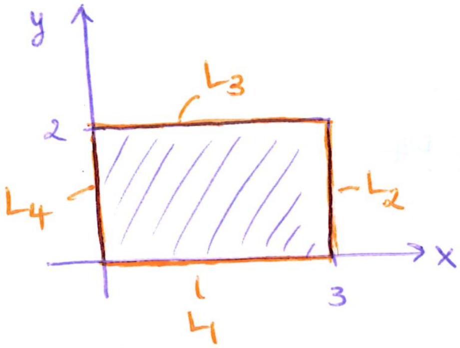
$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$\frac{\partial^2 f}{\partial y^2} = 6y$

$\frac{\partial^2 f}{\partial x \partial y} = 0$

CP (x_0, y_0)	$D(x_0, y_0)$	classification
(0, 1)	-24	saddle point
(0, -1)	24	$f_{xx}(0,-1) < 0 \Rightarrow$ local max.
(1, 1)	48	$f_{xx}(1,1) > 0 \Rightarrow$ local min.
(1, -1)	-48	saddle point
(-1, 1)	48	$f_{xx}(-1,1) > 0 \Rightarrow$ local min.
(-1, -1)	-48	saddle point.

2. Find the global maximum and minimum values of $f(x, y) = x^2 + y^2 - 4xy + 2$ if $0 \le x \le 3$ and $0 \le y \le 2$.



$$\frac{\partial f}{\partial x} = 2x - 4y = 0 \quad (1)$$

$$\frac{\partial f}{\partial y} = 2y - 4x = 0 \quad (2)$$

$(1) \Rightarrow x = 2y$

so $(2) \Rightarrow 2y - 4(2y) = 0$

$\Rightarrow -6y = 0$

$\Rightarrow y = 0 \Rightarrow x = 2(0) = 0$

$\therefore$ The only critical point is $(0,0)$ (which is in the rectangle)

BOUNDARY:

L_1: $y = 0$, $0 \le x \le 3$

$f(x,0) = x^2 + 2$.

$\therefore$ Let $g_1(x) = x^2 + 2$.

$g_1'(x) = 2x = 0 \Rightarrow x = 0$

$\therefore$ testpoints are $\overbrace{\underbrace{(0,0)}_{\text{cp of } g_1(x)}, (3,0)}^{\text{endpoints of } L_1}$

L_2: $x = 3$, $0 \le y \le 2$

$f(3,y) = 9 + y^2 - 12y + 2$

$\therefore$ Let $g_2(y) = 11 + y^2 - 12y$

$g_2'(y) = 2y - 12 = 0$

$\Rightarrow \cancel{y = 6}$

not between 0 and 2

$\therefore$ testpoints are

$\underbrace{(3,0), (3,2)}_{\text{endpoints of } L_2}$

L_3: $y = 2$, $0 \le x \le 3$

$f(x,2) = x^2 + 4 - 8x + 2$.

$\therefore$ Let $g_3(x) = x^2 - 8x + 6$

$g_3'(x) = 2x - 8 = 0$

$\Rightarrow \cancel{x = 4}$

$\therefore$ testpoints are.

$\underbrace{(0,2), (3,2)}_{\text{endpoints of } L_3}$

L_4: $x = 0$, $0 \le y \le 2$.

$f(0, y) = y^2$.

$\therefore$ Let $g_4(y) = y^2$

$g_4'(y) = 2y = 0 \Rightarrow y = 0$

$\therefore$ testpoints are

$\underbrace{(0,0)}_{\text{CP of } g_4(y)}, (0,2)$

endpoints of L_4.

testpoint (x_0, y_0)	$f(x_0, y_0)$
$(0, 0)$	2
$(0, 2)$	6
$(3, 0)$	11
$(3, 2)$	-9

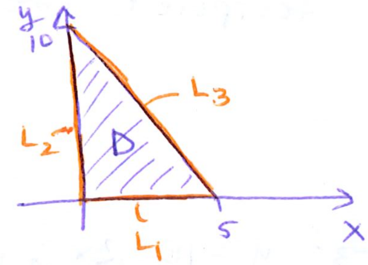
$\therefore$ The global max is 11 and the global min is -9.

3. It's estimated that a company's profit in selling x units of good A and y units of good B is

$$\pi(x, y) = 1500 + 80x - 10x^2 + 40xy + 40y - 60y^2.$$

It costs the company \$200 to produce each unit of good A and \$100 to produce each unit of good B, and they cannot spend more than \$1000 on production costs.

(a) Write down the "physical" domain of the profit function.



$200x + 100y \leq 1000$] budget constraint

$D = \{(x,y) \mid 2x + y \leq 10,\ x \geq 0,\ y \geq 0\}$

(b) How many units of each good should the company produce in order to maximize their profit?

$\pi_x = 80 - 20x + 40y = 0 \Rightarrow 4 - x + 2y = 0$ ①

$\pi_y = 40x + 40 - 120y = 0 \Rightarrow x + 1 - 3y = 0$ ②

① $\Rightarrow x = 4 - 2y$, so ② $\Rightarrow (4-2y) + 1 - 3y = 0$

$5 - 5y = 0 \Rightarrow y = 1$

$\Rightarrow x = 4 - 2(1) = 2.$

$\therefore$ $(2,1)$ is a critical point and $2(2) + (1) = 5 \leq 10$, so it is in D (from part (a)).

Boundary of D:

L_1: $y = 0$, $0 \leq x \leq 5$, $\pi(x, 0) = 1500 + 80x - 10x^2$.

$\therefore$ let $g_1(x) = 1500 + 80x - 10x^2$.

$g_1'(x) = 80 - 20x = 0 \Rightarrow x = 4$ (between 0 and 5 ✓)

$\therefore$ testpoints are $\underbrace{(0,0), (5,0)}_{\text{endpoints of } L_1}$, $\underbrace{(4,0)}_{\text{CP of } g_1(x)}$.

$\underline{L_2}$: $x=0,\ 0 \le y \le 10$, $\pi(0,y) = 1500+40y-60y^2$.

$\therefore$ let $g_2(y) = 1500+40y-60y^2$.

$g_2'(y) = 40-120y = 0 \Rightarrow y = 1/3$ (between 0 and 10 ✓)

$\therefore$ testpoints are $\underbrace{(0,0),(0,10)}_{\text{endpoints of } L_2}, \underbrace{(0,1/3)}_{\text{CP of } g_2(y)}$

$\underline{L_3}$: $y = 10-2x$, $0 \le x \le 5$

$\Rightarrow \pi(x,10-2x) = 1500+80x-10x^2+40x(10-2x)+40(10-2x)$
$-60(10-2x)^2$

$\therefore$ Let $g_3(x) = 1500+80x-10x^2+400x-80x^2+400-80x$
$-6000+2400x-240x^2$

$= -4100+2800x-330x^2$

$g_3'(x) = 2800-660x = 0 \Rightarrow x = \frac{2800}{660} = \frac{140}{33}$

$\Rightarrow y = 10-\frac{280}{33} = \frac{50}{33}$

$\therefore$ testpoints are $\underbrace{(0,10),(5,0)}_{\text{endpoints of } L_3}, \underbrace{(140/33, 50/33)}_{\text{CP of } g_3(x)}$

testpoint (x_0,y_0)	$\pi(x_0,y_0)$
$(2,1)$	1680
$(0,0)$	1500
$(5,0)$	1650
$(4,0)$	1660
$(0,10)$	$-\underline{4100}$
$(0,1/3)$	$1506.\overline{6}$
$(\frac{140}{33},\frac{50}{33})$	$\underline{1839.\overline{39}}$

$\therefore$ global min is $\underline{-4100}$ and global max is $\underline{1839.\overline{39}}$

$\Rightarrow$ They should produce 140/33 (~4) units of good A and 50/33 (~2) units of good B.