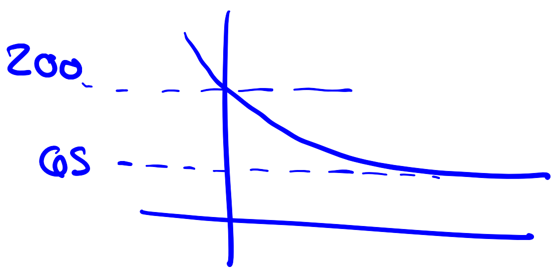
Example 31.1.1. Suppose you're drinking hot apple cider at Koffee. The cider will burn your mouth at 200°, but you know if you wait it will cool down eventually. Your friend's coffee was served at a slightly lower temperature at 180°.

(a) Whose drink do you think is getting cold faster? Are they cooling at the same speed?

(b) If the room is set at 65°. What do you think the graph of the temperature of you drink with respect to time is?



$T(0) = 200$

(c) If the room's temperature is R, and the cider's temperature is T, Newton's law of cooling says

> **Theorem 31.1.2.** *The rate of change of the difference between two temperatures T and R is proportional to the difference between T and R.*

Can you represent this using a mathematical equation?

$\frac{dT}{dt}$ = (proportion)(difference b/w T & R)

$$\boxed{\frac{dT}{dt} = K(T-R)}$$

mathetical model

temp & change in temp

Definition 31.1.3. An equation that contains a variable and its derivative (or derivatives) is called a differential equation. ex. $y = y'$ $\quad y^2 = y'' + x$

Example 31.1.4. Find a model: If a population is flourishing under ideal circumstances and with unlimited resources, its rate of growth is proportional to itself.

$\frac{dP}{dt}$ $\quad P$

$$\frac{dP}{dt} = (\text{proportion}) * (\text{population} \ \cancel{\text{amount}})$$

$$\frac{dP}{dt} = rP$$

constant

$\frac{dc}{dt}$

Example 31.1.5. The concentration of a certain nutrient in a cell changes at a rate proportional to the difference between the concentration of the nutrient inside the cell and the concentration in the surrounding environment. Suppose that the concentration in the surrounding environment is kept constant and is given by N. If the concentration of the nutrient in the cell is greater than N, then the concentration in the cell decreases; if the concentration in the cell is less than N, then the concentration increases. Let $C = C(t)$ be the concentration of the nutrient within the cell. Write a differential equation involving the rate of change of C.

C

N

concentration inside

concentration in surrounding env. (constant)

$$\frac{dc}{dt} = (\text{proportion}) * (\text{diff in concentrations})$$

k N−C

$$\frac{dc}{dt} = k(N-C)$$

if this is negative. that's fine

k will take care of it

(total rate) = (rate in) − (rate out)

0.04

Example 31.1.6. Ten thousand dollars is deposited in a bank account with an annual interest rate of 4% compounded continuously. No further deposits are made. Write a differential equation fitting the situation if money is withdrawn continuously at a rate of $4000 per year. $/yr

proportion

rate in = 0.04 * (Amount in account)

rate out = 4,000

(warning: 4000t is amount)

let $M(t)$ = amount in account

$$\boxed{\begin{aligned} \frac{dM}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= 0.04M - 4000 \\ M(0) &= 10000 \end{aligned}}$$

Example 31.1.7. Consider the differential equation

$$\frac{dM}{dt} = 0.04M - 4000$$

where $M(t)$ is the amount of money in a bank account at time t, where t is given in years. The differential equation reflects the situation in which interest is being paid at a rate of 4% per year compounded continuously and money is being withdrawn at a constant rate of \$4000 per year.

(a) Suppose the initial deposit is \$50000. Will the account be depleted?

$0.04(50000) - 4000 = -2000$

ROC wrt time

decr at first, keeps decr

$\Rightarrow$ acct will be depleted

(b) If money is to be withdrawn at a rate of \$4000 per year, what is the minimum initial investment that assures the account is not depleted?

need to know an M

st $0.04M - 4000 \geq 0$

$\Leftrightarrow M \geq 100{,}000$

(c) If this is a trust fund that is being set up with \$50000 and the idea is that the account should not be depleted, what should the restriction be on the rate of withdrawal? Assume money will be withdrawn at a constant rate.

Now we know

$M = 50{,}000$

solve $0.04(50{,}000) - W \geq 0$

$W \leq 2{,}000$

31.1.1 Extra Problems

Example 31.1.8. The flu is spreading throughout a college dormitory of 300 students. It is highly contagious and long in duration. Assume that during the time period we are modeling no student has recovered and all sick students are still contagious. It is reasonable to assume that the rate at which students are getting ill is proportional to the product of the number of sick students and the number of healthy ones because there must be an interaction between a healthy and a sick student to pass along the disease. Let $S = S(t)$ be the number of sick students at time t . Write a differential equation reflecting the situation.

$$\frac{dS}{dt} = (\text{proportion}) * (\text{sick students}) * (\text{healthy})$$

$$\frac{dS}{dt} = k\,S*H = kS(300-S)$$

Example 31.1.9. The rate at which a certain drug is eliminated from the bloodstream is proportional to the amount of the drug in the bloodstream. A patient now has 45 mg of the drug in his bloodstream. The drug is being administered to the patient intravenously at a constant rate of 5 milligrams per hour. Write a differential equation modeling the situation.

rate in

coming in & going out

(total rate) = (rate in) − (rate out)

$$\frac{dA}{dt} = 5\,\text{mg/hr} - (\text{proportion}) * (\text{amount})$$

$$\boxed{\frac{dA}{dt} = 5 - k * A}$$

fixed rate 5mg/hr

~~general~~

depends on t

$$A(0) = 45$$

initial condition

Example 31.1.10. An object falling through the air undergoes a constant downward acceleration of 32 feet per second per second due to the force of gravity.

(a) Write a differential equation involving $v(t)$, the vertical velocity of the object at time t.

this is change in velocity!

$$\frac{dv}{dt} = -32$$

change in velocity

change ft/s²

(b) Write a differential equation involving $s(t)$, the object's height above the ground.

Since $s'(t) = v(t)$, we know

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = -32$$

remember this notation? It means "second deriv" w.r.t. t

so the diff eqn. is

$$\frac{d^2s}{dt^2} = -32$$

or

$$s'' = -32$$

e.g. $y = y'$

$y = e^x$ is a soln b/c $y' = e^x$

$y' = y$

$e^x = e^x$ ✓

> **Definition 31.2.1.** A function f is a **solution to a differential equation** if it satisfies the differential equation.

Example 31.2.2. Is $y = x^3$ a solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x}?$$

suppose $y = x^3$

then $\frac{dy}{dx} = 3x^2$

is $\frac{dy}{dx} \stackrel{?}{=} \frac{3y}{x}$

$3x^2 \stackrel{?}{=} \frac{3(x^3)}{x} = 3x^2$

$\Rightarrow$ soln!

make sure equation still makes sense!

Example 31.2.3. Is $y = xe^{3x}$ a solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x}?$$

get all derivs you need

$$\frac{dy}{dx} = 3xe^{3x} + e^{3x}$$

make sure eqn balances

$$3xe^{3x} + e^{3x} \stackrel{?}{=} \frac{3xe^{3x}}{x} = 3e^{3x}$$

No!

$3x = 4$

Example 31.2.4. Check that all of the following functions are solutions to the differential equation

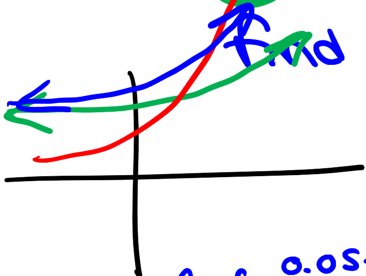
$$\frac{dM}{dt} = 0.05M - 5000$$

Which one has the initial condition that $M(0) = 10000$?

plug in 0 & see if I get 10,000

(a) $M(t) = 2e^{0.05t} + 100000$

find $\frac{dM}{dt}$, and plug in $M(t)$, $\frac{dM}{dt}$



$$\frac{dM}{dt} = \frac{d}{dt}\left(2e^{0.05t} + 100{,}000\right) = 0.1e^{0.05t}$$

$$0.1e^{0.05t} = \boxed{\frac{dM}{dt} = 0.05M - 5000} = 0.05\left(2e^{0.05t} + 100000\right) - 5000$$

must be true

$$= 0.1e^{0.05t} \checkmark$$

$M(0) = 100{,}002$

(b) $M(t) = -90000e^{0.05t} + 100000$

$$\frac{dM}{dt} = -90000 * 0.05e^{0.05t}$$

$$= -4500\,e^{0.05t}$$

$$M(0) = -90000e^{0} + 100000 = 10{,}000$$

(c) $M(t) = 10e^{0.05t} + 100000$

$$\frac{dM}{dt} = 0.5e^{0.05t}$$

$\frac{dy}{dx} = 2$ means slope is two

$\frac{dy}{dx} = x$ means at every point (x,y) $m = x$

e.g. $(1,2)$ $m=1$

Definition 31.2.5. At any point P, we can use a differential equation to find the slope of the tangent line to the solution curve through P. This gives us a rough idea of the shapes of particular solutions. If we plot some of these slopes, we call the result a **slope field.**

Example 31.2.6. Match the following differential equations with their slope fields.

always 2 no matter point

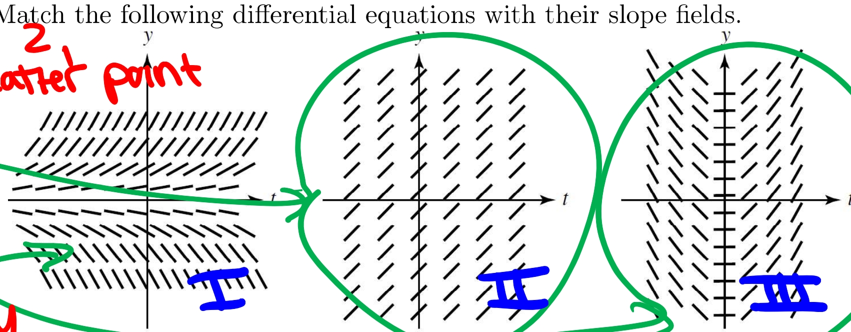
(a) $\frac{dy}{dx} = 2$

slopes don't depend on y

(b) $\frac{dy}{dx} = 2x$

(c) $\frac{dy}{dx} = 2y$

slopes only depend on y



clarification

(a) is II

(b) is III

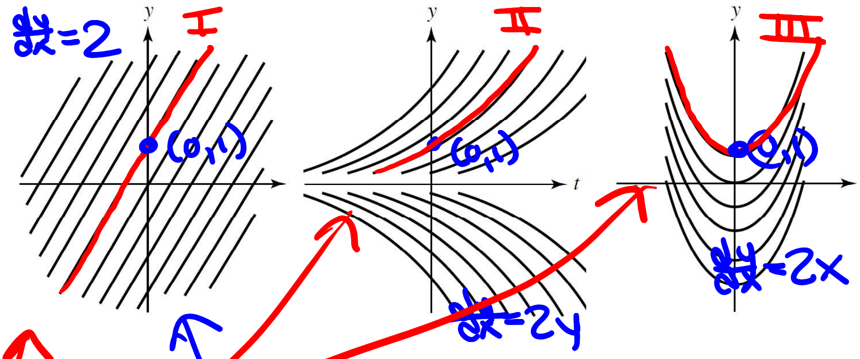
(c) is I

Example 31.2.7. Match the following differential equations with a graph of their solution curves.

(a) $\dfrac{dy}{dx} = 2$ I

(b) $\dfrac{dy}{dx} = 2x$ III

(c) $\dfrac{dy}{dx} = 2y$ II



families of solns

"general solns"

If want a specific one we'd call it a "particular" soln

red lines are "particular" solutions, ones where an initial condition like $y(0)=1$ has been given

Example 31.2.8. Guess and check possible solutions to the following differential equations.

(a) $\frac{dy}{dx} = 2$

$y = ?$

want a fn with derivative 2!

eg. $y = 2x$
$\frac{dy}{dx} = 2$ ✓

$y = 2x + 1$
$\frac{dy}{dx} = 2$

$y = 2x + C$ for any C!
$\frac{dy}{dx} = 2$

(b) $\frac{dy}{dx} = 2x$

what fn has 2x as deriv?

$x^2 + 67$ $\quad x^2 + 10$

$y = x^2 + C$ any C

(c) $\frac{dy}{dx} = 2y$ what function is proportional to its own derivative?
something like e^x?

We tried:

- $y = 2e^x$, but then $\frac{dy}{dx} = 2e^x \Rightarrow \frac{dy}{dx} = y \boxed{\neq 2y}$
- $y = 2e^{x^2}$, but then $\frac{dy}{dx} = 2xe^{x^2} \Rightarrow \frac{dy}{dx} = 2xy \boxed{\neq 2y}$
- $y = e^{2x}$, and this worked! $\frac{dy}{dx} = 2e^{2x} = 2y$ ✓

general solution is $\boxed{y = Ce^{2x}}$

- does $y = e^{2x} + C$ work?

$\Rightarrow \frac{dy}{dx} = 2e^{2x}$ but $2e^{2x} = \frac{dy}{dx} \neq 2y = 2(e^{2x} + C)$

we wanted equality

i.e. if $\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = g(y)$ where f or g are differentiable, then there's always a particular soln.

Theorem 31.2.9 (Existence and Uniqueness)**.** Let (a, b) be a point in the plane.

- Any differential equation $\frac{dy}{dx} = f(x)$ where f is continuous has a unique solution passing through (a, b).
- The same is true if $\frac{dy}{dx} = g(y)$ where g and g' are both continuous.

~~**Example 31.2.10.** An object falling through the air undergoes a constant downward acceleration of 32 feet per second per second due to the force of gravity.~~

~~(a) What is $v(t)$ if the initial velocity is 0?~~

Skip this, we only need to know

$y' = ky \iff y = Ce^{kt}$

~~(b) What is $s(t)$ if the initial position is 100 feet above the ground?~~

Example 31.2.12.

(a) Find a model: If a population is flourishing under ideal circumstances and with unlimited resources, its rate of growth is proportional to itself.

$$\frac{dP}{dt} = kP$$

(b) Let $P(t)$ be population at t years, and k be the proportion. What is the general solution?

general soln

$$P = Ce^{kt}$$

(c) If $k = 0.05$, and $P(0) = 5000$, what's the population after 10 years?

$$P(t) = Ce^{0.05t}$$

$$5000 = P(0) = Ce^{0} = C$$

$$P(t) = 5000e^{0.05t}$$

$$P(10) = 5000e^{0.05 * 10} \approx 8{,}243$$