2.2 # 3, 6, 7, 9, 11, 13, 15, 23, 24, 29, 32, 37

3.) Find the inverse of $\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$.

$$\frac{1}{-21+18}\begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}$$

6.) Use the inverse from exercise 3 to solve the system.

$$7x_1 + 3x_2 = -9$$
$$-6x_1 - 3x_2 = 4$$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}\begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 26/3 \end{bmatrix}$$

7.) Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\vec{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

a.) Find A^{-1} and use it to solve $A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, $A\vec{x} = \vec{b}_3$, & $A\vec{x} = \vec{b}_4$

$$A^{-1} = \frac{1}{12-10}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}_2 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 22 \\ -10 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}_3 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}_4 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 26 \\ -10 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

b.) Solve the equations in part (a) by row reducing $[A\ \vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4]$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix} \begin{matrix} \\ -5R_1+R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \begin{matrix} -R_2+R_1 \\ R_2/2 \end{matrix} \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

9.) True / False

a.) In order for the matrix B to be the inverse of A, the equation $AB = I$ and $BA = I$ must both be true. — a.) True

b.) If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB. — b.) False

2.2

9.) c.) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible. c.) False

d.) If A is an invertible $n \times n$ matrix, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^n$. d.) True

e.) Each elementary matrix is invertible. e.) True

11.) Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

Let $X = A^{-1}B$, then $AX = AA^{-1}B = IB$ since A is invertible, $AA^{-1} = I$

$IB = B$ since I is an $n \times n$ matrix and B is $n \times p$. Therefore $A^{-1}B$ is a solution to $AX = B$. Now we need to show that this solution is unique. Suppose X is an arbitrary solution. Then $AX = B$ and multiplying both sides by A^{-1} gives us $A^{-1}AX = A^{-1}B$ so $X = A^{-1}B$. Therefore $A^{-1}B$ is a unique soln.

13.) Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true in general when A is not invertible?

Multiplying both sides by A^{-1} yields $A^{-1}AB = A^{-1}AC$ which implies $IB = IC$ and so $B = C$. In general this is not true when A is not invertible.

2.2 continued

15.) Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction: If $[A\ B] \sim \cdots \sim [I\ X]$, then $X = A^{-1}B$. (If A is larger than 2×2 then row reduction of $[A\ B]$ is much faster than computing A^{-1} and $A^{-1}B$.)

The elementary row operations that transform A to I are A^{-1}. Applying these to B gives $A^{-1}B$. Therefore $X = A^{-1}B$.

23.) Suppose A is $n \times n$ and the equation $A\vec{x} = \vec{0}$ has only the trivial solution. Explain why A has n pivot columns and A is row equivalent to I_n. (This shows that A must be invertible)

Since there is only the trivial solution, $A\vec{x} = \vec{0}$ must not have any free variables, so A has n pivot columns. Thus there is a sequence of elementary row operations that transform A to I_n. Therefore A and I_n are row equivalent.

24.) Suppose A is $n \times n$ and the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^n$. Explain why A must be invertible.

Since $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^n$, A must have a pivot position in each of the n rows. Therefore A has n pivot columns and A is row equivalent to I_n. Thus A is invertible.

29.) Find the inverse of $A=\begin{bmatrix}1 & -3\\ 4 & -9\end{bmatrix}$ if it exists. Use the algorithm introduced in this section.

$$\left[\begin{array}{cc|cc}1 & -3 & 1 & 0\\ 4 & -9 & 0 & 1\end{array}\right] \begin{array}{c}\\ -4R_1+R_2\end{array} \left[\begin{array}{cc|cc}1 & -3 & 1 & 0\\ 0 & 3 & -4 & 1\end{array}\right] \begin{array}{c}R_2+R_1\\ R_2/3\end{array} \left[\begin{array}{cc|cc}1 & 0 & -3 & 1\\ 0 & 1 & -4/3 & 1/3\end{array}\right] \qquad A^{-1}=\begin{bmatrix}-3 & 1\\ -4/3 & 1/3\end{bmatrix}$$

32.) $A=\begin{bmatrix}1 & 2 & -1\\ -4 & -7 & 3\\ -2 & -6 & 4\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ -4 & -7 & 3 & 0 & 1 & 0\\ -2 & -6 & 4 & 0 & 0 & 1\end{array}\right] \begin{array}{c}\\ 4R_1+R_2\\ 2R_1+R_3\end{array} \left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -1 & 4 & 1 & 0\\ 0 & -2 & 2 & 2 & 0 & 1\end{array}\right] \begin{array}{c}-2R_2+R_1\\ \\ 2R_2+R_3\end{array} \left[\begin{array}{ccc|ccc}1 & 0 & 1 & -7 & -2 & 0\\ 0 & 1 & -1 & 4 & 1 & 0\\ 0 & 0 & 0 & 10 & 2 & 1\end{array}\right]$$

A does not have an inverse.

37.) Let $A=\begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 5\end{bmatrix}$. Construct a 2×3 matrix C (by trial and error) using only 1, -1, and 0 as entries, such that $CA=I_2$. Compute AC and note $AC \neq I_3$.

$$\begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad C=\begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 0\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 5\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 0\end{bmatrix}=\begin{bmatrix}-1 & 3 & -1\\ -2 & 4 & -1\\ -4 & 6 & -1\end{bmatrix} \neq I_3$$