YALE UNIVERSITY

Calculus And Linear Algebra Notes

Economics Applications

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Math 118

Introduction

Linear algebra and multivariable calculus have many applications in the field of economics. For those of you interested in launching careers in government, finance, or consulting after graduation, having a solid grasp of the topics covered in Math 118 is essential for success.

Below are notes on the various calculus and linear algebra topics that will be covered in this class. For those of you who are considering majoring in economics, these notes will cover a few very important concepts that will show up later in your academic journey here at Yale. For the rest of you, I have tried to make these notes enjoyable.

Linear Algebra

Section 1.1 Vectors and Linear Combinations

In traditional undergraduate-level economics classes, the Robinson Crusoe economy is the first model that students will encounter when studying trade theory and the costs and benefits of conducting trade versus maintaining autarky (no trade). The Robinson Crusoe model has several distinct features—in particular, the units of trade which are commonly referred to as "factors" usually consist of homogenous commodities that can be consumed directly without any additional processing (i.e. fruits and vegetables). The following examples will help illustrate how vectors and linear combinations can be applied to the Robinson Crusoe model and trade theory in general.

In the below example, we will be analyzing elves. It is a widely known and readily accepted fact that elves get their magical abilities from following strict dietary guidelines. More specifically, in order for an elf to maintain its magical abilities, it must consume one of three bundles: (i) 3 durians, 2 dragon fruits, and no lychees; (ii) no durians, 4 dragon fruits, and 3 lychees; (iii) no durians, 3 dragon fruits, 5 lychees.

These bundles can be rewritten as column vectors in the form $\begin{bmatrix} Durian \\ Dragon\ Fruit \\ Lychee \end{bmatrix}$:

$$\text{(i) } <\boldsymbol{a}>=\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \text{ or } [3\ \ 2\ \ 0]^{\mathrm{T}} \qquad \text{(ii) } <\boldsymbol{b}>=\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \text{ or } [0\ \ 4\ \ 3]^{\mathrm{T}} \qquad \text{(iii) } <\boldsymbol{c}>=\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \text{ or or } [0\ \ 3\ \ 5]^{\mathrm{T}}$$

<u>Example 1</u>

Suppose an elf named Daniel decides that he will be going on a 120-day trip to explore New Haven and will need to pack a sufficient amount of food for his journey. One possible solution would be for him to consume 40 of each of the three possible bundles during the period. Notice that scalar multiplication works in this case.

Hence, one possible solution:

$$(40)*<\boldsymbol{a}> + (40)*<\boldsymbol{b}> + (40)*<\boldsymbol{c}> = \begin{bmatrix} 120 \\ 80 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ 120 \end{bmatrix} + \begin{bmatrix} 0 \\ 120 \\ 200 \end{bmatrix} = \begin{bmatrix} 120 \\ 360 \\ 320 \end{bmatrix}$$

Alternatively, it would have been fine to simply pick 120 of any of the three bundles:

$$(120)*<\boldsymbol{a}>= \begin{bmatrix} 360 \\ 240 \\ 0 \end{bmatrix} \qquad (120)*<\boldsymbol{b}>= \begin{bmatrix} 0 \\ 480 \\ 360 \end{bmatrix} \qquad (120)*<\boldsymbol{c}>= \begin{bmatrix} 0 \\ 360 \\ 600 \end{bmatrix}$$

Suppose however, that Daniel has a very sensitive nose and greatly dislikes the smell of durian. Additionally, assume that Daniel is allergic to durian and is therefore incapable of eating it. A viable alternative solution for the trip would be:

$$(0)*<\mathbf{a}> + (100)*<\mathbf{b}> + (20)*<\mathbf{c}> = [0\ \ 460\ \ 400]^{\mathrm{T}}$$

Linear Algebra

Section 1.2 Lengths and Dot Products

The dot product of two vectors is denoted:

$$<\boldsymbol{a}_1> \bullet <\boldsymbol{b}_2> = \sum_i a_i b_i$$

It is important to note that the dot product between two vectors produces a single scalar value and is well-defined only when the vectors are of the same dimension. Hence matrix multiplication between a row vector and a column vector can be used as an equivalent representation:

$$<\boldsymbol{a}_1>\bullet<\boldsymbol{b}_2> = \sum_i a_i b_i = [a_1 \quad a_2 \quad \cdots \quad a_{n-1} \quad a_n]\ [b_1 \quad b_2 \quad \cdots \quad b_{n-1} \quad b_n]^{\mathrm{T}}$$

In finance and economics, dot products are extremely useful when performing calculations that involve large market baskets or indices. For those who are unfamiliar with the terminology, a market basket is “a subset of products or securities that is designed to mimic the performance of an overall market. Market baskets contain a fixed selection of items, which are used to track such things as inflation, prices or performance levels. For investors, the market basket is the principal idea behind index funds. A sample of stocks, bonds or other securities are placed in a portfolio that is expected to represent all aspects of the market. This provides investors with a benchmark against which to compare their investment returns.” [1]

Index investing is “a form of passive investing that aims to generate the same rate of return as an underlying market index. Investors that use index investing seek to replicate the performance of a specific index – generally an equity or fixed-income index – by investing in an investment vehicle such as index funds or exchange-traded funds that closely track the performance of these indexes. Proponents of index investing eschew active investment management because they believe that it is impossible to "beat the market" once trading costs and taxes are taken into account. As index investing is relatively passive, index funds usually have lower management fees and expenses than actively managed funds. Lower trading activity may also result in more favorable taxation for index funds as compared with actively managed funds.” [2]

In particular, this has led to the expansion of exchange-traded funds (ETFs), which are investment products that are designed to track the performance of specific market indices such as the S&P 500 and Russell 3000. As the numbers in their names suggest, these indices often contain hundreds, if not thousands, of publicly-traded securities that have all their own corresponding prices. Hence, in order for asset managers to provide daily portfolio performance estimates to their clients, the reporting systems must actively keep track of and perform operations on thousands of pieces of data. Dot products make this possible and efficient.

Example 1

Suppose that there is a portfolio with n publicly-traded securities and that the quantity of each of these securities is represented by $x_{1_t}, x_{2_t}, \cdots, x_{n-1_t}, x_{n_t}$. Further suppose that each of these n securities has an associated end-of-day market price $p_{1_t}, p_{2_t}, \cdots, p_{n-1_t}, p_{n_t}$.

These prices and quantities can be written in vector form:

$$<\boldsymbol{p_t}>= [p_{1_t} \quad p_{2_t} \quad \cdots \quad p_{n-1_t} \quad p_{n_t}]$$

$$<\boldsymbol{x_t}>= [x_{1_t} \quad x_{2_t} \quad \cdots \quad x_{n-1_t} \quad x_{n_t}]$$

The dot product of the two vectors will give a scalar value that represents the end-of-day market value of the portfolio:

$$<\boldsymbol{p_t}> \bullet <\boldsymbol{x_t}> = p_{1_t}x_{1_t} + p_{2_t}x_{2_t} + \ldots + p_{n-1_t}x_{n-1_t} + p_{n_t}x_{n_t}$$

Assuming that there are no cash flows into or out of the portfolio, the daily time-weighted performance can be calculated by taking the market value of the portfolio of the current day and dividing it by the market value of the portfolio from the previous day:

$$\frac{< \boldsymbol{p_t} > \bullet < \boldsymbol{x_t} >}{< \boldsymbol{p_{t-1}} > \bullet < \boldsymbol{x_{t-1}} >} = (1 + return)$$

As a more concrete example, let our portfolio consist of the following:

Security	Quantity (t=0)	Price (t=0)	Quantity (t=1)	Price (t=1)
Stock 1	10 Shares	\$20.20	10 Shares	\$20.40
Stock 2	15 Shares	\$21.12	15 Shares	\$21.00
Stock 3	30 Shares	\$22.22	30 Shares	\$23.45

$$<\boldsymbol{p_0}>= [p_{1_0} \quad p_{2_0} \quad p_{3_0}] = [20.20 \quad 21.12 \quad 22.22]$$

$$<\boldsymbol{x_0}>= [x_{1_0} \quad x_{2_0} \quad x_{3_0}] = [10 \quad 15 \quad 30]$$

$$<\boldsymbol{p_1}>= [p_{1_1} \quad p_{2_1} \quad p_{3_1}] = [20.40 \quad 21.00 \quad 23.45]$$

$$<\boldsymbol{x_1}>= [x_{1_t} \quad x_{2_t} \quad x_{3_t}] = [10 \quad 15 \quad 30]$$

Daily portfolio market value:

$<\boldsymbol{p_0}> \bullet <\boldsymbol{x_0}> =$ \$1185.4

$<\boldsymbol{p_1}> \bullet <\boldsymbol{x_1}> =$ \$1222.5

Daily portfolio performance:

$$\frac{< \boldsymbol{p}_1 > \bullet < \boldsymbol{x}_1 >}{< \boldsymbol{p}_0 > \bullet < \boldsymbol{x}_0 >} = \frac{(20.40 * 10) + (21.00 * 15) + (23.45 * 30)}{(20.20 + 10) + (21.12 * 15) + (22.22 * 30)} = 1.0313$$

Hence the portfolio experienced a one-day return of 3.13%.

Example 2

The Personal Consumption Expenditure (PCE) and Consumer Price Index (CPI) are both price inflation measures that are closely monitored by the Federal Reserve Board of Governors to determine the path of monetary policy in the United States. Below, we will analyze simpler inflation measures such as the Laspeyres and Paasche price indices.

Laspeyres Index[3]:

$$\mathrm{I}_{\mathrm{Laspeyres}} = \frac{<\boldsymbol{p_t}> \bullet <\boldsymbol{x_{t_0}}>}{<\boldsymbol{p_{t_0}}> \bullet <\boldsymbol{x_{t_0}}>}$$

The Laspeyres price index uses a static market basket of goods from the base year, t_0, as a benchmark to weight the prices. The benefit of having a fixed basket is that it saves researchers time and money because they only need to track the changes in prices and not have to worry about constantly updating the basket. The downside of this is that this measure completely ignores both the substitution effect and the introduction of new goods.

Suppose a researcher wishes to use the Laspeyres method to calculate annual inflation. The researcher's first challenge would be to create a hypothetical basket containing all the goods that an average person would consume over the period of one year. Next the researcher would go find and record the prices for each of those goods. After one year, the researcher would go back to record the new prices for those same goods. From then on, the researcher only needs to worry about tracking the changes in the prices of the fixed list of goods in the basket. It is important to note that if average consumption undergoes a shift, then this method would be providing an irrelevant number—for instance, using the average consumption basket from 1850 to calculate inflation today will lead to errors.

Paasche Index[4]:

$$\mathrm{I}_{\mathrm{Paasche}} = \frac{<\boldsymbol{p_t}> \bullet <\boldsymbol{x_t}>}{<\boldsymbol{p_{t_0}}> \bullet <\boldsymbol{x_t}>}$$

The Paasche price index uses a dynamic market basket of goods from the current year, t, to weight the prices. The benefit of this method is that the market basket can be updated to reflect changes in consumption behavior. One major drawback, however, is that this method requires much more time and energy since the basket needs to be updated constantly. Additionally, researchers will need to find historical base year prices for each good that is incorporated. Lastly, this method still ignores the substitution effect, but in a different way—the changes in the basket are incorporated as changes in preferences.

Suppose a researcher wishes to use the Paasche method to calculate inflation for the upcoming year. The researcher's first challenge would be to create a hypothetical basket containing all the goods that the average person consumed over the previous year. Next the researcher would go find and record the current prices as well as the prices from the base year for each of those goods. The researcher would have to complete this process in its entirety each year.

Fun fact: Yale University follows a basket of goods for higher education (HEPI) to calculate inflation. The higher education basket is typically well approximated by CPI + 1%. [5]

Sources:
[1] http://www.investopedia.com/terms/m/market_basket.asp
[2] http://www.investopedia.com/terms/i/index-investing.asp
[3] http://mathworld.wolfram.com/LaspeyresIndex.html
[4] http://mathworld.wolfram.com/PaaschesIndex.htmlhttp://us.spindices.com/
[5] http://investments.yale.edu/index.php/reports/endowment-update

Additional Resources:
http://www.bls.gov/cpi/cpifaq.htm
https://www.commonfund.org/CommonfundInstitute/HEPI/Pages/default.aspx
http://www.msci.com/

Linear Algebra

1.3 Matrices

Example 1

Suppose a student takes five classes that meet biweekly and that these classes are located in Dunham Lab (DL), Linsly Chittenden (LC), Leet Oliver Memorial (LOM), Sheffield-Sterling-Strathcona (SSS), and William L. Harkness (WLH). We can find and record the distance (m) for each trip from the twelve colleges to the five lecture halls in matrix form:

$$\begin{array}{cll}
 & & \text{Ending Location} \\
 & & \begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\
\begin{array}{c}\text{Starting}\\\text{Location}\end{array} &
\begin{array}{l}
\text{Berkeley}\\ \text{Branford}\\ \text{Calhoun}\\ \text{Davenport}\\ \text{Ezra Stiles}\\ \text{Jonathan Edwards}\\ \text{Morse}\\ \text{Pierson}\\ \text{Saybrook}\\ \text{Silliman}\\ \text{Timothy Dwight}\\ \text{Trumbull}
\end{array} &
\begin{bmatrix}
600 & 230 & 650 & 400 & 240\\
700 & 88 & 800 & 550 & 400\\
500 & 300 & 600 & 350 & 240\\
850 & 280 & 900 & 650 & 500\\
800 & 600 & 900 & 650 & 500\\
850 & 130 & 900 & 700 & 500\\
600 & 600 & 700 & 450 & 450\\
950 & 350 & 1000 & 800 & 650\\
700 & 210 & 800 & 550 & 350\\
220 & 600 & 300 & 54 & 130\\
240 & 750 & 300 & 250 & 260\\
650 & 240 & 750 & 500 & 350
\end{bmatrix}
\end{array}$$

If we assume that this student attends all five classes twice a week, rain or shine, and makes round trips to and from each lecture hall, then we can calculate the weekly total distance travelled by the student:

$$\begin{array}{cll}
 & & \text{Ending Location} \\
 & & \begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\
\begin{array}{c}\text{Starting}\\\text{Location}\end{array} &
\begin{array}{l}
\text{Berkeley}\\ \text{Branford}\\ \text{Calhoun}\\ \text{Davenport}\\ \text{Ezra Stiles}\\ \text{Jonathan Edwards}\\ \text{Morse}\\ \text{Pierson}\\ \text{Saybrook}\\ \text{Silliman}\\ \text{Timothy Dwight}\\ \text{Trumbull}
\end{array} &
\begin{bmatrix}
600 & 230 & 650 & 400 & 240\\
700 & 88 & 800 & 550 & 400\\
500 & 300 & 600 & 350 & 240\\
850 & 280 & 900 & 650 & 500\\
800 & 600 & 900 & 650 & 500\\
850 & 130 & 900 & 700 & 500\\
600 & 600 & 700 & 450 & 450\\
950 & 350 & 1000 & 800 & 650\\
700 & 210 & 800 & 550 & 350\\
220 & 600 & 300 & 54 & 130\\
240 & 750 & 300 & 250 & 260\\
650 & 240 & 750 & 500 & 350
\end{bmatrix}
\begin{bmatrix} 4\\4\\4\\4\\4 \end{bmatrix}
=
\begin{bmatrix} 8480\\10152\\7960\\12720\\13800\\12320\\11200\\15000\\10440\\5216\\7200\\9960 \end{bmatrix}
\end{array}$$

Note: these distances are taken from Google Maps and do not include shortcuts or otherwise illegal street crosses.

Example 2

Many of you may have heard about the beta of a portfolio and how it is an important statistic to know. An equally important and related concept is correlation. In finance, correlation matrices with heat map color coding are commonly used by investment managers trying to uncover relationships and trends within the various asset classes, industries, and sectors. The mathematical process is as follows:

$$\mathbf{M}_{Correl} = \begin{bmatrix} Correlation(Asset_{1,1}) & Correlation(Asset_{1,2}) & \cdots & Correlation(Asset_{1,n}) \\ Correlation(Asset_{2,1}) & & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ Correlation(Asset_{n,1}) & & \cdots & Correlation(Asset_{n,n}) \end{bmatrix}$$

Diverse portfolios contain many securities and assets which all behave differently in response to how the economy performs. Uncorrelated assets tend to move in different directions or at least not move in the same direction. The correlation coefficient is a number between -1 and 1, where -1 means complete movement in the opposite direction and 1 means complete movement in the same direction. 0 means no relationship.

It is important to note that correlations change over time, so it is important to keep in mind what time frame you are using when constructing a correlation matrix. The correlation matrix below was constructed using daily closing prices adjusted for splits and dividends over a 7+ year period from 2006 until 2013. The data was pulled from Yahoo! Finance. There is a separate excel file documenting this process for those interested.

	ATT	APPL	XOM	GE	HD	JPM	PG	GOOG	CMG
ATT	1.00	0.43	0.62	0.51	0.55	0.53	0.58	0.44	0.35
APPL	0.43	1.00	0.41	0.45	0.42	0.43	0.33	0.55	0.36
XOM	0.62	0.41	1.00	0.54	0.49	0.48	0.59	0.47	0.34
GE	0.51	0.45	0.54	1.00	0.56	0.66	0.49	0.45	0.37
HD	0.55	0.42	0.49	0.56	1.00	0.59	0.49	0.46	0.46
JPM	0.53	0.43	0.48	0.66	0.59	1.00	0.47	0.47	0.38
PG	0.58	0.33	0.59	0.49	0.49	0.47	1.00	0.38	0.30
GOOG	0.44	0.55	0.47	0.45	0.46	0.47	0.38	1.00	0.38
CMG	0.35	0.36	0.34	0.37	0.46	0.38	0.30	0.38	1.00

As we can see from the heat coloring, it is very obvious that the historical performance of each stock is perfectly correlated to itself; a correctly constructed correlation matrix will always have a diagonal of 1s. An interesting point to note in this example is how JP Morgan and GE's performance are the most correlated among all the available options. This is quite surprising, but keeping in mind that the time frame was from 2006 until 2013, we realize that the 2007-2008 financial crisis would have similarly affected both since GE had a large financial services branch—GE Capital. Lastly, we can see that Procter & Gamble and Chipotle Mexican Grill have the lowest correlation. This makes sense because P&G is in the consumer staples business whereas Chipotle is in the consumer discretionary business, though college students may contest the "discretionary" label.

Linear Algebra

2.1 Vectors and Linear Equations – 2.2 The Idea of Elimination

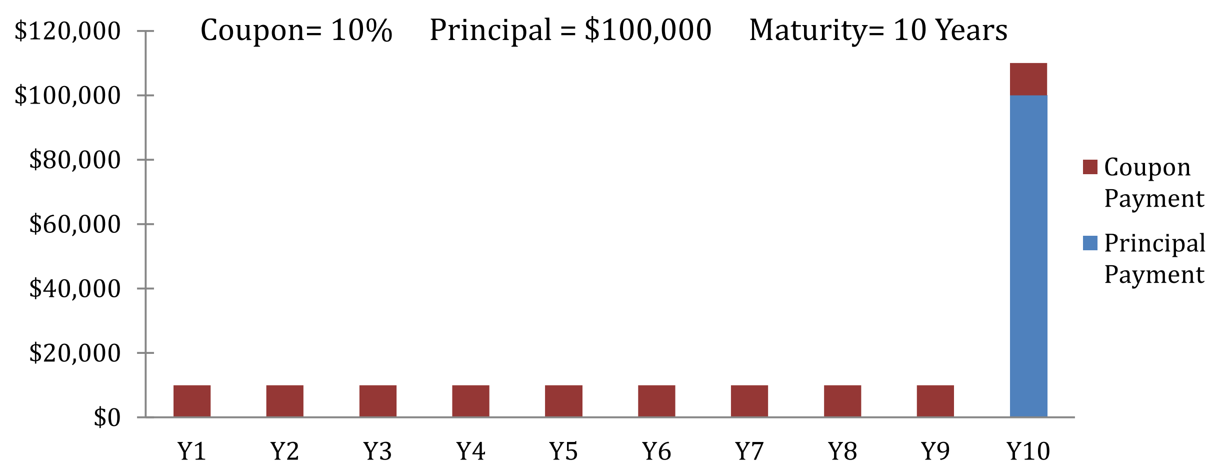
Many people tend to use the performance of equity market indices such as the Dow Jones Industrial Average (DJIA) or the S&P 500 (SPX) as a gauge of the country's overall economic health, which is why the US stock market is often the main focus of major media outlets. As a result, other markets sometimes don't get the attention that they deserve. One such market is the US bond market. It should be noted that the US bond market is roughly three times the size of the U.S. stock market in dollar terms.

Bonds, also referred to as fixed income or debt instruments, have three main characteristics:

I. Coupon (c) – Interest payment made by the bond issuer
II. Principal (p) – Often referred to as Par or Face Value, the principal payment is made as part of the issuer's final payment
III. Maturity (t) – Length of time during which the bond issuer makes payments

Below, we see the stream of payments made by a bond with the following characteristics:

	Coupon	Principal	Maturity
Bond I	10% paid annually	$100,000	10 years



In introductory microeconomics, students learn about the concept of present value, which essentially states that a dollar today is worth at least a dollar tomorrow because a dollar today can be invested today or held until tomorrow.

In other words, if the interest rate for the economy is r% over one period of time, then investing $1 in this economy will lead to the dollar to being worth $(1+r) one period from now. Suppose this investment opportunity never disappears and that it has a constant interest rate of r% over one period of time, then this entire sum could then be re-invested after each period, then the original $1 could potentially grow exponentially and be worth $\$(1+r)^t$ after t-periods from now, assuming the entire amount is continually re-invested.

Below, we see the value of $1 being continually re-invested after each period at an interest rate of r%:

End of Period	1	2	3	…	t-1	t
Total ($)	1+r	$(1+r)^2$	$(1+r)^3$	…	$(1+r)^{t-1}$	$(1+r)^t$

Arranged differently, if there is a constant interest rate r%, then $1 one period from now is worth $\$\left(\frac{1}{(1+r)}\right)$ today and $1 t-periods from now is worth $\$\left(\frac{1}{(1+r)^t}\right)$ today. This is the concept of present value discounting. Below, we see the present value of $1 from each period

End of Period	1	2	3	…	t-1	t
Present Value (P.V)	$\frac{1}{(1+r)}$	$\frac{1}{(1+r)^2}$	$\frac{1}{(1+r)^3}$	…	$\frac{1}{(1+r)^{t-1}}$	$\frac{1}{(1+r)^t}$

In a setting where the interest rate is constant, once we are given an interest rate to discount the cash flows with, calculating the present value of a bond becomes a very straightforward process.

Hence, if we are given a single interest rate to use to discount the stream of cash flows, then we could actually find out the present or fair value at which this bond could be sold today.

In the example below, we will calculate the present value of a bond with the following characteristics:

- Coupon = 10% paid annually
- Principal = $100,000
- Maturity = 10 Years

Assuming that there is a constant interest rate of 5% (note that interest rate is an exogenous variable), then the present value calculation is as follows:

$$\mathrm{P.V.} = \left(\frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{c}{(1+r)^3} + \cdots + \frac{c}{(1+r)^{t-1}} + \frac{c}{(1+r)^t}\right) + \frac{p}{(1+r)^t}$$

*Notice that the coupon payments form a finite geometric series.

$$\mathrm{P.V.} = \left[\left(\frac{\frac{C}{(1+r)}}{1-\frac{1}{(1+r)}}\right) - \left(\frac{\frac{C}{(1+r)^t}}{1-\frac{1}{(1+r)}}\right)\right] + \frac{P}{(1+r)^t}$$

$$\text{P.V.} = \left[\frac{\$10{,}000}{(1.05)^1} + \frac{\$10{,}000}{(1.05)^2} + \frac{\$10{,}000}{(1.05)^3} + \cdots + \frac{\$10{,}000}{(1.05)^9} + \frac{\$10{,}000}{(1.05)^{10}}\right] + \frac{\$100{,}000}{(1.05)^{10}}$$

$$\text{P.V.} = \$138{,}609$$

*The fair market price of this bond would be $138,609, so investors should buy this bond if it is selling for less and sell this bond if it is selling for more.

The examples on the previous page were all based on a constant interest rate environment; however, it should not come as a huge surprise that interest rates tend to fluctuate in the real world. As a result, we must reconcile our previous models to account for varying interest rates.

Let us denote the one period interest rate to be r_i where i is the period. For example, the interest rate for the economy would be $, r_2, r_3, \ldots, r_{t-1}, r_t$ over periods *1, 2, 3, …, t-1,* and *t* respectively.

Below, we see the value of $1 being continually re-invested after each period:

End of Period	1	2	3	…
Total ($)	$(1+r_1)$	$(1+r_1)(1+r_2)$	$(1+r_1)(1+r_2)(1+r_3)$	…

Similarly, the present value of $1 from each period:

End of Period	1	2	3	…
Present Value (P.V)	$\frac{1}{(1+r)}$	$\frac{1}{(1+r_1)(1+r_2)}$	$\frac{1}{(1+r_1)(1+r_2)(1+r_3)}$	…

In the example below, we will calculate the present value of a bond with the following characteristics:

- Coupon = 10% paid annually
- Principal = $100,000 *Note: Principal is often referred to as the par or face value
- Maturity = 2 Years

Assuming that there is an interest rate of 5% the first year and an interest rate of 15% the second year, then the present value calculation is as follows:

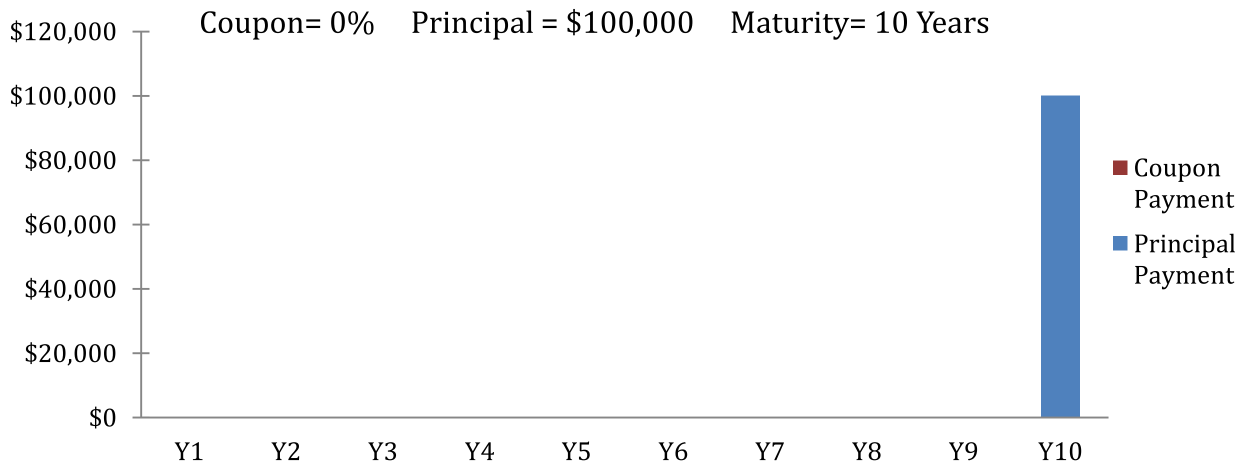
$$\text{P.V.} = \left[\frac{\$10{,}000}{(1.05)} + \frac{\$10{,}000}{(1.05)(1.15)}\right] + \frac{\$100{,}000}{(1.05)(1.15)}$$

$$\text{P.V.} = \$100{,}621$$

More interestingly, we see that \$1 one year from now is currently worth $\$\left(\frac{1}{(1.05)}\right)$ or roughly 95 cents and that \$1 two years from now is currently worth $\$\left(\frac{1}{(1.05)(1.15)}\right)$ or roughly 83 cents. In other words, an investor should be able to make a fair deal and secure a contract paying \$95.23 for \$100 one year from now. In fact, investors often do these kinds of deals.

Zero-coupon bonds:

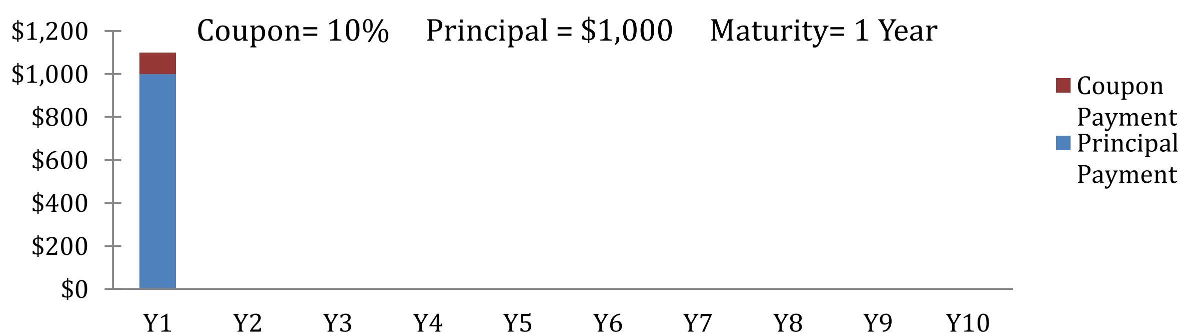
	Coupon	Principal	Maturity
Bond II	0% paid annually	\$100,000	10 years



In financial theory, students will encounter zero-coupon bonds, or zeroes. As the name suggests, these are bonds that have no coupon payments and only pay the principal value in the final year of the bond's life. Investors often use US Treasury bonds of various maturities to figure out how to appropriately price the present value of \$1 in the future.

<u>Example 1</u>
Suppose that the U.S. Treasury is issuing a bond with the following characteristics:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	\$1,000	1 year	\$1,001



Let us denote the present value of \$1 in the future at the end of period t as (z_t). We can then rewrite the information in the table for Bond C as a linear equation:

$$1{,}100z_1 = \$1{,}001$$

Solving this equation, we get z_1 to be \$0.91. This can be interpreted to mean that the present value of a dollar one year from now is 91 cents.

Example 2

Suppose that the U.S. Treasury is issuing the following two bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	\$1,000	1 year	\$1,001
Bond B	15% paid annually	\$1,000	2 year	\$1,002

This can be written as a system of linear equations:

$$1{,}100z_1 = \$1{,}001$$
$$150z_1 + 1{,}150z_2 = \$1{,}002$$

Notice that we can rewrite this system of linear equations in vector format such that the equations correspond to a lower triangular matrix:

$$\begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1{,}001 \\ 1{,}002 \end{bmatrix}$$

Solving using back substitution, we get z_1 to be \$0.91 and z_2 to be approximately \$0.75. This means that the present value of a dollar one year from now is 91 cents and that the present value of a dollar two years from now is 75 cents.

Example 3

Suppose that the U.S. Treasury is issuing the following four bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	\$1,000	1 year	\$1,001
Bond B	15% paid annually	\$1,000	2 year	\$1,002
Bond C	18% paid annually	\$10,000	3 year	\$10,003
Bond D	20% paid annually	\$100,000	4 year	\$100,004

This can be written as a system of linear equations:

$$1{,}100z_1 = \$1{,}001$$
$$150z_1 + 1{,}150z_2 = \$1{,}002$$
$$1{,}800z_1 + 1{,}800z_2 + 11{,}800z_3 = \$10{,}003$$
$$20{,}000z_1 + 20{,}000z_2 + 20{,}000z_3 + 120{,}000z_4 = \$100{,}004$$

Notice that we can rewrite this system of linear equations in vector format such that the equations correspond to a lower triangular matrix:

$$\begin{bmatrix} 1{,}100 & 0 & 0 & 0 \\ 150 & 1{,}150 & 0 & 0 \\ 1{,}800 & 1{,}800 & 11{,}800 & 0 \\ 20{,}000 & 20{,}000 & 20{,}000 & 120{,}000 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1{,}001 \\ 1{,}002 \\ 10{,}003 \\ 100{,}004 \end{bmatrix}$$

Solving using back substitution, we get:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} .91 \\ .75 \\ .59 \\ .45 \end{bmatrix}$$ *Note that these are truncated to two decimal places without rounding

These can be interpreted to mean: the present value of a dollar one year from now is $0.91; the present value of a dollar two years from now is $0.75; the present value of a dollar three years from now is $0.59; the present value of a dollar four years from now is $0.45.

<u>Food for Thought</u>
Consider the following:

In one year's time, a three-year bond with a 10% coupon paid annually and $100 face value will become a two-year bond with a 10% coupon paid annually and a $100 face value. In general, an (N)-year bond with principal P and coupon C% will become a (N-T)-year bond with principal P and coupon C% after T years.

Hence, imagine that there are two bonds. One is a 2-year bond with 5% coupon paid annually and $100 principal that was just issued and the other is a 5-year bond with the exact same 5% coupon and $100 principal that was issued three years ago by the same issuer. These two bonds are essentially identical and should be priced exactly the same, however, there are sometimes factors that cause them to be priced differently.

Additional Resources:
http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield
http://www.bloomberg.com/markets/rates-bonds/government-bonds/us/

Linear Algebra

2.5 Inverse matrices

The inverse of a matrix A is denoted A^{-1} and has several important properties which are covered in class. Two of those very important properties to note for now include: not all matrices have inverses; when matrices do have inverses, $AA^{-1}=I$ and $A^{-1}A=I$.

In the previous example dealing with bonds, we had set up equations in the form Ax=b (A was the matrix of coefficients representing coupons, x was the column vector of variables representing future dollars, and b was the column vector of bond prices) and then solved using back substitution.

While this method provided us with the correct solution, we could have used inverse matrices to solve this same system:

$$Ax=b$$

$$A^{-1}Ax= A^{-1}b$$

$$Ix= A^{-1}b$$

$$x= A^{-1}b$$

While solving for the inverse matrix A^{-1} might seem tedious and not worthwhile when dealing with smaller matrices, it is important to realize that elimination only worked in getting us an answer because A was invertible. In fact, in many cases where the matrix of coefficients is larger, it is more useful to know whether or not a unique solution exists, and that depends entirely on whether or not A has an inverse matrix. In the cases where we have an underdetermined system, there will be infinitely many solutions; in cases where we have an overdetermined system, we will have to use other methods to approximate the best fitting solution.

Let us harken back to the section 1.1 example with Daniel the elf. Recall that elves need to follow strict dietary guidelines in order to maintain their magical abilities. More specifically, there are three bundles from which to choose from and are listed below in this format: $\begin{bmatrix} Durian \\ Dragon\ Fruit \\ Lychee \end{bmatrix}$.

(i) $<\boldsymbol{a}>=\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ (ii) $<\boldsymbol{b}>=\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ (iii) $<\boldsymbol{c}>=\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

Example 1

Suppose we know that Daniel brought 0 durians, 460 dragon fruits, and 400 lychees on his trip. How many of each bundle did he bring?

To answer this question, we can use inverse matrices! First, we need to set up the equation in Ax=b form:

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 460 \\ 400 \end{bmatrix}$$

Next, we need to find the inverse of matrix A which turns out to be:

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix}$$

Multiplying both sides of the equation by A^{-1}, we get:

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix} \begin{bmatrix} 0 \\ 460 \\ 400 \end{bmatrix}$$

Simplifying:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 20 \end{bmatrix}$$

Example 2

Suppose we have another elf named Miki who is not allergic to durians. She is visiting New Haven as well and has brought 660 durians, 3080 dragon fruits, and 3190 lychees.

We can answer this question using inverse matrices again! Setting up the equation in Ax=b form:

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 660 \\ 3080 \\ 3190 \end{bmatrix}$$

Since the bundles have not changed, the coefficient matrix has remained the same, which means that the inverse of matrix A is the same:

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix}$$

Multiplying both sides of the equation by A^{-1}, we get:

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-10}{33} & \frac{5}{11} & \frac{-3}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \end{bmatrix} \begin{bmatrix} 660 \\ 3080 \\ 3190 \end{bmatrix}$$

Simplifying:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 220 \\ 330 \\ 440 \end{bmatrix}$$

<u>Example 3</u>
For this next example, we will deviate from the topic of elves and talk about business.

Suppose that Bob works in a broom factory. This factory has machines that use 4 different types of inputs represented by $i_1, i_2, i_3,$ and i_4 to make 4 different types of brooms represented by $b_1, b_2, b_3,$ and b_4.

By now, we should be familiar with the idea of writing these bundles in vector form. In this case, we will write the input vectors in the form $\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$:

(i) $<b_1> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ (ii) $<b_2 = \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \end{bmatrix}$ (iii) $<b_3 = \begin{bmatrix} 0 \\ 1 \\ 9 \\ 1 \end{bmatrix}$ (iv) $<b_4> = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix}$

Ned, who is Bob's manager, orders a shipment of raw inputs to be used for a large, time-sensitive order for their best client. Apparently the client's marketing team did a very good job and the brooms have become extremely popular. In fact, they are slowly sweeping the nation. Ned, however, rushes off for a technology-free getaway and forgets to tell Bob the details of the order—namely how many of each broom the client ordered. Caught in a very obvious predicament of not being able to contact his boss and also not wanting to lose his best customer, Bob begins to fret. That is, until Bob realizes that he can use inverse matrices!

Setting up the equation in Ax=b form:

$$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 9 & 9 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 628 \\ 942 \\ 6594 \\ 2198 \end{bmatrix}$$

*The fact that author was born on February 11, 1992 and will graduate in 2014 is purely coincidental

Calculating the inverse matrix A^{-1} and multiplying both sides to get $A^{-1}Ax = A^{-1}b$:

$$\begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{17}{18} & \frac{-17}{18} & \frac{1}{9} & \frac{-1}{18} \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 9 & 9 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{17}{18} & \frac{-17}{18} & \frac{1}{9} & \frac{-1}{18} \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 628 \\ 942 \\ 6594 \\ 2198 \end{bmatrix}$$

*Please remember that order matters in matrix multiplication

Simplifying:

$Ix = A^{-1}b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 314 \\ 314 \\ 314 \\ 314 \end{bmatrix}$$

$x = A^{-1}b$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 314 \\ 314 \\ 314 \\ 314 \end{bmatrix}$$

Linear Algebra

2.7 Transposes and Permutations

<u>Example 1</u>
Suppose we wish to know the walking distance from each of the five lecture halls to one another. We can again record the distances in matrix form:

$$\mathbf{A} = \begin{array}{cc} & \text{Ending Location} \\ \text{Starting Location} & \begin{array}{c|c} & \begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\ \begin{array}{c} \text{DL} \\ \text{LC} \\ \text{LOM} \\ \text{SSS} \\ \text{WLH} \end{array} & \begin{bmatrix} 0 & 800 & 100 & 160 & 350 \\ 800 & 0 & 900 & 650 & 450 \\ 100 & 900 & 0 & 260 & 450 \\ 160 & 650 & 260 & 0 & 190 \\ 350 & 450 & 450 & 190 & 0 \end{bmatrix} \end{array} \end{array}$$

Suppose we take the transpose of this matrix A:

$$\mathbf{A^T} = \begin{array}{cc} & \text{Starting Location} \\ \text{Ending Location} & \begin{array}{c|c} & \begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\ \begin{array}{c} \text{DL} \\ \text{LC} \\ \text{LOM} \\ \text{SSS} \\ \text{WLH} \end{array} & \begin{bmatrix} 0 & 800 & 100 & 160 & 350 \\ 800 & 0 & 900 & 650 & 450 \\ 100 & 900 & 0 & 260 & 450 \\ 160 & 650 & 260 & 0 & 190 \\ 350 & 450 & 450 & 190 & 0 \end{bmatrix} \end{array} \end{array}$$

Notice in this case that the matrix has the special property of being symmetric. We would expect this because the walking distances should be the same regardless of direction.

<u>Example 2</u>
Suppose that this student's parents decide to visit on Family Weekend and want to drive around and see the various lecture halls. Since New Haven is littered with one-way streets, we would expect the driving distances from one lecture hall to another to form a non-symmetric matrix:

$$\mathbf{B} = \begin{array}{cc} & \text{Ending Location} \\ \text{Starting Location} & \begin{array}{c|c} & \begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\ \begin{array}{c} \text{DL} \\ \text{LC} \\ \text{LOM} \\ \text{SSS} \\ \text{WLH} \end{array} & \begin{bmatrix} 0 & 1350 & 170 & 490 & 260 \\ 1190 & 0 & 910 & 660 & 550 \\ 170 & 1070 & 0 & 240 & 430 \\ 910 & 820 & 240 & 0 & 180 \\ 1220 & 2120 & 1231 & 1290 & 0 \end{bmatrix} \end{array} \end{array}$$

Taking the transpose of matrix B:

$$
\mathbf{B}^{T} = \text{Ending Location} \quad
\begin{array}{c}
\\
\text{DL} \\ \text{LC} \\ \text{LOM} \\ \text{SSS} \\ \text{WLH}
\end{array}
\begin{array}{c}
\text{Starting Location} \\
\begin{array}{ccccc} \text{DL} & \text{LC} & \text{LOM} & \text{SSS} & \text{WLH} \end{array} \\
\begin{bmatrix}
0 & 1190 & 170 & 910 & 1220 \\
1350 & 0 & 1070 & 820 & 2120 \\
170 & 910 & 0 & 240 & 1231 \\
490 & 660 & 240 & 0 & 1290 \\
260 & 550 & 430 & 180 & 0
\end{bmatrix}
\end{array}
$$

Notice that this matrix is not symmetric and hence taking the transpose gives us a different result.

Sources: https://maps.google.com/

Linear Algebra

3.1 Spaces of Vectors

In class, you will be introduced to the four fundamental subspaces: column space; nullspace; row space; left nullspace. Eventually, you will come to realize that these four spaces are inherently entwined, but for now, we will focus on analyzing the column space of a matrix.

<u>Example 1</u>
Let us take a moment to remember the elves and their eating habits. The three bundles from which they can choose from and are listed below:

(i) $<\boldsymbol{a}>=\begin{bmatrix}3\\2\\0\end{bmatrix}$ (ii) $<\boldsymbol{b}>=\begin{bmatrix}0\\4\\3\end{bmatrix}$ (iii) $<\boldsymbol{c}>=\begin{bmatrix}0\\3\\5\end{bmatrix}$

Recall that our buddy, Daniel the elf, simply can't eat durians. Upon first glance, it becomes obvious that he can only pick and choose from a combination of bundle b and bundle c. Writing this as a matrix, we have:

$$\mathbf{A}=\begin{bmatrix}0 & 0\\4 & 3\\3 & 5\end{bmatrix}$$

More importantly, in Ax=b form:

$$\begin{bmatrix}0 & 0\\4 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}b\\c\end{bmatrix}=\begin{bmatrix}Durian\\Dragon\ Fruit\\Lychee\end{bmatrix}$$

In this case, we see that the resulting number of fruits in Daniel's bundle will consist of a linear combination of bundles b and c. Hence, we would say that the vectors $\begin{bmatrix}0\\4\\3\end{bmatrix}, \begin{bmatrix}0\\3\\5\end{bmatrix}$ span the column space of matrix A. In more mathematical terms:

$$C_1\begin{bmatrix}0\\4\\3\end{bmatrix}+C_2\begin{bmatrix}0\\3\\5\end{bmatrix}=\begin{bmatrix}0\\4C_1+3C_2\\3C_1+5C_2\end{bmatrix}$$

*This is actually not the best example since the values for durian are zero, which means that we have 2 equations with 2 unknowns as opposed to 3equations and 2 unknowns. Usually, when we have a matrix with more rows than columns, we will have what is called an overdetermined system. In this case, there will either be one solution or no solution.

Linear Algebra

3.2 Nullspace of A: Solving Ax=0

<u>Example 1</u>
In previous sections, we had always assumed that elves need to consume durians, dragon fruits, and lychees as part of a healthy, magical diet. Recent research, however, shows that lychees may actually be a placebo and, as a result, while they may taste good, they are not a necessary. This means that an ordinary elf can forget about lychees and consume any one of the following bundles of durians and dragon fruits on a given day and still maintain its magical abilities.

(i) <***a***>=$\begin{bmatrix}3\\2\end{bmatrix}$ (ii) <***b***>=$\begin{bmatrix}0\\4\end{bmatrix}$ (iii) <***c***>=$\begin{bmatrix}0\\3\end{bmatrix}$

Writing this as a matrix, we have:

$$\mathbf{A} = \begin{bmatrix}3 & 0 & 0\\2 & 4 & 3\end{bmatrix}$$

More importantly, in Ax=0 form:

$$\begin{bmatrix}3 & 0 & 0\\2 & 4 & 3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

The nullspace, as discussed in class and in the text, is the collection of all vectors that satisfy Ax=0. In this case, the rank of matrix A is 2, which means that A exhibits full row rank. This guarantees the existence of a solution, but not the uniqueness as we had seen when dealing with previous examples.

Using Gauss-Jordan elimination, we get:

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 3/4 & 0\end{bmatrix}$$

This means that the nullspace contains all multiples of $\begin{bmatrix}0\\-3\\4\end{bmatrix}$ since this vector satisfies Ax=0.

In more mathematical terms, the null space is spanned by: c $\begin{bmatrix}0\\-3\\4\end{bmatrix}$ where c is a scalar.

Linear Algebra

3.3 Rank and the Reduced Row Form

Example 1
Suppose that the U.S. Treasury is issuing the following four bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	\$1,000	1 year	\$1,001
Bond B	15% paid annually	\$1,000	2 year	\$1,002
Bond C	18% paid annually	\$10,000	3 year	\$10,003
Bond D	20% paid annually	\$100,000	4 year	\$100,004

This can be written as a system of linear equations:

$$\begin{aligned} 1{,}100z_1 &= \$1{,}001 \\ 150z_1 + 1{,}150z_2 &= \$1{,}002 \\ 1{,}800z_1 + 1{,}800z_2 + 11{,}800z_3 &= \$10{,}003 \\ 20{,}000z_1 + 20{,}000z_2 + 20{,}000z_3 + 120{,}000z_4 &= \$100{,}004 \end{aligned}$$

Notice that we can rewrite this system of linear equations in vector format:

$$\begin{bmatrix} 1{,}100 & 0 & 0 & 0 \\ 150 & 1{,}150 & 0 & 0 \\ 1{,}800 & 1{,}800 & 11{,}800 & 0 \\ 20{,}000 & 20{,}000 & 20{,}000 & 120{,}000 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1{,}001 \\ 1{,}002 \\ 10{,}003 \\ 100{,}004 \end{bmatrix}$$

Previously, we noticed that these equations corresponded to a lower triangular matrix and solved the equations using back substitution. Armed with new tools, we can now convert this system of equations into an augmented matrix and use Gauss-Jordan elimination get it into reduced row-echelon form (rref):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & .9100 \\ 0 & 1 & 0 & 0 & .7526 \\ 0 & 0 & 1 & 0 & .5940 \\ 0 & 0 & 0 & 1 & .4572 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} .91 \\ .75 \\ .59 \\ .45 \end{bmatrix} \quad * \text{Note that these are truncated to two decimal places without rounding}$$

These can still be interpreted to mean: the present value of a dollar one year from now is \$0.91; the present value of a dollar two years from now is \$0.75; the present value of a dollar three years from now is \$0.59; the present value of a dollar four years from now is \$0.45.

Notice that we can input the augmented matrix into a graphing calculator and arrive at the solutions in one step by using the reduced row echelon form function. We were unable to do this previously.

Example 2

Suppose that the U.S. Treasury is issuing the following two bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	$1,000	1 year	$950
Bond B	15% paid annually	$1,000	3 year	$950

This can be written as a system of linear equations:

$$1{,}100z_1 = \$950$$
$$150z_1 + 150z_2 + 1{,}150z_3 = \$950$$

Notice that we can rewrite this system of linear equations in vector format:

$$\begin{bmatrix} 1{,}100 & 0 & 0 \\ 150 & 150 & 1{,}150 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \end{bmatrix}$$

Getting this matrix into reduced row echelon form, we get that:

$$\begin{bmatrix} 1 & 0 & 0 & .8636 \\ 0 & 1 & 7.6666 & 5.4697 \end{bmatrix} \longrightarrow \begin{array}{rl} z_1 & = .8636 \\ z_2 + 7.6666\, z_3 & = 5.4697 \end{array}$$

One solution could be:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} .86 \\ .83 \\ .60 \end{bmatrix} * \text{Note that these are truncated to two decimal places without rounding}$$

Another viable solution:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} .86 \\ .63 \\ .63 \end{bmatrix} * \text{Note that these are truncated to two decimal places without rounding}$$

We started with a system of 3 variables and 2 independent equations, and as a result, we were guaranteed the existence but not the uniqueness of our solutions.

Example 3
Suppose that the U.S. Treasury is selling the following three bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	\$1,000	1 year	\$950
Bond B	15% paid annually	\$1,000	2 year	\$950
Bond C	20% paid annually	\$1,000	2 year	\$1,010

This can be written as a system of linear equations:

$$1{,}100z_1 = \$950$$
$$150z_1 + 1{,}150z_2 = \$950$$
$$200z_1 + 1{,}200z_2 = \$1{,}010$$

Notice that we can rewrite this system of linear equations in vector format:

$$\begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \\ 200 & 1{,}200 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \\ 1{,}010 \end{bmatrix}$$

Getting this matrix into reduced row echelon form, we get that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{array}{rcl} z_1 & = & 0 \\ z_2 & = & 0 \\ (0)\, z_1 + (0)\, z_2 & = & 1 \end{array} \longrightarrow \text{Nonsense.}$$

This is what is known as an over-determined system. We would normally be able to solve this equation if we had just Bond A and Bond B or even Bond A and Bond C, but having both Bond B and Bond C leads to problems since they claim different values for z_2. We will learn how to best approximate the solution to this problem in the next chapter.

Linear Algebra

3.4 The Complete Solution to Ax=b

Example 1
Suppose that the U.S. Treasury is issuing the following four bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	$1,000	1 year	$1,001
Bond B	15% paid annually	$1,000	2 year	$1,002
Bond C	18% paid annually	$10,000	3 year	$10,003
Bond D	20% paid annually	$100,000	4 year	$100,004

Writing this system of linear equations in Ax=b format:

$$\begin{bmatrix} 1{,}100 & 0 & 0 & 0 \\ 150 & 1{,}150 & 0 & 0 \\ 1{,}800 & 1{,}800 & 11{,}800 & 0 \\ 20{,}000 & 20{,}000 & 20{,}000 & 120{,}000 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1{,}001 \\ 1{,}002 \\ 10{,}003 \\ 100{,}004 \end{bmatrix}$$

Using Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & .9100 \\ 0 & 1 & 0 & 0 & .7526 \\ 0 & 0 & 1 & 0 & .5940 \\ 0 & 0 & 0 & 1 & .4572 \end{bmatrix}$$

A particular solution:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} .91 \\ .75 \\ .59 \\ .45 \end{bmatrix} \quad * \text{ Note that these are truncated to two decimal places without rounding}$$

Notice that this matrix is of full rank, hence the nullspace can only be $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in order to satisfy Ax=0.

The complete solution $\mathrm{x} = \mathrm{x_p} + \mathrm{x_n}$ to $\mathrm{Ax} = \mathrm{b}$ is: $\begin{bmatrix} .91 \\ .75 \\ .59 \\ .45 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ where c is a scalar. In this case, the particular solution is unique and is the complete solution.

Example 2

Suppose that the U.S. Treasury is issuing the following two bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	$1,000	1 year	$950
Bond B	15% paid annually	$1,000	3 year	$950

Writing this system of linear equations in Ax=b format:

$$\begin{bmatrix} 1{,}100 & 0 & 0 \\ 150 & 150 & 1{,}150 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \end{bmatrix}$$

Getting this matrix into reduced row echelon form, we get that:

$$\begin{bmatrix} 1 & 0 & 0 & .8636 \\ 0 & 1 & 7.6666 & 5.4697 \end{bmatrix} \longrightarrow \begin{array}{rcl} z_1 & = & .8636 \\ z_2 + 7.6666\, z_3 & = & 5.4697 \end{array}$$

One particular solution:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} .86 \\ .83 \\ .60 \end{bmatrix} * \text{Note that these are truncated to two decimal places without rounding}$$

Finding the nullspace via Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7.6666 & 0 \end{bmatrix}$$

This means that the nullspace contains all multiples of $\begin{bmatrix} 0 \\ -23 \\ 3 \end{bmatrix}$ since this vector satisfies Ax=0.

The complete solution $\mathrm{x} = \mathrm{x_p} + \mathrm{x_n}$ to $\mathrm{Ax} = \mathrm{b}$ is: $\begin{bmatrix} .86 \\ .83 \\ .60 \end{bmatrix} + \mathrm{c} \begin{bmatrix} 0 \\ -23 \\ 3 \end{bmatrix}$ where c is a scalar.

Example 3

Suppose that the U.S. Treasury is selling the following three bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	$1,000	1 year	$950
Bond B	15% paid annually	$1,000	2 year	$950
Bond C	20% paid annually	$1,000	2 year	$1,010

Writing this system of linear equations in Ax=b format:

$$\begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \\ 200 & 1{,}200 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \\ 1{,}010 \end{bmatrix}$$

Getting this matrix into reduced row echelon form, we get that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Nonsense.}$$

In this case, we have an over-determined system, and we are unable to find the complete solution. We will come back to this problem once we have more tools.

Example 4

A junior researcher elf at the Elfish Institute of Technology is claiming to have made a new groundbreaking discovery—the identification of a fourth consumable bundle, d.

(i) $\langle \boldsymbol{a} \rangle = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ (ii) $\langle \boldsymbol{b} \rangle = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ (iii) $\langle \boldsymbol{c} \rangle = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$ (iv) $\langle \boldsymbol{d} \rangle = \begin{bmatrix} 1.5 \\ 2.5 \\ 2.5 \end{bmatrix}$

Writing this as a matrix, we have:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 & 1.5 \\ 2 & 4 & 3 & 2.5 \\ 0 & 3 & 5 & 2.5 \end{bmatrix}$$

In Ax=0 form:

$$\begin{bmatrix} 3 & 0 & 0 & 1.5 \\ 2 & 4 & 3 & 2.5 \\ 0 & 3 & 5 & 2.5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Jordan elimination, we get:

$$\begin{bmatrix} 1 & 0 & 0 & .5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & .5 & 0 \end{bmatrix}$$

Recall that the nullspace is the collection of all vectors that satisfy Ax=0. In this case, the rank of matrix A is 3, which means that A exhibits full row rank. This guarantees the existence of a solution, but not the uniqueness as we had seen when dealing with previous examples.

Further analysis shows that the nullspace contains all multiples of $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ since this vector satisfies Ax=0.

In more mathematical terms, the null space is spanned by: c $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ where c is a scalar.

In less mathematical terms, the fourth bundle d is actually not a new discovery but rather a 50/50 combination of the first and third bundles, so the junior researcher elf needs to go back to the drawing board.

Linear Algebra

3.5 Independence, Basis, and Dimension

The concepts of independence, basis, and dimension are related and, in this section of notes, we will briefly review the definition of each as stated in Strang's *Linear Algebra* textbook and then we will go through a couple of examples.

<u>Definition</u>

- Independence: Independent vectors are vectors $v_1, v_2, \cdots, v_k$ such that no combination $c_1v_1, c_2v_2, \cdots, c_kv_k$= zero vector unless all $c_i = 0$. If the v's are the columns of **A**, the only solution to **A**x=0 is x=0.

- Basis: Basis for a vector space consists of independent vectors $v_1, v_2, \cdots, v_d$ whose linear combinations give every vector in the space.

- Dimension: The dimension of a vector space is equal to the number of vectors in any basis for that vector space.

<u>Example 1</u>
From the elf example in section 3.1, recall that Daniel cannot eat durians and, as a result, can only chose between two out of the three bundles. Hence we have in Ax=b form:

$$\begin{bmatrix} 0 & 0 \\ 4 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} Durian \\ Dragon\ Fruit \\ Lychee \end{bmatrix} \rightarrow \mathbf{A} = \begin{bmatrix} 0 & 0 \\ 4 & 3 \\ 3 & 5 \end{bmatrix}$$

Independence: We notice that the two column vectors $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$ are independent.

Basis: One possible basis for **A** is $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$. Note that there are many possible basis vectors.

Dimension: In this example, the matrix **A** is a 3x2 matrix and would therefore fall in the vector space of all 3x2 matrices. More relevantly, the matrix **A** has 3-dimensional column vectors. In other words, the column vectors of matrix **A** are in R^3 .

Example 2

From the first bond example in section 3.4, recall that we had four bonds being issued and were able to write the system of linear equations in Ax=b format:

$$\begin{bmatrix} 1{,}100 & 0 & 0 & 0 \\ 150 & 1{,}150 & 0 & 0 \\ 1{,}800 & 1{,}800 & 11{,}800 & 0 \\ 20{,}000 & 20{,}000 & 20{,}000 & 120{,}000 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1{,}001 \\ 1{,}002 \\ 10{,}003 \\ 100{,}004 \end{bmatrix}$$

In this case, we have that:

$$\mathbf{A} = \begin{bmatrix} 1{,}100 & 0 & 0 & 0 \\ 150 & 1{,}150 & 0 & 0 \\ 1{,}800 & 1{,}800 & 11{,}800 & 0 \\ 20{,}000 & 20{,}000 & 20{,}000 & 120{,}000 \end{bmatrix}$$

Independence: We notice that the four column vectors $\begin{bmatrix} 1{,}100 \\ 150 \\ 1{,}800 \\ 20{,}000 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1{,}150 \\ 1{,}800 \\ 20{,}000 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 11{,}800 \\ 20{,}000 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 120{,}000 \end{bmatrix}$ are all independent.

Basis: One possible basis for **A** is $\begin{bmatrix} 1{,}100 \\ 150 \\ 1{,}800 \\ 20{,}000 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1{,}150 \\ 1{,}800 \\ 20{,}000 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 11{,}800 \\ 20{,}000 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 120{,}000 \end{bmatrix}$. Remember that there are many possibilities for this.

Dimension: In this example, the matrix **A** is a 4x4 matrix and would therefore fall in the vector space of all 4x4 matrices. More relevantly, the matrix **A** has 4-dimensional column vectors. In other words, the column vectors of matrix **A** are in R^4.

Linear Algebra

3.6 Dimensions of the four subspaces
The four fundamental subspaces are column space, nullspace, row space, and left nullspace. In a m x n matrix with rank r, the dimension of the spaces are as follow:

- Column space = r
- Nullspace = n – r
- Row space = r
- Left nullspace = m – r

<u>Example 1</u>
Taking our elf example from 3.1 of the notes, we can analyze the dimensions of the four subspaces for A:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 4 & 3 \\ 3 & 5 \end{bmatrix}$$

The columns of this matrix are independent and we can determine the rank of this 3x2 matrix to be 2.

The column space has dimension 2 and is spanned by $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

The nullspace has dimension 0 and is spanned by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The rowspace has dimension 2 and is spanned by $\begin{bmatrix} 4 \\ 3 \end{bmatrix}^T$ and $\begin{bmatrix} 3 \\ 5 \end{bmatrix}^T$

The left nullspace has dimension 1 and is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

<u>Example 2</u>
Taking our elf example from 3.2 of the notes, we can analyze the dimensions of the four subspaces for A:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

The rows of this matrix are independent and we can determine the rank of this 2x3 matrix to be 2.

The column space has dimension 2 and is spanned by $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}0\\4\end{bmatrix}$

The nullspace has dimension 1 and is spanned by $\begin{bmatrix}0\\-3/4\\1\end{bmatrix}$

The rowspace has dimension 2 and is spanned by $\begin{bmatrix}3\\0\\0\end{bmatrix}^T$ and $\begin{bmatrix}2\\4\\3\end{bmatrix}^T$

The left nullspace has dimension 0 and is spanned by $\begin{bmatrix}0\\0\end{bmatrix}$

Linear Algebra

4.2 Projections & 4.3 Least Squares Approximations

Projections are commonly used in linear algebra and, in general, a projection matrix takes the entire space and maps it to a lower dimensional subspace. In the context of economics and finance, linear projection matrices are particularly useful when it comes to linear regression and the field of econometrics.

<u>Example 1</u>

For this first least square approximations example, we will analyze the dollar value of a hypothetical stock against time.

Year (t)	Stock Price (p)
1	10
2	20
3	40
4	60
5	70

Suppose we believe that the price of our hypothetical stock is directly influenced by the year and is modeled by (Stock Price) = $B_0 + B_1$t. We can rewrite the data from the table above in matrix form:

Ax=b:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} B0 \\ B1 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 40 \\ 60 \\ 70 \end{bmatrix}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$x=$\mathbf{A}^{\mathrm{T}}$b (Multiply both left-hand sides by$\mathbf{A}^{\mathrm{T}}$)

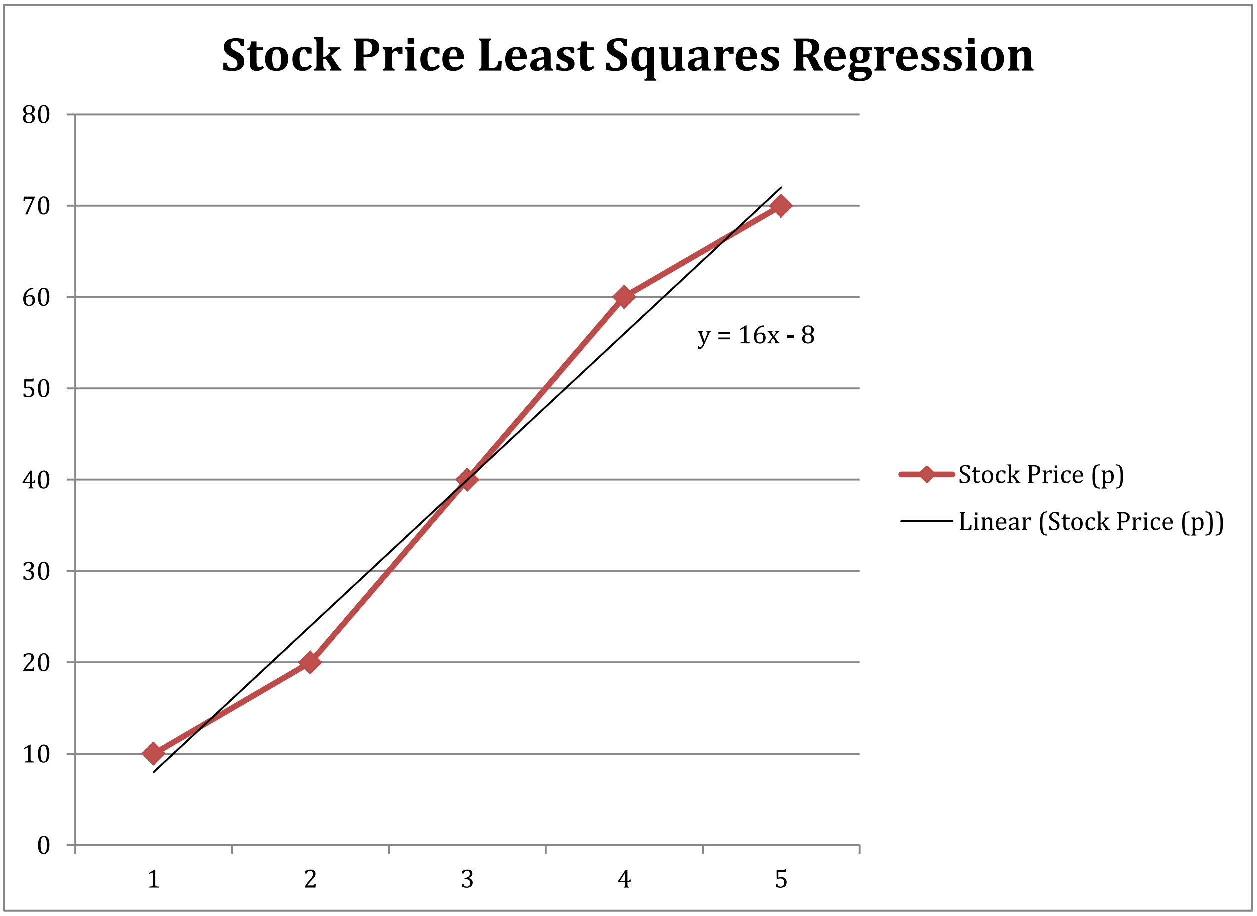
$\hat{x}$=$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ $\mathbf{A}^{\mathrm{T}}$b (Multiply both left-hand sides by $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$)

We can quickly calculating

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} = \begin{bmatrix} 11/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

$$\hat{x} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\ \mathbf{A}^{\mathrm{T}}\mathrm{b} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$



Stock Price Least Squares Regression
80
70
60
50
40
30
20
10
0
1
2
3
4
5
y = 16x - 8
Stock Price (p)
Linear (Stock Price (p))

Example 2

For this second example, we will analyze the historical dollar value of SPY against time. The data is pulled from Yahoo! Finance and is simply the adjusted closing price of the index at the beginning of every January over the five-year period ending in 2013. The SPY is an exchange-traded fund that aims to track the S&P 500.

Year (t)	Stock Price (p)
1	74.98
2	99.46
3	121.52
4	126.61
5	147.53

Suppose we believe that the price of SPY is directly influenced by the year and can be modeled by (Price) = $B_0 + B_1$t. We can rewrite the data from the table above in matrix form:

$\mathbf{A}$x=b:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} B0 \\ B1 \end{bmatrix} = \begin{bmatrix} 74.98 \\ 99.46 \\ 121.52 \\ 126.61 \\ 147.53 \end{bmatrix}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$x=$\mathbf{A}^{\mathrm{T}}$b (Multiply both left-hand sides by$\mathbf{A}^{\mathrm{T}}$)

x̂=$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ $\mathbf{A}^{\mathrm{T}}$b (Multiply both left-hand sides by $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$)

We can quickly calculating

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} = \begin{bmatrix} 11/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

$$\hat{x} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\, \mathbf{A}^{\mathrm{T}}b = \begin{bmatrix} 62.345 \\ 17.225 \end{bmatrix}$$


Price
y = 17.225x + 62.345
160
140
120
100
80
60
40
20
0
1
2
3
4
5
Price
Linear (Price)

Example 3

In this example, we will cover one of the previous bond examples where we were unable to find a solution to an over-determined system.

Suppose that the U.S. Treasury is selling the following three bonds:

	Coupon	Principal	Maturity	Price
Bond A	10% paid annually	$1,000	1 year	$950
Bond B	15% paid annually	$1,000	2 year	$950
Bond C	20% paid annually	$1,000	2 year	$1,010

Writing this system of linear equations in Ax=b format:

$$\begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \\ 200 & 1{,}200 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \\ 1{,}010 \end{bmatrix}$$

Getting this matrix into reduced row echelon form, we get that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Nonsense.}$$

In this case, we were unable to find a unique solution using our traditional methods, however, given our new tools, we can approximate an answer.

$\mathbf{A}$x=b:

$$\begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \\ 200 & 1{,}200 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 950 \\ 950 \\ 1{,}010 \end{bmatrix}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$x=$\mathbf{A}^{\mathrm{T}}$b (Multiply both left-hand sides by$\mathbf{A}^{\mathrm{T}}$)

$\hat{\mathrm{x}}=(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\,\mathbf{A}^{\mathrm{T}}\mathrm{b}$ (Multiply both left-hand sides by $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$)

We can quickly calculating

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1{,}100 & 150 & 200 \\ 0 & 1{,}150 & 1{,}200 \end{bmatrix} \begin{bmatrix} 1{,}100 & 0 \\ 150 & 1{,}150 \\ 200 & 1{,}200 \end{bmatrix} = \begin{bmatrix} 1{,}272{,}500 & 412{,}500 \\ 412{,}500 & 2{,}762{,}500 \end{bmatrix}$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} = \begin{bmatrix} 221/267{,}610{,}000 & -33/267{,}610{,}000 \\ -33/267{,}610{,}000 & 509/1{,}338{,}050{,}000 \end{bmatrix}$$

$$\hat{\mathrm{x}}=(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\,\mathbf{A}^{\mathrm{T}}\mathrm{b} = \begin{bmatrix} 0.8633 \\ 0.7052 \end{bmatrix}$$

Calculus

12.1 Functions of Two Variables

As students, we were first introduced to functions of a single variable. Examples of functions of a single variable include: $f(x) = x$; $f(x) = \sin(x)$; $f(x) = e^x$. We would often rewrite the function using dependent and independent variables, so in the case of $f(x) = x$, we would write $y = x$, with "y" as our dependent variable and "x" as our independent variable. As we learned in later math courses, these functions could be used to represent various types of problems, such as how to convert from one currency to another. By allowing functions to take in more than one variable input, we are able to expand what we are able to model.

Functions of two variables, for instance, use two variable inputs that help determine the value of a single dependent variable. Examples of functions of two variables include: $f(x, y) = x + y$; $f(x, y) = \sin(xy) + \cos(y)$; $f(x, y) = e^{xy}$. Consider the United States' economy as an example. The country's annual Gross Domestic Product (GDP), which is the value of all goods and services that it produces in a single year, is often used as an indicator of its economic health. The Cobb-Douglas production function is one of the most famous models, and it estimates GDP using a function of two variables, capital and labor.

<u>Example 1</u>

The Cobb-Douglas Production function is represented by the following equation:

$$f(K, L) = AK^{\alpha}L^{\beta}$$

Typically in economics textbooks, this function will be rewritten as follows:

$$Y = AK^{\alpha}L^{\beta}$$

In this case, Y, which represents GDP, is the dependent variable, and it is a function of two variables, capital (K) and labor (L). A, α, and β are positive constants. We will discuss in the following sections both the history and the special properties of the Cobb-Douglas production function.

Calculus

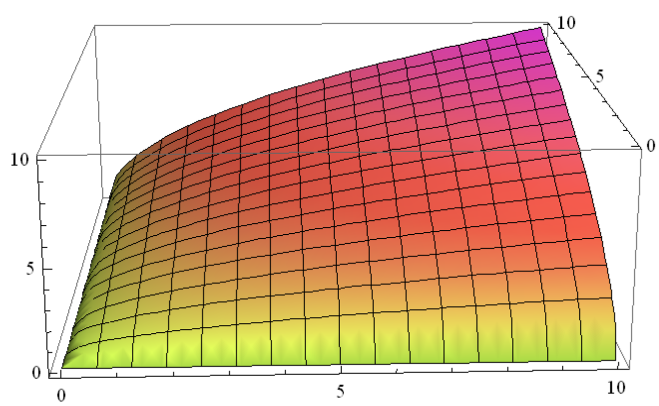
12.2 Graphs of Functions of Two Variables & 12.3 Contour Diagrams

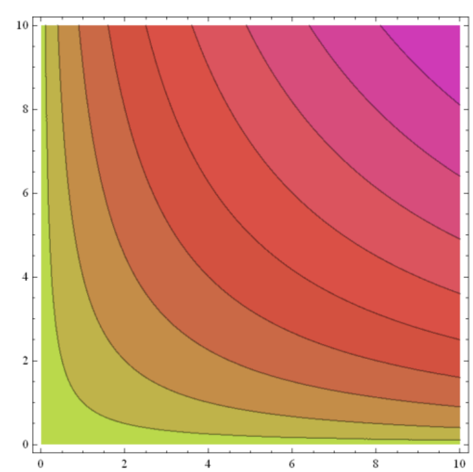
<u>Cobb-Douglas Function</u>
The Cobb-Douglas production function is one of the most widely used functions in economics. During the 1920s and 30s, Senator Paul Douglas and mathematician Charles Cobb worked together to model the relationship between inputs and outputs in order to analyze annual U.S. aggregate production.

The Cobb-Douglas function is traditionally used in the context of production with two independent variables, capital input (K) and labor input (L), though it is also used in general equilibrium theory as utility functions as opposed to production functions. These variables are raised to constant powers α and β to represent the relative weighting of capital versus labor—we will discuss this in the final section of the notes. Oftentimes, there is also a constant A that is used to capture changes in technology that affect both inputs.

The two graphs below are generated by plotting $Y=(K^{1/2})(L^{1/2})$ in Mathematica. Since it is technically not possible for outputs to be less than zero, it is common to only graph the positive region with $K\geq 0$ and $L\geq 0$. When analyzing two graphs below, you will notice that there are clearly outlined contour lines that correspond to various fixed levels of Y. Interpreting this in the context of the Cobb-Douglas production function, we see that each contour line corresponds to a level of total output that is equal for all given baskets of capital and labor inputs. These contour lines are commonly referred to as isoquants, meaning equal quantities.

Lastly, the reason that economists like using the Cobb-Douglas function of two variables is because it is monotonically increasing in both inputs, and, for function $\alpha + \beta \leq 1$, the function is concave.

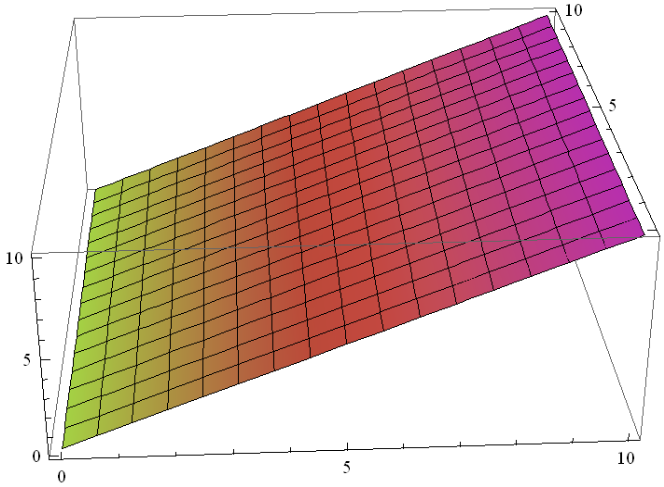
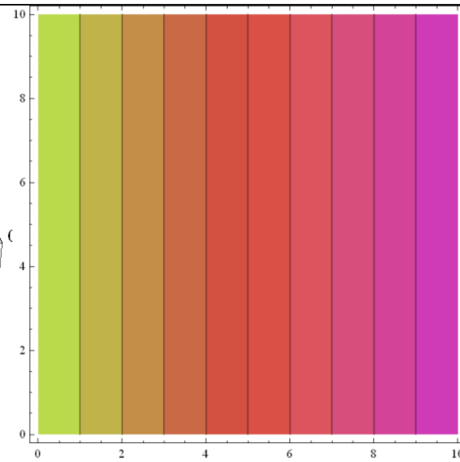
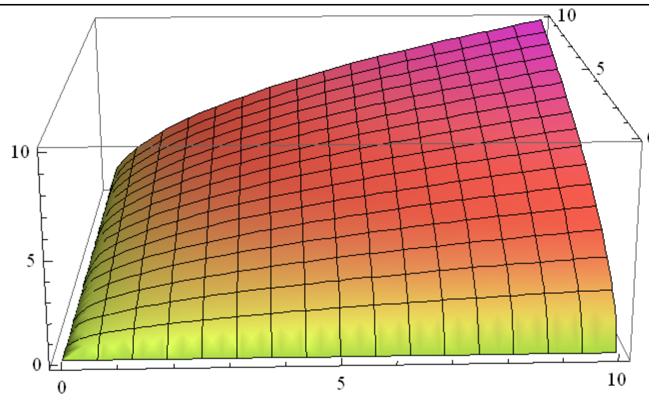
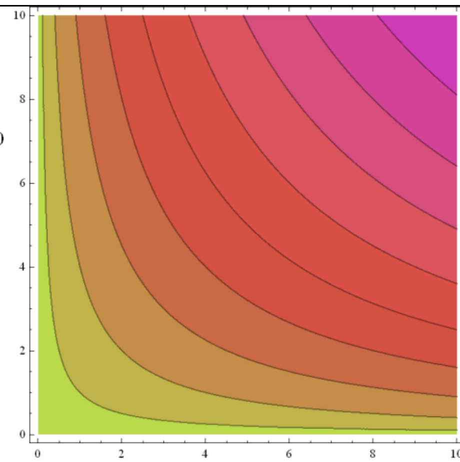
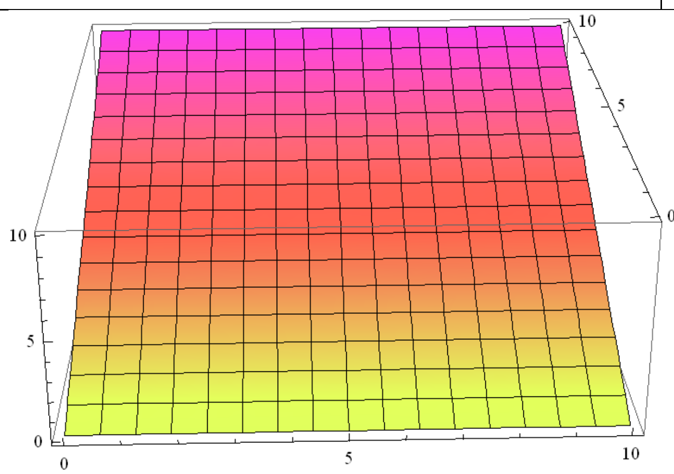
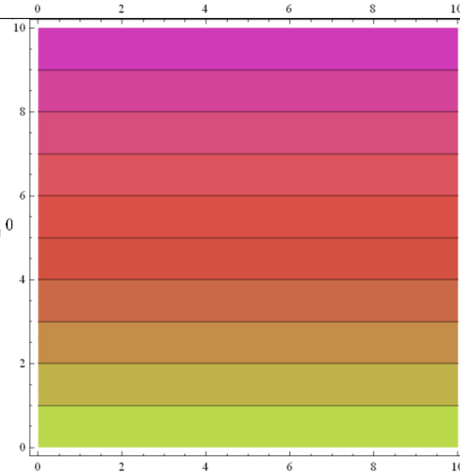




The next page shows Cobb Douglas functions at various levels of α and β.

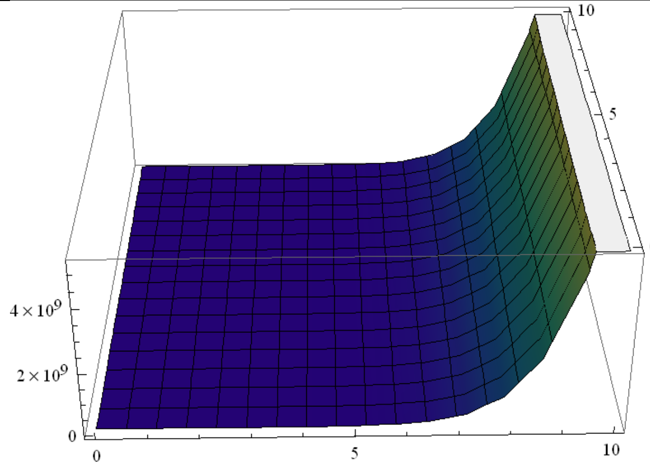
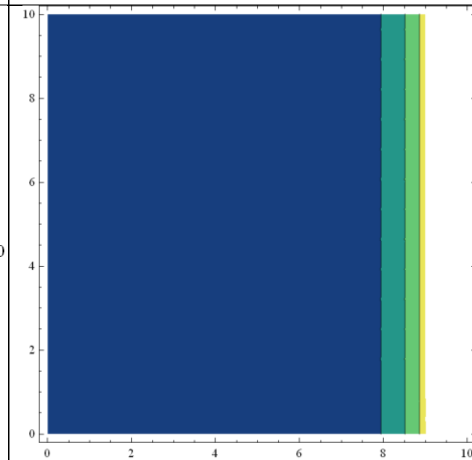
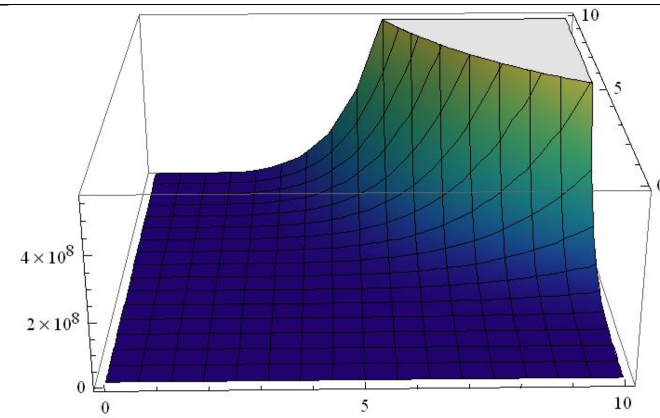
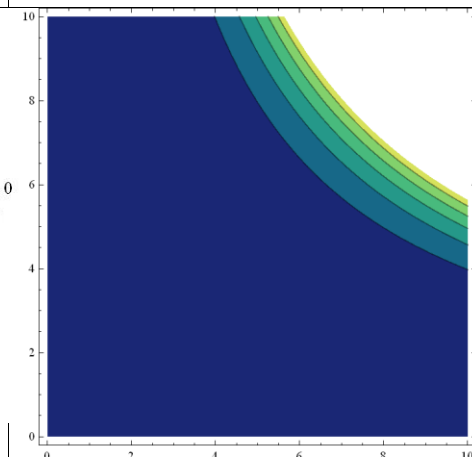
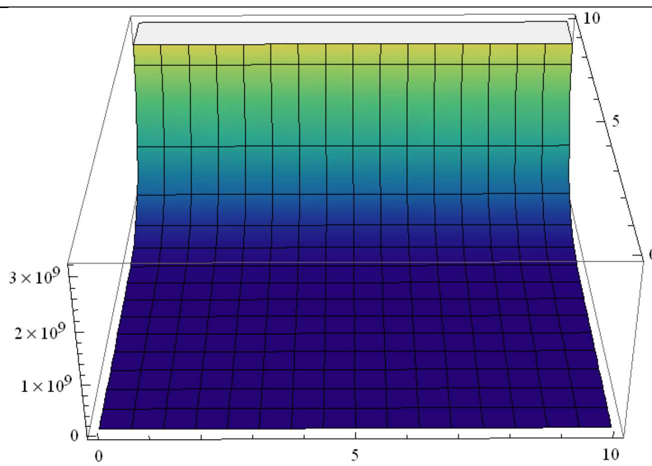
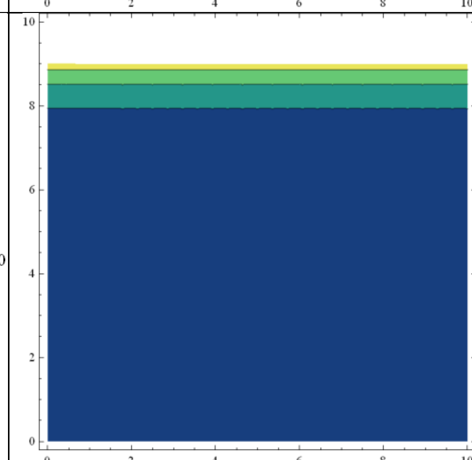
Category I

$\alpha+\beta=1$ implies constant returns to scale, which means that doubling the inputs doubles the outputs. These functions are said to be homogeneous of degree 1.

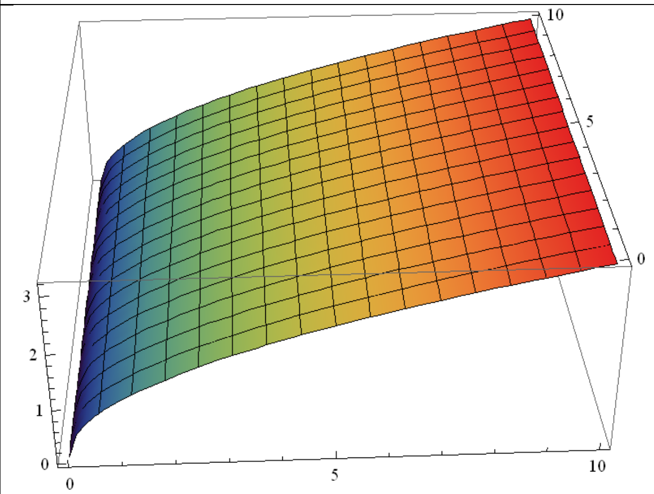
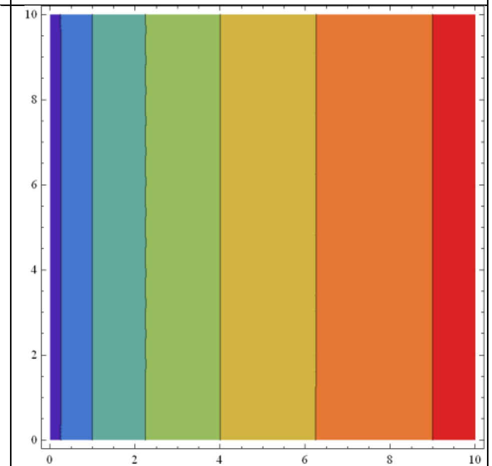
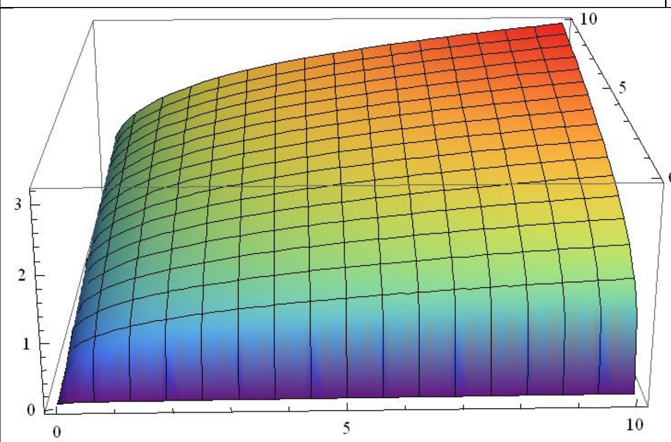
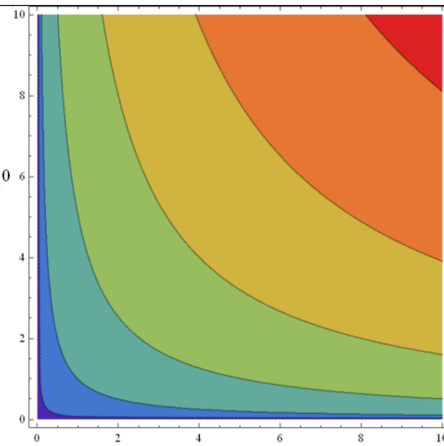
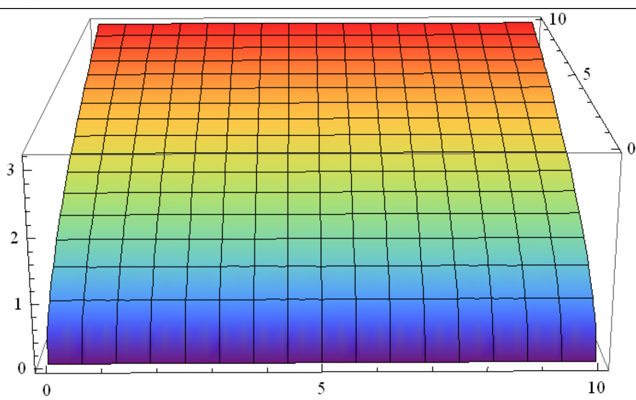
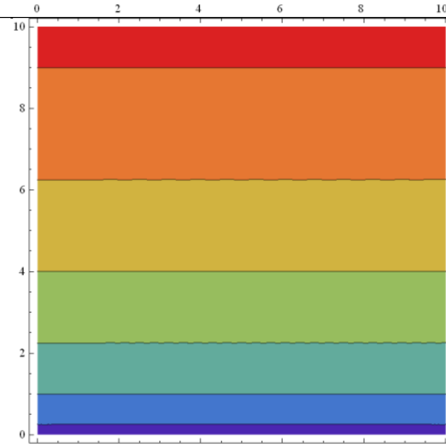
$Y=K^{\alpha}L^{\beta}$	3D Plot	Contour Plot
$\alpha=1$, $\beta=0$		
$\alpha=0.5$, $\beta=0.5$		
$\alpha=0$, $\beta=1$		

Category II

α+β>1 implies that a doubling of inputs leads to a more than doubling of outputs.

	3D Plot	Contour Plot
α=10, β=0		
α=5, β=5		
α=0, β=10		

Category III
α+β<1 leads to a less than doubling of outputs when inputs are doubled.

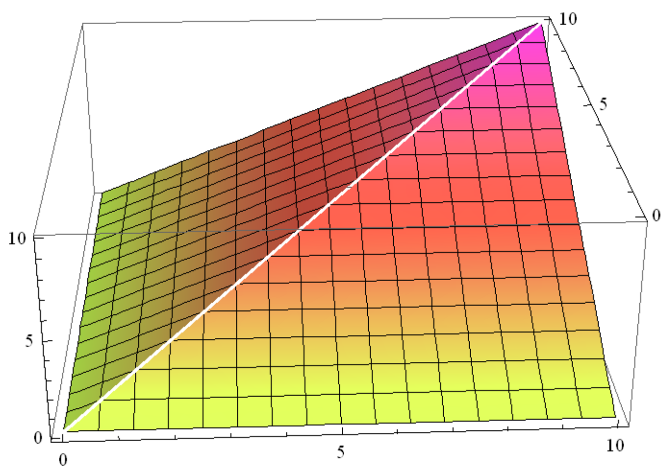
	3D Plot	Contour Plot
α=10, β=0		
α=5, β=5		
α=0, β=10		

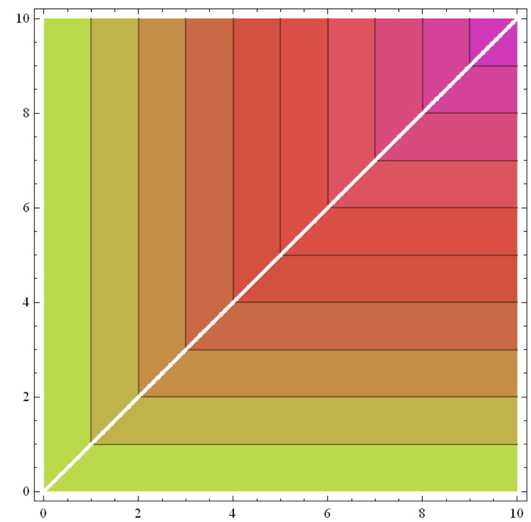
Leontief Function

Professor Wassily Leontief of Harvard University was a Nobel Prize winning economist known for his work involving the linkage of input-output tables to analyze the flow of goods. In other words, Professor Leontief was attempting to analyze how an increase in demand for an end product, such as an automobile, would cascade through the supply chain.

The Leontief production function can be interpreted as a function that identifies the limiting factor in situations involving perfect complements. For those unfamiliar with the concept of perfect complements, the best example would be of right and left shoes; people want exactly one right shoe for every left shoe.

The graphs below are generated by plotting Y=min(K,L) in Mathematica. Since it is again technically not feasible for inputs to be less than zero, it is common to only graph the positive region. When analyzing the two graphs, you will notice that the contour lines correspond to various fixed levels of Y. As with any production function, these contour lines are referred to as isoquants.





Sources
http://en.wikipedia.org/wiki/Cobb%E2%80%93Douglas_production_function and
http://en.wikipedia.org/wiki/Leontief_production_function

Calculus

12.4 Linear Functions

In the case of functions of a single variable, linear functions represent lines. We can generalize our definition of linear functions as polynomial functions of degree 0 or 1. Linear functions are popular in economics because they can be used to model situations where output grows at a constant rate. The general form for linear functions is
$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n + b$

<u>Example 1</u>
The general form of linear functions of a single variable is $f(x)=ax+b$

Currency conversion can be modeled as a collection of functions of a single variable. As of March 12, 2014, the conversion from the United States dollar (USD) to seven other major currencies including European euro (EUR), Japanese yen (JPY), Great Britain pound (GBP), Swiss franc (CHF), Canadian dollar (CAD), Australian dollar (AUD), Hong Kong dollar (HKD).

$f_{EUR}(USD) = 0.7219$ USD
$f_{JPY}(USD) = 103.0300$ USD
$f_{GBP}(USD) = 0.6015$ USD
$f_{CHF}(USD) = 0.8782$ USD
$f_{CAD}(USD) = 1.1113$ USD
$f_{AUD}(USD) = 1.1154$ USD
$f_{HKD}(USD) = 7.7621$ USD

Note that this could also be written in matrix form:

	USD	EUR	JPY	GBP	CHF	CAD	AUD	HKD
USD	1	0.7219	103.03	0.6015	0.8782	1.1113	1.1154	7.7621
EUR	1.3852	1	142.72	0.8333	1.2167	1.5395	1.5452	10.7529
JPY	0.0097	0.0070	1	0.0058	0.0085	0.0108	0.0108	0.0753
GBP	1.6625	1.2000	172.4138	1	1.46	1.8476	1.8542	12.9044
CHF	1.1387	0.8219	117.6471	0.6849	1	1.2654	1.2701	8.8384
CAD	0.8998	0.6496	92.5926	0.5412	0.7903	1	1.0037	6.8846
AUD	0.8965	0.6472	92.5926	0.5393	0.7873	0.9963	1	6.9591
HKD	0.1288	0.0930	13.2802	0.0775	0.1131	0.1453	0.1437	1

Example 2

The general form of linear functions of two variables is $f(x, y) = ax + by + c$.

The constant elasticity of substitution (CES) production function is represented by the following equation:

$$f(K, L) = A(\theta(\alpha K)^{\gamma} + (1-\theta)(\beta L)^{\gamma})^{1/\gamma}$$

In this case, the function output is the GDP, and it is a function of two inputs, capital (K) and labor (L). A, α, and β are positive constants. θ is between 0 and 1 and represents the share that is allocated to capital. γ is a parameter such that $(s-1)/s$ is the elasticity of substitution. The special cases are $\gamma = -\infty$, $\gamma = 0$, and $\gamma = 1$.

$\gamma = -\infty$ in the limit leads to the case where the CES function is the Leontief function.

$\gamma = 0$ in the limit leads to the case where the CES function is the Cobb-Douglas function.

$\gamma = 1$ leads to the case where the CES function is a linear function represented by $f(K, L) = A(\theta(\alpha K) + (1-\theta)(\beta L))$

Sources

http://en.wikipedia.org/wiki/Constant_elasticity_of_substitution
http://www.bloomberg.com/markets/currencies/

Calculus

12.5 Functions of three variables

Combining the ideas from the previous calculus sections, we can analyze functions of three variables. Examples of functions of three variables include: f(x, y, z) = x + y + z; f(x, y, z) = sin(xz) + cos(yz) ; f(x, y, z) = e^{xyz}. Let us consider the Cobb-Douglas production function below using multifactor inputs.

<u>Example 1</u>

The Cobb-Douglas Production function, as discussed earlier, is represented by the following equation:

$$f(A, K, L) = AK^{\alpha}L^{\beta}$$

Previously, A was an exogenous constant, but in this case, A is a variable. The function output, Y, which represents GDP, is the dependent variable, and it is a function of three variables, capital (K), labor (L), and total factor productivity (A). α and β are positive constants.

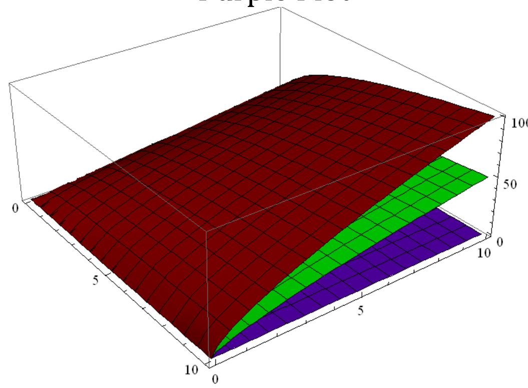
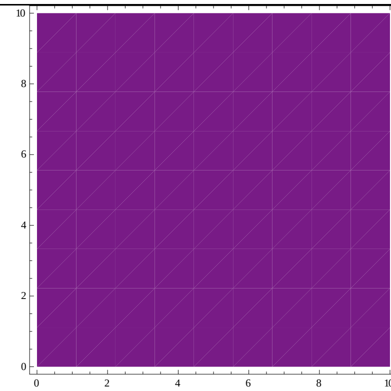
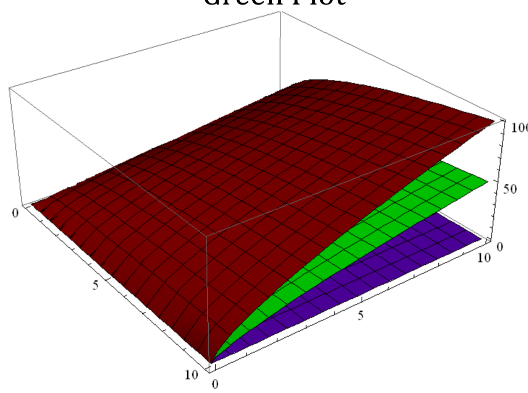
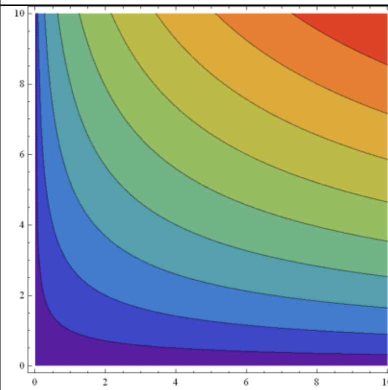
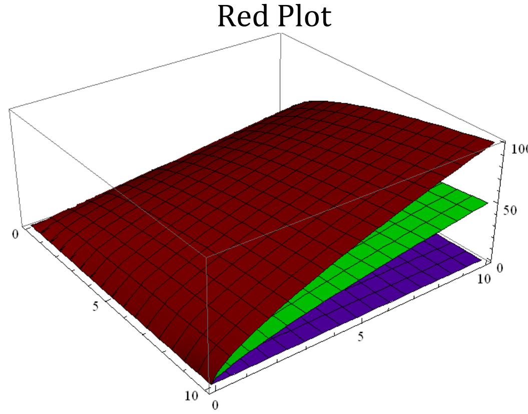
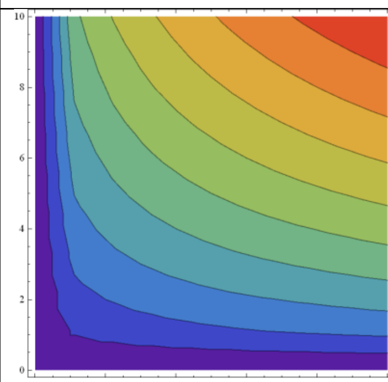
Technology innovations occur over time and are captured by the variable A. As these new discoveries are made, they affect the productivity of both labor and capital input.

We can use Mathematica to help us visualize these functions. The contour graphs on the following page can be generated using the following command:

```
Manipulate[ContourPlot[(A) ∗ (K)^(1/3) ∗
(L)^(2/3), {K, 0, 10}, {L, 0, 10}, ColorFunction → "Rainbow"], {A, 0, 10}]
```

The 3D plots can be generated using the following command:

```
Plot3D[Evaluate@Table[(A) ∗ (K)^(1/3) ∗
(L)^(2/3), {A, {0, 5, 10}}], {K, 0, 10}, {L, 0, 10}, PlotStyle → {Purple, Green, Red}]
```

	3D Plot	Contour Plot
A=0 α fixed at 1/3 β fixed at 2/3	Purple Plot 	
A=5 α fixed at 1/3 β fixed at 2/3	Green Plot 	
A=10 α fixed at 1/3 β fixed at 2/3	Red Plot 	

Calculus

14.1 Partial Derivative & 14.2 Computing the Partial Derivative Algebraically

Partial derivatives are especially useful in dealing with functions of several variables. When taking the partial derivative of the function with respect to desired variable, we keep all of the other variables constant. In economics, this is known as Ceteris Paribus.

Example 1

Partial derivatives with respect to capital and labor in the Cobb-Douglas model:

$$f(K, L) = AK^{\alpha}L^{\beta}$$

Partial derivative with respect to Capital

$$\frac{\alpha A L^{\beta}}{K^{1-\alpha}}$$

Partial derivative with respect to Labor

$$\frac{\beta A K^{\alpha}}{L^{1-\beta}}$$

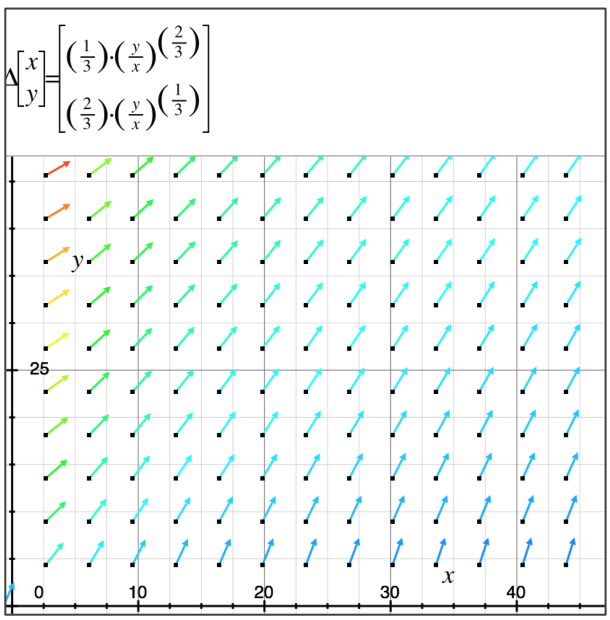
Calculus

14.4 Gradients and Directional Derivatives in the Plane

Example 1
If we analyze the following Cobb-Douglas function, we can see how the gradients are perpendicular to the indifference curves and point towards the upper right corner.

$$f(K, L) = K^{1/3}L^{2/3}$$

The following plot is generated using the Grapher application for Macs. I have replaced K with x and L with y.

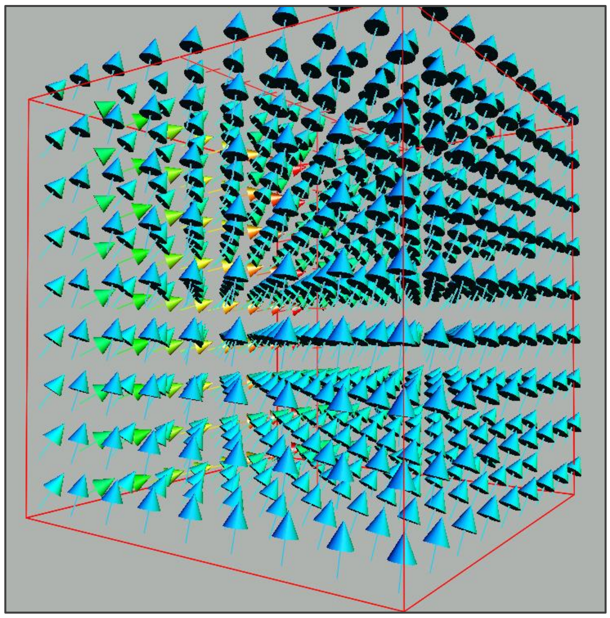


Calculus

14.5 Gradients and the Directional Derivatives in Space

Example 1
Using the same example as the previous section, we can now analyze the 3D vector field.

$$f(K, L) = K^{1/3}L^{2/3}$$



Calculus

14.6 Chain rule

In economics, there are many instances where the optimization process involves functions that would normally be a hassle to foil out and differentiate.

Example 1

The derivation of the Slutsky equation, which relates compensated demand to normal uncompensated demand, relies on the chain rule. The Slutsky equation breaks down changes in demand into a substitution and income effects. Wikipedia offers a mathematical derivation.

Source

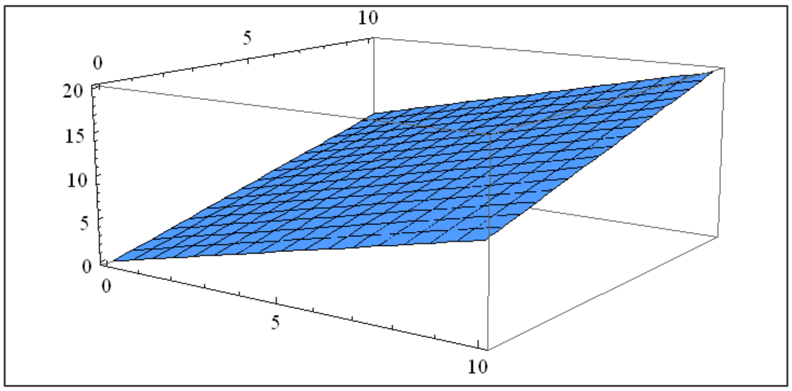
http://en.wikipedia.org/wiki/Slutsky_equation

Calculus

15.1 Local Extrema

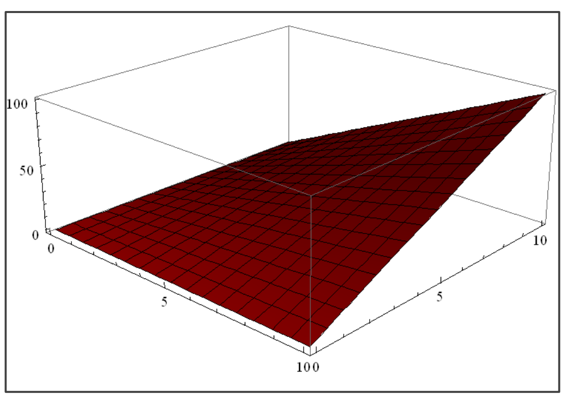
<u>Example 1</u>
CES function with γ = 1 leads to the case where the CES function is a linear function represented by f(K, L) = A(θ(αK) + (1-θ)(βL)). This is represented by the plane z = x + y

$$\mathbf{Plot3D}[x + y, \{x, 0, 10\}, \{y, 0, 10\}]$$



<u>Example 2</u>
The Cobb-Douglas function with α and β both equal to one means that a doubling of inputs leads to a more than doubling of outputs. This is represented by the function z = xy

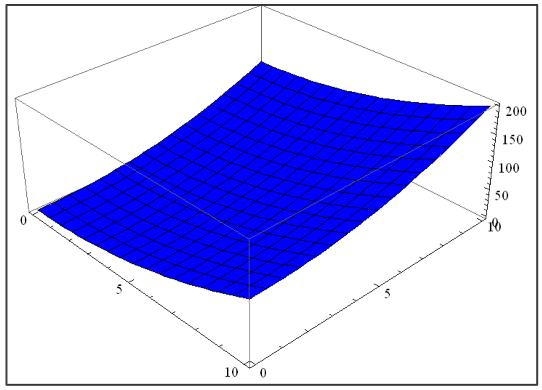
$$\mathbf{Plot3D}[x * y, \{x, 0, 10\}, \{y, 0, 10\}, \mathbf{PlotStyle} \to \mathbf{Red}]$$



Example 3

Many times, firm profit maximization relies on maximizing quadratic revenue functions.

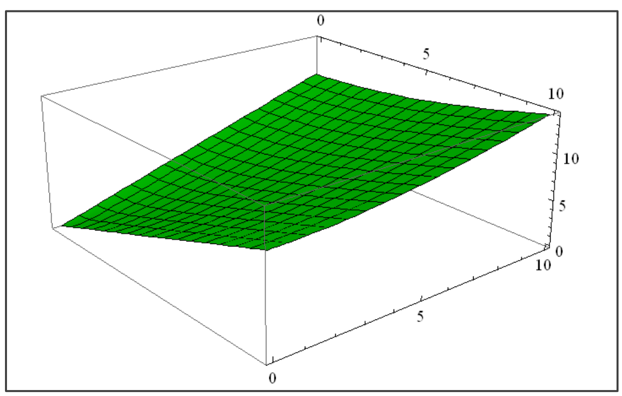
Plot3D[x^2 + y^2, {x, 0, 10}, {y, 0, 10}, PlotStyle → Blue]



Example 4

In general equilibrium theory, the utility function of an individual for two goods is often represented by the following function.

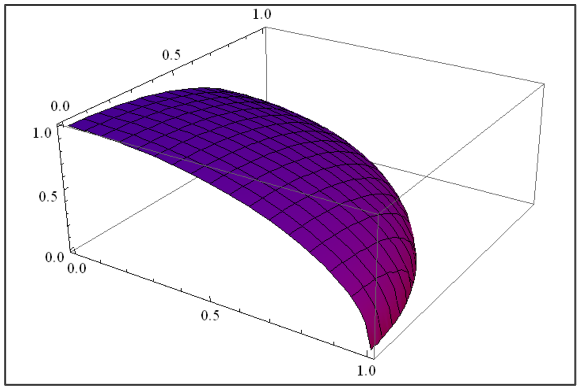
Plot3D[(x^2 + y^2)^(1/2), {x, 0, 10}, {y, 0, 10}, PlotStyle → Green]



Example 5

Spheres are often used to represent production possibility frontiers.

$$\mathbf{Plot3D}[(\mathbf{1}-x\verb|^|\mathbf{2}-y\verb|^|\mathbf{2})\verb|^|(\mathbf{1/2}),\{x,\mathbf{0},\mathbf{1}\},\{y,\mathbf{0},\mathbf{1}\},\mathbf{PlotStyle}\to\mathbf{Purple}]$$



Calculus

15.2 Optimization & 15.3 Constrained Optimization: Lagrange Multipliers
Lagrange multipliers are used throughout economics. Cobb-Douglas functions are often maximized subject to budget constraints, and we will cover the general format below.

<u>Example 1</u>

Production Function: $\mathrm{Y(K,L)} = \mathrm{K}^{\alpha}\mathrm{L}^{\beta}$
Budget Constraint: $\mathrm{P}_L\mathrm{L} + \mathrm{P}_K\mathrm{K} = \mathrm{C}$

P_L is the wage
P_K is the rental rate for capital
C is the available budget

Assumptions—goods. Monotonic increasing production (i.e. positive marginal productivity). As a result we will always deal with strict equality where we use all available resources.

Lagrange: $\mathrm{K}^{\alpha}\mathrm{L}^{\beta} - \lambda(\mathrm{P}_K\mathrm{K} + \mathrm{P}_L\mathrm{L} - \mathrm{C})$
1. $\alpha\mathrm{K}^{\alpha-1}\mathrm{L}^{\beta} - \lambda\mathrm{P}_L = 0$
2. $\beta\mathrm{K}^{\alpha}\mathrm{L}^{\beta-1} - \lambda\mathrm{P}_K = 0$
3. $\mathrm{P}_K\mathrm{K} + \mathrm{P}_L\mathrm{L} - \mathrm{C} = 0$

Derivatives with respect to K, L, and λ

Rewrite:
1. $\alpha\mathrm{K}^{\alpha-1}\mathrm{L}^{\beta} = \lambda\mathrm{P}_K$
2. $\beta\mathrm{K}^{\alpha}\mathrm{L}^{\beta-1} = \lambda\mathrm{P}_L$

Divide equation 1 by equation 2:

$$\frac{\alpha\mathrm{K}^{\alpha-1}\mathrm{L}^{\beta}}{\beta\mathrm{K}^{\alpha}\mathrm{L}^{\beta-1}} = \frac{\lambda\mathrm{P}_K}{\lambda\mathrm{P}_L}$$

Simply to

$$\frac{\alpha\mathrm{L}}{\beta\mathrm{K}} = \frac{\mathrm{P}_K}{\mathrm{P}_L}$$

Rewrite:

4. $\mathrm{P}_K\mathrm{K} = \frac{\alpha\mathrm{P}_L\mathrm{L}}{\beta}$

Substitute above into (3):

$$\mathrm{P}_L\mathrm{L} + \frac{\alpha\mathrm{P}_L\mathrm{L}}{\beta} = \mathrm{C}$$

Solve for L
$\beta P_L L + \alpha P_L L = \beta C$
$(\alpha + \beta) P_L L = \beta C$

$L = \frac{\left(\frac{\beta}{\alpha+\beta}\right)(C)}{(P_L)}$

Analysis of the result leads to the observation that the portion of the budget that will be spent on each of the two inputs will depend on the α and β of the two inputs.
In other words:

$\left(\frac{\alpha}{\alpha+\beta}\right)$ on capital and $\left(\frac{\beta}{\alpha+\beta}\right)$ on labor.