# 4.2 # 3, 6, 11, 14, 17, 19, 21, 24, 25, 33, 34

3.) $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$ Find an explicit description of Nul A by listing vectors that span the null space.

Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right]$$

$x_3, x_4$ free

$x_1 = 2x_3 - 4x_4$

$x_2 = -3x_3 + 2x_4$

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The spanning set of Nul A is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

6.) $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2+1} \begin{bmatrix} 1 & 0 & 5 & -6 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3, x_4, x_5$ free

$x_1 = -5x_3 + 6x_4 - x_5$

$x_2 = 3x_3 - x_4$

$$\vec{x} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The spanning set of Nul A is

$$\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

11.) Either use an appropriate theorem to show that the given set W is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} s-2t \\ 3+3s \\ 3s+t \\ 2s \end{bmatrix} : s, t \text{ real} \right\}$$

If W were a vector space, it would contain the zero vector, but the second entry is zero when $s=-1$ and the last entry is zero when $s=0$. Therefore this set doesn't contain the zero vector. W is not a vector space.

14.) (same directions as #11)

$$\left\{ \begin{bmatrix} -s+3t \\ s-2t \\ 5s-t \end{bmatrix} : s,t \text{ real} \right\}$$

$\vec{w} = s\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ Since any $\vec{w}$ in $W$ can be written this way,

$W = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \right\} = \text{Col}\begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix}$. Since the column space of a matrix is a subspace, $W$ is a vector space.

17.) a.) Find $k$ such that Nul A is a subspace of $\mathbb{R}^k$
b.) Find $k$ such that Col A is a subspace of $\mathbb{R}^k$

$A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$ A is $4 \times 2$ so Nul A is a subspace of $\mathbb{R}^2$ $(k=2)$ and Col A is a subspace of $\mathbb{R}^4$ $(k=4)$.

19.) $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$ A is $2 \times 5$ so Nul A is a subspase of $\mathbb{R}^5$ and Col A is a subspace of $\mathbb{R}^2$.

21.) With A as in #17, find a non-zero vector in Nul A and a nonzero vector in Col A.

$\begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$ is a non-zero vector in Col A since Col A = Span$\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\}$.

Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 6 & -4 & 0 \\ -3 & 2 & 0 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2$ is free $\vec{x} = x_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$x_1 = 2/3\, x_2$.

All we have to do to find a non-zero vector in Nul A is choose a nonzero value for $x_2$ and find $\vec{x}$.

For example if $x_2 = 3$ $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

# 4.2 Continued

**24.)** $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ Determine if $\vec{w}$ is in Col A. Is $\vec{w}$ in Nul A?

$$\left[\begin{array}{cccc|c} 10 & -8 & -2 & -2 & 2 \\ 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 6 & 0 & 0 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since $A\vec{x} = \vec{w}$ is consistent, $\vec{w}$ is in Col A.

$$\begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 20-16+0-4 \\ 0+4+0-4 \\ 2-2+0+0 \\ 2+2+0-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $A\vec{w} = \vec{0}$, $\vec{w}$ is in Nul A.

**25.)** True/False A is an $m \times n$ matrix.

a.) The null space of A is the soln set of $A\vec{x} = \vec{0}$. TRUE

b.) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$. FALSE

c.) The column space of A is the range of the mapping $\vec{x} \mapsto A\vec{x}$. TRUE

d.) If the equation $A\vec{x} = \vec{b}$ is consistent, then Col A is $\mathbb{R}^m$. FALSE

e.) The Kernel of a linear transformation is a vector space. True

f.) Col A is the set of all vectors that can be written as $A\vec{x}$ for some $\vec{x}$. TRUE

4.2

33.) Let $M_{2\times2}$ be the vector space of all $2\times2$ matrices and define $T: M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a.) Show that $T$ is a linear transformation.

$$T(A+B) = (A+B) + (A+B)^T = A + B + A^T + B^T = (A+A^T) + (B+B)^T$$
$$= T(A) + T(B)$$

$$T(cA) = cA + (cA)^T = cA + c(A^T) = c(A+A^T) = cT(A).$$

b.) Let $B$ be any element of $M_{2\times2}$ such that $B^T = B$. Find an $A$ in $M_{2\times2}$ such that $T(A) = B$.

If $T(A) = B$ then $A + A^T = B$. Suppose $A = \frac{1}{2}B$, then $A^T = (\frac{1}{2}B)^T = \frac{1}{2}B^T$

and $T(A) = A + A^T = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B = B.$

c.) Show that the range of $T$ is the set of $B$ in $M_{2\times2}$ with the Property that $B^T = B$.

In part (b) we showed if $B = B^T$, then $B$ is in the range of $T$.
Now we show the other direction ie. If $B$ is in the range of $T$ then $B = B^T$.

Suppose $B = A + A^T$ then $B^T = (A + A^T)^T = A^T + A = B.$

d.) Describe the Kernel of $T$.

The Kernel is the set of all $A$ such that $T(A) = 0$.

$$A + A^T = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a, d = 0 \text{ and } b = -c$$

Kernel of $T$ is:

$$\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ is real} \right\}$$

4.2 continued

34) Define $T: C[0,1] \to C[0,1]$ as follows: For $\vec{f} \in C[0,1]$, let $T(\vec{f})$ be the antiderivative $\vec{F}$ of $\vec{f}$ such that $\vec{F}(0)=0$. Show that $T$ is a linear transformation and describe the Kernel of $T$.

Let $\vec{f}, \vec{g}$ be elements in $C[0,1]$.

$T(\vec{f}+\vec{g})$ is the antiderivative of $\vec{f}+\vec{g}$, from calculus we know this is the antiderivative of $\vec{f}$ plus the antiderivative of $\vec{g}$.

So $T(\vec{f}+\vec{g}) = \vec{F}+\vec{G}$ such that $(\vec{F}+\vec{G})(0)=0$.

Then $T(\vec{f}+\vec{g}) = T(\vec{f}) + T(\vec{g})$. Similarly,

$T(c\vec{f}) = cT(\vec{f})$.

The Kernel of $T$ is the set of all functions $\vec{f}$ whose antiderivative is zero and $\vec{F}(0)=0$. Therefore $\vec{f}=\vec{0}$. The Kernel of $T$

is $\{\vec{0}\}$.