MATH 111, Intro to Functions and Calc II, S2020, Condensed Lecture Notes

Taken in part from
An Integrated Approach to Functions and their Rates of Change
Gottlieb

Notes compiled by Bobby McDonald
Yale University, 2020

NOTE: I will update these notes as often as I can with the topics and examples (which will be worked out by hand in a separate document) we cover in class.

Contents

Chapter 16

Taking the Derivative of Composite Functions

16.1 The chain rule

16.1. Goals
- review derivatives
- express derivatives of composite functions $f(g(x))$

Groups 16.1.1. What are the derivatives of the following functions:
- x^n
- e^x
- b^x
- $ln(x)$
- $\log_b(x)$
- $f(x) \pm g(x)$
- $f(x)g(x)$
- $f(x)/g(x)$

Think, Pair, Share 16.1.2. What does the derivative represent?

Question 16.1.3. Suppose we're selling ice cream. The price that we set depends on the demand, that is, $p = f(u)$ where u is measured in people who want ice cream, and P is measured in dollars. Of course, the demand of our ice cream depends on the temperature, so we also have $u(t)$ where t is degrees Fahrenheit. So we guess, our revenue *really* depends on the temperature outside. How could we find the rate of change of our revenue with respect to temperature?

Theorem 16.1.4. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Question 16.1.5. How does this look with prime notation? What's nice about Leibnitz notation?

Groups 16.1.6. Suppose the price of our ice cream in dollars is

$$P(x) = 3x + 1,$$

where x is the demand in people. and the demand is $x = 1 - t^2$ where t is temperature measured in degrees Fahrenheit.

(a) write p as a function of t

(b) find $p'(t)$ (called the *marginal profit*), and its units

Example 16.1.7. What are the derivatives of $f(kx)$ and $f(x + k)$? Why does this make sense geometrically?

Groups 16.1.8. Write the functions as a composition

(a) $(x^2 + 1)^{10}$

(b) $\ln(x^2 + 2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.9. Find the derivatives of

(a) $(x^2+1)^{10}$

(b) $\ln(x^2+2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.10. Suppose the population of frogs in a pond is e^{g}, where g is the temperature of the pond in Celcius and the average temperature in the month of February is $0.25t+14$ where t is in days. What is the rate of change of frogs with respect to time?

Groups 16.1.11. Suppose a rectangle is inscribed inside the ellipse

$$\frac{x^2}{9} + 4y = 1.$$

What's the largest possible area of such a rectangle?

16.2 The derivative of x^n for any real number n

16.2. Goals

- prove the power rule a different way
- find the derivative of b^x a different way
- find derivatives of quotients a different way

Example 16.2.1. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Question 16.2.2. Can we recover the derivative of b^x for $b > 0$ using only the chain rule?

Groups 16.2.3. Now, let's consider x^n.

(a) rewrite x^n using $\ln x$ and e^x.

(b) find the derivative of the function you found in (a)

(c) what did you prove?

Question 16.2.4. Can we find the derivative of $f(x)/g(x)$ without using the quotient rule?

16.3 Using the chain rule

16.3. Goals

- do more with the chain rule
- derive a formula for functions of multiple compositions
- understand the importance of simplification

Theorem 16.3.1. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Example 16.3.2. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Groups 16.3.3. Differentiate $e^{\sqrt{x^3+1}}$.

Think, Pair, Share 16.3.4. Can we find a general rule for the derivative of $f(g(h(x)))$?

Example 16.3.5. Differentiate

$$\ln\left(\sqrt{\frac{1+x}{(1-x)^3}}\right).$$

Example 16.3.6. Differentiate

$$\frac{8^{x^2+1}}{\left(2^x\right)^x}$$

Example 16.3.7. Find the derivative of

$$\left(\frac{x^4-2x^2+1}{x^2-1}\right)^5.$$

(Note: The notes I gave you last class has x^4+2x^2+1 in the numerator. Can you still simplify?)

Groups 16.3.8. Does xe^{x^2} have a maximum?

Chapter 17

Implicit Differentiation and its Applications

17.1 The derivative of x^x

17.1. Goals

- finding the derivative of x^x
- logarithmic differentiation

Example 17.1.1.

(a) What are the derivatives of x^n and b^x? Do either of these rules work for x^x when $x > 0$?

(b) Can we make it so that x is not a power? (hint: do you remember your log rules?)

Example 17.1.2. Find the derivative of $y = x^x$.

Definition 17.1.3. The method we just used, by taking logs of both sides and using the chain rule, is called **logarithmic differentiation**.

17.2 Logarithmic differentiation

17.2. Goals

- using logarithmic differentiation

Example 17.2.1. Find the tangent to the curve $f(x) = (x^2 + 1)^x$ at $x = 0$.

Example 17.2.2.

(a) what is the domain of $(x-1)^{1-x^2}$?

(b) on this domain, find $f'(2)$.

Question 17.2.3. What are the properties of logarithms that we know?

Example 17.2.4. Find the derivative of $y = 2x^{e^x}$

Example 17.2.5. Find the derivative of $y = \dfrac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$

17.2.1 Extra Examples

Example 17.2.6. Find the derivatives of

(a) $\dfrac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$

(b) $e^{2x}(x^2+3)^5(2x^2+1)^3$

(c) $\left(e^{x-1}\right)^{x+1}$

17.3 Implicit differentiation

17.3. Goals

- using the ideas of the previous section to find $\frac{dy}{dx}$ of implicitly defined functions

Spot the mistake 17.3.1. Find $\frac{dy}{dx}$ for the circle

$$x^2+y^2=1$$

Example 17.3.2. Find $\frac{dy}{dx}$ for the circle

$$x^2+y^2=1$$

Example 17.3.3. What kinds of information can we use to sketch the graph of a curve?

Example 17.3.4. Sketch a graph of the curve $y^2=x^3-x$ (don't worry about concavity)

Example 17.3.5. Find all points where the tangent to

$$x^3+y^3=1$$

is horizontal or vertical

Procedure 17.3.6 (using implicit differentiation)**.**

Example 17.3.7. Find the slope of the tangent to

$$x^3+y^3=6xy$$

at the point $(3,3)$.

Example 17.3.8. Find the absolute maximum and minimum y-values of the ellipse

$$2x^2+4xy+3y^2=6.$$