6.4 # 1, 3, 7, 9, 11, 17, 19

1.) The given set is a basis for a subspace W. Use the Gram-Schmidt Process to produce an orthogonal basis for W.

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$$
$\vec{x}_1, \vec{x}_2$

Set $\vec{v}_1 = \vec{x}_1$ and $\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{24+6}{10} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$

orthogonal basis for W: $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$

3.) $$\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$$
$\vec{x}_1 \quad \vec{x}_2$

$\vec{v}_1 = \vec{x}_1$, $\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{8+5+2}{4+25+1} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$

orthogonal basis for W: $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$

7.) Find an orthonormal basis of the subspace spanned by the vectors in #3.

$\|\vec{v}_1\| = \sqrt{4+25+1} = \sqrt{30}$

$\|\vec{v}_2\| = \sqrt{54/4} = \frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6}$

orthonormal basis: $\left\{ \begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$

9.) Find an orthogonal basis for the column space of each matrix.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3$

There is a pivot in every column, so the cols of A form a basis for Col A. So to find an orthogonal basis we use Gram-Schmidt.

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{-15+1-5-21}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3+1+2+24}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{-5+1-10-56}{100} \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

11.) $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \\ \vec{x}_1 \;\; \vec{x}_2 \;\; \vec{x}_3 \end{array} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Again all three columns form a basis for the Column space.

$\vec{v}_1 = \vec{x}_1$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \frac{2-1-4-4+2}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

orthogonal basis for Col A:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

17.) True/False. All vectors and subspaces are in $\mathbb{R}^n$.

a.) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for W, then multiplying $\vec{v}_3$ by a scalar c gives a new orthogonal basis, $\{\vec{v}_1, \vec{v}_2, c\vec{v}_3\}$.

b.) The Gram-Schmidt process produces from a linearly indep. set $\{\vec{x}_1, \dots, \vec{x}_p\}$ an orthogonal set $\{\vec{v}_1, \dots, \vec{v}_p\}$ with the property that for each k, the vectors $\vec{v}_1, \dots, \vec{v}_k$ span the same subspace as that spanned by $\vec{x}_1, \dots, \vec{x}_k$.

c.) If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.

a.) FALSE b.) TRUE c.) TRUE

19.) Suppose $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show that if the cols. of A are linearly indep., then R must be invertible.

Suppose $\vec{x}$ is a soln to $R\vec{x} = \vec{0}$. We would like to show that $\vec{x}$ must be the trivial soln. If $R\vec{x} = \vec{0}$, then $QR\vec{x} = Q\vec{0} = \vec{0}$ which implies $A\vec{x} = \vec{0}$. The columns of A are linearly indep. so $\vec{x} = \vec{0}$ is the only soln. Therefore R is invertible.