1.9 The Matrix of a Linear Transformation

McDonald Fall 2018, MATH 2210Q, 1.9Slides

1.9 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in***: 1, 5, 13, 19, 23, 26, 34
- Recommended: 2, 15, 20, 32

Whenever a function is decribed geometrically or in words, we usually want to find a formula. In linear algebra, the same will be true for linear transformations. It turns out that *every* linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Example 1.9.1. Suppose that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -8 \\ 6 \end{bmatrix}$$

Find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$, and a matrix, A, such that $T(\mathbf{x}) = A\mathbf{x}$.

Definition 1.9.2. The **identity matrix**, I_n, is the $n \times n$ matrix with ones on the diagonal [$\diagdown$], and zeros everywhere else. For example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remark 1.9.3. The key to finding the matrix for a linear transformation is to see what it does I_n.

Theorem 1.9.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the indentity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

Definition 1.9.5. The matrix A in Theorem 1.9.4 is called the **standard matrix** for T.

Example 1.9.6. If $r \geq 0$, find the standard matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{x} \mapsto r\mathbf{x}$.

Example 1.9.7. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that rotates each point counter clockwise about the origin through an angle α. Find the standard matrix for T.

The following definitions should sound familiar.

Definition 1.9.8. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$. T is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

Remark 1.9.9. T being *onto* is an *existence* question: for every $\mathbf{b}$ in $\mathbb{R}^m$, does an $\mathbf{x}$ exist such that $T(\mathbf{x}) = \mathbf{b}$? T being *one-to-one* is a *uniqueness* question: for every $\mathbf{b}$ in $\mathbb{R}^m$, if there is a solution to $T(\mathbf{x}) = \mathbf{b}$, is it unique?

Example 1.9.10. Let T be the transformation whose standard matrix is

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 4 & 3 \\ -2 & 0 & 1 \end{bmatrix}$$

Is T one-to-one? Is T onto?

Theorem 1.9.11. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Theorem 1.9.12. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then:*

(a) T is one-to-one if and only if the columns of A are linearly independent;

(b) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

Example 1.9.13. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the transformation that brings $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to $\begin{bmatrix} 2x_1 + 4x_4 \\ x_1 + x_2 + 3x_4 \\ -2x_1 + x_3 - 4x_4 \end{bmatrix}$.

Find a standard matrix for T and determine if T is one-to-one. Is T onto?

Example 1.9.14. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation that brings $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$.

Find a standard matrix for T and determine if T is one-to-one. Is T onto?

Example 1.9.15. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If T is onto, what can you say about m and n? If T is one-to-one, what can you say about m and n?

Example 1.9.16. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that first reflects points through the horizontal x_1-axis, and then reflects them through the line $x_2 = x_1$. Find the standard matrix of T.

Remark 1.9.17. The following tables, taken from Lay's Linear Algebra book, illustrate common geometric linear transformations of the plane. Each shows the transformation of the unit square.

TABLE 1 Reflections

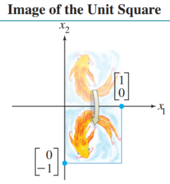
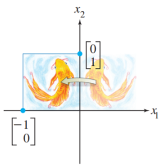
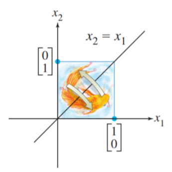
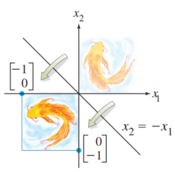
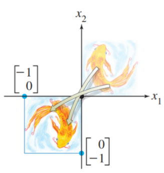
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

TABLE 2 Contractions and Expansions

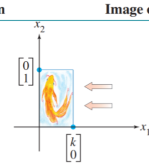
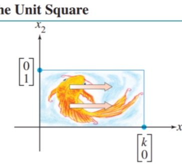
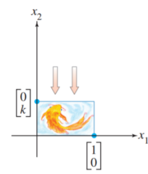
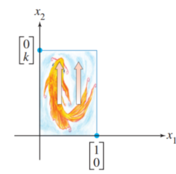
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

TABLE 3 Shears

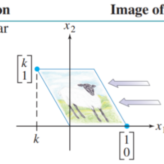
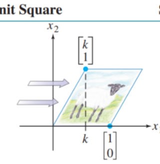
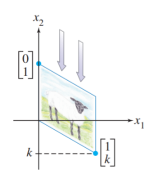
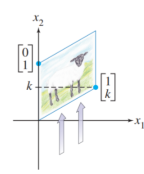
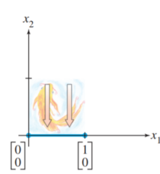
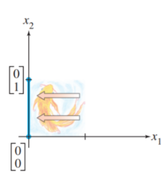
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$