6.5 # 3, 5, 7, 9, 11, 17, 19, 21

3.) Find a least-squares solution of $A\vec{x}=\vec{b}$ by (a) constructing the normal equations for $\hat{x}$ and (b) solving for $\hat{x}$.

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

a.) The normal equations are the system given by $A^TA\hat{x} = A^T\vec{b}$

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}\vec{x} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

b.) Solving this for $\hat{x}$, $\left[\begin{array}{cc|c} 6 & 6 & 6 \\ 6 & 42 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \end{array}\right]$ $\boxed{\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}}$

5.) Describe all least-squares solutions of $A\vec{x}=\vec{b}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$x_1 + x_3 = 5$

$x_2 - x_3 = -3$

$$\boxed{\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

7.) Compute the least-squares error associated w/ the least-squares solution found in Exercise 3.

The least-squares error is $\|A\hat{x} - \vec{b}\|$.

$$A\hat{x} - \vec{b} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \qquad \|A\hat{x} - \vec{b}\| = \sqrt{20}$$

9.) Find (a) the orthogonal projection of $\vec{b}$ onto Col A and (b) a least-squares solution of $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$\vec{a}_1 \cdot \vec{a}_2 = 5 + 3 - 8 = 0$ Since the columns of A are orthogonal, $\hat{b} = \frac{\vec{b} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1}\vec{a}_1 + \frac{\vec{b} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2}\vec{a}_2$

(a) $$\hat{b} = \frac{4-6+6}{1+9+4}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{20-2-12}{25+1+16}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(b) $$\hat{x} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$$

11.) $$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{a}_1 \cdot \vec{a}_2 = -5 + 6 - 1 = 0$

$\vec{a}_1 \cdot \vec{a}_3 = 4 + 1 - 5 = 0$

$\vec{a}_2 \cdot \vec{a}_3 = -5 + 5 = 0$

(a) $$\hat{b} = \frac{36}{54}\begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ -5 \\ 1 \\ -1 \end{bmatrix} + \frac{9}{27}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

(b) $$\hat{x} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

6.5 continued

17.) True/False. A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$.

a.) The general least-squares problem is to find an $\vec{x}$ that makes $A\vec{x}$ as close as possible to $\vec{b}$.

b.) A least-squares solution of $A\vec{x}=\vec{b}$ is a vector $\hat{x}$ that satisfies $A\hat{x}=\hat{b}$ where $\hat{b}$ is the orthogonal projection of $\vec{b}$ onto $\text{Col}\,A$.

c.) A least-squares solution of $A\vec{x}=\vec{b}$ is a vector $\hat{x}$ such that $\|\vec{b}-A\vec{x}\| \leq \|b-A\hat{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$.

d.) Any solution of $A^TA\vec{x}=A^T\vec{b}$ is a least-squares solution of $A\vec{x}=\vec{b}$.

e.) If the columns of A are linearly independent, then the equation $A\vec{x}=\vec{b}$ has exactly one least-squares solution.

a.) True b.) True c.) False d.) True e.) True

19.) Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\vec{x}$ in $\mathbb{R}^n$ satisfies $A\vec{x}=\vec{0}$ if and only if $A^TA\vec{x}=\vec{0}$. This will show that $\text{Nul}\,A = \text{Nul}\,A^TA$.

a.) Show that if $A\vec{x}=\vec{0}$, then $A^TA\vec{x}=\vec{0}$.

b.) Suppose $A^TA\vec{x}=\vec{0}$. Explain why $\vec{x}^TA^TA\vec{x}=\vec{0}$ and use this to show that $A\vec{x}=\vec{0}$.

a.) If $A\vec{x}=\vec{0}$, then $A^TA\vec{x}=A^T\vec{0}=\vec{0}$. In other words if $\vec{x}$ is in $\text{Nul}\,A$, then $\vec{x}$ is in $\text{Nul}(A^TA)$. Thus $\text{Nul}\,A$ is contained in $\text{Nul}(A^TA)$.

b.) If $A^TA\vec{x}=\vec{0}$, then $\vec{x}^TA^TA\vec{x}=\vec{x}^T\vec{0}=\vec{0}$. Since $\vec{x}^TA^T=(A\vec{x})^T$ we have shown $(A\vec{x})^T(A\vec{x})=\vec{0}$ or in other words $\|A\vec{x}\|^2=0$. This proves $A\vec{x}=\vec{0}$ therefore $\text{Nul}(A^TA)$ is contained in $\text{Nul}\,A$.

By combining (a) and (b) we have shown $\text{Nul}\,A=\text{Nul}(A^TA)$

21.) Let A be an $m \times n$ matrix whose columns are linearly independent. (Careful, A may not be square.)

a.) Use Exercise 19 to show that A^TA is invertible.

b.) Explain why A must have at least as many rows as columns.

c.) Determine the rank of A

a.) If the columns of A are linearly independent, then $A\vec{x}=\vec{0}$ has only the trivial solution, so $\text{Nul } A = \{\vec{0}\}$. By #19 $\text{Nul} A = \text{Nul}(A^TA) = \{\vec{0}\}$. Since $\text{Nul}(A^TA) = \{\vec{0}\}$, A^TA is invertible.

b.) Since the n linearly independent columns of A belong to $\mathbb{R}^m$, m cannot be less than n.

c.) Since the columns of A are linearly independent, these columns form a basis for $\text{Col } A$, so $\text{rank } A = n$.