4.6 #1, 2, 5, 7, 10, 13, 19, 24, 27, 28

1.) Without Calculations, list rank A and dim Nul A. Then find bases for Col A, Row A and Nul A.

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Two pivot cols $\Rightarrow$ dim Col A = 2

dim Col A = rank A So rank A = 2.

rank A + dim Nul A = # of cols of A

dim Nul A = 4 − 2 = 2

basis for Col A : $\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$, basis for Row A : $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix} \right\}$

basis for Nul A : we need to reduce to RREF.

$$A \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2/(-2)} \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So when $A\vec{x} = \vec{0}$

$x_1 = x_3 - 5x_4$

$x_2 = \frac{5}{2}x_3 - 3x_4$

x_3, x_4 free

basis for Nul A: $\left\{ \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.) $$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A = 3

dim Nul A = 5 − 3 = 2

basis for Col A : $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$

basis for Row A : $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$

basis for Nul A

$$A \sim \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3/(-5)]{-4R_2+R_1} \begin{bmatrix} 1 & 3 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_3+R_2]{2R_3+R_1} \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -3x_2 - 3x_4$

$x_3 = x_4$

$x_5 = 0$

x_2, x_4 free

basis for Nul A: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

5.) If a 4×7 matrix A has rank 3, find dim Nul A, dim Row A and rank A^T.

$\dim \text{Nul} A = 7 - 3 = 4$

$\dim \text{Row} A = \text{rank} A = 3$

$\text{rank} A^T = \dim \text{Col} A^T = \dim \text{Row} A = 3$

7.) Suppose a 4×7 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^3$? Explain.

Col A is a subspace of $\mathbb{R}^4$ and since $\dim \text{Col} A = 4$, we must have Col $A = \mathbb{R}^4$.
Nul A is a subspace of $\mathbb{R}^7$, it has dimension 3, but that doesn't mean Nul $A = \mathbb{R}^3$.

10.) If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of Col A?

$\dim \text{Col} A + \dim \text{Nul} A = 7$

$\dim \text{Col} A = 7 - 5 = 2$

13.) If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A?

In both cases the largest number of pivots we can have is 5, so this is the largest possible rank of A.

19.) Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of equations?

$A\vec{x} = \vec{0}$ where A is 5×6 has Nul A with dimension 1 by the first sentence. Therefore Col $A = 6 - 1 = 5$. Since Col A is a subspace of $\mathbb{R}^5$ with dimension 5, Col $A = \mathbb{R}^5$. Therefore yes, the system $A\vec{x} = \vec{b}$ will have a solution for every $\vec{b} \in \mathbb{R}^5$.

4.6 continued

24.) Is it possible for a non homogeneous system of seven equations in six unknowns to have a unique soln for some right hand side of constants? Is it possible for such a system to have a unique soln for every right hand side? Explain.

$A\vec{x} = \vec{b}$ where A is a 7×6 matrix. In order to have a unique solution for some $\vec{b}$ on the right hand side, we must have a pivot in every column. So we must have rank $A = 6$ for this to be possible. Since A can have at most 6 pivots, rank $A \leq 6$ and so rank $A = 6$ when $\text{Nul}\, A = 0$. So the first statement is possible. Since $\text{Col}\, A$ is a subspace of $\mathbb{R}^7$ and $\dim \text{Col}\, A = \text{rank}\, A \leq 6$, there exists a $\vec{b} \in \mathbb{R}^7$ such that the system is inconsistent. Therefore the second statement is not possible.

27.) A is $m \times n$. Which of the subspaces Row A, Col A, Nul A, Row A^T, Col A^T, Nul A^T are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are in this list?

$\mathbb{R}^n$: Row A, Nul A, Col A^T

$\mathbb{R}^m$: Col A, Row A^T, Nul A^T

Row A^T = Col A and Col A^T = Row A

So there are 4 distince subspaces in the list Row A, Col A, Nul A, Nul A^T

28.) Justify the equalities: (A is $m \times n$)
a.) $\dim \text{Row}\, A + \dim \text{Nul}\, A = n$
b.) $\dim \text{Col}\, A + \dim \text{Nul}\, A^T = m$

a.) $\dim \text{Row}\, A = \dim \text{Col}\, A = \text{rank}\, A$
Since $\text{rank}\, A + \dim \text{Nul}\, A = n$, substitution gives us $\dim \text{Row}\, A + \dim \text{Nul}\, A = n$.

b.) Since A^T is $n \times m$, $\text{rank}\, A^T + \dim \text{Nul}\, A^T = m$ and $\text{rank}\, A^T = \dim \text{Col}\, A^T = \dim \text{Row}\, A^T = \dim \text{Col}\, A$. Therefore $\dim \text{Col}\, A + \dim \text{Nul}\, A^T = m$ by substitution.