## 4.1 Orthogonality

**4.1. Key Ideas**

- Subspaces $V$ and $W$ are orthogonal if every $\mathbf{v}$ in $V$ is orthogonal to every $\mathbf{w}$ in $W$.

**Question 4.1.1.** How do we know when two vectors are *orthogonal* to each other?

- if in $\mathbb{R}^2$

$\begin{bmatrix} a \\ b \end{bmatrix}$ is perp to $k\begin{bmatrix} -b \\ a \end{bmatrix}$

$y = \frac{b}{a}x$

$y = \frac{-a}{b}x$

$by = -ax$

- otherwise if $u \cdot v = 0$ then vectors perpendicular

$$\boxed{u \cdot v = \|u\| \|v\| \cos\theta}$$

**Definition 4.1.2.** Two vectors in $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

**Definition 4.1.3.** A vector **u** in $\mathbb{R}^n$ is orthogonal to a subspace $V$ of $\mathbb{R}^n$ if and only if it's orthogonal to every vector in $V$.

**In English...**

to check if $\vec{u}$ is orthogonal to $V$

$\vec{u} \cdot \vec{v} = 0$ for all $\vec{v}$ in $V$

really only need to check for all $v$ in a spanning set for $V$.

**Example 4.1.4.** Show that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is orthogonal to the column space of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

just check

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 - 4 + 3 = 0 \checkmark$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 - 10 + 6 = 0 \checkmark$$

$$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ is orthogonal to Col } A$$

**Definition 4.1.5.** Two subspaces $V$ and $W$ of $\mathbb{R}^n$ are orthogonal if and only if every vector in $V$ is orthogonal to every vector in $W$.

**In English...**

given $V = \text{Span}\{v_1, \dots, v_p\}$
$W = \text{Span}\{w_1, \dots, w_j\}$

need to check each pair of $v_i, w_k$

**Example 4.1.6.** Find all of the vectors that are orthogonal to the column space of

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

if $\vec{x}$ is orthogonal to Col(A)

then $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \vec{x} = 0 \quad \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \cdot \vec{x} = 0$

$$1 \cdot x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right] \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \vec{x} = \vec{0}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$$\sim \begin{cases} \boxed{x_1 = x_3 + 2x_4} \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_4$$

$$\text{soln set} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

all orth to Col A

**Definition 4.1.7.** If $A$ is an $m \times n$ matrix, the **transpose** of $A$ is the $n \times m$ matrix, denoted $A^T$, whose columns are formed from the corresponding rows of $A$.

take columns make them rows

**Example 4.1.8.** Let $A = \begin{bmatrix} a & b & d \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \\ 6 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 3 & 1 & 2 & -6 \end{bmatrix}$.

Find $A^T$, $B^T$, and $C^T$.

e.g. $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

$A^T = \begin{bmatrix} a & b & d \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$

$B^T = \begin{bmatrix} 8 & 4 \\ 5 & 5 \\ 6 & 2 \end{bmatrix}^T = \begin{bmatrix} 8 & 5 & 6 \\ 4 & 5 & 2 \end{bmatrix}$

$C^T = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 3 & 1 & 2 & -6 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 \\ -2 & 1 \\ 1 & 2 \\ 3 & -6 \end{bmatrix}$

in the last example, we found vectors orth to $\boxed{\text{Col}A}$ by solving $A^T\vec{x} = \vec{0}$ i.e. finding $\boxed{\text{Nul}(A^T)}$

Col$A$ is always orth to Nul$(A^T)$

**Observation 4.1.9.** Using this notation, sometimes it's convenient to write the dot product as matrix multiplication

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1*1 + 2*2 + 3*3$$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} = 5 + 2 + 35 = 42$$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}^T \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 1 + 5 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 42 \end{bmatrix}$$

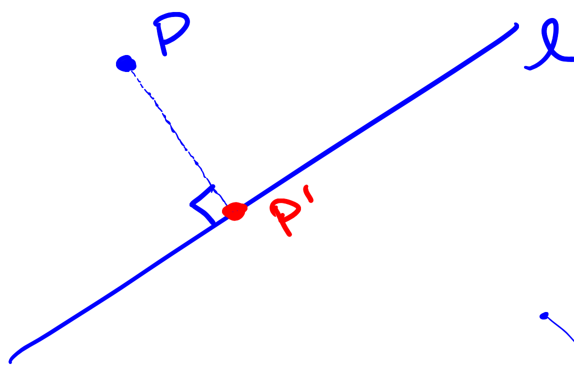
$$= 42$$

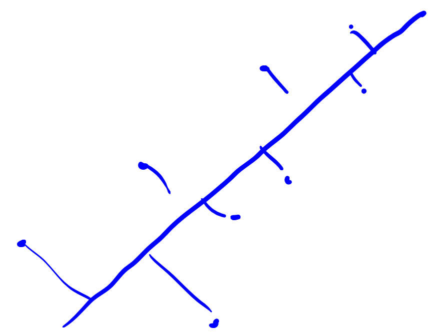
## 4.2 Projections

**4.2. Key Ideas**

- projection of a point onto a line
- projection of a point onto a space

**Question 4.2.1.** Given a point $P$ and a line $\ell$, how can we find the closest point on $\ell$ to $P$?



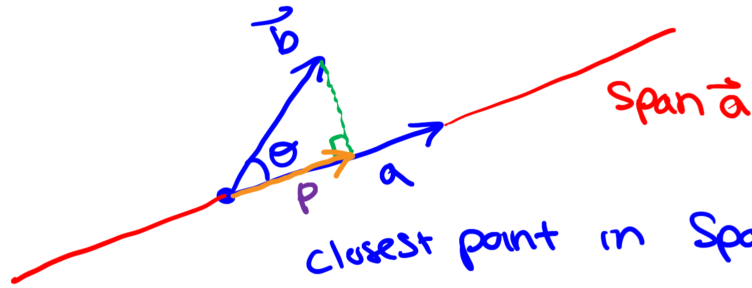



recall $u \cdot v = \|u\| \|v\| \cos\theta$



**Question 4.2.2.** Given two vectors **b** and **a** in $\mathbb{R}^n$, suppose we're interested in the closest point in Span(**a**) to **b**. How is this like our line and point example above?

line




closest point in Span $\vec{a}$ to $\vec{b}$

trig $\Rightarrow$

$\cos\theta = \frac{\|p\|}{\|b\|} \iff \|p\| = \|b\| \cos\theta$

$\iff \|a\| \|p\| = \|a\| \|b\| \cos\theta$

$\|a\| \|p\| = a \cdot b$

$\iff \|p\| = \frac{a \cdot b}{\|a\|}$

Know $\|\vec{p}\|$ want to know $\vec{p}$

want vector of length $\|\vec{p}\|$ in direction of $\vec{a}$

scalar

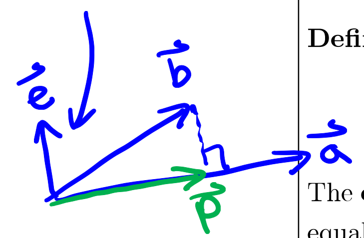
$$\vec{p} = \|\vec{p}\| \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a} \cdot \vec{b}}{\|a\|} \frac{\vec{a}}{\|a\|} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} * \vec{a}$$

$$\left(\sqrt{a_1^2 + a_2^2 + \cdots}\right)^2 = a_1^2 + a_2^2 + \cdots$$

$\|\vec{e}\| =$ distance from $\vec{b}$ to Span$(\vec{a})$

$\vec{e} = \vec{b} - \vec{p}$ error






**Definition 4.2.3.** The **orthogonal projection** of **b** onto the span of **a** is

$$\mathbf{p} = \text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a} = \boxed{\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}}\mathbf{b}.$$

rearrange

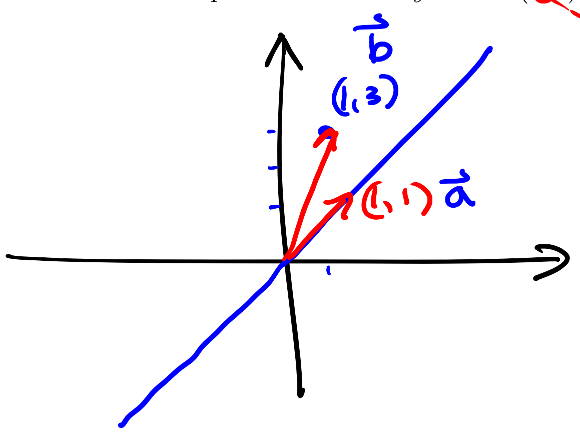
called projection matrix

The **error** is the vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$. It's the vector that's perpendicular to **a** and has length equal to the distance from **b** to **p**.

$$\text{The matrix } P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} \text{ is called the } \textbf{projection matrix.}$$

**Example 4.2.4.** Find the closest point on the line $y = x$ to $(1, 3)$. (7.5, 7.5)

(10, 5)




closest point to $\vec{b}$ on Span$(\vec{a})$ is $\vec{p}$ (projection)

$$\vec{p} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}} * \begin{bmatrix}1\\1\end{bmatrix} = \frac{4}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

$$\|\vec{e}\| = \|\vec{p} - \vec{b}\| = \left\| \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}1\\3\end{bmatrix} \right\| = \left\| \begin{matrix}1\\-1\end{matrix} \right\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

distance from (1,3) to y=x

Projection mtx

$a = \begin{bmatrix}1\\1\end{bmatrix}$ $\quad b = \begin{bmatrix}1\\3\end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}} = \frac{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}}{2} = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$$

e.g. $P\vec{b} = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

another ex

$\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}\begin{bmatrix}10\\5\end{bmatrix} = \begin{bmatrix}7.5\\7.5\end{bmatrix}$

hint: always be symmetric about

**Example 4.2.5.** Find the projection matrix $P$ onto the line through $\mathbf{a} = (1, 2, 2)$, and use this to find the point on the line closest to $(1, 1, 1)$.

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}1&2&2\end{bmatrix}}{\begin{bmatrix}1&2&2\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}} = \frac{\begin{bmatrix}1&2&2\\2&4&4\\2&4&4\end{bmatrix}}{9}$$

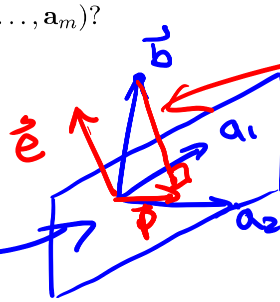
$$P = \begin{bmatrix}1/9&2/9&2/9\\2/9&4/9&4/9\\2/9&4/9&4/9\end{bmatrix}$$

$$P\vec{b} = \begin{bmatrix}1/9&2/9&2/9\\2/9&4/9&4/9\\2/9&4/9&4/9\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix} \leftarrow \text{closest point on } \operatorname{Span}\left(\begin{bmatrix}1\\2\\2\end{bmatrix}\right) \text{ to } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

how far away from line

$$\|\vec{e}\| = \|\vec{p} - \vec{b}\| = \left\|\begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}\right\| = \left\|\begin{bmatrix}-4/9\\1/9\\1/9\end{bmatrix}\right\| = \frac{\sqrt{2}}{3}\ ?$$

**Question 4.2.6.** Given a collection of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$, and $\mathbf{b}$ in $\mathbb{R}^n$, how could we find the closest point to $\mathbf{b}$ in $\text{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$?




Still want something perp

$\vec e = \vec p - \vec b \Rightarrow$ perpendicular

WTF $\hat x$ such that

$$\boxed{\vec p = A\hat x} = \vec a_1 \hat x_1 + \vec a_2 \hat x_2 + \cdots + \vec a_n \hat x_n$$

in col(A) projecting onto column space of $A = [\vec a_1\ \vec a_2 \cdots \vec a_n]$

$\vec e = \vec b - \vec p$ is orthogonal to Col(A)

so $\vec a_i \cdot \vec e = a_i^T \vec e = 0$ for all $\vec a_i$

$$a_1^T \vec e = a_1^T(b - \vec p) = a_1^T(\vec b - A\vec x) = 0$$
$$a_2^T(\vec b - A\vec x) = 0$$
$$a_3^T(\vec b - A\vec x) = 0$$
$$\vdots$$
$$a_n^T(\vec b - A\vec x) = 0$$

i.e. we have n-equations can rewrite as

$$\begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} (\vec b - A\vec x) = \vec 0$$

$$A^T(\vec b - A\vec x) = \vec 0$$

$$\boxed{A^T A\vec x = A^T \vec b}$$

if $\vec p = A\hat x$ is closest point in Col(A) to $\vec b$, then $\boxed{\hat x \text{ is a soln to } A^T A\hat x = A^T\vec b}$

**Definition 4.2.7.** The combination $\mathbf{p} = \widehat{x}_1\mathbf{a}_1 + \cdots + \widehat{x}_m\mathbf{a}_m = A\widehat{\mathbf{x}}$ closest to $\mathbf{b}$ comes from

know

$$A^T A\widehat{\mathbf{x}} = A^T\mathbf{b}.$$

know

find

The **projection** of $\mathbf{b}$ onto the subspace spanned by the $\mathbf{a}_i$ is

$$\mathbf{p} = A\widehat{\mathbf{x}} = A(A^T A)^{-1}A^T\mathbf{b}.$$

Again, $P = A(A^T A)^{-1}A^T$ is called the **projection matrix**. The **error** is the vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Its length is equal to the distance from $\mathbf{b}$ to $\mathbf{p}$.

**Example 4.2.8.** Find the closest point to $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ in the plane spanned by $\underset{a_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ and $\underset{a_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

1. find $A^T$, $A^TA$, and $A^T\vec{b}$
2. solve $A^TA\hat{x} = A^T\vec{b}$
3. $\vec{p}$ will be product $A\hat{x}$

① $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$$\boxed{A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

② Solve $A^TA\hat{x} = A^T\vec{b}$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}\hat{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \sim \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right]$$

so $\hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

③ Find $\vec{p} = A\hat{x}$

$$\vec{p} = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$