## 20.4 Solving Trigonometric Equations

**Goals**

- using properties of sine, cosine and tangent to solve for an unknown variable
- using inverse trigonometric functions to solve for an unknown variable

**Example 20.4.1.** Solve for $x$.

$$4\cos x + 1 = -1$$

want the actual angle(s)

$$4\cos x + 1 = -1$$
$$\iff 4\cos x = -2$$
$$\iff \cos x = -\frac{1}{2}$$

$x = \frac{2\pi}{3}, \frac{4\pi}{3} \leftarrow$ two of them

$x = \frac{2\pi}{3} + 2n\pi$ or $\frac{4\pi}{3} + 2n\pi$

by unit circle

**Definition 20.4.2.** A trigonometric equation is an equation in which the variable to be solved for is inside a trigonometric function.

**Strategy 20.4.3.** If we can solve the equation to $\sin u = k$, $\cos u = k$ or $\tan u = k$, then we can use inverse trig functions to help solve trigonometric equations!

$\sin\theta(2\sin\theta - 1) = 1$

**Example 20.4.4.** Solve for $\theta$.

$$2\sin^2(\theta) - \sin(\theta) = 1$$

$$2\sin^2\theta - \sin\theta - 1 = 0$$

if $u = \sin\theta$

then $\underbrace{2u^2 - u - 1} = 0$

$\iff (2u+1)(u-1) = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

either $2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6} + 2n\pi$ or $\frac{11\pi}{6} + 2n\pi$

or $\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \frac{\pi}{2} + 2\pi n$

so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{7\pi}{6} + 2n\pi$ or $\frac{11\pi}{6} + 2n\pi$

**Strategy 20.4.5.** Don't forget the Pythagorean identity!

$$\sin^2(x) + \cos^2(x) = 1$$

**Example 20.4.6.** Solve for $\theta$.

$$2\underbrace{\cos^2(\theta)}_{1-\sin^2\theta} + \sin(\theta) = 1$$

$$\iff 2(1-\sin^2\theta) + \sin\theta = 1$$

$$2 - 2\sin^2\theta + \sin\theta - 1 = 0$$

$$-2\sin^2\theta + \sin\theta + 1 = 0$$

same as last

$$\boxed{2\sin^2\theta - \sin\theta - 1 = 0}$$

$\theta = \frac{\pi}{2} + 2n\pi$, or

$\frac{7\pi}{6} + 2n\pi$, or

$\frac{11\pi}{6} + 2n\pi$

$x^2 = 1$

$x = -1$ or $1$

**Example 20.4.7.** Find all $x \in [-2\pi, 2\pi]$ such that

$$\tan x \sec x = \tan x.$$

factor / ~~use identities~~ to solve

$\tan u = k$
$\sec u' = k'$

$\tan x \sec x - \tan x = 0$

$\tan x(\sec x - 1) = 0$

$\tan x = 0$
$x = n\pi$

~OR~

$\sec x = 1$

$\frac{1}{\cos x} = 1$

$\iff \cos x = 1$

$x = 2n\pi$

or $\tan x = 0$ when $x = n\pi$ and then divide

$\frac{\tan x \sec x}{\tan x} = \frac{\tan x}{\tan x}$

$\sec x = 1$

$\pi \quad 2\pi$

$n\pi$ is sometimes $2k\pi$ for some $k$ (when $n$ even)

So $\tan x \sec x = \tan x$

$\iff x = n\pi$

$x = \{-2\pi, -\pi, 0, \pi, 2\pi\}$

**Example 20.4.8.** Find all $x \in [0, 2\pi]$ such that

$$\cos(2x) = \frac{\sqrt{3}}{2}.$$

$$\cos(u) = \frac{\sqrt{3}}{2} \iff u = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$

$u = 2x$

$$2x = \pi/6 \iff x = \frac{\pi}{12}$$

$$2x = \frac{11\pi}{6} \iff x = \frac{11\pi}{12}$$

$$2x = \frac{13\pi}{6} \iff x = \frac{13\pi}{12}$$

$$2x = \frac{23\pi}{6} \iff x = \frac{23\pi}{12}$$

is in interval

## 20.6 Trigonometric Identities

**Goals**

- $\sin(A+B) \neq \sin A + \sin B$
- sum formulas
- using the sum formulas to derive other trig identities
- using trig identities to solve trig equations

**Think, Pair, Share 20.6.1.** Is $\sin(A+B) = \sin(A) + \sin(B)$? Why or why not?

try $A = \pi/2$ $B = \pi/2$

$\sin(A+B) = \sin(\pi/2 + \pi/2) = \sin(\pi) = 0$

$\sin(A) + \sin(B) = \sin(\pi/2) + \sin(\pi/2) = 2$

$\neq$

**Proposition 20.6.2** (Addition Formulas)**.** *The correct identities are*

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

**Example 20.6.3.** From the addition formulas, verify the double-angle formulas

$A = x \quad B = x$

(a) $\sin 2x = 2\sin x \cos x$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\sin(x+x) = \sin x \cos x + \sin x \cos x$

$= 2\sin x \cos x$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

(b) $\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = \cos(x+x) = \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

(c) $\cos 2x = 2\cos^2 x - 1$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2(x) - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

(d) $\cos 2x = 1 - 2\sin^2 x$

check this at home

**Example 20.6.4.** From the last two formulas for $\cos 2x$, derive the power-reducing formulas

① $\cos 2x = 2\cos^2 x - 1$
② $\cos 2x = 1 - 2\sin^2 x$

(e) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

solve ① for $\cos^2 x$

(f) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

solve eqn ② for $\sin^2 x$

**Example 20.6.5.** From the formula $\sin^2 x + \cos^2 x = 1$, derive the following

(e) $\tan^2 x + 1 = \sec^2 x$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

divide by $\sin^2 x$

(f) $1 + \cot^2 x = \csc^2 x$

### 20.6.1 Extra Examples

$x^2 = 1 \quad x = \pm 1$

**Example 20.6.6.** Solve.

$\cos 2x = 2\cos^2 x - 1$

(a) $\cos^2 x - \cos 2x = 0$ for $x \in (-\infty, \infty)$

$\cos^2 x - (2\cos^2 x - 1) = -\cos^2 x + 1 = 0$

$\Leftrightarrow \boxed{\cos^2 x = 1}$

or $\cos 2x = \cos^2 x - \sin^2 x$

$\cancel{\cos^2 x} - (\cancel{\cos^2 x} - \sin^2 x) = 0$

$\boxed{\sin^2 x = 0}$

$\Downarrow$

$\cos^2 x = 1$

$\Leftrightarrow \cos x = \pm 1$

$\cos x = 1$

at $x = 0, 2\pi, 4\pi$

$\cos x = -1$

at $\pi, 3\pi, 5\pi$

so

$\cos^2 x - \cos 2x = 0$

when $x = n\pi$

(b) $2 \sin x \cos x = \frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$

$\Leftrightarrow \sin 2x = \frac{\sqrt{3}}{2}$

(c) $\sin^2 x + \cos 2x = 0$ for $x \in [0, 2\pi]$

sub something using an identity

(d) $-\cos^2 x + \frac{1}{2} \sin x + 1 = 0$ for $x \in [0, 2\pi]$