5.4 # 1, 3, 6, 7, 10, 15, 16, 23, 25

1.) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$ be bases for V and W respectively. Let $T: V \to W$ be a linear transformation s.t $T(\vec{b}_1) = 3\vec{d}_1 - 5\vec{d}_2$, $T(\vec{b}_2) = -\vec{d}_1 + 6\vec{d}_2$ and $T(\vec{b}_3) = 4\vec{d}_2$. Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$.

$$\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$$

3.) Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$, let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a basis for a vector space V, and let $T: \mathbb{R}^3 \to V$ be a linear transformation s.t $T(x_1, x_2, x_3) = (2x_3 - x_2)\vec{b}_1 - (2x_2)\vec{b}_2 + (x_1 + 3x_3)\vec{b}_3$

a.) Compute $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$

$T(\vec{e}_1) = T(1,0,0) = 0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3$

$T(\vec{e}_2) = T(0,1,0) = \vec{b}_1 - 2\vec{b}_2 + 0\vec{b}_3$

$T(\vec{e}_3) = T(0,0,1) = 2\vec{b}_1 + 0\vec{b}_2 + 3\vec{b}_3$

b.) Compute $[T(\vec{e}_1)]_{\mathcal{B}}, [T(\vec{e}_2)]_{\mathcal{B}}, [T(\vec{e}_3)]_{\mathcal{B}}$

$[T(\vec{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [T(\vec{e}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, [T(\vec{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

c.) Compute the matrix for T relative to $\mathcal{E}$ an $\mathcal{B}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

6.) Let $T: \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\vec{p}(t)$ into the polynomial $\vec{p}(t) + 2t^2\vec{p}(t)$.

a.) Find the image of $\vec{p}(t) = 3 - 2t + t^2$. $3 - 2t + t^2 + 2t^2(3 - 2t + t^2)$

$= 2t^4 - 4t^3 + 7t^2 - 2t + 3$

b.) Show that T is a linear transformation.

Let $\vec{p}$ & $\vec{q}$ be polynomials in $\mathbb{P}_2$ and c be scalar.

$cT(\vec{p}) = c(\vec{p} + 2t^2\vec{p}) = c\vec{p} + 2ct^2\vec{p}$ and $T(c\vec{p}) = c\vec{p} + 2t^2(c\vec{p}) = c\vec{p} + 2ct^2\vec{p}$ ✓

$T(\vec{p}+\vec{q}) = (\vec{p}+\vec{q}) + 2t^2(\vec{p}+\vec{q})$ and $T(\vec{p}) + T(\vec{q}) = \vec{p} + 2t^2\vec{p} + \vec{q} + 2t^2\vec{q}$ ✓

$= \vec{p} + \vec{q} + 2t^2\vec{p} + 2t^2\vec{q}$

c.) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$. $\mathcal{C}$

$T(1) = 1 + 2t^2$ $T(t) = t + 2t^3$ $T(t^2) = t^2 + 2t^4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

7.) Assume the mapping $T: \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$ is linear. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

$T(1) = 3 + 5t \quad T(t) = -2t + 4t^2 \quad T(t^2) = t^2$

$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \quad T(t^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

10.) Define $T: \mathbb{P}_3 \to \mathbb{R}^4$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(-2) \\ \vec{p}(3) \\ \vec{p}(1) \\ \vec{p}(0) \end{bmatrix}$.

a.) Show that T is a lin. trans.

b.) Find the matrix of T relative to the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$.

a.) $cT(\vec{p}) = \begin{bmatrix} c\vec{p}(-2) \\ c\vec{p}(3) \\ c\vec{p}(1) \\ c\vec{p}(0) \end{bmatrix} = T(c\vec{p})$ ✓

$T(\vec{p} + \vec{q}) = \begin{bmatrix} (\vec{p}+\vec{q})(-2) \\ (\vec{p}+\vec{q})(3) \\ (\vec{p}+\vec{q})(1) \\ (\vec{p}+\vec{q})(0) \end{bmatrix} = \begin{bmatrix} \vec{p}(-2) + \vec{q}(-2) \\ \vec{p}(3) + \vec{q}(3) \\ \vec{p}(1) + \vec{q}(1) \\ \vec{p}(0) + \vec{q}(0) \end{bmatrix}$ ✓

$T(\vec{p}) + T(\vec{q}) = \begin{bmatrix} \vec{p}(-2) \\ \vec{p}(3) \\ \vec{p}(1) \\ \vec{p}(0) \end{bmatrix} + \begin{bmatrix} \vec{q}(-2) \\ \vec{q}(3) \\ \vec{q}(1) \\ \vec{q}(0) \end{bmatrix} = \begin{bmatrix} \vec{p}(-2) + \vec{q}(-2) \\ \vec{p}(3) + \vec{q}(3) \\ \vec{p}(1) + \vec{q}(1) \\ \vec{p}(0) + \vec{q}(0) \end{bmatrix}$

b.) $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, T(t) = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, T(t^2) = \begin{bmatrix} 4 \\ 9 \\ 1 \\ 0 \end{bmatrix}, T(t^3) = \begin{bmatrix} -8 \\ 27 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 3 & 9 & 27 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5.4 Continued

15.) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. Diagonalize A. The characteristic polynomial is $(1-\lambda)(-4-\lambda) - 6 =$
$= \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2)$ So $-5, 2$ are eigenvalues

$\lambda = -5$
$(A+5I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -\frac{1}{3}x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ Use $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$ as a basis for the eigenspace.

$\lambda = 2$
$(A-2I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 2x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$ $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ Thm 8

16.) $A = \begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}$ $(4-\lambda)(5-\lambda) - 2 = \lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6)$, $\lambda = 3, 6$

$\lambda = 3$
$(A-3I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 2x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathcal{B} = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\lambda = 6$
$(A-6I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

23.) If $B = P^{-1}AP$ and $\vec{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\vec{x}$ is an eigenvector of B corresponding to λ.

Verify. We want to show $B(P^{-1}\vec{x}) = \lambda(P^{-1}\vec{x})$

$BP^{-1}\vec{x} = (P^{-1}A\underbrace{P)P^{-1}}_{I}\vec{x} = P^{-1}A\vec{x}$ since $\vec{x}$ is an eigenvector of A, $A\vec{x} = \lambda\vec{x}$
$= P^{-1}\lambda\vec{x}$
$= \lambda P^{-1}\vec{x}$

25.) The trace of a square matrix A is the sum is the sum of the diagonal entries of A and is denoted by $\operatorname{tr} A$.
It can be verified that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two $n \times n$ matrices F and G. Show that if A, and B are similar then $\operatorname{tr} A = \operatorname{tr} B$.

A is similar to B if there exists an invertible matrix P s.t. $A = PBP^{-1}$.

$\operatorname{tr} A = \operatorname{tr}(PBP^{-1}) = \operatorname{tr}(P^{-1}(PB)) = \operatorname{tr}(IB) = \operatorname{tr} B$.