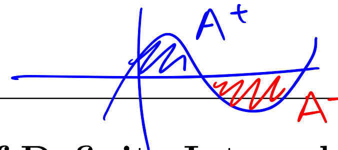
$$\int_a^b f(x)\,dx = \text{signed area "under" } f = A^+ - A^-$$



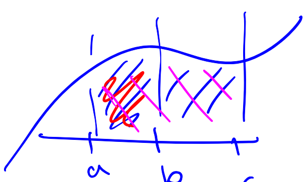
22.4 Properties of Definite Integrals

Goals

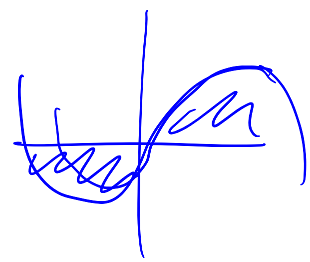
- constat factor, dominance, endpoint reversal, additive integrand, splitting interval, and symmetry properties of the definite integral

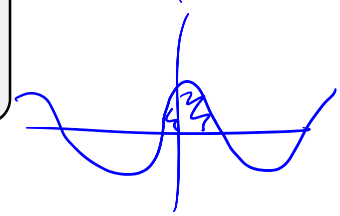
Theorem 22.4.1. We have the following properties of definite integrals.

1. $\int_a^b kf(x)\;dx = k\int_a^b f(x)\;dx$ (constant factor property)
2. If $f \le g$ on the interval $[a,b]$, then $\int_a^b f(x)\;dx \le \int_a^b g(x)\;dx$ (dominance property)
3. $\int_a^b (f(x)+g(x))\;dx = \int_a^b f(x)\;dx + \int_a^b g(x)\;dx$ (additive integrand property)
4. If $a \le b \le c$, then $\int_a^c f(x)\;dx = \int_a^b f(x)\;dx + \int_b^c f(x)\;dx$ (splitting interval property)
5. $\int_{-a}^a f(x)\;dx = 0$ if f is odd; $\int_{-a}^a f(x)\;dx = 2\int_0^a f(x)\;dx$ if f is even. (symmetry property)
6. $\int_b^a f(x)\;dx = -\int_a^b f(x)\;dx$ (endpoint reversal property)



$$\int_a^c = \int_a^b + \int_b^c$$





$$\int_5^2 (x^2+1)\,dx = -\int_2^5 (x^2+1)\,dx$$

Question 22.4.2. Can we use area interpretations to justify some of the integral properties?

Example 22.4.3. Write a single integral of the form $\int_a^b f(x)\,dx$ if

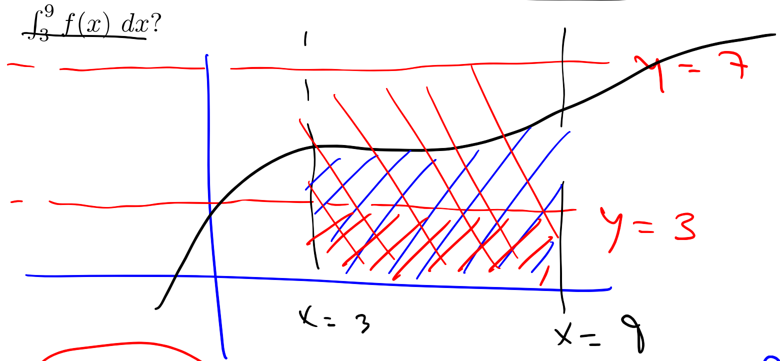
$$\int_a^b f(x)\,dx = \underline{\int_{-2}^{2} f(x)\,dx} + \underline{\int_{2}^{5} f(x)\,dx} - \underline{\int_{-2}^{-1} f(x)\,dx}$$

$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$$

$$= \int_{-2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

$$= \cancel{\int_{-2}^{-1} f\,dx} + \int_{-1}^{5} f\,dx - \cancel{\int_{-2}^{-1} f\,dx} = \int_{-1}^{5} f(x)\,dx$$

Example 22.4.4. Suppose f is a continuous function such that $6 \leq f(x) \leq 7$ on the interval $[3, 9]$. What are the largest and smallest possible values for $\int_3^9 f(x)\, dx$?

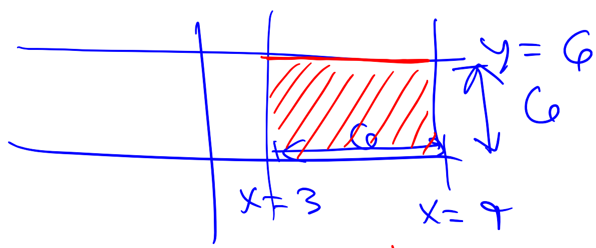


$$\int_3^9 f(x)\, dx$$

$$\int_3^9 6\, dx \leq \int_3^9 f(x)\, dx \leq \int_3^9 7\, dx$$

$6 * (9 - 3)$

$\boxed{6 * 9 - 6 * 3}$



$A = 36$

$b - a = 9 - 3 = 6$

$$36 \leq \int_3^9 f(x)\, dx \leq 42$$

22.4.1 Extra examples

Example 22.4.5. Use one of the properties of the definite integral to evaluate

$$\int_{-1}^{1} \boxed{x\sqrt{4-x^2}}\,dx$$

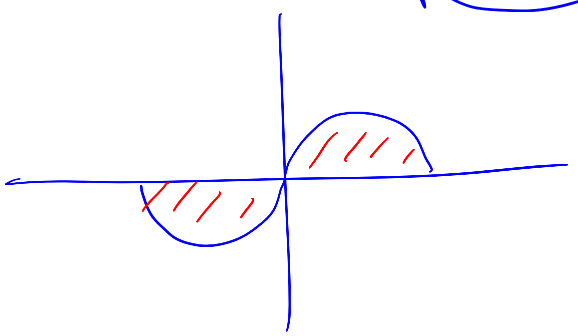
is this odd / even

$$f(x) = x\sqrt{4-x^2}$$

$$f(-x) = (-x)\sqrt{4-(-x)^2}$$

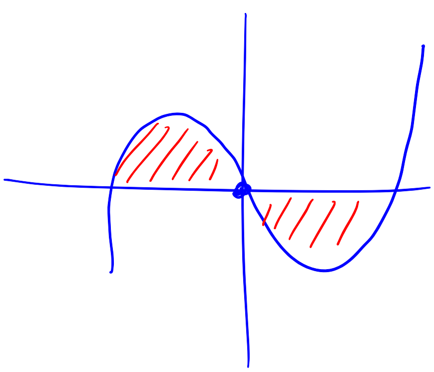
$$= -x\sqrt{4-x^2}$$

$$= -f(x)$$



odd

e.g. x^3, x^5, x^7

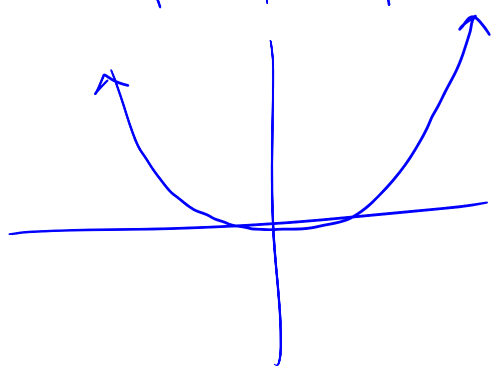


$f(-x) = -f(x)$

$\Rightarrow$ odd

even

$x^2, x^4, x^6, \cos x$



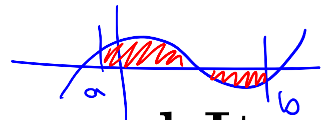
$f(-x) = f(x)$

$\Rightarrow$ even

Continuing

$$\int_{-1}^{1} x\sqrt{1-x^2} = 0$$

$\int_a^b f(x)\,dx$ = signed area b/w f and x-axis from a to b



Chapter 23

The Area Function and Its Characteristics

23.1 An Introduction to the Area Function

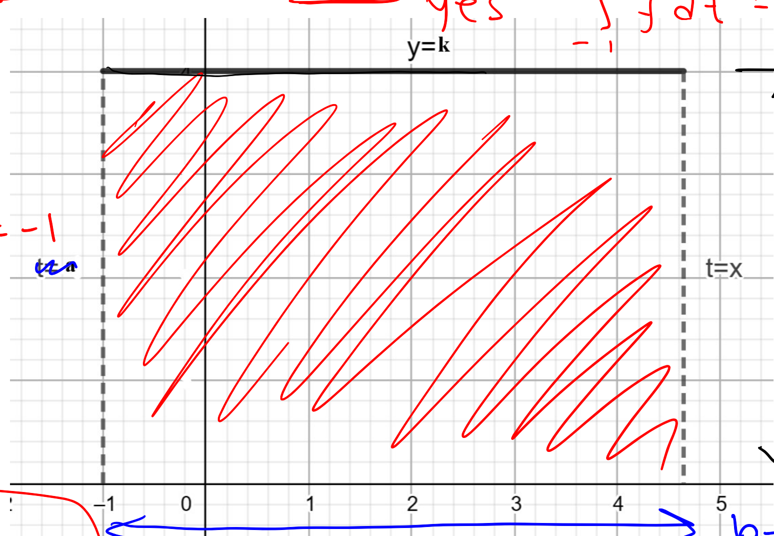
Goals

- define area function

Question 23.1.1. Suppose f is the constant function $f(t) = k$ for some constant k. ~~The case when $k = 4$ is shown below.~~ Is $\int_{-1}^{x} f(t)\ dt = \int_{-1}^{x} k\ dt$ itself a function? Can it be given by a formula?

yes

yes $\int_{-1}^{x} f\,dt = kx + k$



t = −1

k

b − a = x − (−1) = $\boxed{x+1}$

is $\int_{a=-1}^{b=x} f(t)\,dt$ a function of x?

= area under $f(t) = k$ from $t = -1$ to $t = x$

= $lw = k * (x+1) = kx + k = \int_{-1}^{x} k\,dt$

$\int_a^x k\,dt = (x-a)k = \boxed{kx} - ka$

Definition 23.1.2. For a continuous function f and constant a in the domain of f, we define ${}_aA_f(x)$ to be

prescript subscript the function I'm integrating

$$ {}_aA_f(x) = \int_a^x f(t)\, dt. $$

lower limit of int

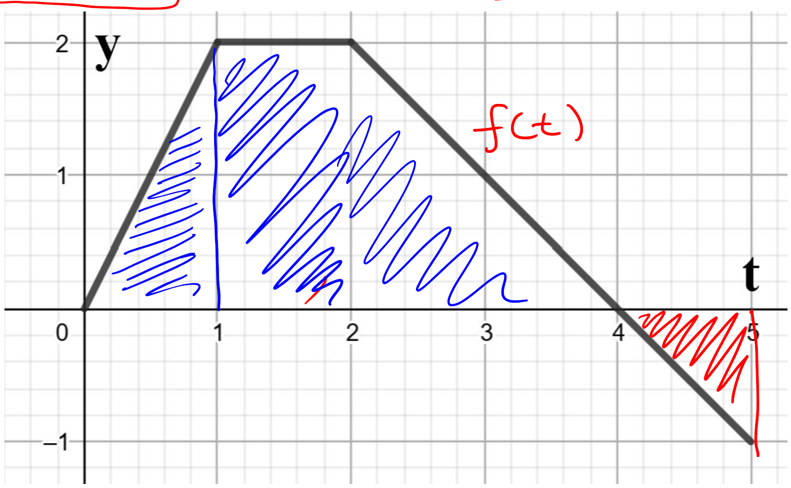
This function represents the signed area under the function f, so we call it **the Area Function**.

when we're lazy, and obv what f and a are

$$A(x) = \int_a^x f(t)\, dt$$

23.2 Characteristics of the Area Function

On your own 23.2.1. If $f(t)$ is a function whose graph is shown below, and ${}_0A_f(x) = \int_0^x f(t)\,dt$, find $A(0)$, $A(1)$, $A(2)$ and $A(4)$ and $A(5)$.



$A(1) = {}_0A_f(1) = 1$

$A(2) = 3$

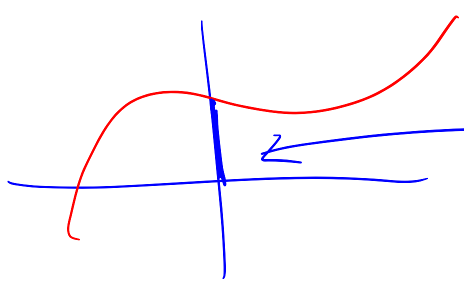
$A(3) = 4.5$

$A(4) = 5$

$A(5) = A^+ - A^- = 5 - 0.5 = 4.5$

$A(0) = {}_0A_f(0) =$ area under f from $x=0$ to $x=0$

$= 0$



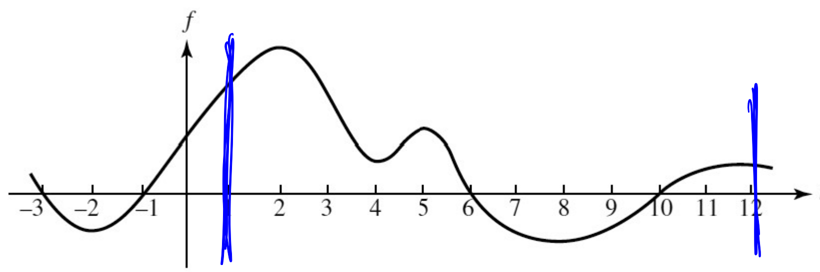
width is zero

in all cases no matter f or a

${}_aA_f(a) = \int_a^a f(t)\,dt = 0$

definition

Example 23.2.2. Let ${}_1A_f(x) = \int_1^x f(t)\, dt$, where the graph of f is drawn below and the domain of A_f is restricted to $1 \le x \le 12$.

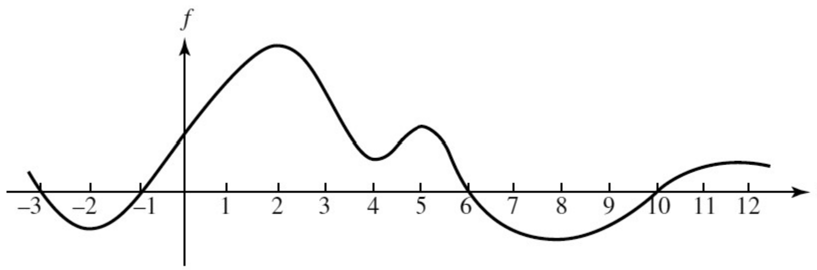


(a) On what interval(s) is the function ${}_1A_f(x)$ increasing? What characteristic of f ensures that f ensures ${}_1A_f(x)$ is increasing?

as we go along, adding area until six, and subtracting from $[6, 10)$ and adding again from $[10, 12]$

$A(x)$ = area, increasing from $[0, 6]$ then again from $[10, 12]$

- if $f > 0 \Rightarrow A(x)$ increasing
- if $f < 0 \Rightarrow A(x)$ is decrease



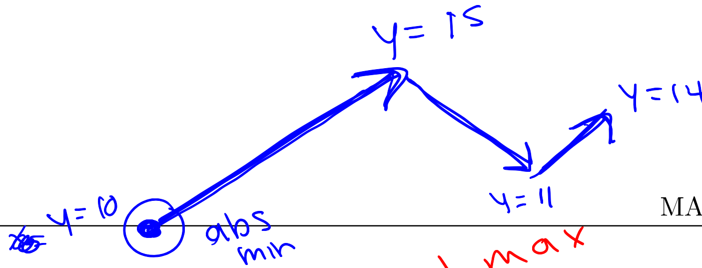
(b) On what interval(s) is the function ${}_1A_f(x)$ decreasing? What characteristic of f ensures that f ensures ${}_1A_f(x)$ is decreasing?

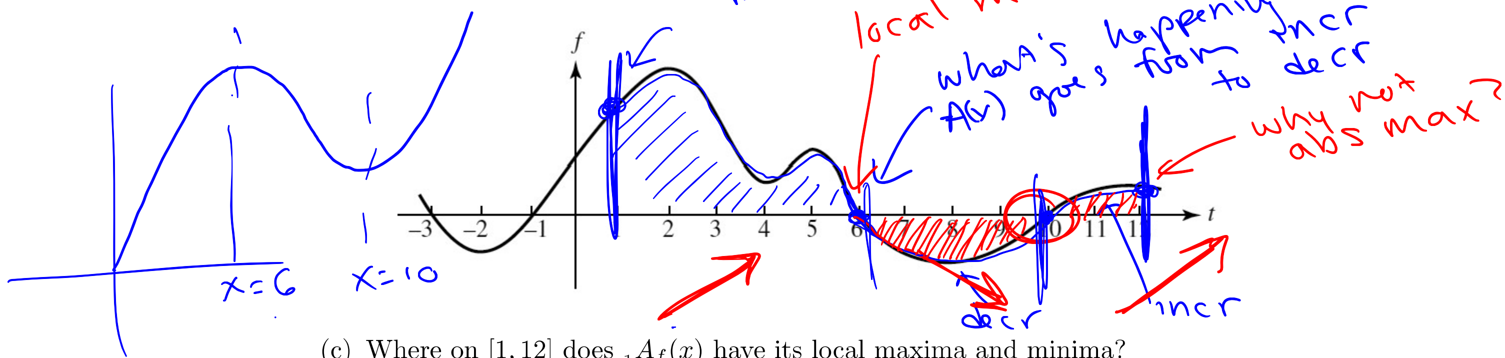
see above

- $f > 0 \implies A(x)$ is incr
- $f < 0 \implies A(x)$ is decr
- f incr $\implies A(x)$ is CU

  $f' > 0 \implies A(x)$ is CU
- $f' < 0 \implies A(x)$ is CD

seems like f is somehow the derivative of $A(x)$





(c) Where on $[1, 12]$ does ${}_1A_f(x)$ have its local maxima and minima? Where does ${}_1A_f(x)$ have its *absolute* maxima and minima?

local max at $x = 6$ b/c switch from incr to decr

local min at $x = 10$

potential abs maxes are endpts or local min/max

so could have a maximum $x = 6$, $x = 12$

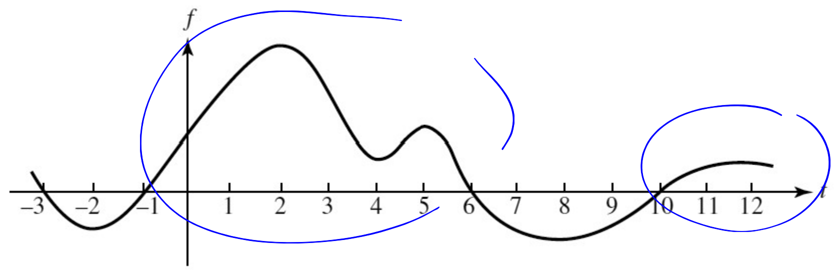
CLAIM $x = 12$ is not abs max why?

the area we subtract from 6 to 10 is less than the area we add back

$\Rightarrow$ at $x = 12$ $A(x)$ is smaller than at $x = 6$

abs min at $x = 1$

b/c we incr a lot, then decr a little to get to local min at $x = 10$



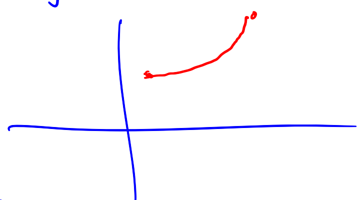
(d) On what interval(s) is the function ${}_1A_f(x)$ concave up? Concave down? What characteristic of f ensures that ${}_1A_f(x)$ is concave up or down?

CASE 1 f is positive

if f is incr,

$f > 0 \Rightarrow A$ incr

f incr $\Rightarrow A$ incr faster and faster

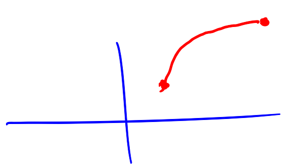


$\Rightarrow$ concave up

if f decr

$f > 0 \Rightarrow A$ incr

f decr $\Rightarrow A$ incr at slower slower rate



$\Rightarrow$ concave down

CASE 2 (analogously)

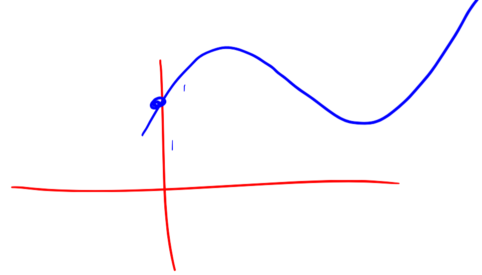
$f < 0$ and incr $\Rightarrow$ concave up

$f < 0$ and decr $\Rightarrow$ concave down

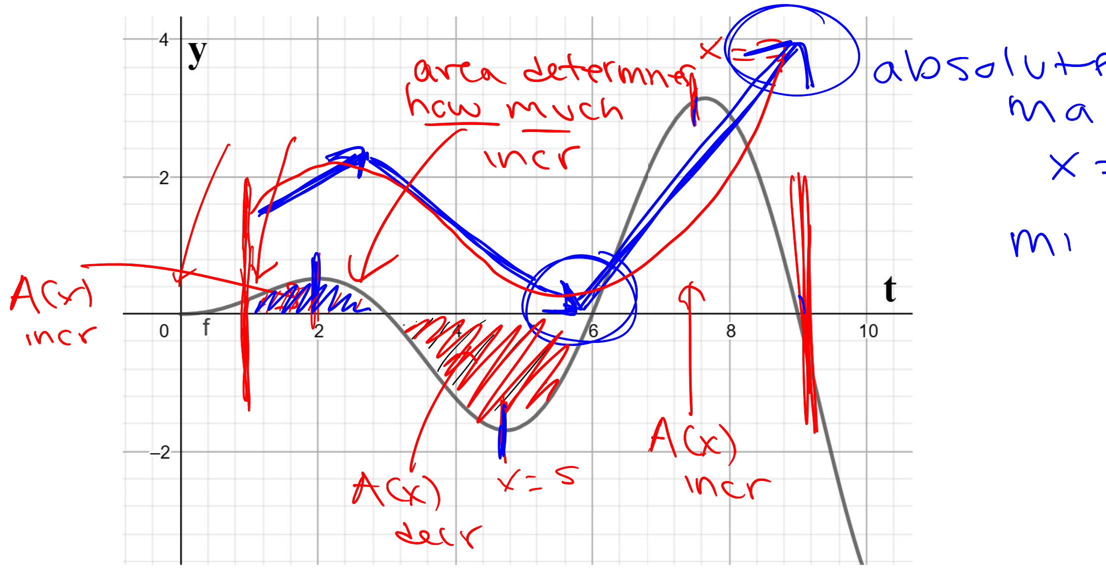
in both cases

f incr $\Rightarrow A(x)$ concave up

f decr $\Rightarrow A(x)$ concave down



Groups 23.2.3. Let ${}_0A_f(x) = \int_0^x f(t)\, dt$ where the graph of f is drawn below. (Note: the roots of f are $t = 0, 3, 6, 9$)



(a) On what interval(s) is the function ${}_1A_f(x)$ increasing? $[0,3]$, $[6,9]$

(b) On what interval(s) is the function ${}_1A_f(x)$ decreasing? $[3,6]$

(c) Where on $[0, 9]$ does ${}_1A_f(x)$ have its local maxima and minima?

(d) Where on $[0, 9]$ does ${}_1A_f(x)$ have its *absolute* maxima and minima?

(e) On what interval(s) is the function ${}_1A_f(x)$ concave up? Concave down?

c) $A(x)$ has local min at $x = 6$
and local max at $x = 9$

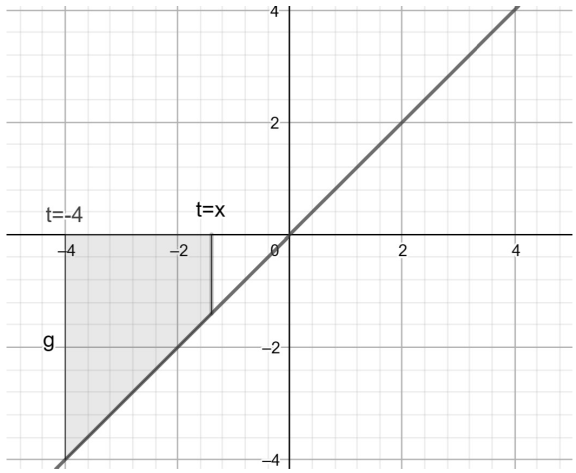
d)

e) concavity: CU on $[0,2]$, $[5,7]$
CD on $[2,5]$, $[7,9]$

$f(t) = t$

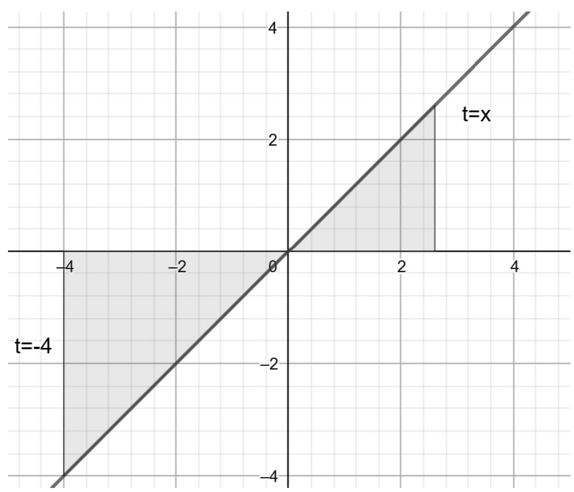
. **Groups 23.2.4.** Let $\cancel{f(t) = x}$ and $A(x) = {}_{-4}A_f(x) = \int_{-4}^{x} f(t)\ dt$.

(a) Find a formula for $A(x)$ when the domain of $A(x)$ is $[-4, 0]$



same for Tues

(b) Find a formula for $A(x)$ when the domain of $A(x)$ is $x \geq 0$.
(Note: you should get the same formula as in part (a)!)



(c) What is a general formula for ${}_{-4}A_f(x) = \int_{-4}^{x} f(t)\ dt$

(d) What is the formula for $\int_{-2}^{x} f(t)\ dt$? How is it related to $\int_{-4}^{x} f(t)\ dt$

.

Observation 23.2.5. Let a and k be constants. If $f(t) = k$, then

$$ {}_aA_f(x) = \int_a^x k\ dt = kx + C, \text{ for some constant } C, $$

If $f(t) = t$, then

$$ {}_aA_f(x) = \int_a^x t\ dt = \frac{x^2}{2} + C, \text{ where } C = ka. $$

$\frac{d}{dx} \frac{x^2}{2} = x$

$f(t) = t$
$f(x) = x$

Question 23.2.6. If $f(t) = t^2$, can you predict the solution to

$$ {}_aA_f(x) = \int_a^x t^2 \; dt? $$

Observation 23.2.7. We summarize the following properties of the area function ${}_aA_f(x)$

- area functions f (i.e. for different a) are all vertical translates
- $f > 0$ means that A_f is increasing
- $f < 0$ means that A_f is decreasing
- f increasing means that A_f is concave up
- f decreasing means that A_f is concave down