NOTE: These slides contain *both* Section 1.4 and 1.5.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

McDonald Fall 2018, MATH 2210Q 1.4 & 1.5 Slides

1.4 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 1, 3, 13, 17, 19, 22, 23, 25
- Extra Practice: 4, 7, 9, 11, 31

The definition below lets us rephrase some of the concepts from Section 1.3 by viewing linear combinations of vectors as the product of a matrix and a vector

Definition 1.5.1. If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$,** denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Remark 1.5.2. $A\mathbf{x}$ is only defined the number of columns of A equals the number of entries in $\mathbf{x}$.

Example 1.5.3. Find the following products:

(a) $\begin{array}{c} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \\ a_1 \quad a_2 \quad a_3 \end{bmatrix}$ $\begin{array}{c} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4 + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(b) $\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 9 & 6 \end{bmatrix}}_{\text{3 colms}} \left.\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\text{2 rows}$

$\Longrightarrow$ undefined

Example 1.5.4. Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} x_3 = \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

NOTICE:

$$\begin{bmatrix} 2 & 3 & 4 \\ & \ast & \\ & \ast & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \\ \end{bmatrix}$$

just a sum of products (sometimes called "dot product")

$$\begin{bmatrix} & \ast & \\ -1 & 5 & -3 \\ & \ast & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ast \\ -x_1 + 5x_2 - 3x_3 \\ \ast \end{bmatrix}$$

Procedure 1.5.5 (Row Vector Rule)**.** If $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example 1.5.6. Compute

(a) $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + (-3)7 \\ 8 \cdot 4 + 0 \cdot 7 \\ -5 \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$

(b) $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{called "identity matrix"}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Example 1.5.7. Write the system below as $A\mathbf{x} = \mathbf{b}$ for some A and $\mathbf{b}$.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

$$\begin{pmatrix} x_1 + 2x_2 - 3x_3 \\ -5x_2 + 3x_3 \end{pmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ -5x_2 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -5 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 3 \end{bmatrix} x_3$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Definition 1.5.8. The equation $A\mathbf{x} = \mathbf{b}$ is called a **matrix equation.**

three diff ways to think about a system

Theorem 1.5.9. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then*

$$A\mathbf{x} = \mathbf{b} \quad (\text{with } \vec{x} \in \mathbb{R}^n)$$

we'll typically assume this

has the same solutions as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n$$

which has the same solutions as the system of linear equations with augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Corollary 1.5.10. *The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

Example 1.5.11. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $A\mathbf{x} = \mathbf{b}$ consistent for all b_1, b_2, b_3?

$A\vec{x} = \vec{b}$ consist $\iff$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \text{ const}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 0 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{bmatrix}$$

only constent if $0 = b_1 - \frac{1}{2}b_2 + b_3$

eqn of a plane

Theorem 1.5.12. *Let A be an $m \times n$ matrix. Then the following statements are either all true, or all false.*

(a) *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
(b) *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*
(c) *The columns of A span $\mathbb{R}^m$.*
(d) *A has a pivot position in every row.*

careful, this is talking about the coefficient matrix

Example 1.5.13. If $A = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 11 \\ 3 \end{bmatrix}$, for what $\mathbf{x}$ is $A\mathbf{x} = \mathbf{b}$ consistent?

$$Ax=b \sim \begin{bmatrix} 2 & 0 & -2 & 2 \\ 2 & 3 & 4 & 11 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -2 & 2 \\ 0 & 3 & 6 & 9 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & -2 & 2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

consistent $\Longleftrightarrow$ $\begin{matrix} x_1 & & -x_3 = 1 \\ & x_2 & +2x_3 = 3 \end{matrix}$ $\Longleftrightarrow$ $\begin{cases} x_1 = 1 + x_3 \\ x_2 = 3 - 2x_3 \end{cases}$

$$\Longleftrightarrow \vec{x} = \begin{bmatrix} 1+x_3 \\ 3-2x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} x_3 \\ -2x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

called parametric vector form

We end this section with some important properties of $A\mathbf{x}$, which we will use throughout the course.

Theorem 1.5.14. *If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar:*

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
(b) $A(c\mathbf{u}) = c(A\mathbf{u})$.

1.6 Solutions Sets of Linear Systems

1.5 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 5, 11, 15, 19, 23, 30, 32
- Extra Practice: 2, 6, 18, 22, 27

like in last ex.

In this section, we will use vector notation to give explicit and geometric descriptions of solution sets of linear systems. We begin by defining a special type of system.

Definition 1.6.1. A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix, and $\mathbf{x}$ is in $\mathbb{R}^m$, and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Remark 1.6.2. The equation $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely $\mathbf{x} = \mathbf{0}$, called the **trivial solution**. We will be interested in finding **non-trivial solutions**, where $\mathbf{x} \neq \mathbf{0}$.

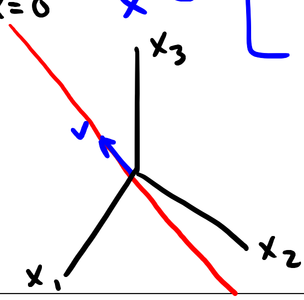
Example 1.6.3. Determine if the following homogeneous system has a nontrivial solution, and describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{aligned} x_1 &= \tfrac{4}{3}x_3 \\ x_2 &= 0 \\ x_3 &\text{ free} \end{aligned}$$

$A\vec{x}=0$

$$\vec{x} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 \vec{v} \quad \text{where } \vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$



Soln set $\text{Span}\{\vec{v}\}$

$A\vec{x}=0$ is a line thru origin

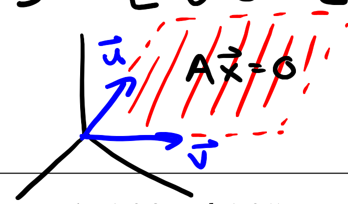
Proposition 1.6.4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.*

Example 1.6.5. Describe all solutions to the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0. \iff \begin{cases} x_1 = \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 + 0x_3 \\ 0x_2 + x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{5}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{u}} + x_3 \underbrace{\begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{v}}$$

soln set = Span$\{u, v\}$



this time it's a plane!

Definition 1.6.6. The answers in 1.6.3 and 1.6.5 are **parametric vector equations.** Sometimes, to emphasize that the parameters vary over all real numbers, we write

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \text{ for } s, t \in \mathbb{R}.$$

In both examples, we say that the solution is in **parametric vector form.**

Example 1.6.7. Describe all solutions of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

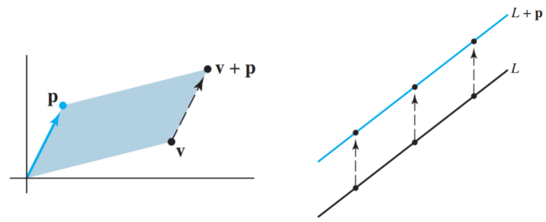
$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{=\vec{p}} + x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

same $\vec{v}$ from ex. 3

Definition 1.6.8. We can think of vector addition as *translation*. Given $\mathbf{p}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* v in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by p** to $\mathbf{v}+\mathbf{p}$. If each point on a line L is translated by a vector $\mathbf{p}$, the result is a line parallel to L.



For $t \in \mathbb{R}$, we call $\mathbf{p} + t\mathbf{v}$ **the equation of the line parallel to v through p.**

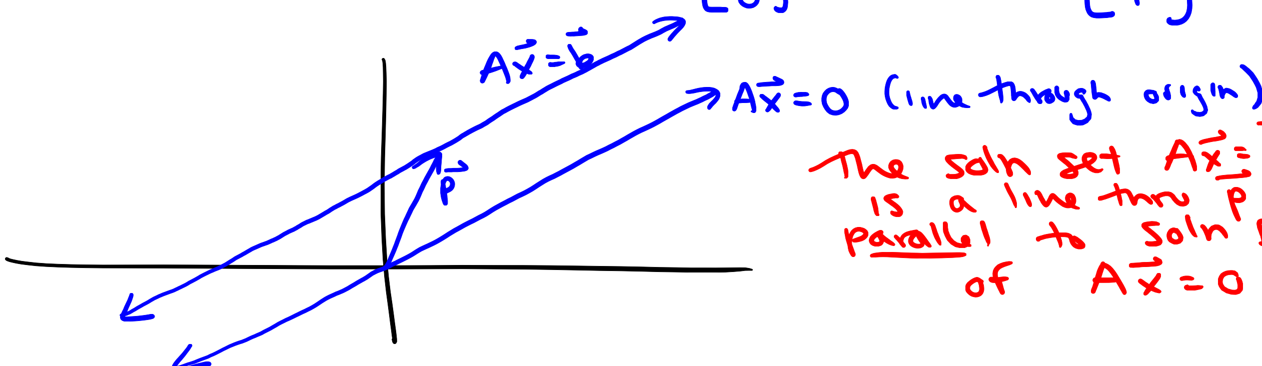
Example 1.6.9. Use this observation to describe the relationships between the solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ using the A and $\mathbf{b}$ from Examples 1.6.3 and 1.6.7.

line

soln to $A\vec{x} = 0$ was $t\vec{v}$

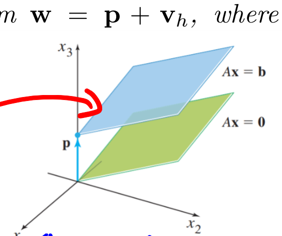
soln to $A\vec{x} = b$ was $\vec{p} + t\vec{v}$

where $\vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$



The soln set $A\vec{x} = \vec{b}$ is a line thru $\vec{p}$ parallel to soln set of $A\vec{x} = 0$

Theorem 1.6.10. *Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vecotrs of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.*



"parallel solns"

i.e. if $A\vec{x} = \vec{b}$ has soln, then soln set can be obtained by translating soln set of $A\vec{x} = \vec{b}$

can usually imagine as a nonzero point or line/plane not going thru origin

Procedure 1.6.11. To write a solution set in parametric vector form

1. Row reduce the augmented matrix to RREF
2. Express each basic variable in terms of any free variables
3. Write $\mathbf{x}$ as a vector whose entries depend on the free variables (if there are any)
4. Decompose $\mathbf{x}$ into a linear combination of vectors using free variables as parameters

Example 1.6.12. Describe and compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ if

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -5 - 4x_3 \\ x_2 = 3 + 3x_3 \\ x_3 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -5 - 4x_3 \\ 3 + 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$t \in \mathbb{R}$

parallel line through $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} x_1 = -4x_3 \\ x_2 = 3x_3 \\ x_3 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

line