MATH 118
Written Assignment #2

For the written homework assignment, you are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence of your solutions. Your solutions must have your name written on the top of the first page.

Your written assignment should be submitted as a pdf electronically through Canvas. Please email Pam if you are experiencing any technical difficulties.

1. Determine the values of a (a is just a constant) so that the following system has (i) no solution, (ii) a unique solution, (iii) more than one solution.

$$\begin{aligned} x + y - z &= -1 \\ 2x + 3y + az &= 3 \\ x + ay + 3z &= 2 \end{aligned}$$

2. Determine the values of a (a is just a constant) so that the following system has (i) no solution, (ii) a unique solution, (iii) more than one solution.

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$$

3. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, B = \begin{bmatrix} 7 & -1 & 5 \\ 0 & -3 & 3 \\ 1 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Calculate each of the following, or explain why the operation doesn't make sense.

(a) A^2
(b) AB
(c) AC
(d) BA
(e) B^2
(f) BC
(g) CA
(h) CB
(i) C^2

4. Let $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

(a) Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$.
(b) Describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

5. A large apartment building is to be built using modular construction techniques. The arrangement of apartments on each floor is to be chosen from one of three basic floor plans. Plan A has 3 three-bedroom units, 7 two-bedroom units, and 8 one-bedroom units on a floor. Plan B has 4 three-bedroom units, 4 two-bedroom units, and 8 one-bedroom units on a floor. Plan C has 5 three-bedroom units, 3 two-bedroom units, and 9 one-bedroom units on a floor.

   Is it possible to design the building so that it has exactly 66 three-bedroom units, 74 two-bedroom units, and 136 one-bedroom units? If so, is there more than one way to do it? Explain your answer.

6. The economist Wassily Leontief won a Nobel prize in 1973 for his work on determining how changes in one economic sector may affect other sectors (input-output analysis), and much of this was developed using linear algebra. In this problem, we'll look at a simple example of an input-output model.

   The economy can be divided into productive sectors (*i.e.* sectors which produce goods/services) and the open sector (*i.e.* the sector that consumes goods/services but does not produce them). Suppose the economy consists of three productive sectors: manufacturing, agriculture and services, each of which itself consumes portions of the outputs of the other sectors. For each unit of output produced by a sector, we can measure the inputs that it consumes. This is summarized below:

	Inputs consumed per unit output by:		
Input source	Manufacturing	Agriculture	Services
Manufacturing	.50	.40	.20
Agriculture	.20	.30	.10
Services	.10	.10	.30

   (a) The *intermediate demand* on each sector is the demand on that sector by the productive sectors (*i.e.* the number of units of their goods required by all of the productive sectors). Suppose manufacturing produces x units of output, agriculture produces y units of output, and services produces z units of output. Write down a matrix C such that $C\mathbf{x}$ gives the intermediate demand vector if $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

   (b) Suppose the open sector consumes 50 units of manufacturing, 30 units of agriculture, and 20 units of services. Let $\mathbf{d} = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$. Interpret each piece of the matrix equation

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}.$$

   (c) The matrix equation above can be rearranged to $(I_3 - C)\mathbf{x} = \mathbf{d}$. Find the matrix $I_3 - C$ by subtracting "element-wise".

(d) Now solve the matrix equation in (c) by row reduction (or by finding $(I_3 - C)^{-1}$). Explain the meaning of your answer.

7. In Canada, federal elections happen every four years and there are three main political parties: the Conservative Party, the Liberal Party and the New Democratic Party (NDP). Using the results of past elections, it's estimated that:

   - A Conservative voter in the previous election has a 70% of voting Conservative again, a 20% chance of voting Liberal and a 10% chance of voting NDP.
   - A Liberal voter from the previous election is estimated to have a 10% chance of Conservative, a 60% chance of voting Liberal again and a 30% chance of voting NDP.
   - An NDP voter from the previous election is estimated to have a 10% chance of voting Conservative, a 40% chance of voting Liberal and a 50% chance of voting NDP again.

   Suppose that, in the 2015 election, 35% of the voting population voted Conservative, 50% voted Liberal, and 15% voted NDP.

   (a) Calculate

   $$\begin{bmatrix} .70 & .10 & .10 \\ .20 & .60 & .40 \\ .10 & .30 & .50 \end{bmatrix} \begin{bmatrix} .35 \\ .50 \\ .15 \end{bmatrix}$$

   and interpret its meaning.

   (b) Make a prediction for the outcome of the 2023 election.