Lecture 7 The Multivariable Chain Rule

Stewart 14.1, McCallum 12.3, 12.5

- chain rule part 1
- chain rule part 2
- chain rule general case
- implicit differentiation

Question 7.1. What do we remember about the chain rule from single variable calculus?

composition

$\frac{d}{dx} f(u(x))$

$y = f(u)$ and

$u = g(x)$

$\frac{d}{dx} f(u(x)) = f'(u(x))u'(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{\cancel{du}} \frac{\cancel{du}}{dx}}$$

$\frac{dz}{dx}\ \frac{dx}{dt}$

$y = f(u)$

$u = g(x)$

$\Rightarrow y$ is a fn of x

Example 7.2. If $z = xy$, where $x = t^2$ and $y = \sin t$, find $\frac{dz}{dt}\big|_{t=\pi}$.

(this notation means $\frac{dz}{dt}$ when $t = \pi$).

$z = xy \qquad x = t^2 \qquad y = \sin t$

$\frac{dz}{dt} = ?$

$\boxed{z = xy = t^2 \sin t}$ ← really, z is a fn of t

$\frac{dz}{dt} = 2t\sin t + t^2\cos t$

product rule!

$\frac{dz}{dt}\Big|_{t=\pi} = 2(\pi)\overset{0}{\cancel{\sin\pi}} + \pi^2\overset{-1}{\cancel{\cos(\pi)}}$

$= -\pi^2$

① way to do this is to rewrite z as a fn of t

x is a fn of one var so $\frac{d}{dt}$ is okay

Theorem 7.3. (Chain Rule Case 1) If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\cancel{\partial x}}\frac{\cancel{dx}}{dt} + \frac{\partial z}{\cancel{\partial y}}\frac{\cancel{dy}}{dt}.$$

$\Rightarrow$ z is a fn of t

$f(x(t), y(t))$

b/c $f(x,y)$ is a fn of two vars we need ∂

Example 7.4. Find the tangent line at $t = \pi$ of $f(x, y) = xy$, where

$$x = t^2 \quad \boxed{y = \sin t.}$$

CR part 1 $\quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy) = x$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2) = 2t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= y\,2t + x\cos t$$

$$= 2t \cdot \sin t + t^2 \cos t$$

$z = xy \quad x = t^2 \quad y = \sin t$

want tangent line at $t = \pi$ to xy

need point + direction

$$\langle t^2, \sin t, t^2 \sin t \rangle$$

$$P = \langle \pi^2, 0, 0 \rangle$$

direction: $\left\langle \frac{dx}{dt}\Big|_\pi, \frac{dy}{dt}\Big|_\pi, \frac{dz}{dt}\Big|_\pi \right\rangle = \langle 2\pi, -1, -\pi^2 \rangle$

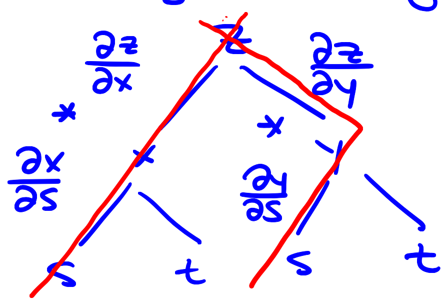
$$\boxed{\mathcal{L} = \langle \pi^2, 0, 0 \rangle + s\langle 2\pi, -1, -\pi^2 \rangle}$$

Theorem 7.5. (Chain Rule Case 2) If $z = r(x, y)$ where $x = f(s, t)$ and $y = g(s, t)$, then

z is a function of s, t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\cancel{\partial x}}\frac{\cancel{\partial x}}{\partial s} + \frac{\partial z}{\cancel{\partial y}}\frac{\cancel{\partial y}}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \boxed{\frac{\partial z}{\cancel{\partial x}}}\frac{\cancel{\partial x}}{\partial t} + \boxed{\frac{\partial z}{\cancel{\partial y}}}\frac{\cancel{\partial y}}{\partial t}$$

remember using tree diagram



$\frac{\partial z}{\partial s}$ = sum of all path products from z to s

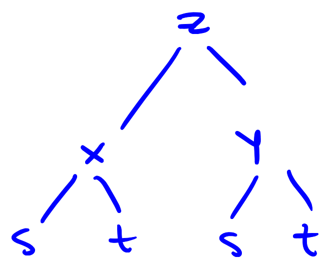
$$= \frac{\partial z}{\partial x} * \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} * \frac{\partial y}{\partial s}$$

Example 7.6. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

① could write

$$z = e^x \sin y = e^{st^2} \sin(s^2 t)$$

find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$



$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

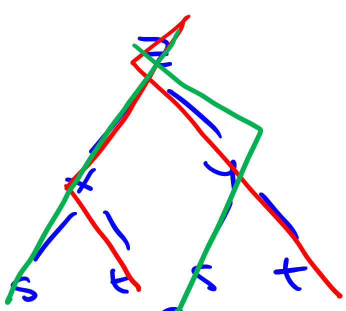
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(st^2) = t^2$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(st^2) = 2st$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(s^2 t) = 2st$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(s^2 t) = s^2$$



$\frac{\partial z}{\partial t}$ = sum of all path prod from z to t

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

plug in

$$= e^x \sin y \, 2st + e^x \cos y \, s^2$$

$x = st^2$

$$= e^{st^2} \sin(s^2 t)\, 2st + e^{st^2} \cos(s^2 t)\, s^2$$

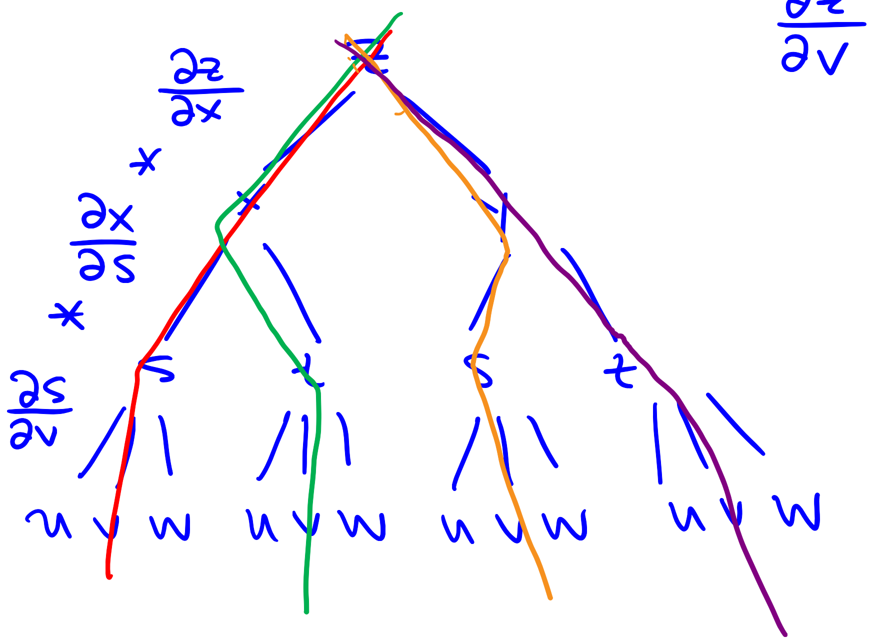
$y = s^2 t$

$\frac{\partial z}{\partial s}$ = sum of all path prod from z to s

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Example 7.7. Sometimes we might have z as a function of several variables which are themselves functions of several variables, which are in turn...
For example, suppose $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, and finally $s = \phi(u, v, w)$ and $t = \psi(u, v, w)$. Draw a tree diagram, and find $\cancel{\frac{\partial z}{\partial z}}$. $= 1$

$\frac{\partial z}{\partial v}$



$$\frac{\partial z}{\partial v} = \text{sum of all paths from } z \text{ to } v$$

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial v}$$

~~**Example 7.8.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 1$, and interpret $\partial z/\partial x$ at $(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ geometrically.~~

Question 7.9. What do we remember about implicit differentiation from single variable calculus? Use $x^2 + y^2 = 1$ as an example.

given $x^2+y^2 = 1$

find $\frac{dy}{dx}$

"implicit" means $y = y(x)$

$\frac{d}{dx} y = y'(x)$

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx} 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{\frac{\partial}{\partial x}(x^2+y^2)}{\frac{\partial}{\partial y}(x^2+y^2)} = -\frac{F_x}{F_y}}$$

where $F(x,y) = x^2+y^2$

$z + x^2 + 2y^2 + z^3 = 0$

Theorem 7.10. Suppose instead of a function $z = f(x, y)$, we are given z implicitly by an equation $F(x, y, z) = 0$. Then by the chain rule

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}} = -\frac{\cancel{\partial F}/\partial x}{\cancel{\partial F}/\partial z} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}} = -\frac{\cancel{\partial F}/\partial y}{\cancel{\partial F}/\partial z}.$$

formulas

find

F_x
F_y
F_z

solve for zero

$x^2+y^2+z^2-1=0$

$F(x,y,z)=x^2+y^2+z^2-1$

Example 7.11. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\boxed{x^2+y^2+z^2=1,}$ and interpret $\partial z/\partial x$ at $(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ geometrically.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

where $F(x,y,z)$

$$F_x = \frac{\partial}{\partial x}\left(x^2+y^2+z^2-1\right) = 2x$$

$$F_y = \frac{\partial}{\partial y}\left(x^2+y^2+z^2-1\right) = 2y$$

$$F_z = \frac{\partial}{\partial z}\left(x^2+y^2+z^2-1\right) = 2z$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z} = -x/z$$

at $P(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ $\quad \frac{\partial z}{\partial x} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$

$\Rightarrow$ slope of the tangent line at P in the direction of x is $-\sqrt{3}$

$\frac{\partial z}{\partial y}$ = deriv w/ everything but y constant

$\frac{\partial z}{\partial y}$

$$\frac{d}{dx}\left(x^2+y^2\right) = 2x+2yy'$$

$$\boxed{\frac{\partial}{\partial x}\left(x^2+y^2\right) = 2x}$$