4.1 # 1, 3, 8, 12, 13, 15, 17, 22, 23, 31, 32

1.) Let V be the first quadrant in the xy-plane; that is, let
$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

a.) If $\vec{u}$ and $\vec{v}$ are in V, is $\vec{u}+\vec{v}$ in V? Why

b.) Find a specific vector $\vec{u}$ in V and a specific scalar c such that $c\vec{u}$ is not in V. (This is enough to show V is not a vector space)

a.) Yes. The entries in $\vec{u}$ and $\vec{v}$ are non negative, so their sum is non negative too which means $\vec{u}+\vec{v}$ is in V.

b.) Many answers, but if $c=-1$ and $\vec{u}$ is in V, then $c\vec{u}$ is not in V.

3.) Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2+y^2 \leq 1 \right\}$. Find a specifice example - two vectors or a vector & a scalar - to show that H is not a subspace of $\mathbb{R}^2$.

Many answers. If $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{u}$ is in H, but $4\vec{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ is not in H. Therefore H is not closed under scalar multiplication and is not a subspace of $\mathbb{R}^2$.

8.) Determine if the set of all polynomials in $\mathbb{P}_n$ such that $\vec{p}(0)=0$ is a subspace of $\mathbb{P}_n$.

Call our set H. The zero polynomial is in H. If $\vec{p} \in H$, then for any c
$c\vec{p}(0) = c(0) = 0$, so $c\vec{p}(0) \in H$. If $\vec{p}$ and $\vec{q}$ are in H, then
$(\vec{p}+\vec{q})(0) = \vec{p}(0)+\vec{q}(0) = 0+0=0$, so $\vec{p}+\vec{q} \in H$. Thus H is a subspace.

4.1

12.) Let W be the set of all vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$.

Show that W is a subspace of $\mathbb{R}^4$.

If $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$, then $W = \text{Span}(\vec{u}, \vec{v})$.

(s and t are the coefficients in the linear combination of $\vec{u}$ & $\vec{v}$)

Since $\vec{u}, \vec{v}$ are in $\mathbb{R}^4$, $\text{Span}(\vec{u}, \vec{v})$ is a subspace of $\mathbb{R}^4$. Thus W is a subspace of $\mathbb{R}^4$.

13.) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a.) Is $\vec{w}$ in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

b.) How many vectors are in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

c.) Is $\vec{w}$ in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Why?

a.) No, 3

b.) Infinitely many

c.) In other words, is $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \mid \vec{w}]$ consistent?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \underset{\substack{\text{many row} \\ \text{operations} \\ \text{later}}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Yes, $\vec{w}$ is in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

15.) Let W be the set of all vectors of the form $\begin{bmatrix} 2a+3b \\ -1 \\ 2a-5b \end{bmatrix}$, where a, b are arbitrary real numbers. Find a set S of vectors that spans W or give an example to show W is not a vector space.

The zero vector is not in W, so W is not a vector space.

4.1 continued

17.) (Same directions as #15)

$$\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \qquad W = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$$

$$S = \{\vec{u}, \vec{v}, \vec{w}\}$$

22.) Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$. Determine if H is a subspace of $M_{2\times 4}$.

The zero matrix is in H. If A, B are in H then $F(A+B) = FA + FB = 0$. so $F(A+B) \in H$. For a scalar c, $F(cA) = c(FA) = c(0) = 0$.

Therefore H is a subspace of $M_{2\times 4}$

23.) True/False

a.) If $\vec{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\vec{f}(t) = 0$ for some t, then $\vec{f}$ is the zero vector in V.

b.) A vector is an arrow in three-dimensional space.

c.) A subset H of a vector space V is a subspace of V if the zero vector is in H.

d.) A subspace is also a vector space.

a.) False b.) False c.) False d.) True

4.1

31.) Let $\vec{u}$ and $\vec{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\vec{u}$ & $\vec{v}$. Explain why H also contains Span$\{\vec{u}, \vec{v}\}$. This shows that Span$\{\vec{u}, \vec{v}\}$ is the smallest subspace of V that contains both $\vec{u}$ & $\vec{v}$.

A subspace containing $\vec{u}$ & $\vec{v}$ must contain all linear combinations of $\vec{u}$ and $\vec{v}$ and thus contains Span$\{\vec{u}, \vec{v}\}$.

32.) Let H and K be subspaces of a vector space V. The intersection of H and K, written $H \cap K$, is the set of $\vec{v} \in V$ that belong to both H and K. Show that $H \cap K$ is a subspace of V. Then give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Since H & K are vector spaces, $\vec{0} \in H$, $\vec{0} \in K$, so $\vec{0} \in H \cap K$.
Suppose $\vec{u}, \vec{v} \in H \cap K$. Then for a scalar c, $c\vec{u} \in H$ and $c\vec{u} \in K$ so $c\vec{u} \in H \cap K$. Also, since $\vec{u}, \vec{v} \in H \cap K$, $\vec{u}, \vec{v} \in H$ and $\vec{u}, \vec{v} \in K$, therefore $\vec{u} + \vec{v} \in H$ and $\vec{u} + \vec{v} \in K$ since H, K are both vector spaces. Thus $\vec{u} + \vec{v} \in H \cap K$. Hence $H \cap K$ is a vector space.

In $\mathbb{R}^2$, let H be the x-axis, K be the y-axis. Then H & K are subspaces, but their union is not because it is not closed under addition. For example $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in K$, but $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $H \cup K$. So in general, the union of two vector spaces is not a vector space.