MATH 111, Intro to Functions and Calc II, S2020, Lecture Notes

Taken in part from
An Integrated Approach to Functions and their Rates of Change
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NOTE: I will update these notes as often as I can with the topics and examples (which will be worked out by hand in a separate document) we cover in class.

Contents

Chapter 16

Taking the Derivative of Composite Functions

16.1 The chain rule

16.1. Goals

- review derivatives
- express derivatives of composite functions $f(g(x))$

Groups 16.1.1. What are the derivatives of the following functions:

- x^n
- e^x
- b^x
- $ln(x)$
- $\log_b(x)$
- $f(x) \pm g(x)$
- $f(x)g(x)$
- $f(x)/g(x)$

Think, Pair, Share 16.1.2. What does the derivative represent?

Question 16.1.3. Suppose we're selling ice cream. The price that we set depends on the demand, that is, $p = f(u)$ where u is measured in people who want ice cream, and P is measured in dollars. Of course, the demand of our ice cream depends on the temperature, so we also have $u(t)$ where t is degrees Fahrenheit. So we guess, our revenue *really* depends on the temperature outside. How could we find the rate of change of our revenue with respect to temperature?

Theorem 16.1.4. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Question 16.1.5. How does this look with prime notation? What's nice about Leibnitz notation?

Groups 16.1.6. Suppose the price of our ice cream in dollars is

$$P(x) = 3x + 1,$$

where x is the demand in people. and the demand is $x = 1 - t^2$ where t is temperature measured in degrees Fahrenheit.

(a) write p as a function of t

(b) find $p'(t)$ (called the *marginal profit*), and its units

Example 16.1.7. What are the derivatives of $f(kx)$ and $f(x+k)$? Why does this make sense geometrically?

Groups 16.1.8. Write the functions as a composition

(a) $(x^2+1)^{10}$

(b) $\ln(x^2+2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.9. Find the derivatives of

(a) $(x^2+1)^{10}$

(b) $\ln(x^2+2)$

(c) e^{3x^2}

(d) $\ln(x^2)$

Example 16.1.10. Suppose the population of frogs in a pond is e^{g}, where g is the temperature of the pond in Celcius and the average temperature in the month of February is $0.25t + 14$ where t is in days. What is the rate of change of frogs with respect to time?

Groups 16.1.11. Suppose a rectangle is inscribed inside the ellipse

$$\frac{x^2}{9} + 4y = 1.$$

What's the largest possible area of such a rectangle?

16.2 The derivative of x^n for any real number n

16.2. Goals

- prove the power rule a different way
- find the derivative of b^x a different way
- find derivatives of quotients a different way

Example 16.2.1. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Question 16.2.2. Can we recover the derivative of b^x for $b > 0$ using only the chain rule?

Groups 16.2.3. Now, let's consider x^n.

(a) rewrite x^n using $\ln x$ and e^x.

(b) find the derivative of the function you found in (a)

(c) what did you prove?

Question 16.2.4. Can we find the derivative of $f(x)/g(x)$ without using the quotient rule?

16.3 Using the chain rule

16.3. Goals

- do more with the chain rule
- derive a formula for functions of multiple compositions
- understand the importance of simplification

Theorem 16.3.1. The derivative of $f(g(x))$ with respect to x is

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt} \qquad \text{or} \qquad \big(f(g(x))\big)' = f'(g(x))g'(x).$$

Example 16.3.2. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

Groups 16.3.3. Differentiate $e^{\sqrt{x^3+1}}$.

Think, Pair, Share 16.3.4. Can we find a general rule for the derivative of $f(g(h(x)))$?

Example 16.3.5. Differentiate
$$\ln\left(\sqrt{\frac{1+x}{(1-x)^3}}\right).$$

Example 16.3.6. Differentiate
$$\frac{8^{x^2+1}}{\left(2^x\right)^x}$$

Example 16.3.7. Find the derivative of
$$\left(\frac{x^4-2x^2+1}{x^2-1}\right)^5.$$

(Note: The notes I gave you last class has x^4+2x^2+1 in the numerator. Can you still simplify?)

Groups 16.3.8. Does xe^{x^2} have a maximum?

Chapter 17

Implicit Differentiation and its Applications

17.1 The derivative of x^x

17.1. Goals

- finding the derivative of x^x
- logarithmic differentiation

Example 17.1.1.

(a) What are the derivatives of x^n and b^x? Do either of these rules work for x^x when $x > 0$?

(b) Can we make it so that x is not a power? (hint: do you remember your log rules?)

Example 17.1.2. Find the derivative of $y = x^x$.

Definition 17.1.3. The method we just used, by taking logs of both sides and using the chain rule, is called **logarithmic differentiation**.

17.2 Logarithmic differentiation

17.2. Goals

- using logarithmic differentiation

Example 17.2.1. Find the tangent to the curve $f(x) = (x^2 + 1)^x$ at $x = 0$.

Example 17.2.2.

(a) what is the domain of $(x - 1)^{1-x^2}$?

(b) on this domain, find $f'(2)$.

Question 17.2.3. What are the properties of logarithms that we know?

Example 17.2.4. Find the derivative of $y = 2x^{e^x}$

Example 17.2.5. Find the derivative of $y = \dfrac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$

17.2.1 Extra Examples

Example 17.2.6. Find the derivatives of

(a) $\dfrac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$

(b) $e^{2x}(x^2+3)^5(2x^2+1)^3$

(c) $\left(e^{x-1}\right)^{x+1}$

17.3 Implicit differentiation

17.3. Goals

- using the ideas of the previous section to find $\frac{dy}{dx}$ of implicitly defined functions

Spot the mistake 17.3.1. Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.2. Find $\frac{dy}{dx}$ for the circle

$$x^2 + y^2 = 1$$

Example 17.3.3. What kinds of information can we use to sketch the graph of a curve?

Example 17.3.4. Sketch a graph of the curve $y^2 = x^3 - x$ (don't worry about concavity)

Example 17.3.5. Find all points where the tangent to

$$x^3 + y^3 = 1$$

is horizontal or vertical

Procedure 17.3.6 (using implicit differentiation)**.**

Example 17.3.7. Find the slope of the tangent to

$$x^3 + y^3 = 6xy$$

at the point $(3, 3)$.

Example 17.3.8. Find the absolute maximum and minimum y-values of the ellipse

$$2x^2 + 4xy + 3y^2 = 6.$$

17.4 Related rates

17.4. Goals

- use implicit differentiation to relate rates of change

Example 17.4.1. You're cruising at 60 miles per hour, on a road where the speed limit is 40 miles per hour. 9 feet ahead of you behind a tree 12 feet off the road, a cop stands with a radar gun and clocks your speed. Will you get pulled over?

Strategy 17.4.2 (Relating rates)**.**

1. draw and label a diagram (try using suggestive notation, e.g. A for area, r for radius)
2. find the rates you are trying to relate, say $\frac{dx}{dt}$ and $\frac{dy}{dt}$
3. determine which is the rate you know, and which you're trying to find, say $\frac{dy}{dt}$
4. find an equation relating x and y
5. take a derivative with respect to t, using implicit differentiation for x and y
6. solve for the rate you need to know
7. plug in known values for x, y and $\frac{dx}{dt}$

Example 17.4.3. A 5 foot ladder leans against the side of a building. You grab the base of the ladder and begin sliding it away from the wall at a constant rate of 0.5 feet per second.

(a) How fast is the ladder sliding down when the base of the ladder is 3 feet away from the wall?

(b) What happens to the rate of change of the height of the ladder as the base gets farther away?

Example 17.4.4. You are blowing up a balloon at a constant rate of 3 square inches per second, how fast is the radius of the balloon changing when the radius of the balloon is 2 inches?

(a) Draw a diagram, and determine what rates you are trying to relate.
What are your knowns and unknowns?

(b) What equation relates the two rates?

(c) Take a derivative of your equation from (b).

(d) Using your knowns and unknowns, how fast is the radius changing when $r = 2$?

(e) What happens to the rate of change of the radius as more air is blown into the balloon?

Example 17.4.5. A water bottle is made of a cylinder of roughly 4 inches in diameter and six inches high, capped with the top half of a sphere of radius 2. Water is flowing from a faucet into the bottle at a rate of 2 cubic inches per second.

(a) When the water in the bottle is between 0 and 6 inches of height (when it's in the cylinder), show that the height of water is changing at a constant rate.

(b) From 6 to 8 inches of height, the volume of the water in the bottle is

$$V = \pi\left(48 + 4y - \frac{y^3}{3}\right)$$

where $y+6$ is the height of the water in the bottle (we'll use y to make our calculations easier). Show that the rate of change of the water increases as the height increases.

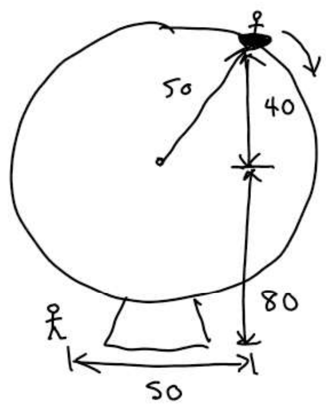
17.4.1 Extra problems

Example 17.4.6. At 10:00 a.m. a fishing boat leaves the dock in Vancouver and heads due south at 20 mph. At the same moment a ferry located 60 miles directly west of the dock is traveling toward the dock at 25 mph. Assume that the boats maintain their speeds for the next two hours.

(a) How fast is the distance between the boats changing at 11:00 a.m.?

(b) Are the boats getting farther apart or closer together?

Example 17.4.7. You are on a Ferris wheel, 50 feet in diameter. After going over the top (i.e. you're going down now), at a height of 40 feet from the center of the Ferris wheel, your height is changing at a rate of 8 feet per second.



(a) How fast are you moving horizontally from the away from the base of the Ferris wheel?

(b) In the diagram below (drawn with my very bad backup pen), a person stands 120 feet below and 50 feet behind you, waiting to ride. How fast are you moving toward or away from them?

Chapter 31

Differential Equations

31.1 Indroduction to Modeling with Differential Equations

31.1. Goals

- what is a model?
- what is a differential equation?

Example 31.1.1 (Newton's Law of Cooling)**.** Suppose you're drinking hot apple cider at Koffee. The cider will burn your mouth at 200°, but you know if you wait it will cool down eventually. Your friend's coffee was served at a slightly lower temperature at 180°.

(a) Whose drink do you think is getting cold faster? Are they cooling at the same speed?

(b) If the room is set at 65°. What do you think the graph of the temperature of you drink with respect to time is?

(c) If the room's temperature is R, and the cider's temperature is T, Newton's law of cooling says

> **Theorem 31.1.2.** *The rate of change of the difference between two temperatures T and R is proportional to the difference between T and R.*

Can you represent this using a mathematical equation?

Definition 31.1.3. An equation that contains a variable and its derivative (or derivatives) is called a differential equation.

Example 31.1.4 (Population biology)**.** Find a model: If a population is flourishing under ideal circumstances and with unlimited resources, its rate of growth is proportional to itself.

Example 31.1.5 (Cellular biology)**.** The concentration of a certain nutrient in a cell changes at a rate proportional to the difference between the concentration of the nutrient inside the cell and the concentration in the surrounding environment. Suppose that the concentration in the surrounding environment is kept constant and is given by N. If the concentration of the nutrient in the cell is greater than N, then the concentration in the cell decreases; if the concentration in the cell is less than N, then the concentration increases. Let $C = C(t)$ be the concentration of the nutrient within the cell. Write a differential equation involving the rate of change of C.

Example 31.1.6 (Economics)**.** Ten thousand dollars is deposited in a bank account with an annual interest rate of 4% compounded continuously. No further deposits are made. Write a differential equation fitting the situation if money is withdrawn continuously at a rate of \$4000 per year.

Example 31.1.7. Consider the differential equation

$$\frac{dM}{dt} = 0.04M - 4000$$

where $M(t)$ is the amount of money in a bank account at time t, where t is given in years. The differential equation reflects the situation in which interest is being paid at a rate of 4% per year compounded continuously and money is being withdrawn at a constant rate of \$4000 per year.

(a) Suppose the initial deposit is \$50000. Will the account be depleted?

(b) If money is to be withdrawn at a rate of \$4000 per year, what is the minimum initial investment that assures the account is not depleted?

(c) If this is a trust fund that is being set up with \$50000 and the idea is that the account should not be depleted, what should the restriction be on the rate of withdrawal? Assume money will be withdrawn at a constant rate.

31.1.1 Extra Problems

Example 31.1.8 (Epidemiology)**.** The flu is spreading throughout a college dormitory of 300 students. It is highly contagious and long in duration. Assume that during the time period we are modeling no student has recovered and all sick students are still contagious. It is reasonable to assume that the rate at which students are getting ill is proportional to the product of the number of sick students and the number of healthy ones because there must be an interaction between a healthy and a sick student to pass along the disease. Let $S = S(t)$ be the number of sick students at time t . Write a differential equation reflecting the situation.

Example 31.1.9 (Medicine)**.** The rate at which a certain drug is eliminated from the bloodstream is proportional to the amount of the drug in the bloodstream. A patient now has 45 mg of the drug in his bloodstream. The drug is being administered to the patient intravenously at a constant rate of 5 milligrams per hour. Write a differential equation modeling the situation.

Example 31.1.10 (Physics)**.** An object falling through the air undergoes a constant downward acceleration of 32 feet per second per second due to the force of gravity.

(a) Write a differential equation involving $v(t)$, the vertical velocity of the object at time t.

(b) Write a differential equation involving $s(t)$, the object's height above the ground.

31.2 Solutions to Differential Equations

31.2. Goals

- solving differential equations
- slope fields
- existence and uniqueness

Definition 31.2.1. A function f is a **solution to a differential equation** if it satisfies the differential equation.

Example 31.2.2. Is $y = x^3$ a solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x}?$$

Example 31.2.3. Is $y = xe^{3x}$ a solution to the differential equation

$$\frac{dy}{dx} = \frac{3y}{x}?$$

Example 31.2.4. Check that all of the following functions are solutions to the differential equation

$$\frac{dM}{dt} = 0.05M - 5000$$

Which one has the initial condition that $M(0) = 10000$?

(a) $M(t) = 2e^{0.05t} + 100000$

(b) $M(t) = -90000e^{0.05t} + 100000$

(c) $M(t) = 10e^{0.05t} + 100000$

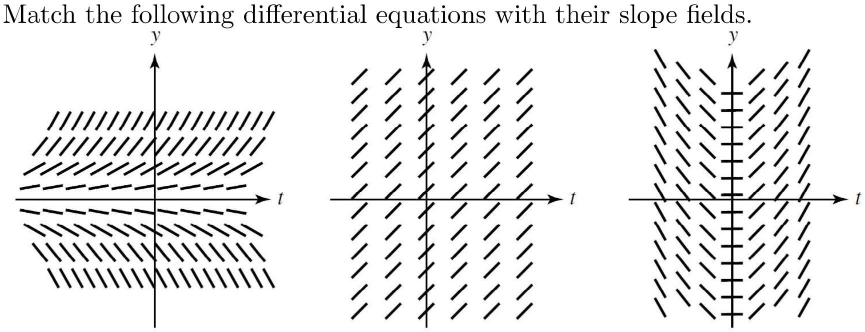
> **Definition 31.2.5.** At any point P, we can use a differential equation to find the slope of the tangent line to the solution curve through P. This gives us a rough idea of the shapes of particular solutions. If we plot some of these slopes, we call the result a **slope field.**

Example 31.2.6. Match the following differential equations with their slope fields.

(a) $\dfrac{dy}{dx} = 2$

(b) $\dfrac{dy}{dx} = 2x$

(c) $\dfrac{dy}{dx} = 2y$

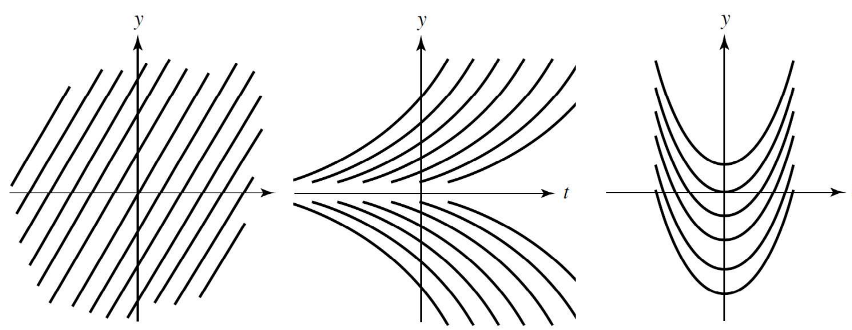


Example 31.2.7. Match the following differential equations with a graph of their solution curves.

(a) $\dfrac{dy}{dx} = 2$

(b) $\dfrac{dy}{dx} = 2x$

(c) $\dfrac{dy}{dx} = 2y$



Example 31.2.8. Guess and check possible solutions to the following differential equations.

(a) $\dfrac{dy}{dx} = 2$

(b) $\dfrac{dy}{dx} = 2x$

(c) $\dfrac{dy}{dx} = 2y$

Theorem 31.2.9 (Existence and Uniqueness)**.** Let (a, b) be a point in the plane.

- Any differential equation $\frac{dy}{dx} = f(x)$ where f is continuous has a unique solution passing through (a, b).
- The same is true if $\frac{dy}{dx} = g(y)$ where g and g' are both continuous.

Example 31.2.10 (Physics)**.** An object falling through the air undergoes a constant downward acceleration of 32 feet per second per second due to the force of gravity.

(a) What is $v(t)$ if the initial velocity is 0?

(b) What is $s(t)$ if the initial position is 100 feet above the ground?

Observation 31.2.11. If $\frac{dy}{dx} = f(x)$, then the solution to the differential equation is $y = F(x) + C$, where $F(x)$ is some function whose derivative is $f(x)$.

Example 31.2.12 (Population biology)**.**

(a) Find a model: If a population is flourishing under ideal circumstances and with unlimited resources, its rate of growth is proportional to itself.

(b) Let $P(t)$ be population at t years, and k be the proportion.
Check that $P = Ce^{kt}$ is a solution for all C.

(c) If $k = 0.05$, and $P(0) = 5000$, what's the population after 10 years?

Observation 31.2.13. If $\frac{dy}{dx} = ky$, then $y = Ce^{kx}$, where $C = y(0)$.

31.2 Solutions to Differential Equations (continued)

31.2. Goals

- solving differential equations
- slope fields
- existence and uniqueness

Example 31.2.1. Determine whether each function is a solution to $y'' - y = 0$

(a) Ce^{x}

(b) Ce^{-x}

Example 31.2.2. Determine whether each function is a solution to the equation $xy' - 2y = 0$. If either is a solution, find the particular solution whose graph passes through the point $(1, 3)$.

(a) Cx^2

(b) $x^2 + C$

Example 31.2.3. .

(a) Match each differential equation to its slope field.

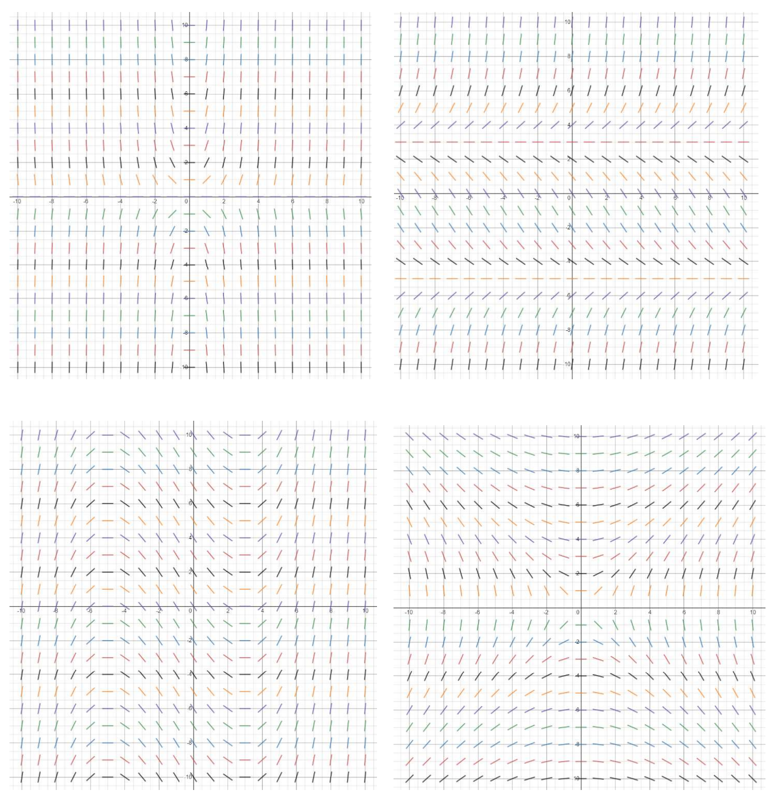
(b) On each graph, find the particular solution through the origin.

(i) $\frac{dy}{dx} = xy$

(ii) $\frac{dy}{dx} = x/y$

(iii) $\frac{dy}{dx} = (x-3)(x+5)$

(iv) $\frac{dy}{dx} = (y-3)(y+5)$



31.3 Qualitative Analysis

31.3. Goals
- more about slope fields
- qualitative analysis of solutions to differential equations

Example 31.3.1. Suppose a hot or cold beverage is put in a room that is kept at 65 degrees. Then the rate of change of the temperature of the beverage is

$$\frac{dT}{dt} = k(65 - T),$$

where t is a positive constant.

(a) What must the temperature of the beverage be in order for its temperature to remain constant.

(b) For what temperatures is the beverage cooling down?

(c) Sketch representative solution curves corresponding to a variety of intitial conditions.

Definition 31.3.2. An equilibrium solution to a differential equation is a solution that is constant for all values of the independent variable. If $\frac{dy}{dt} = f(y)$, then the equilibrium solutions can be found by solving $f(y)$ for y.

Example 31.3.3. Do a qualitative analysis of the solutions to the differential equation

$$\frac{dy}{dx} = (y-1)(y-3)$$

Sketch representatives of the family of solutions.

Definition 31.3.4. Equilibrium can be classified as stable, unstable, or semistable. The equilibrium at $y = 1$ is called a **stable equilibrium**. The equilibrium at $y = 3$ is called an **unstable equilibrium**.

Example 31.3.5. Find and classify the equilibrium solutions of

$$\frac{dx}{dt} = x^2 - x$$

Sketch representatives of the family of solutions with x as the vertical axis and t the horizontal.

Example 31.3.6. An industrial plant produces radioactive material at a constant rate of 4 kilograms per year. The radioactive material decays at a rate proportional to the amount present and has a halflife of 20 years.

(a) Write a differential equation whose solution is $R(t)$, the amount of material present t years after this practice begins.

(b) Sketch some representative solutions corresponding to different initial values of R. Include the equilibrium solution. Can we predict the level of radioactive material in the long run?

Example 31.3.7. Remember, under perfect conditions, population grows at a rate proportional to itself. Suppose the number of fish in a lake grows according to the equation

$$\frac{dP}{dt} = 0.0005P.$$

Recall that the solution family to this differential equation is $P = Ce^{0.0005t}$. Graph some particular solutions to this curve. What are the equilibrium? What's the maximum population? Is this reasonable?

Definition 31.3.8. Really, resources like space or food would limit the size of the population to some amount of fish, say L. It turns out a more reasonable model is

$$\frac{dP}{dt} = kP(L - P) = kLP - kP^2.$$

This is called **logistic population growth**, and L is called the **carrying capacity.**

Example 31.3.9. Suppose the number of fish in a lake grows according to the equation

$$\frac{dP}{dt} = 0.45P - 0.0005P^2.$$

(a) What is the lake's carrying capacity for fish? Is it a stable equilibrium?

(b) What size is the fish population when it is growing most rapidly?

31.3.1 Extra Problems

Example 31.3.10. Recent crime in the town of Willimantic, CT, has caused people to move to another town at a rate of N thousand people per year, where N is a constant. The rate of change of the population of the town can be modeled by the differential equation

$$\frac{dP}{dt} = 0.05P - N,$$

where $P = P(t)$ is the number of people in the town in hundreds.

(a) If P(0) $= 20,000$, what is the largest rate of moving out the town can support in the long run?

(b) How big must the population of the town be to support the loss of 1000 people per year?

Example 31.3.11. Before the crime rate spike, population in Willimantic grew at a rate of

$$\frac{dP}{dt} = 125P - .005P^2$$

(a) What is the carrying capacity of Willimantic?

(b) When will the population be growing the most rapidly?