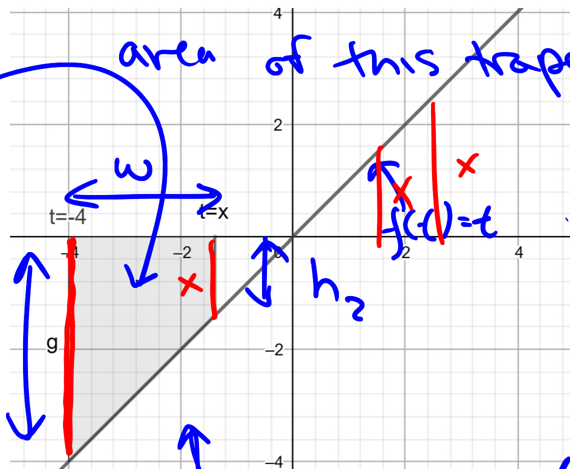
. **Groups 23.2.4.** Let $f(t) = x$ and $A(x) = {}_{-4}A_f(x) = \int_{-4}^{x} f(t)\, dt$.

(a) Find a formula for $A(x)$ when the domain of $A(x)$ is $[-4, 0]$

only way to find $\int_{-4}^{x} t\,dt$ is to use area

area of this trapezoid



$h_1 = -4$

$w = b - a$

$= x - (-4) = x + 4$

$h_2 = x$

$A = \frac{1}{2}(h_1 + h_2)w$

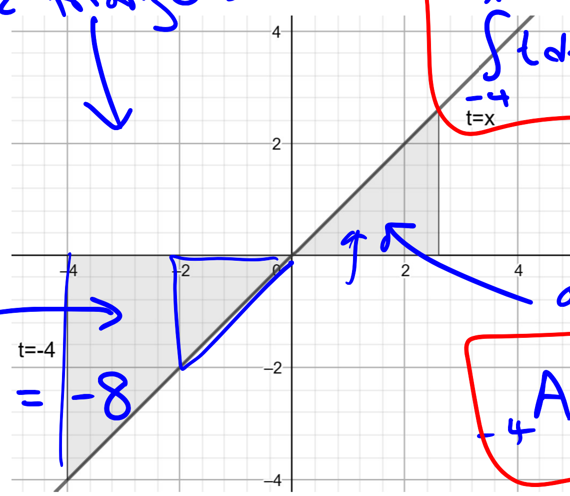
$= \frac{1}{2}(x-4)(x+4)$

$\int_{-4}^{x} t\,dt = \frac{1}{2}x^2 - 8$ on $[4, 0]$

(b) Find a formula for $A(x)$ when the domain of $A(x)$ is $x \geq 0$.

(Note: you should get the same formula as in part (a)!)

2 triangles



area $= \frac{1}{2}x^2$

area $= -\frac{1}{2} 4 * 4 = -8$

${}_{-4}A_f(x) = \frac{1}{2}x^2 - 8$

(c) What is a general formula for ${}_{-4}A_f(x) = \int_{-4}^{x} f(t)\, dt$

$\frac{1}{2}x^2 - 8 = \int_{-4}^{x} t\,dt$

(d) What is the formula for $\int_{-2}^{x} f(t)\, dt$? How is it related to $\int_{-4}^{x} f(t)\, dt$

$\int_{-2}^{x} t\,dt = \frac{1}{2}x^2 - 1$

this only differ by a constant

.

Observation 23.2.5. Let a and k be constants. If $f(t) = k$, then

$$ {}_aA_f(x) = \boxed{\int_a^x k\ dt = kx + C}, \text{ for some constant } C, $$

If $f(t) = t$, then

$$ {}_aA_f(x) = \boxed{\int_a^x t\ dt = \frac{x^2}{2} + C}, \text{ where } \cancel{C = ka}. $$

$\int_a^x t^2 dt = \frac{t^3}{3} + C$?

$\frac{d}{dx} kx = k$

$\frac{d}{dx} \frac{x^2}{2} = x$

Question 23.2.6. If $f(t) = t^2$, can you predict the solution to

$$ {}_aA_f(x) = \int_a^x t^2 \, dt? $$

Observation 23.2.7. We summarize the following properties of the area function ${}_aA_f(x)$

- area functions f (i.e. for different a) are all vertical translates
- $f > 0$ means that A_f is increasing
- $f < 0$ means that A_f is decreasing
- f increasing means that A_f is concave up
- f decreasing means that A_f is concave down

the are all props of derivatives

$f'(x)$ $f(x)$

23.3 The Fundamental Theorem of Caclulus

Goals (for 23.3 and 24.1

- FTC part one
- derivative "undoes" the integral
- FTC part two
- applications of FTC to definite integrals

Theorem 23.3.1. If f is continuous on $[a, b]$ and $c \in [a, b]$, then

$$ {}_cA_f(x) = \int_c^x f(t)\, dt \quad x \in [a, b]$$

is differentiable on (a, b) and $\frac{d}{dx}\, {}_cA_f(x) = f(x)$.

i.e.

$$\frac{d}{dx}\int_5^x (t^2+1) = x^2+1$$

$$\frac{d}{dx}\int_2^x (e^t \cos t) = e^x \cos x$$

Seems like $\frac{d}{dx}$ undoes $\int_2^x$

derivative is "inverse" of integral

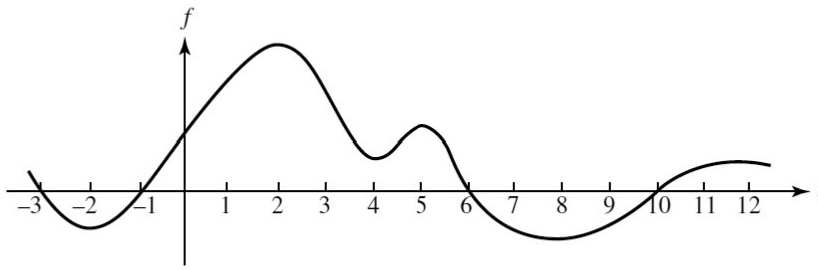
$$\sqrt{x^2} = x$$

Theorem 23.3.2. If f is continuous on $[a, b]$, then $\int_c^x f(t)\ dt = f(x)$ is differentiable on (a, b) and

$$\boxed{\frac{d}{dx}\int_c^x f(t)\ dt = f(x)}\text{ for any } c \in [a, b].$$

same f !

Example 23.3.3. Let $F(x) = \int_0^x f(t)\ dt$ and $G(x) = \int_1^x f(t)\ dt$, where f is the function graphed below. What are the relative minimums, maximums, and inflection points of F? What about G? How are the graphs of F and G related?



Chapter 24

The Fundamental Theorem of Caclulus

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

24.1 Definite Integrals and FTC

Definition 24.1.1. A function F is an **antiderivative** of f its derivative is f. In other words, F is an antiderivative of f if $F' = f$.

$\frac{x^4}{4}$ is an antideriv of x^3

b/c $\frac{d}{dx}\frac{x^4}{4} = x^3$

Example 24.1.2. Find antiderivatives of the following. In other words, in each case, find a function F such that $F'(x) = f(x)$. Then, find another. What must be true about any two antiderivatives of a function?

(a) $3x^2$ (b) x^2

a) WTF a function $F(x)$
st. $F'(x) = 3x^2$
maybe x^3?
$F(x) = x^3 \Rightarrow F'(x) = \frac{d}{dx}x^3 = 3x^2$ ✓
so x^3 is an antideriv of $3x^2$
also x^3+10 b/c $\frac{d}{dx}(x^3+10) = 3x^2+0$
x^3+ constant is always an antideriv of $3x^2$

b) WTF function $F(x)$ st. $F'(x) = x^2$
maybe x^3 b/c power rule
$\frac{d}{dx}x^3 = 3x^2 \neq x^2$
but try $\frac{1}{3}x^3$ $\frac{d}{dx}\frac{1}{3}x^3 = \frac{3}{3}x^2 = x^2$ ✓
so $\frac{1}{3}x^3 + C$ is an antideriv of x^2
any constant

Example 24.1.3. Let k and n be constants, and $b > 0$. Complete the table below by thinking of a function F whose derivative is f.

$f(x)$	$F(x)$ such that $F'(x) = f(x)$	$f(x)$	$F(x)$ such that $F'(x) = f(x)$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$ (power rule for anti)	$\cos(x)$	$\sin x$
$\frac{1}{x}$	$\ln x + C$	$\sin(x)$	$-\cos x$
0	any constant C	$\sec^2(x)$	$\tan x$
k	kx	$\sec(x)\tan(x)$	$\sec x$
e^x	e^x	$\frac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\frac{b^x}{\ln b}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

if I know $\frac{d}{dx} f(x) = f'(x)$

then $f(x)$ is an antideriv of $f'(x)$

WTF antideriv x^n

try $\frac{1}{n+1} x^{n+1}$ $\qquad$ $\frac{d}{dx} \frac{x^{n+1}}{n+1} = \frac{n+1}{n+1} x^{n+1-1}$

$= x^n$ ✓

antid x^{-1} is $\frac{x^{-1+1}}{-1+1}$? $\frac{x^0}{0}$ ✗

$$F(x)\Big]_a^b = F(b) - F(a)$$

FUNDAMENTAL theorem of Calculus

Theorem 24.1.4. Let f be continuous on $[a, b]$. If F is an antiderivative of f, that is, if $F' = f$, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

F is anti deriv of f

e.g. $$\int_1^2 x\,dx = \left[\frac{x^2}{2} + C\right]_1^2 = \underbrace{\left(\frac{(2)^2}{2} + \cancel{C}\right)}_{2} - \underbrace{\left(\frac{1^2}{2} + \cancel{C}\right)}_{0.5}$$

$.5$

an antideriv of x^1 is $\frac{x^2}{2} + C$

Example 24.1.5. Compute $\displaystyle\int_1^3 3x^2\ dx$.

FTC

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F' = f$$

$$\int_1^3 3x^2\,dx = \text{any antideriv of } 3x^2 \Big]_1^3$$

$$= x^3 \Big]_1^3 = 3^3 - 1^3 = 26$$

$$F(x)\Big]_a^b = F(b) - F(a)$$

eg.

$$x^2\Big]_1^7 = 7^2 - 1^2$$

$$e^x\Big]_0^{\ln 5} = e^{\ln 5} - e^0 = 4$$

Example 24.1.6. Compute $\int_0^1 x^2\, dx$.

$$\int_0^1 x^2\, dx = F(1) - F(0)$$

where $F'(x) = x^2$

$F(x) = \frac{x^3}{3}$ has $F'(x) = x^2$

$$\int_0^1 x^2\, dx = \frac{x^3}{3}\Big]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Example 24.1.7. Compute $\int_1^e \frac{2}{x}\, dx$.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b)-F(a)$$

where $F' = f$

need antideriv of $\frac{2}{x}$

try $2\ln x$

$$\frac{d}{dx} 2\ln x = 2\frac{d}{dx}\ln x = \frac{2}{x} \checkmark$$

$$\int_1^e \frac{2}{x}\,dx = 2\ln x\Big]_1^e = 2\ln(e) - 2\ln(1) = 2$$

antideriv?

Example 24.1.8. Suppose that water enters a reservoir at a rate of $r(t) = 40,000 + 60,000 \cos t$ gallons per month, where t is measured in months. What is the net change in water level in the first two months?

net change from a to b of r(t)

is $\int_a^b r(t)\,dt$

WTF $\int_0^2 (40,000 + 60,000 \cos t)\,dt = F(2) - F(0)$

want antideriv of $40,000 + 60,000 \cos t$

~recall~

$\frac{d}{dx}(f \pm g) = \frac{d}{dx} f \pm \frac{d}{dx} g$

$\Rightarrow$ split antiderivs over sums

$\frac{d}{dx}(kf) = k \frac{d}{dx} f$

$\Rightarrow$ pull constants out

$\int_a^b kf\,dx$

$= k \int_a^b f\,dx$

CONCLUSION: I can evaluate antideriv term by term

$r(t) = 40000 + 60000 \cos t$

an antideriv $40,000t + 60000 * (\text{antideriv of } \cos t)$

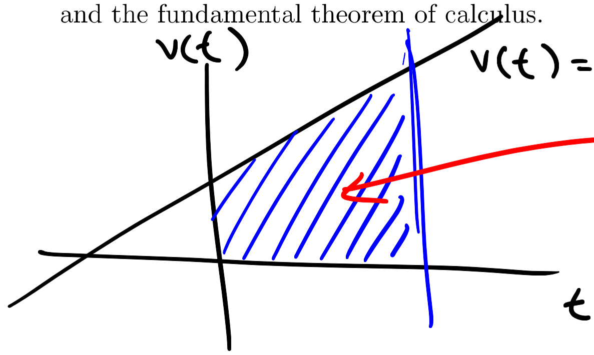
$= 40000t + 60000 \sin t$

$\int_0^2 (40000 + 60000 \cos t)\,dt = 40000t + 60000 \sin t \Big]_0^2$

$= (40000(2) + 60000 \cancel{\cos 2}\ \sin 2) - 0$

$= 80000 + 60000 \cancel{\cos 2}\ \sin 2$ gallons

Example 24.1.9. A cheetah starts running away from you in a straight line. In the first 5 seconds, its velocity is given by $\boxed{v(t) = 2t + 40}$ ~~feet per second~~ mph. Find the distance the cheetah runs in the first 5 seconds in two ways: using geometry and the fundamental theorem of calculus.



$v(t) = 2t + 40$

area = distance traveled

Check

distance cheetah runs = area under $2t+40$ from 0 to 5

$$= \int_0^5 (2t+40)\,dt$$

$$\int_0^5 (2t+40) = t^2 + 40t \Big]_0^5 = 25 + 200 - 0$$

$$= 225$$

verify that using geometry gives same answer