3.3 Cramer's Rule, Volume, and Linear Transformations

McDonald Fall 2018, MATH 2210Q, 3.3 Slides

3.3 Homework: Read section and do the reading quiz. Start with practice problems.

• Hand in: TBD.

• Recommended: TBD.

3.3.1 Cramer's rule

Theorem 3.3.1 (Cramer's Rule). Let A be an invertible $n \times n$ matrix. For any \mathbf{b} in \mathbb{R}^n , the unique solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n$$

where

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad \mathbf{b} \quad \mathbf{a}_{i+1} \quad \cdots \quad \mathbf{a}_n]$$

Example 3.3.2. Use Cramer's rule to solve the system

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

Example 3.3.3. Consider the following system of equations where $a \neq 0$. Prove that if $a \neq 24$ then the system has exactly one solution. What does the solution set look like?

$$\begin{cases} ax - 6y = -1 \\ 4x - y = 3 \end{cases}$$

1

Definition 3.3.4. The adjugate (or classical adjoint) of A, is

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{T} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Where $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Theorem 3.3.5 (Inverse Formula). Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 3.3.6. Find the inverse of the matrix

$$A = \left[\begin{array}{rrr} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{array} \right]$$

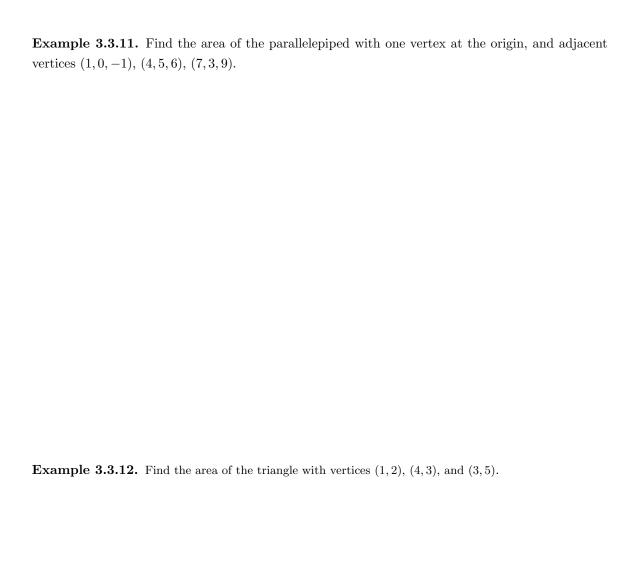
Remark 3.3.7. This formula for the inverse is really useful for theoretical calculations, but in almost all cases, our algorithm of reducing to the identity is *much* more efficient.

3.3.2 Determinants as Area or Volume

Theorem 3.3.8. If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Example 3.3.9. Calculate the area of the parallelogram with vertices (0,0), (1,2), (2,3) and (3,5).

Example 3.3.10. Find the area of the parallelogram with vertices (-2, -2), (0, 3), (4, -1) and (6, 4).



3.3.3 Linear Transformations

Theorem 3.3.13. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in \mathbb{R}^2 , then

$$\{area\ of\ T(S)\} = |\det A| \cdot \{area\ of\ S\}$$

If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation determined by a 3×3 matrix A, and S is a parallelepiped in \mathbb{R}^3 , then

$$\{volume\ of\ T(S)\} = |\det A| \cdot \{volume\ of\ S\}$$

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$. Find the area of T(S).

3.3.4 Additional Notes and Problems