MATH 118
Planes in $\mathbb{R}^3$ and functions of 2 variables

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1. Find a non-zero vector perpendicular to the plane $2x - y + 3z = 5$.

$$\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

2. Find the equation of the plane which passes through the point $P(0,1,1)$ and is perpendicular to the line given by $\mathbf{r}(t) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The line is in the direction of $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, the plane passing through $(0,1,1)$ perpendicular to the line has the equation

$$1\cdot(x-0) + 1\cdot(y-1) + 1\cdot(z-1) = 0$$

OR $x + y + z = 2$.

3. Find the equation of the plane which passes through the point $P(2,-1,1)$ and contains the non-parallel vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$.

Let $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -3 & 1 \end{vmatrix} = \hat{i}(1-3) + \hat{j}(-1-2) + \hat{k}(-6-1) = \begin{bmatrix} -2 \\ -3 \\ -7 \end{bmatrix}$

Therefore, the equation of the plane is

$$-2(x-2) - 3(y+1) - 7(z-1) = 0$$

4. Find and sketch the domain for each of the following functions.

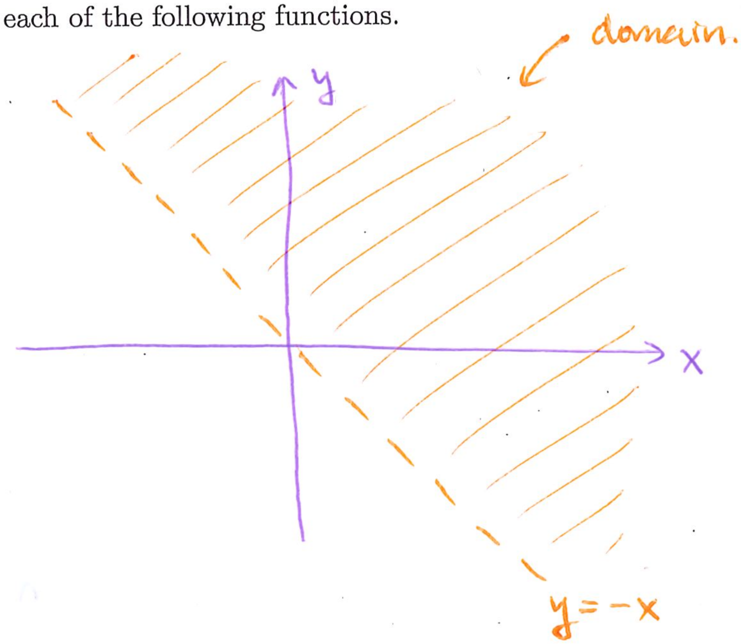
(a) $f(x, y) = \dfrac{\sin(xy)}{\sqrt{x+y}}$

Need $x+y>0$

$\Leftrightarrow y > -x$

$\therefore$ The domain is

$\{(x,y) \mid y > -x\}$



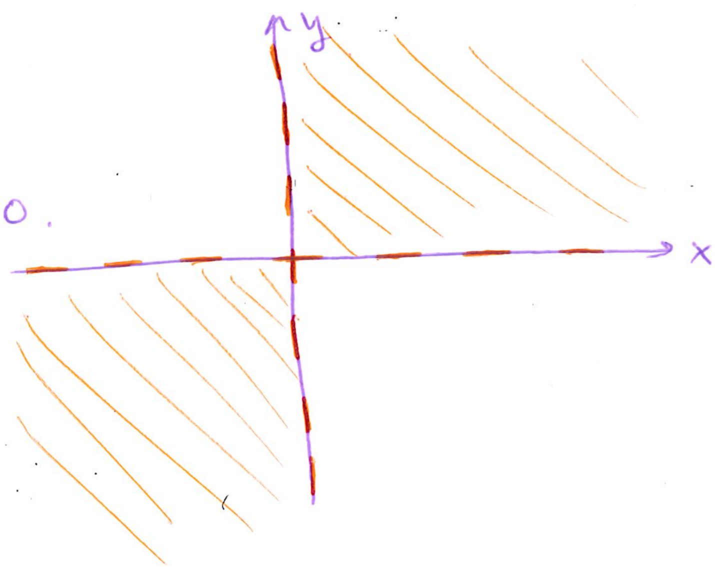
(b) $g(x, y) = \ln\left(\dfrac{x}{y}\right)$

Need $\dfrac{x}{y} > 0$

$\Leftrightarrow x>0, y>0$ or $x<0, y<0$.

$\therefore$ domain is

$\{(x,y) \mid \frac{x}{y} > 0\}$



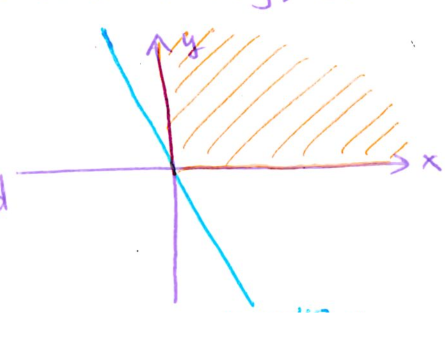
5. When x and y represent physical quantities, sometimes there are additional restrictions on the domain due to physical limitations of the quantities. In such a case we will refer to the result as the "physical domain" of the function.

If a company sells x units of good A and y units of good B, their revenue is $R(x, y) = 40x+22y$. What is the "physical domain" of $R(x, y)$?

x and y must both be non-negative and $R(x,y)$ should be non-negative. $\Rightarrow 40x+22y \geq 0$

$\Rightarrow \underbrace{y \geq -\frac{40}{22}x}_{\text{already guaranteed by insisting } x,y \geq 0.}$

$\boxed{\text{domain is } \{(x,y) \mid x \geq 0, y \geq 0\}}$



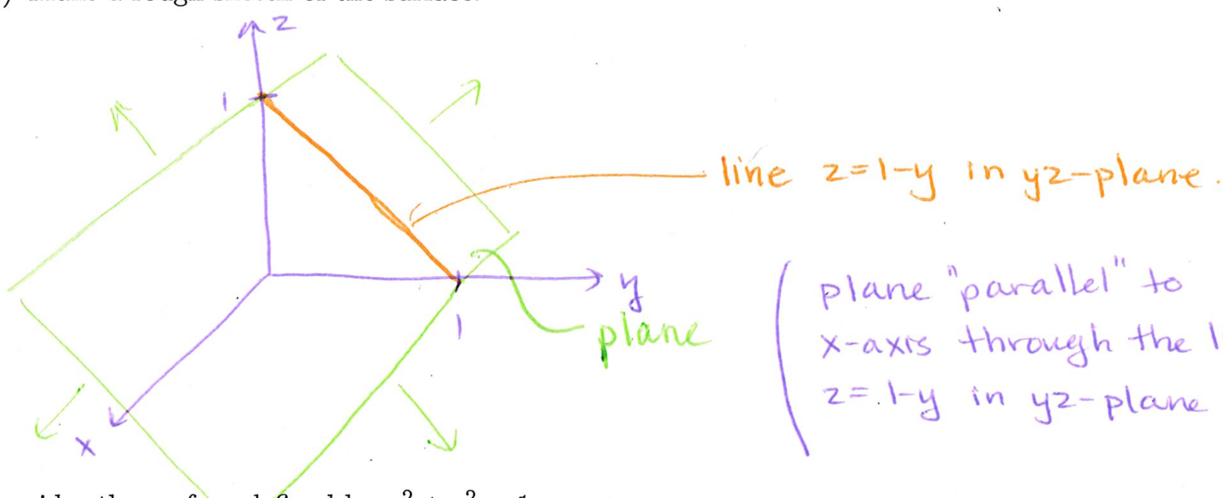
6. Consider the surface defined by $z = 1 - y$.

(a) What kind of surface is defined by the equation?

A plane! $\underbrace{z = 1-y}$ is a linear equation in x, y and z.

same as $z = 1 - y + 0 \cdot x$

(b) Make a rough sketch of the surface.



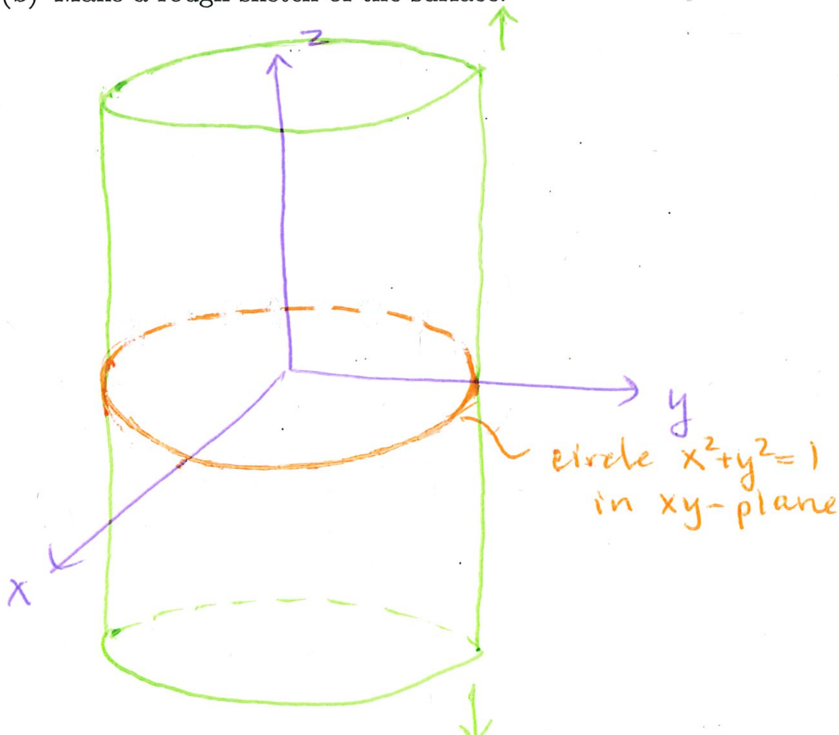
(plane "parallel" to x-axis through the line $z = 1-y$ in yz-plane)

7. Consider the surface defined by $x^2 + y^2 = 1$.

(a) What kind of surface is defined by the equation?

A circular cylinder!

(b) Make a rough sketch of the surface.



MATH 118
Contour plots and partial derivatives

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1. For each of the following functions, make a contour plot for $f(x, y)$ and use it to help you sketch the graph of the surface $z = f(x, y)$.

   (a) $f(x, y) = \sqrt{x^2 + y^2}$

Set $f(x,y) = k$: $\sqrt{x^2+y^2} = k \Rightarrow x^2+y^2 = k^2$

$k=0$: $x^2+y^2=0 \Rightarrow x=0, y=0$.

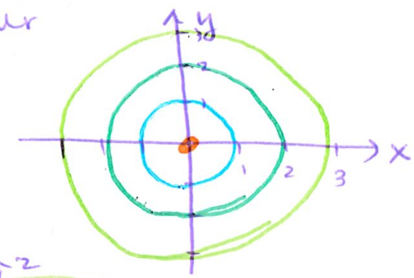
$k=1$: $x^2+y^2=1 \Rightarrow$ circle of radius 1

$k=2$: $x^2+y^2=4 \Rightarrow$ " 2

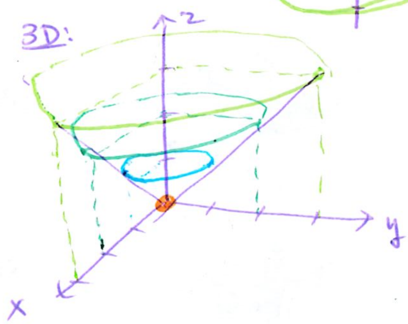
$k=3$: $x^2+y^2=9 \Rightarrow$ " 3

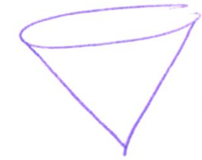
(no solution for $k<0$)

contour plot



3D:





CONE

   (b) $f(x, y) = x^2 + y^2$

Set $f(x,y) = k$: $x^2+y^2 = k$

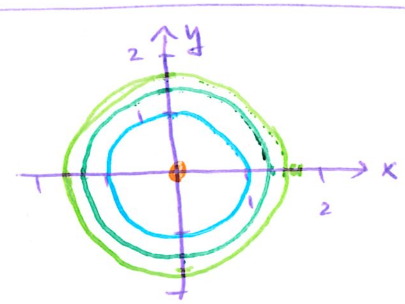
$k=0$: $x^2+y^2=0 \Rightarrow x=0, y=0$

$k=1$: $x^2+y^2=1 \Rightarrow$ circle or radius 1

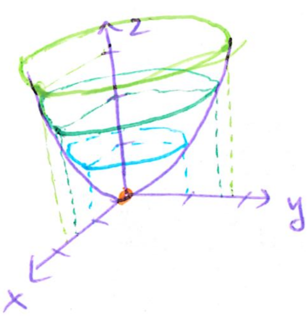
$k=2$: $x^2+y^2=2 \Rightarrow$ " $\sqrt{2} \sim 1.4$

$k=3$: $x^2+y^2=3 \Rightarrow$ " $\sqrt{3} \sim 1.7$

CONTOUR PLOT



3D:



PARABOLOID

2. The contour plots in question 1 are very similar. Write a sentence or two describing the difference in the corresponding surfaces.

The paraboloid gets steeper faster (level curves get closer together) whereas the cone gets steeper at a constant rate (level curves are equally spaced out

closer lever curves $\Rightarrow$ steeper surface

3. Match the following equations with their graphs and contour plots.

$f(x, y) = e^{x-y}$	Graph C	Contour E
$f(x, y) = e^{x}$	Graph F	Contour A
$f(x, y) = \cos(x - y)$	Graph D	Contour D
$f(x, y) = \ln(x^2 + y^2)$	Graph E	Contour F
$f(x, y) = \frac{x-y}{1+x^2+y^2}$	Graph B	Contour C
$f(x, y) = \cos(x^2 + y^2)$	Graph A	Contour B

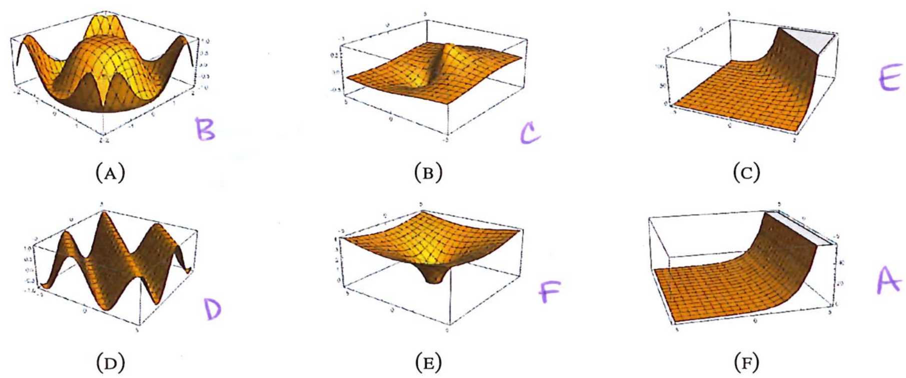


FIGURE 1. The graphs for problem

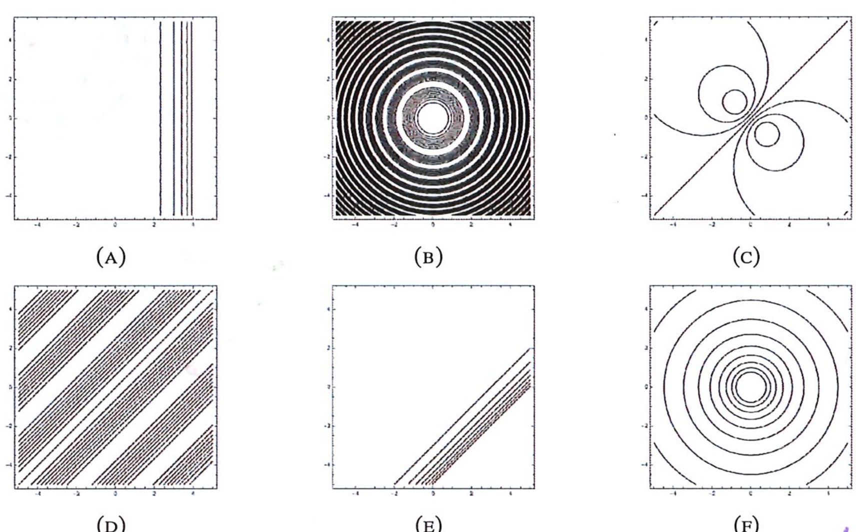


FIGURE 2. The contours for problem

(SEE ATTACHED PAGES FOR FULL SOLUTIONS)

STRATEGY: ① Visually match graphs to contour plots

② Make contour plots for functions and match to given contour plots.

③ Use ① and ② to match functions to graphs.

4. Calculate the partial derivatives of each of the following functions.

(a) $f(x, y) = 1 + x \sin(xy)$

$\frac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy)$

$\frac{\partial f}{\partial y} = x^2 \cos(xy)$

(b) $g(x, y, z) = \frac{z}{1 + x^2} + e^{xyz}$

$\frac{\partial g}{\partial x} = -z(1+x^2)^{-2} \cdot 2x + yze^{xyz} = -\frac{2xz}{(1+x^2)^2} + yze^{xyz}$

$\frac{\partial g}{\partial y} = xze^{xyz}$

$\frac{\partial g}{\partial z} = \frac{1}{1+x^2} + xye^{xyz}$

5. Recall your solutions to question 2.

(a) Use your contour plots to determine the sign of $f_x(1, 1)$ and $f_y(1, 1)$ for each function.

1(a)
$f_x(1,1) > 0$
$f_y(1,1) > 0$

(b)
$f_x(1,1) > 0$
$f_y(1,1) > 0$

(b) Do your answer to (a) agree with your sketches of the surfaces $z = f(x, y)$ in each case?

Yes, the surface slopes upward at (1,1) in the positive x and y-directions

(c) Now calculate $f_x(1, 1)$ and $f_y(1, 1)$ in each case and verify that the result agrees with your answers to the questions above.

(a) $f(x,y) = \sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \Rightarrow f_x(1,1) = \frac{1}{\sqrt{2}} > 0$

$\frac{\partial f}{\partial y} = \frac{y}{\ldots} \Rightarrow f_y(1,1) = \frac{1}{\ldots} > 0$

(b) $f(x,y) = x^2+y^2$

$\frac{\partial f}{\partial x} = 2x \Rightarrow f_x(1,1) = 2 > 0$

$\frac{\partial f}{\partial y} = 2y \Rightarrow f_y(1,1) = 2 > 0.$

$f(x,y) = e^{x-y}$

Contour plot: Set $e^{x-y} = k$ ($\Rightarrow k>0$, otherwise no solution).

$\Rightarrow x - y = \ln(k)$

$\Rightarrow y = x - \ln(k)$.

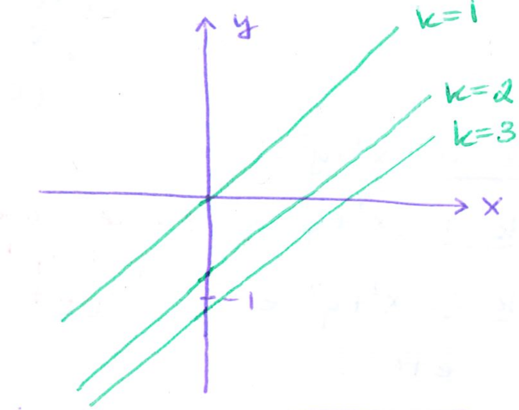
$k=1$: $y = x$ (line of slope 1 through $(0,0)$).

$k=2$: $y = x - \ln(2)$ (" through $(0, \underbrace{-\ln(2)}_{<0})$).

$k=3$: $y = x - \ln(3)$ (" $(0, \underbrace{-\ln(3)}_{<0})$)

etc...

$\Rightarrow$ all y-intercepts have $y \le 0$



looks like (E).

$f(x,y) = e^x$

Contour plot: Set $e^x = k$ ($\Rightarrow k>0$).

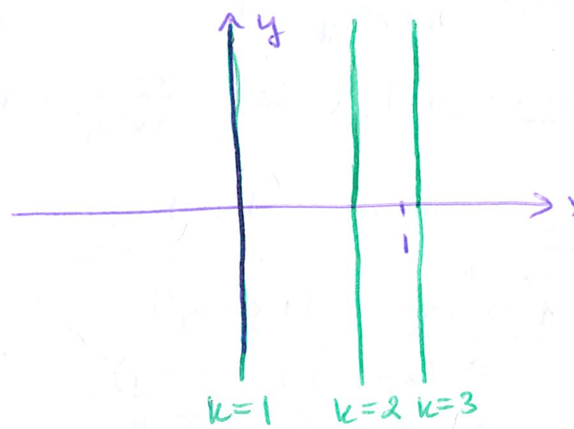
$\Rightarrow x = \ln(k)$.

$k=1$: $x = 0$ (vertical line)

$k=2$: $x = \ln(2)$ (")

$k=3$: $x = \ln(3)$ (")

etc...



looks like (A).

looks like (D)

$f(x,y) = \cos(x-y)$

Contour plot: Set $\cos(x-y) = k$

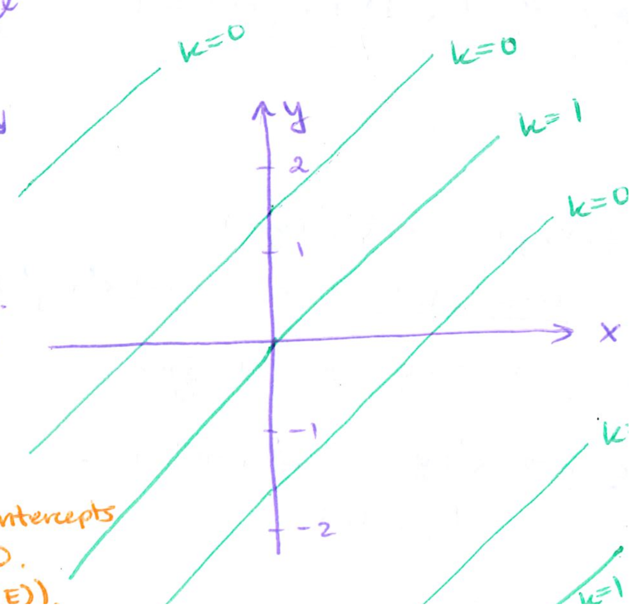
$k=0$: $\cos(x-y) = 0 \Rightarrow x - y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Rightarrow y = x \mp \frac{\pi}{2}, x \mp \frac{3\pi}{2}, x \mp \frac{5\pi}{2}, \ldots$

lines with slope 1 with y-int

$\mp\pi/2, \mp 3\pi/2, \mp 5\pi/2, \ldots$

$k=1$: $\cos(x-y) = 1 \Rightarrow y = x, x \pm 2\pi, x \pm 4\pi, \ldots$ } some y-intercepts are >0. (unlike (E)).



$f(x,y) = \ln(x^2+y^2)$

Contour plot: Set $\ln(x^2+y^2) = k$

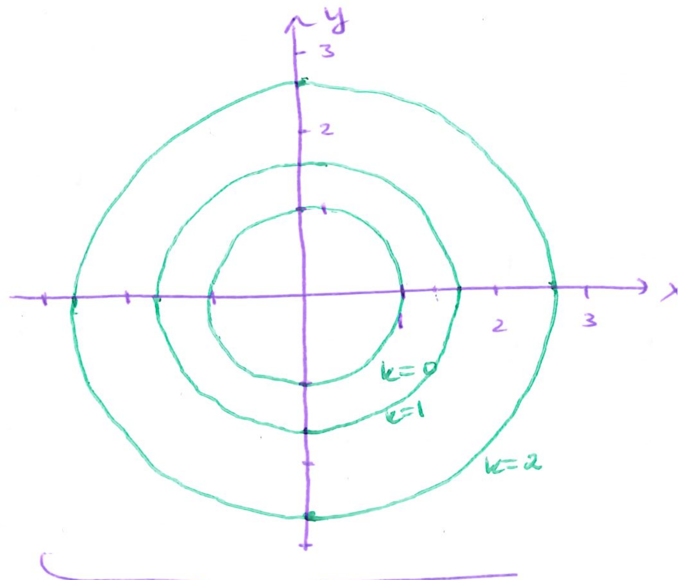
$\Rightarrow \underbrace{x^2+y^2 = e^k}_{\text{circle of radius } \sqrt{e^k} \text{ centred at } (0,0).}$

$k=0$: $x^2+y^2=\underline{1}$

$k=1$: $x^2+y^2=\underline{e}$ circles are getting further apart.

$k=2$: $x^2+y^2=\underline{e^2}$

etc...



(not (B)) looks like (F).

$f(x,y) = \dfrac{x-y}{1+x^2+y^2}$ (Easiest to do this one by process of elimination).

Contour plot: Set $\dfrac{x-y}{1+x^2+y^2} = k$.

$k=0 \Rightarrow x=y$ (line)

no solution for $k=\pm1, \pm2, \ldots$

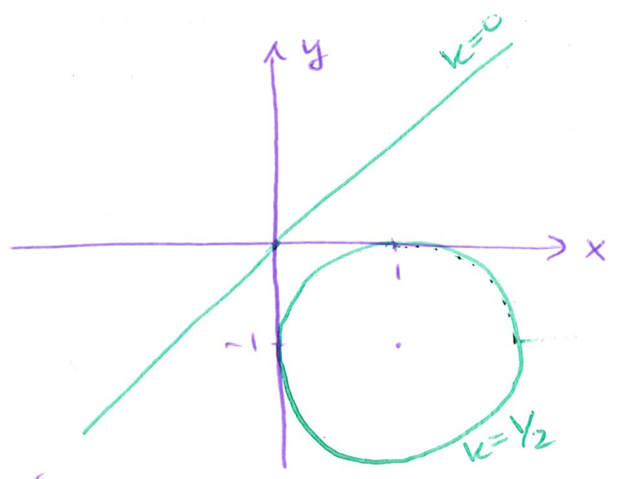
$k=\frac{1}{2} \Rightarrow \dfrac{x-y}{1+x^2+y^2} = \dfrac{1}{2}$

$\Rightarrow 2x-2y = 1+x^2+y^2.$

$\Rightarrow x^2-2x+y^2+2y+1=0$

complete the squares

$(x-1)^2+(y+1)^2=1.$ (circle centred at $(1,-1)$).



looks like (C)

$f(x,y) = \cos(x^2+y^2)$

Contour plot: Set $\cos(x^2+y^2) = k$.

$k=0 \Rightarrow x^2+y^2 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$k=1 \Rightarrow x^2+y^2 = 0, 2\pi, 4\pi, 6\pi \ldots$

(circles)

looks like (B)