Lecture 5 Derivatives

Stewart 14.1, McCallum 12.3, 12.5

Lecture 5. Key Ideas In differential calculus, we learned how to us derivatives to find tangent lines to a curve. A surface may have many tangent lines at a point, but there is only one tangent *plane*.

- understand and compute partial derivatives
- understand and compute the gradient of a function
- find tangent planes to a surface
- understand and compute second order partials
- understand and compute the Hessian of a funciton

Lecture 5.1 The partial derivative

Definition 5.1.

- The **partial derivative of** $f(x, y)$ **with respect to** x, denoted $f_x(x, y)$ or $\frac{\partial f}{\partial x}$ is the function gotten by holding y constant and differentiating with respect to x.
- The **partial derivative of** $f(x, y)$ **with respect to** y, denoted $f_y(x, y)$ or $\frac{\partial f}{\partial y}$ is the function gotten by holding x constant and differentiating with respect to y.

f_x is the IROC of f when we move in the dir of pos x-axis

- f_x is the slope of tangent at P parallel to the yz-plane
- f_y is the slope of tangent at P parallel to the xz-plane

Example 5.2. Find the partial derivatives of $f(x, y) = x^2 \sin(y) + x$.

$$f_x = \frac{\partial}{\partial x}\left(x^2 \sin y + x\right) = 2x \sin y + 1$$

$$f_y = \frac{\partial}{\partial y}\left(x^2 \sin y + x\right) = x^2 \cos y + 0$$

constant

Find a line tangent to $f(x,y)$ parallel to xz-plane at point $(2,0)$

$$f_y(2,0) = 2^2 \cos(0) = 4$$

$$f(2,0) = (2,0,2)$$

parallel to $\langle 0,1,4 \rangle$

IROC in z

change in y dir

$$L_y(t) = \langle 2,0,2 \rangle + t\langle 0,1,4 \rangle$$

Example 5.3. Find the partial derivatives of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

f_x = deriv obtained by holding y, z const and diff wrt x.

f_y = everything but y const

f_z = " " z const

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{x^2 + \boxed{y^2 + z^2}}\right) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} * 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

constant

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2y = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Definition 5.4. The **gradient** of a function $f(x, y)$, denoted $\nabla f(x, y)$ is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

e.g. $f(x,y) = e^{xy}$ $\quad f_x = ye^{xy}$ $\quad f_y = xe^{xy}$

$\nabla f = (ye^{xy}, xe^{xy})$

$\nabla f(0,0) = (0,0)$

$\nabla f(P)$ gives the direction of the steepest ROC of f.

$\|\nabla f(P)\|$ is magnitude of that change.

Example 5.5. Find the gradient of $x^2 \sin(y)$ at the point $(1, 0)$.

$$\nabla f(P) = \langle f_x(P), f_y(P) \rangle$$

$$f_x = \frac{\partial}{\partial x} x^2 \sin y = 2x \sin y$$

$$f_y = \frac{\partial}{\partial y} x^2 \sin y = x^2 \cos y$$

$$\nabla f(x, y) = \langle 2x \sin y, x^2 \cos y \rangle$$

$$\nabla f(1, 0) = \langle 2(1) \sin(0), (1)^2 \cos(0) \rangle = \langle 0, 1 \rangle$$

Example 5.6. Show that

> **Theorem 5.7.** The tangent plane to $f(x, y)$ at the point $P(x_0, y_0, z_0)$ has equation $z - z_0 = f_x(P)(x - x_0) + f_y(P)(y - y_0)$.

need a point and a normal

point

a plane tangent to f at $\boxed{P}$ has to contain all lines tangent to f at P.

tangent in y dir

tangent in x-dir

i.e. $\mathcal{L}_y(t) = \langle x_0, y_0, z_0\rangle + t\langle 0, 1, f_y\rangle$

$\mathcal{L}_x(t) = \langle x_0, y_0, z_0\rangle + t\langle 1, 0, f_x\rangle$

a perpendicular to this plane is $\begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} = \langle f_x, f_y, -1\rangle$

normal

$f_x(x - x_0) + f_y(y - y_0) - (z - z_0)$ $\blacksquare$

Example 5.8. Find the tangent plane to $f(x, y) = x^2 \sin(y) + x$ at $(x, y) = (1, 0)$

tangent plane has eqn

$$\boxed{z - z_0 = f_x(x - x_0) + f_y(y - y_0)}$$

$$z_0 = f(1,0) = 1^2 \sin(0) + 1 = 1$$

$$\boxed{P = (1,0,1)}$$

$$Df = \langle \underbrace{2x\sin y + 1}_{f_x}, \underbrace{x^2 \cos y}_{f_y} \rangle$$

$$Df(1,0) = \langle 2(1)\sin(0) + 1, (1)\cos(0) \rangle = \langle 1, 1 \rangle$$

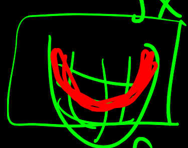
$$\text{normal} = \langle 1, 1, -1 \rangle$$

eqn. $z - 1 = (x - 1) + y \implies z = x + y$

Definition 5.9. A **second order partial derivative** of a function $f(x, y)$ is one that is a partial derivative of f_x or f_y. We have

$$f_{xx} = \frac{\partial}{\partial x} f_x \quad f_{yy} = \frac{\partial}{\partial y} f_y \quad f_{xy} = \frac{\partial}{\partial y} f_x \quad f_{yx} = \frac{\partial}{\partial x} f_y$$

fx represents the incr/decr of the intersection of f with a plane parallel to yz-plane



fy reps the incr/decr " " xz-plane

fxx reps concavity of intersection of f w/ plane parallel to yz-plane

fyy reps concavity of intersection of f w/ plane // to xz-plane

Example 5.10. Compute the second partials of $f(x, y) = x^2 \sin(y) + x$.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(2x \sin y + 1)$$

$$= 2 \sin y$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(2x \sin y + 1) = 2x \cos y$$

$$f_{yy} = \frac{\partial}{\partial y}(x^2 \cos y) = -x^2 \sin y$$

$$f_{yx} = \frac{\partial}{\partial x}(x^2 \cos y) = 2x \cos y$$

mixed partials $\quad f_{xy} = f_{yx}$

Theorem 5.11. If f_{xy} and f_{yx} are defined and continuous near a point P, then $f_{xy}(P) = f_{yx}(P)$.

useful if

$f_{xxxxxxy}$ of $f(x,y) = x^{100}$

$f_{xxxxxxy} = f_{yxxxxxx}$

$\left(\frac{\partial}{\partial x} f\right)_{xxxxxy}$

or by theorem

$\left(\frac{\partial}{\partial y} f\right)_{xxxxxx} = (0)_{xxxxxx} = 0$

Definition 5.12. The **Hessian** of a function $f(x,y)$ is

tells us if f CU or CD

$$\text{Hess}(f) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \boxed{f_{xx}f_{yy} - (f_{xy})^2}.$$

determinant

$f_{xx} > 0 \implies f$ is CU in x-dir

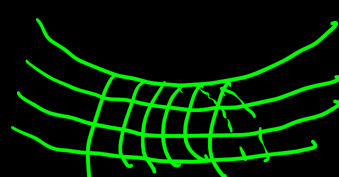
$f_{xx} < 0 \implies$ CD

$f_{yy} > 0 \implies f$ is CU in y-dir

$f_{yy} < 0 \implies f$ is CD in y-dir

$f(x,y) = x^2 - y^2$

if $f_{xx}f_{yy} < 0 \implies f_{xx}$ and f_{yy} have different sign



in one dir CU in other CD

Example 5.13. Find the Hessians of $f(x) = x^2 + y^2$ and $g(x) = x^2 - y^2$ and evaluate them both at $(x, y) = (0, 0)$.

geogebra

$f(x,y) = x^2 + y^2$

$f_x = 2x$

$f_{xx} = 2$

$f_y = 2y$

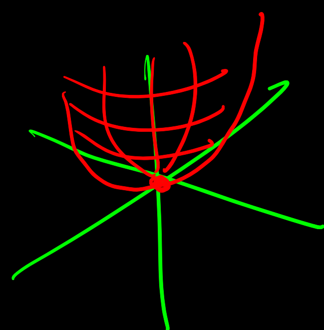
$f_{yy} = 2$

$f_{xy} = 0$

$f_{yx} = 0$

$Hess(f) = f_{xx} f_{yy} - (f_{xy})^2$

$= 2 * 2 = 4$



$g(x,y) = x^2 - y^2$

$f_x = 2x$

$f_y = -2y$

$f_{xx} = 2$

$f_{yy} = -2$

$f_{xy} = f_{yx} = 0$

$Hess(f) = f_{xx} f_{yy} - (f_{xy})^2$

$= 2 * (-2) - 0 = -4$