Question 19.2.5. How do we stretch, shrink, shift, or flip the graph of a function?

$y = f(x)$

$y = f(x) + b$ shift up of b units

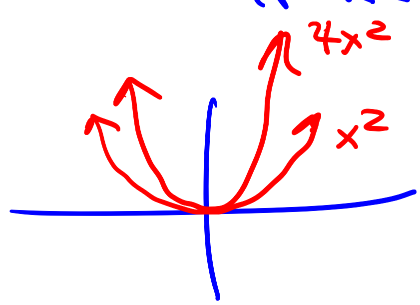
$y = f(x-a)$ shifts right a units

$y = kf(x) \Leftarrow$ vertical stretch if $k>1$
compression $k<1$

$y = f(kx) \leftarrow$ horizontal stretch if $k<1$
and compression if $k>1$

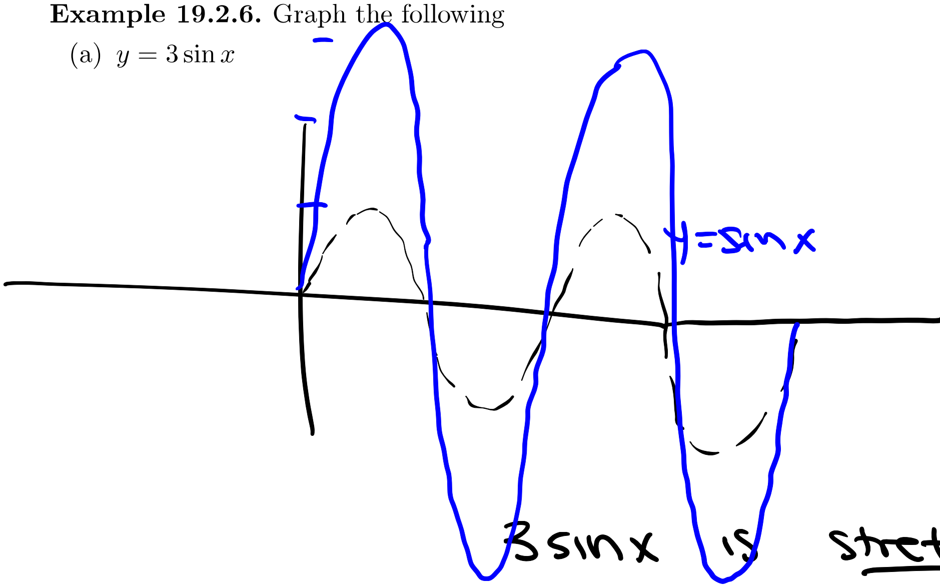
$y = x^2$

$(2x)^2 = 4x^2$



Example 19.2.6. Graph the following

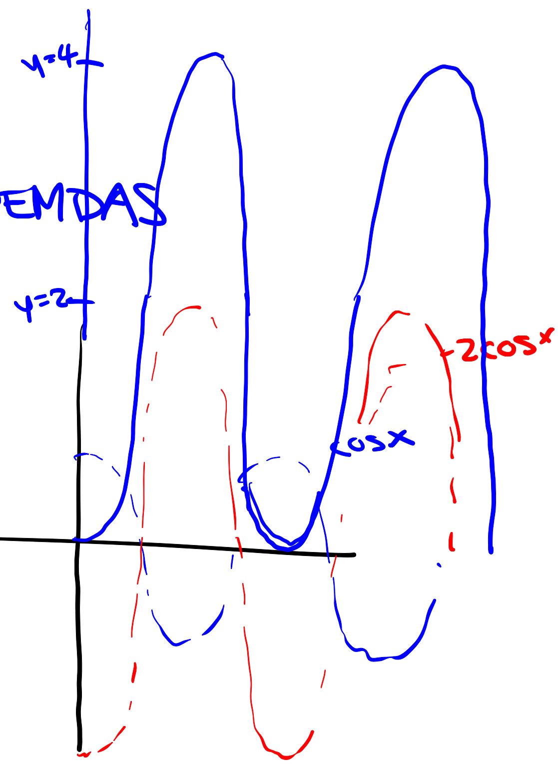
(a) $y = 3\sin x$



3sinx is stretched to have amp = 3

(b) $y = -2\cos x + 2$

PEMDAS



balance line
y=2
amplitude = 2
period = 2π

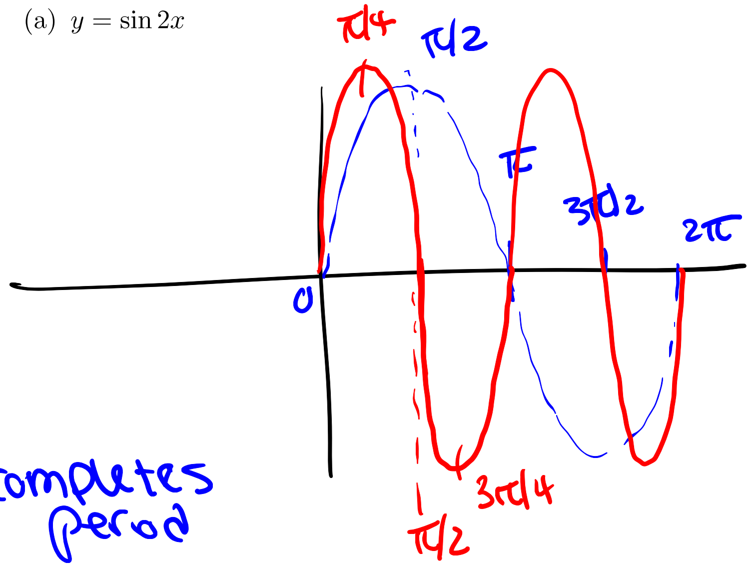
Observation 19.2.7.

- if $y = A\sin Bx + K$, or
- if $y = A\cos Bx + K$,

then $\begin{cases} \text{the balance value is } K \\ \text{the amplitude is } |A| \\ \text{the period is} \frac{2\pi}{|B|} \end{cases}$

Example 19.2.8. Graph the following

(a) $y = \sin 2x$



period

$\frac{2\pi}{2} = \pi$

completes period

$0 \leq 2x \leq 2\pi$

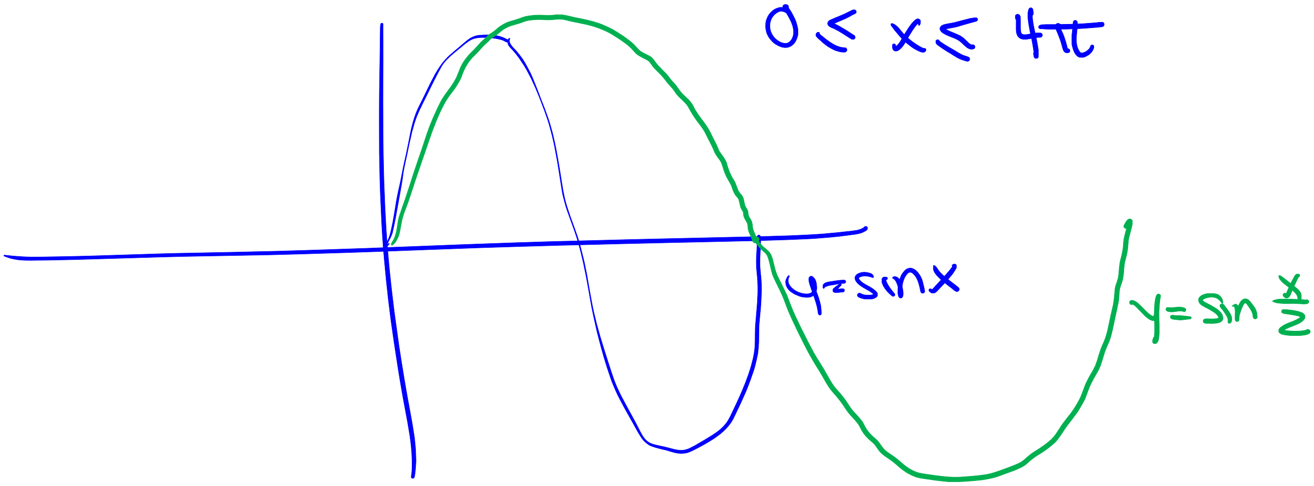
$0 \leq x \leq \pi$

$\sin \frac{x}{2}$ completes period in

$0 \leq \frac{x}{2} \leq 2\pi$

$0 \leq x \leq 4\pi$

(b) $y = \sin \frac{x}{2}$



period = 4π

Example 19.2.9. Graph the following

(a) $y = 5\cos(x + \pi/2)$

ELFS

(b) $y = \sin(2x - \pi) - 1$

ELFS

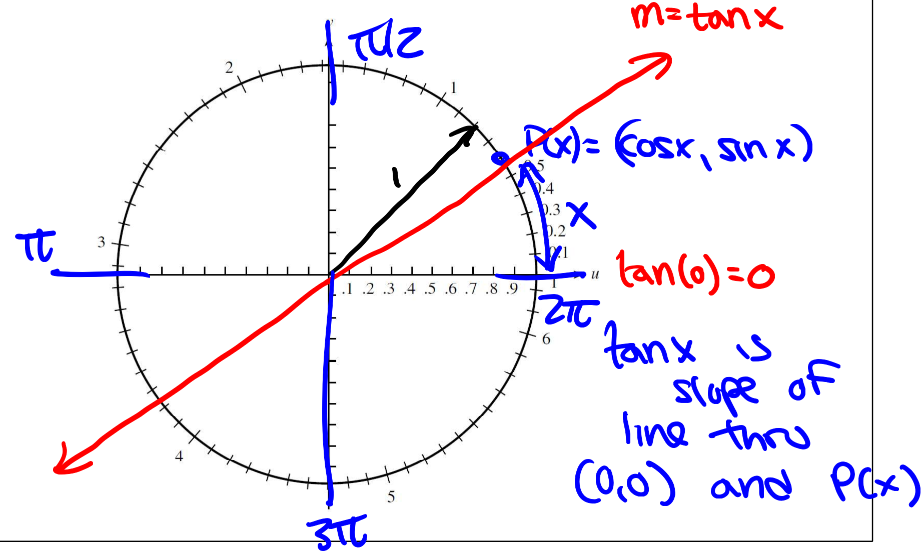
Observation 19.2.10.

- the graph of $A\sin\big(B(x-C)\big)$ is the graph of $A\sin Bx$ shifted C units to the right
- the graph of $A\cos\big(B(x-C)\big)$ is the graph of $A\cos Bx$ shifted C units to the right

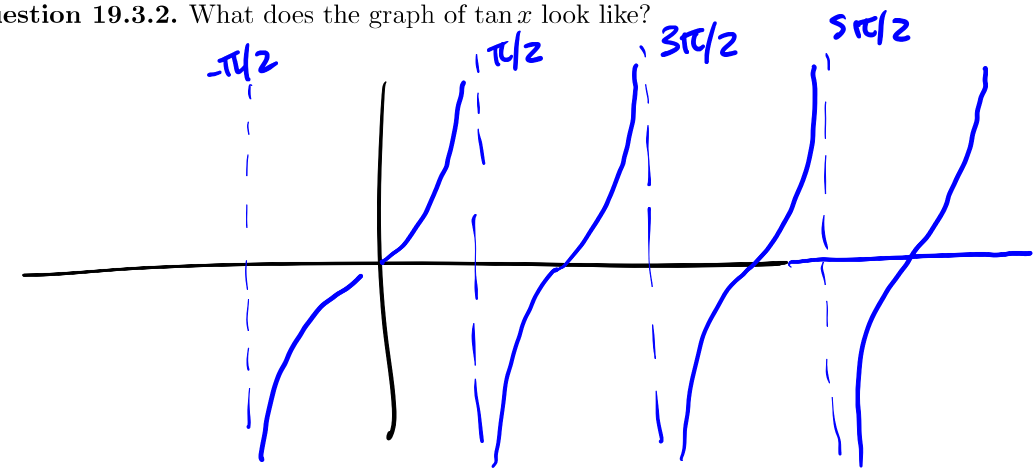
19.3 The Tangent Function

Observation 19.3.1. The function $\tan x$, called the tangent function, is defined as

$$\tan(x) = \frac{\sin x}{\cos x}$$



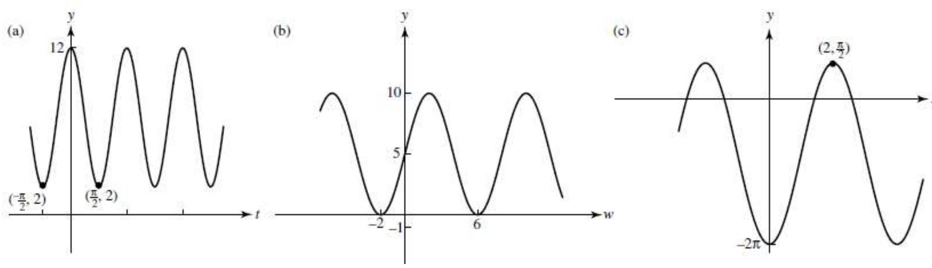
Question 19.3.2. What does the graph of $\tan x$ look like?



period = π

19.3.1 Extra examples

Example 19.3.3. Find equations for the following sinusoidal functions.



Example 19.3.4. A steadily spinning Ferris wheel with a radius of 10 meters makes one counter-clockwise revolution every 2 minutes. Placing the origin of a coordinate system at the center of the vertical wheel, consider the position of a seat that is at the point $(10, 0)$ at time $t = 0$.

(a) Plot the height of the seat as a function of t, t in minutes.

(b) Plot the horizontal position of the seat as a function of time, t in minutes.

(c) If the Ferris wheel slows down to one revolution every 3 minutes, what is the new equation?

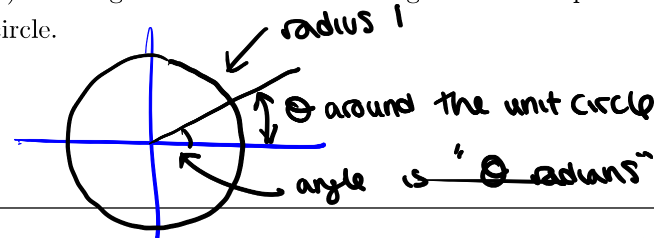
19.4 Angles and Arc Lengths

Goals

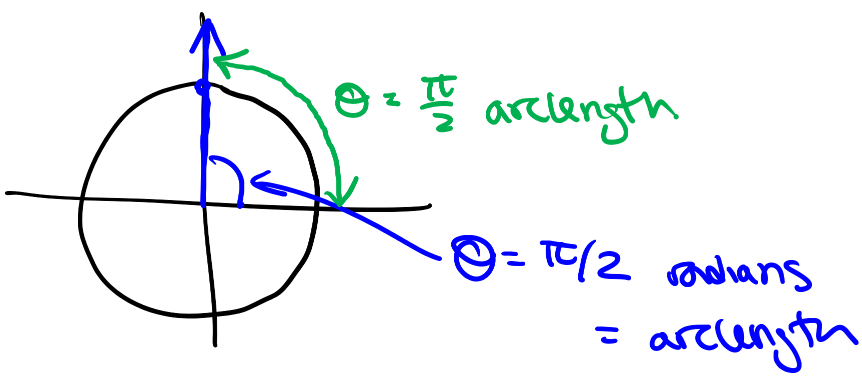
- a new way to talk about angles
- converting between degrees and radians

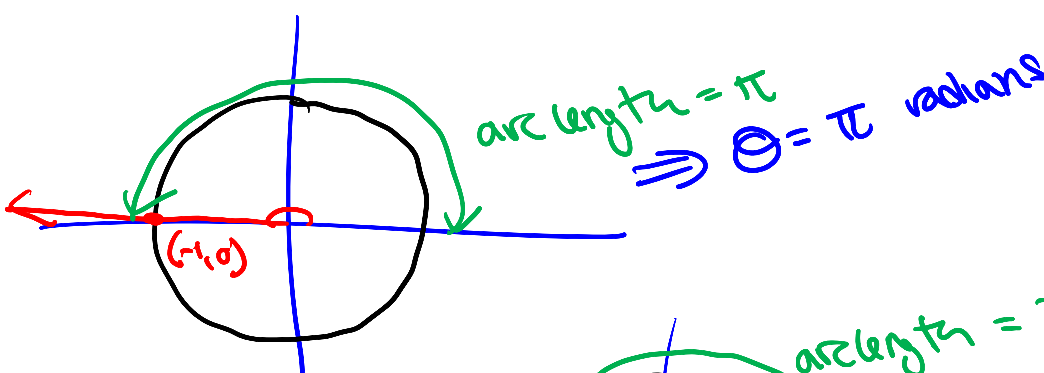
Definition 19.4.1 (Radians)**.** An angle of θ **radians** is the angle that corresponds to an arc length of θ on the unit circle.

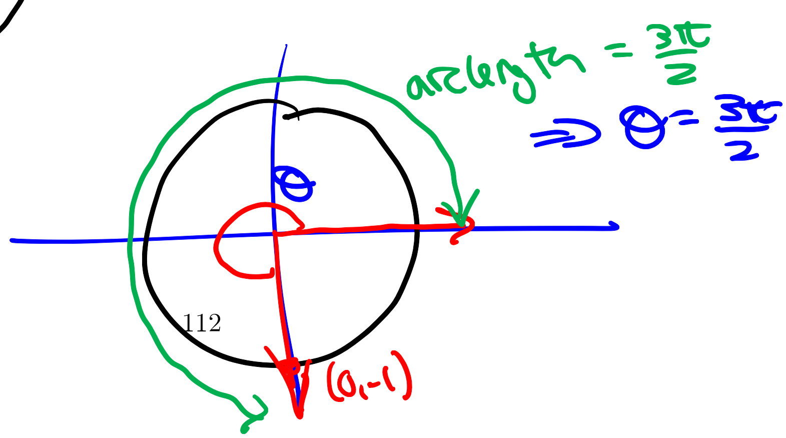
In English...

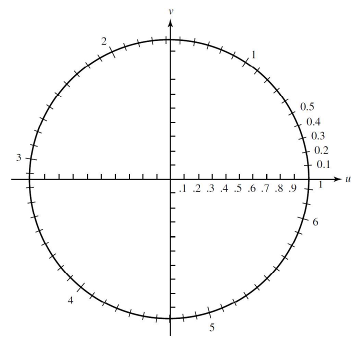


Example 19.4.2. Find radian measures for the angle associated to the points $(0, 1)$, $(-1, 0)$ and $(0, -1)$ on the unit circle.

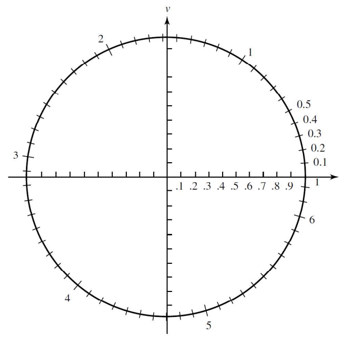




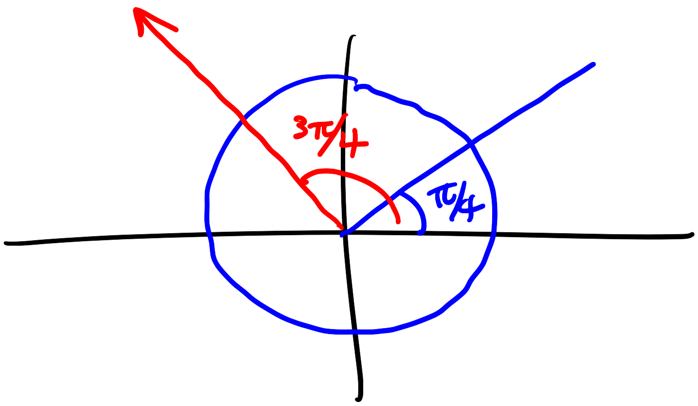




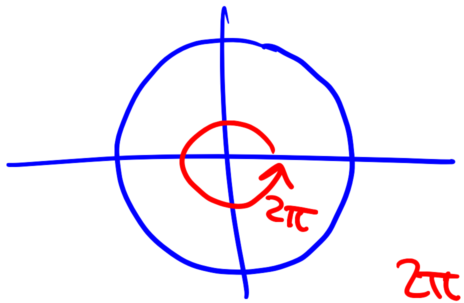
Example 19.4.3. Label an angle of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ on the unit circle below.



Lable an angle of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ on unit circle



Question 19.4.4. How many radians are there in a circle? How many radians are there in 180°?



2π radians = 360 degrees

Observation 19.4.5.

$$180° = \pi \text{ radians}$$

Example 19.4.6. Convert from degrees to radians or radians to degrees.

(a) $45°$

(b) $-30°$

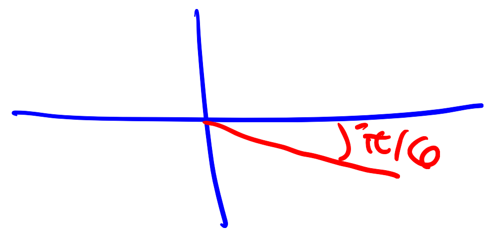
(c) $\frac{3\pi}{2}$ radians

(d) -2 radians

180 deg = π radians

a) $45\cancel{°} * \frac{\pi}{180\cancel{°}} = \frac{45}{180}\pi = \frac{\pi}{4}$

b) $-30\cancel{°} * \frac{\pi}{180\cancel{°}} = -\frac{\pi}{6}$



c) $\frac{3\cancel{\pi}}{2}\cancel{\text{rad}} * \frac{180°}{\cancel{\pi}\,\cancel{\text{rad}}} = 270°$

d) $-2\,\cancel{\text{rad}} * \frac{180°}{\pi\,\cancel{\text{rad}}} = \frac{-360°}{\pi} \approx -115°$

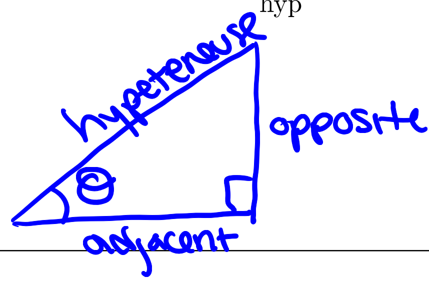
20.1 Right-Triangle Trigonometry

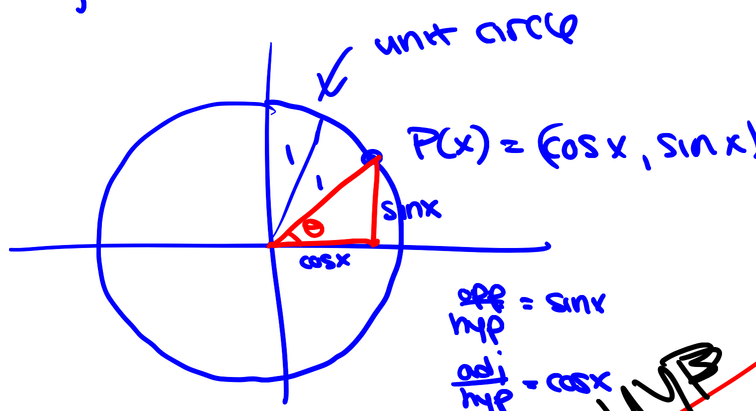
Goals

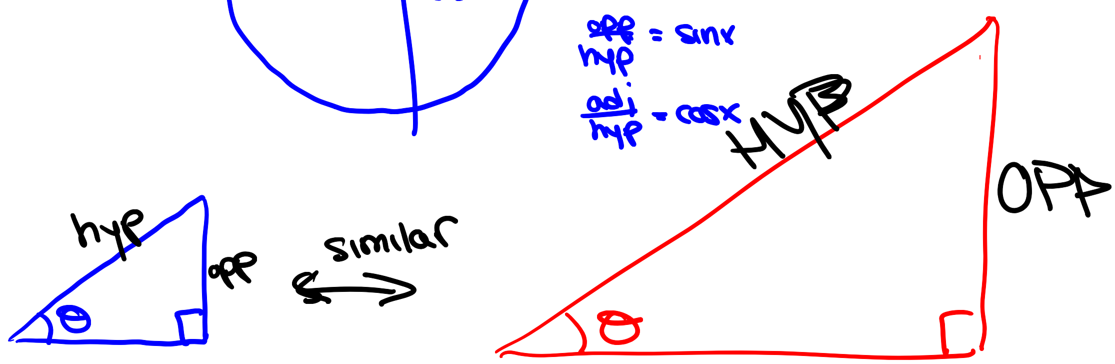
- compare definitions of trig functions
- word problems
- special triangles

Definition 20.1.1. Given the triangle drawn below,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$







similar Δ
$\Rightarrow$ sides are scaled

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\text{OPP}}{\text{HYP}}$$

Definition 20.1.2. We have three more trig functions to define.

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cancel{\tan\theta =} \frac{\text{adj}}{\text{opp}}$$

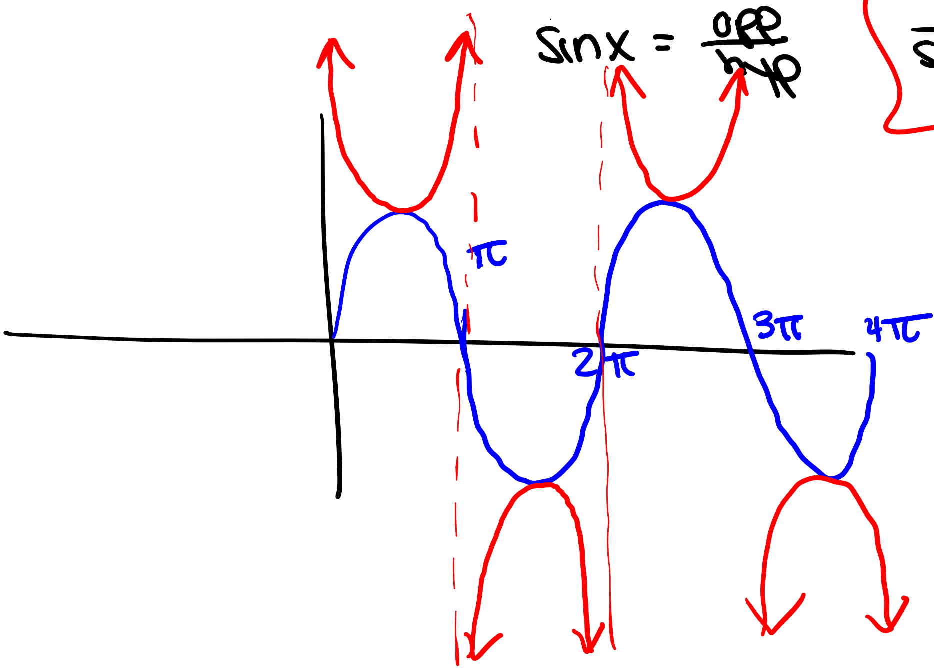
$\cot = \frac{\text{adj}}{\text{opp}}$

Example 20.1.3. Graph $\csc x$, $\sec x$ and $\cot x$.

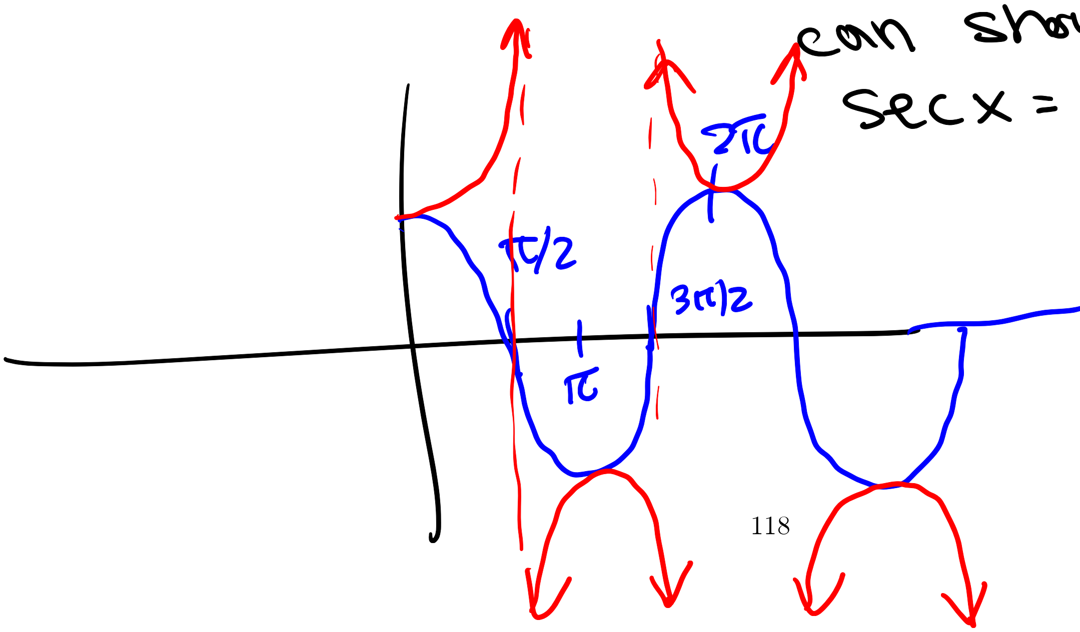
graph $y = \csc x = \frac{1}{\sin x}$

$\sin x = \frac{\text{opp}}{\text{hyp}}$

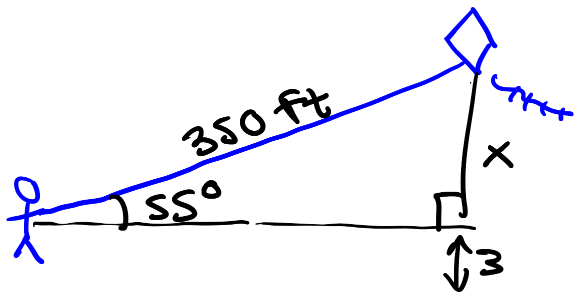
$\frac{1}{\sin x} = \frac{1}{\frac{\text{opp}}{\text{hyp}}} = \frac{\text{hyp}}{\text{opp}} = \csc x$



can show $\sec x = \frac{1}{\cos x}$



Example 20.1.4. A little girl flying a kite on a taut 350-foot string asks her father for the height of her kite. Her father estimates the angle of elevation of the kite to be 55°. Give an estimate for the height of the kite. Assume that the girl is holding the string 3 feet above the ground and her father is measuring the angle of elevation from this height.



$h = x + 3$

$$\sin(55) = \frac{x}{350} = \frac{\text{opp}}{\text{hyp}}$$

$$x = 350\sin(55) = 350 \times 0.82 \approx 290$$

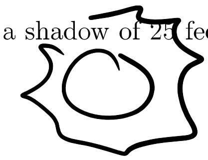
$x = 290$ ft

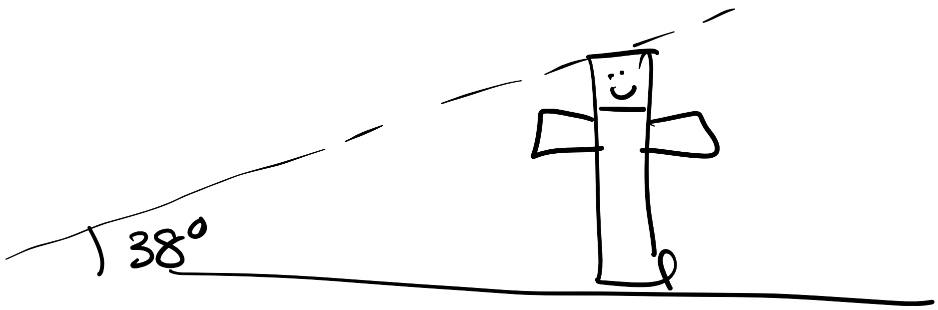
$h = x + 3 = 293$

Definition 20.1.5. **Angle of elevation** refers to the angle from the horizontal up to an object; **angle of depression** refers to the angle from the horizontal down to an object.

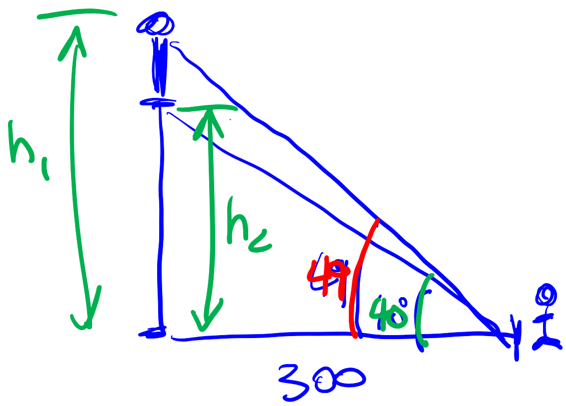
20.1.1 Extra Examples

Example 20.1.6. When the sun is 38° above the horizon, a totem pole casts a shadow of 25 feet. How tall is the totem pole?

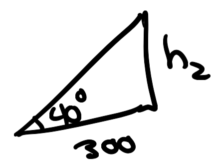




Example 20.1.7. A clock tower sits on top of a tall building. From a point 300 feet from the base of the building the angle of elevation to the base of the clock tower is 40° and the angle of elevation to the top of the tower is 49°. How tall is the clock tower?



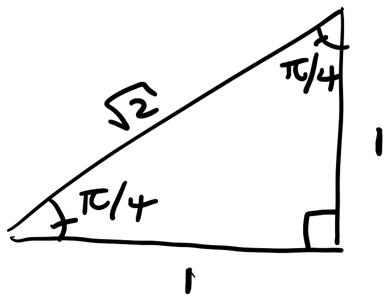
$h_1 =$

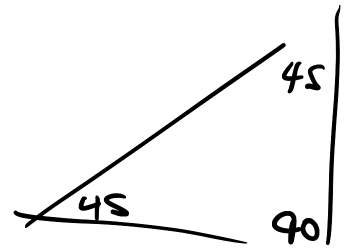


20.2 Special Triangles

find missing sides/angles

Example 20.2.1. Solve the triangle (find the rest of its sides and angles) and use it to find $\sin(\pi/4)$, $\cos(\pi/4)$ and $\tan(\pi/4)$.





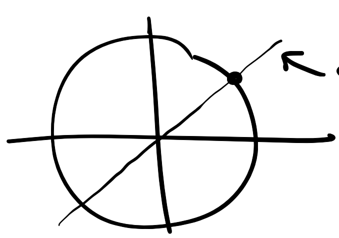
$1^2 + 1^2 = c^2$

$c = \sqrt{2}$

$\sin \pi/4 = \frac{opp}{hyp} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}} = \boxed{\frac{1}{\sqrt{2}}}$

$\cos \pi/4 = \frac{adj}{hyp} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan \pi/4 = \frac{\sin(\pi/4)}{\cos(\pi/4)} = 1 = \frac{opp}{adj} = \frac{1}{1}$



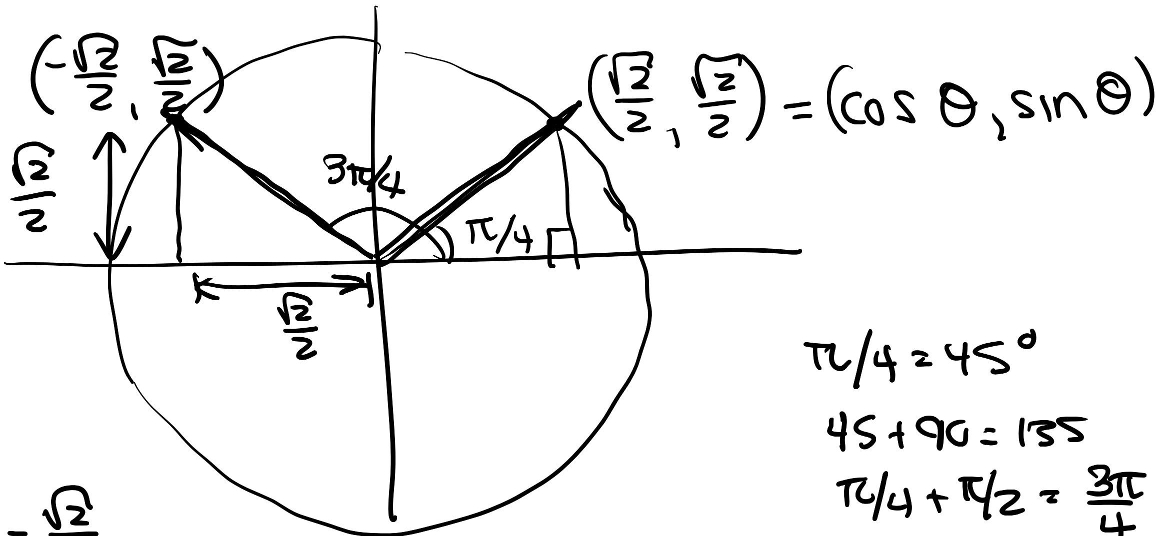
slope = 1
$\Rightarrow$ tangent 1

$\pi/4 - \pi/4 - \pi/2$ triangle

Observation 20.2.2 (45-45-90 Triangle).

$$\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2 \qquad \tan(\pi/4) = 1.$$

Remark 20.2.3. Now we can find sine, cosine, and tangent for any multiple of $\pi/4$.



$\pi/4 = 45^\circ$

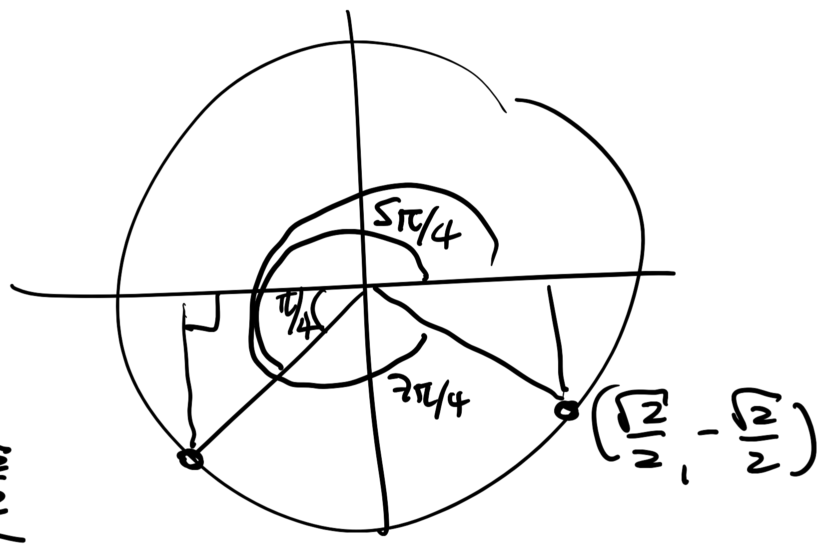
$45 + 90 = 135$

$\pi/4 + \pi/2 = \frac{3\pi}{4}$

$\cos(3\pi/4) = -\frac{\sqrt{2}}{2}$

$\sin(3\pi/4) = \frac{\sqrt{2}}{2}$

$\tan(3\pi/4) = -1$



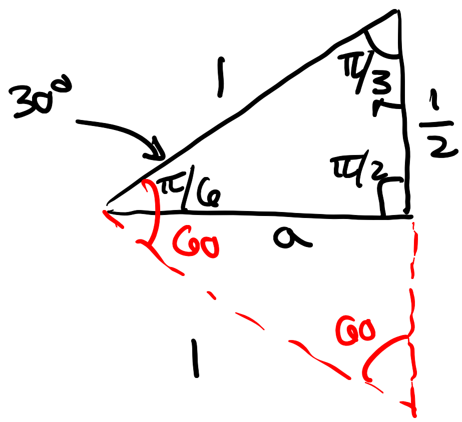
$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

Example 20.2.4. Solve the triangle to find sine, cosine, and tangent at $\theta = \pi/6$ and $\theta = \pi/3$.



$$a^2 + \left(\tfrac{1}{2}\right)^2 = 1$$

$$a^2 = 1 - \tfrac{1}{4}$$

$$a = \sqrt{3/4} = \frac{\sqrt{3}}{2}$$

$$\sin(\pi/6) = \frac{\text{opp}}{\text{hyp}} = \frac{1/2}{1} = \frac{1}{2}$$

$$\cos \pi/6 = \frac{\sqrt{3}}{2}$$

$$\sin(\pi/3) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\cos(\pi/3) = \frac{1}{2}$$

$$\tan \pi/6 = \frac{\sin \pi/6}{\cos \pi/6} = \frac{1/2}{\sqrt{3}/2} = \frac{\cancel{2}}{\cancel{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan \pi/3 = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

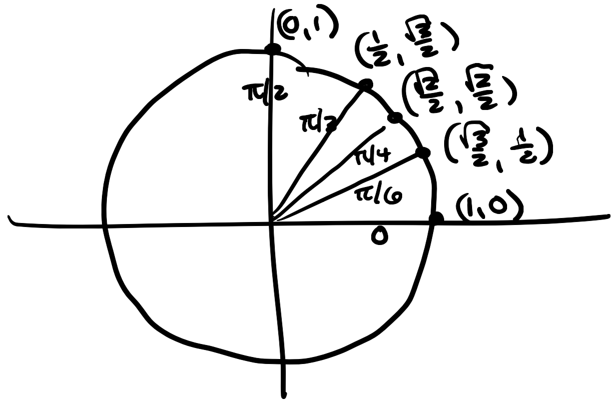
Observation 20.2.5 (30-60-90 Triangle)**.**

$$\sin(\pi/6) = \frac{1}{2} \qquad \cos(\pi/6) = \frac{\sqrt{3}}{2} \qquad \tan(\pi/6) = \frac{\sqrt{3}}{3}$$

$$\sin(\pi/3) = \frac{\sqrt{3}}{2} \qquad \cos(\pi/3) = \frac{1}{2} \qquad \tan(\pi/3) = \sqrt{3}$$

Remark 20.2.6. Now we can find sine, cosine, and tangent for any multiple of $\pi/6$.

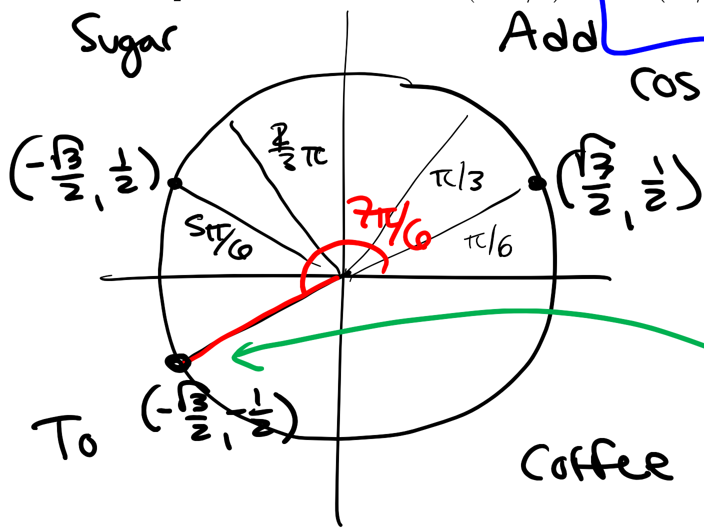
Observation 20.2.7 (Memory Aid).



θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

ELFS

Example 20.2.8. Evaluate $\cos(-5\pi/6)$ and $\sin(3\pi/4)$.



$$\cos\left(2\pi + -5\pi/6\right) = \cos\left(-5\pi/6\right)$$
$$= \cos\left(7\pi/6\right)$$

$$\boxed{\cos\left(x + 2\pi\right) = \cos x}$$

$$\cos\left(7\pi/6\right) = -\frac{\sqrt{3}}{2}$$
$$\cos\left(-5\pi/6\right)$$

A ← all +
S – sin pos
T – tangent pos
C – cos pos

20.2.1 Extra Examples

Example 20.2.9. You're interested in knowing the height of a very tall tree. You position yourself so that your line of sight to the top of the tree makes a 60° angle with the horizontal. You measure the distance from where you stand to the base of the tree to be 45 feet. How tall is the tree? (Assume that your eyes are five feet above the ground.)