Definition 2.3.15. Steps 1-4 above are called the **forward phase** of the row reduction algorithm. Step 5 is called the **backward phase.**

Example 2.3.16. Find the general solution of a linear system whose augmented matrix can be reduced to the matrix below.

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} \text{eqn 1} \\ \text{eqn 2} \\ \text{eqn 3} \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 \quad - 5x_3 = 1$

$x_2 + x_3 = 4$

$0 = 0$

already in RREF

pivot colm $\Rightarrow x_1, x_2$ basic

not pivot C $\Rightarrow$ "x_3 free

$x_1 = 1 + 5x_3$

$x_2 = 4 - x_3$

$x_3 =$ whatever

$x_3 = 0$

$x_1 = 1 + 5(0) = 1$

$x_2 = 4 - 0 = 4$

$x_3 = 1$

$x_1 = 6$

$x_2 = 3$

for <u>any</u> choice of x_3 we get unique x_1, x_2

> **Definition 2.3.17.** The variables corresponding to pivot columns of a matrix are called **basic variables**, the other variables are called **free variables**.

Remark 2.3.18. Whenever a system is consistent, the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

Example 2.3.19. Find the general solution of a system whose augmented matrix is reduced to

x_2, x_4 free

$$\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \iff \begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 = -4 \\ 2x_3 - 8x_4 - x_5 = 3 \\ x_5 = 7 \end{cases}$$

x_1 x_3 x_5

basic

$$\begin{cases} x_1 = -4 - 6x_2 - 2x_3 + 5x_4 + 2x_5 \\ x_2 = \text{free} \\ x_3 = (3 + 8x_4 + x_5)/2 \\ x_4 = \text{free} \\ x_5 = 7 \end{cases}$$

Example 2.3.20. Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + \ \ 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + \ \ 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

free

$\Rightarrow$ inf many solns

Suppose a system reduces to

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right] \leftarrow 0 = 8$$

NOT TRUE

x_2, x_4 - free

but

$\Rightarrow$ NO SOLN

Theorem 2.3.21. *A linear system is consistent if and only if the rightmost column of the augmented matrix is* not *a pivot column. If a linear system is consistent, then the solution set contains either*

(i) a unique solution, where there are no free variables, or

(ii) infinitely many solutions, when there is at least one free variable.

Using the theorem, and the rest of this section, we have the following procedure to find and describe all the solutions of a linear system.

Procedure 2.3.22 (Using Row Reduction to Solve a Linear System)**.**

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to write the matrix in echelon form. If the system is inconsistent, stop, there are no solutions; otherwise, go to the next step.
3. Use the row reduction algorithm to write the matrix in reduced echelon form.
4. Write the system of equations corresponding to the reduced matrix.
5. Solve each basic variable in terms of any free variables.

2.4 Rules for matrix operations

2.4. Key Ideas

- The (i, j) entry of AB is the dot product of row i of A with column j of B.
- An $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix, and uses mnp separate multiplications.
- $A(BC) = (AB)C$, but $AB \neq BA$ in general

main diag

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Definition 2.4.1. If A is an $m \times n$ matrix (m rows and n columns), then the entry in the ith row and jth column of A, typically denoted a_{ij}, is called the (i, j)**-entry** of A. We write $A = [a_{ij}]$ using this notation. Columns of A are vectors in $\mathbb{R}^m$, usually denoted $\mathbf{a}_1, \ldots \mathbf{a}_n$. We often write

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The **main diagonal** of $A = [a_{ij}]$ is the entries $a_{11}, a_{22}, a_{33}, \ldots$. A **zero matrix** is one whose entries are all zero. The **identity matrix** is a $n \times n$ square matrix with ones on the main diagonal and zeros everywhere else, usually denoted I_n

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 2.4.2. Two matrices are **equal** if they have the same size and their corresponding entires are equal. If A and B are matrices of the same size, then the **sum** $A + B$ is the matrix whose entries are the sums of the corresponding entries in A and B.

Example 2.4.3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 5 & -6 \end{bmatrix}$.

Find $A + B$, $B + A$, and $\boxed{A + C.}$ $\longleftarrow$ not defined

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} x & x+y \\ x+2y & 4y \end{bmatrix} \iff \begin{aligned} x &= 1 \\ x+y &= 2 \\ x+2y &= 3 \\ 4y &= 4 \end{aligned}$$

$$A+B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+5 & 3+6 \\ -4+7 & 5+(-8) & -6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 5+2 & 6+3 \\ * & * & * \end{bmatrix} = B+A$$

ANALOGOUS TO VECTOR

Definition 2.4.4. If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose entries are r times the corresponding entries of A. Notationally, $-A$ stands for $(-1)A$, and $A - B = A + (-1)B$.

Example 2.4.5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $2B$ and $A - (2B)$.

$$2B = 2\begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 & 2\cdot 5 & 2\cdot 6 \\ 2\cdot 7 & 2(-8) & 2\cdot 9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 & 12 \\ 14 & -16 & 18 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 1 - 2\cdot 4 & 2 - 2\cdot 5 & 3 - 2\cdot 6 \\ * & * & * \end{bmatrix}$$

$$2(A+B) \stackrel{?}{=} 2A + 2B$$

$$= \begin{bmatrix} 2(a_{11}+b_{11}) & * \\ * & * \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2a_{11} + 2b_{11} & * \\ * & * \end{bmatrix}$$

Theorem 2.4.6. *Let A, B, and C be matrices of the same size, and r and s be scalars.*

a. $A + B = B + A$	*d.* $r(A + B) = rA + rB$
b. $(A + B) + C = A + (B + C)$	*e.* $(r + s)A = rA + sA$
c. $A + 0 = A$.	*f.* $r(sA) = (rs)A$.

Definition 2.4.7. If A is an $m \times n$ matrix, and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \cdots & A\mathbf{b}_p \end{bmatrix}.$$

Remark 2.4.8. If the number of columns of A doesn't match the number of rows of B, then the product AB is *undefined.*

Example 2.4.9. Compute AB and BA, when $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -8 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ 2 & -8 & 3 \end{bmatrix} = \begin{bmatrix} \overset{A\vec{b}_1}{\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}} & \overset{A\vec{b}_2}{\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -8 \end{bmatrix}} & \overset{A\vec{b}_3}{\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \end{bmatrix}$$

vector in $\mathbb{R}^2$

$$= \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 6+2 \\ -9+8 \end{bmatrix} & \begin{bmatrix} 10-8 \\ -15-32 \end{bmatrix} & \begin{bmatrix} 2+3 \\ -3+12 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 & 2 & 5 \\ -1 & -47 & 9 \end{bmatrix}$$

ANOTHER WAY $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$

$$\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ 2 & -8 & 3 \end{bmatrix} = \begin{bmatrix} 6+2 & 10-8 & 2+3 \\ -9+8 & -15-32 & -3+12 \end{bmatrix}$$

(i,j)-entry of AB is ith row of A dotted with jth column of B

dot product!

Procedure 2.4.10 (Row-Column Rule for AB)**.** If the product AB is defined, then the (i, j)-entry ~~of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB,~~ and A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

Example 2.4.11. With A and B from Example 2.4, compute AB using the row-column rule.

Theorem 2.4.12. *Let A be an $m \times n$ matrix, and let B and C have the right sizes so that the following sums and products are defined.*

a. $A(BC) = (AB)C$

b. $A(B + C) = AB + AC$

c. $(B + C)A = BA + CA$.

d. $r(AB) = (rA)B = A(rB)$ (for any scalar r)

e. $I_m A = A = A I_n$

Example 2.4.13. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$.

(a) Find AB and BA.

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 16-15 & 8-15 \\ * & * \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 0 \\ -10 & 15 \end{bmatrix}$$

in general $AB \neq BA$

(b) Find AC.

$$AC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$AB = AC$ but $B \neq C$

(c) Find AD.

$AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $A \neq 0$ and $D \neq 0$

$y = z \Leftrightarrow xy = xz$

$B = C \Rightarrow AB = AC$

Watchout! 2.4.14. Here are some important warnings for matrix multiplication:

1. In general, $AB \neq BA$.
2. Cancellation laws *do not hold* for multiplication; $CA = CB$ (or $AC = BC$) does not mean $A = B$.
3. If $AB = 0$, this *does not mean* $A = 0$ or $B = 0$.

Definition 2.4.15. If A is an $m \times n$ matrix, the **transpose** of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A.

Example 2.4.16. Let $A = \begin{bmatrix} a & b & d \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \\ 6 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 3 & 1 & 2 & -6 \end{bmatrix}$.

Find A^T, B^T, and C^T.

every number x has an inverse $y = x^{-1}$ s.t. $xy = 1$

e.g. $5 \cdot \frac{1}{5} = 1$

$\pi \cdot \frac{1}{\pi} = 1$

2.5 Inverse matrices

2.5. Key Ideas

- The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.
- A is invertible if and only if it has n pivots
- If $A\mathbf{x} = 0$ for a nonzero vector $\mathbf{x}$, then A has no inverse
- The inverse of AB is $B^{-1}A^{-1}$, and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
- Reducing $\begin{bmatrix} A & I \end{bmatrix}$ to reduced row echelon form gives $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

Definition 2.5.1. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I, \text{ where } I = I_n \text{ is the identity matrix.}$$

In this case, C is called the **inverse** of A. A matrix that is *not* invertible is called a **singular matrix**, and an invertible matrix is called a **non-singular matrix**.

Remark 2.5.2. Suppose B and C were both inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

It turns out, that if A has an inverse, it's unique. We call this unique inverse A^{-1}.

Example 2.5.3. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Show that $C = A^{-1}$

check $AC = I$

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} * \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A = C^{-1}$ $C = A^{-1}$

$x x^{-1} = 1$

$x \frac{1}{x} = 1$

Theorem 2.5.4. *Invertible matrices have the following three properties.*

1. *If A is an invertible matrix, then A^{-1} is invertible, and $(A^{-1})^{-1} = A$.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. ~~*If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*~~

switches order

Theorem 2.5.5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If $ad - bc \neq 0$, then A is invertible and*

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Example 2.5.6. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

a b c d

$$ad - bc = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Check

$$AA^{-1} = I$$

$$3x = 4$$

$$x = 1 \cdot x = 3^{-1} 3x = 3^{-1} \cdot 4 = \frac{4}{3}$$

$$x = \frac{4}{3}$$

$A\vec{x} = \vec{b}$ and A has an inverse,

then $A^{-1}A\vec{x} = A^{-1}\vec{b}$

$$\boxed{\vec{x} = I\vec{x} = A^{-1}\vec{b}}$$

google "reshish"

$4x = 3 \Leftrightarrow x = \frac{3}{4}$

Theorem 2.5.7. *If A ~~is an inver~~tible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$.*

Example 2.5.8. Solve the system

$$x_1 + 2x_2 = 1$$
$$3x_1 + 4x_2 = 2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

if A invertible then $\vec{x} = A^{-1}\vec{b}$

check $ad - bc \neq 0$

$4 \cdot 1 - 2 \cdot 3 = -2$ ✓

$\Rightarrow$ invertible

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

i.e. $\vec{x} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$