4.5 # 1, 4, 8, 11, 14, 21, 23, 26, 29

1.) For each subspace, a) find a basis for the subspace b) State the dimension

$$\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s,t \text{ in } \mathbb{R} \right\}$$ This subspace is the span of $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Since S is linearly independent, S is a basis for the subspace.
The dimension of the subspace is 2.

4.) $$\left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p,q \in \mathbb{R} \right\}$$ This subspace is the span of $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Since S is linearly independent, S is a basis for the subspace.
The dimension of the subspace is 2.

8.) $\{(a,b,c,d) : a-3b+c=0\}$

This subspace is the span of

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3b-c \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ No free variables

Since this has only the trivial solution, S is linearly independent.
The subspace has dimension 3.

11.) Find the dimension of the subspace spanned by the vectors

$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix}, \begin{bmatrix}5\\2\\2\end{bmatrix}$ $\quad \begin{bmatrix}1&3&-2&5\\0&1&-1&2\\2&1&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}$

We are asked to find the dimension of the column space. Since columns 1, 2 & 4 are pivot columns, $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}5\\2\\2\end{bmatrix}\right\}$ is a basis for this space so the space has dimension 3.

14.) Determine the dimensions of Nul A and Col A.

$A = \begin{bmatrix}1&2&-4&3&-2&6&0\\0&0&0&1&0&-3&7\\0&0&0&0&1&4&-2\\0&0&0&0&0&0&1\end{bmatrix}$

There are four pivot columns, so the dimension of Col A is 4. There are 3 free variables, so the dimension of Nul A is 3.

21.) The first four Hermite polynomials are $1, 2t, -2+4t^2, -12t+8t^3$

Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

The standard basis of $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$.

$A = \begin{bmatrix}1&0&-2&0\\0&2&0&-12\\0&0&4&0\\0&0&0&8\end{bmatrix}$. The columns of A are coordinate vectors, Since there are 4 pivots, the columns are linearly independent. Since there are 4 and $\dim \mathbb{P}_3 = 4$, they form a basis.

23.) B is the basis of $\mathbb{P}_3$ consisting of Hermite polynomials from exercise 21.

Let $\vec{p}(t) = -1 + 8t^2 + 8t^3$. Find the coordinate vector of p relative to B.

$\left[\begin{array}{cccc|c}1&0&-2&0&-1\\0&2&0&-12&0\\0&0&4&0&8\\0&0&0&8&8\end{array}\right] \begin{array}{l}\\R_2/2\\R_3/4\\R_4/8\end{array} \left[\begin{array}{cccc|c}1&0&-2&0&-1\\0&1&0&6&0\\0&0&1&0&2\\0&0&0&1&1\end{array}\right] \begin{array}{l}2R_3+R_1\\6R_4+R_2\\\\\\\end{array} \left[\begin{array}{cccc|c}1&0&0&0&3\\0&1&0&0&6\\0&0&1&0&2\\0&0&0&1&1\end{array}\right]$

$[\vec{p}]_B = \begin{bmatrix}3\\6\\2\\1\end{bmatrix}$

4.5 continued

26.) Let H be an n dimensional subspace of an n dimensional vector space V. Show that $H=V$.

Case 1: $n=0$. If $\dim H = 0$ then $H=\{\vec{0}\}$ and similarly since $\dim V = 0$, $V=\{\vec{0}\}$. So $H=V=\{\vec{0}\}$.

Case 2: $n>0$. Suppose $S=\{\vec{v}_1,\ldots,\vec{v}_n\}$ is a basis for H. Since S is a linearly independent set in V with exactly n vectors, it is a basis for V. Therefore $H=V=\text{Span } S$.

29.) a.) If there exists a set $\{\vec{v}_1,\ldots,\vec{v}_p\}$ that spans V, then $\dim V \leq p$.

b.) If there exists a linearly independent set $\{\vec{v}_1,\ldots\vec{v}_p\}$ in V, then $\dim V \geq p$.

c.) If $\dim V = p$, then there exists a spanning set of $p+1$ vectors in V.

a.) True

b.) True

c.) True