7.2 # 1, 5, 8, 11, 13, 19, 21, 27

1) Compute the quadratic form $\vec{x}^T A \vec{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

a.) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5x_1 + \frac{1}{3}x_2 \\ \frac{1}{3}x_1 + x_2 \end{bmatrix} =$

$$= x_1(5x_1 + \tfrac{1}{3}x_2) + x_2(\tfrac{1}{3}x_1 + x_2) = 5x_1^2 + (2/3)x_1x_2 + x_2^2$$

b.) $\vec{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ $\quad \vec{x}^T A \vec{x} = 5(6)^2 + (2/3)(6)(1) + 1^2 = 185$

c.) $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{x}^T A \vec{x} = 5(1^2) + (2/3)(1)(3) + (3^2) = 16$

5) Find the matrix of the quadratic form. Assume $\vec{x}$ is in $\mathbb{R}^3$.

a.) $\underbrace{8x_1^2 + 7x_2^2 - 3x_3^2}_{\text{on diagonal}} - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$ $\quad \begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$

½ in a_{12}, a_{21} positions

b.) $4x_1x_2 + 6x_1x_3 - 8x_2x_3$ $\quad \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{bmatrix}$

8.) Let A be the matrix of the quadratic form $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$. The eigenvalues of A are 3, 9, 15. Find an orthogonal matrix P s.t. the change of variable $\vec{x} = P\vec{y}$ transforms $\vec{x}^T A \vec{x}$ into a quadratic form with no cross product term. Give P and the new quadratic form.

$A = \begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}$ $\quad \lambda = 3 \quad A - \lambda I = \begin{bmatrix} 6 & -4 & 4 \\ -4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} x_3$

Normalize: $\begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

8.) continued

$\lambda = 9$ $\quad A - \lambda I = \begin{bmatrix} 0 & -4 & 4 \\ -4 & -2 & 0 \\ 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3$

Normalize: $\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$\lambda = 15$ $\quad A - \lambda I = \begin{bmatrix} -6 & -4 & 4 \\ -4 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} x_3$

Normalize: $\begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ Then $A = PDP^{-1}$ where $P = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$

and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix}$.

this is orthogonal because A is symmetric which implies $P^{-1} = P^T$

The change of variable is $\vec{x} = P\vec{y}$ and so the quadratic form

is $\vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T \underbrace{P^T A P}\vec{y} = \vec{y}^T D \vec{y} =$

$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 + 9y_2^2 + 15y_3^2$

7.2 continued

11.) Classify the quadratic form. Then make a change of variable $\vec{x} = P\vec{y}$ that transforms the quadratic form into one with no cross product term. Write the new quadratic form. Construct P using methods from section 7.1.

$2x_1^2 + 10x_1x_2 + 2x_2^2$ the matrix for this quadratic form is $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

The eigenvalues are: $\det(A - \lambda I) = (2-\lambda)(2-\lambda) - 25 = 0$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$(\lambda - 7)(\lambda + 3) = 0 \qquad \lambda = -3, 7$$

Since the eigenvalues are both positive and negative, the quadratic form is $\boxed{\text{indefinite}}$.

Next, we orthogonally diagonalize A.   Basis for eigenspace

$\lambda = 7 \quad A - \lambda I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 = x_2$, x_2 free $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\lambda = -3 \quad A - \lambda I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 = -x_2$, x_2 free $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

We normalize these vectors to find an orthonormal basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$. So $A = PDP^{-1}$ where $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix}$

$\vec{x} = P\vec{y}$ is the change of variable so

$$\vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T P^T A P \vec{y} = y^T \underbrace{P^T P}_{I} D \underbrace{P^{-1} P}_{I} y = y^T D \vec{y} =$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7y_1 & -3y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 7y_1^2 - 3y_2^2$$

13.) Same directions as #11

$x_1^2 - 6x_1x_2 + 9x_2^2$

$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$ $\det(A-\lambda I) = (1-\lambda)(9-\lambda) - 9 = 0$

$\lambda^2 - 10\lambda = 0$

$\lambda(\lambda - 10) = 0$ $\lambda = 0, 10$

$\boxed{\text{Positive definite}}$

$\lambda = 0$ $A - \lambda I = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $x_1 = 3x_2$, x_2 free. Basis for eigenspace $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

Normalize these

$\lambda = 10$ $A - \lambda I = \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ $x_1 = -1/3 x_2$, x_2 free $\left\{ \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right\}$

orthonormal basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \right\}$. So $A = PDP^{-1}$

where $P = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$.

$\vec{x} = P\vec{y}$ $\vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T P^T P D P^{-1} P \vec{y} = \vec{y}^T D \vec{y}$

$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 10y_2^2$

19.) What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\vec{x} = (x_1, x_2)$ and $\vec{x}^T\vec{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$?

$$\begin{aligned} 5x_1^2 + 8x_2^2 &= 5x_1^2 + 5x_2^2 + 3x_2^2 \\ &= 5(x_1^2 + x_2^2) + 3x_2^2 \\ &= 5 \cdot 1 + 3x_2^2 \end{aligned}$$

Since $x_1^2 \geq 0$ (squares are positive) $x_2^2 = 1 - x_1^2$, the largest possible value for x_2^2 is 1.

So the largest possible value of $5x_1^2 + 8x_2^2$ is $5 + 3(1) = \underline{\underline{8}}$.

21.) True/False. All matrices are $n \times n$ and vectors are in $\mathbb{R}^n$

a.) The matrix of a quadratic form is a symmetric matrix.

b.) A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.

c.) The principal axes of a quadratic form $\vec{x}^T A \vec{x}$ are eigen vectors of A.

d.) A positive definite quadratic form Q satisfies $Q(x) > 0$ for all $\vec{x}$ in $\mathbb{R}^n$.

e.) If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\vec{x}^T A \vec{x}$ is positive definite.

f.) A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

a.) True   b.) True   c.) True   d.) False   e.) True   f.) True

27.) Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A+B$ are all positive.

Since the eigenvalues are positive, the quadratic forms, $\vec{x}^T A \vec{x}$ and $\vec{x}^T B \vec{x}$ are positive definite. ie. $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq 0$ and $\vec{x}^T B \vec{x} > 0$ for all $\vec{x} \neq 0$. So for $\vec{x} \neq 0$,

$\vec{x}^T (A+B) \vec{x} = \vec{x}^T A \vec{x} + \vec{x}^T B \vec{x} > 0$, so $A+B$ is positive definite.

This implies that the eigenvalues of $A+B$ are positive.