2.3 # 1, 3, 5, 8, 11, 13, 15, 17, 26, 28, 35, 40 (challenge)

1.) Determine if the matrix is invertible.

$$\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix} \updownarrow \begin{bmatrix} -3 & -6 \\ 5 & 7 \end{bmatrix} \xrightarrow{R_1/(-3)} \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix} \xrightarrow{-5R_1+R_2} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$ has 2 pivot positions

$\Rightarrow$ invertible

3.) $\begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 8 & 5 & -3 \end{bmatrix}$ $A^T = \begin{bmatrix} 3 & -3 & 8 \\ 0 & -4 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ has 3 pivot positions, so A^T is invertible $\Rightarrow$ A invertible

5.) $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 0 & -4 \\ -4 & 0 & 7 \end{bmatrix}$ Can't have a pivot position in the second column

$\Rightarrow$ not invertible.

8.) $\begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has 4 pivot positions $\Rightarrow$ invertible

11.) True / False (A is $n \times n$ matrix)

a.) If the equation $A\vec{x} = \vec{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

b.) If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

c.) If A is an $n \times n$ matrix, then the eqn $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.

d.) If the eqn $A\vec{x} = \vec{0}$ has a nontrivial soln, then A has fewer than n pivot positions.

e.) If A^T is not invertible, then A is not invertible.

a.) TRUE b.) TRUE c.) FALSE d.) TRUE e.) TRUE

2.3

13.) An $m \times n$ upper triangular matrix is one whose entries below the main diagonal are 0's. When is a square upper triangular matrix invertible?

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & & * \\ 0 & & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}$$

For an upper triangular matrix to be invertible, all of its diagonal entries must be non-zero so that it has n pivot positions.

15.) Is it possible for a 4×4 matrix to be invertible when its columns do not span $\mathbb{R}^4$? Why or why not.

No this would contradict the equivalence in the Invertible Matrix theorem (IMT).

17.) Can a square matrix with two identical columns be invertible?

No, if it had two identical columns, its columns would be linearly dependent and by the IMT the matrix would not be invertible.

26.) Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of an $n \times n$ matrix A are linearly independent.

If the columns of A are linearly independent, then A is invertible. Then $A^2 = AA$ is also invertible, so its columns span $\mathbb{R}^n$.

28.) Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is B.

Let X be the inverse of AB. Then $XAB = I$ and $(XA)B = I$.

Then by the IMT, B is invertible.

2.3 continued

35.) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Explain why T is both one-to-one and onto $\mathbb{R}^n$. Use equations (1) and (2). Then give a second explanation using one or more theorems.

T is invertible if $\exists$ an S s.t. for all $\vec{x}$ in $\mathbb{R}^n$

(1) $S(T(\vec{x})) = \vec{x}$

(2) $T(S(\vec{x})) = \vec{x}$

First explanation:

One-to-one: Suppose $T(\vec{x}_1) = T(\vec{x}_2)$ then since T is invertible, there exists an S s.t. $S(T(\vec{x}_1)) = \vec{x}_1$ and $S(T(\vec{x}_2)) = \vec{x}_2$.

Therefore $T(\vec{x}_1) = T(\vec{x}_2) \Rightarrow S(T(\vec{x}_1)) = S(T(\vec{x}_2)) \Rightarrow \vec{x}_1 = \vec{x}_2$. ✓

Onto: Suppose $\vec{x} \in \mathbb{R}^n$ then $S(\vec{x}) \in \mathbb{R}^n$. Then $T(S(\vec{x})) = \vec{x}$. ✓

Second explanation:

Since T is invertible, its standard matrix A is invertible. Then by the IMT $T(\vec{x}) = A\vec{x}$ is one-to-one and onto.

40.) (challenge) Suppose T and S satisfy the invertibility equations (1) and (2), where T is a linear transformation. Show directly that S is a linear transformation. (Hint: Given $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, let $\vec{x} = S(\vec{u})$, $y = S(\vec{v})$. Then $T(\vec{x}) = \vec{u}$, $T(\vec{y}) = \vec{v}$. Why? Apply S to both sides of $T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$. Also consider $T(c\vec{x}) = cT(\vec{x})$.)

$$S(\vec{u}+\vec{v}) = S(T(\vec{x}) + T(\vec{y})) = S(T(\vec{x}+\vec{y})) = \vec{x}+\vec{y} = S(\vec{u}) + S(\vec{v})$$

$$S(c\vec{u}) = S(cT(\vec{x})) = S(T(c\vec{x})) = c\vec{x} = cS(\vec{u})$$

Since $S(\vec{u}+\vec{v}) = S(\vec{u}) + S(\vec{v})$ and $S(c\vec{u}) = cS(\vec{u})$, S is a linear transformation.