4.3 # 3, 4, 8, 10, 14, 15, 21, 23, 24, 29, 30, 31

3.) Determine whether the sets are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$.

$$\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\-4\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix} \qquad \begin{bmatrix}1&3&-2\\0&1&-1\\-3&-4&1\end{bmatrix} \sim \begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}$$

Since we have an $n\times n$ matrix without n pivots, by the IMT, the set is linearly dependent and do not span $\mathbb{R}^3$. The set is not a basis for $\mathbb{R}^3$.

4.) $$\begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\end{bmatrix}, \begin{bmatrix}-8\\5\\4\end{bmatrix} \qquad \begin{bmatrix}2&2&-8\\-1&-3&5\\1&2&4\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

We have an $n\times n$ matrix with n pivots, so by IMT, the columns are lin. indep and span $\mathbb{R}^3$. The set is a basis for $\mathbb{R}^3$.

8.) $$\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix} \qquad \begin{bmatrix}1&0&2&0\\-2&3&-1&0\\3&-1&5&-1\end{bmatrix} \sim \begin{bmatrix}1&0&2&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

(IMT not applicable) Since there is a pivot in every row the columns of A span $\mathbb{R}^3$. Since there is a free variable, the columns are not lin. indep. So the set is not a basis for $\mathbb{R}^3$.

10.) Find a basis for the null space of the matrix.

$$\begin{bmatrix}1&1&-2&1&5\\0&1&0&-1&-2\\0&0&-8&0&16\end{bmatrix} \sim \begin{bmatrix}1&0&0&2&-9\\0&1&0&-1&10\\0&0&1&0&-2\end{bmatrix}$$

x_4, x_5 free

$x_1 = -2x_4 + 9x_5$

$x_2 = x_4 - 10x_5$

$x_3 = 2x_5$

$$\vec{x} = x_4\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}9\\-10\\2\\0\\1\end{bmatrix}$$

Basis for the null space:

$$\left\{\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}9\\-10\\2\\0\\1\end{bmatrix}\right\}$$

14.) Assume A is row equivalent to B. Find bases for Nul A and Col A.

$$A = \begin{bmatrix}1&2&3&-4&8\\1&2&0&2&8\\2&4&-3&10&9\\3&6&0&6&9\end{bmatrix} \qquad B = \begin{bmatrix}1&2&0&2&5\\0&0&3&-6&3\\0&0&0&0&-7\\0&0&0&0&0\end{bmatrix}$$

echelon form

$$\sim \begin{bmatrix}1&2&0&2&0\\0&0&1&-2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

x_2, x_4 free

$x_1 = -2x_2 - 2x_4$

$x_3 = 2x_4$

$x_5 = 0$

$$\vec{x} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\0\\2\\1\\0\end{bmatrix}$$

Basis for Nul A is:

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\2\\1\\0\end{bmatrix}\right\}$$

Basis for Col A: We know from B the pivot columns are the 1st, 3rd, 5th columns. So looking at A we get the Basis for Col A:

$$\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-3\\0\end{bmatrix}, \begin{bmatrix}8\\8\\9\\9\end{bmatrix}\right\}$$

4.3

15.) Find a basis for the space spanned by the given vectors.

$\begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-2\\-8\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix}$

We want to find a basis for Col A.

$$A = \begin{bmatrix}1 & 0 & 2 & 2 & 3\\ 0 & 1 & -2 & -1 & -1\\ -2 & 2 & -8 & 10 & -6\\ 3 & 3 & 0 & 3 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & 0 & 0\\ 0 & 1 & -2 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix} \right\}$

21.) True / False

a.) A single vector by itself is linearly dependent.

b.) If $H = \text{Span}\{\vec{b}_1, \dots \vec{b}_p\}$, then $\{\vec{b}_1, \dots \vec{b}_p\}$ is a basis for H.

c.) The columns of an invertible nxn matrix form a basis for $\mathbb{R}^n$.

d.) A basis is a spanning set that is as large as possible.

e.) In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

a.) False b.) False c.) True d.) False e.) False

23.) Suppose $\mathbb{R}^4 = \text{Span}\{\vec{v}_1, \dots \vec{v}_4\}$. Explain why $\{\vec{v}_1, \dots \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

Let $A = [\vec{v}_1 \dots \vec{v}_4]$. Then A is a 4×4 matrix. Since its columns span $\mathbb{R}^4$, the columns of A are linearly independent by the IMT. Since $\{\vec{v}_1, \dots \vec{v}_4\}$ is linearly independent and spans $\mathbb{R}^4$, it is a basis for $\mathbb{R}^4$.

24.) Let $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why $\mathcal{B}$ must be a basis for $\mathbb{R}^n$.

Let $B = [\vec{v}_1 \dots \vec{v}_n]$. B is an n×n matrix whose columns are linearly independent. By the IMT, the columns of B span $\mathbb{R}^n$. Therefore $\mathcal{B}$ is linearly independent, spans $\mathbb{R}^n$ and hence $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

4.3 continued

29.) Let $S=\{\vec{v}_1,\dots\vec{v}_K\}$ be a set of K vectors in $\mathbb{R}^n$, with $K<n$. Use a theorem from Section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

Let $A=[\vec{v}_1 \dots \vec{v}_K]$. Since A has more rows than columns, there can't be a pivot in every row. Therefore by thm 4 in 1.4 the columns of A do not span $\mathbb{R}^n$. Thus S is not a basis for $\mathbb{R}^n$.

30.) Let $S=\{\vec{v}_1,\dots\vec{v}_K\}$ be a set of K vectors in $\mathbb{R}^n$, with $K>n$. Use a theorem from chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Since S has more vectors than entries, S is linearly dependent by theorem 8 in 1.7. Therefore S cannot be a basis for $\mathbb{R}^n$.

31.) Let V, W be vector spaces, $T: V \to W$ be a linear transformation and $\{\vec{v}_1,\dots\vec{v}_p\}$ is a subset of V. Show that if $\{\vec{v}_1,\dots\vec{v}_p\}$ is linearly dependent in V, then the set of images, $\{T(\vec{v}_1),\dots T(\vec{v}_p)\}$ is linearly dependent in W.

Since $\{\vec{v}_1,\dots\vec{v}_p\}$ is linearly dependent, there exist scalars $c_1,\dots c_{p-1}$ such that $\vec{v}_p = c_1\vec{v}_1 + \dots + c_{p-1}\vec{v}_{p-1}$. Applying the linear trans. T to this gives us $T(\vec{v}_p)=T(c_1\vec{v}_1+\dots+c_{p-1}\vec{v}_{p-1}) = c_1T(\vec{v}_1)+\dots+c_{p-1}T(\vec{v}_{p-1})$

therefore $\{T(\vec{v}_1),\dots T(\vec{v}_p)\}$ is linearly dependent.

(This also shows if $\{T(\vec{v}_1),\dots T(\vec{v}_p)\}$ is linearly independent, then the original set $\{\vec{v}_1,\dots\vec{v}_p\}$ is linearly independent.