1.8 # 2, 4, 8, 9, 13, 15, 17, 21, 26, 31

2.) Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$. Find $T(\vec{u})$ and $T(\vec{v})$.

$$T(\vec{u}) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \qquad T(\vec{v}) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a \\ \frac{1}{3}b \\ \frac{1}{3}c \end{bmatrix}$$

4.) $T(\vec{x}) = A\vec{x}$. Find a vector $\vec{x}$ whose image under T is $\vec{b}$ and determine whether $\vec{x}$ is unique.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 0 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -6 \\ 0 & -1 & 0 & 7 \\ 0 & 1 & -3 & -4 \end{array}\right] \xrightarrow[R_2+R_3]{R_2(-1)} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & -3 & 3 \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -20 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & -3 & 3 \end{array}\right] \xrightarrow[R_3/(-3)]{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} -17 \\ -7 \\ -1 \end{bmatrix}$ is unique.

8.) How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\vec{x}) = A(\vec{x})$?

$$\underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{7\times 5} \underbrace{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}}_{5\times 1} = \underbrace{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}}_{7\times 1}$$

7 rows, 5 columns

1.8

9.) Find all $\vec{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\vec{x} \mapsto A\vec{x}$ for the matrix $A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 2 & -4 & 4 & -4 & 0 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{cccc|c} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 2 & -6 & 6 & 0 \end{array}\right] \xrightarrow[-2R_2+R_3]{3R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -4 & 10 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \rightarrow x_4 = 0$$

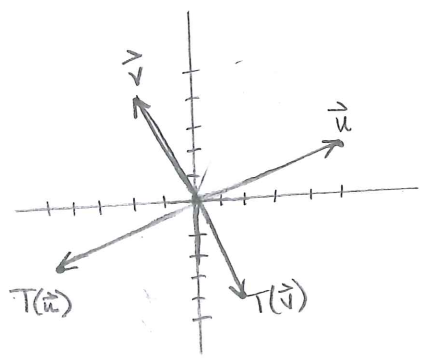
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

13.) Use a rectangular coordinate system to plot $\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and their images under the given transformation T.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{u}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

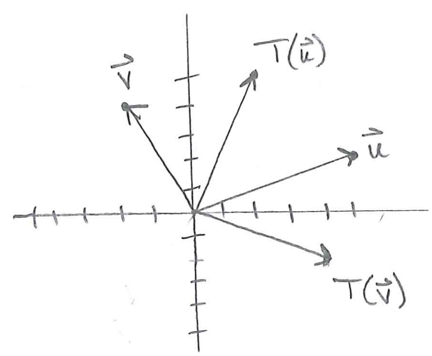
$$T(\vec{v}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$



15.) $$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$T(\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



1.8 continued

17.) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and maps $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $2\vec{u}$, $3\vec{v}$ and $2\vec{u}+3\vec{v}$.

$$T(2\vec{u}) = 2T(\vec{u}) = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$T(3\vec{v}) = 3T(\vec{v}) = 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$T(2\vec{u}+3\vec{v}) = T(2\vec{u}) + T(3\vec{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

21.) True/False

a.) A linear transformation is a special type of function.

b.) If A is a 3×5 matrix and T is a transformation defined by $T(\vec{x}) = A\vec{x}$, then the domain of T is $\mathbb{R}^3$.

c.) If A is an $m \times n$ matrix, then the range of the transformation $\vec{x} \mapsto A\vec{x}$ is $\mathbb{R}^m$.

d.) Every linear transformation is a matrix transformation.

e.) A transformation T is linear if and only if $T(c_1\vec{v} + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2$ in the domain of T and all scalars c_1, c_2.

a.) True b.) False c.) False d.) False e.) True

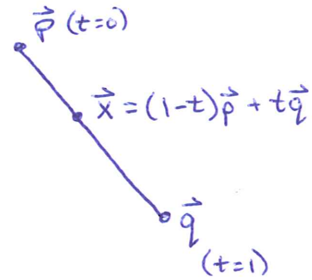
26a) a.) Show that the line through vectors $\vec{p}$ and $\vec{q}$ in $\mathbb{R}^n$ may be written in parametric form $\vec{x} = (1-t)\vec{p} + t\vec{q}$

\# 22 in 1.5 shows the line M through $\vec{p}$ and $\vec{q}$ is parallel to $\vec{q} - \vec{p}$

So $\vec{x} = \vec{p} + t(\vec{q} - \vec{p}) = (1-t)\vec{p} + t\vec{q}$

1.8

26.) b.) The line segment from $\vec{p}$ to $\vec{q}$ is the set of points of the form $(1-t)\vec{p} + t\vec{q}$ for $0 \leq t \leq 1$. Show that a linear transformation T maps this line segment onto a line segment or onto a single point.



T is linear, so for $0 \leq t \leq 1$

$T(\vec{x}) = T((1-t)\vec{p} + t\vec{q}) = (1-t)T(\vec{p}) + tT(\vec{q})$.

Therefore T maps the line segment to a line segment from $T(\vec{p})$ to $T(\vec{q})$ or a point if $T(\vec{p}) = T(\vec{q})$.

31.) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set, one vector can be written as a linear combination of the others. Let's say $\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$.

Then $T(\vec{v}_3) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$ by linearity. Thus $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.