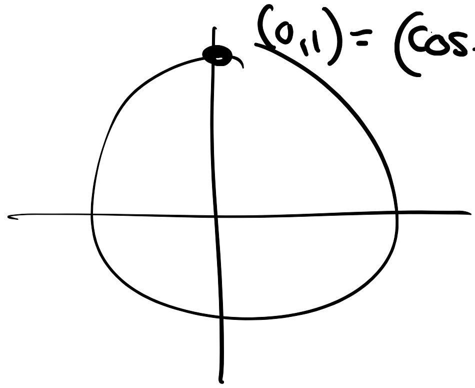
20.3 Inverse Trigonometric Functions

Goals
- inverse trigonometric functions

Example 20.3.1. Find the exact angle θ between 0 and 2π such that $\sin(\theta) = 1$



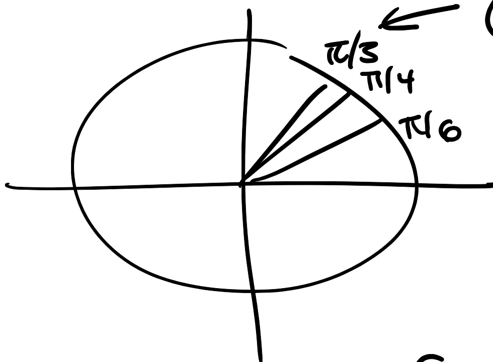
$(0,1) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$

if $\sin\theta = 1$
and $0 \le \theta \le 2\pi$
then $\theta = \frac{\pi}{2}$

problem is there's more than 1

Example 20.3.2. Find the exact angle θ between 0 and 2π such that $\sin(\theta) = \frac{\sqrt{3}}{2}$

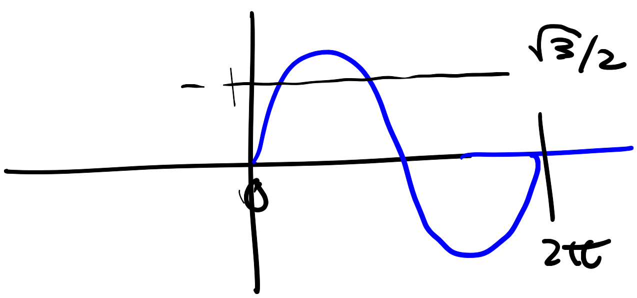
$\left(\cos \pi/3, \sin \pi/3\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



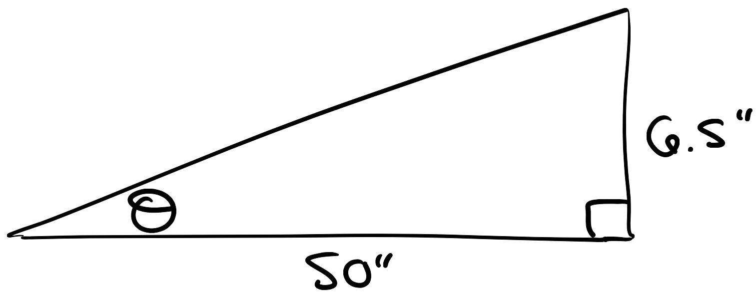
cf

$\sin \theta = \frac{\sqrt{3}}{2}$ (and θ b/w 0 and 2π)

$\Rightarrow \theta = \frac{\pi}{3}$



Example 20.3.3. Suppose we're making a ramp for a building to improve accessibility. ADA regulations say that ramps should have a slope of no more than 5 degrees. The ramp needs to be 6.5 inches high, and we only have 50 inches of length to fit it. Is there enough room?



what θ gives $\tan\theta = \frac{6.5}{50} = \frac{\text{opp}}{\text{adj}}$?

Question 20.3.4. Let's review inverse functions!

(a) What does an inverse function f^{-1} do to f? In other words, what is $f^{-1}(f(x))$?

cancelation $\begin{cases} f^{-1}(f(x)) = x \\ f(f^{-1}(x)) = x \end{cases}$

(b) What are some examples of inverse functions you know?

inverse $\neq$ reciprical

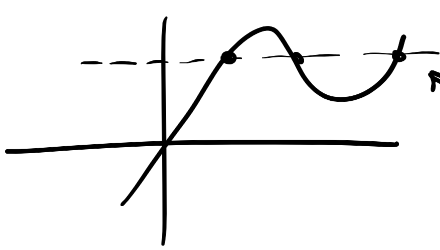
$x^2 \sim \sqrt{x}$

$e^x \sim \ln x$

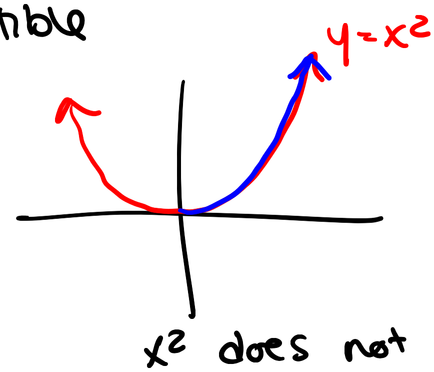
$4x \sim \frac{1}{4}x$

(c) What do we have to know about f to know if it has an inverse?

has to be one to one

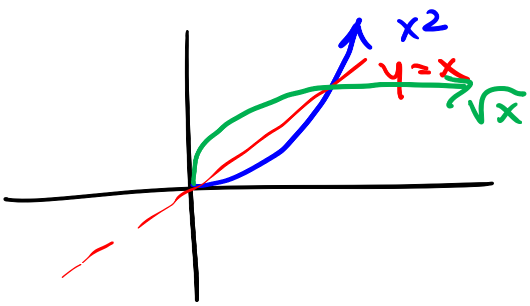


not invertible



(d) How are the graphs of $f^{-1}(x)$ and $f(x)$ related?

it's reflection ~~about~~ $y = x$

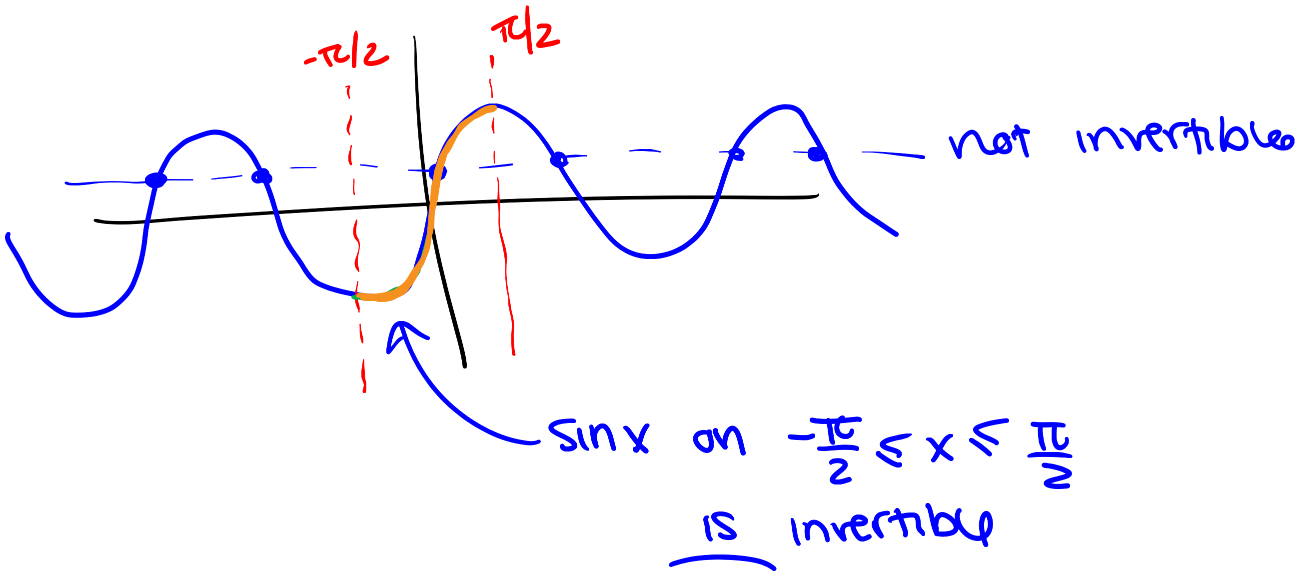


x^2 does not have inverse

but x^2 on $x \geq 0$ does

$\sqrt{x}$

Question 20.3.5. Suppose we want to make an inverse function for $\sin(x)$. What's the problem?



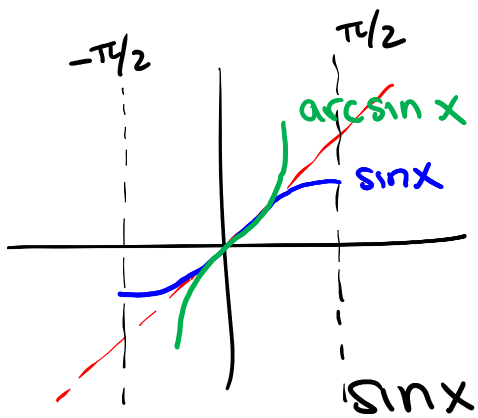
$\sin^{-1}(x) \neq (\sin x)^{-1} = \csc x$

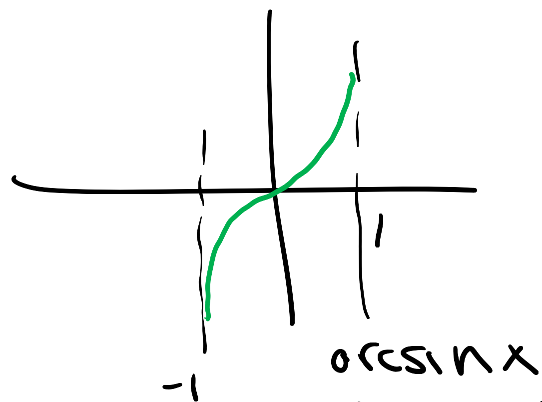
Definition 20.3.6 (Arcsine)**.** If we restrict the domain of $\sin(x)$ to $-\pi/2 \leq x \leq \pi/2$, we get a one-to-one function, so it has an inverse. We'll call the inverse function the **arcsine**, denoted $\arcsin(x)$ or $\sin^{-1}(x)$. We have

In English...

$\arcsin(x)$ is the angle Θ that gives $\sin\Theta = x$

Example 20.3.7. What does the graph of $\boxed{\arcsin(x)}$ look like. What are its domain and range?





domain $= [-\pi/2, \pi/2] \Longrightarrow$ range $= [-\pi/2, \pi/2]$

range $= [-1, 1] \Longrightarrow$ domain $= [-1, 1]$

$\arcsin(2)$

"what angle Θ gives $\sin\Theta = 2$?"

undefined!

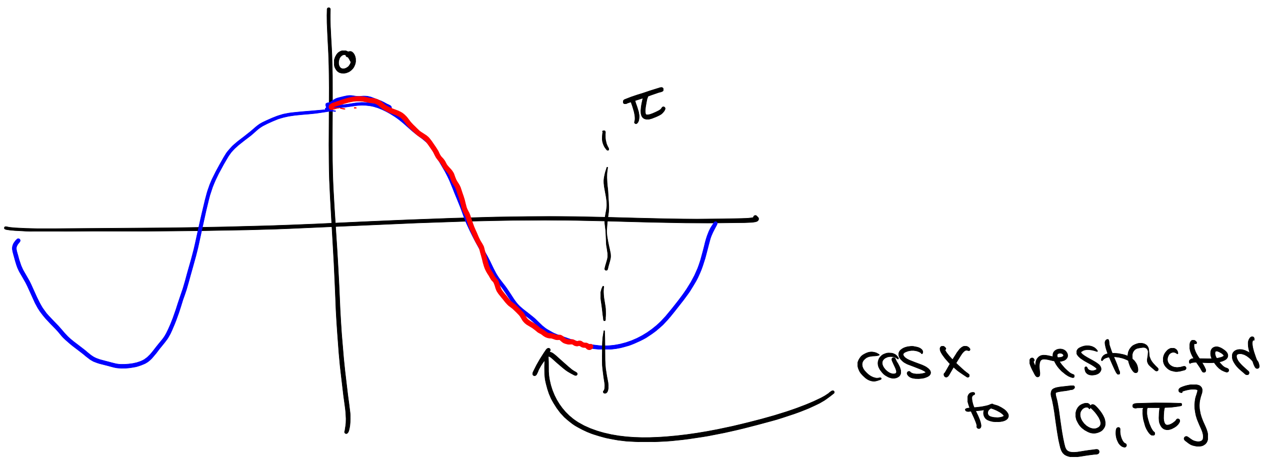
Definition 20.3.8 (Arccosine)**.** If we restrict the domain of $\cos(x)$ to $[0, \pi]$, it has an inverse function **arccosine**, denoted $\arccos(x)$ or $\cos^{-1}(x)$.

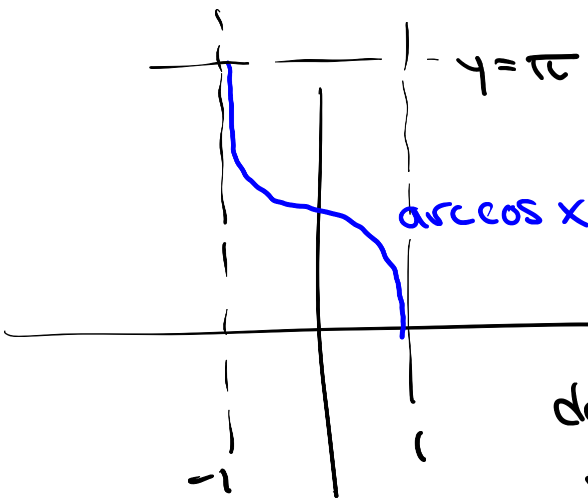
$$\arccos(\cos(x)) = x \text{ for all } x \text{ in } [0, \pi]$$

In English...

arccos(x) gives θ such that $\cos(\theta) = x$

Example 20.3.9. What does the graph of $\arccos(x)$ look like. What are its domain and range?





domain = $[-1, 1]$

range = $[0, \pi]$

arccos(0) = the angle θ between $0, \pi$ that gives $\cos\theta = 0$

$\Longrightarrow \arccos(0) = \pi/2$

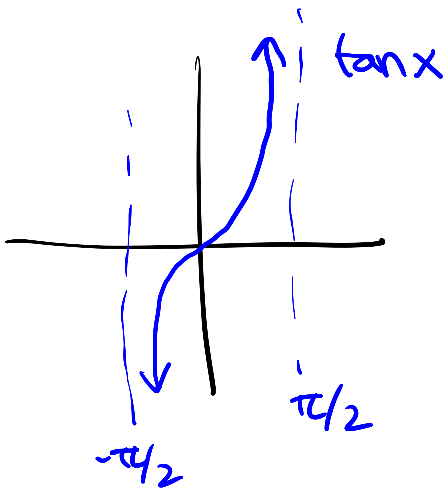
Definition 20.3.10 (Arctangent)**.** If we restrict the domain of $\tan(x)$ to $[-\pi/2, \pi/2]$, it has an inverse function **arctangent**, denoted $\arctan(x)$ or $\tan^{-1}(x)$.

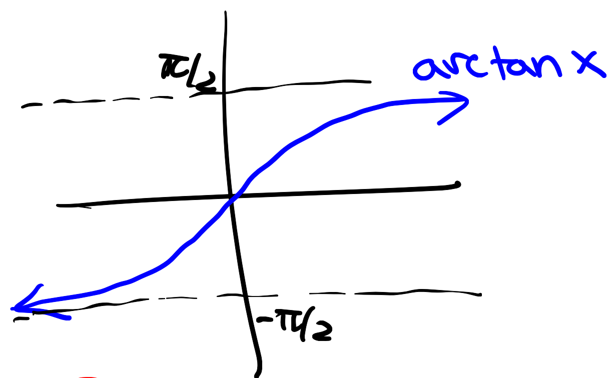
$$\arctan(\tan(x)) = x \text{ for all } x \text{ in } [-\tfrac{\pi}{2}, \pi/2]$$

In English...

arctan x gives angle b/w $-\pi/2$ and $\pi/2$ s.t. $\tan\theta = x$

Example 20.3.11. What does the graph of $\arctan(x)$ look like. What are its domain and range?





$R = [-\pi/2, \pi/2]$

$D = [-\infty, \infty]$

Example 20.3.12. Simplify the following

(a) arccos(1)

= what angle gives $\cos(\theta) = 1$
b/w $[0, \pi]$

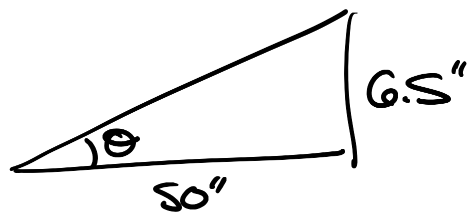
takes x in $[-1, 1]$

outputs θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

(b) $\arcsin(\frac{\sqrt{2}}{2}) = \theta$

what angle θ gives $\sin\theta = \frac{\sqrt{2}}{2}$
b/w $[-\pi/2, \pi/2]$

Example 20.3.13. Suppose we're making a ramp for a building to improve accessibility. ADA regulations say that ramps should have a slope of no more than 5 degrees. The ramp needs to be 6.5 inches high, and we only have 50 inches of length to fit it. Is there enough room?



want $\theta \leq 5$ degrees

$$\tan\theta = \frac{6.5}{50}$$

$$\theta = \arctan\left(\frac{6.5}{50}\right) \approx 7.41^\circ$$

(too steep)

$$\arctan x = \tan^{-1} x \neq \frac{1}{\tan x}$$

am I even allowed to evaluate arcsine at $\frac{\sqrt{3}}{2}$

Example 20.3.14. Simplify the following

(a) $\sin(\arcsin(\frac{\sqrt{3}}{2})) = \frac{\sqrt{3}}{2} \approx 0.85$

$\sin(\sin^{-1}(x)) = x$ if $-1 \leq x \leq 1$

(b) $\arcsin(\sin(2\pi)) = 0$

$\arcsin(\sin x) = x$ only if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\sin(2\pi) = 0$

$\arcsin(\sin 2\pi) = \arcsin(0) =$

$\arcsin(0) =$ angle where $\sin(\theta) = 0$ b/w $-\pi/2$ to $\pi/2$

$\arcsin(0) = 0$

(c) $\sin(\arcsin(2\pi))$

$\sin(\arcsin x) = x$ when $-1 \leq x \leq 1$

so not true

$\arcsin(2\pi) \stackrel{?}{=} 0$

$\arcsin x$ is angle $-\pi/2 \leq \theta \leq \pi/2$ where $\sin\theta = x$

i.e. $\sin\theta = 2\pi$ never!

i.e. $\boxed{\arcsin(2\pi) \text{ is undefined!}}$

$\arccos x \neq \frac{1}{\cos x}$

Example 20.3.15. Simplify the following

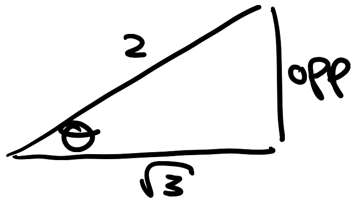
$\sin^2 x + \cos^2 x = 1$

$\sin x = \sqrt{1-\cos^2 x}$

(a) $\sin(\arccos(\frac{\sqrt{3}}{2}))$

$\sin x = \frac{opp}{hyp}$ $\arccos\left(\frac{adj}{hyp}\right) = \theta$

$\Rightarrow \boxed{\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{3}}{2}}$



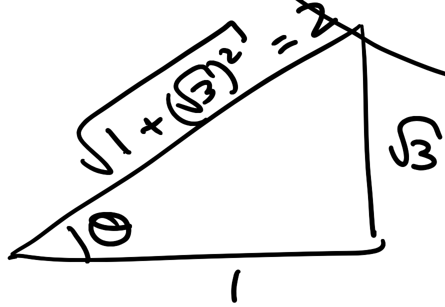
$\boxed{\sin\theta = \frac{opp}{hyp} = \frac{opp}{2} = \frac{1}{2}}$

$(opp)^2 + (\sqrt{3})^2 = 2^2 \Rightarrow opp = \sqrt{4-3} = 1$

(b) $\cos(\arctan(\sqrt{3}))$

could just evaluate $\arctan(\sqrt{3}) = \frac{\pi}{6}$

$\sin(\frac{\pi}{6}) = \frac{1}{2}$



set $\arctan(\sqrt{3}) = \theta$ and we want to know $\cos(\theta)$

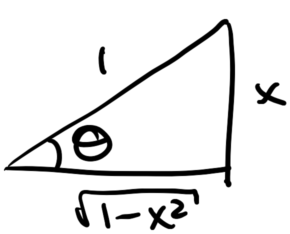
$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{3}}{1}$

so $\cos\theta = \frac{adj}{hyp} = \frac{1}{2}$

(c) $\cos(\arcsin(x))$

Let $\theta = \arcsin(x)$

$\Rightarrow \sin\theta = x = \frac{x}{1} = \frac{opp}{hyp}$



want $\cos\theta = \sqrt{1-x^2}$

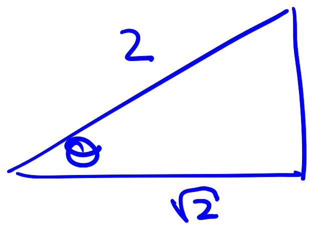
$\boxed{\cos(\arcsin(x)) = \sqrt{1-x^2}}$

20.3.1 Extra Examples

Example 20.3.16. Simplify the following

(a) $\sin(\arccos(\frac{\sqrt{2}}{2}))$ $\theta = \arccos\left(\frac{\sqrt{2}}{2}\right)$

$\cos\theta = \frac{\sqrt{2}}{2}$



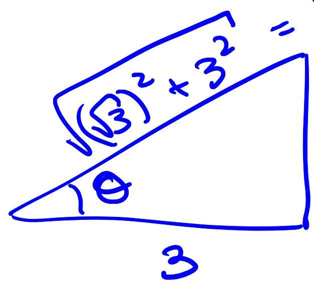
$\sqrt{2^2 - (\sqrt{2})^2} = \sqrt{2}$

$\boxed{\sin\theta = \frac{\sqrt{2}}{2}}$

(b) $\sin(\arctan(\frac{\sqrt{3}}{3}))$

$\theta = \arctan\left(\frac{\sqrt{3}}{3}\right) \Rightarrow \tan(\theta) = \frac{\sqrt{3}}{3}$

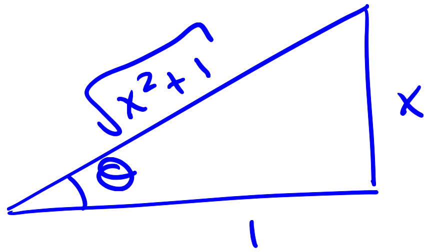
$= \sqrt{12} = 2\sqrt{3}$



$\sin(\theta) = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$

(c) $\sin(\arctan(x))$

$\arctan x = \theta \Rightarrow \tan\theta = x$



$\sin\theta = \frac{x}{\sqrt{x^2+1}}$