1.9 # 1, 2, 5, 13, 15, 20, 23, 26, 32, 34

1.) Assume T is linear, find its standard matrix.

$T: \mathbb{R}^2 \to \mathbb{R}^4$, $T(\vec{e}_1) = (3,1,3,1)$, $T(\vec{e}_2) = (-5,2,0,0)$, $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

2.) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(\vec{e}_1) = (1,4)$, $T(\vec{e}_2) = (-2,9)$, $T(\vec{e}_3) = (3,-8)$ where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the columns of the 3×3 identity matrix.

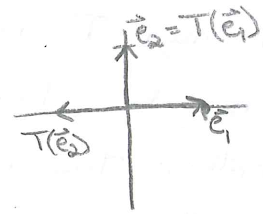
$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$$

5.) $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates points about the origin through $\pi/2$ radians (counterclock)

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $T(\vec{e}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $T(\vec{e}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

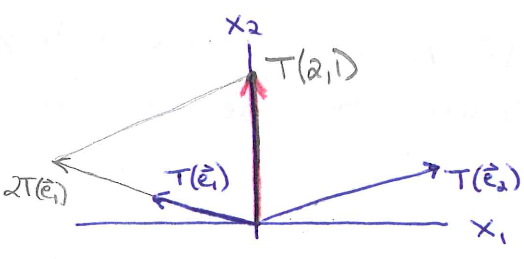
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



13.) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\vec{e}_1)$ and $T(\vec{e}_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2,1)$.

$T(2,1) = 2T(1,0) + T(0,1)$

$= 2T(\vec{e}_1) + T(\vec{e}_2)$



15.) Fill in the missing entries of the matrix assuming that the equation holds for all values of the variables.

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

1.9

20.) Show that T is a linear transformation by finding a matrix that implements the mapping.

$T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 3x_1 + 4x_3 - 2x_4 \qquad \begin{bmatrix} 3 & 0 & 4 & -2 \end{bmatrix}$$

23.) True/False

a.) A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

b.) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle φ then T is a linear transformation.

c.) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

d.) A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\vec{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

e.) If A is a 3×2 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ cannot be one to one.

a.) True b.) True c.) False d.) False e.) False

1.9 continued

26.) Determine if $T: \mathbb{R}^3 \to \mathbb{R}^2$ $T(\vec{e}_1) = (1,4)$, $T(\vec{e}_2) = (-2,9)$, $T(\vec{e}_3) = (3,-8)$ where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are columns of the 3x3 identity matrix is

a.) one-to-one. No b.) onto. Yes

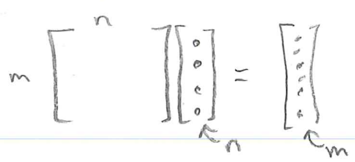
$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix} \xrightarrow{-4R_1 + R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$$

There is a pivot position in each row, so $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$. Therefore T is onto.

Since there are more columns than rows, there is always a free variable and therefore there is more than one solution for each $\vec{b}$. So, T is not one-to-one.

32.) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the statement to make it true.

"T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff A has $\underline{m}$ pivot columns."



A has a pivot position in each row iff $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$

34.) Let $S: \mathbb{R}^p \to \mathbb{R}^n$ and $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that $\vec{x} \mapsto T(S(\vec{x}))$ is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$.

$$\begin{aligned} T(S(c\vec{u} + d\vec{v})) &= T(cS(\vec{u}) + dS(\vec{v})) && \text{since } S \text{ is linear} \\ &= cT(S(\vec{u})) + dT(S(\vec{v})) && \text{since } T \text{ is linear} \end{aligned}$$