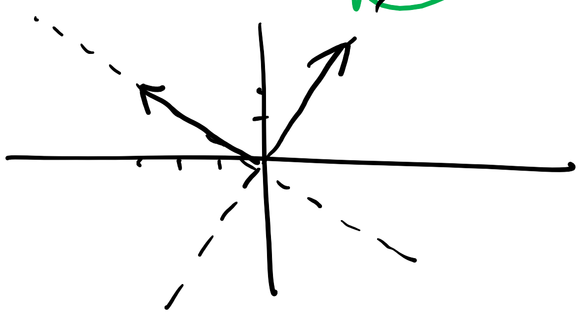
$y = \frac{b}{a}x$

Groups 1.2.12 Show that $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (-3, 2)$ meet at right angles. Hint: we've already seen that (a, b) lives on the line $ay = bx$.



$\vec{v}$ lives on $y = -\frac{2}{3}x$

$\vec{u}$ lives on $y = \frac{3}{2}x$

(a, b), $(-b, a)$

$\langle a, b\rangle \cdot \langle -b, a\rangle = -ab + ab = 0$

$\langle a, b\rangle \cdot \langle -kb, ka\rangle$

$= kab - kab = 0$

$\langle 1, 2\rangle \cdot \langle 3(-2), 6(1)\rangle$

$= 1\cdot(-6) + 2\cdot(6) = 6$

Groups 1.2.14 Find a nonzero vector in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u} = (1, 2, 3)$.

let $\langle x_1, x_2, x_3 \rangle$
be perpendicular to $\langle 1, 2, 3 \rangle$
then

$$\langle 1, 2, 3 \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

$$x_1 + 2x_2 + 3x_3 = 0$$

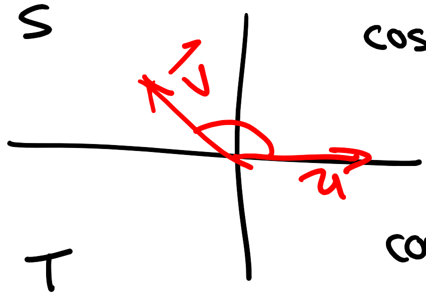
e.g. $\langle 1, 1, -1 \rangle$
$\langle 2, -1, 0 \rangle$

Question 1.2.16 What does this tell us about the *sign* of the dot product $\mathbf{u} \cdot \mathbf{v}$?

$$0 < \vec{u} \cdot \vec{v} = \|u\| * \|v\| * \cos\theta$$

$$0 < \cos\theta$$

S



$\cos\theta > 0$
A

T

$\cos\theta > 0$
C

$\cos\theta > 0$

$\Rightarrow$ acute

$\Rightarrow u \cdot v > 0$

$\Leftrightarrow \theta < 90^\circ$
acute

$u \cdot v < 0$

$\Leftrightarrow \theta > 90^\circ$
obtuse

$< 180^\circ$

1.3 Matrices

1.3. Key Ideas

- linear equations and vector equations
- solving simple systems of equations
- matrices
- **Matrix times vector:** $A\mathbf{x} =$ linear combination of the columns of A with x_i as weights.

1.3.1 Linear equations

Example 1.3.1 Suppose we are buying and selling candy, again. Remember, gum costs \$1.00 for a pack, chocolate is \$0.75 a bar, and hard candies are \$1.50 for a roll. Suppose

- Monday, we sell 10 packs of gum and 20 chocolate bars and buy 10 rolls of hard candy,
- Tuesday, we buy 10 packs of gum, sell 10 chocolate bars, and buy/sell no hard candies,
- Wednesday, we buy/sell no packs of gum, buy 4 chocolate bars, and buy/sell no hard candies.

What is our net profit?

on quiz...

Monday TUES

$$\begin{bmatrix} 1.00 \\ 0.75 \\ 1.50 \end{bmatrix} \quad \begin{bmatrix} 10 \\ 20 \\ -10 \end{bmatrix} \quad \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix}$$

prices

dot prod of

$$\boxed{1.00 * (10) + 0.75\,(20) + 1.50\,(-10) = 10}$$

$$1.00 * (-10) + 0.75\,(10) + 1.50\,(0) = -2.50$$

$$1.00 * 0 + 0.75\,(-4) + 1.50\,(0) = -3$$

$$1.00 * \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix} + 0.75 \begin{bmatrix} 20 \\ 10 \\ -4 \end{bmatrix} + 1.50 \begin{bmatrix} -10 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -2.50 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} \text{prod A mon} \\ \text{prod A tues} \\ \text{prod A wed} \end{bmatrix}$$

Example 1.3.4 Which of the following are linear equations?

1. $4x_1 - 5x_2 + 2 = x_1$
2. $x_2 = 2(\sqrt{6} - x_1) + x_3$ $+2x_1 + x_2 - x_3 = 2\sqrt{6}$
3. $4x_1 - 5x_2 = x_1x_2$ not linear
4. $x_2 = 2\sqrt{x_1} - 6$

not linear

$y = mx + b$

$xy = 1$

Example 1.3.6 Is $(5, 6.5, 3)$ in the solution set (the set of all solutions) of the system

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - \ 4x_3 &= -7 \end{aligned}$$

i.e. is $(5, 6.5, 3)$ in <u>both</u> lines (i.e. in intersection)

$$2(5) - 6.5 + 1.5(3) \stackrel{?}{=} 8 \quad \checkmark$$

$$5 \qquad - 4(3) \stackrel{?}{=} -7 \quad \checkmark$$

Example 1.3.9 What are the solution sets of the following systems?

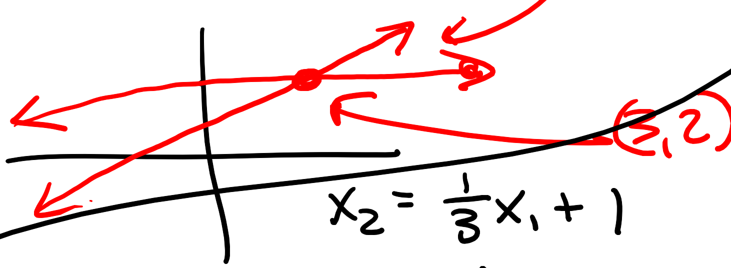
(a) $x_1 - 2x_2 = -1$
$-x_1 + 3x_2 = 3$

(b) $x_1 - 2x_2 = -1$
$-x_1 + 2x_2 = 3$

(c) $x_1 - 2x_2 = -1$
$2x_1 - 4x_2 = -2$

$-2x_2 = -1 - x_1$

$x_2 = \frac{1}{2}x_1 + \frac{1}{2}$

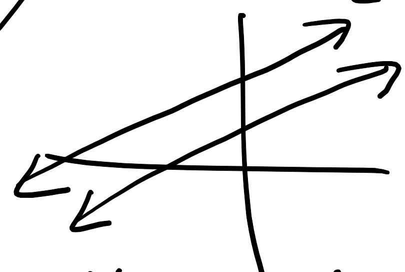


$x_2 = \frac{1}{3}x_1 + 1$

ONE SOLN

$x_2 = \frac{1}{2}x_1 + \frac{1}{2}$

$x_2 = \frac{1}{2}x_1 + \frac{3}{2}$



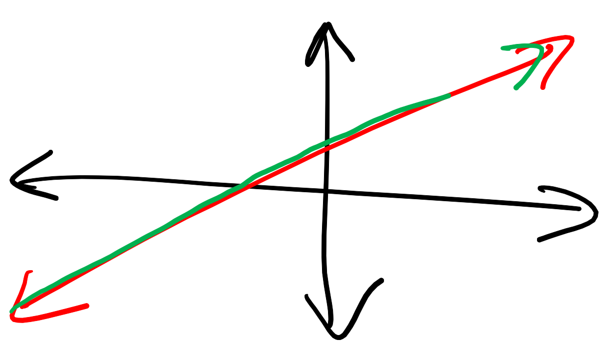
NO SOLN

$x_2 = \frac{1}{2}x_1 + \frac{1}{2}$

$x_2 = \frac{2}{4}x_1 + \frac{2}{4} = \frac{1}{2}x_1 + \frac{1}{2}$

all points

$(x_1, \frac{1}{2}x_1 + \frac{1}{2})$



Example 1.3.10 What is the solution set of the following system? If we fix b_1, b_2, b_3, how many solutions will it have?

$$\begin{aligned} x_1 \qquad\qquad &= b_1 \implies x_1 = b_1 \\ -x_1 + x_2 \qquad &= b_2 \implies x_2 = b_2 + x_1 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

$$= b_1 + b_2$$

$$x_3 = b_3 + x_2$$

$$= b_1 + b_2 + b_3$$

Substitution

Example 1.3.11 How can we interpret solutions to systems of equations with three variables geometrically?

1.3.2 Matrices

$$\begin{matrix} 1.00(10) & + & 0.75(20) & + & 1.50(-10) \\ 1.00(-10) & + & 0.75(10) & + & 1.50(0) \\ 1.00(0) & + & 0.75(-4) & + & 1.50(0) \end{matrix}$$

$$= 1.00 \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix} + 0.75 \begin{bmatrix} 20 \\ 10 \\ -4 \end{bmatrix} + 1.50 \begin{bmatrix} -10 \\ 0 \\ 0 \end{bmatrix}$$

Example 1.3.15 Compute the product $A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1.3.3 Linear equations and matrices

Example 1.3.17 Compute the product $A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 1.3.18 What if $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given, but $\mathbf{x}$ is unknown? How could we find $\mathbf{x}$ if we're told

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example 1.3.19 What is $\mathbf{x}$ if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Chapter 2

Solving Linear Equations

2.1 Vectors and linear equations

2.1. Key Ideas

- systems of equations can have no, one, or many solutions
- a system of equations with at least one solution is called consistent
- systems can be solved using back substitution

Example 2.1.1 How many solutions do each of the following systems have?

(a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 - 4x_2 &= -2 \end{aligned}$$

Example 2.1.3 Are the following system consistent?

(a) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

(b) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(c) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 - 4x_2 &= -2 \end{aligned}$$

Example 2.1.5 Determine if the following system of equations is consistent.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

2.2 The idea of elimination

2.2. Key Ideas

- A linear system becomes upper triangular after elimination.
- We subtract ℓ_{ij} times equation j from equation i to make the (i, j) entry zero, where

$$\ell_{ij} = \frac{(i, j) \text{ entry}}{\text{pivot in row } j}.$$

- The upper triangular system is solved by back substitution.

Example 2.2.1 In Section 1.3 we determined whether the following systems were consistent using geometric and substitution arguments. Is there an algebraic way to do this without substitution?

(a) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$

(b) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$

(c) $\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 - 4x_2 &= -2 \end{aligned}$

Example 2.2.3 Determine if the following system of equations is consistent without substitution

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Example 2.2.4 Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 6x_2 + 4x_3 &= 1 \end{aligned}$$

2.3 Elimination using matrices

2.3. Key Ideas

- we can record information about a system in a matrix
- we can use elementary row operations to reduce matrices
- row reduction algorithm and how to use it to solve a system of equations

Example 2.3.3 Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

using a matrix.

Example 2.3.6 Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

Example 2.3.8 Which of the following is in echelon form? Reduced echelon form?

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.3.11 Label the pivot positions and pivot columns of the matrices above.

Example 2.3.12 Row reduce the matrix A to echelon form and locate pivot columns.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Example 2.3.14 Apply elementary row operations to transform the following matrix into echelon form, and then reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Example 2.3.16 Find the general solution of a linear system whose augmented matrix can be reduced to the matrix below.

$$\left[\begin{array}{ccr|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 2.3.19 Find the general solution of a system whose augmented matrix is reduced to

$$\left[\begin{array}{ccccc:c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

Example 2.3.20 Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + \ \ 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + \ \ 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Using the theorem, and the rest of this section, we have the following procedure to find and describe all the solutions of a linear system.

2.4 Rules for matrix operations

2.4. Key Ideas

- The (i, j) entry of AB is the dot product of row i of A with column j of B.
- An $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix, and uses mnp separate multiplications.
- $A(BC) = (AB)C$, but $AB \neq BA$ in general

Example 2.4.3 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 5 & -6 \end{bmatrix}$.
Find $A + B$, $B + A$, and $A + C$.

Example 2.4.5 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $2B$ and $A - 2B$.

Example 2.4.9 Compute AB and BA, when $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -8 & 3 \end{bmatrix}$.

Example 2.4.11 With A and B from Example 2.4.9, compute AB using the row-column rule.

Example 2.4.13 Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$.

(a) Find AB and BA.

(b) Find AC.

(c) Find AD.

Example 2.4.16 Let $A = \begin{bmatrix} a & b & d \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \\ 6 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 3 & 1 & 2 & -6 \end{bmatrix}$.

Find A^T, B^T, and C^T.