3.2 The null space, solutions to $A\mathbf{x} = 0$

3.2. Key Ideas

- The null space $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$. It contains all solutions to $A\mathbf{x} = \mathbf{0}$.
- If $n > m$, then A has at least one column without pivots, giving a special solution. So there are nonzero vectors in $\mathrm{Nul}(A)$.

Definition 3.2.1. A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix, and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Remark 3.2.2. The equation $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely $\mathbf{x} = \mathbf{0}$, called the **trivial solution**. We will be interested in finding **non-trivial solutions**, where $\mathbf{x} \neq \mathbf{0}$.

Example 3.2.3. Determine if the following homogeneous system has a nontrivial solution, and describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

free var $\Rightarrow$ inf many solns

$$\sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{cases} x_1 - 4/3 x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

soln

$$\boxed{\begin{cases} x_1 = 4/3 x_3 \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}}$$

$$\boxed{\vec{x} =} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{x} = s \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

(parametric vectors)

$\sim$ or $\sim$

$$\boxed{\mathrm{Span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}} = \mathrm{Span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \right\}$$

line thru origin

Example 3.2.5. Describe all solutions to the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

$10x_1 = 3x_2 + 2x_3$

$x_1 = \frac{3}{10}x_2 + \frac{2}{10}x_3$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/10\, x_2 + \frac{2}{10}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/10\, x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/10\, x_3 \\ 0 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2/10 \\ 0 \\ 1 \end{bmatrix}$

$= \text{Span}\left\{ \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/10 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 3 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right\}$

$= \left\{ a\begin{bmatrix} 3 \\ 10 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} : a, b \text{ are in } \mathbb{R} \right\}$

plane ~~thro~~ w/ points

$(3, 10, 0), (1, 0, 5), (0, 0, 0)$

Example 3.2.6. What do our solutions to the previous two examples look like geometrically? In general, if A is a matrix with three columns, what could the solution set possibly look like?

$$A = \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} * & * & * & 0 \\ * & * & * & 0 \\ & \vdots & & \vdots \\ * & * & * & 0 \end{array}\right]$$

1-free var $\Rightarrow$ line thru origin

2 free var $\Rightarrow$ plane thru origin

no free var $\Rightarrow$ point $(0,0,0)$

$0=0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Definition 3.2.7. The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul}\, A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Example 3.2.8. Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Show that $\mathbf{u}$ is in Nul A.

WTShow that $\vec{u}$ is a soln to $A\vec{x} = \vec{0}$

$$A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \text{ \underline{is} in Nul}(A)$$

What if I want to know all of the vectors in Nul(A)?

$$\text{Nul}(A) = \{\vec{x} : A\vec{x} = \vec{0},\ \vec{x} \text{ is in } \mathbb{R}^n,\ A\ m\times n\}$$

$$= \left\{ \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \text{solve } \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \end{array}\right]$$

Example 3.2.9. For a matrix with 3 columns, what does Nul(A) look like geometrically?

$$\text{Nul}(A) = \{\text{all solns to } A\vec{x} = \vec{0}\}$$

$$= \{\text{all vectors in } \mathbb{R}^3 \text{ s.t. } \begin{bmatrix} * & * & * \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\}$$

same plane, line, or point thru origin.

Example 3.2.11. Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

$\text{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^5 \text{ s.t. } A\vec{x} = \vec{0} \}$

solve $\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$

$$\sim \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & & & \end{array} \end{array}$$

$$\sim \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 \text{ free} \\ x_3 = -2x_4 + 2x_5 \\ x_4 \text{ free} \\ x_5 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

coeff of x_2 coeff of x_4 coeff of x_5

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{x} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix}$$

Question 3.2.12. If a matrix A has more columns than rows, what can you say about Nul(A)?

is it trivial?

is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ only soln?

Nul(A) is nontrivial if and only if there are free vars in $A\vec{x}=\vec{0}$

$$m\left\{\underbrace{\begin{bmatrix} & & & \\ & & & \end{bmatrix}}_{n}\right.$$

If $m < n$ then always have free vars.

How are the null space and column space of a matrix related? In the next example, we'll see that the two spaces are very different. **Example 3.2.13.** Consider the following matrix.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$

(a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

$k=3$

for $m \times n$ mtx Col(A) is always a subspace of $\mathbb{R}^m$

(b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

$\text{Nul}\, A = \{ \vec{x} : A\vec{x} = \vec{0} \}$

for $m \times n$ mtx Nul A is always a subspace of $\mathbb{R}^4$

(c) Find a nonzero vector in Col A, and a nonzero vector in Nul A.

$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ it's easy to find in Col A

need to solve $A\vec{x} = 0$

$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right] \sim \text{Span}\left\{ \begin{bmatrix} -9 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) Is $\mathbf{u} = (3, -2, -1, 0)$ in Nul A? Could it be in Col A?

check

No! Too many entries

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \end{array}$$

$A\vec{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ * \end{bmatrix}$

(e) Is $\mathbf{v} = (3, -1, 3)$ in Col A? Could it be in Nul A?

is $\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} x_4$

$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & 13/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{array}\right] \sim \begin{aligned} x_1 + 9x_3 &= 5 \\ x_2 - 5x_3 &= 13/17 \\ x_4 &= 1/17 \end{aligned}$

$x_1 = 5 - 9x_3$
$x_2 = 13/17 + 5x_3$
x_3 free
$x_4 = 1/17$

$5a_1 + 13/17\, a_2 + \frac{1}{17} a_4 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
~~**8.** Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.~~	~~**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.~~

3.4 The complete solution to $A\mathbf{x} = \mathbf{b}$

3.4. Key Ideas

- One particular solution $\mathbf{x}_p$ has all free variables equal to zero.
- The pivot variables are determined after the free variables are chosen.
- The full solution to $A\mathbf{x} = \mathbf{b}$ is the solution set to $A\mathbf{x} = \mathbf{0}$ shifted by x_p.

In this section, we will use vector notation to give explicit and geometric descriptions of solution sets of linear systems.

Example 3.4.1. Determine if the following homogeneous system has a nontrivial solution, and describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

reduce the mtx

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

check

$$\boxed{\begin{cases} x_1 = 4/3\, x_3 \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}}$$

soln set

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$= \boxed{\text{Span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$ span

$$\boxed{\vec{x} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} t}$$

parametric vector form

> **Proposition 3.4.2.** *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.*

Example 3.4.3. Describe all solutions of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 4/3 x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

free variable $\Longrightarrow$ inf many soln

check

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3 x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} x_3$$

parametric vector form

$A\vec{x} = \vec{b}$ is a line

alt: $\vec{x} = \vec{p} + t\vec{v}$

is this Span $\left\{ \cancel{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}} \right\}$?

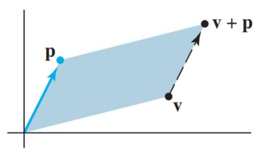
$x_3 = 0$

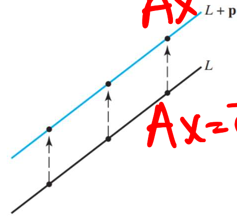
$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is in Span,

$x_3 = 3$ $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

Definition 3.4.5. We can think of vector addition as *translation.* Given $\mathbf{p}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* v in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by p** to $\mathbf{v}+\mathbf{p}$. If each point on a line L is translated by a vector $\mathbf{p}$, the result is a line parallel to L.

$A\vec{x}=\vec{b}$

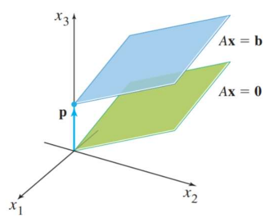




$A\vec{x}=\vec{0}$

For $t \in \mathbb{R}$, we call **$\mathbf{p}+t\mathbf{v}$ the equation of the line parallel to v through p.**

Theorem 3.4.6. *Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{x}_p$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{x}_p + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.*



ex. previous $A\vec{x}=\vec{0}$ had soln

$L = \text{Span}\left\{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}\right\}$

$L = \left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} t : t \text{ is real} \right\}$

soln to $A\vec{x}=\vec{b}$

$w = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} t$

Procedure 3.4.7. To write a solution set in parametric vector form

1. Row reduce the augmented matrix to RREF
2. Express each basic variable in terms of any free variables
3. Write $\mathbf{x}$ as a vector whose entries depend on the free variables (if there are any)
4. Decompose $\mathbf{x}$ into a linear combination of vectors using free variables as parameters

Example 3.4.8. Describe and compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ if

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$

2 to find soln $A\vec{x} = \vec{0}$

reduce $[A \,\vdots\, \vec{0}]$

1 to find soln $A\vec{x} = \vec{b}$

reduce $[A \,\vdots\, \vec{b}]$