Example 16.1.7. What are the derivatives of $f(kx)$ and $f(x+k)$? Why does this make sense geometrically?

done outside of class:

$(f(kx))' = kf'(x)$

this makes sense b/c the function is growing k times faster!

$(f(k+x))' = f'(k+h)$

this makes sense b/c the function grows at the same rate, just shifted

Example 16.1.9. Find the derivatives of

(a) $(x^2+1)^{10}$
(b) $\ln(x^2+2)$
(c) e^{3x^2}
(d) $\ln(x^2)$

a) 1) decompose

$u(x) = x^2+1$ $\begin{pmatrix} x^2+1 \\ x^{10} \end{pmatrix}$ $\leftarrow$ $10x^9 \cancel{(x^2+1)}$ $2x$

$f(u) = u^{10}$ $\leftarrow$ $f' = 10u^9$

$\frac{d}{dx}(f(u(x))) = f'(u(x))\, u'(x) = 10(x^2+1)^9\, 2x$

b) $\ln(x^2+2)$

$u(x) = x^2+2$

$f(u) = \ln(u)$

$\frac{dy}{dx} = \underbrace{\frac{1}{x^2+2}}_{f'(u(x))} 2x$

$\rightarrow u'(x) = 2x$

$\rightarrow f'(u) = \frac{1}{u} = \frac{1}{x^2+2}$

c) e^{3x^2} try to write as (de)composition

$u(x) = 3x^2 \rightsquigarrow 6x$

$f(u) = e^u \rightsquigarrow e^u = e^{3x^2}$

$\frac{d}{dx}(e^{3x^2}) = 6xe^{3x^2}$

d) $\ln(x^2)$ $\leftarrow$? $(\ln x)(x^2)$

$u(x) = x^2 \rightsquigarrow u'(x) = 2x$

$f(u) = \ln(u) \rightsquigarrow f'(u) = \frac{1}{u}$

$(\ln(x^2))' = \frac{2x}{u} = \frac{2x}{x^2} = \frac{2}{x}$

~or~

$\ln(x^2) = 2\ln(x)$

$\frac{d}{dx} 2\ln x = \frac{2}{x}$

Example 16.1.10. Suppose the population of frogs in a pond is e^g, where g is the temperature of the pond in Celcius and the average temperature in the month of February is $0.25t + 14$ where t is in days. What is the rate of change of frogs with respect to time?

done outside class:

$f(g) = e^g \qquad g(t) = 0.25t + 14$

$f'(g) = e^g \qquad g'(t) = 0.25$

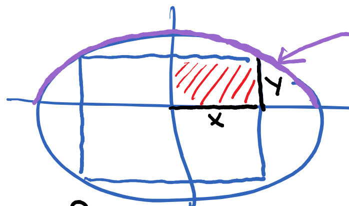
$f'(g(t))g'(t) = e^{g(t)}0.25 = e^{0.25t+14} * 0.25$

Groups 16.1.11. Suppose a rectangle is inscribed inside the ellipse

$$\frac{x^2}{9} + 4y^2 = 1.$$

What's the largest possible area of such a rectangle?

HINT:



top is $y = \frac{1}{2}\sqrt{1 - \frac{x^2}{9}}$

$= \frac{1}{6}\sqrt{9 - x^2}$

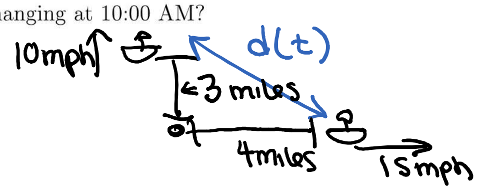
The area of the whole rectangle is 4 times area of the little red rectangle, so since area = $l * w$

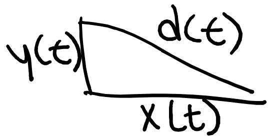
$A = 4xy = \frac{2}{3} x \sqrt{9 - x^2}$

this is what we want to maximize.

16.2.2 Planned examples from class

Example 16.2.2. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?





$$d(t) = \sqrt{x(t)^2 + y(t)^2}$$

$$d'(t) = \left(\sqrt{x(t)^2 + y(t)^2}\right)$$

$$\frac{d}{dt}(x(t))^2$$

$$f(u) = \sqrt{u} = \frac{1}{2}u^{-1/2}$$

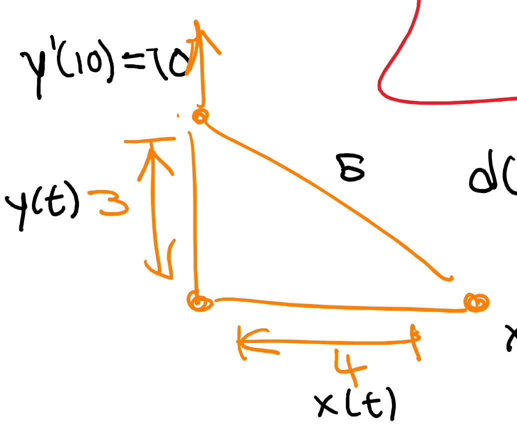
$$u(t) = x(t)^2 + y(t)^2$$

$$u'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

$$\frac{d}{dx}(f(x))^2$$

$$2f(x)f'(x)$$

$$\boxed{\frac{d}{dt}(d(t)) = \frac{1}{2}u^{-1/2}(2x(t)x'(t) + 2y(t)y'(t)) = \frac{2x(t)x'(t) + 2y(t)y'(t)}{2\sqrt{x(t)^2 + y(t)^2}}}$$



$$d(10) = \frac{2(4)(15) + 2(3)(10)}{2\sqrt{3^2 + 4^2}} = 18 \text{ mph}$$

$$x'(t) = 15$$

Question 16.2.3. Can we recover the derivative of b^x for $b > 0$ using only the chain rule?

we skipped these questions, but here they are

$$b^x = e^{\ln(b^x)} = e^{x\ln b}$$

$$\text{so } \frac{d}{dx}(b^x) = \ln b \, e^{x\ln b} = \ln b * b^x$$

ch rule

Groups 16.2.4. Now, let's consider x^n.

(a) rewrite x^n using $\ln x$ and e^x.

(b) find the derivative of the function you found in (a)

(c) what did you prove?

a) $x^n = e^{\ln(x^n)} = e^{n\ln x}$

b)

$$\frac{d}{dx}(x^n) = \frac{d}{dx} e^{n\ln x} \underset{\text{ch rule}}{=} \frac{n}{x} e^{n\ln x} = \frac{n}{x} x^n = n x^{n-1}$$

c) ... the power rule

Question 16.2.5. Can we find the derivative of $f(x)/g(x)$ without using the quotient rule?

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) * g(x)^{-1}\right)$$

prod rule

$$= f'(x)g(x)^{-1} - f(x)\left(-1g(x)^{-2}g'(x)\right)$$

ch rule

$-1+2=1$

$$= \frac{\cancel{g(x)^2}}{g(x)^2}\left(f'(x)g(x)^{\cancel{-1}} - f(x)\cancel{g(x)^{-2}}g'(x)\right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

16.3.2 Planned examples from class

Example 16.3.2. Two ships leave a port, one heading due East, and the other due North. At 10:00 AM, the first ship is four miles East, traveling at 15 miles per hour, and the second ship is 3 miles North, traveling at 10 miles per hour. At what rate of change is the distance between the ships changing at 10:00 AM?

we did this, above, in 16.2.2

Groups 16.3.3. Differentiate $e^{\sqrt{x^3+1}}$.

$f(u) = e^u \rightarrow f'(u) = e^u$

$u(x) = \sqrt{x^3+1}$

$u(v) = \sqrt{v} \longrightarrow \frac{1}{2}v^{-1/2}$

$v(x) = x^3+1 \longrightarrow 3x^2$

chain

$\left(f(u(x))\right)' = f'(u)\underbrace{u'(x)} = f'(u)\,u'(v)\,v'(x)$

$\frac{df}{dx} = e^u \frac{1}{2}v^{-1/2}\,3x^2 = e^{\sqrt{x^3+1}}\,\frac{1}{2}(x^3+1)^{-1/2}\,3x^2$

Think, Pair, Share 16.3.4. Can we find a general rule for the derivative of $f(g(h(x)))$?

$$(f(g(h(x))))'$$

ch rule

$$= f'(g(h(x)))\,(g(h(x)))'$$

$$= f'(g(h(x)))\,g'(h(x))\,h'(x)$$

in Liebonitz notation:

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$$

Example 16.3.5. Differentiate
$$\ln\left(\sqrt{\frac{1+x}{(1-x)^3}}\right).$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{(1-x)^3}\right)$$
$$= \frac{1}{2}\left(\ln(1+x) - \ln\left((1-x)^3\right)\right)$$
$$= \frac{1}{2}\left(\ln(1+x) - 3\ln(1-x)\right)$$
$$f'(x) = \frac{d}{dx}\left(\quad \text{mess} \quad\right)$$
check
$$= \frac{1}{2}\left(\frac{1}{1+x} + \frac{3}{1-x}\right)$$

Example 16.3.6. Differentiate

$$\frac{8^{x^2+1}}{(2^x)^x} = \frac{8^{x^2+1}}{2^{x^2}} = \frac{8\cdot 8^{x^2}}{2^{x^2}}$$

$$= 8\cdot\left(\frac{8}{2}\right)^{x^2} = 8\cdot 4^{x^2}$$

$$(a^x)^y = a^{xy}$$

$$a^{x+y} = a^x a^y$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

$$\frac{d}{dx}(\text{junk}) = \frac{d}{dx}\, 8\cdot 4^{x^2}$$

$$8\cdot \frac{d}{dx}\, 4^{x^2}$$ chain rule

$$u(x) = x^2$$

$$f(u) = 4^u$$

$$\frac{d}{dx}(\text{junk}) = 16x \ln 4\ 4^{x^2}$$