Appendix F. L'Hôpital's Rule

Goals

- review indeterminate forms
- L'Hôpital's Rule
- summation notation

Example F.1. Find $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1}$ and $\lim_{x\to\infty} \frac{x^2-2x+1}{x^2-1}$.

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$ if both defined

$$\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \boxed{\frac{[0]}{[0]}} = \lim_{x\to 1} \frac{(x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$$

not good

$\Rightarrow$ factor

$$\lim_{x\to 1} \frac{x-1}{x+1} = \frac{0}{2} = 0 \checkmark$$

doesn't mean limit is DNE

$$\lim_{x\to\infty} \frac{x^2-2x+1}{x^2-1} = \boxed{\frac{\infty}{\infty}} = 1 = \lim_{x\to\infty} \frac{1-\cancel{2/x}^{\,0}+\cancel{1/x^2}^{\,0}}{1-\cancel{1/x^2}^{\,0}}$$

more work to do

compare "dominating" coeff

$\frac{[0]}{[0]}$ and $\frac{[\infty]}{[\infty]}$ are called "indeterminate"

Example F.2. Find $\lim_{x \to 1} \frac{\ln(x)}{1-x^2}$

$$\lim_{x \to 1} \frac{\ln(x)}{1-x^2} = \frac{\lim_{x \to 1} (\ln x)}{\lim_{x \to 1} (1-x^2)} = \frac{0}{0}$$

cannot factor these!

more work

Theorem F.3. If f and g are differentiable near $x = c$ (or ∞), and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \lim_{x\to c} f(x) = \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \qquad \left(\text{similarly } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}\right)$$

the limit of $\frac{f}{g}$ as x approches c

is the same as $\lim_{x\to c} \frac{f'}{g'}$

e.g.

$$\lim_{x\to 1} \frac{\ln x}{1-x^2} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{1/x}{-2x} = \frac{1/1}{-2(1)} = -\frac{1}{2}$$

Question F.4. Why does L'Hôpital's Rule work?

we're given $f(x), g(x)$

s.t. $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

linearization $\Rightarrow$

$f(x) \approx f'(x)(x-c) + \cancel{f(c)}$ 0 tangent line at $x=c$

(when x is very close to c)

$g(x) \approx g'(x)(x-c) + \cancel{g(c)}$ 0

$$\lim_{x \to c} \frac{f(x)}{g(x)} \approx \lim_{x \to c} \frac{f'(x)\cancel{(x-c)}}{g'(x)\cancel{(x-c)}} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$\Rightarrow$ L'H rule

Example F.5. Find $\lim_{x\to 2} \dfrac{e^{x^2} - e^4}{x-2}$

check num/denom

$$\lim_{x\to 2} \frac{e^{x^2}-e^4}{x-2} = \frac{e^4-e^4}{2-2} = \frac{0}{0}$$

so this is indeterminate

$\Rightarrow$ use L'H rule

$$\boxed{\begin{aligned} \frac{d}{dx}\left(e^{x^2}-e^4\right) &= 2xe^{x^2} \\ \frac{d}{dx}(x-2) &= 1 \end{aligned}}$$

$$\lim_{x\to c} \frac{e^{x^2}-e^4}{x-2} = \lim_{x\to c} \frac{2xe^{x^2}}{1} = \frac{2(2)e^{2^2}}{1} = 4e^4$$

WARNING not quotient rule

Example F.6. Find $\lim_{x\to\infty} \dfrac{3x-2}{e^{x^2}}$

$$\frac{\lim_{x\to\infty}(3x-2)}{\lim_{x\to\infty} e^{x^2}} = \frac{\infty}{\infty} \leftarrow \text{indeterminate}$$

$\Longrightarrow$ use L'H rule

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}(3x-2)}{\frac{d}{dx} e^{x^2}} = \lim_{x\to\infty} \frac{3}{\underbrace{2xe^{x^2}}_{\to\infty}} = 0$$

Example F.7. Find $\lim_{x\to 0} \dfrac{\sin(x)}{x+1}$

can't use L'H rule!
not indeterminate

$$\lim_{x\to 0} \frac{\sin x}{x+1} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} x+1} = \frac{0}{1} = 0$$

WARNING cannot use L'H here unless $\frac{0}{0}$ $\frac{\infty}{\infty}$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} x+1 = 1$$

$$0 = \lim_{x\to 0} \frac{\sin x}{x+1} \neq \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

Example F.8. Find $\lim_{x\to 0} \frac{x^{100}}{x^{100}-x^{99}} = \frac{0}{0}$

$$\overset{L'H}{\Longrightarrow} \cancel{\lim_{x\to 0} \frac{100x^{99}}{100x^{99}-99x^{98}}} = \frac{0}{0}$$

$$\overset{L'H}{\Longrightarrow} \cancel{\lim_{x\to 0} \frac{9900x^{98}}{9900x^{98}-99*98x^{97}}}$$

$$\lim_{x\to 0} \frac{x^{100}}{x^{100}-x^{99}} = \lim_{x\to 0} \frac{x}{x-1} = \frac{0}{-1} = 0$$

factor instead

ALWAYS try factoring first!

Example F.9. Find $\lim_{x\to\infty} xe^{-x}$ ← indeterminate

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) * \lim_{x\to c} g(x)$$
(if both exist)

$$\lim_{x\to\infty} xe^{-x} = \infty * 0$$ ← indeterminate

$$\lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

INDETERMINATE FORMS

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty * 0$$

rewrite as quotient and use L'H rule.

Example F.10. Find $\lim\limits_{x\to 0^+} x\ln x$ $= 0 * (-\infty)$

indeterminate

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{(1/\ln x)} \quad (1)$$

or (2)

$$= \lim_{x\to 0^+} \frac{\ln x}{(1/x)}$$

use L'H on one of these

derivatives are okay to take but get messy

both are easy and don't get messy

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-x^{-2}}$$

$$= \lim_{x\to 0^+} -\frac{1}{x}x^2$$

$$= \lim_{x\to 0^+} (-x) = 0$$

Example F.11. Find $\lim_{x\to\infty} x^{1/x}$

$$\lim_{x\to c}\left(f(x)\right)^{g(x)} = \left(\lim_{x\to c} f(x)\right)^{\left(\lim_{x\to c} g(x)\right)}$$

if both exist

$$\lim_{x\to\infty} x^{1/x} = (\infty)^0$$

is this one?

$\boxed{\text{indeterminate form}}$ IDK

$$L = \lim_{x\to\infty} x^{1/x}$$

$$\ln(L) = \ln\left(\lim_{x\to\infty} x^{1/x}\right) = \lim_{x\to\infty} \ln\left(x^{1/x}\right)$$

$$= \lim_{x\to\infty} \frac{1}{x}\ln(x)$$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\ln(L) = 0$$

$$\iff L = e^0 = 1$$

Question F.12. What are the indeterminate forms we've looked at and how do we find their limits?

$\frac{0}{0}$, $\frac{\infty}{\infty}$ } apply L'H direction

$0 * \infty$ ← rewrite as quotient then apply L'H

∞^0, 0^0, 1^∞ } use a logarithm to say

$$\ln(L) = \ln\left(\lim_{x \to c} f(x)^{g(x)}\right)$$

$$= \underline{\lim_{x \to c} g(x) \ln(f(x))}$$

Appendix F. Extra examples

Example F.13.

(a) Find $\lim_{x\to 0} \frac{e^{3x} - 1 - 3x}{e^{x^2} - \cos x}$

(b) Find $\lim_{x\to 0} \ln x \tan x$

(c) Find $\lim_{x\to\infty} (1 + 3/x)^x$ $= \boxed{1^{\infty}}$ can't tell by inspection

$\frac{\infty}{\infty}$ $\frac{0}{0}$

$L = \lim_{x\to\infty} (1 + 3/x)^x$ ∞^0

$\ln L = \lim_{x\to\infty} \ln\left(\left(1 + 3/x\right)^x\right) = \lim_{x\to\infty} \underline{x \ln\left(1 + \frac{3}{x}\right)}$

$\infty \cdot 0$

L'H $= \lim_{x\to\infty} \frac{\ln(1 + 3/x)}{x^{-1}}$