Elementry Differential Geometry

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1 Differentiable Manifolds

The field of information geometry mostly requires only the theory of the locally characterizable properties of manifolds, and only fundamental ideas and methodologies of differential geometry are needed.

Intuitively, a manifold S is a 'set with a coordinate system', whose elements are called 'points'. By coordinate system we mean a one-to-one mapping from S (or its subset) to R^n, thus we can use vector of n real numbers to specify points in S (n is called dimension of S).

A coordinate system has S as its domain is a global coordinate system, while there are some manifolds do not have global coordinate systems (surface of a sphere for example).

Let S be a manifold and $\phi : S \to R^n$ be a coordinate system for S. Each ϕ maps each point p in S to n real numbers:

$$\phi(p) = [\xi^1(p), ...\xi^n(p)] = [\xi^1, ..., \xi^n]$$

each ξ^i can be viewed as a function which maps a point into its i^{th} coordinate, we can $\xi^i, i = 1, ..., n$ coordinate functions. We shall write coordinate system ϕ in the way $\phi = [\xi^1, ...\xi^n] = [\xi^i]$.

If there are two coordinate systems for S, $\phi = [\xi^i]$ and $\psi = [\rho^i]$. The transformation on R^n given by

$$\psi \circ \phi^{-1} : [\xi^1, ..., \xi^n] \to [\rho^1, ..., \rho^n]$$

is called coordinate transformation.

Definition 1. *Let S be a set, if there exists a set of coordinate systems A for S which satisfies conditions below, we call (S, A) an n dimensional C^∞ differentiable manifold, or simply manifold.*

1. *Each element ϕ of A is a one-to-one mapping from S to some open subset of R^n;*
2. *For all $\phi \in A$, given any one-to-one mapping ψ from S ro R^n, the following holds:*

$$\psi \in A \Leftrightarrow \psi \circ \phi^{-1} \textit{is a } C^\infty \textit{ diffeomorphism};$$

By C^∞ diffeomorphism we mean $\psi \circ \phi^{-1}$ and its inverse $\phi \circ \psi^{-1}$ are both C^∞ (infinitely many times differentiable). Infinitely differentiable is not necessary actually, we may consider this notation as 'sufficiently smooth'.

Let $f : S \to R$ be a function on a manifold S, and $\phi = [\xi^i]$ be a coordinate system for S. We have $f(p) = \bar{f}(\xi^1, ..., \xi^n)$, where $\bar{f} = f \circ \phi^{-1}$, i.e. $\bar{f}$ is a function of coordinates and has domain $\phi(S)$.

Suppose $\bar{f}$ is partially differentiable at each point in $\phi(S)$, then the partial derivative $\frac{\partial}{\partial \xi^i}\bar{f}(\xi^1, ..., \xi^n)$ is also a function on $\phi(S)$. We can define the partial derivative of f to be

$$\frac{\partial f}{\partial \xi^i} = \frac{\partial \bar{f}}{\partial \xi^i} \circ \phi : S \to R$$

We use $(\frac{\partial f}{\partial \xi^i})_p$ to denote the partial derivative at point p.

When $\bar{f} = f \circ \phi^{-1}$ is C^∞, we can f a C^∞ function on S. This definition does not depend on the coordinate systems. Partial derivatives of a C^∞ function is also C^∞.

Let's denote the class of C^∞ functions on S by $F(S)$, or simply F. For all $f, g \in F$ and a real number c, define the sum $f + g$ as $(f + g)(p) = f(p) + g(p)$, the scaling cf as $(cf)(p) = cf(p)$, and the product $f\dot{g}$ as $(f\dot{g})(p) = f(p)\dot{g}(p)$, these functions are also members of F.

Coordinate functions are clearly C^∞ on manifolds. Let $[\xi^i]$ and $[\rho^i]$ be two coordinate systems on manifold S, we have

$$\frac{\partial \xi^i}{\partial \rho^j}\frac{\partial \rho^j}{\partial \xi^k} = \frac{\partial \rho^i}{\partial \xi^j}\frac{\partial \xi^j}{\partial \rho^k} = \delta^i_k$$

where $\delta^i_k = 1$ if $i = k$ and 0 otherwise. Note that Einstein's convention is used.

Let S and Q be manifolds with coordinate systems $\phi : S \to R^n$, $\rho :\to R^m$. A mapping $\lambda : S \to Q$ is said to be C^∞ or smooth if $\psi \circ \lambda \circ \phi^{-1}$ is a C^∞ mapping from R^n to R^m. A necessary and sufficient condition is that $f \circ \lambda \in F(S)$ for all $f \in F(Q)$.

2 Tangent Vectors and Tangent Spaces

Consider a one-to-one function $\gamma : I \to S$ from some interval $I \subset R$ to S. By defining $\gamma^i(t) = \xi^i(\gamma(t))$ we may express the point $\gamma(t)$ using coordinates as $\bar{\gamma}(t) = [\gamma^1(t), ..., \gamma^n(t)]$. If $\bar{\gamma}(t)$ is C^∞ for $t \in I$, we call γ a C^∞ curve on S. This definition is independent of coordinate system choice.

Let $f \in F$ be a C^∞ function on S and consider value of $f(\gamma(t))$ on the curve. Using coordinate, we have $f(\gamma(t)) = \bar{f}(\bar{\gamma}(t)) = \bar{f}(\gamma^1(t), ..., \gamma^n(t))$. The directional derivative of f along curve γ is:

$$\frac{d}{dt}f(\gamma(t)) = (\frac{\partial \bar{f}}{\partial \xi^i})_{\bar{\gamma}(t)}\frac{\gamma^i(t)}{dt} = (\frac{\partial f}{\partial \xi^i})_{\gamma(t)}\frac{\gamma^i(t)}{dt}$$

Partial derivative is simply a directional derivative along a coordinate axis, the operator $(\frac{\partial}{\partial \xi^i})_p$ is the 'tangent vector' at point p of the i^{th} coordinate curve. If we define an operator $(\frac{d\gamma}{dt})_p$ which maps $f \in F$ to $\frac{d}{dt}f(\gamma(t))|_{t=a}$ as tangent vector, then the above formula can be rewritten as:

$$(\frac{d\gamma}{dt})_p = \frac{d\gamma^i(t)}{dt}|_{t=a}(\frac{\partial}{\partial \xi^i})_p$$

Consider all the curves pass through point p, we denote the set of all the tangent vectors corresponding to these curves by T_p or $T_p(S)$.

$$T_p(S) = \{c^i(\frac{\partial}{\partial \xi^i})_p,\ [c^1, ..., c^n] \in R^n\}$$

this forms a linear space with dimension n. We cal $T_p(S)$ and its elements the tangent space and tangent vectors of S at point p. We call $(\frac{\partial}{\partial \xi^i})_p$ the natural basis of coordinate system $[\xi^i]$.

Let $D \in T_p$ be some tangent vector, $f, g \in F, a, b \in R$, then

$$D(af + bg) = aD(f) + bD(g), D(f \cdot g) = f(p)D(g) + g(p)D(f)$$

Let $\lambda : S \to Q$ be a smooth mapping from manifold S to manifold Q, then the linear mapping $(d\lambda)_p : T_p(S) \to T_{\lambda(p)}(Q)$ is differential of λ at p. When S and Q have coordinate systems $[\xi^i], [\rho^i]$ respectively, we have:

$$(d\lambda)_p((\frac{\partial}{\partial \xi^i})_p) = (\frac{\partial(\rho^j \circ \lambda)}{\partial \xi^i})_p(\frac{\partial}{\partial \rho^j})_{\lambda(p)}$$

3 Vector Fields and Tensor Fields

4 Submanifolds

5 Riemannian Metrics

6 Affine Connections and Covariant Derivatives

7 Flatness

8 Autoparallel Submanifolds

9 Projection of Connections and Embedding Curvature

References