for independence
for confidence
for creativity
for insight

Circular functions 3

Graphs of circular functions

teacher version

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	**graphs $y = \cos x$ and the like**
Relationships between circular functions	$\sin(90^\circ - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A+B) = \sin A \cos B + \cos A \sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on
Differentiating circular functions	radians, and tangents to graphs
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

The main difficulty with the unit circle approach to circular functions is that the axes on the graph with the unit circle are not the same as the axes on the graphs of the functions. For example, with $y = \sin x$, the angle on the circle becomes the x coordinate on the graph, but the y coordinate on the circle is still the y coordinate on the graph, whereas on the graph $y = \cos x$, it is the x coordinate on the circle that becomes the y coordinate on the graph. I have tried so many ways to introduce the graphs over the years, most of which do the job but leave understanding just a little hazy. The questions here are designed to demonstrate to my students just how the circle and the graphs are related; or look at it another way, how to generate the graphs from the unit circle definitions. This seems to me rather more worthwhile than making a table of values with (or without) a calculator, as it shows just why the curve is the shape that it is.

Without some kind of exercise along these lines, it's quite possible to leave school either without having seen the unit circle definitions of sin, cos, and tan, or without understanding the shape of the curve, or both. This makes working with these functions a skill without any understanding, almost magic.

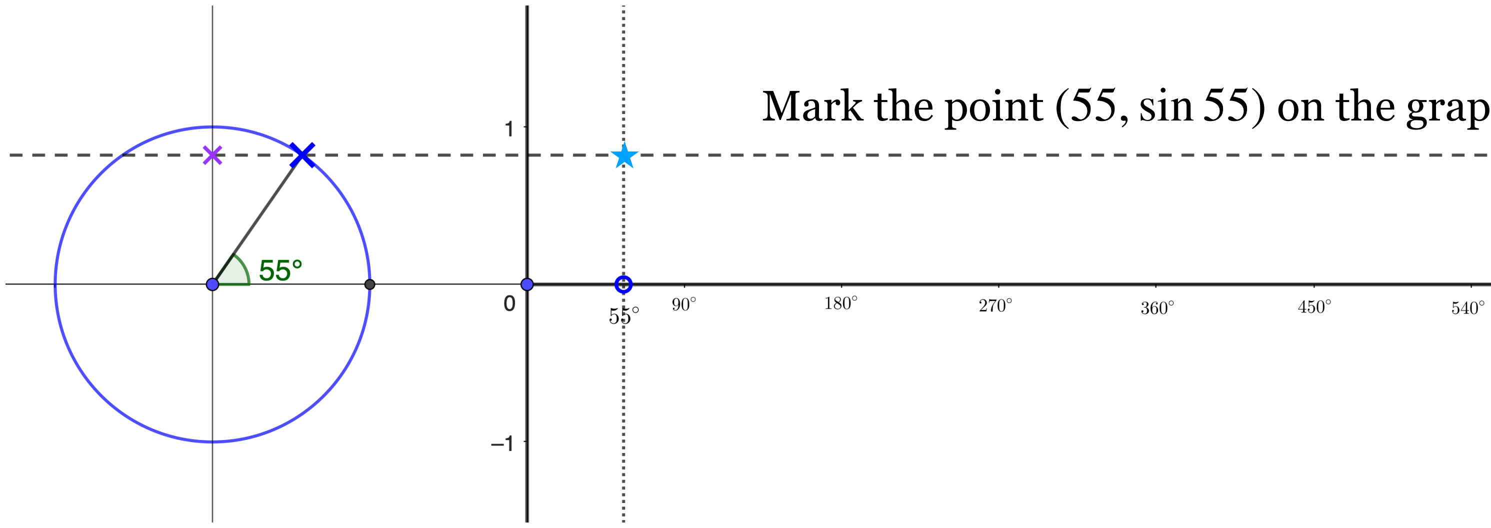
Of course this relationship is far easier to demonstrate with dynamic geometry or animation, and this is a valuable approach. Even this, though, still shows students something that they can nod their heads to without gaining an embodied (that is, the strongest possible) sense of why things are the way they are.

In the first sequence, your students may well get the principle right away or they may need a little time to understand the diagrams and what the question is asking. Give them space to work this out, or challenge them to work it out for themselves, rather than pointing it out.

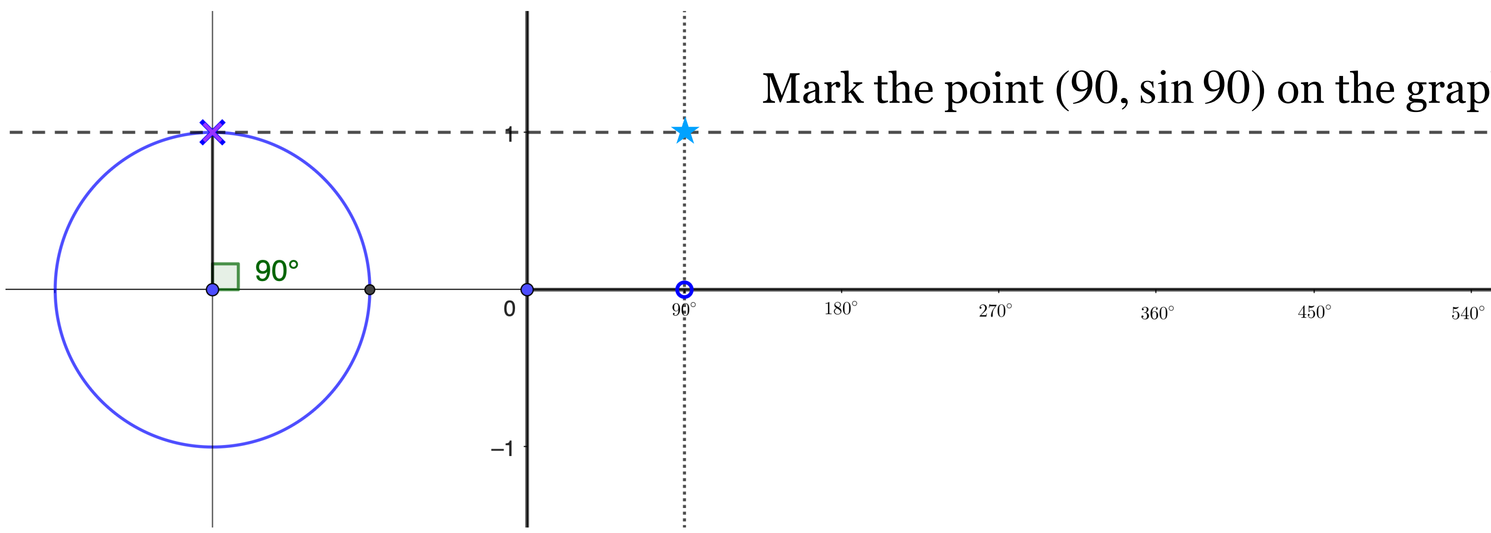
In the cos sequence, it's a bit harder, but ask them to think about the relationship between the lengths of the pink lines on the circles and the lengths of the pink lines on the graphs. Soon enough, someone will spot what's going on!

In the tan sequence, spotting the relationship between the pink lines is much harder and you may well need to lead them more actively toward understanding that length of the pink line on the graph represents the gradient of the pink line on the unit circle.

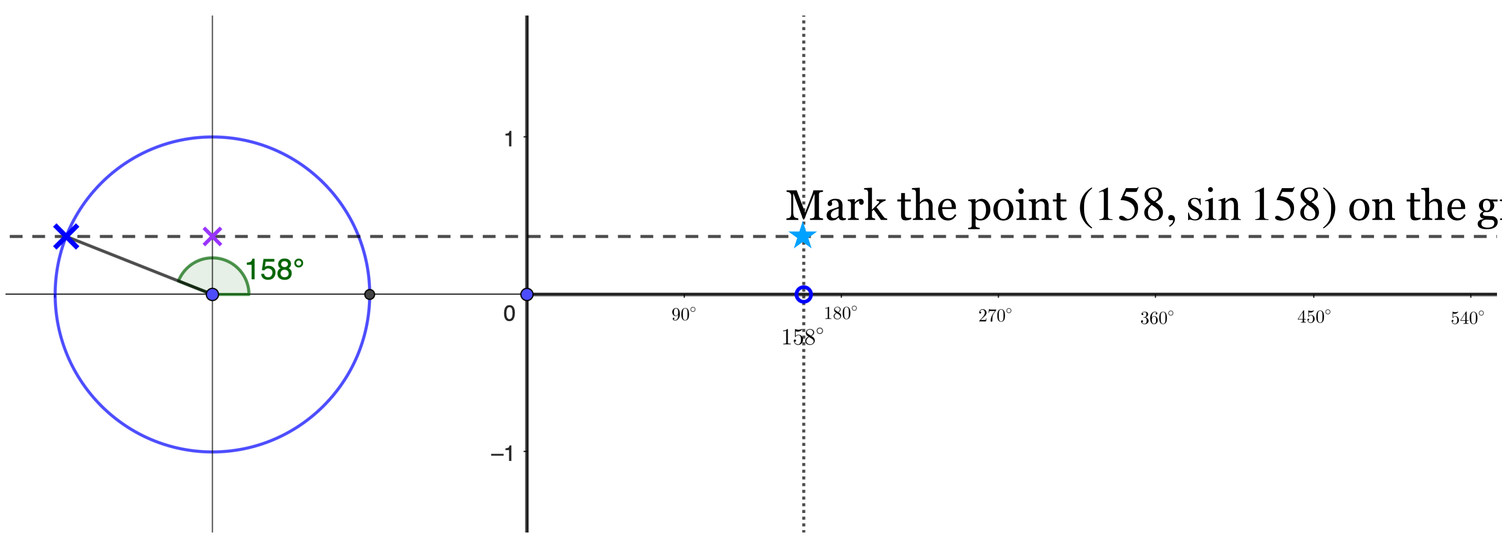
Mark the point (55, sin 55) on the graph



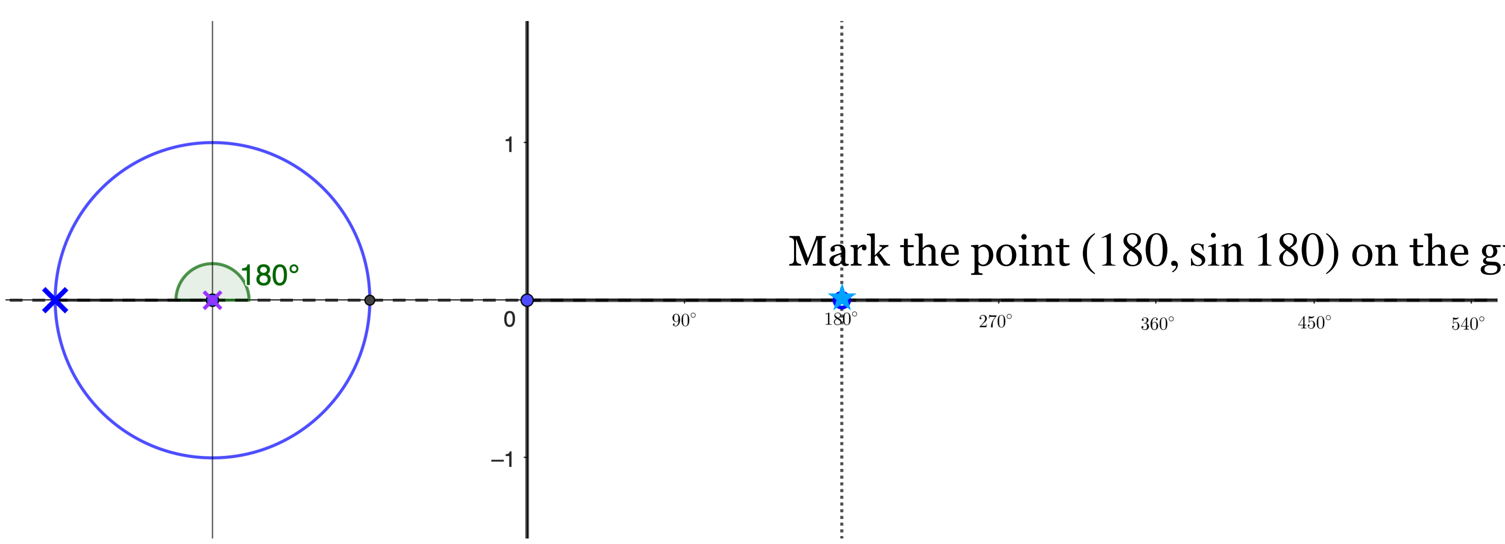
Mark the point (90, sin 90) on the graph

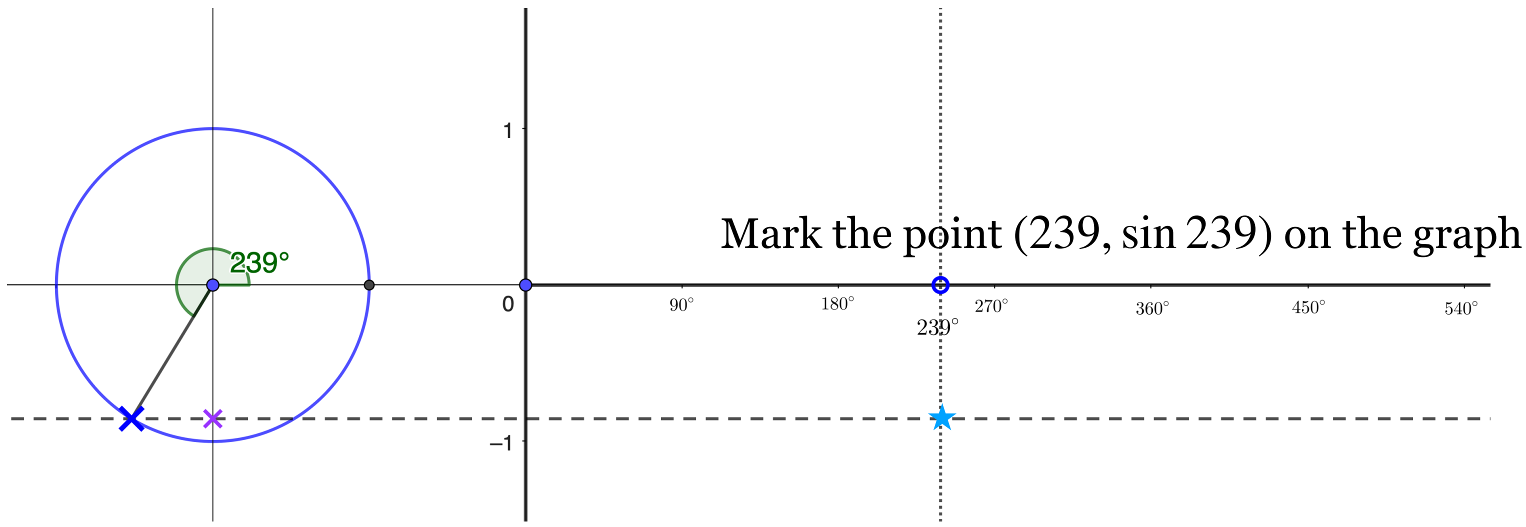


Mark the point (158, sin 158) on the graph

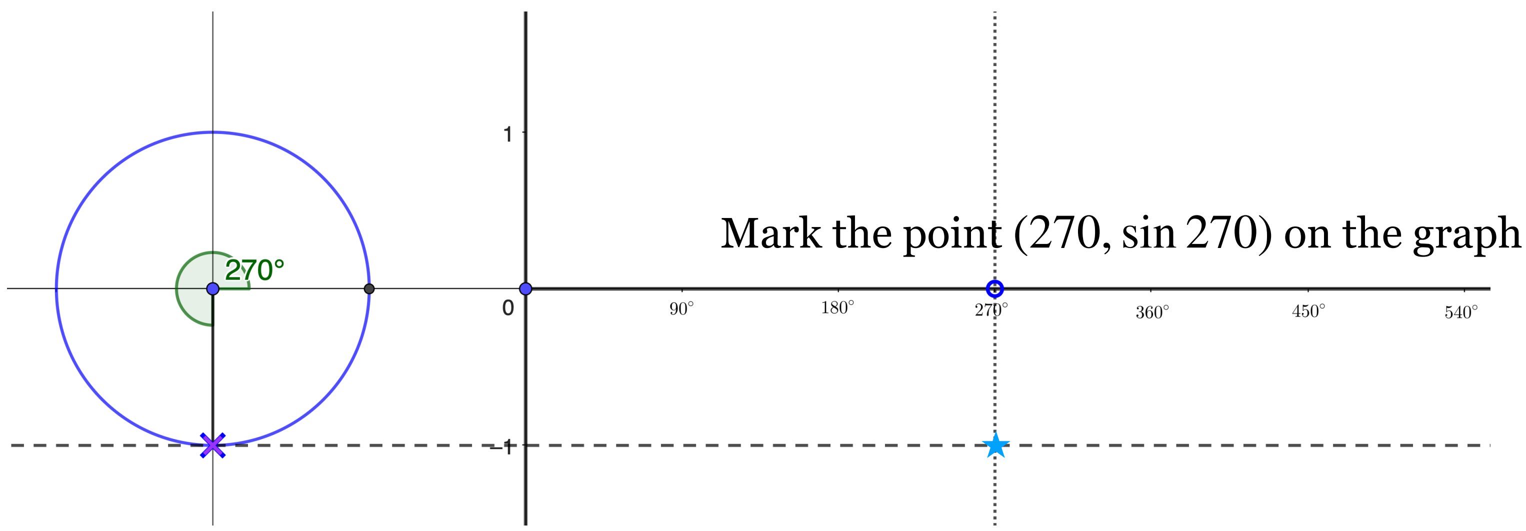


Mark the point (180, sin 180) on the graph

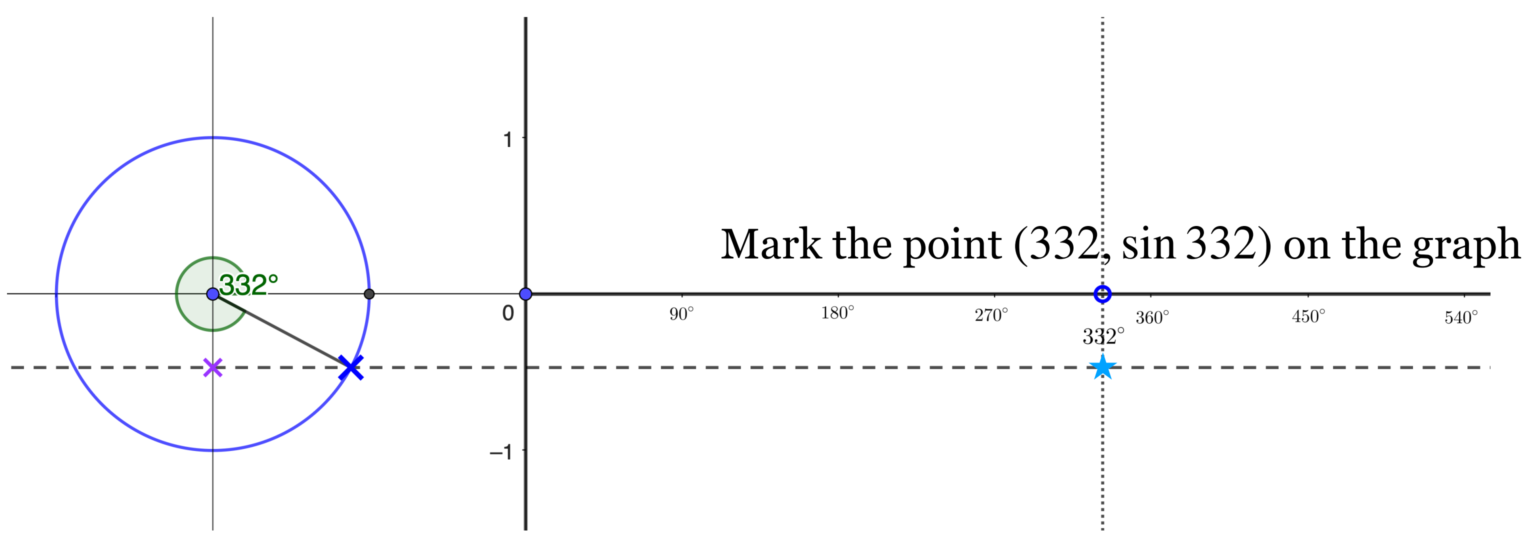




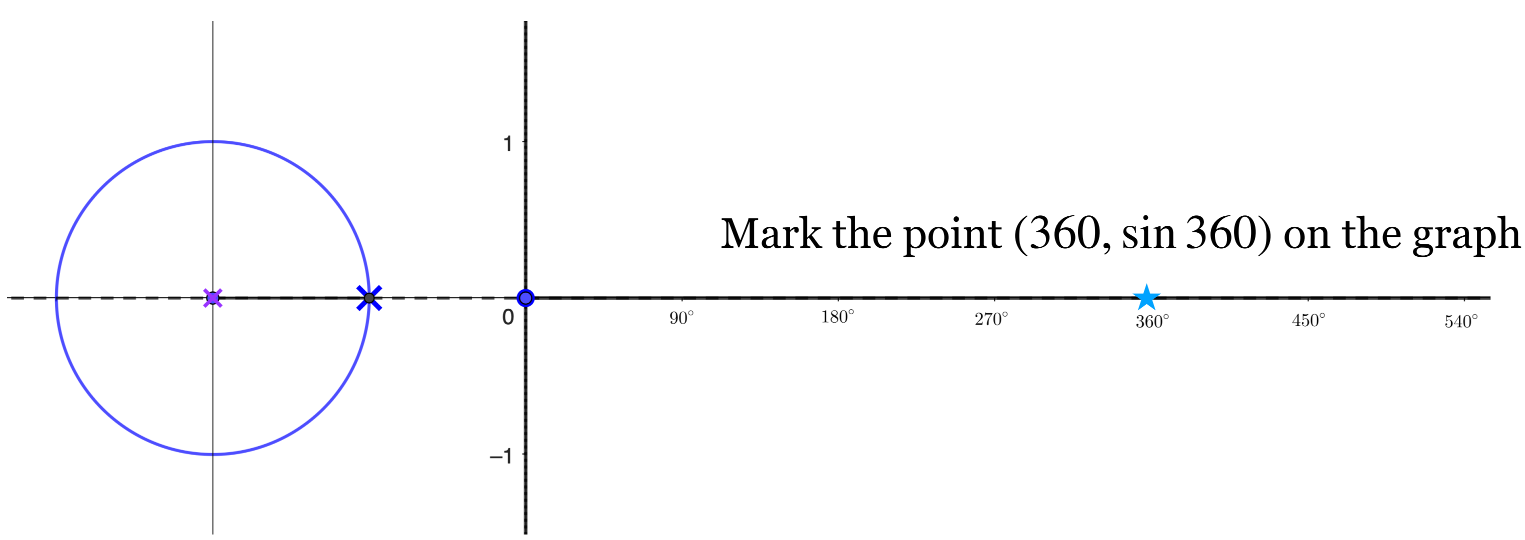
Mark the point (239, sin 239) on the graph



Mark the point (270, sin 270) on the graph

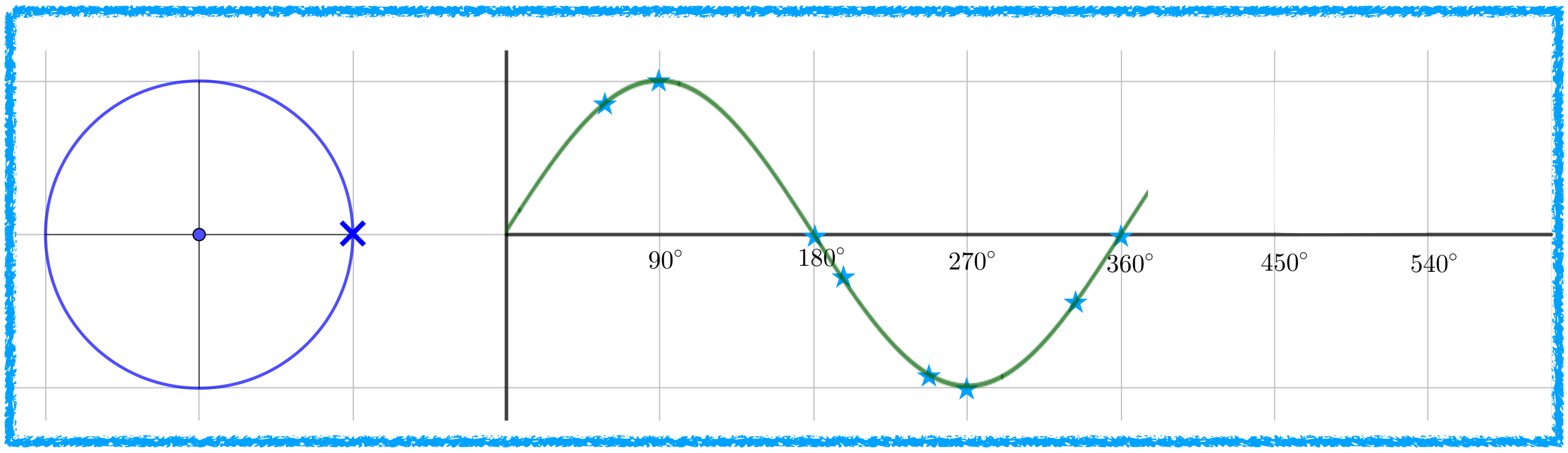


Mark the point (332, sin 332) on the graph

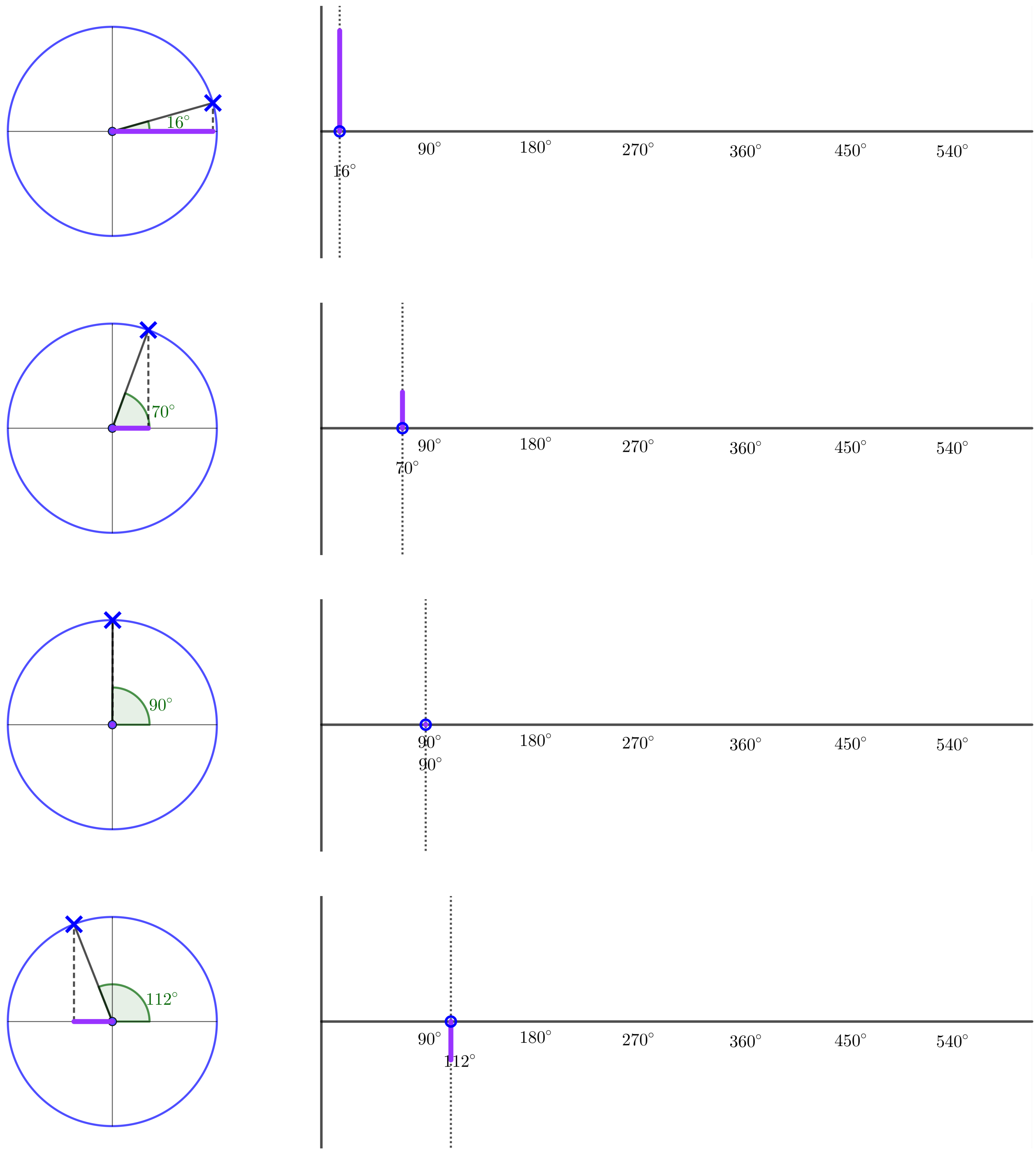


Mark the point (360, sin 360) on the graph

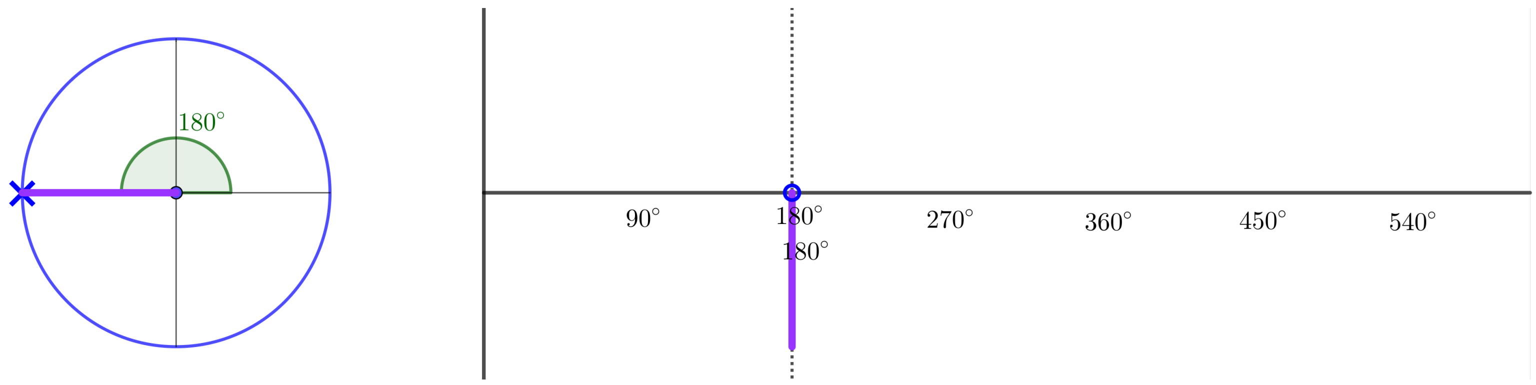
Use these points as a guide to draw the graph $y = \sin x$.

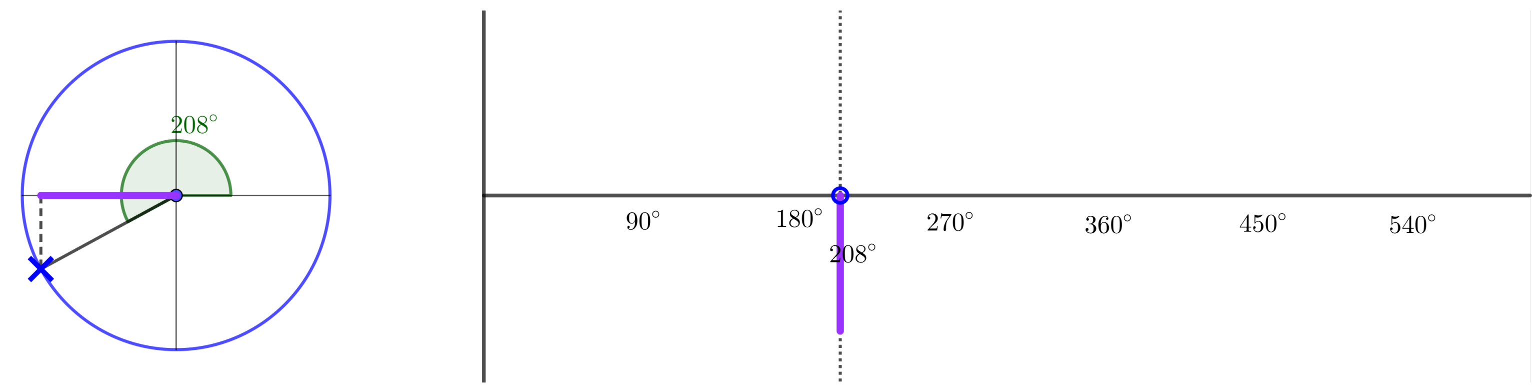


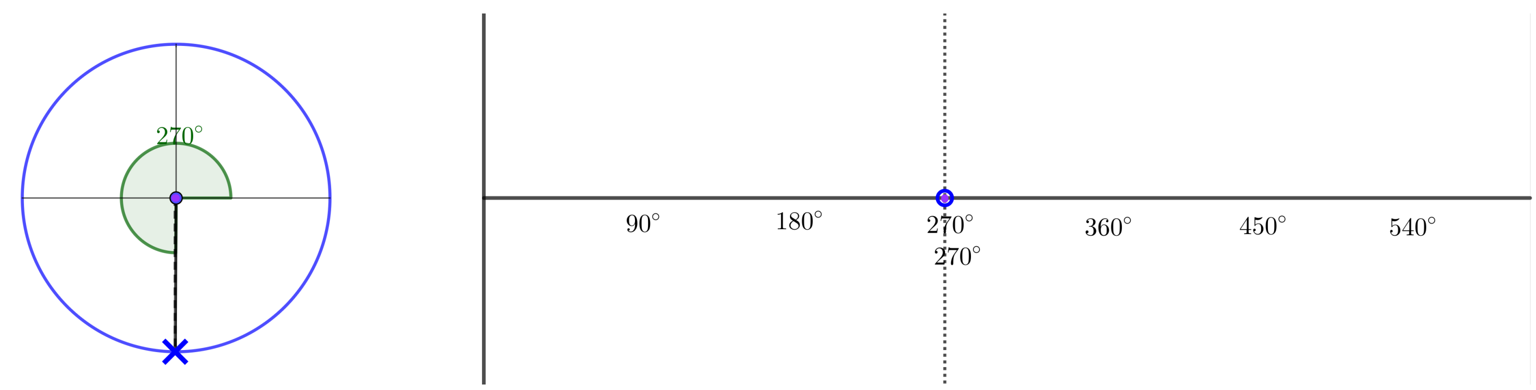
Look at the next sequence of images, and think about the relationship between the purple line segment on the left and the purple line segment on the right.

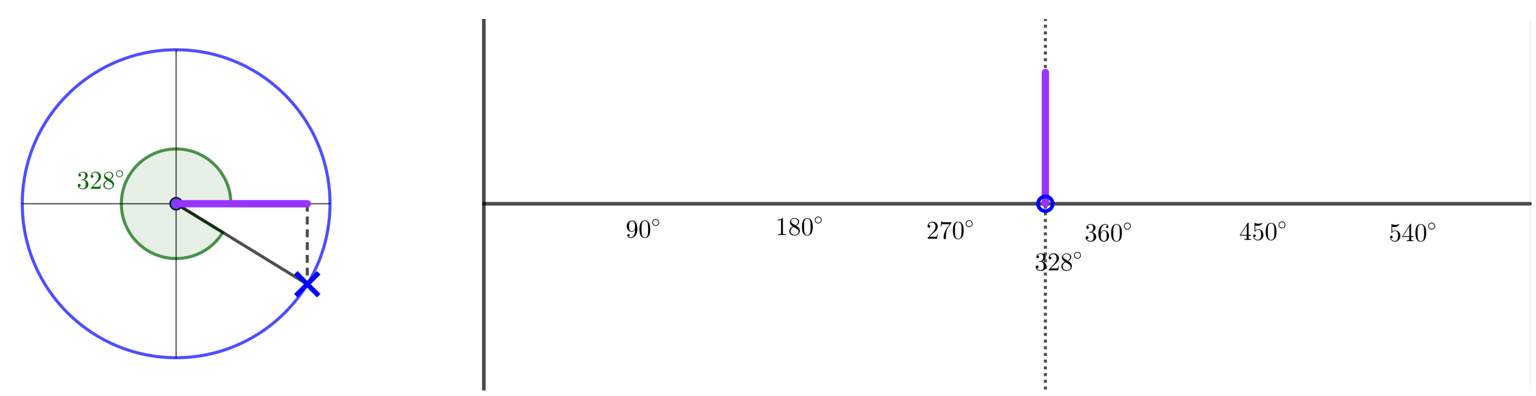


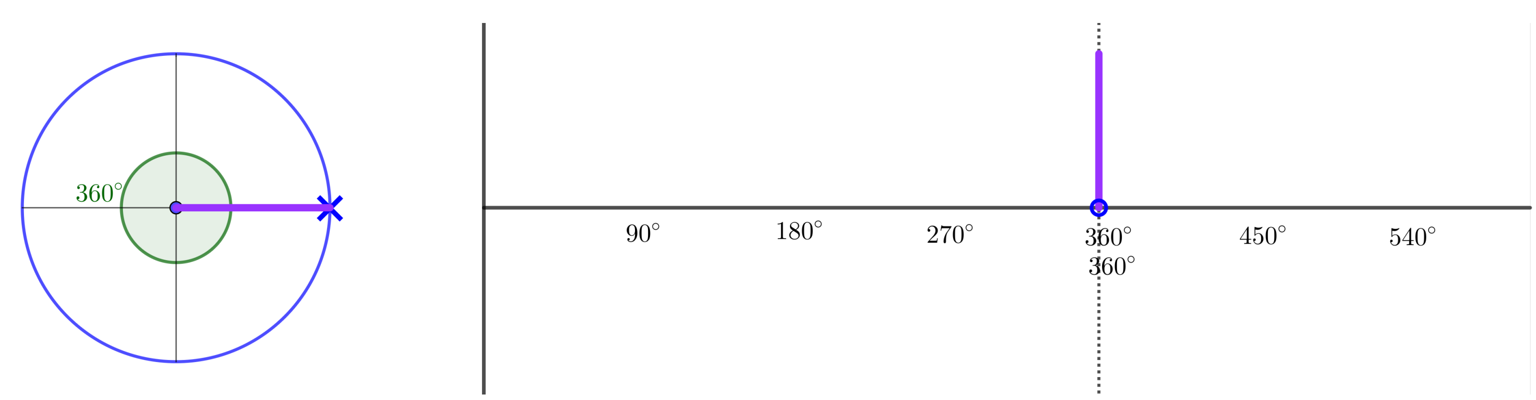
Demonstrating how the x coordinates on the unit circle become the y coordinates on the graph is a bit trickier. The signed length of the segment on the left is the same as the signed length of that on the right.


180°
90°
180°
270°
360°
450°
540°
180°

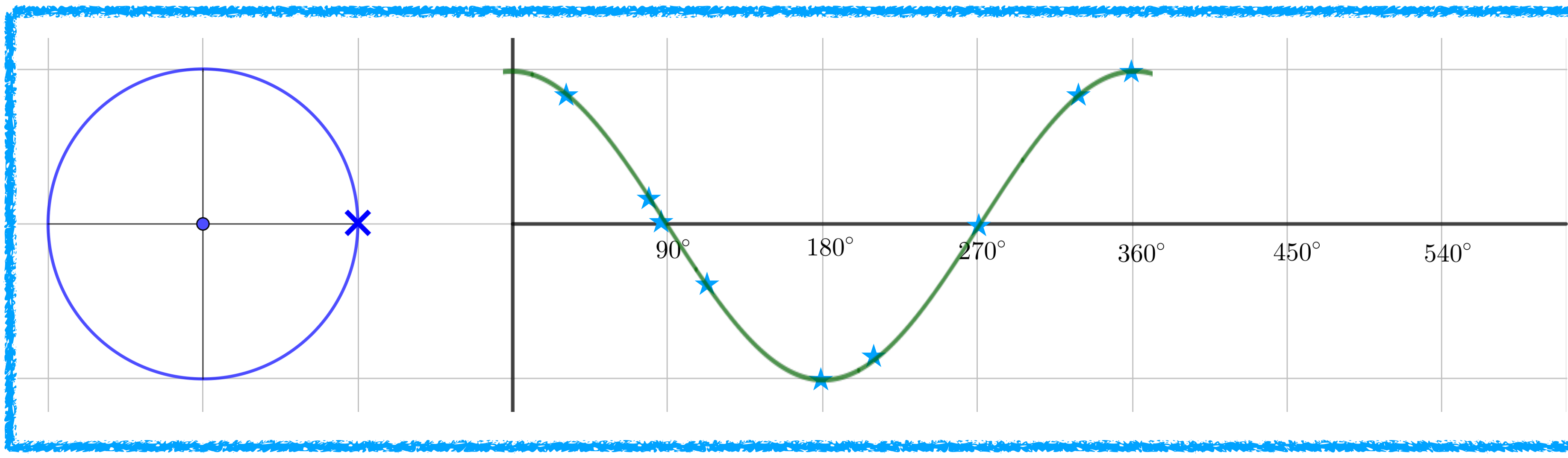

208°
90°
180°
270°
360°
450°
540°
208°


270°
90°
180°
270°
360°
450°
540°
270°

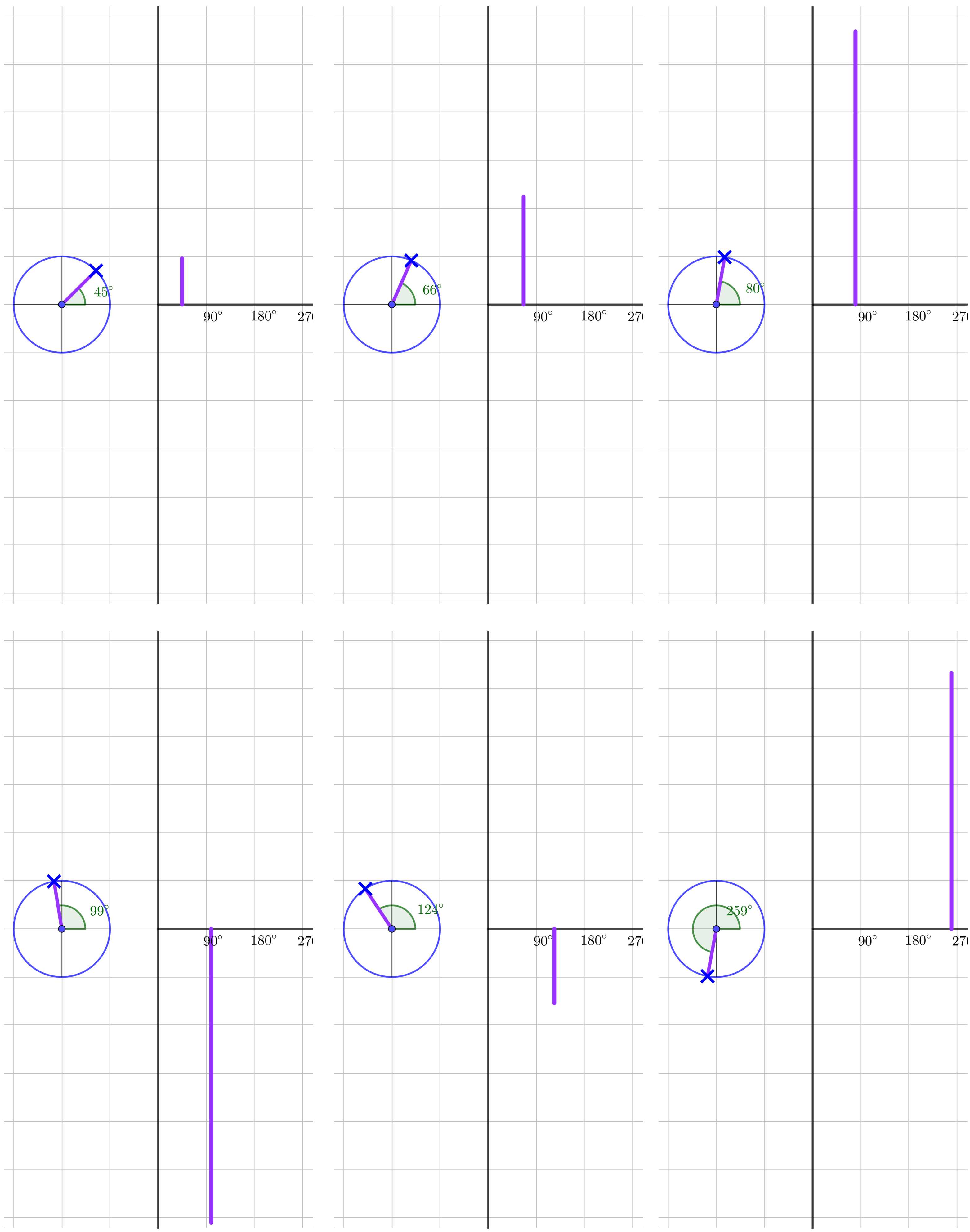

328°
90°
180°
270°
360°
450°
540°
328°


360°
90°
180°
270°
360°
450°
540°
360°

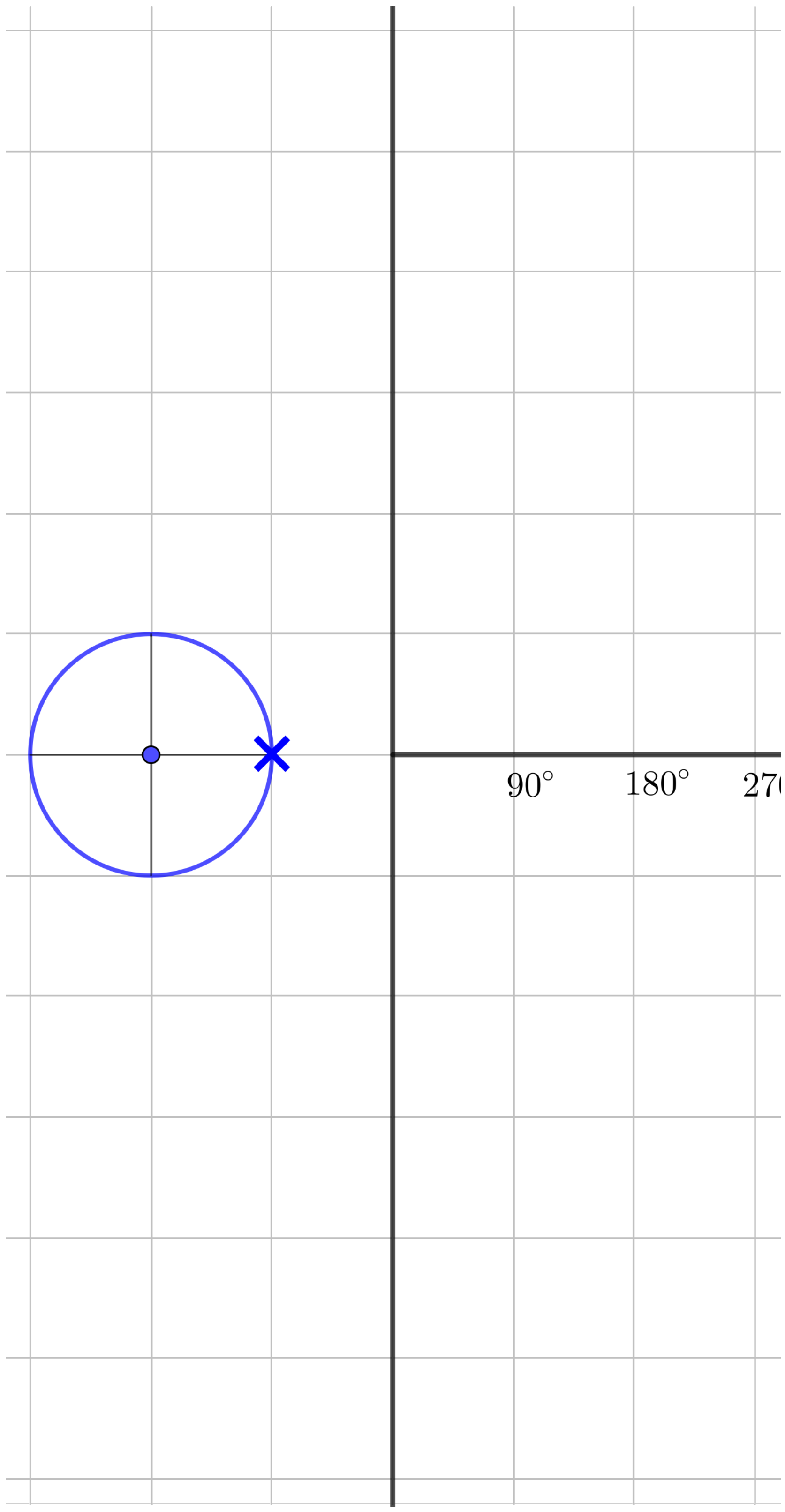
Use these points as a guide to draw the graph $y = \cos x$.



Look at this sequence of images, and describe the relationship between the purple line segment on the left and the purple line segment on the right.



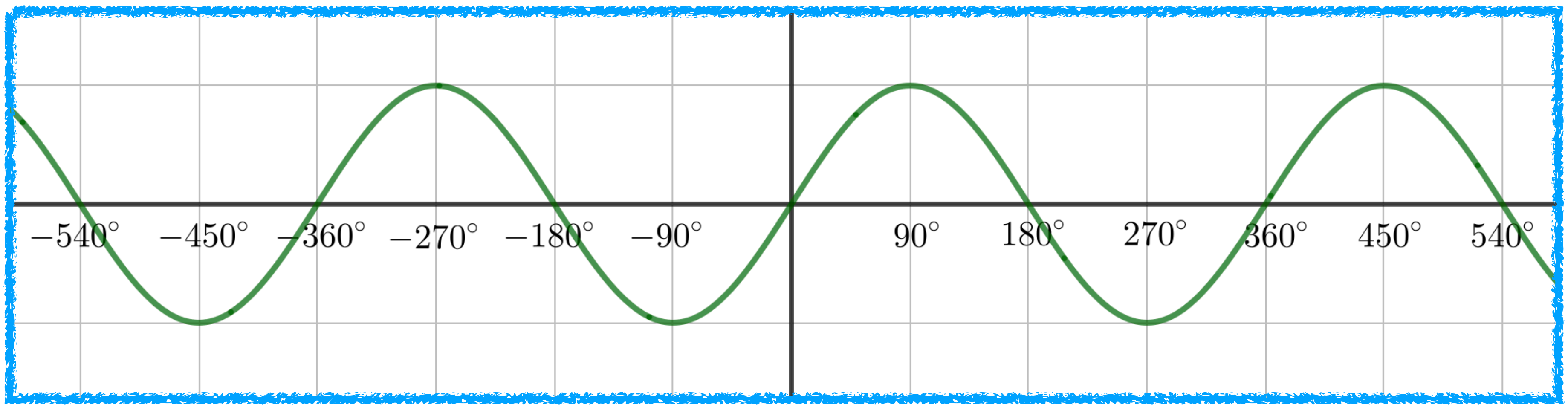
Use this idea to draw the graph $y = \tan x$.



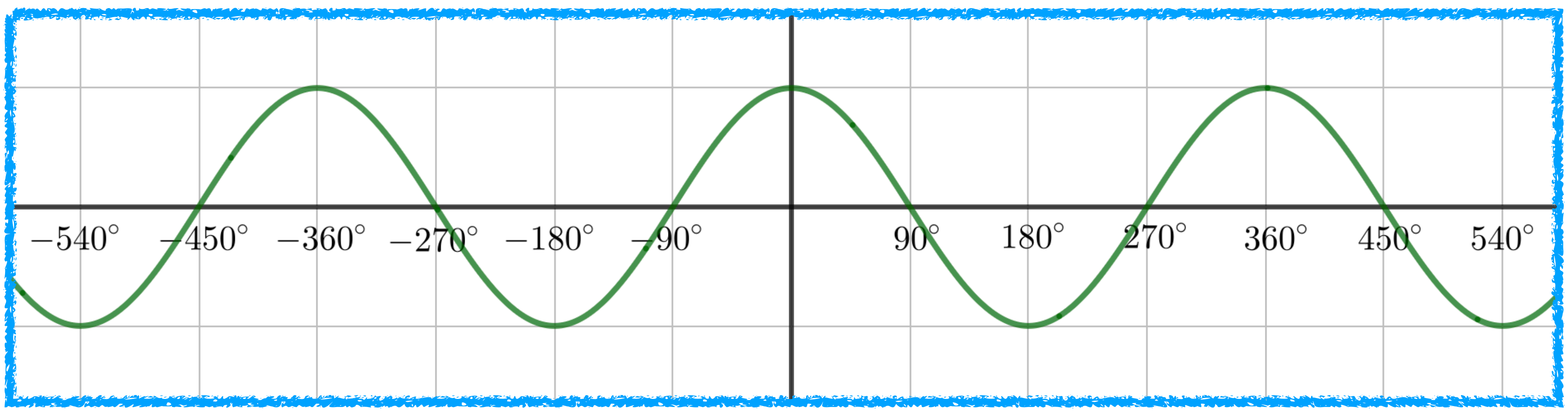
In the last sequence of diagrams, it is the gradient of the segment on the left that determines the signed length of the segment on the right. This should clarify the reason for the behaviour around the asymptotes: as the angle gets closer to 90 from below, the gradient gets large without limit. As the angle gets closer to 90 from above, the gradient gets increasingly large in the negative direction.

The value of this section is in understanding the relationship between the graph and the gradient of the segment in the unit circle.

Draw the graph $y = \sin x$.

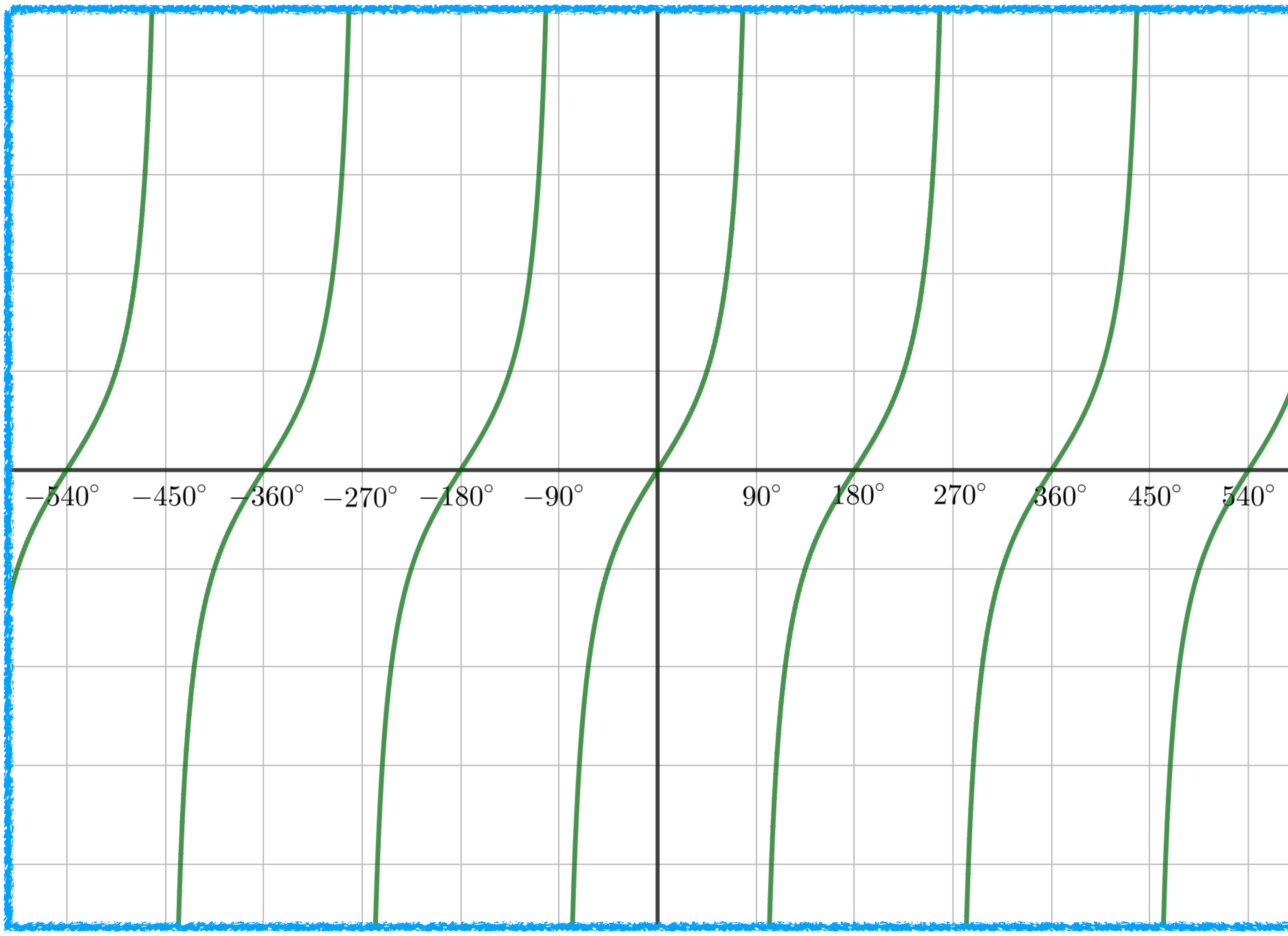


Draw the graph $y = \cos x$.

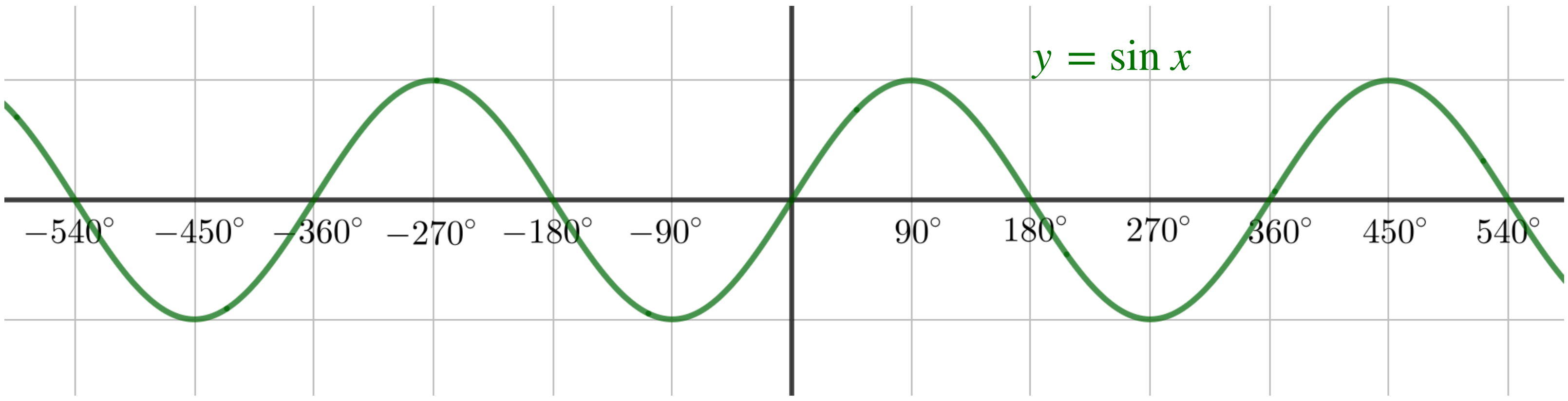


So far, we have only drawn the graphs for positive values of x, so here we add in the negative values by continuing the pattern, checking that we are correct by looking at the unit circle from time to time.

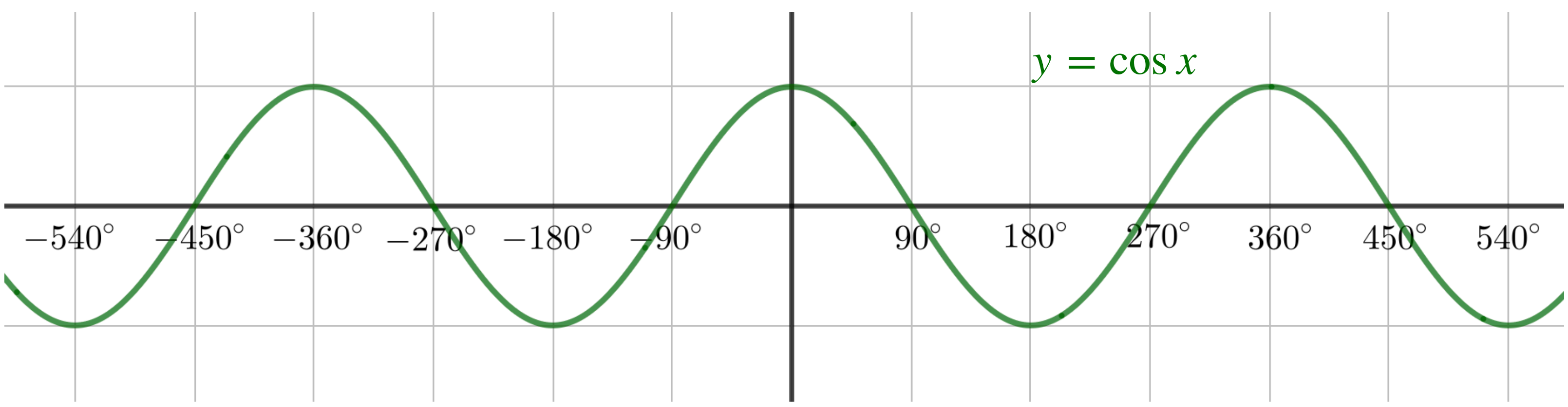
Draw the graph $y = \tan x$.



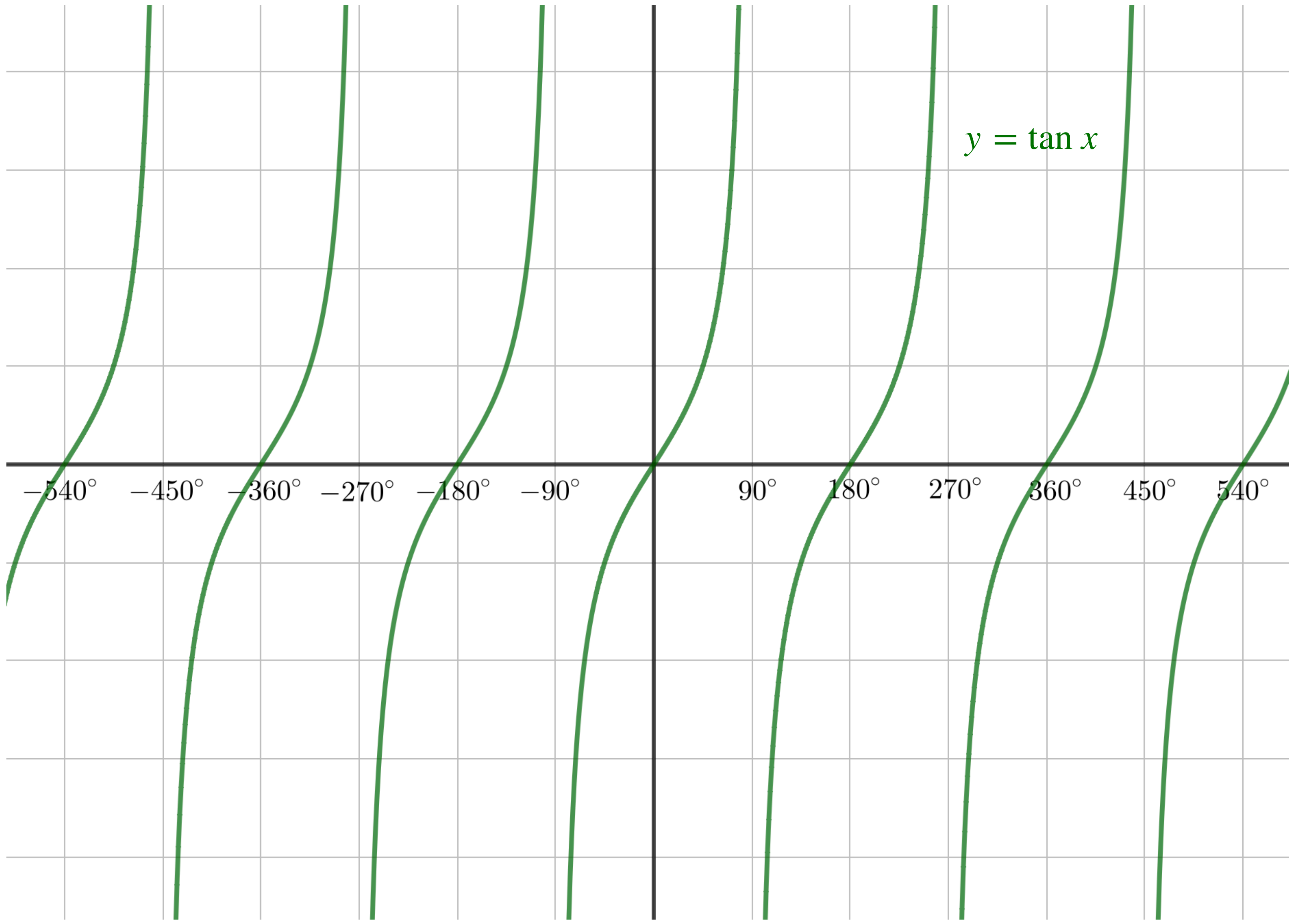
Describe the symmetry of each of these graphs.



Reflect in both axes or in the origin. Or reflect in x axis and the line $x = 180n$ for any integer n.
Or reflect in any points $(180n, 0)$



Reflect in the y axis or in the line $x = 180n$ for any integer n



Reflect in both axes or in the origin. Or reflect in x axis and the line $x = 180n$ for any integer n. Or reflect in any points $(180n, 0)$