Reciprocal circular functions

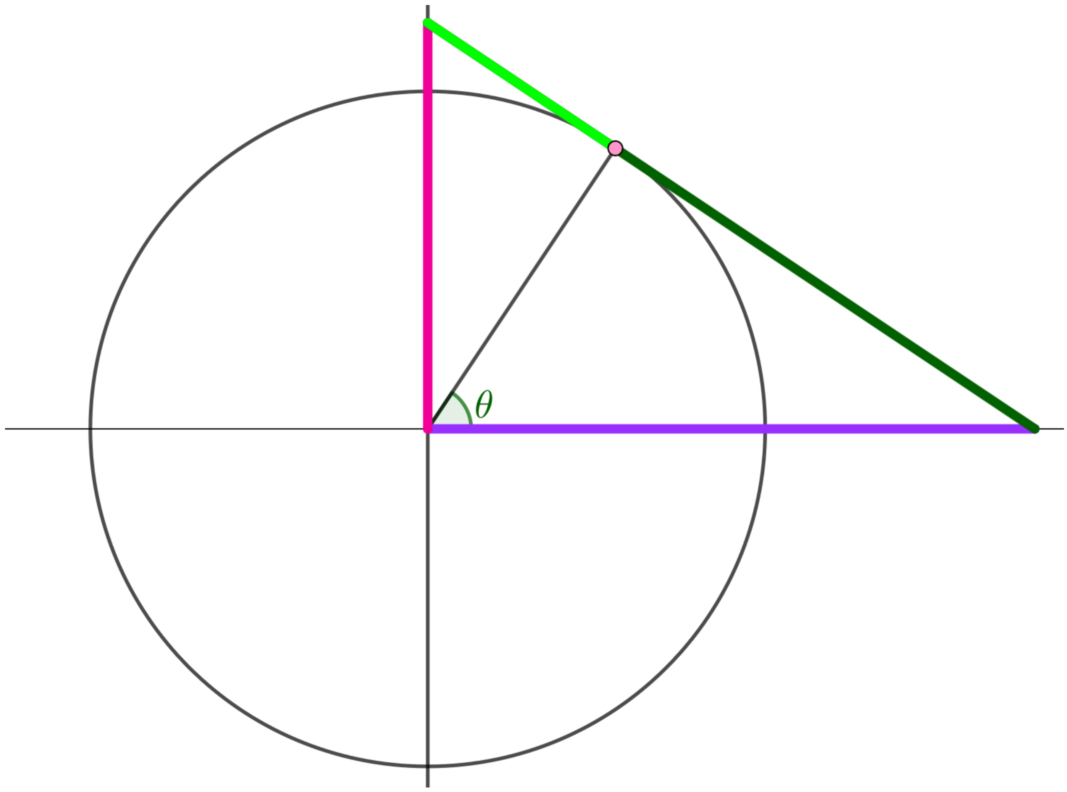
sec, cosec (or csc) and cotan (or cot) get far less attention than sin, cos, and tan, but they also have interesting interpretations on the unit circle, which I introduce using this diagram.

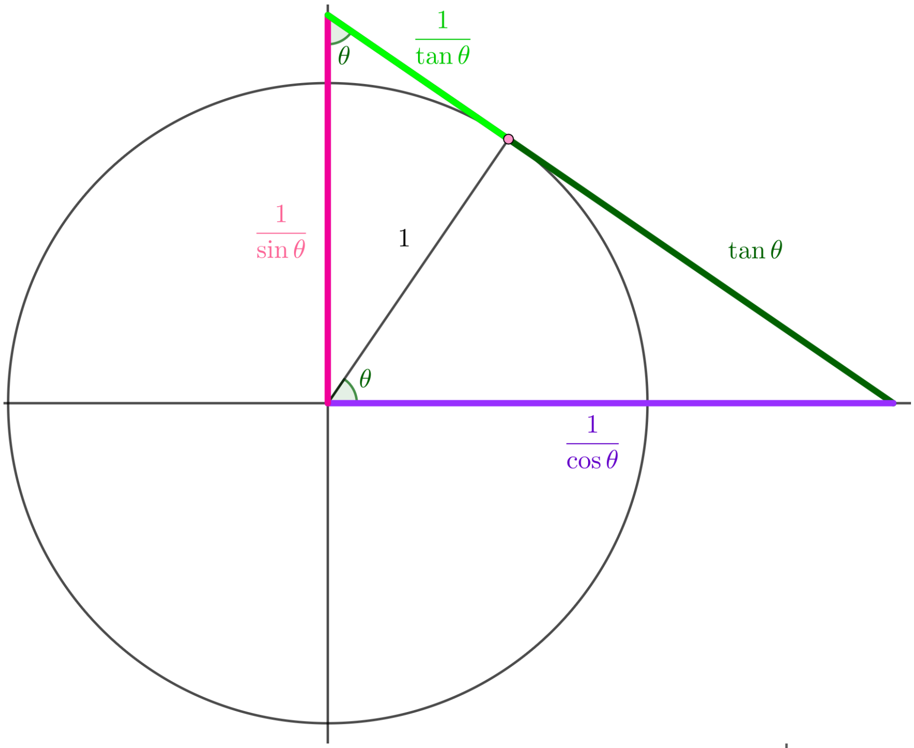
If you were to look at the diagram on the next page and say to me: “what’s the point of a diagram like this? Why not just tell them what sec, cosec, and cotan are?” I would say: “you are quite right, there is no real point, go ahead and miss it out.” I don’t really believe this, though. It all helps to build up a really strong idea of the circular functions, and it’s just plain interesting.

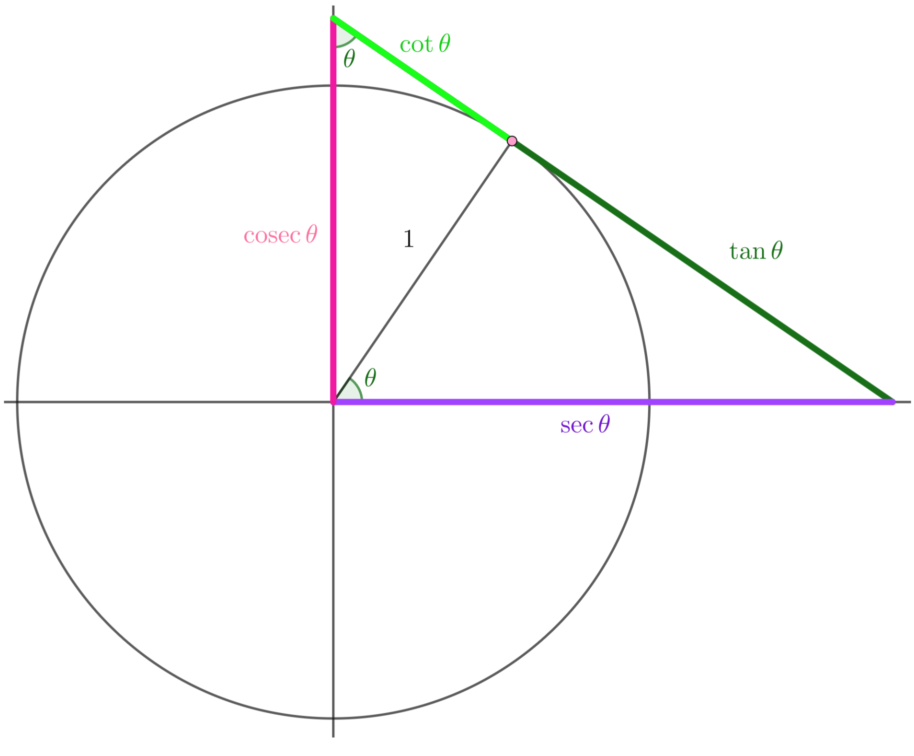
We tend to sketch their graphs by starting from sin cos and tan and sketching their inverses, but here too we can relate the functions to the unit circle. And here again, you might ask what the point is, and I would give the same answer.

This is a circle radius 1.

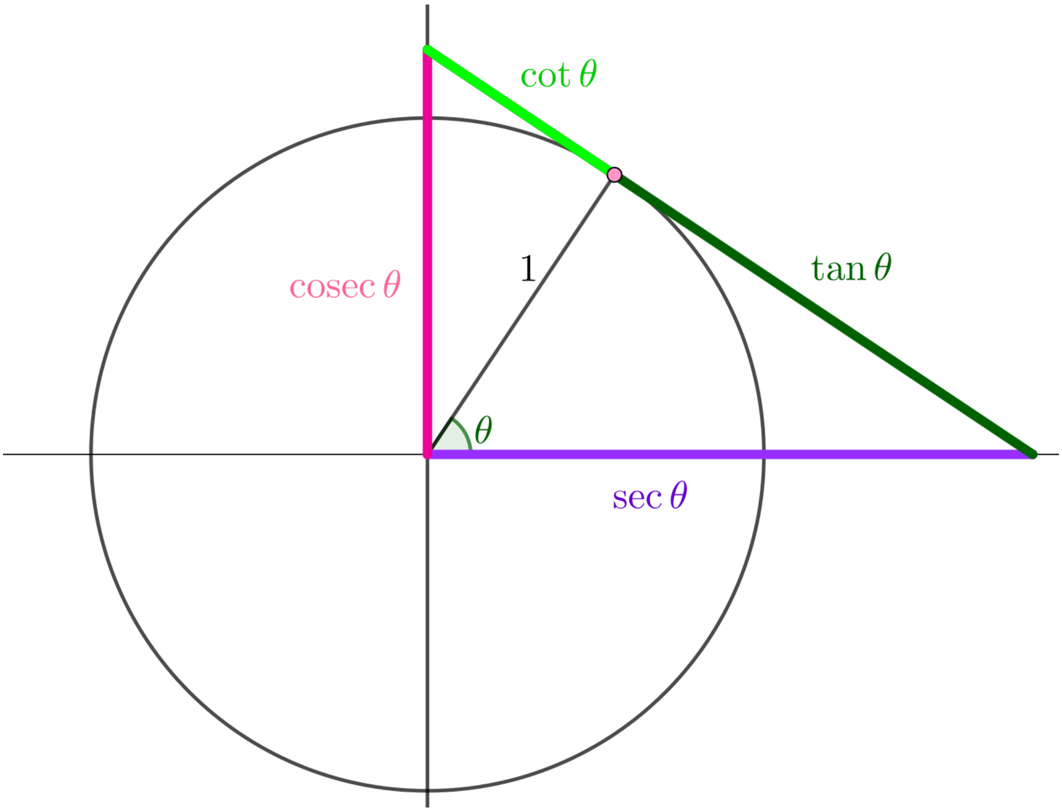
What are the lengths of the highlighted segments?







I suggest starting with the top diagram, which is what your students' work should look like.
Use this to introduce the new names in the right-hand triangle.



There are three right-angled triangles here. What would Pythagoras say about each of them?

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\operatorname{cosec}^2\theta + \sec^2\theta = (\tan\theta + \cot\theta)^2$

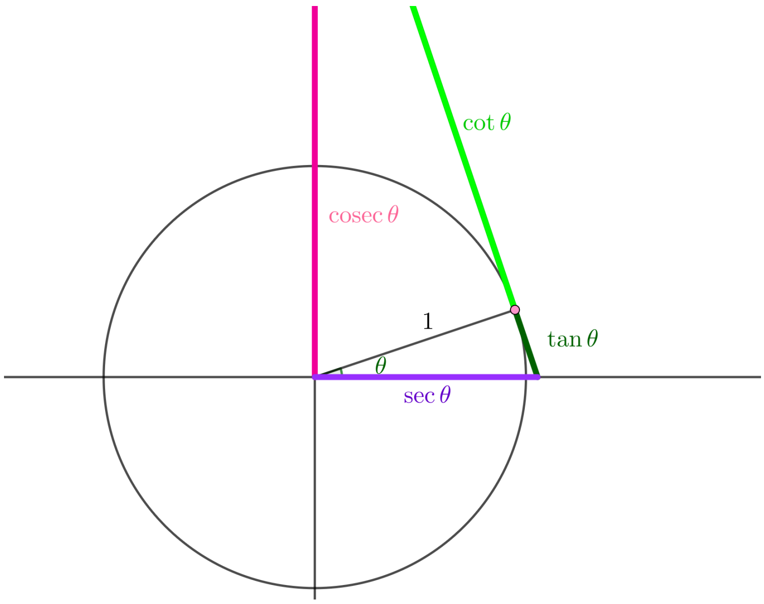
The first two of these can easily be derived from the identity

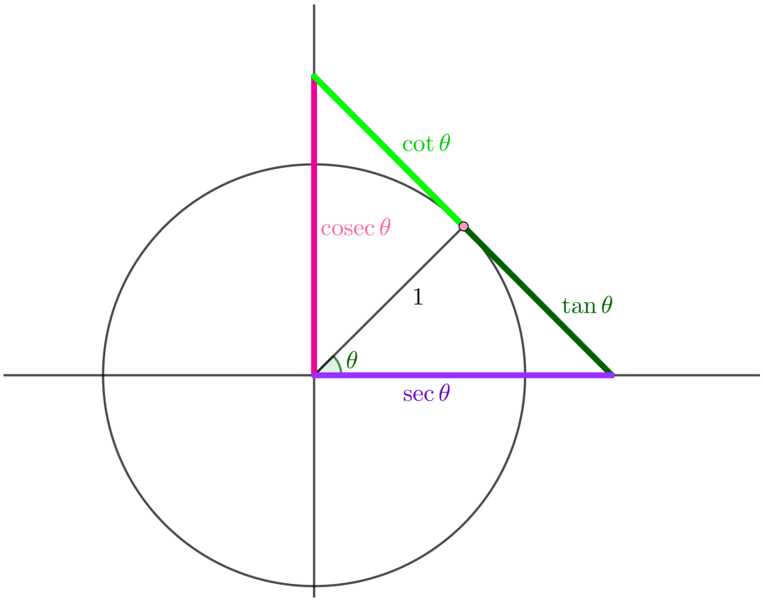
$\cos^2\theta + \sin^2\theta = 1$

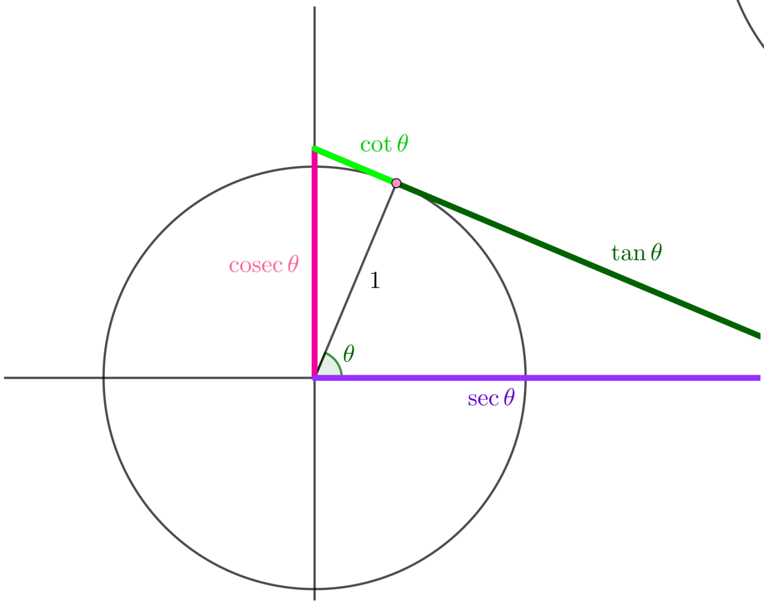
by dividing through either by $\cos^2\theta$ or $\sin^2\theta$.

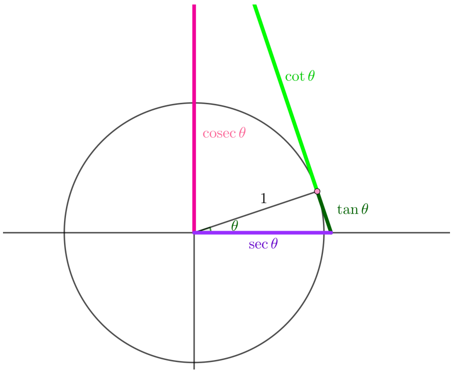
The last is a bit exotic: its not one that have ever taught or bothered to learn.

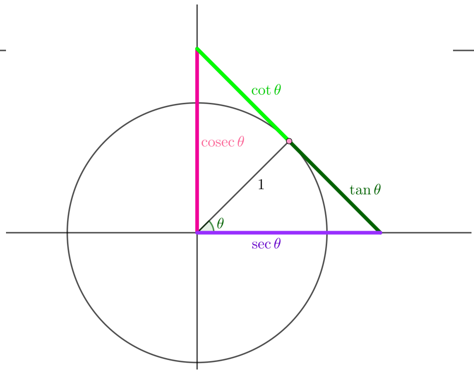
Use this sequence of diagrams as a guide to help you answer the questions on the next page:

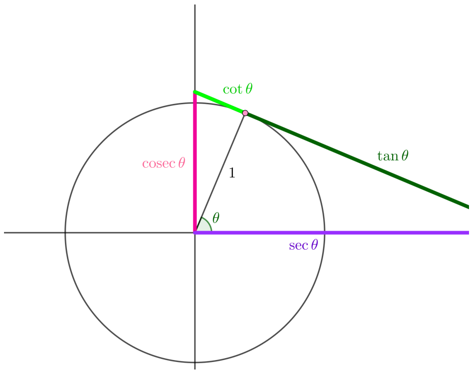












What happens to $\sec\theta$ as $\theta \to 0$? $\sec\theta \to 1$ as $\theta \to 0$?

What is the minimum value of $\sec\theta$? minimum value of $\sec\theta = 1$?

What happens to $\sec\theta$ as $\theta \to 90°$? $\sec\theta \to \infty$ as $\theta \to 90°$?

What happens to $\operatorname{cosec}\theta$ as $\theta \to 0$? $\operatorname{cosec}\theta \to \infty$ as $\theta \to 0$?

What is the minimum value of $\operatorname{cosec}\theta$? minimum value of $\operatorname{cosec}\theta = 1$?

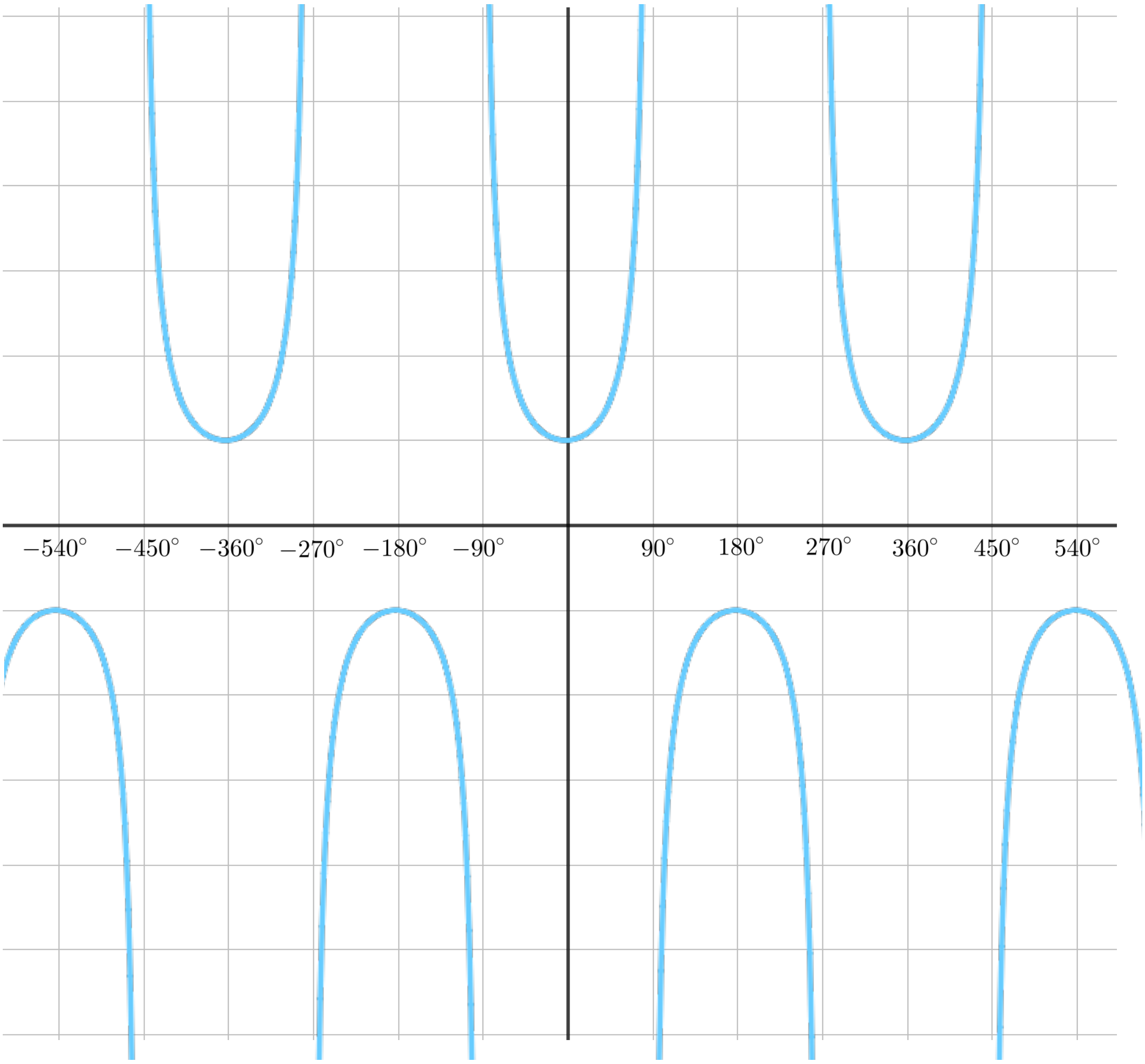
What happens to $\operatorname{cosec}\theta$ as $\theta \to 90°$? $\operatorname{cosec}\theta \to 1$ as $\theta \to 90°$?

What happens to $\cot\theta$ as $\theta \to 0$? $\cot\theta \to \infty$ as $\theta \to 0$?

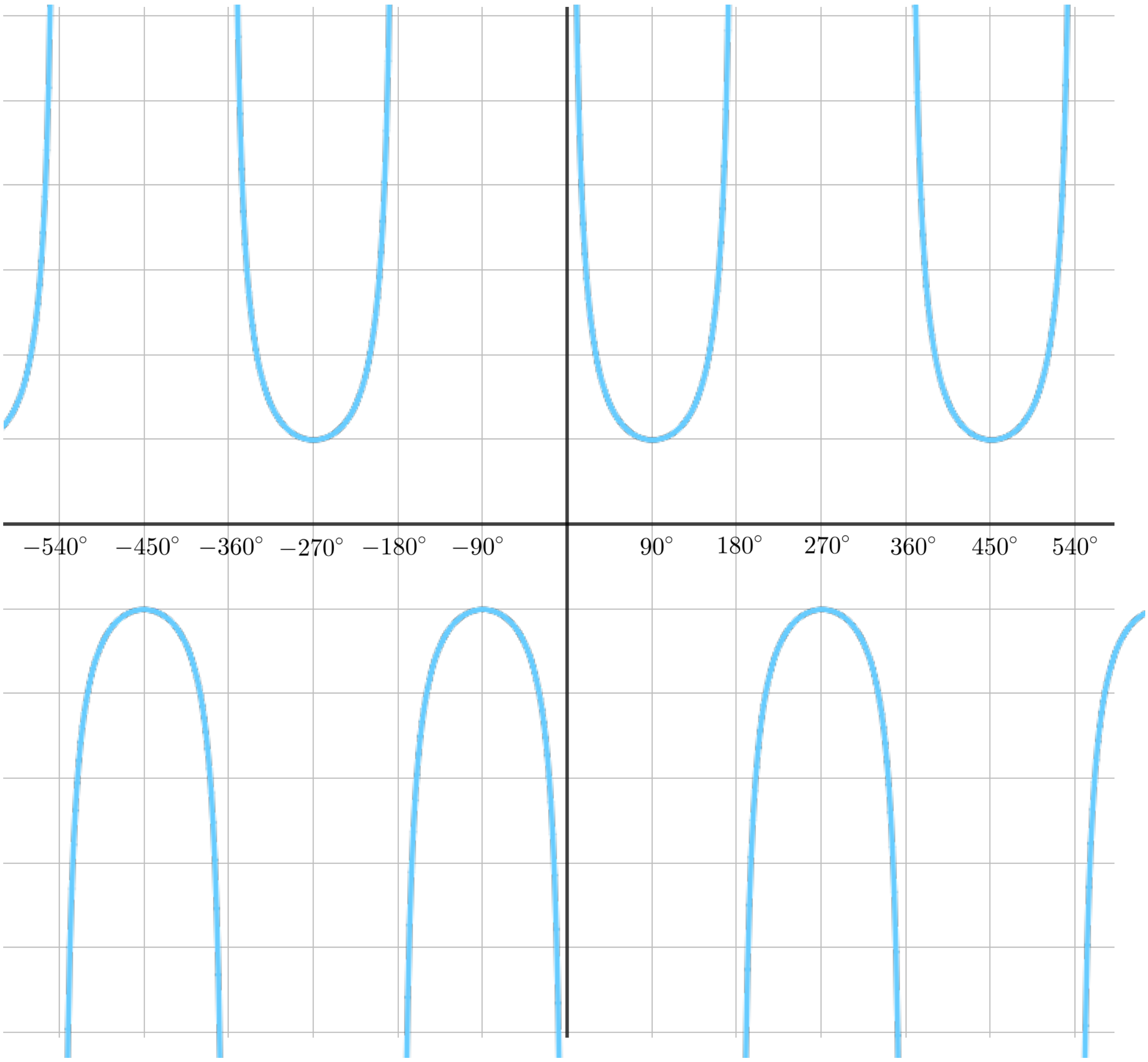
What is $\cot 45°$? minimum value of $\cot 45° = 1$?

What happens to $\cot\theta$ as $\theta \to 90°$? $\cot\theta \to 0$ as $\theta \to 90°$?

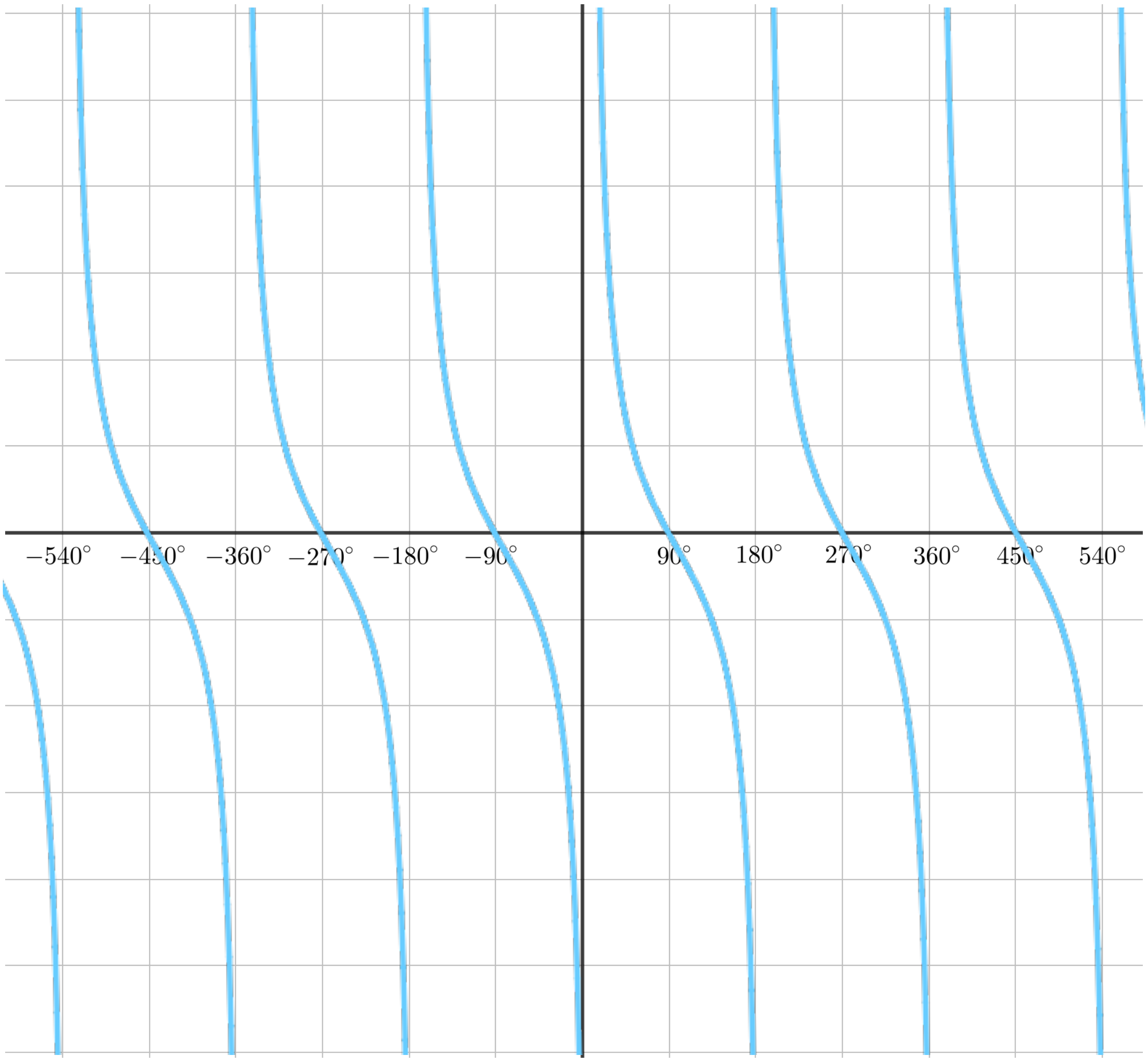
Draw the graph $y = \sec x$.



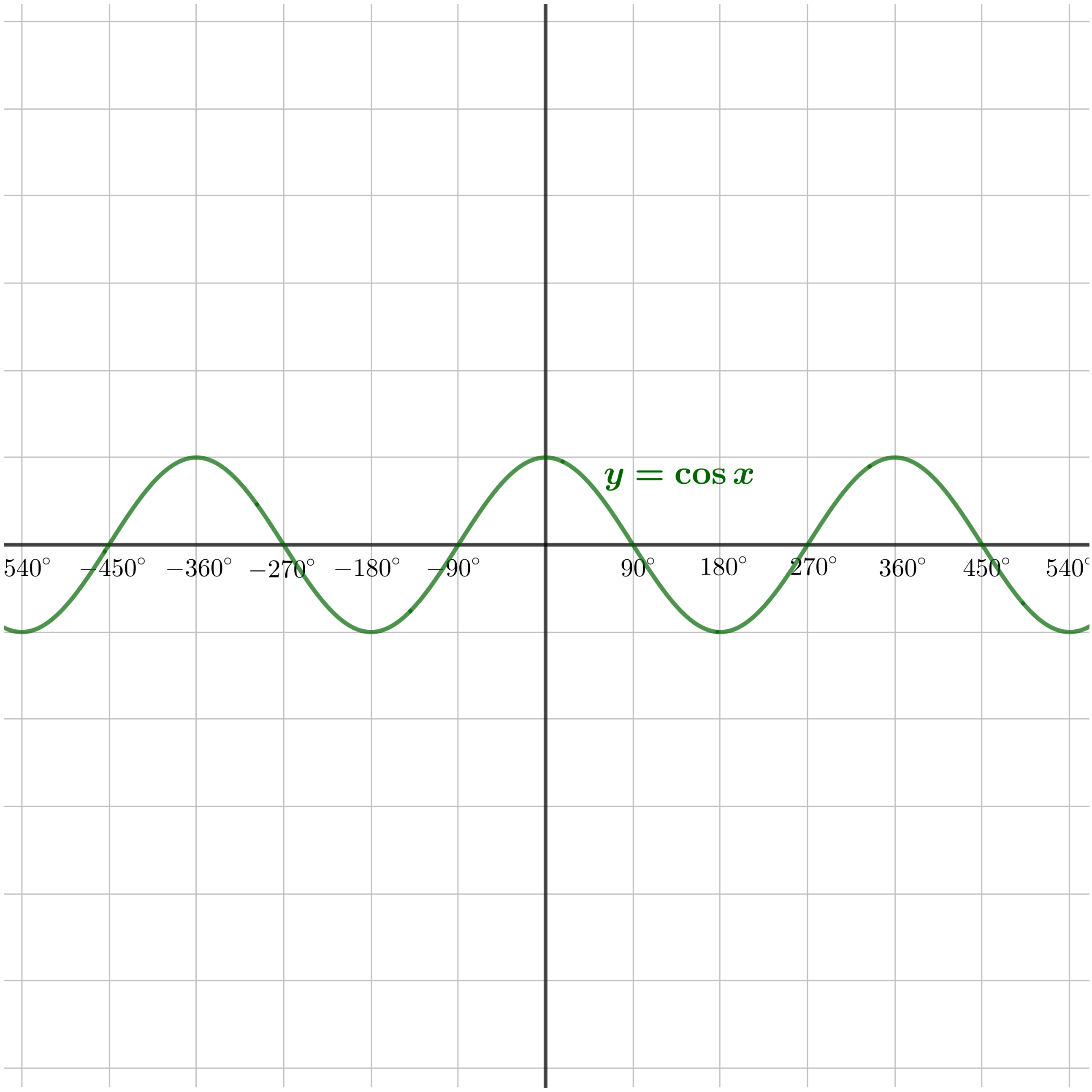
Draw the graph $y = \operatorname{cosec} x$.

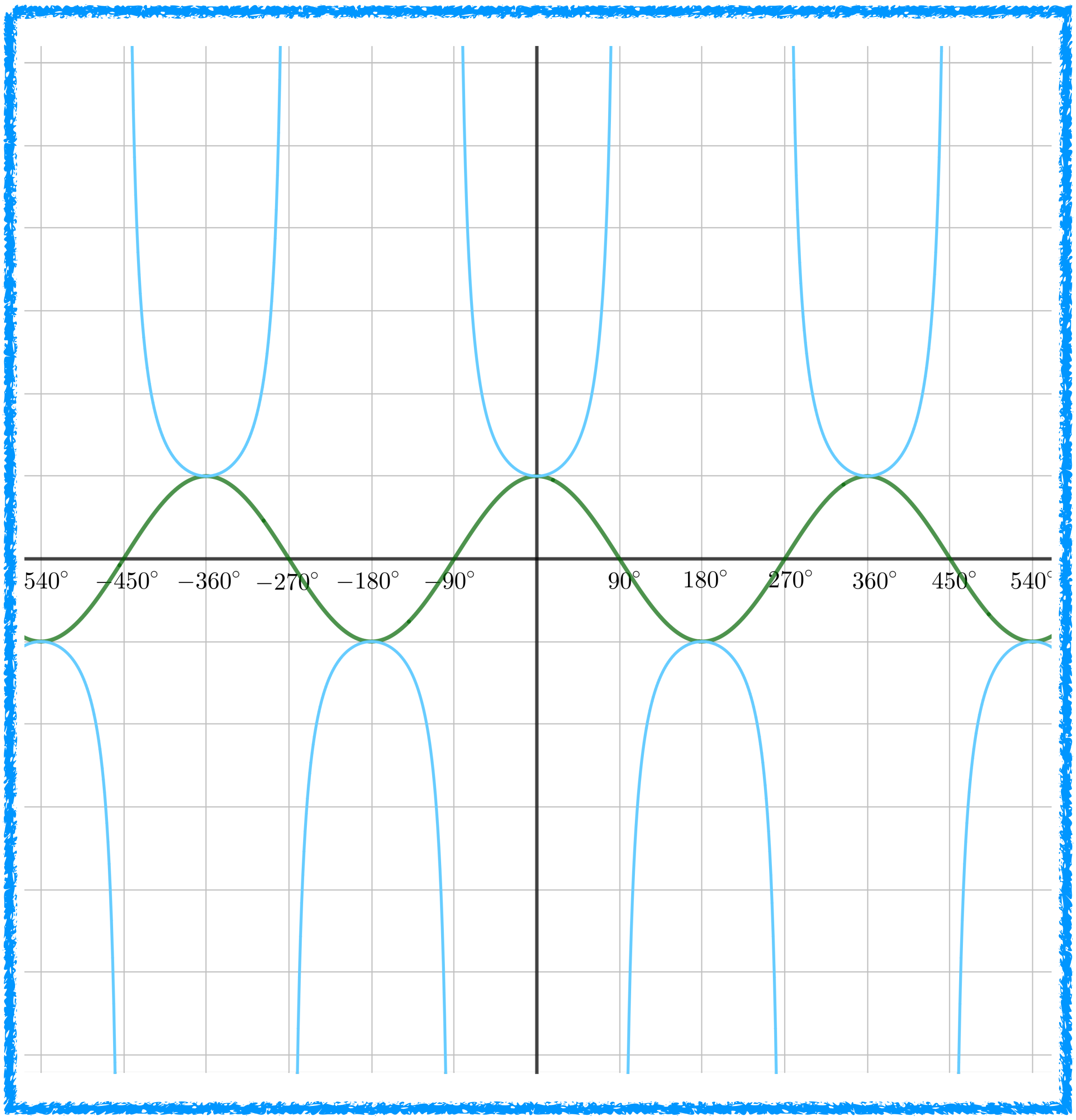


Draw the graph $y = \cot x$.



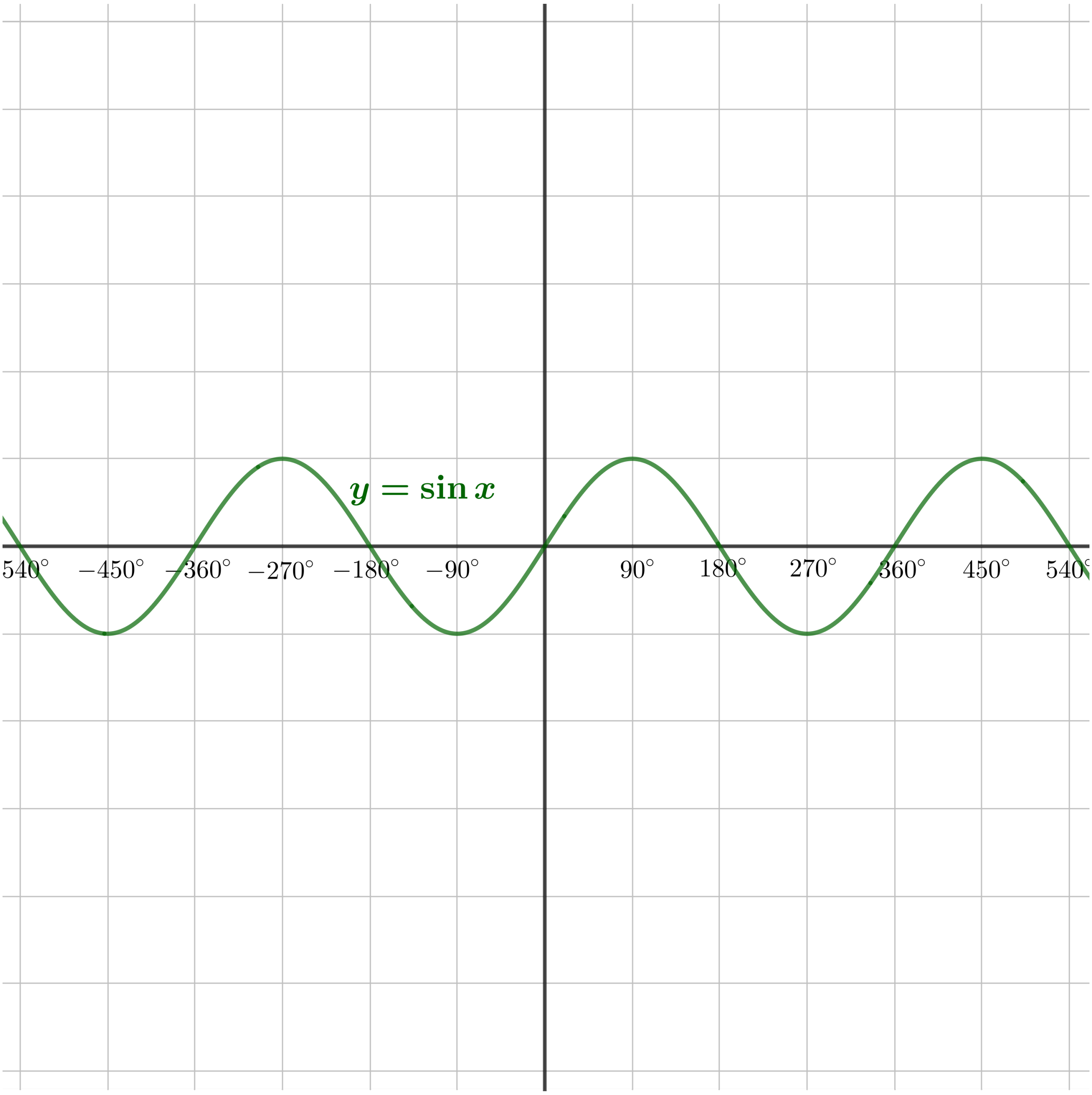
Draw the graph $y = \sec x$ again and describe its relationship to the graph $y = \cos x$.

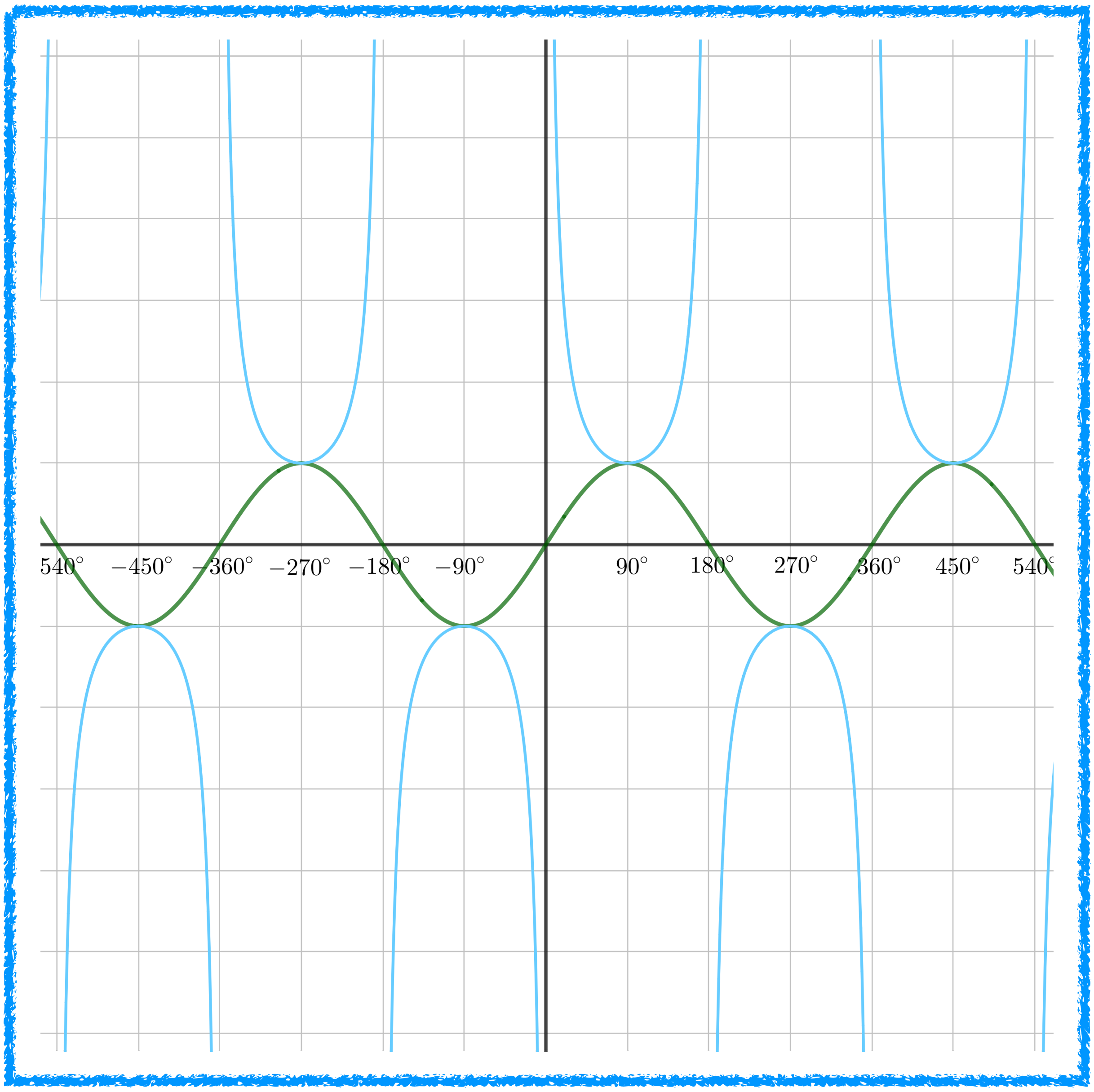




One graph is just the reciprocal of the other.

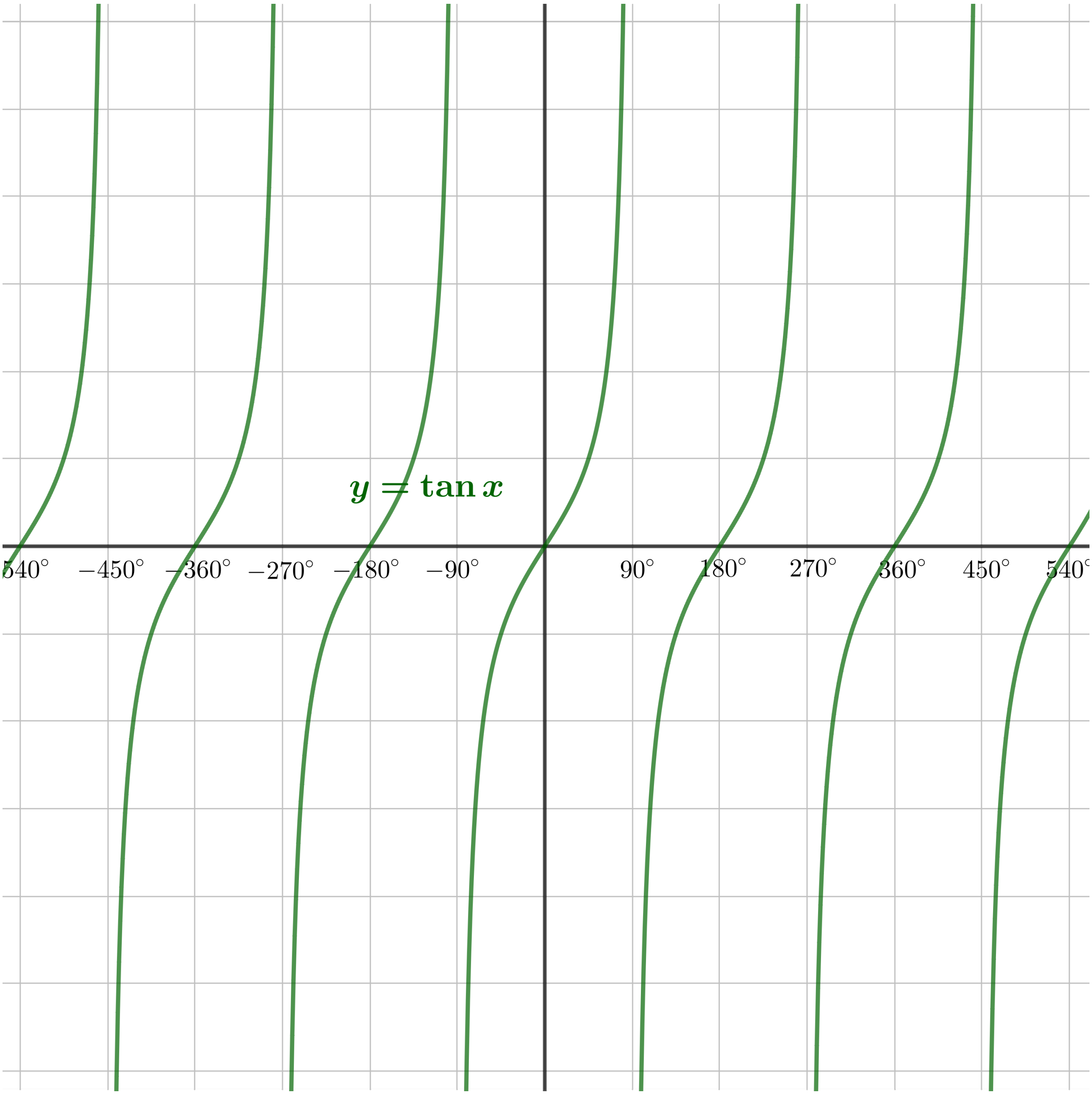
Draw the graph $y = \operatorname{cosec} x$ again and describe its relationship to the graph $y = \sin x$.

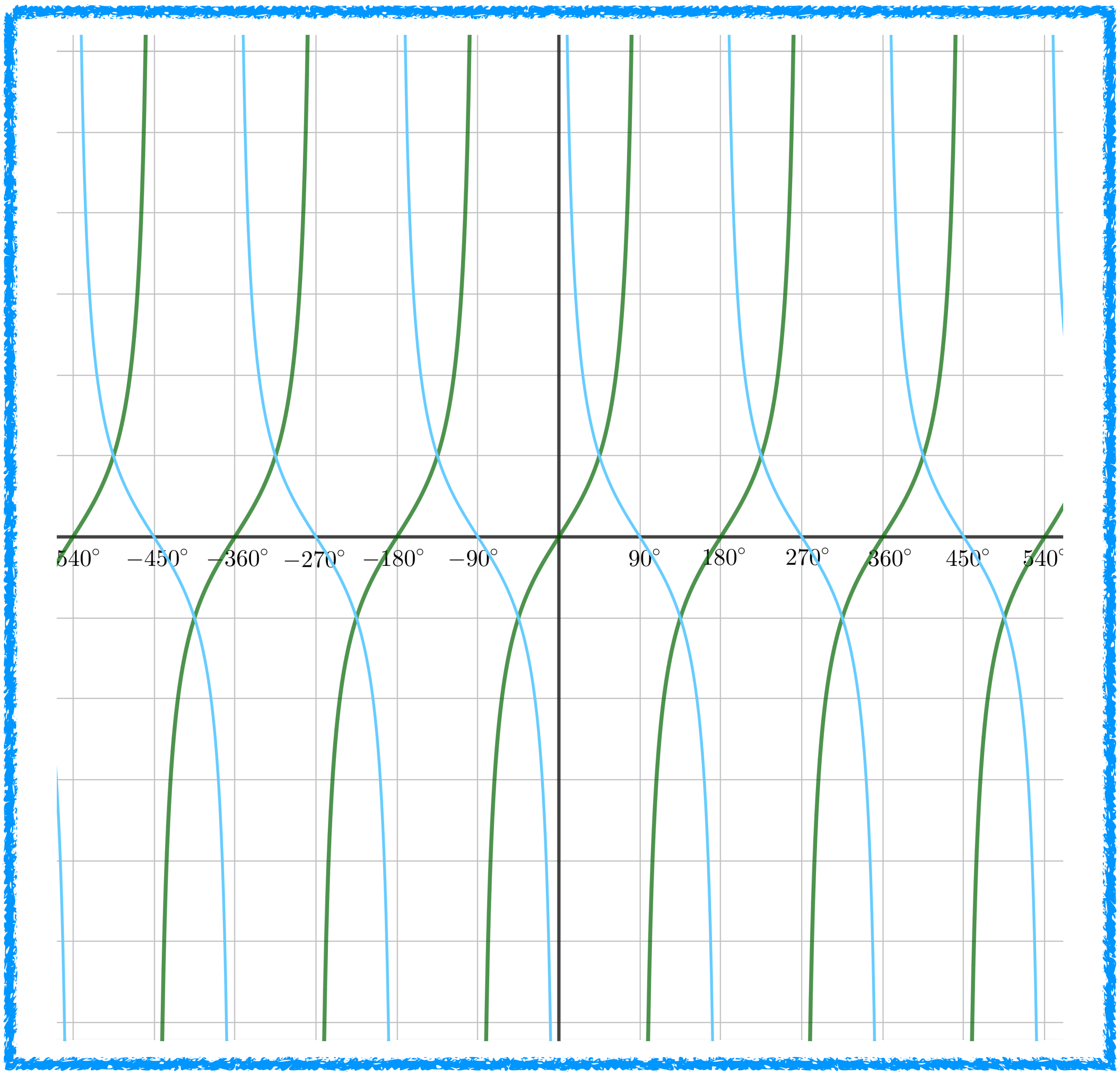




−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

Draw the graph $y = \cot x$ again and describe its relationship to the graph $y = \tan x$.





One graph is the reciprocal of the other, but there is more going on:

they cross at odd multiples of 45°

they are reflections of each other in, for example, $x = 45°$

Show that

$$\frac{1}{\cot\theta} + \cot\theta = \sec\theta\,\mathrm{cosec}\theta$$

whenever θ is not a multiple of 90°.

Proving identities involving these reciprocal circular functions can usually be approached in two ways:

- convert the sec, cosec, and cotan to sin, cos, and tan, and then prove as before, or
- use the identities

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\mathrm{cosec}\,\theta}{\sec\theta}, \quad 1+\tan^2\theta = \sec^2\theta, \quad \text{and } 1+\cot^2\theta = \mathrm{cosec}^2\,\theta$$

We have already done this example earlier by the first method. For the second, try this:

$$\begin{aligned}\frac{1}{\cot\theta} + \cot\theta &= \frac{1+\cot^2\theta}{\cot\theta} \\ &= \frac{\mathrm{cosec}^2\,\theta}{\cot\theta} \\ &= \mathrm{cosec}^2\,\theta \times \frac{\sec\theta}{\mathrm{cosec}\,\theta} \\ &= \sec\theta\,\mathrm{cosec}\,\theta\end{aligned}$$