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Circular functions 1

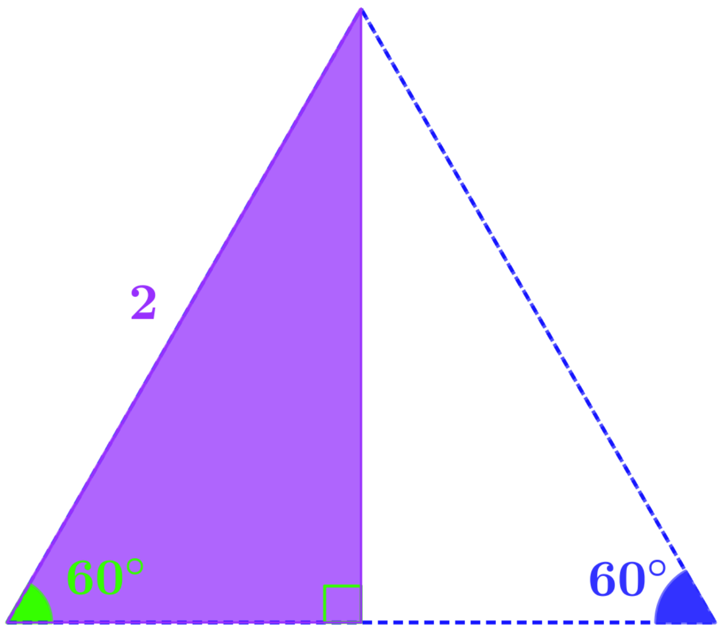
Defining the circular functions

sin, cos, tan and the unit circle

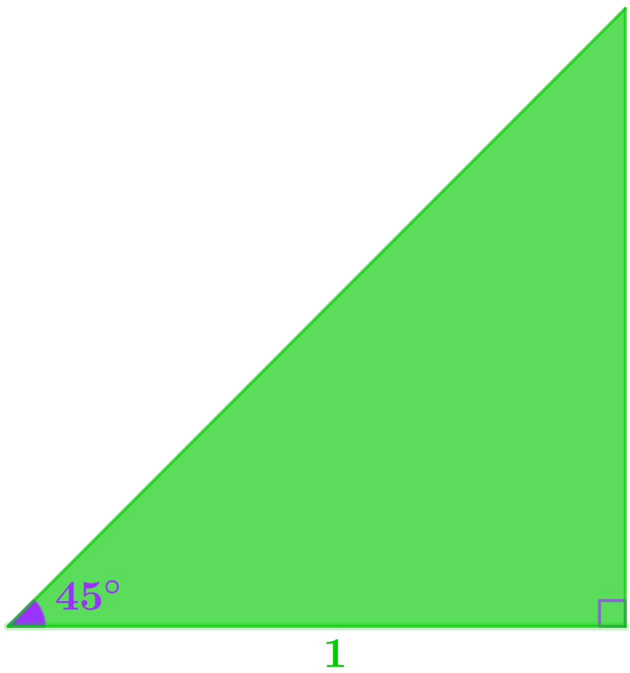
Circular functions

Defining the circular functions	**sin, cos, tan and the unit circle**
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90° - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45°\right)$ and so on
Differentiating circular functions	radians, and tangents to graphs
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

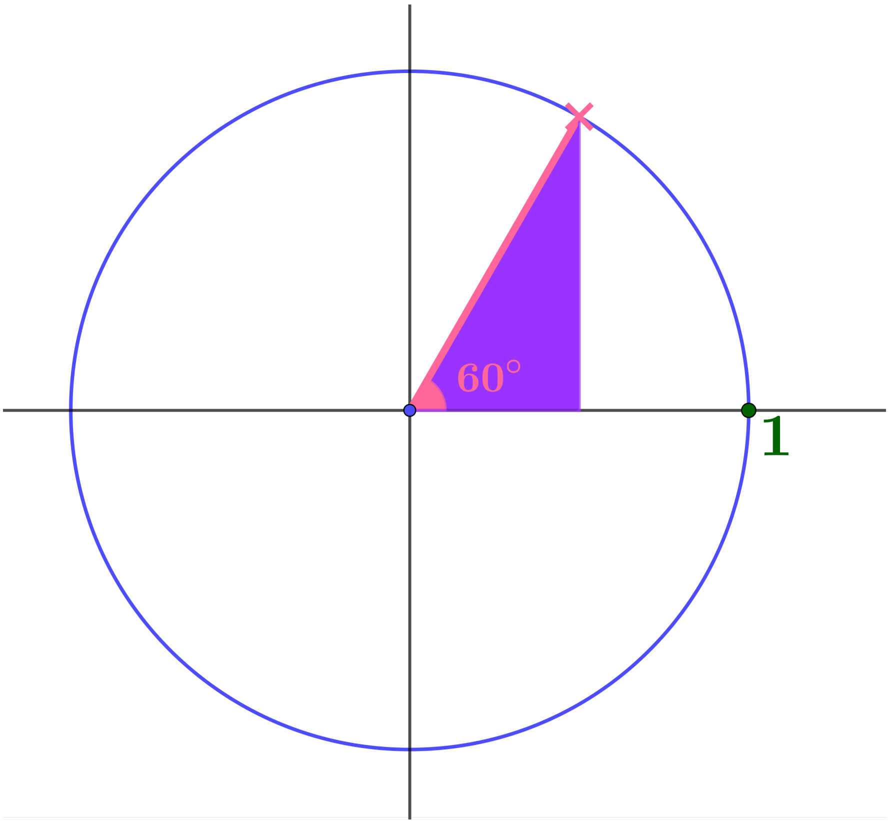
Use this diagram to find sin, cos, and tan of 60° and 30°.



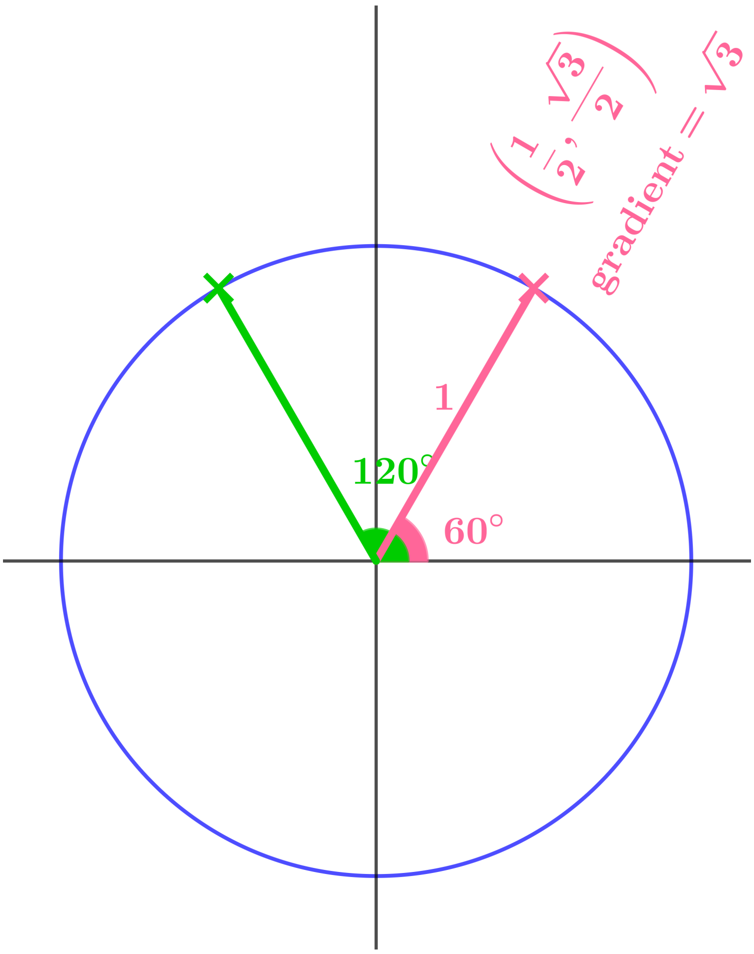
Use this diagram to find sin, cos, and tan of 45°.



What are the coordinates of the pink point and the gradient of the pink radius?



What are the coordinates of the green point and the gradient of the green radius?



Suggest values for sin, cos, and tan of 120°.

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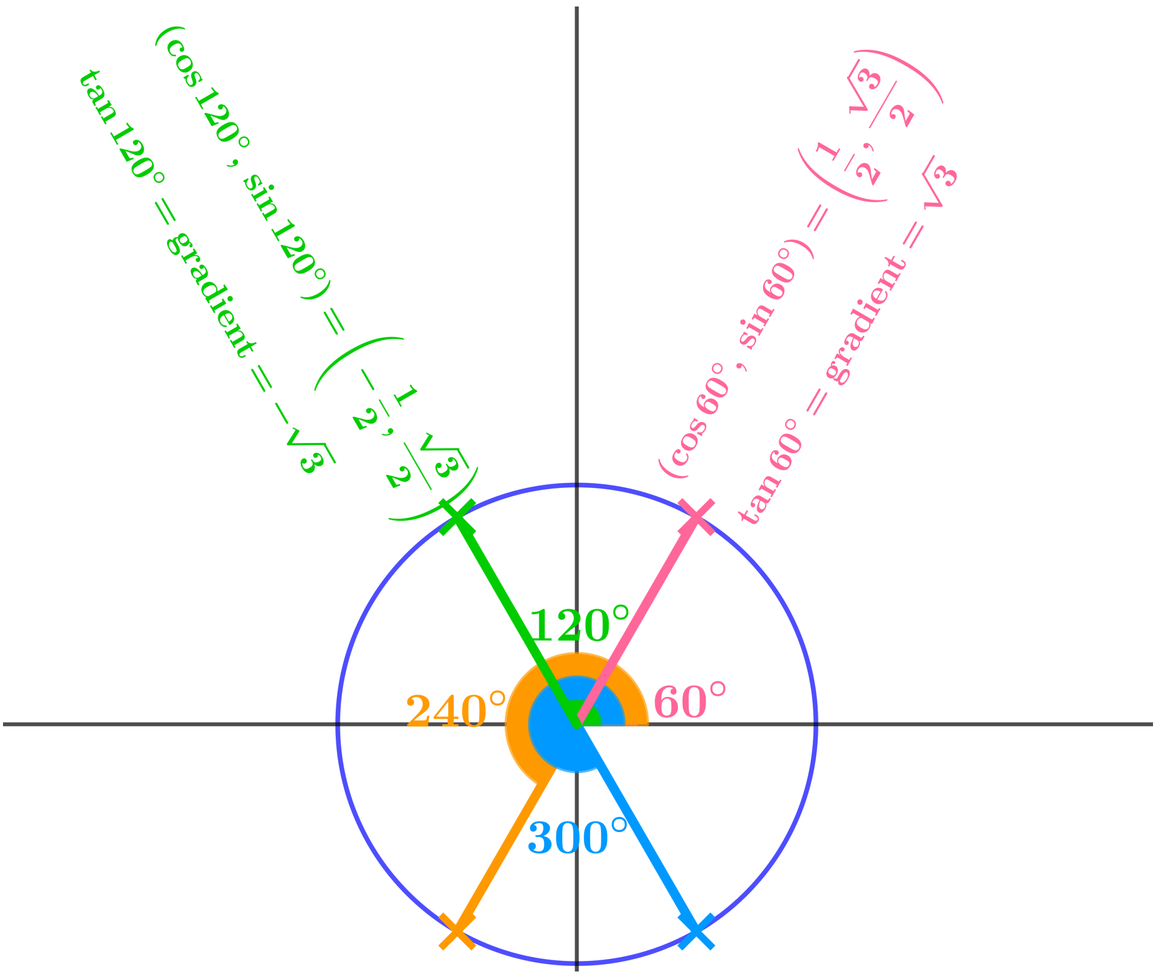
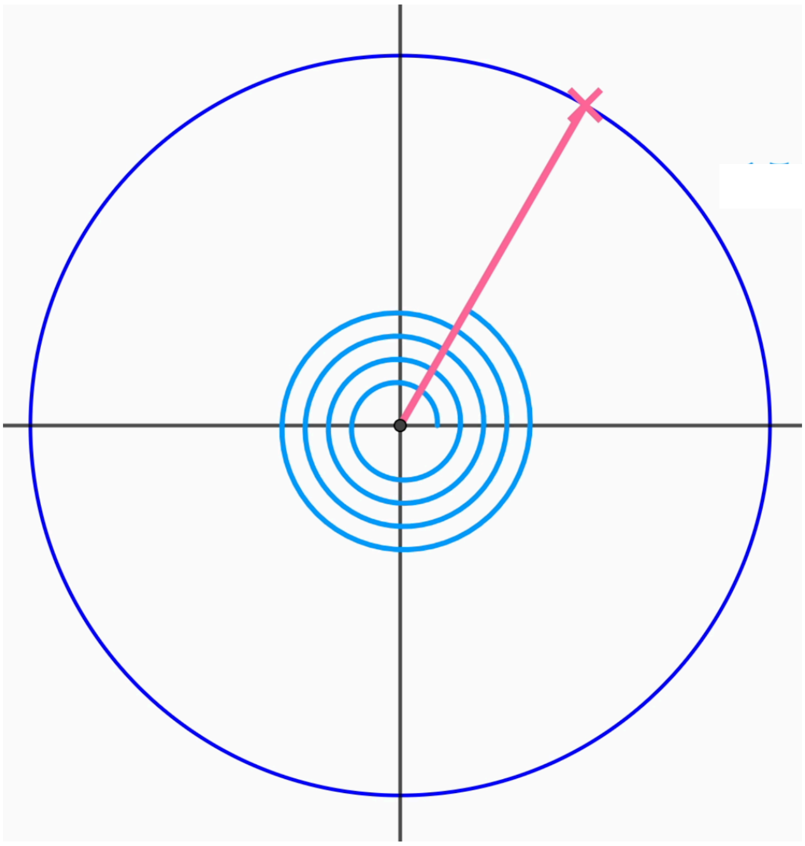
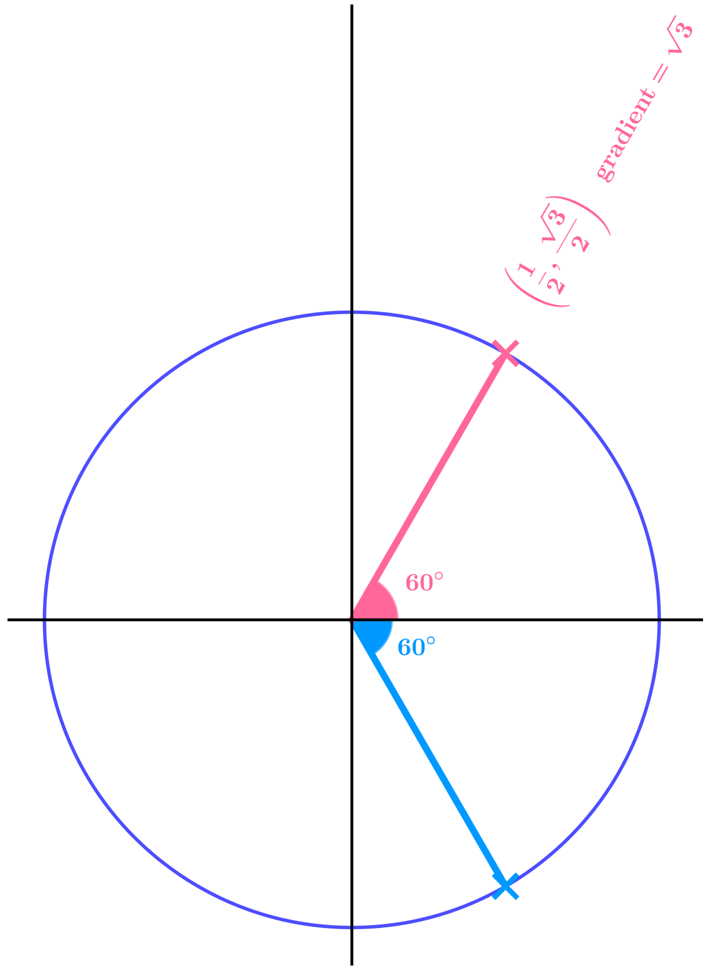
Suggest values for sin, cos, and tan of 240° and 300°.



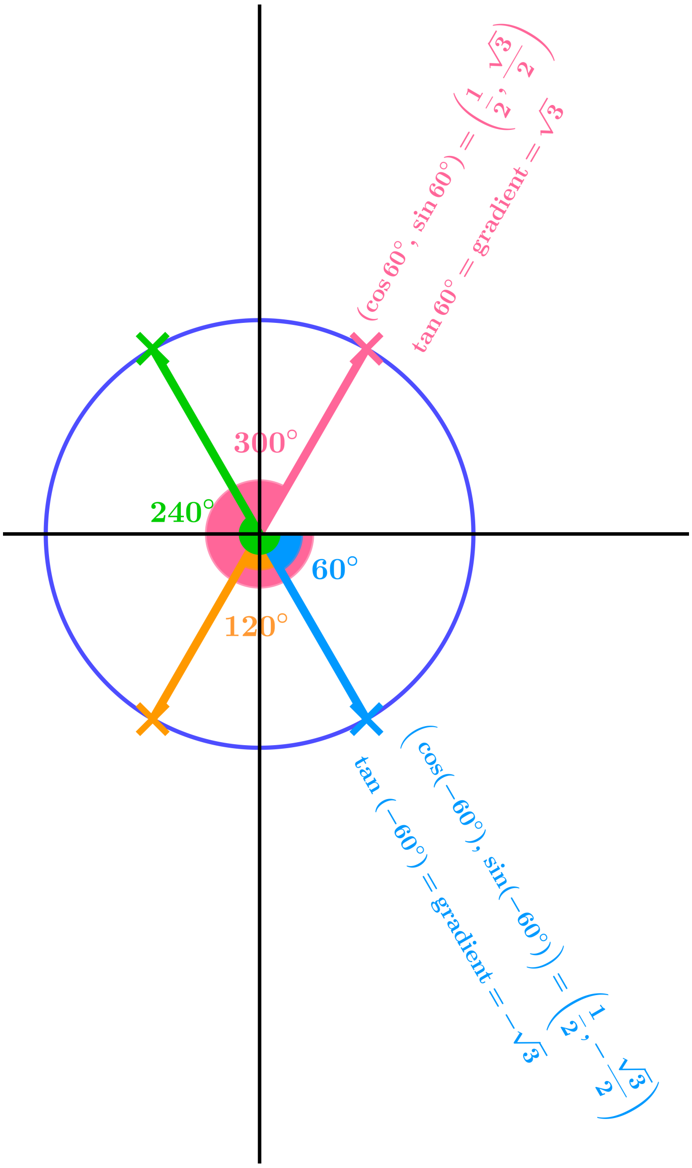
What are sin, cos, tan of 420°, 780°, 1140°, 1500° ?



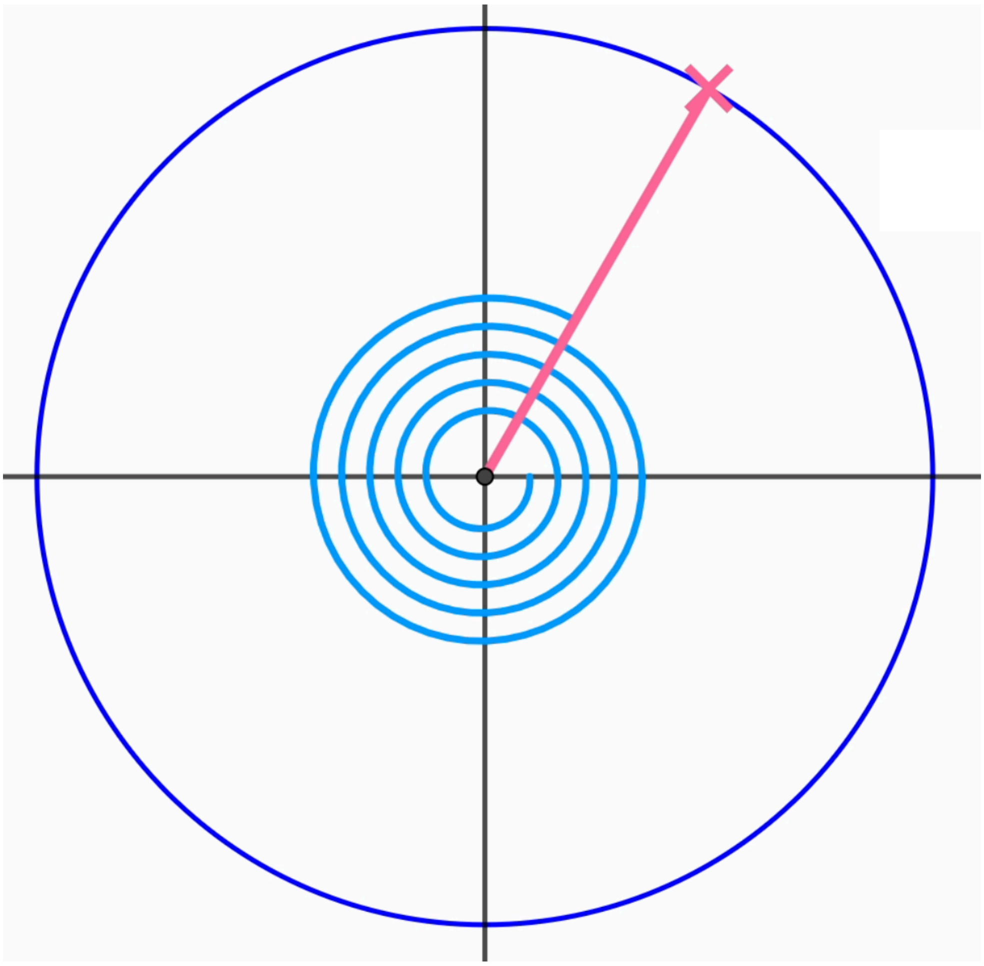
Suggest values for sin, cos, tan of -60° ?

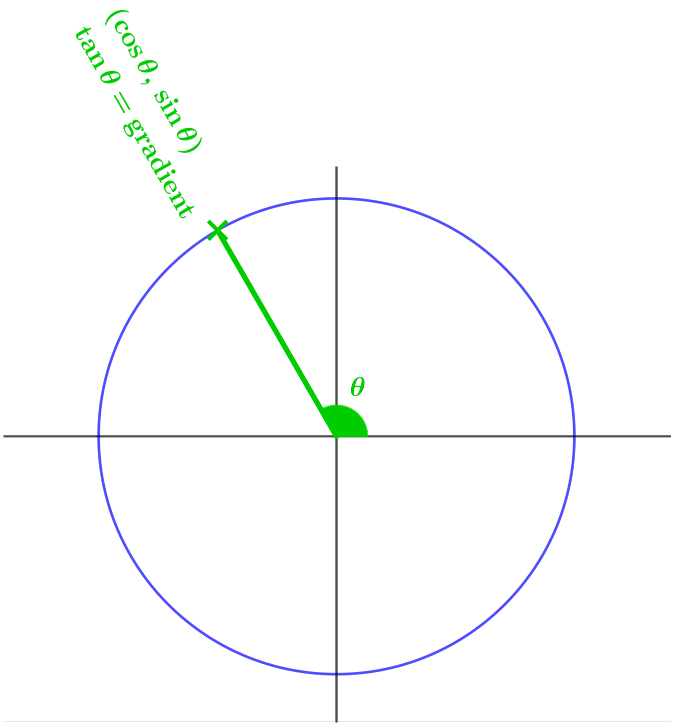


Suggest values for sin, cos, tan of -120° -240° -300° .

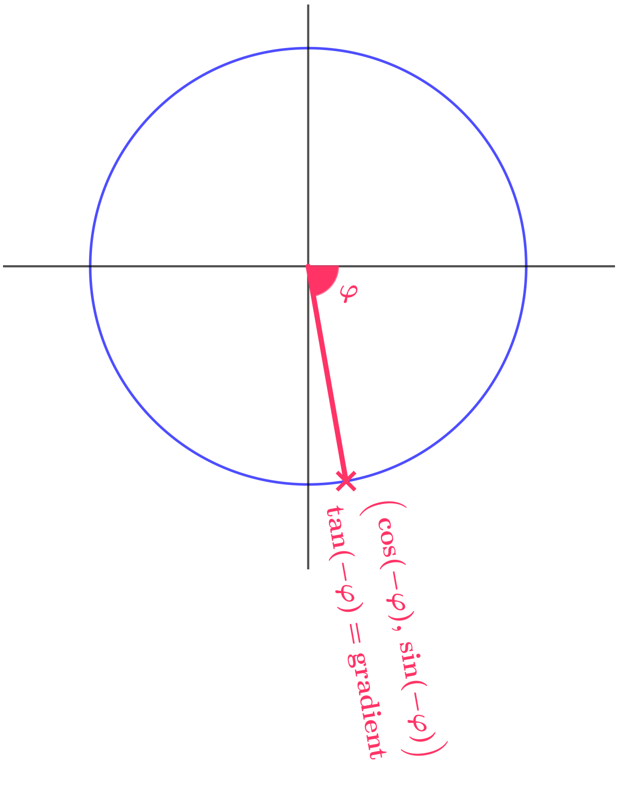


Suggest values for sin, cos, tan of -660°, -1020° -1380° -1740° .



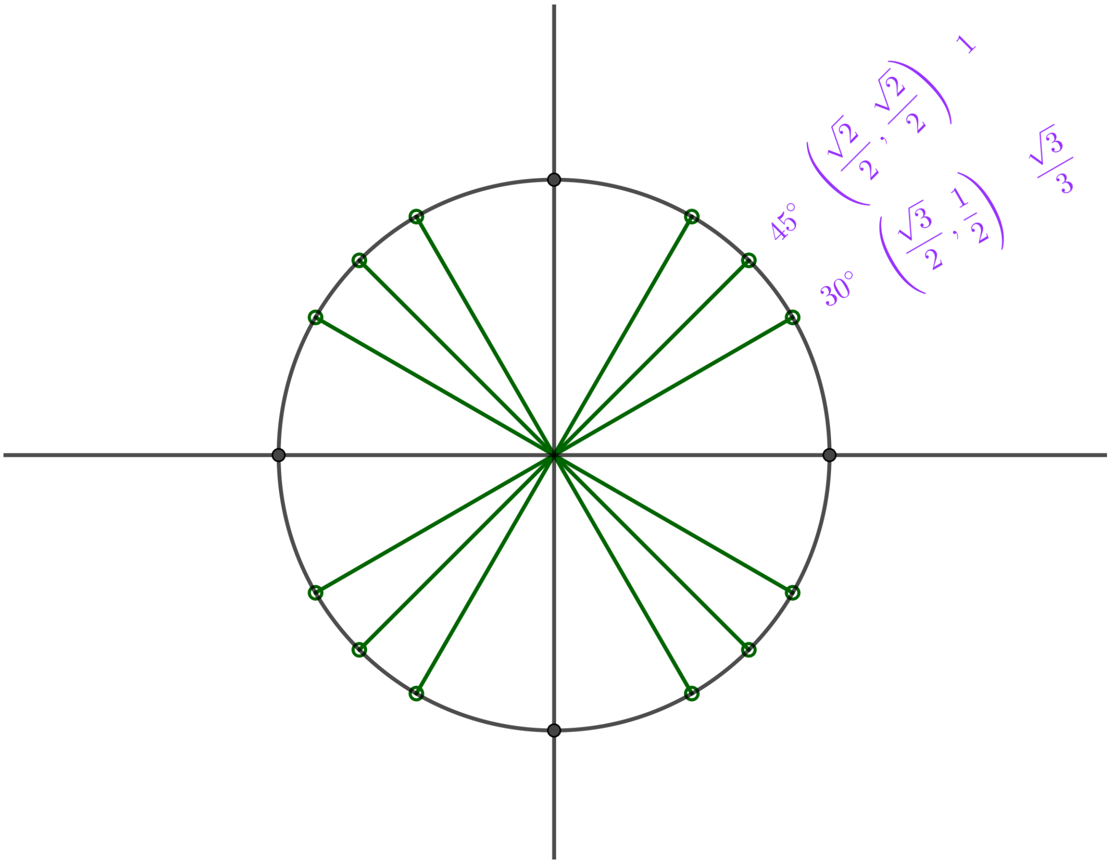


$(\cos\theta, \sin\theta)$
$\tan\theta = \text{gradient}$
θ



φ
$\left(\cos(-\varphi), \sin(-\varphi)\right)$
$\tan(-\varphi) = \text{gradient}$

Complete this diagram.



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Circular functions 2

Solving circular functions equations

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	**like $\sin\theta = 0.4$**
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90^\circ - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$
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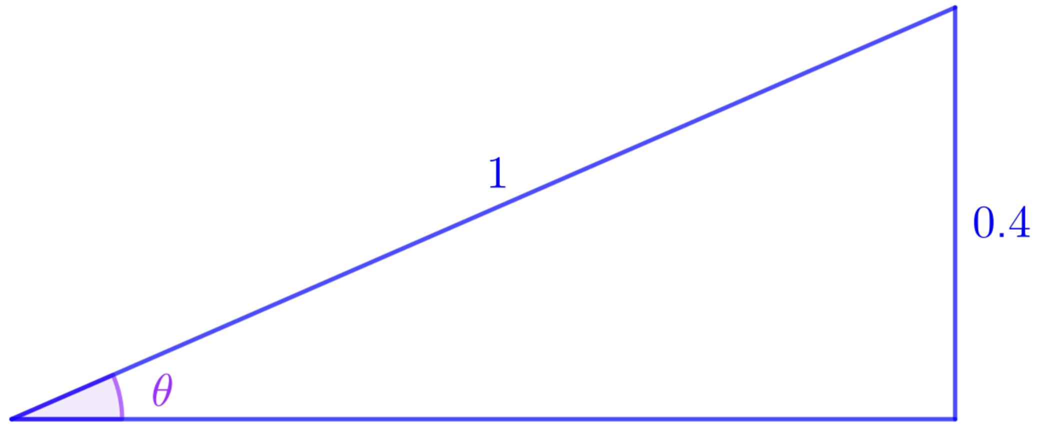
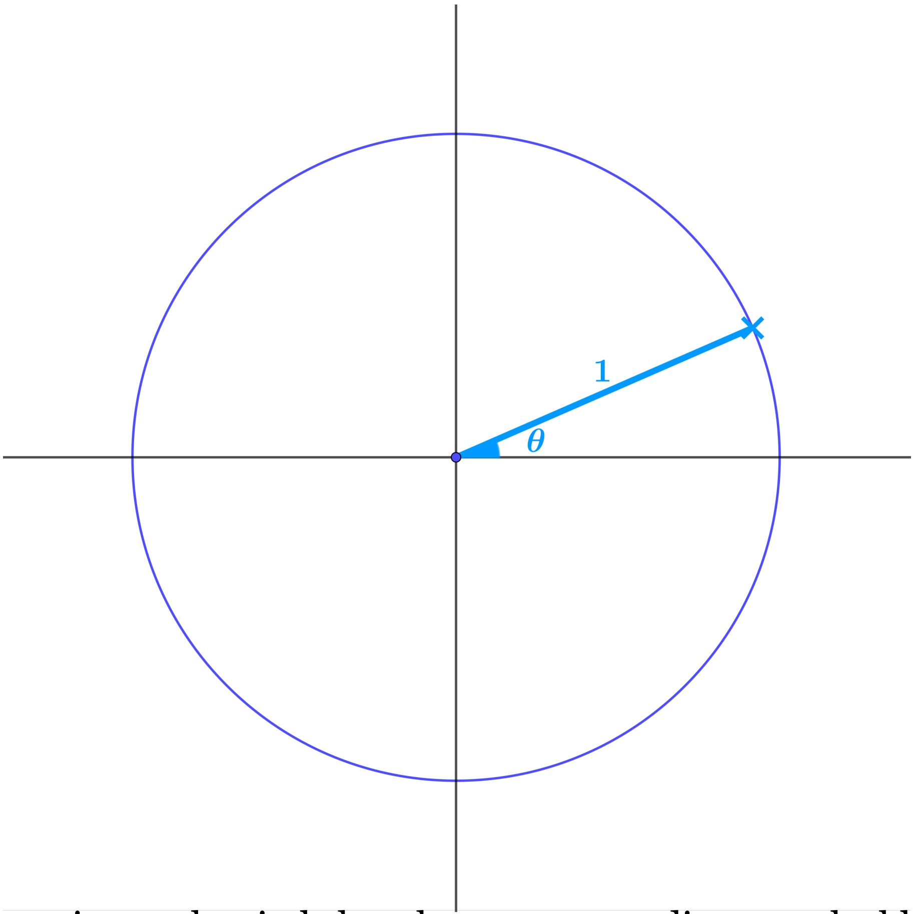
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Solving equations with circular functions

Use your calculator to find the angle θ.

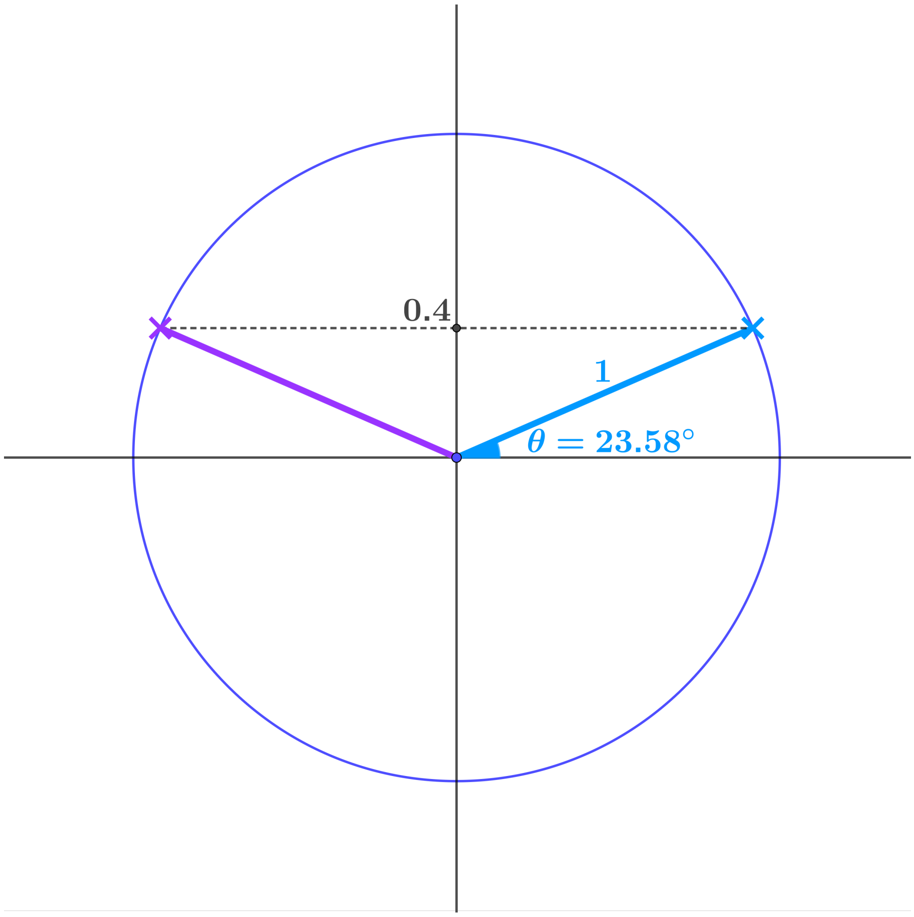


If the y coordinate of the blue point is 0.4, find the angle θ.

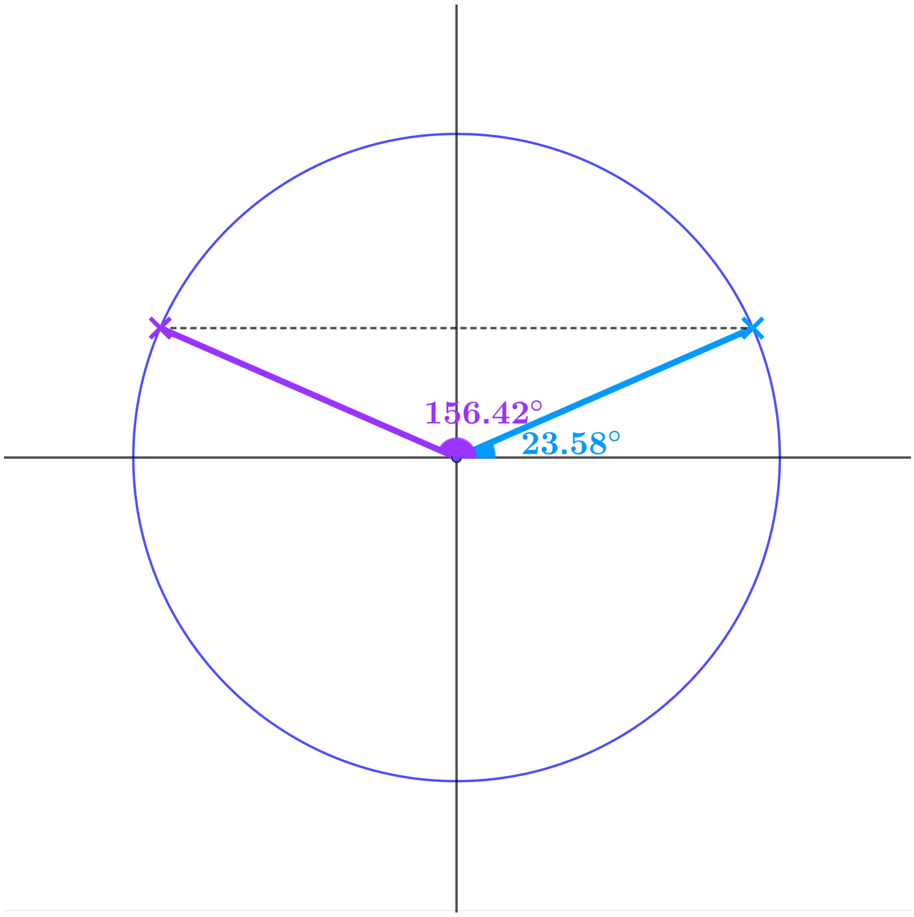


What other point on the circle has the same y coordinate as the blue point?

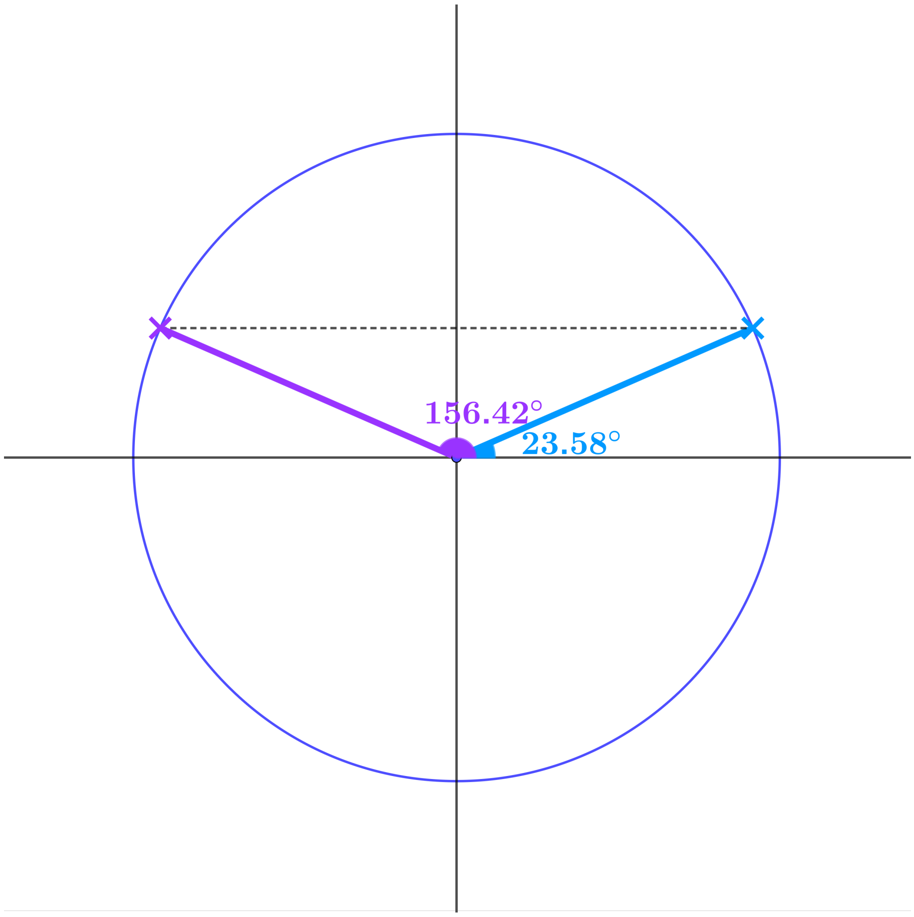
What other positive angle between 0° and 360° is a solution of $\sin\theta = 0.4$?



What negative angles between $-360°$ and $0°$ are solutions of $\sin\theta = 0.4$?



Solve the equation $\sin\theta = 0.4$



If α is any solution of the equation $\sin\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

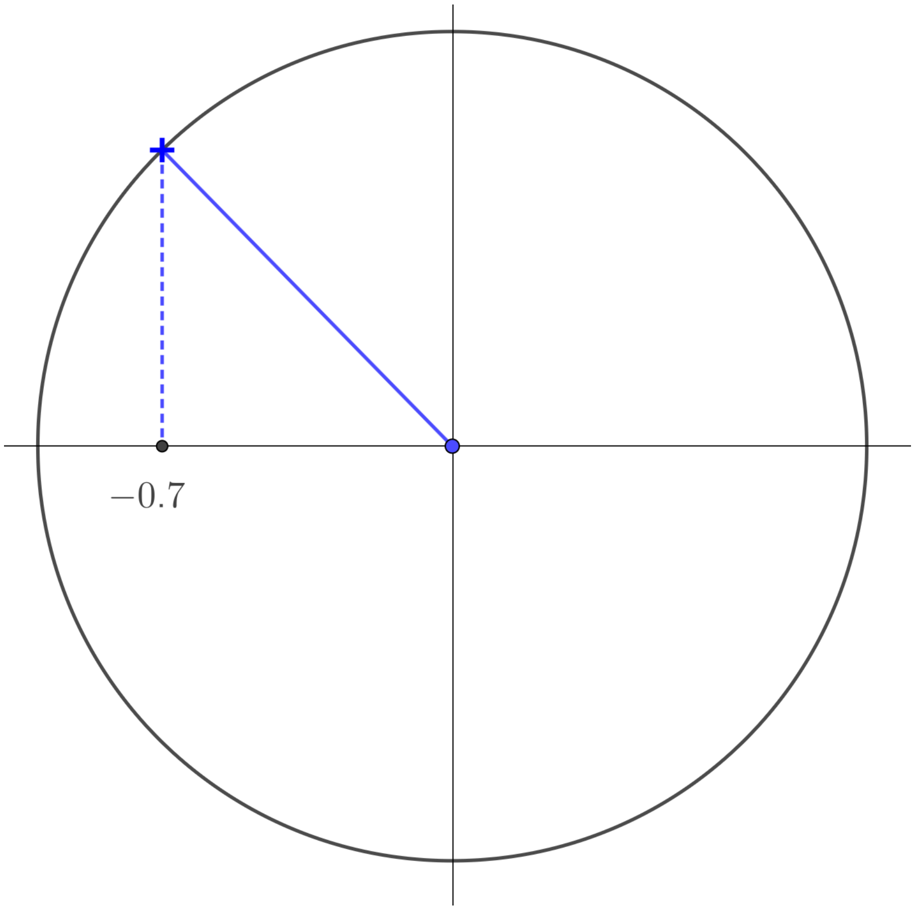
$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

Adapt the previous method to solve the equation $\cos\theta = -0.7$.



If α is any solution of the equation $\cos\theta = k$, which of the following are also solutions of the equation:

$$180 - \alpha$$

$$180 + \alpha$$

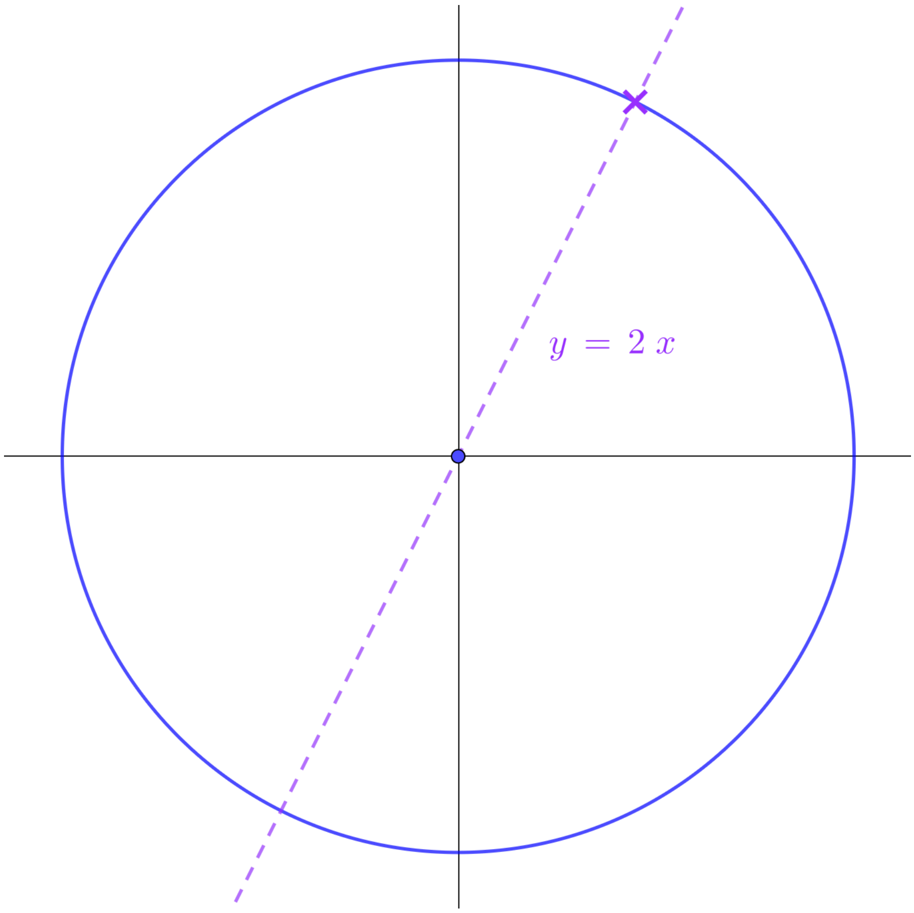
$$-\alpha$$

$$\alpha + 360$$

$$\alpha - 360$$

Use this diagram and a calculator to solve the equation

$$\tan\theta = 2$$



If α is any solution of the equation $\tan\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

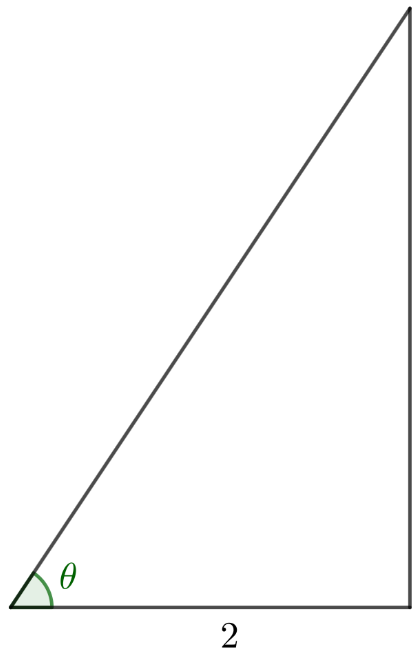
$180 + \alpha$

$-\alpha$

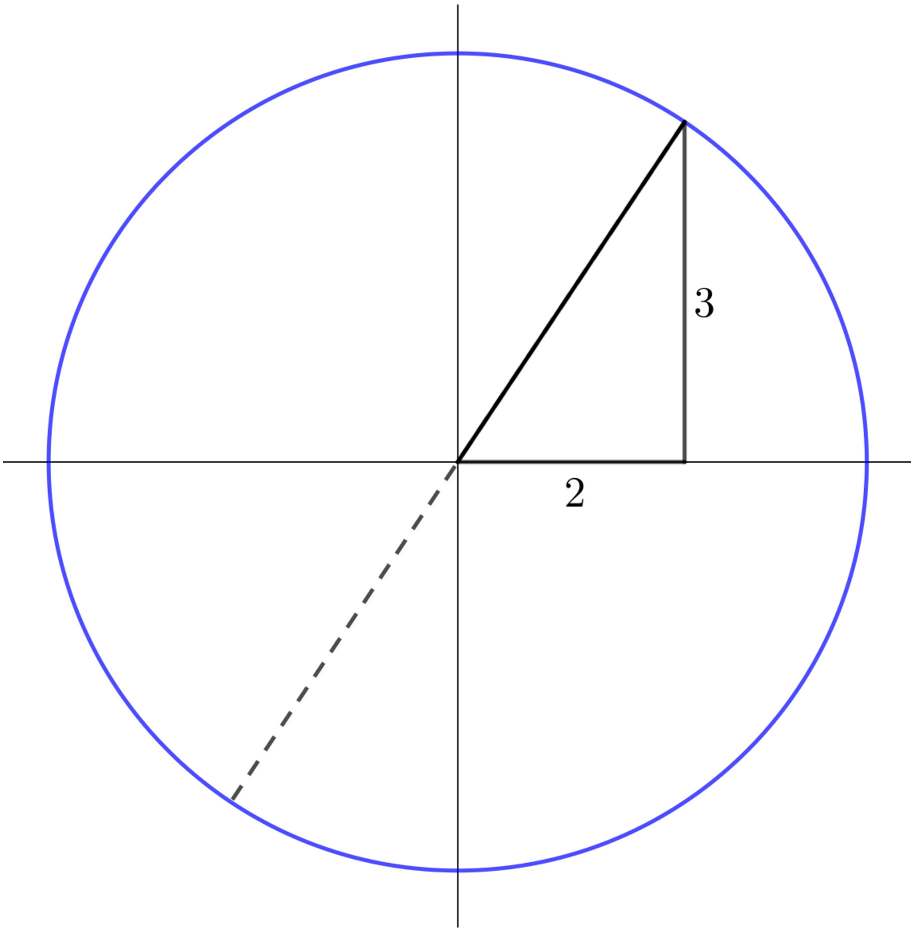
$\alpha + 360$

$\alpha - 360$

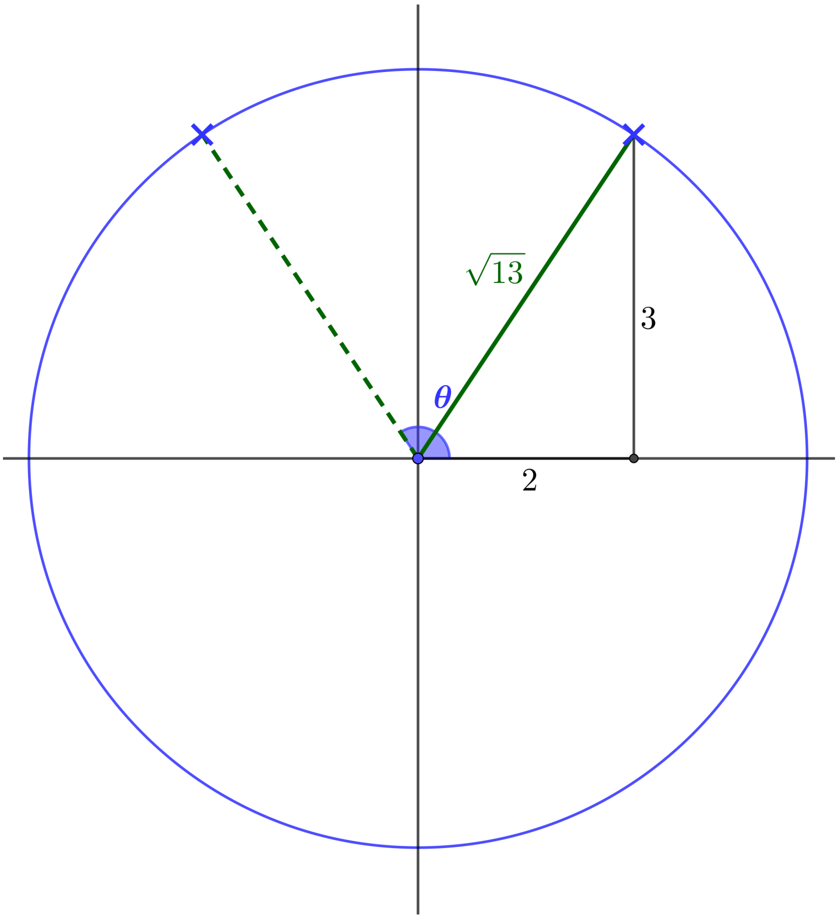
If $\tan\theta = \dfrac{3}{2}$, find $\sin\theta$ and $\cos\theta$.



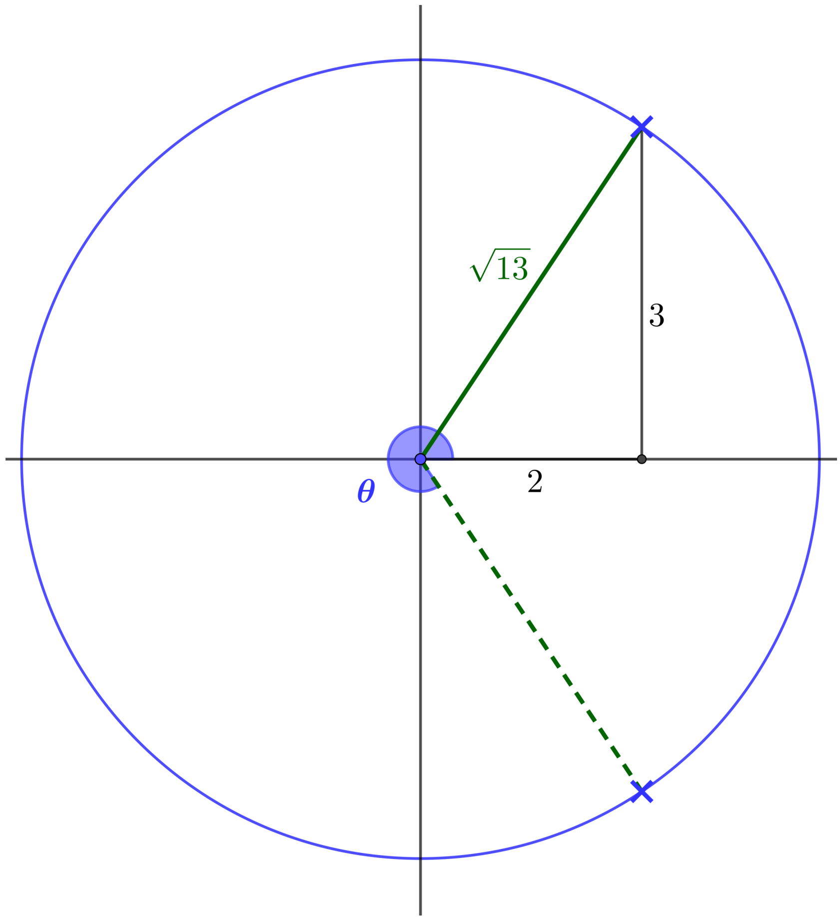
If $\tan\theta = \dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.



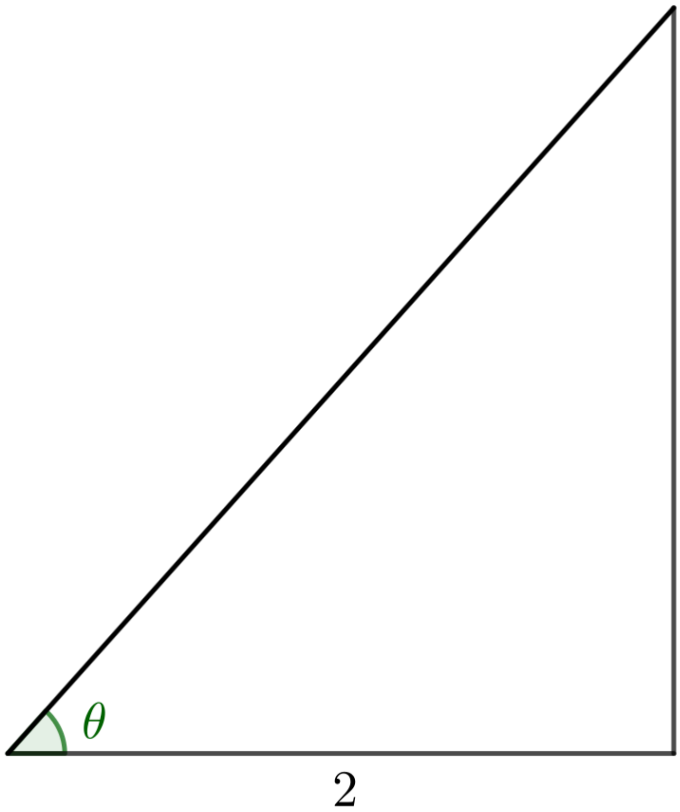
If $\tan\theta = -\dfrac{3}{2}$, and θ is obtuse, find $\sin\theta$ and $\cos\theta$.



If $\tan\theta = -\dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.



If $\cos\theta = \dfrac{2}{3}$, find $\tan\theta$ and $\sin\theta$.



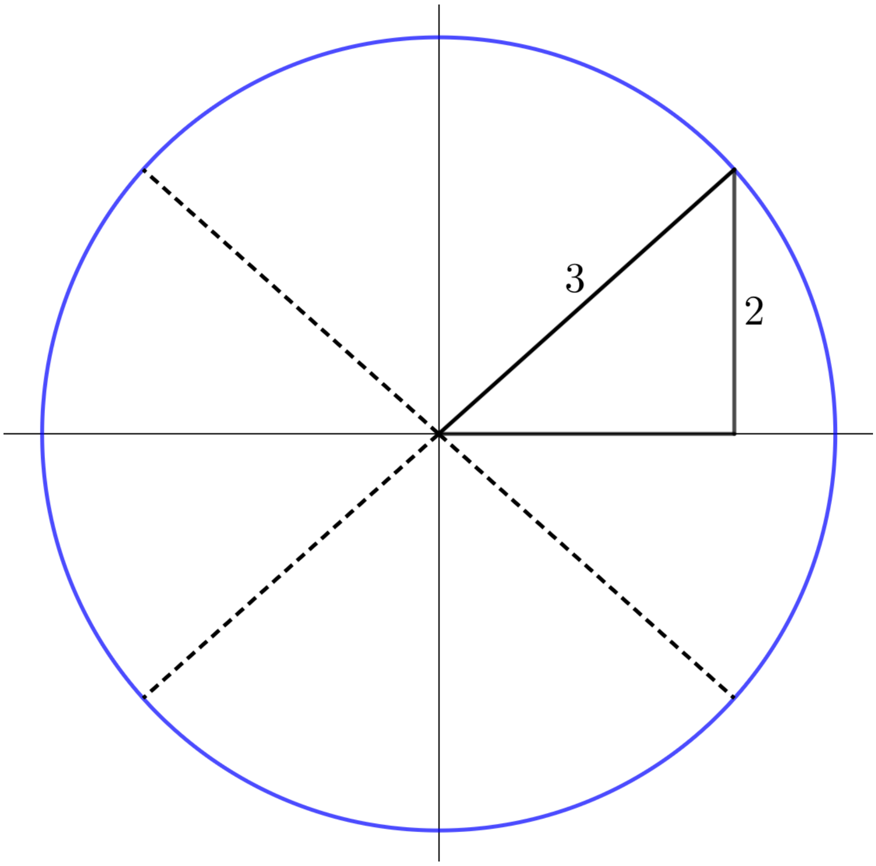
Find $\tan\theta$ and $\sin\theta$ when:

$\cos\theta = \frac{2}{3}$, and θ is between 270° and 360°

$\cos\theta = -\frac{2}{3}$, and θ is between 180° and 270°

$\cos\theta = -\frac{2}{3}$, and θ is obtuse.

Find $\tan\theta$ and $\cos\theta$ when $\sin\theta = \pm\dfrac{2}{3}$ for the various possible values of θ.



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Circular functions 3

Graphs of circular functions

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
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Graphing the circular functions	**graphs $y = \cos x$ and the like**
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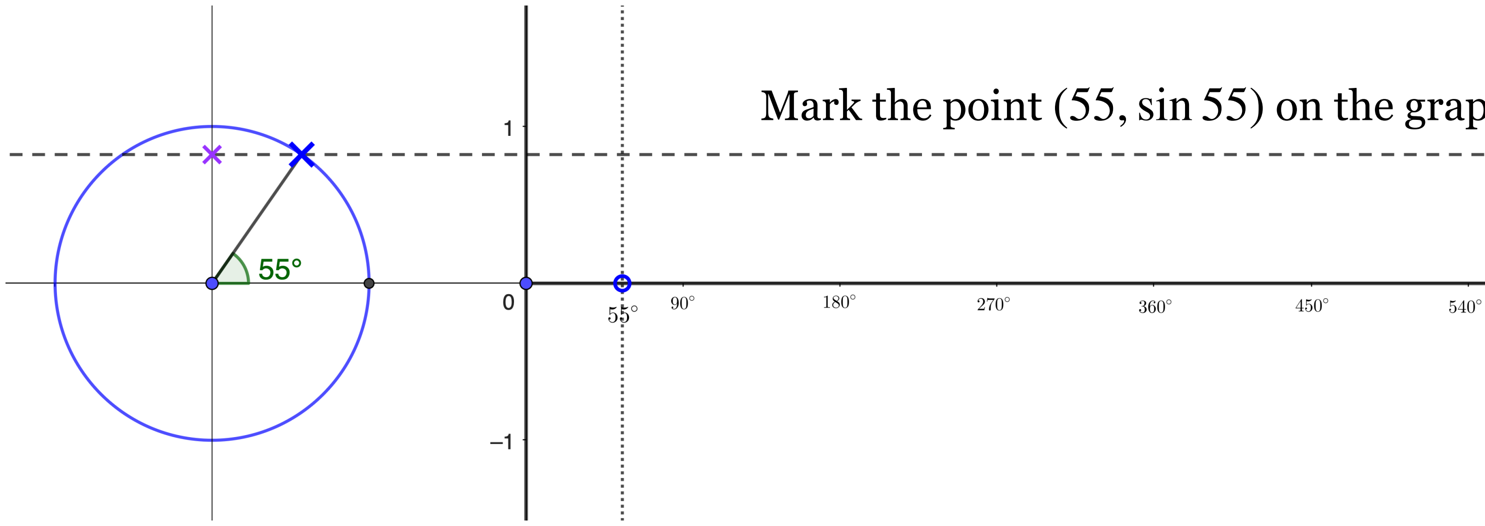
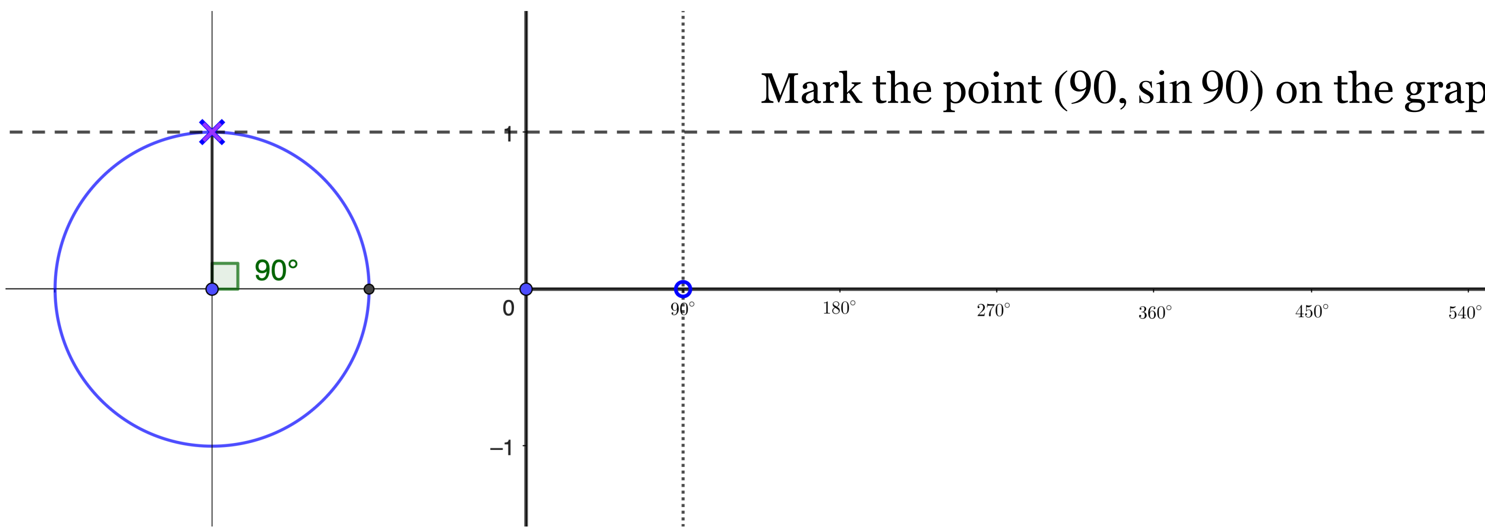
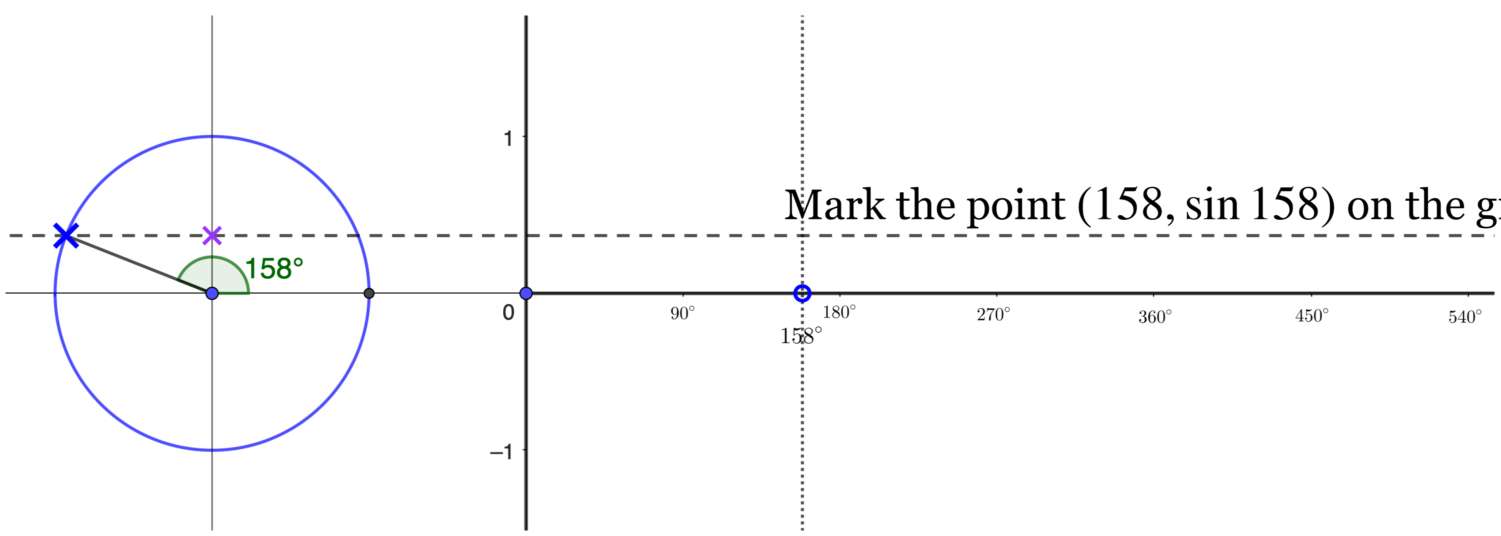


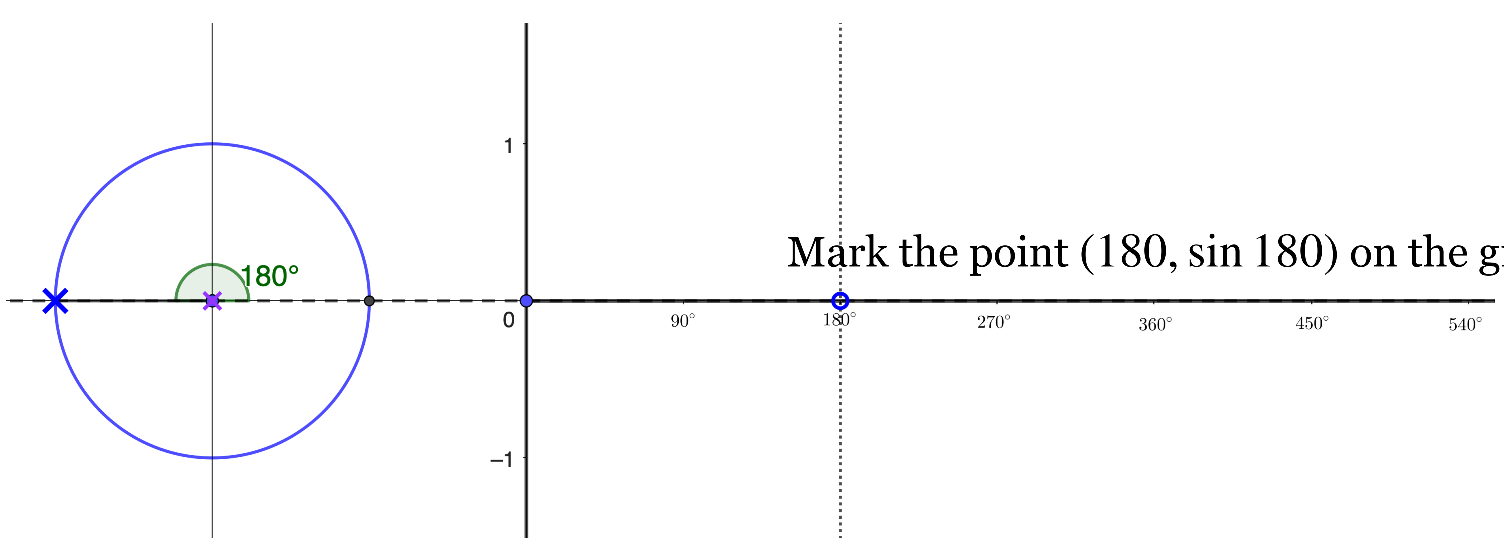
Mark the point (55, sin 55) on the graph



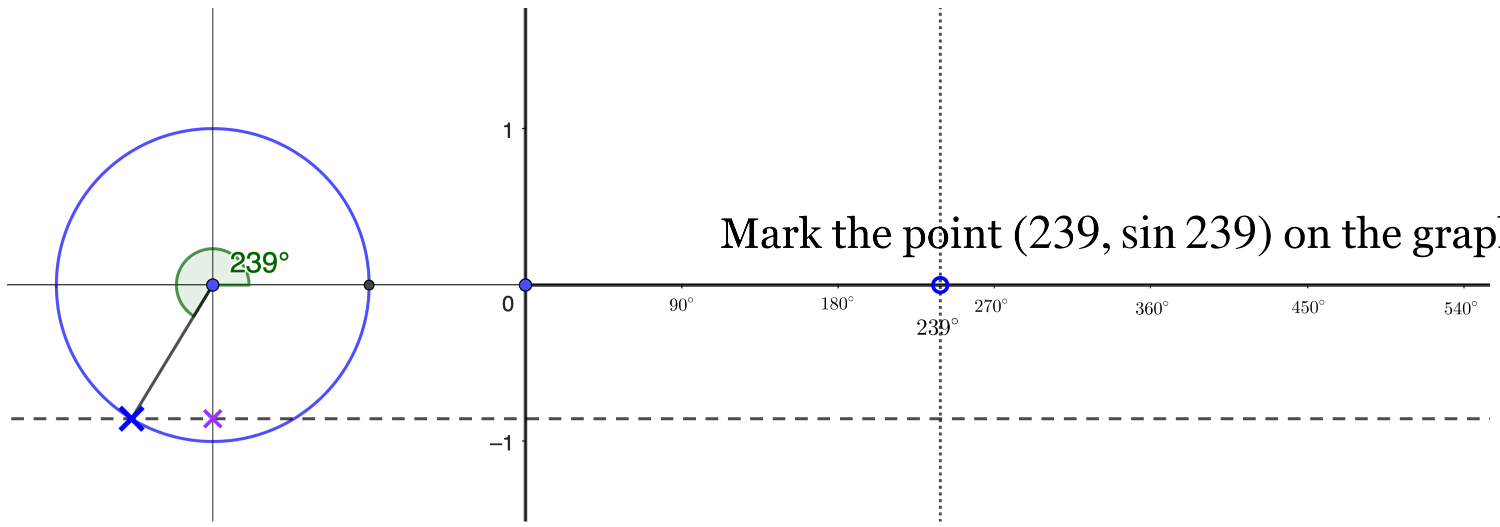
Mark the point (90, sin 90) on the graph



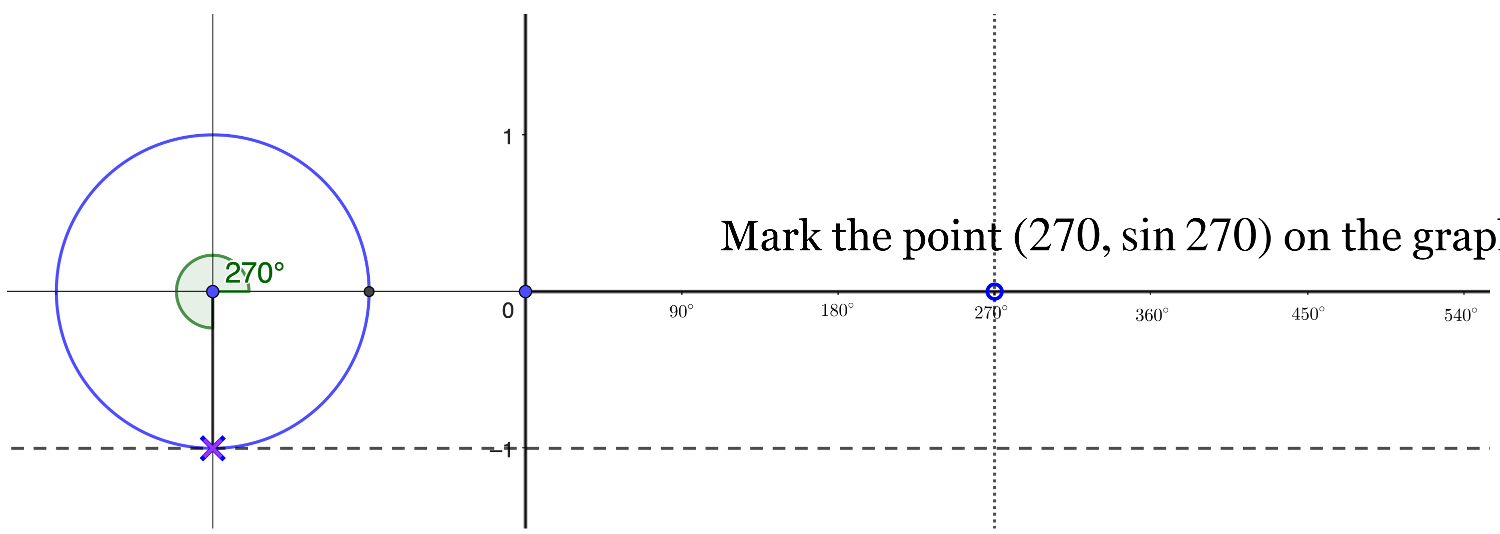
Mark the point (158, sin 158) on the graph



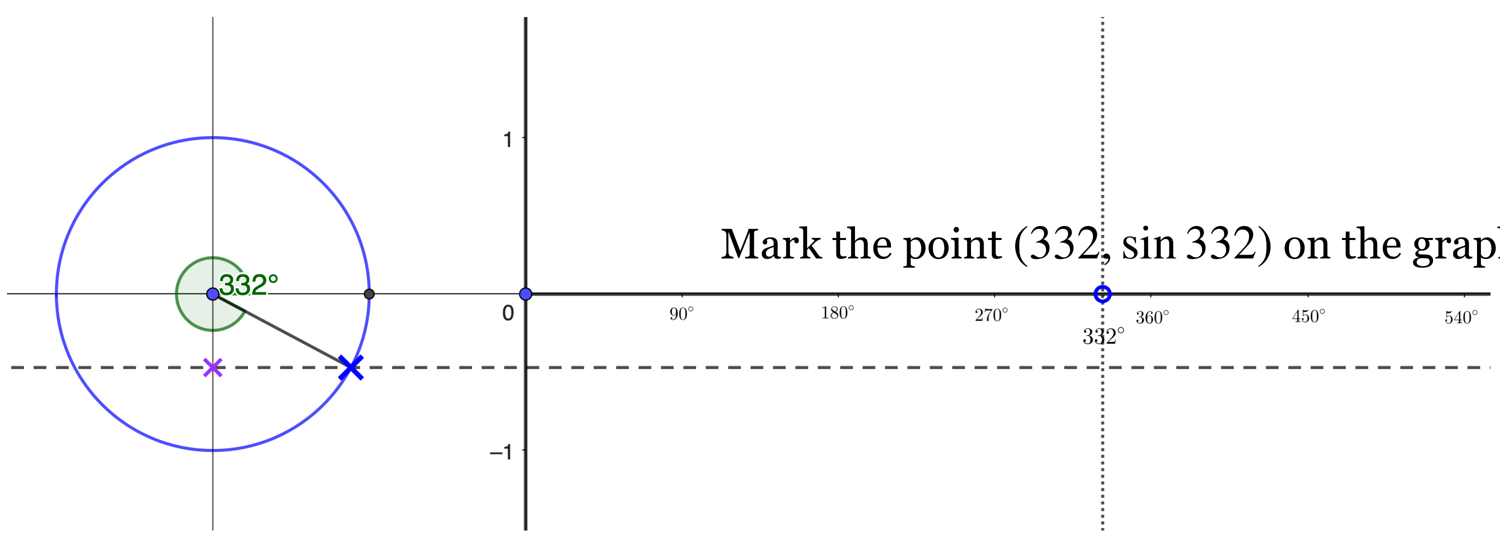
Mark the point (180, sin 180) on the graph



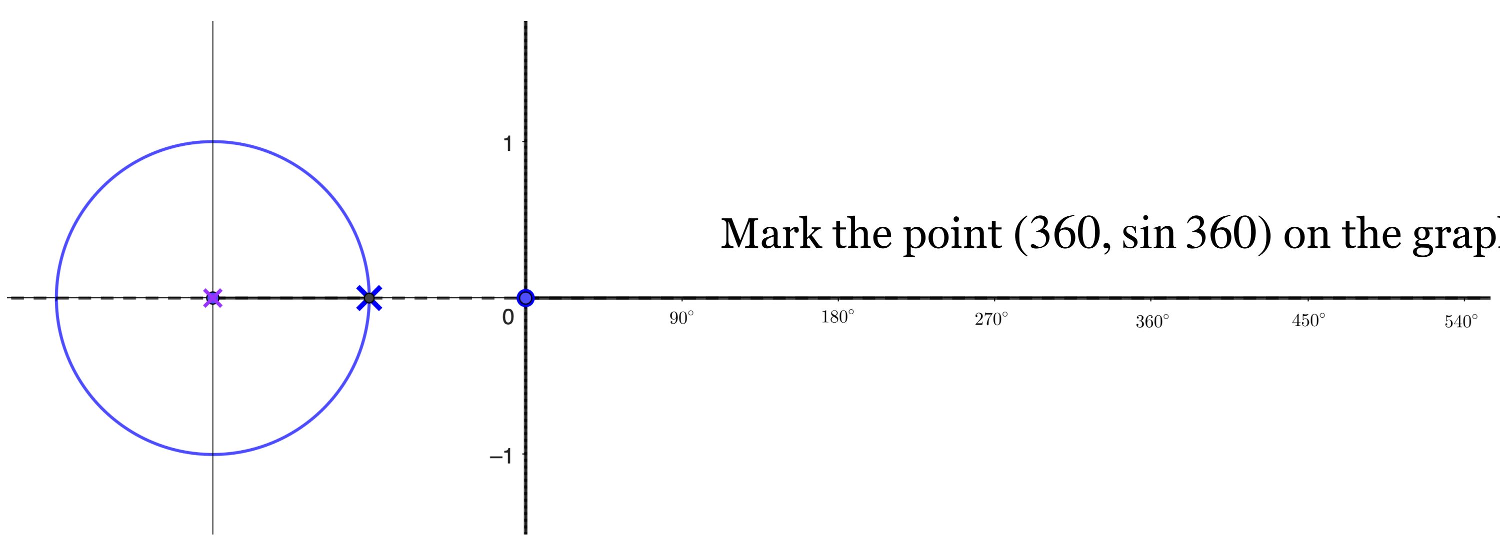
Mark the point (239, sin 239) on the graph



Mark the point (270, sin 270) on the graph

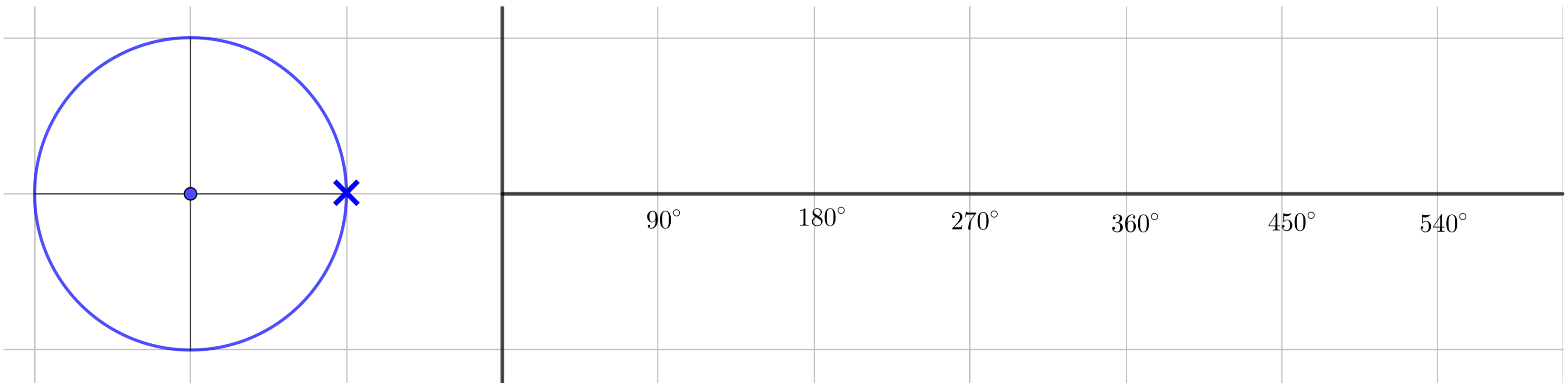


Mark the point (332, sin 332) on the graph

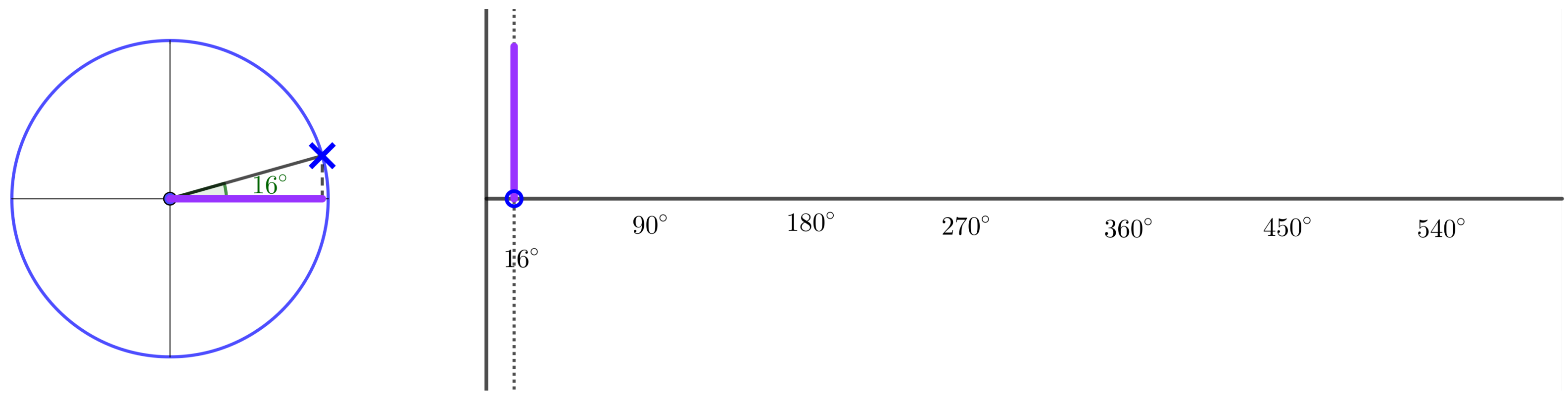


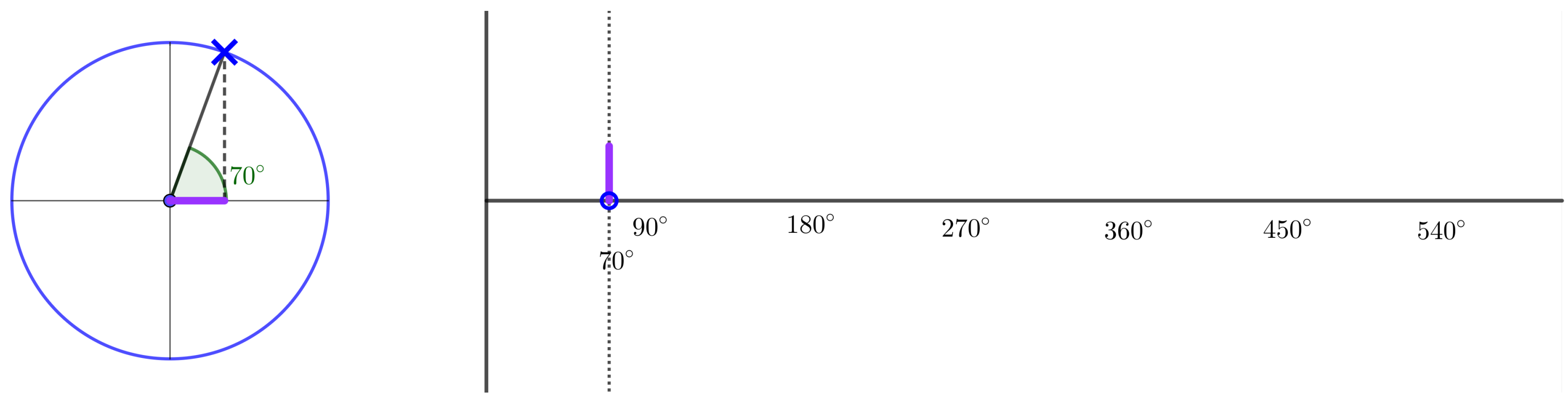
Mark the point (360, sin 360) on the graph

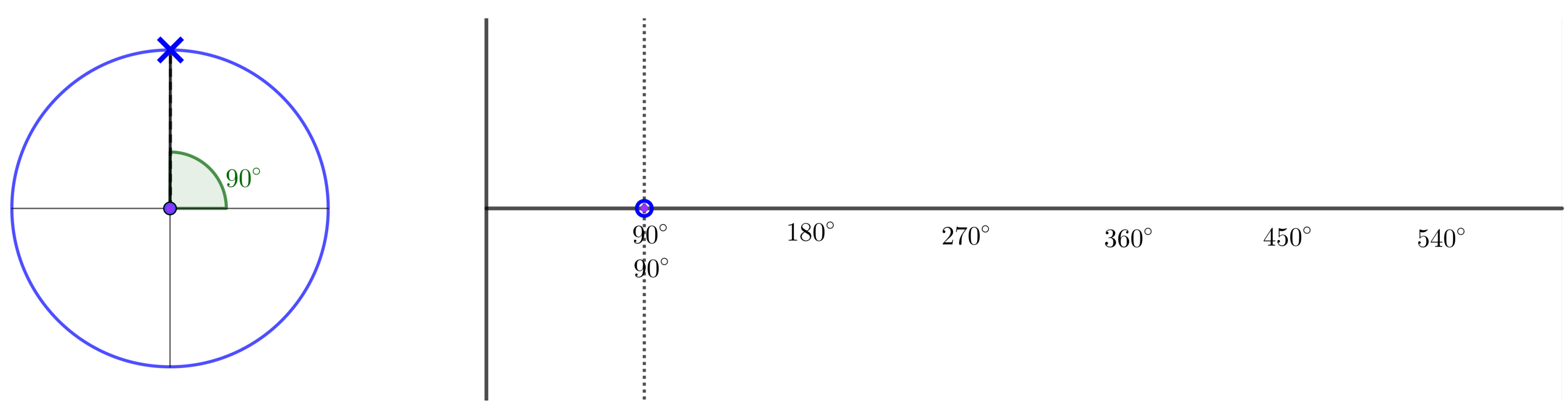
Use these points as a guide to draw the graph $y = \sin x$.

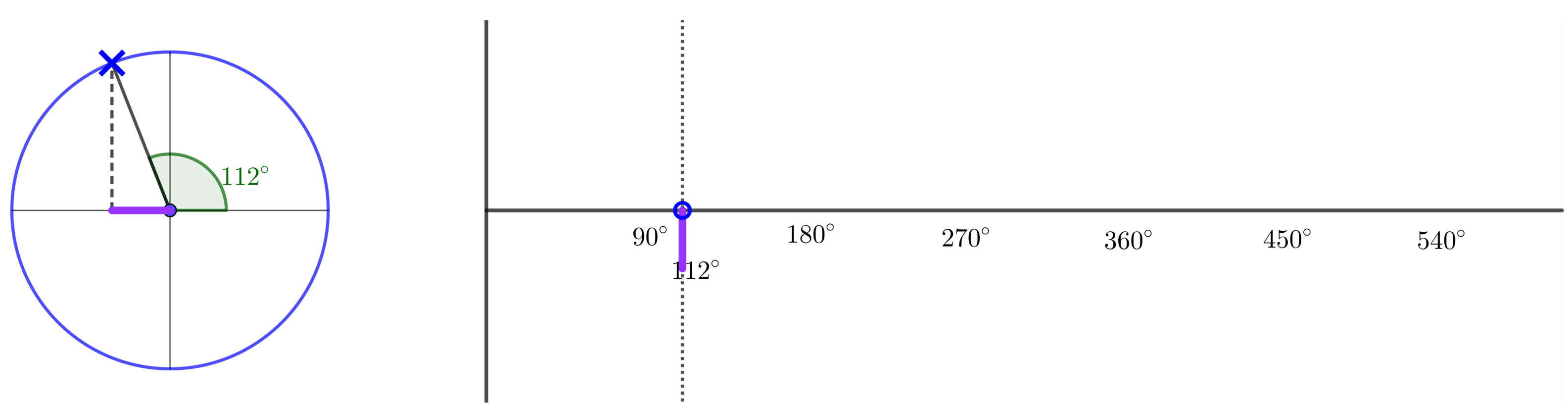


Look at the next sequence of images, and think about the relationship between the purple line segment on the left and the purple line segment on the right.

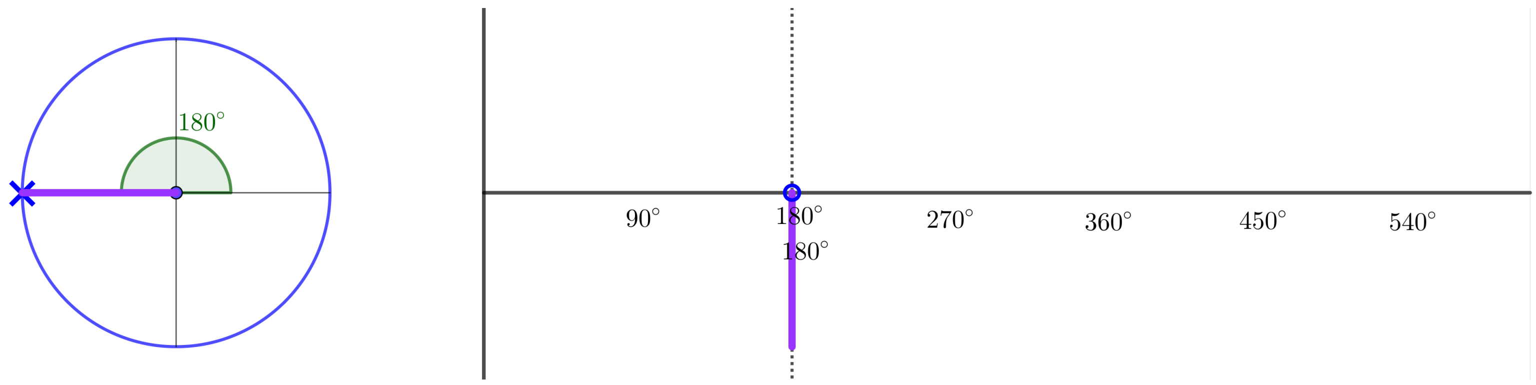


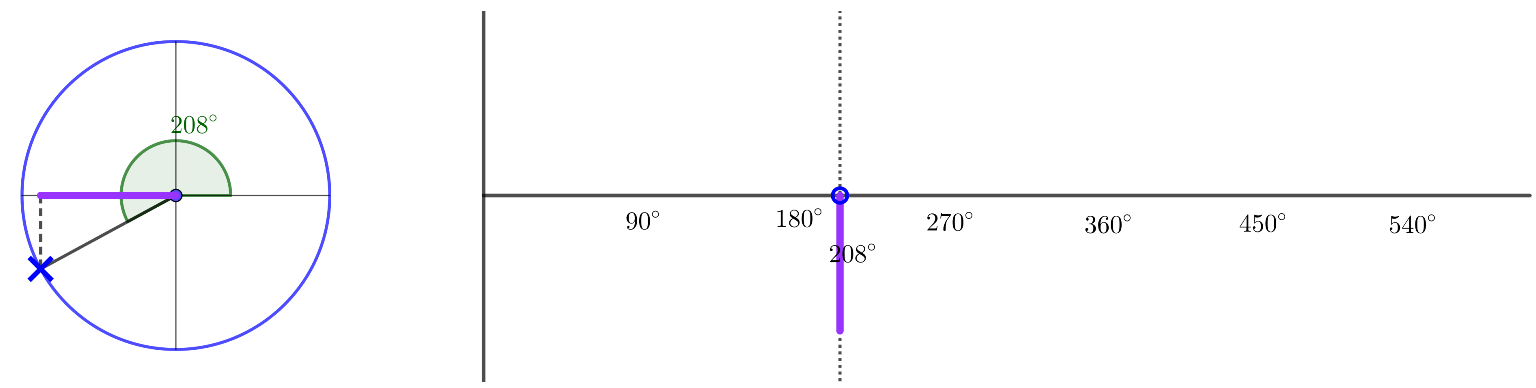


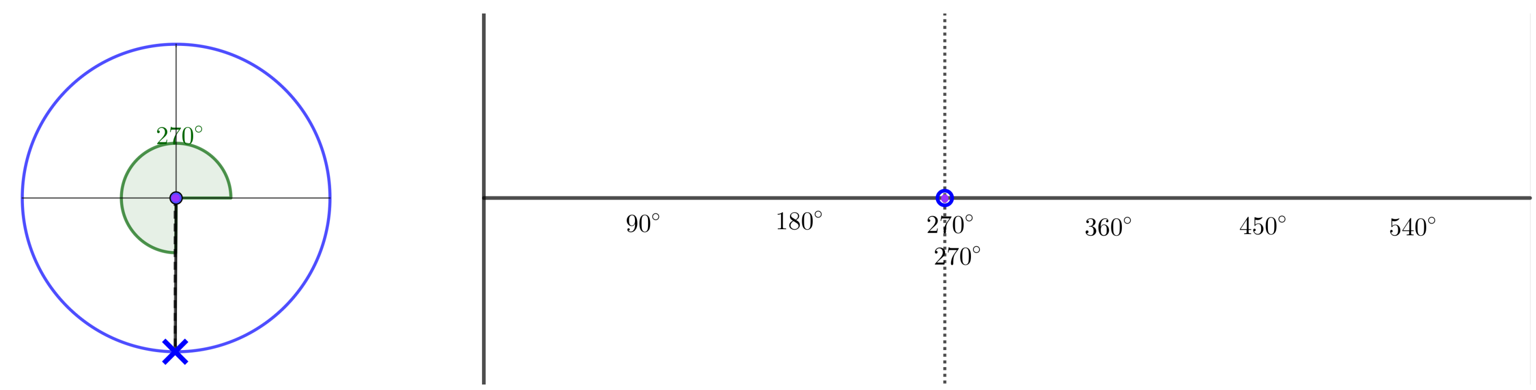


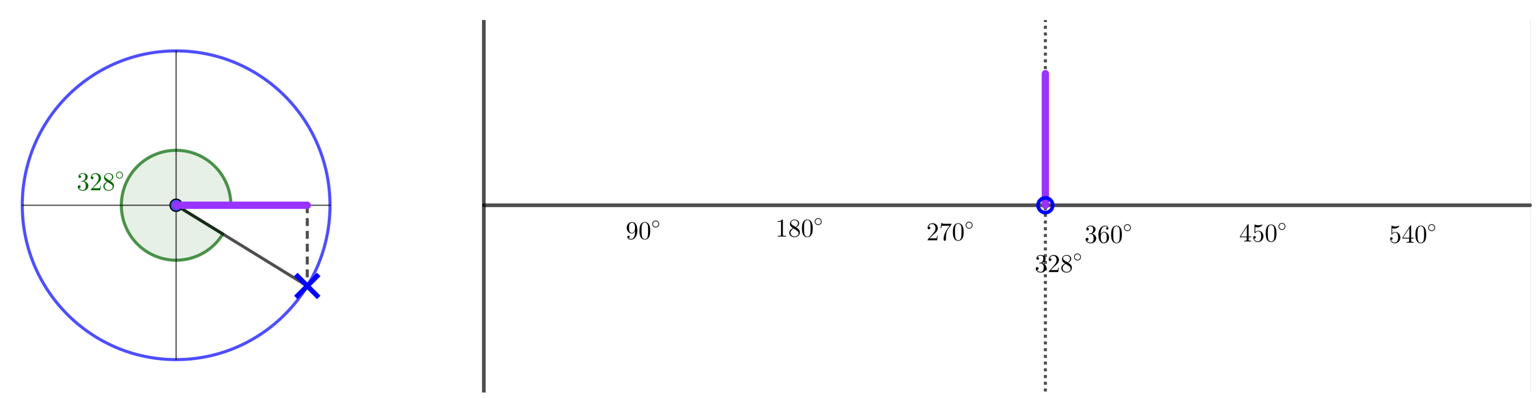


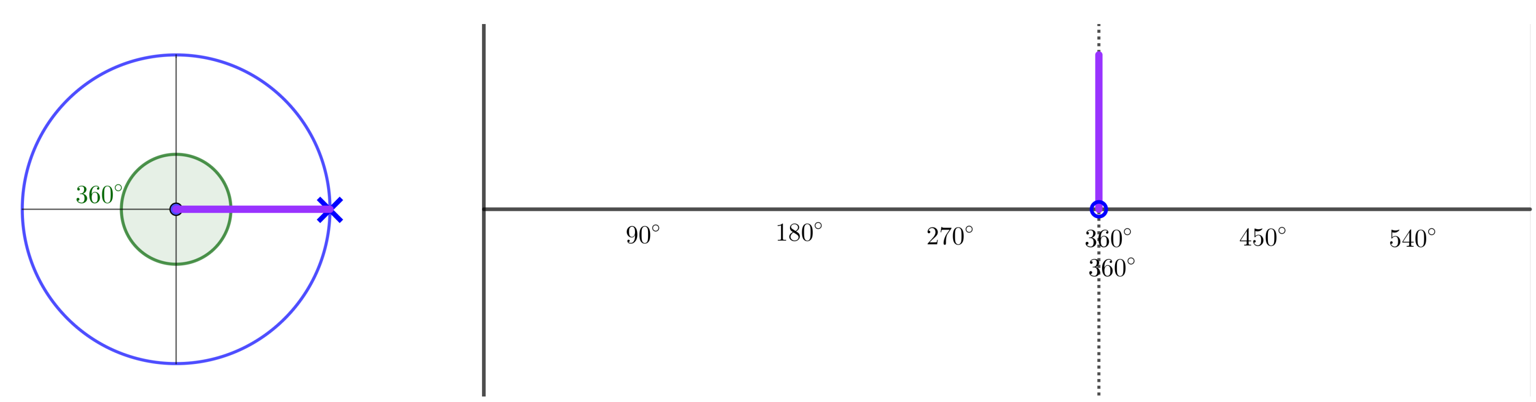
Demonstrating how the x coordinates on the unit circle become the y coordinates on the graph is a bit trickier. The signed length of the segment on the left is the same as the signed length of that on the right.


180°
90°
180°
270°
360°
450°
540°
180°

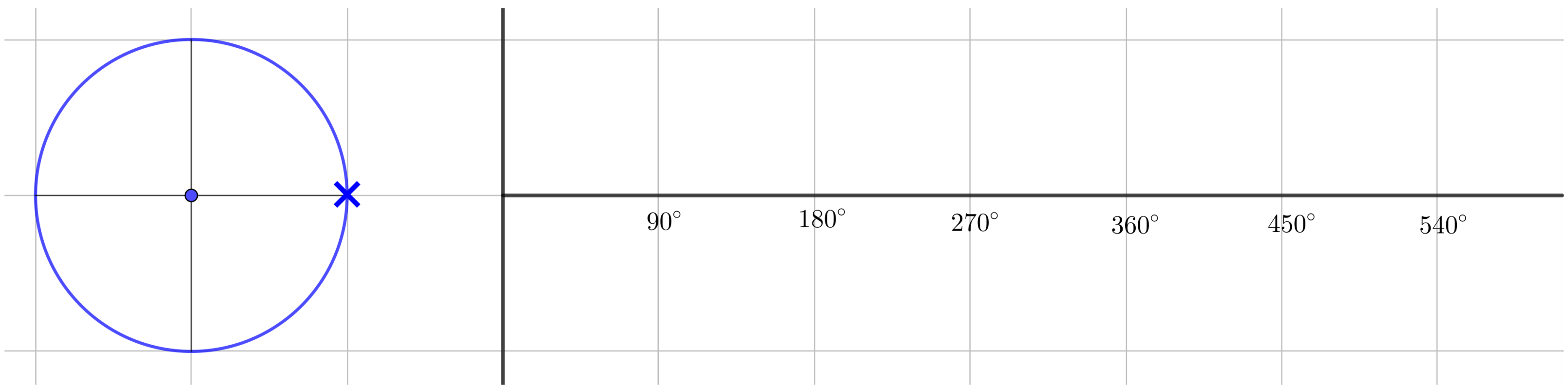

208°
90°
180°
270°
360°
450°
540°
208°


270°
90°
180°
270°
360°
450°
540°
270°

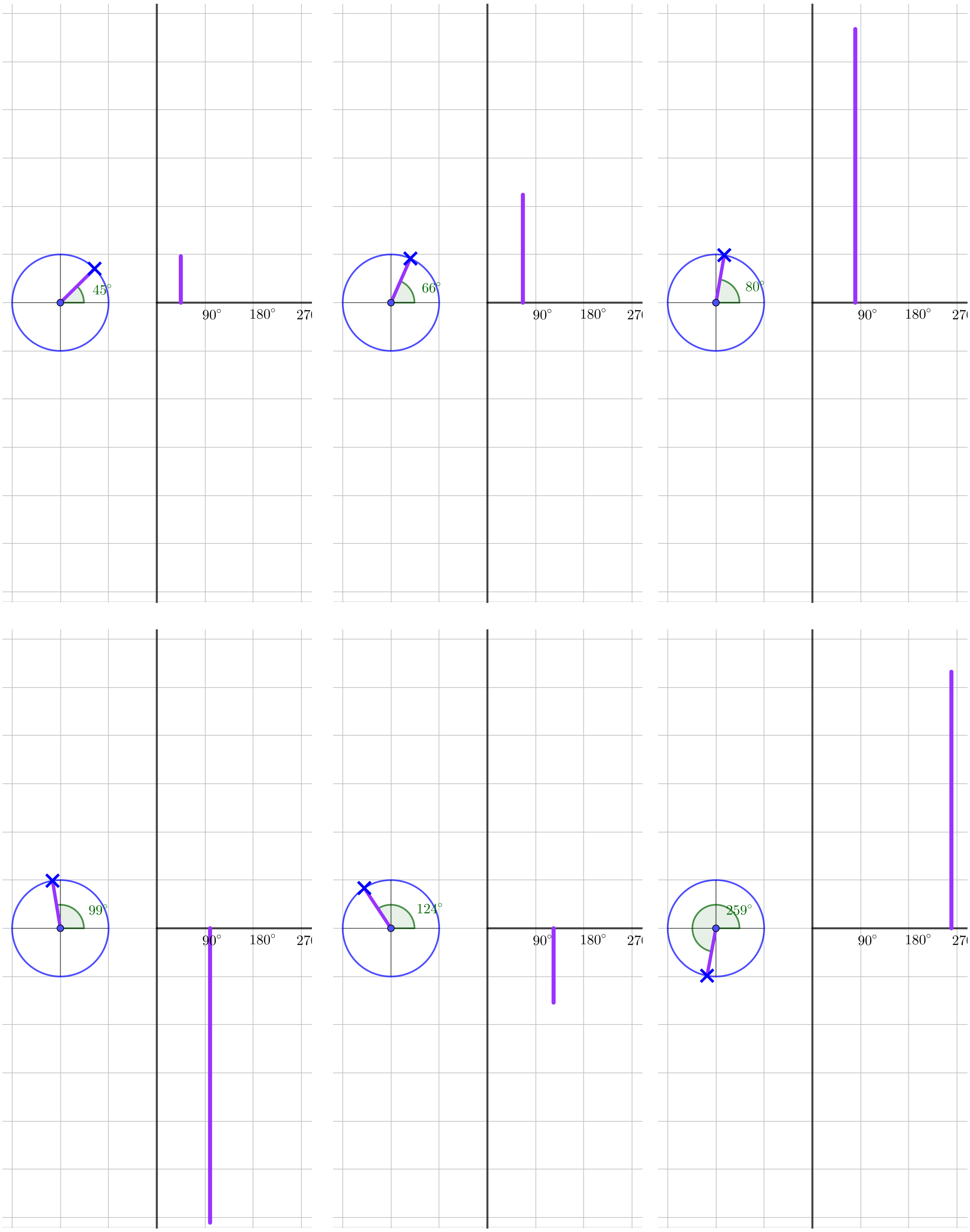

328°
90°
180°
270°
360°
450°
540°
328°


360°
90°
180°
270°
360°
450°
540°
360°

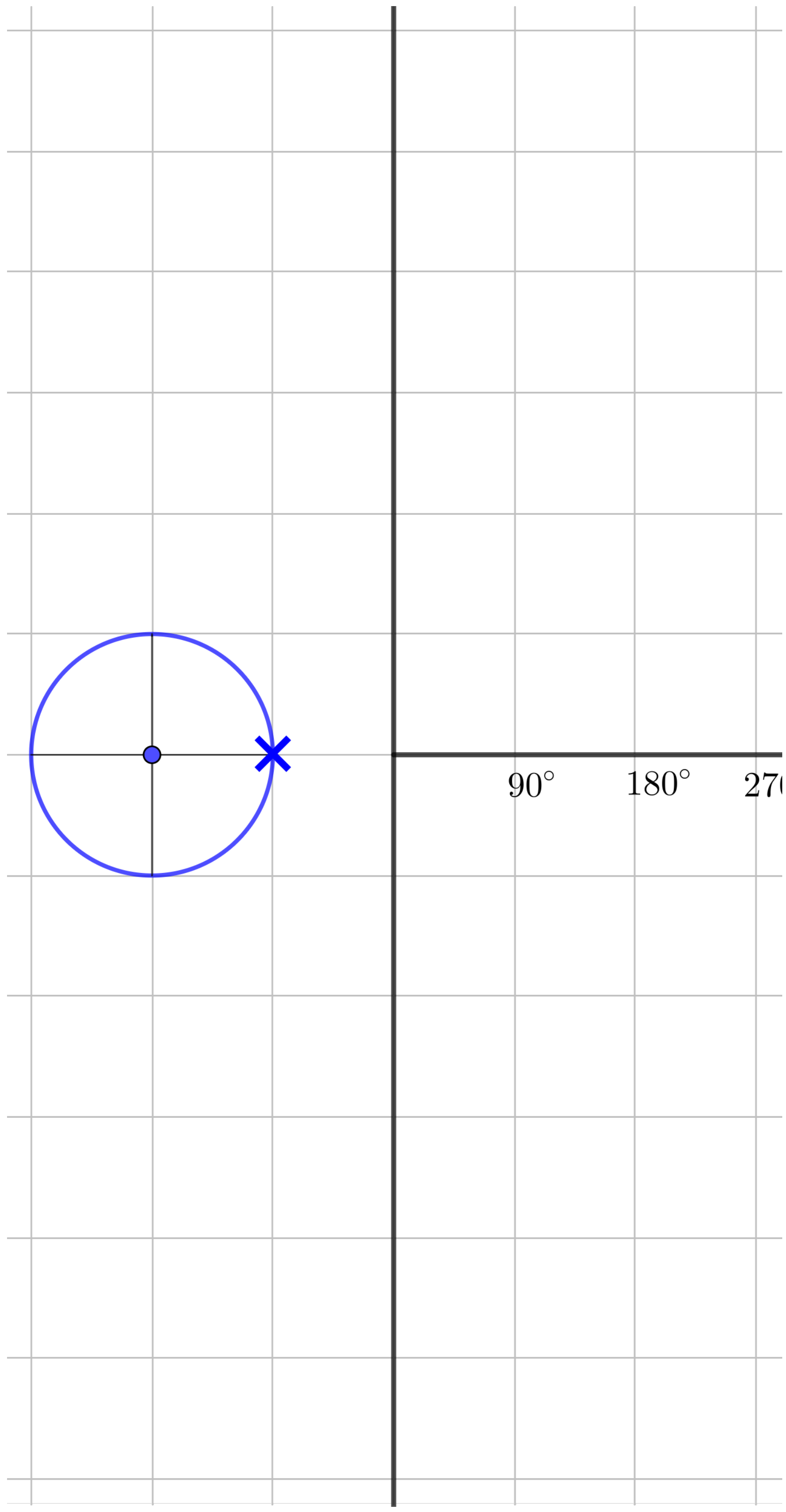
Use these points as a guide to draw the graph $y = \cos x$.



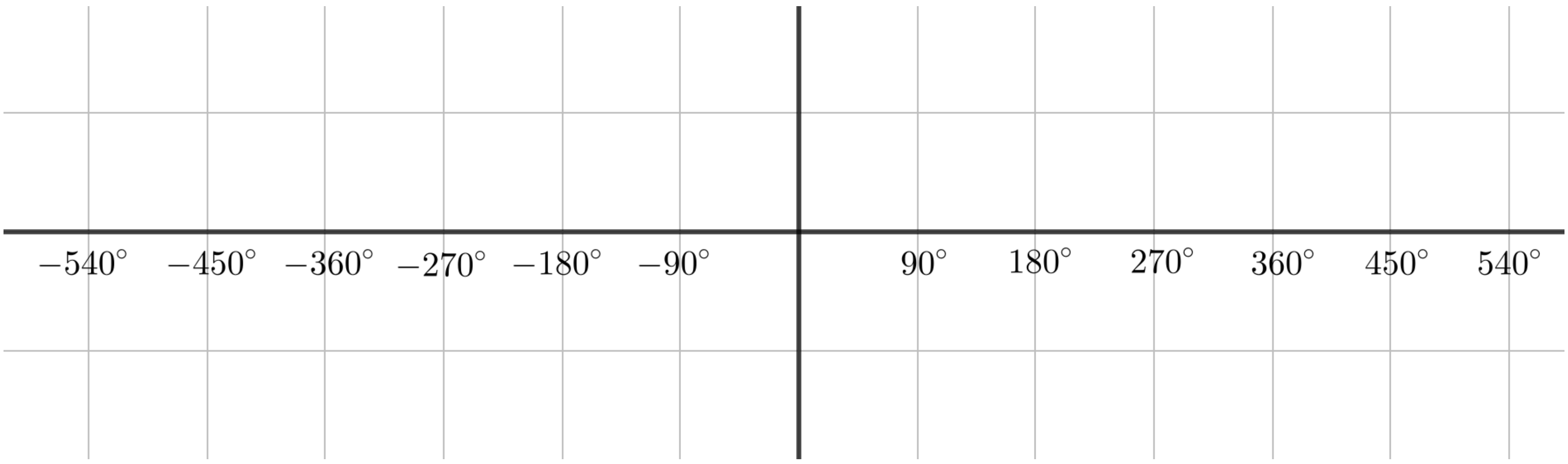
Look at this sequence of images, and describe the relationship between the purple line segment on the left and the purple line segment on the right.



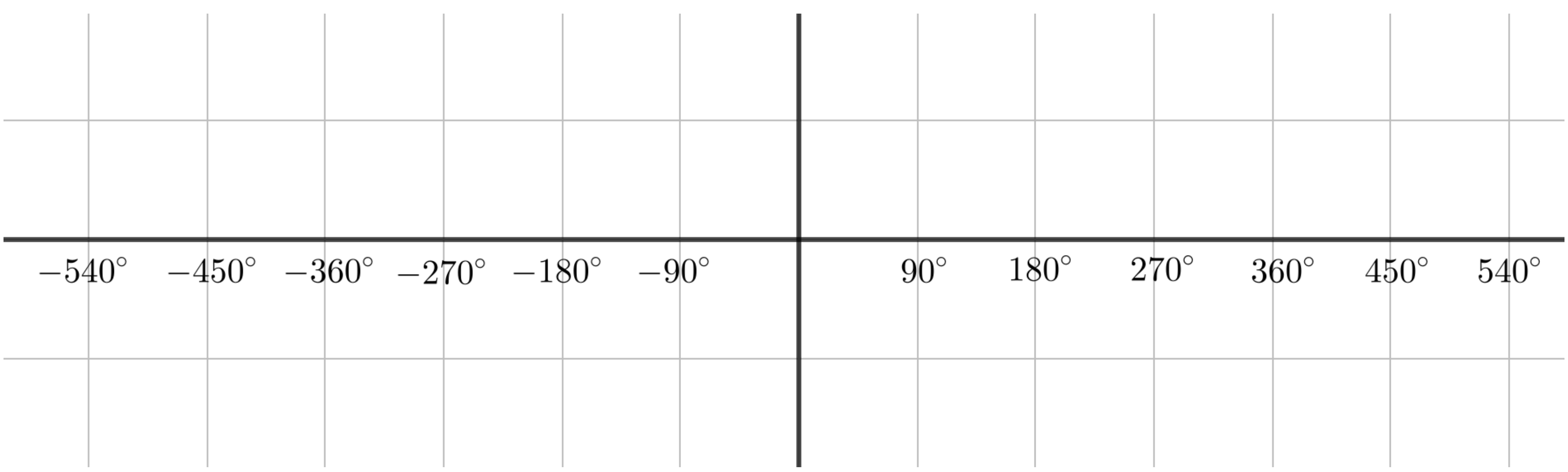
Use this idea to draw the graph $y = \tan x$.



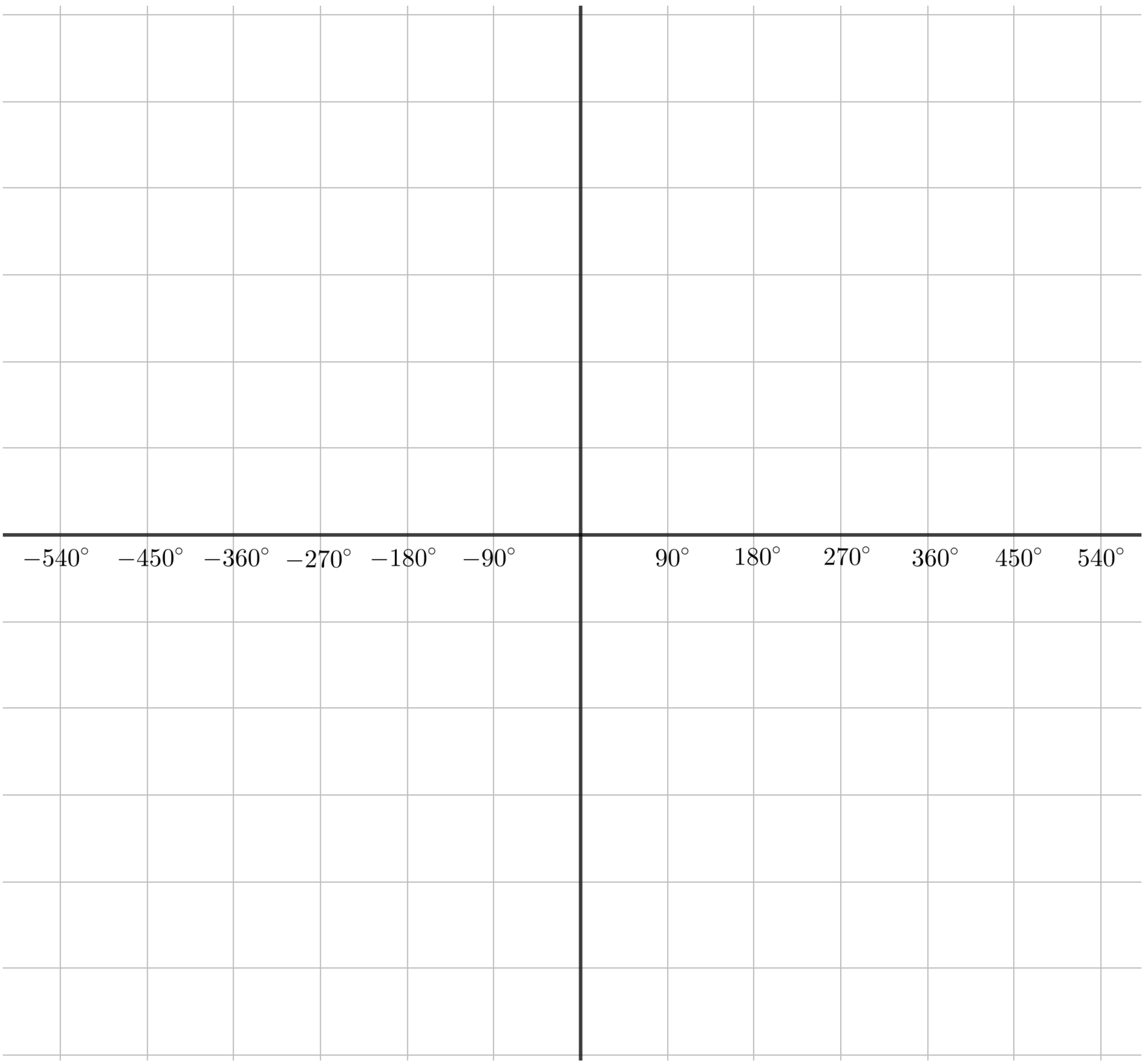
Draw the graph $y = \sin x$.



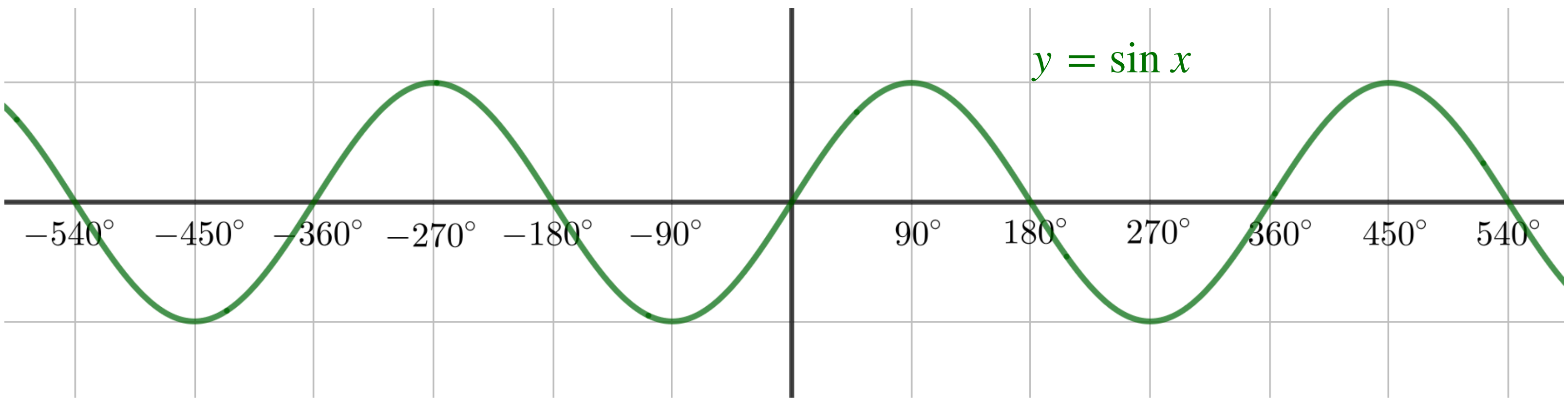
Draw the graph $y = \cos x$.

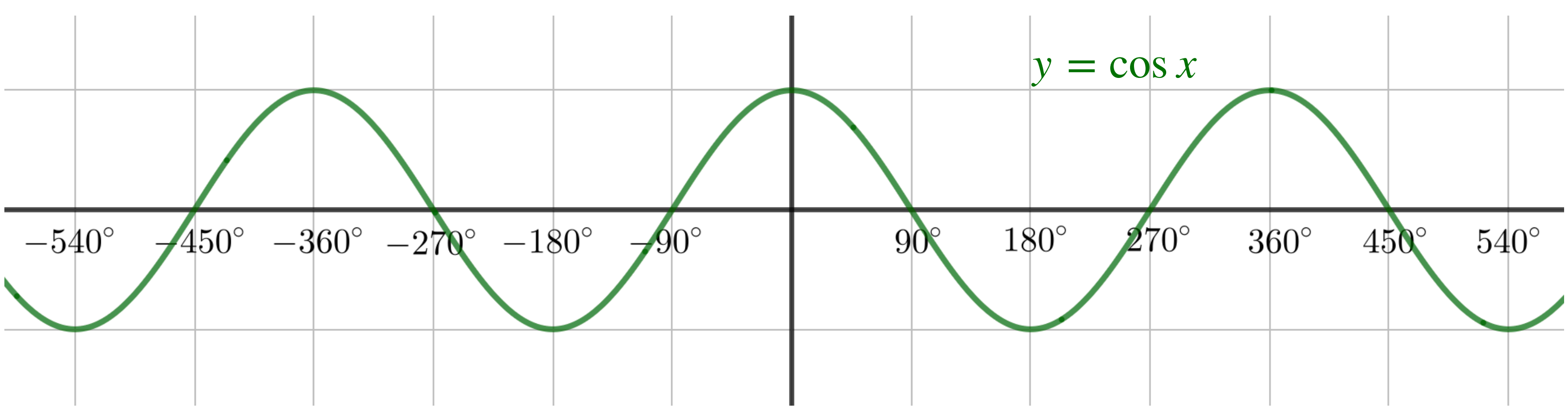


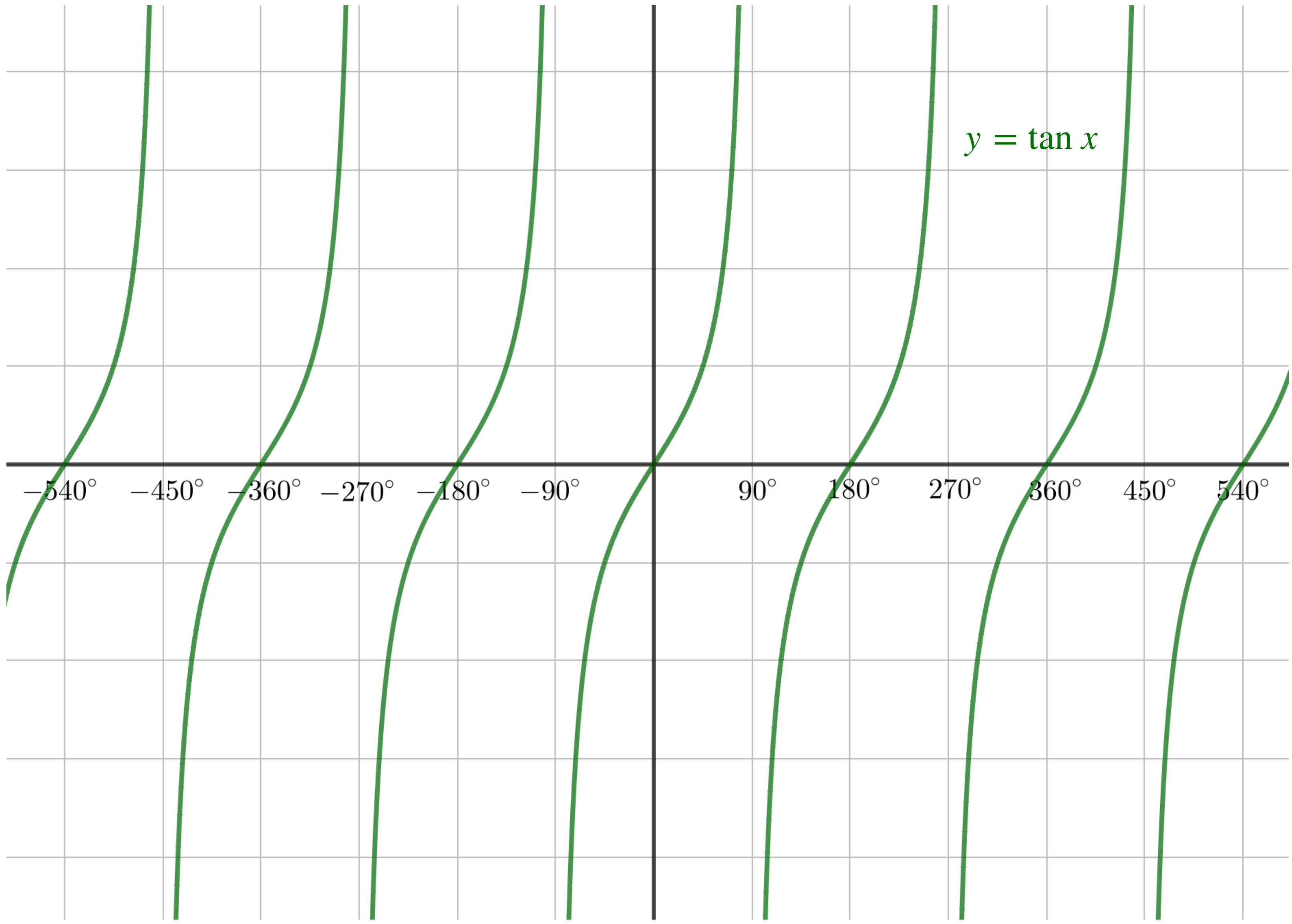
Draw the graph $y = \tan x$.



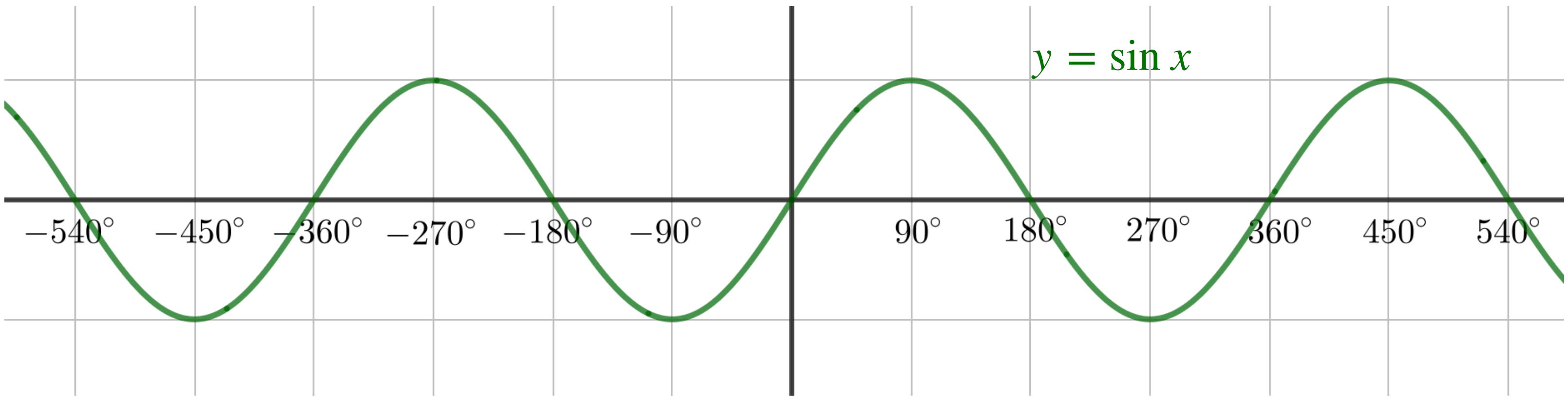
Describe the symmetry of each of these graphs.

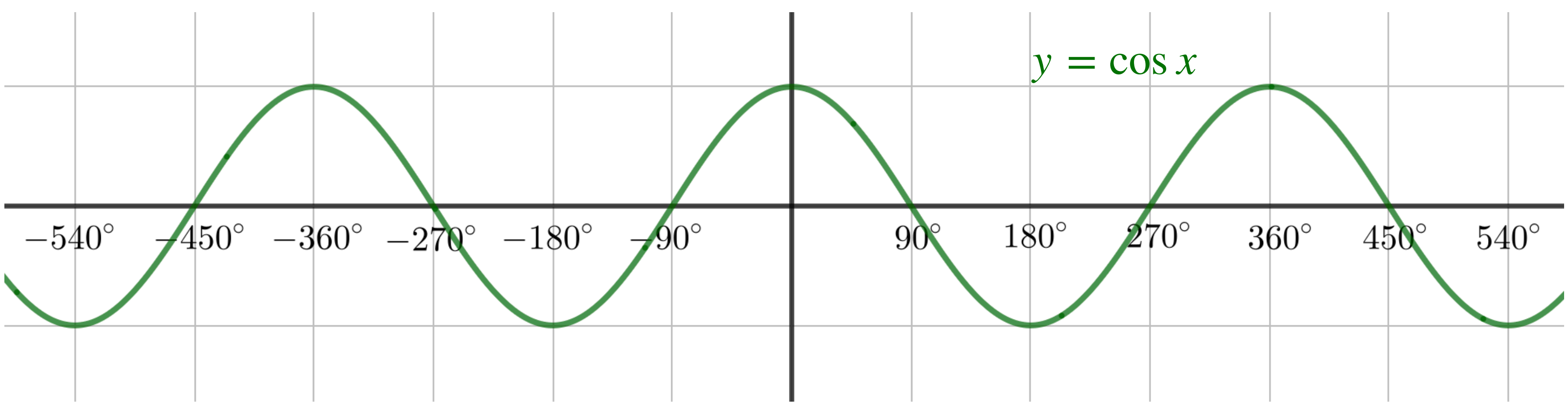





$y = \tan x$
$-540°$
$-450°$
$-360°$
$-270°$
$-180°$
$-90°$
$90°$
$180°$
$270°$
$360°$
$450°$
$540°$

Describe the symmetry of each of these graphs.

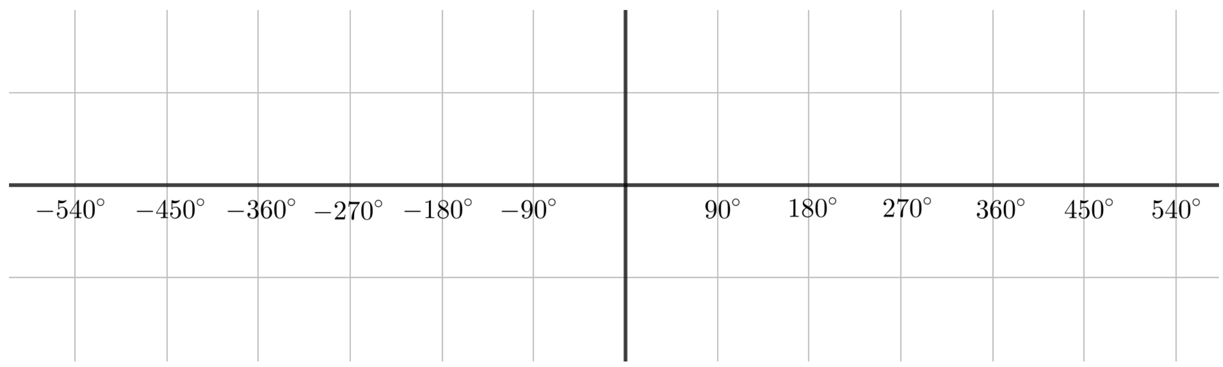




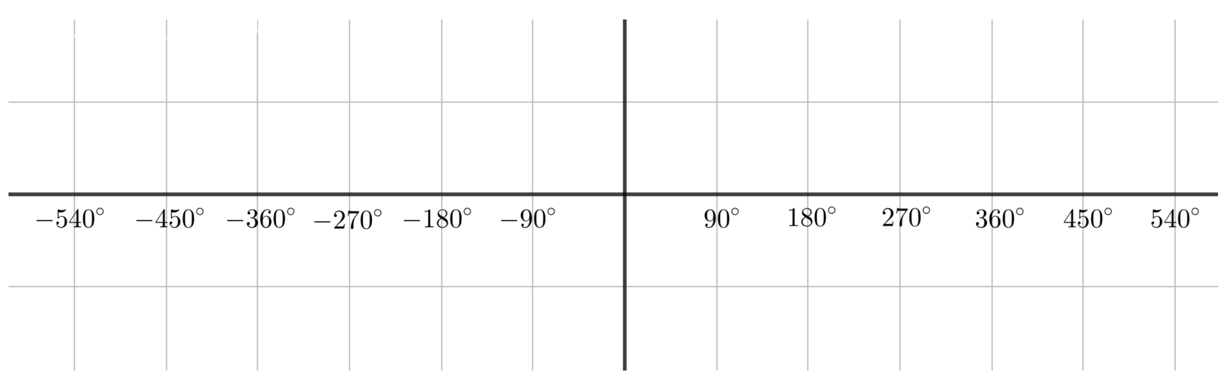
Relationships between circular functions

Draw the following graphs:

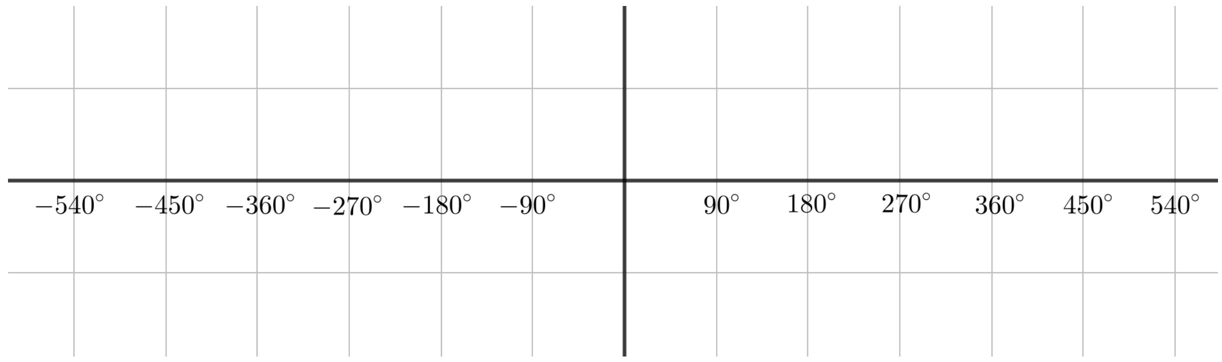
$y = -\sin x$



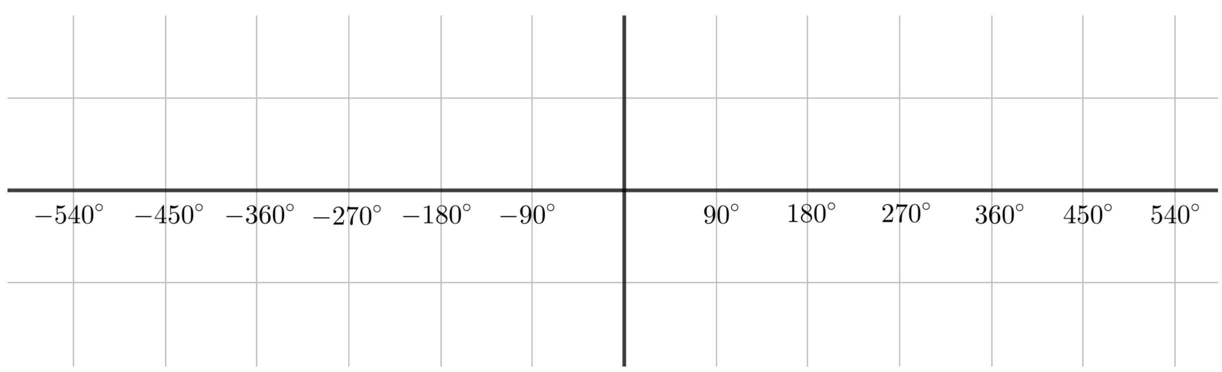
$y = \sin(-x)$



$y = -\cos x$


−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

$y = \cos(-x)$


−540°
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−180°
−90°
90°
180°
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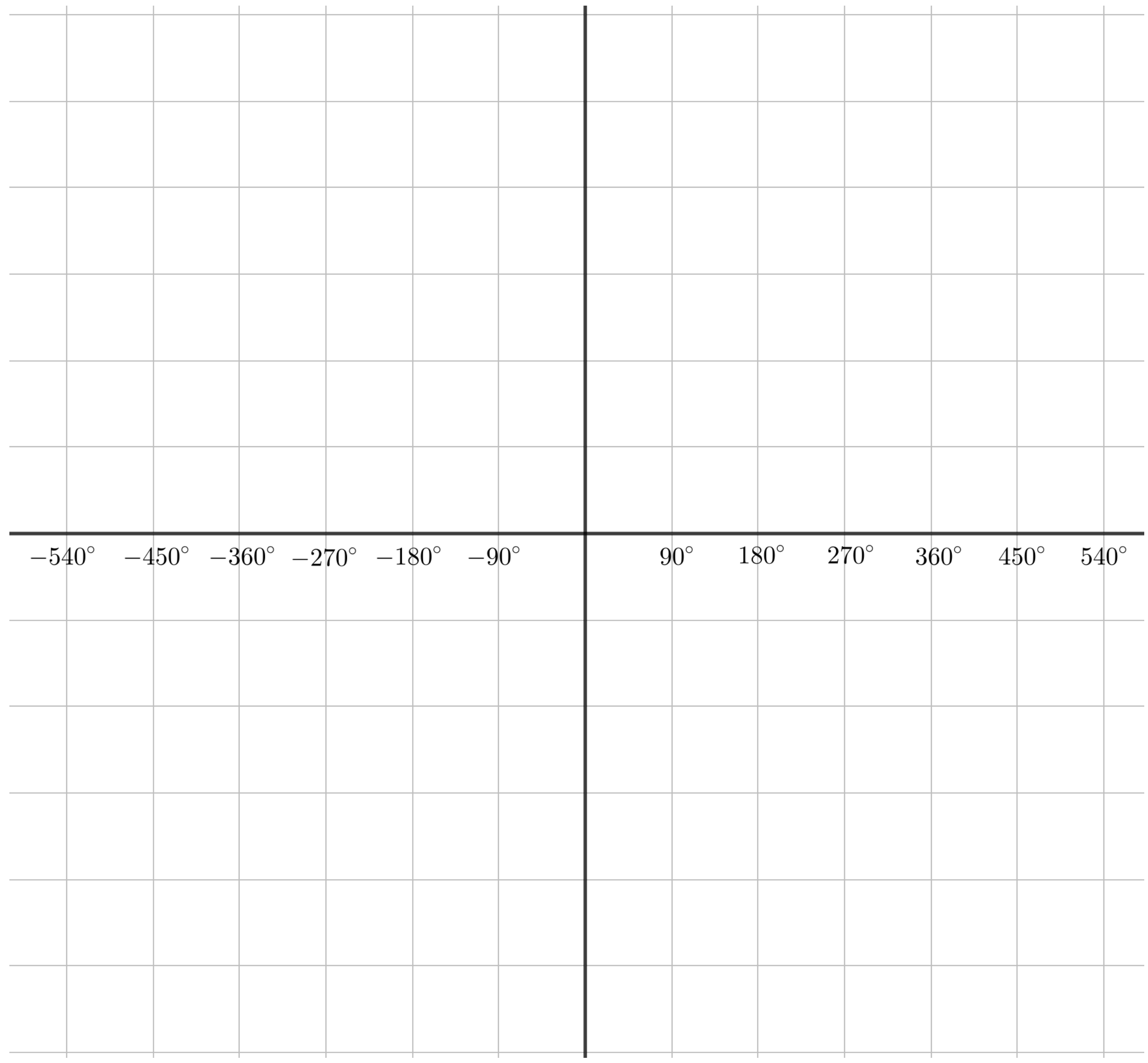
360°
450°
540°

$y = -\tan x$


−540°
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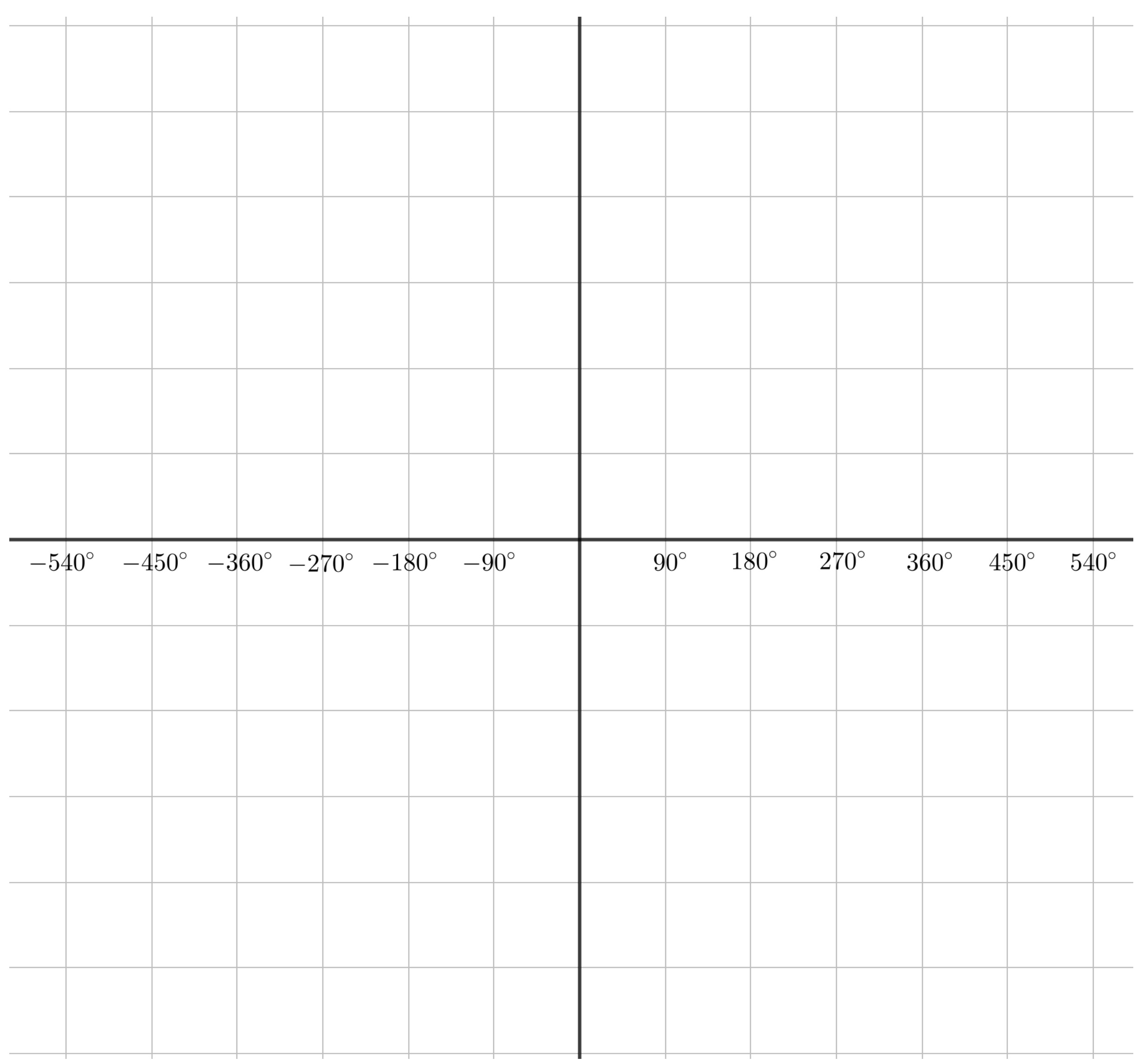
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

$y = \tan(-x)$


−540°
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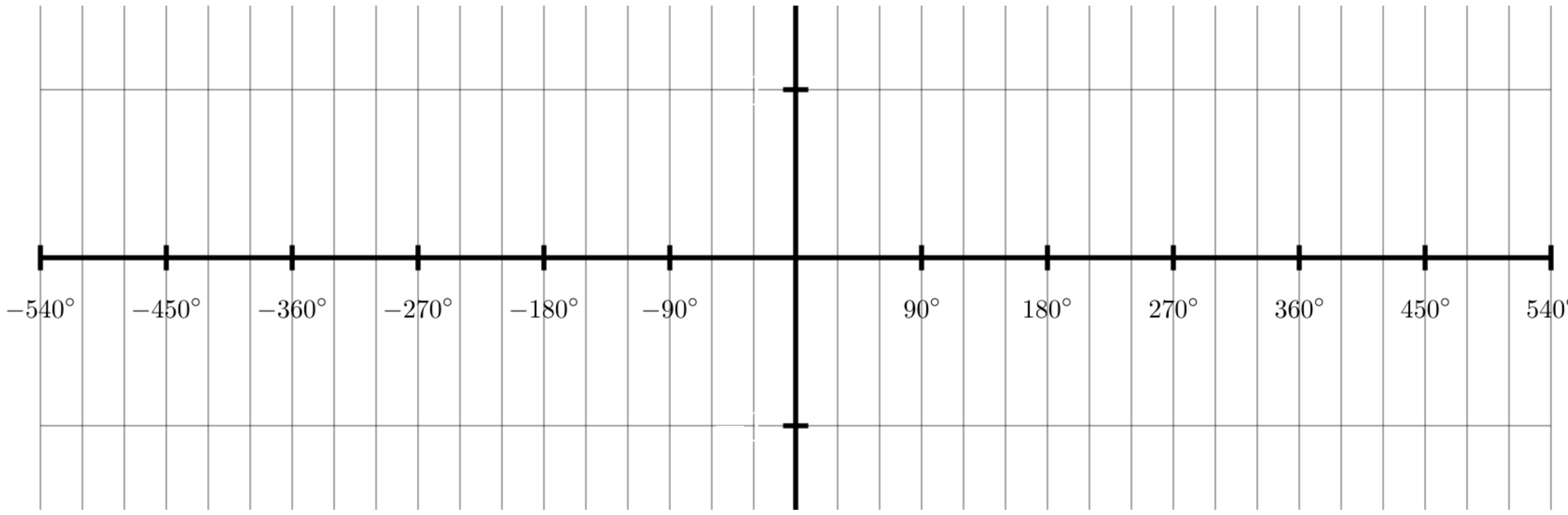
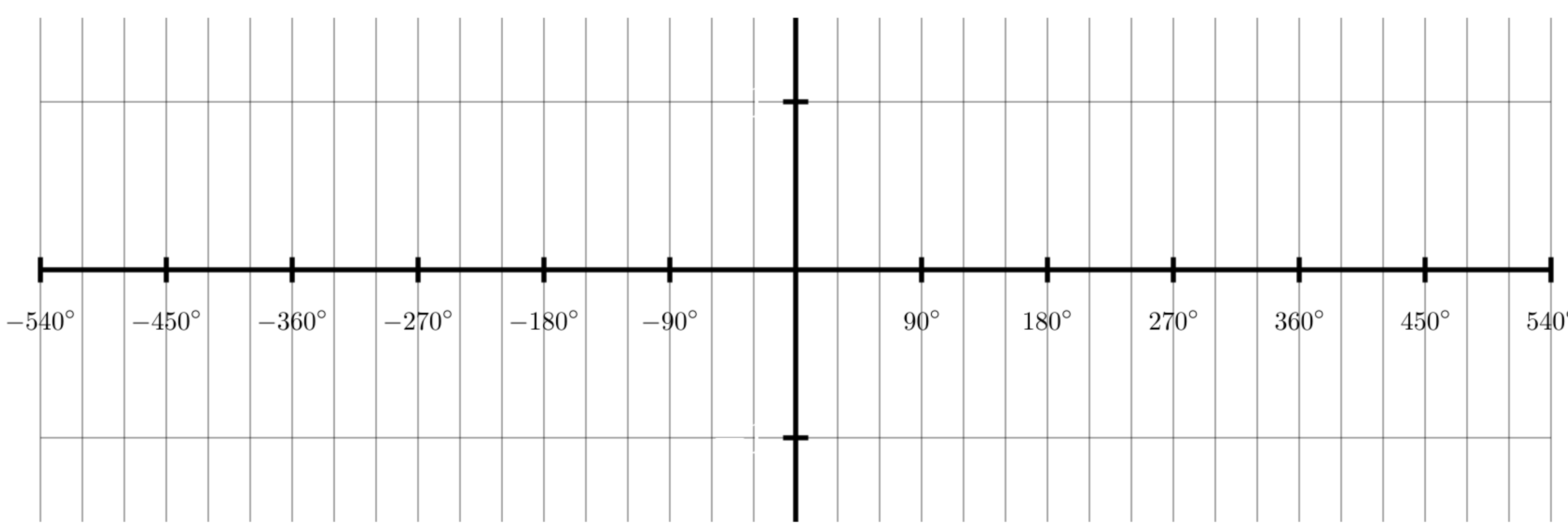
90°
180°
270°
360°
450°
540°

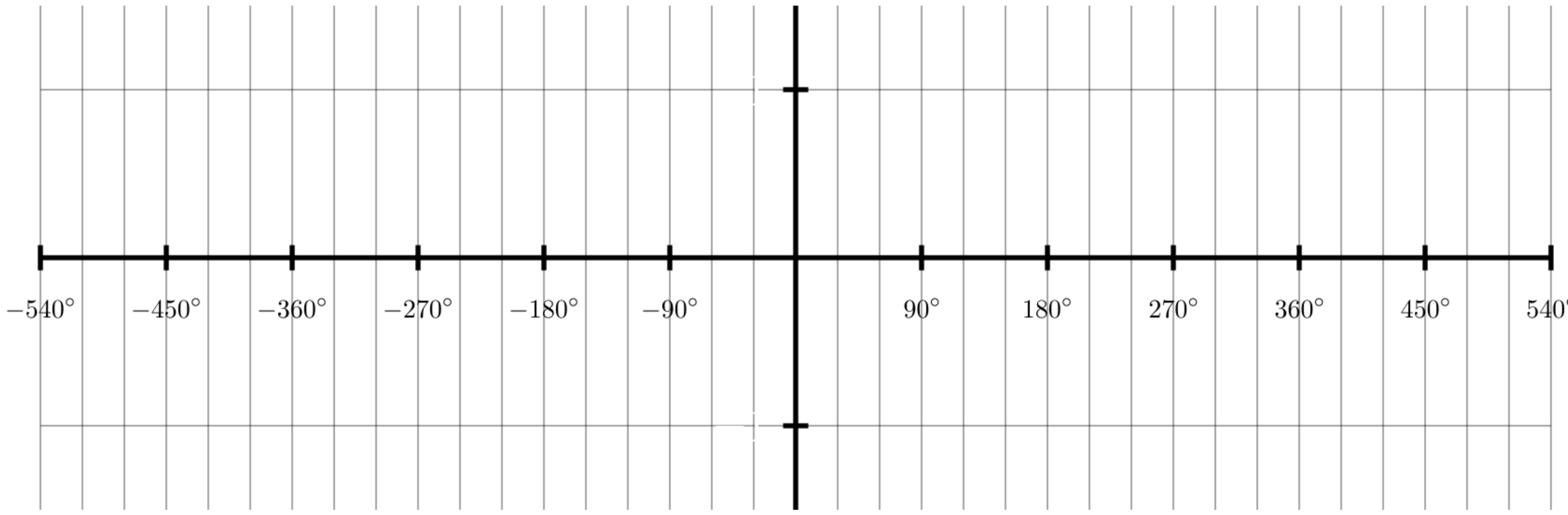
$y = \sin(90° - x)$



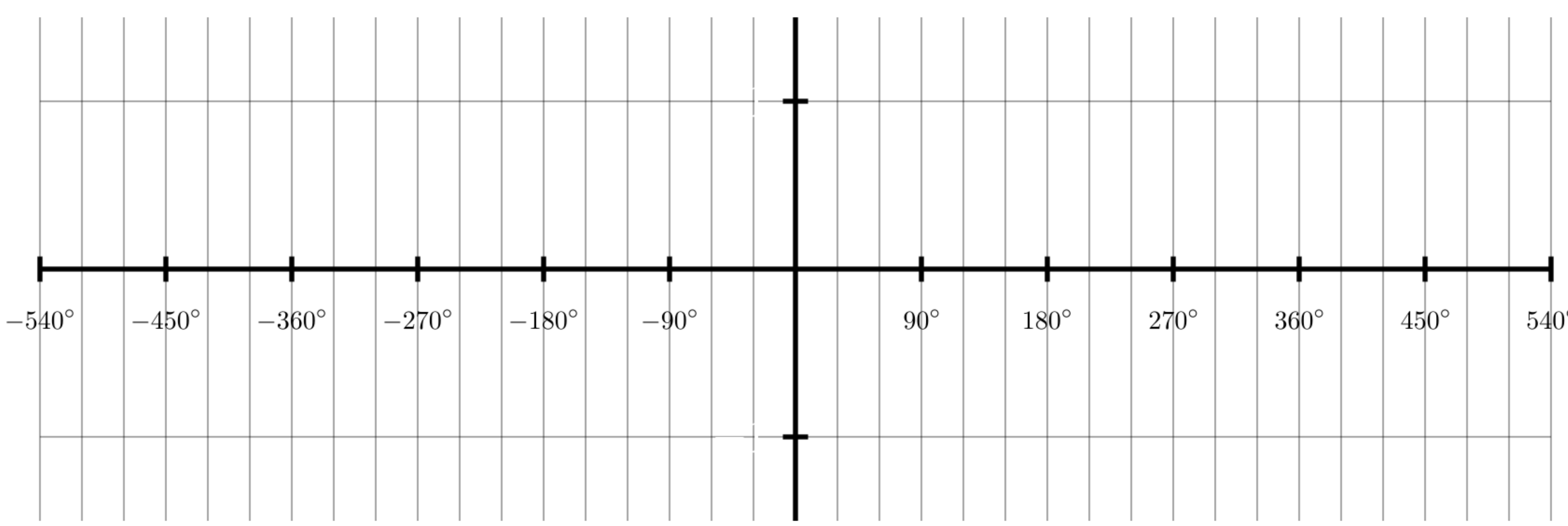
$y = \cos(90° - x)$



$y = \sin(90^\circ + x)$



$y = \cos(90^\circ + x)$



Use these graphs to find some relationships between sin and cos.

Show that

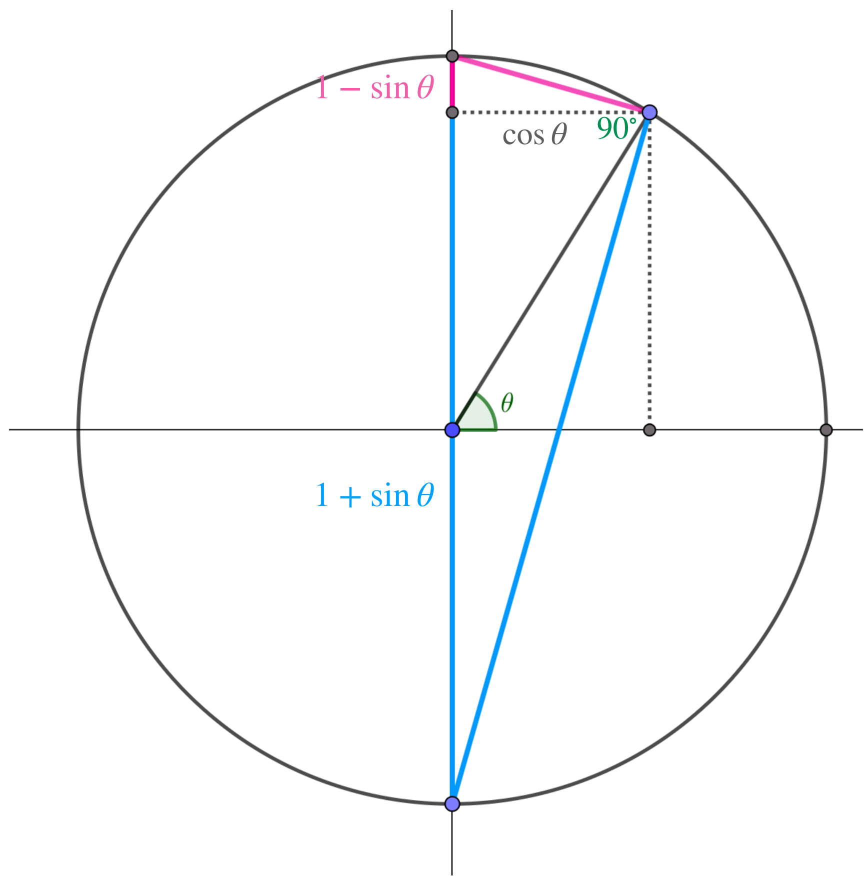
$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

whenever θ is not a multiple of 90°.

Show that

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$$

whenever θ is not a multiple of 90°.


$1-\sin\theta$
$\cos\theta$
$90°$
θ
$1+\sin\theta$

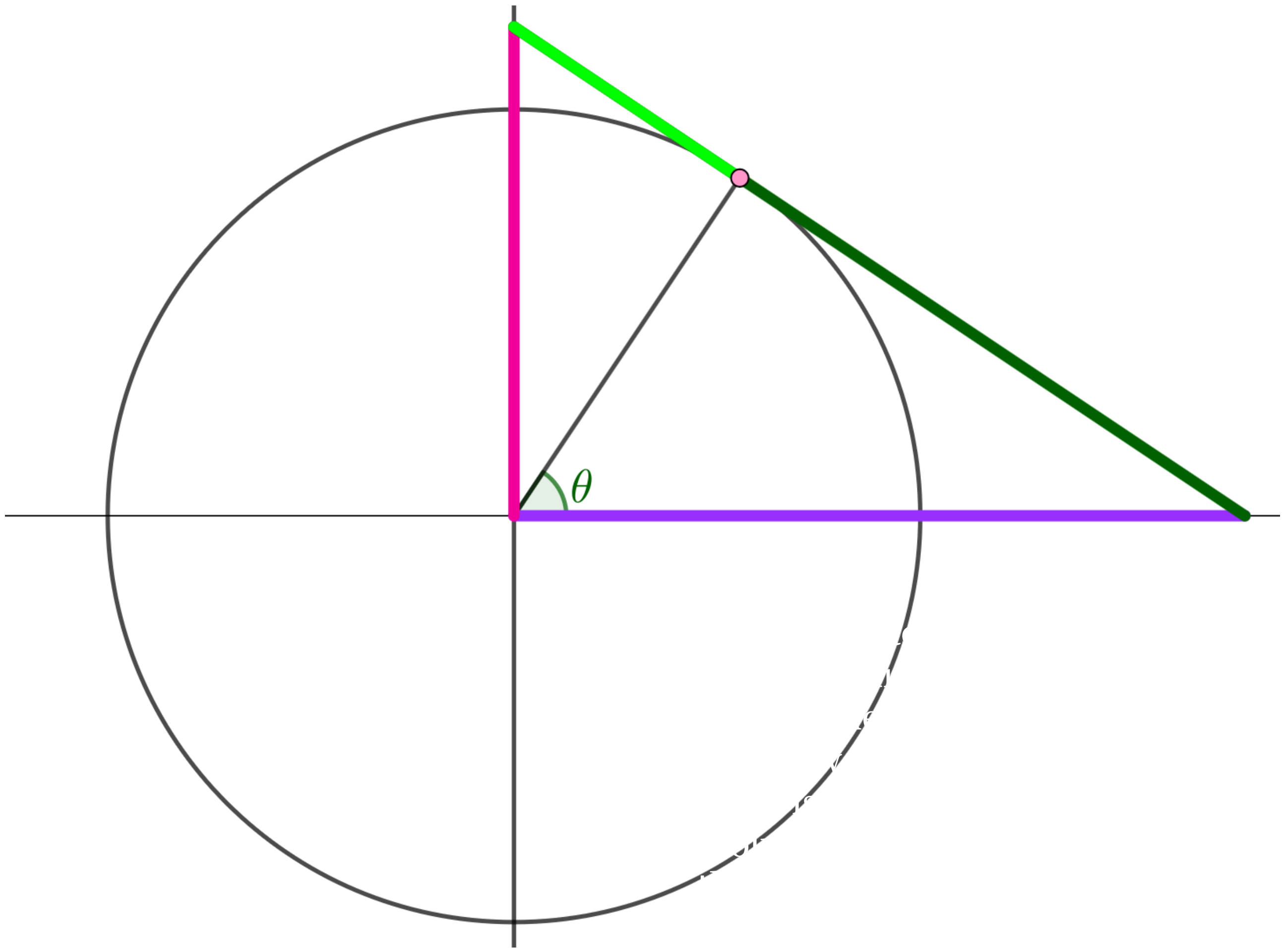
Show that

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} \equiv \frac{1 + \sin\theta}{\cos\theta}$$

Show that

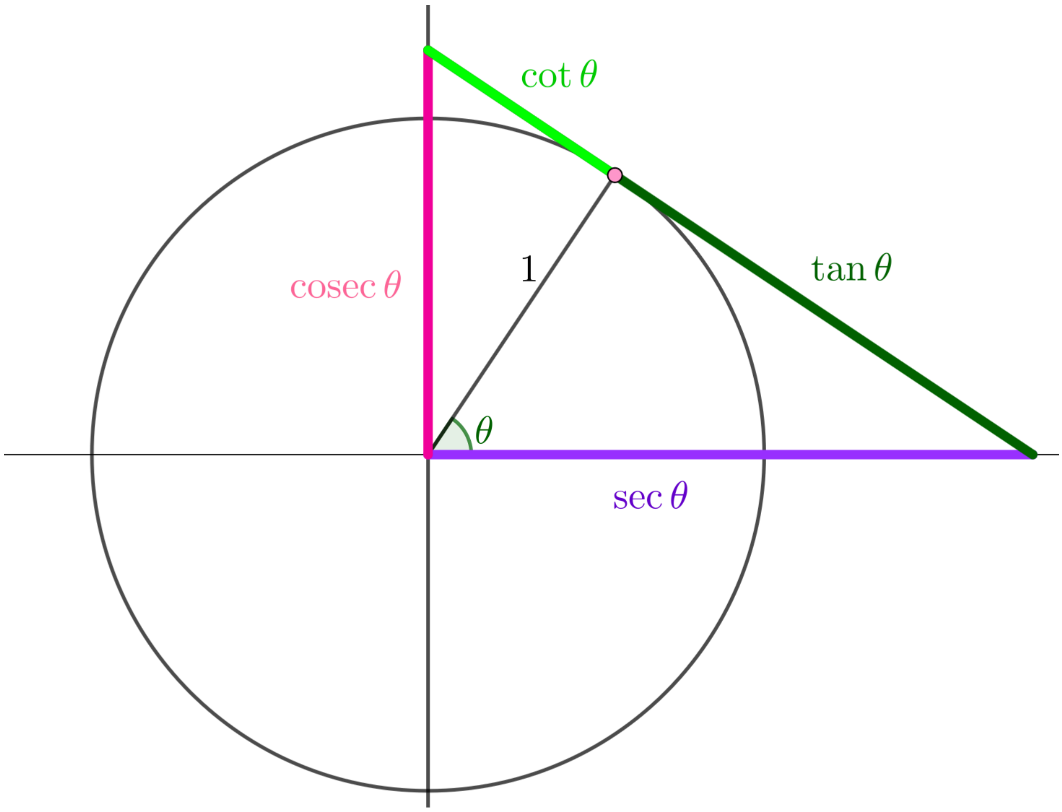
$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} \equiv \frac{1 + \sin\theta}{\cos\theta}$$

Reciprocal circular functions

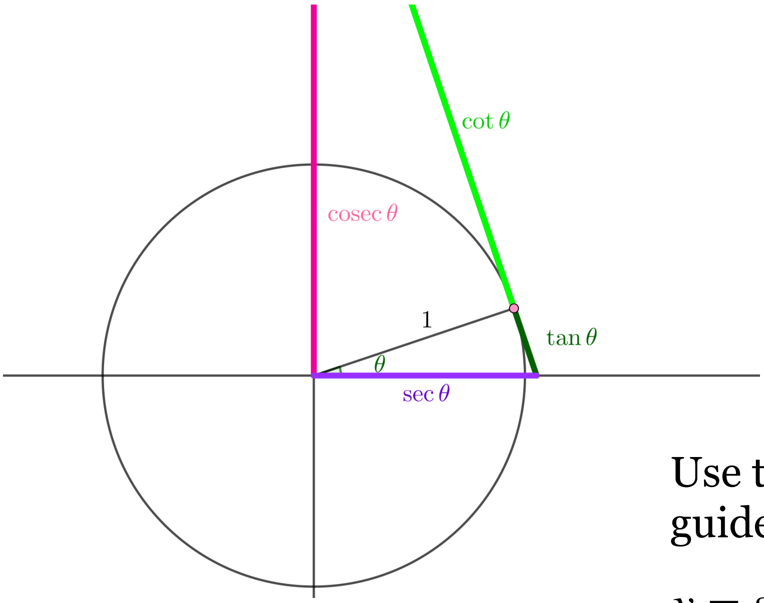


This is a circle radius 1.

What are the lengths of the highlighted segments?



There are three right-angled triangles here. What would Pythagoras say about each of them?

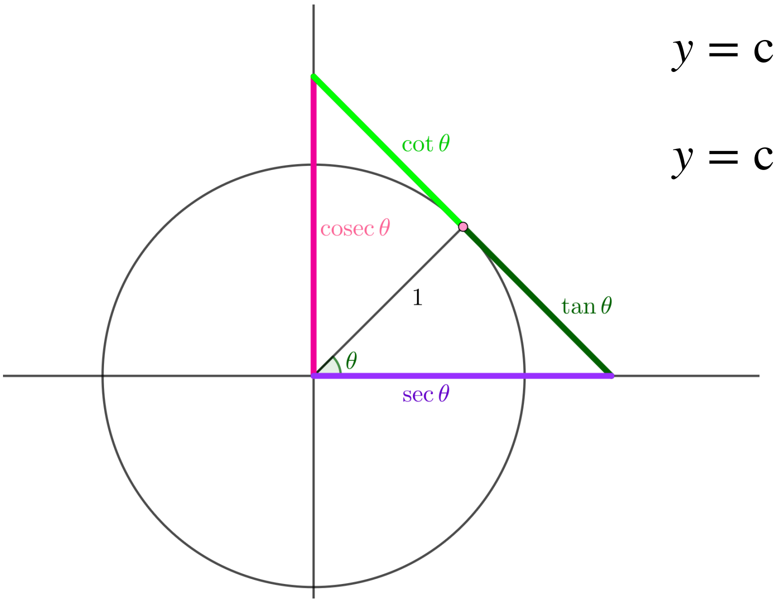


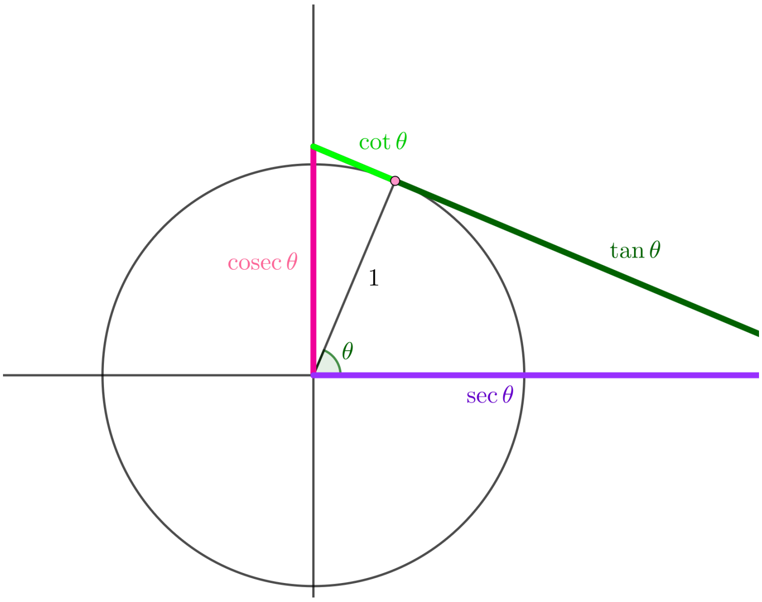
Use this sequence of diagrams as a guide to help you draw the graphs

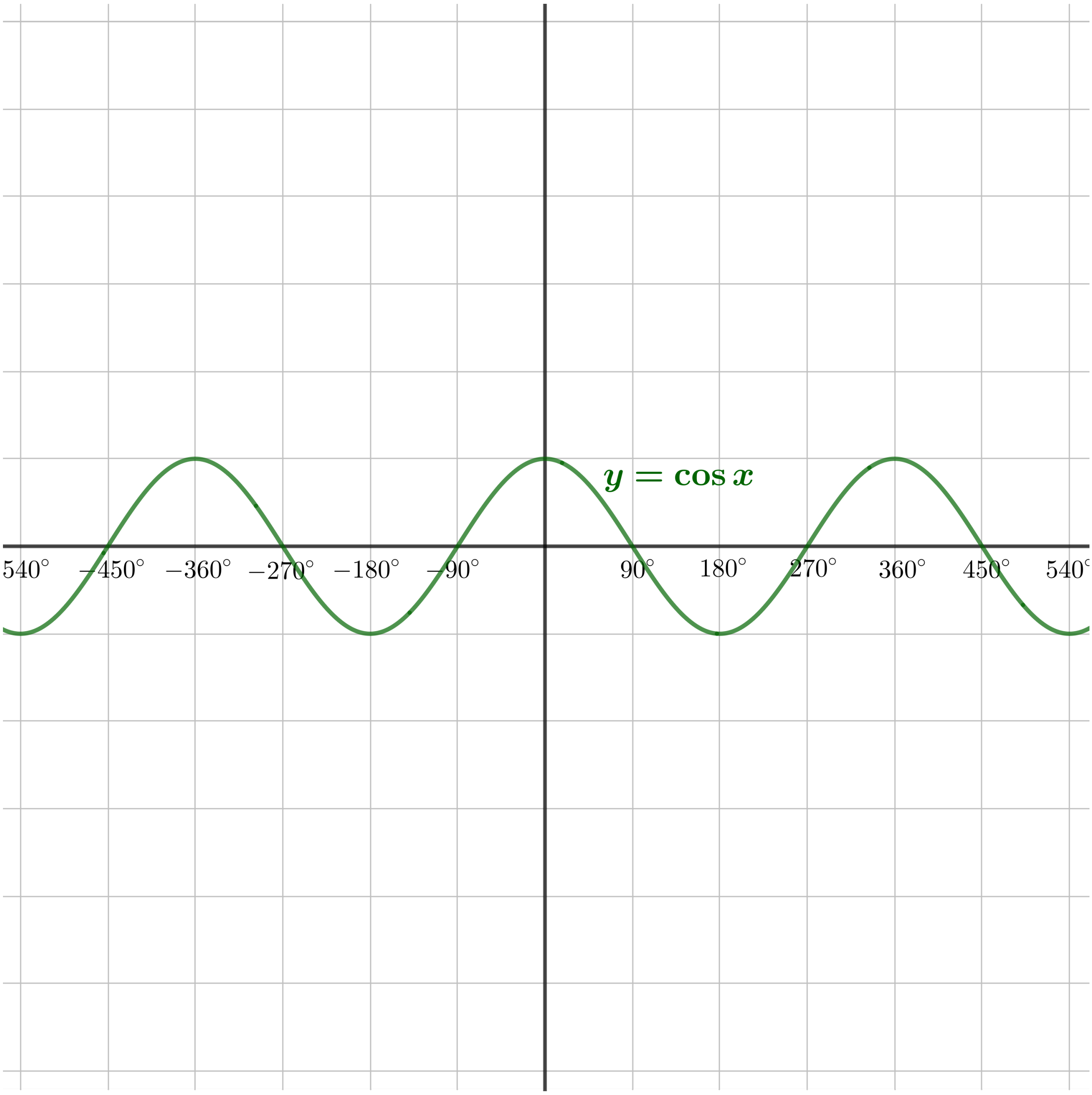
$y = \sec x$

$y = \operatorname{cosec} x$

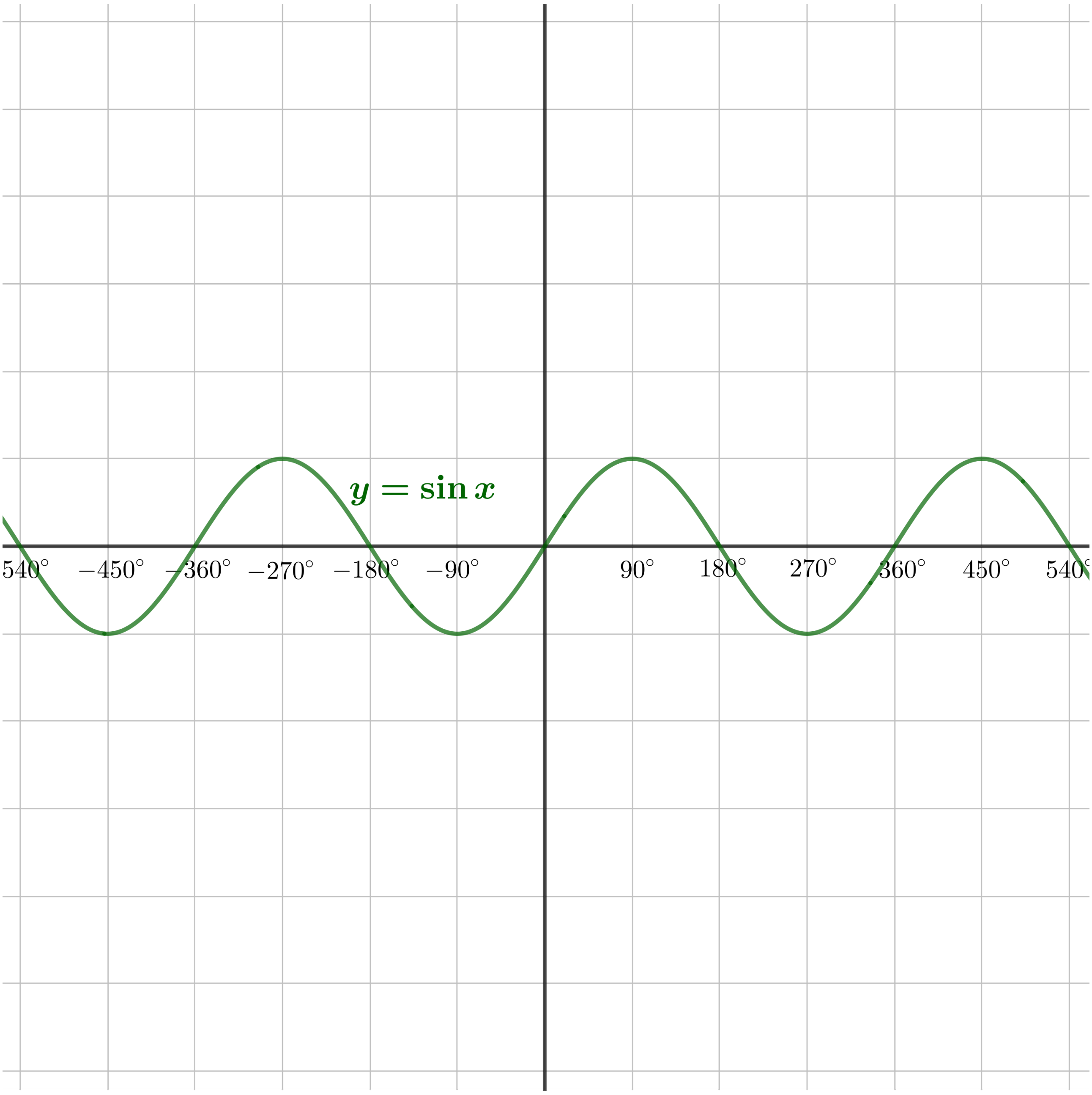
$y = \cot x$

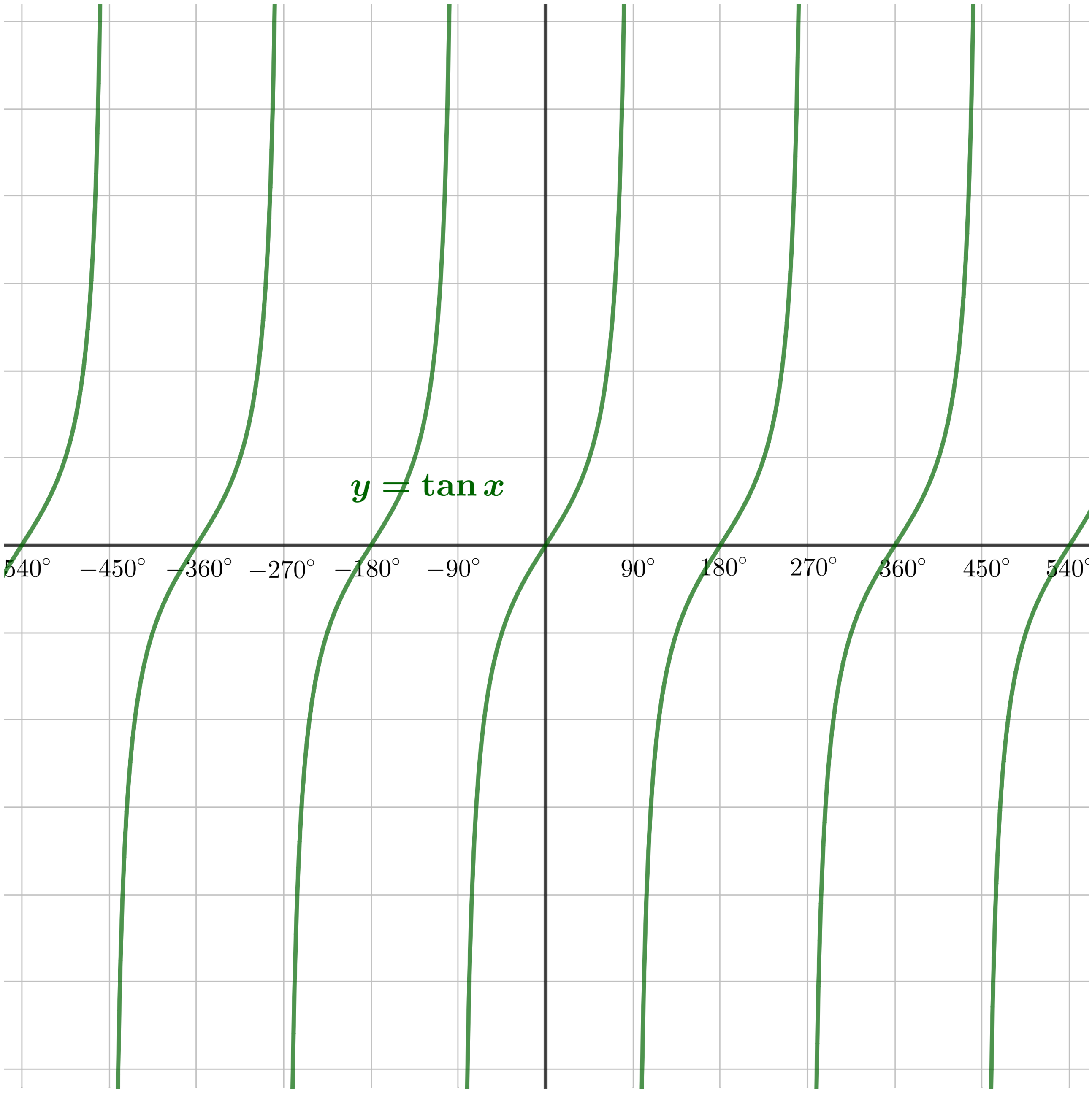






$y = \cos x$
$-540°$
$-450°$
$-360°$
$-270°$
$-180°$
$-90°$
$90°$
$180°$
$270°$
$360°$
$450°$
$540°$


$y = \sin x$
$-540°$
$-450°$
$-360°$
$-270°$
$-180°$
$-90°$
$90°$
$180°$
$270°$
$360°$
$450°$
$540°$


$y = \tan x$
540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

Show that

$$\frac{1}{\cot\theta} + \cot\theta = \sec\theta\,\mathrm{cosec}\theta$$

whenever θ is not a multiple of 90°.

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for confidence
for creativity
for insight

Circular functions 6

$\sin(A + B)$ etc

teacher version

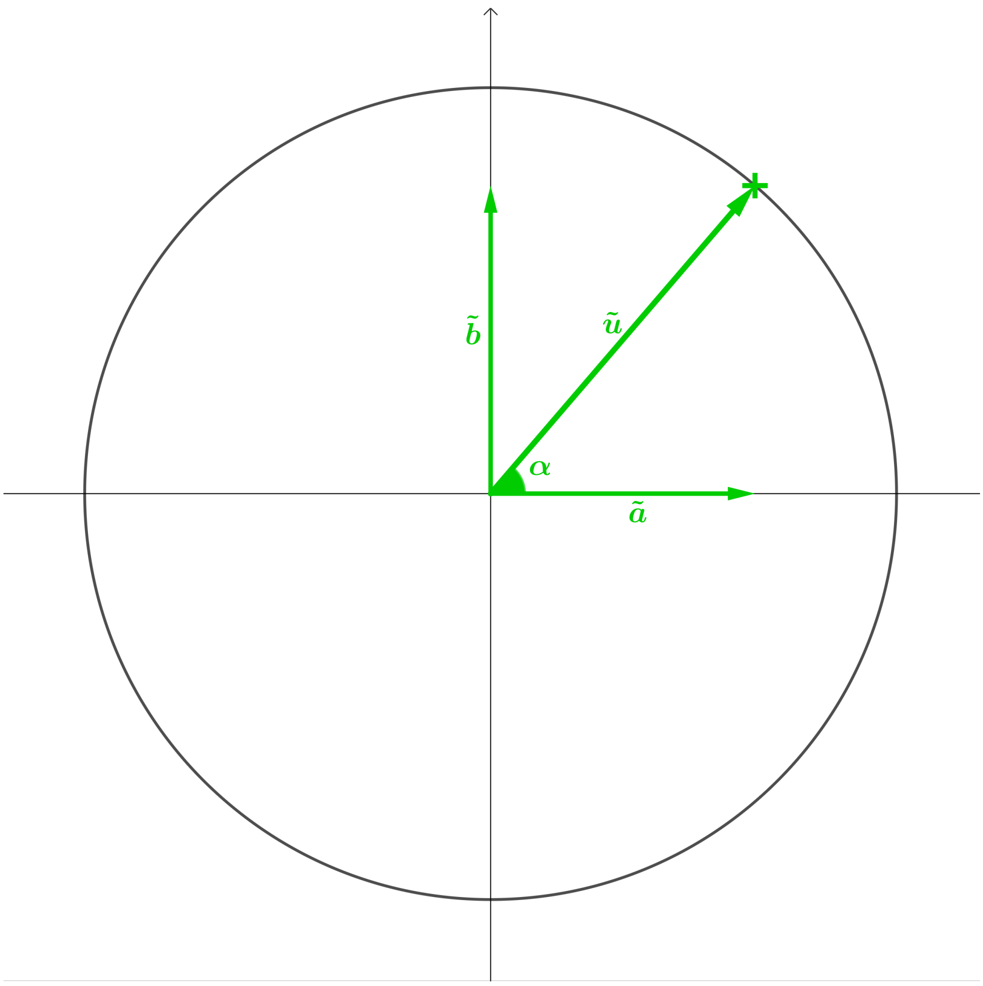
Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90^\circ - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	**formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$**
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on
Differentiating circular functions	radians, and tangents to graphs
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Here is a circle radius 1.

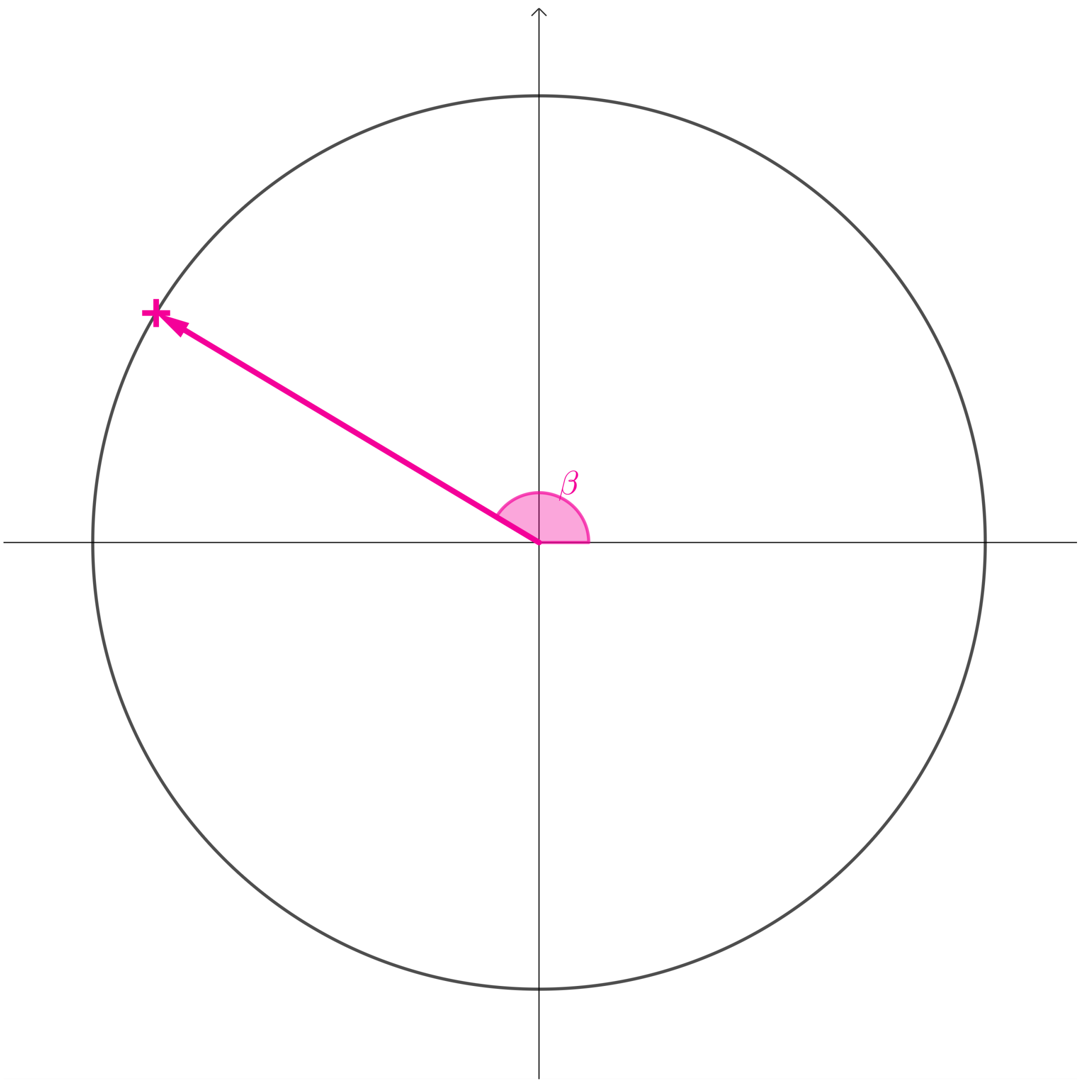
What are the coordinates of the green cross?

Write each of the vectors as column vectors in terms of α.



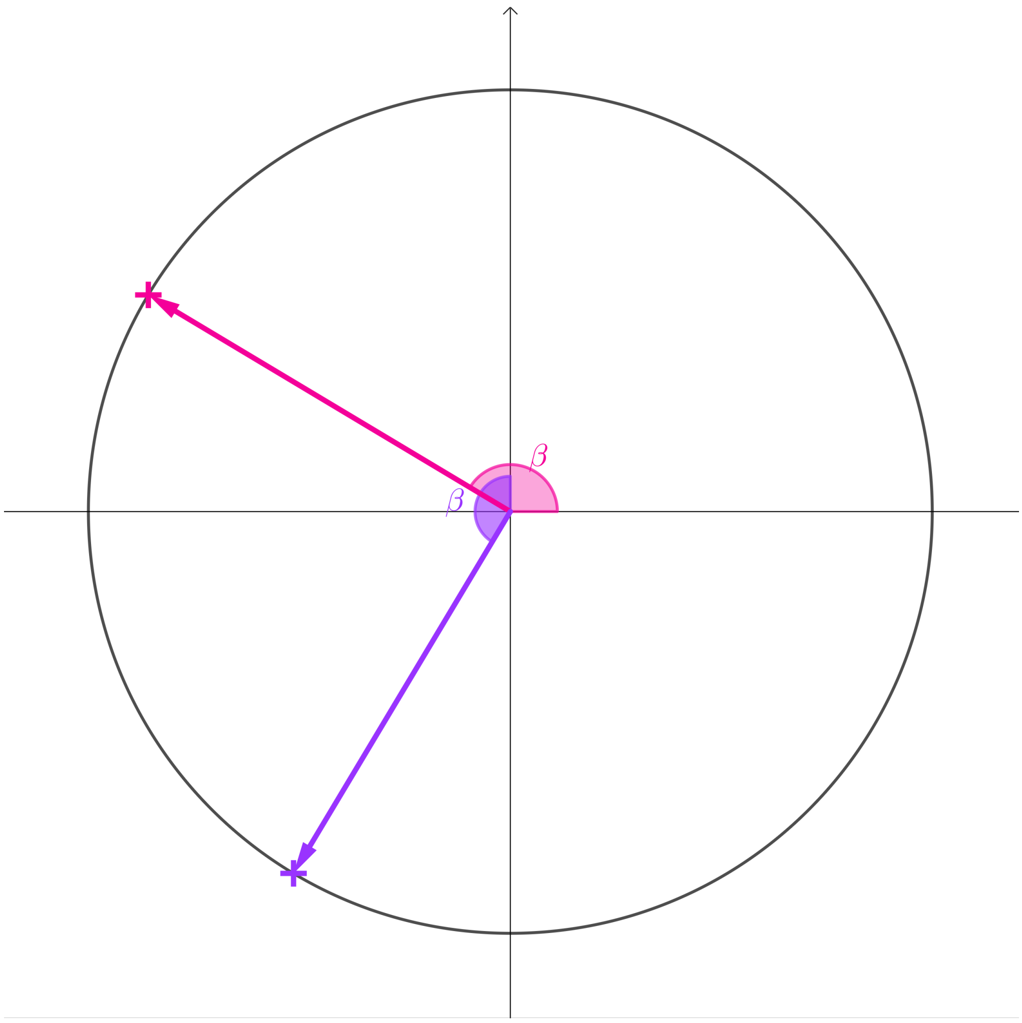
What are the coordinates of the red cross?

Write the red vector as column vector in terms of β.



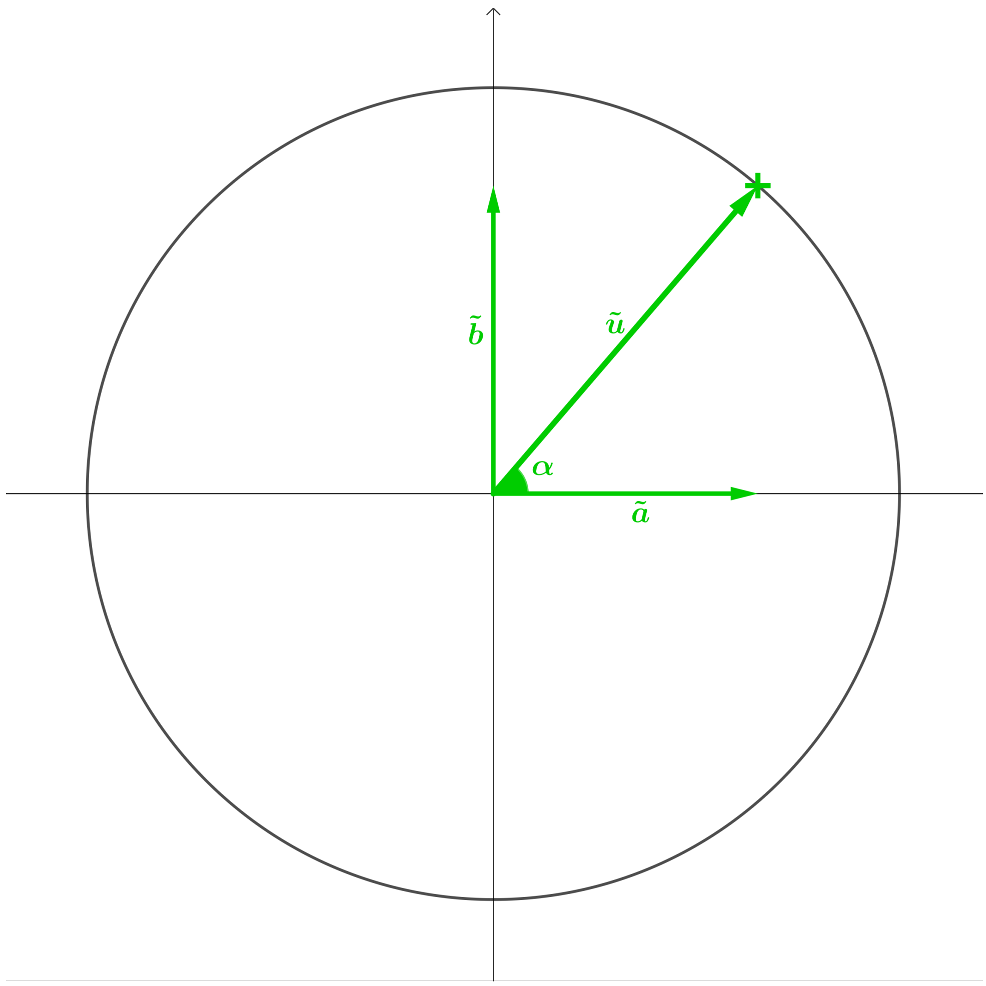
What are the coordinates of the darker cross?

Write the purple vector as column vector in terms of β.



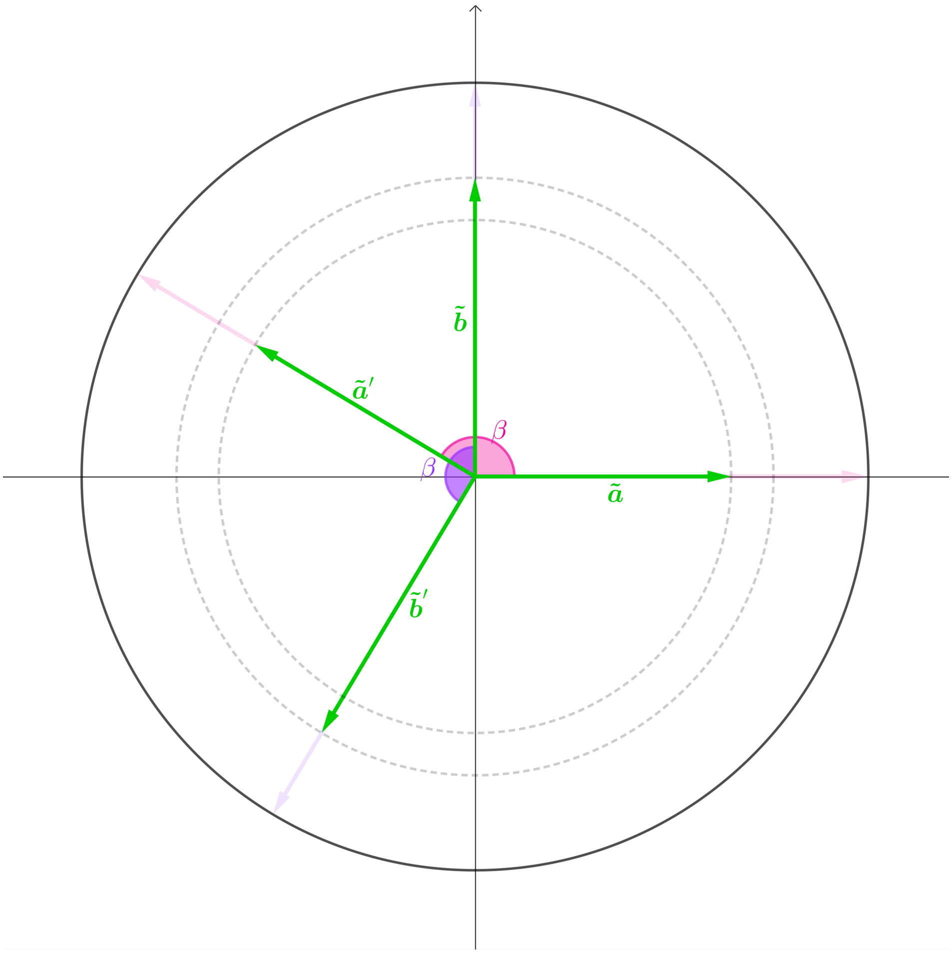
Quick reminder:

Write each of the vectors as column vectors in terms of α.

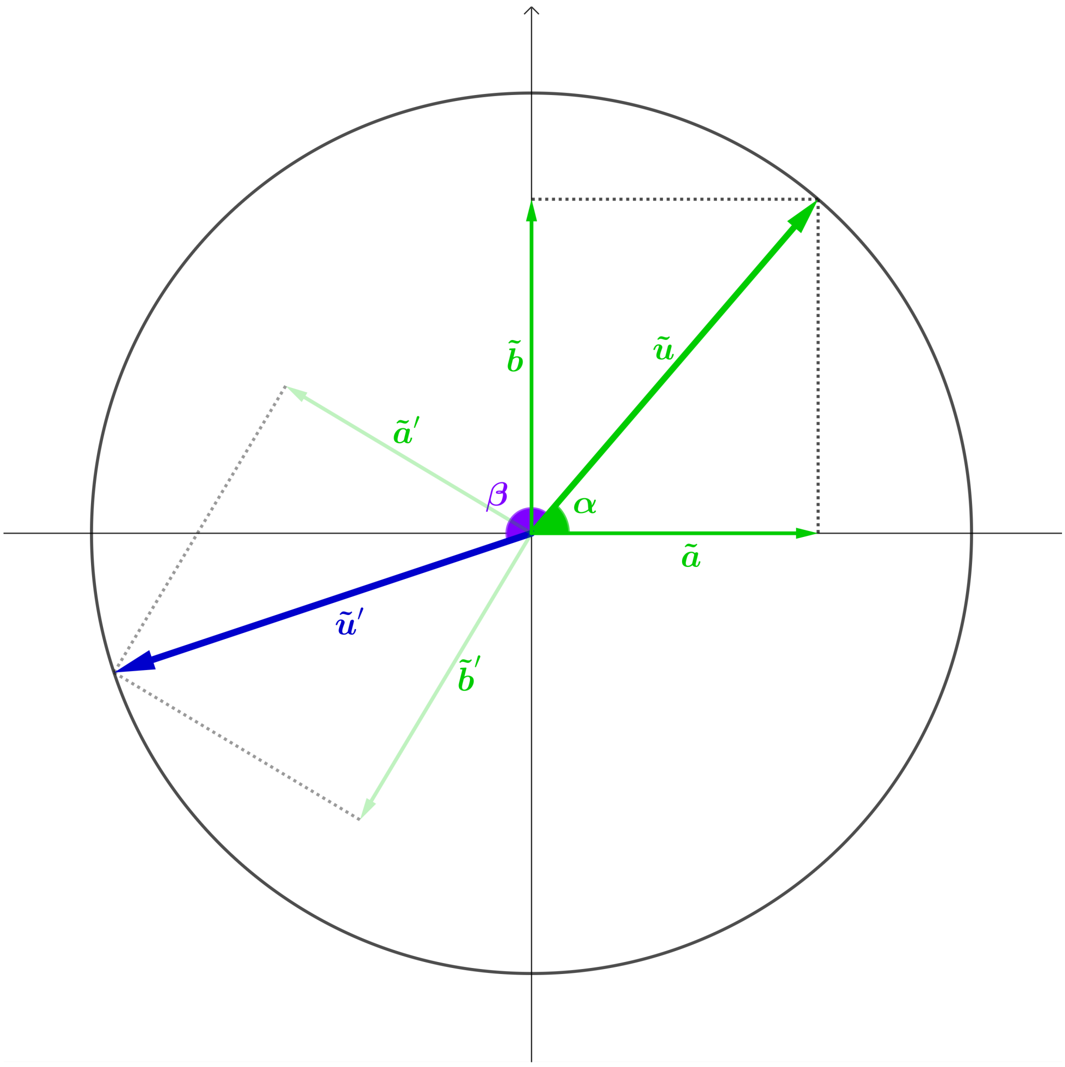


What are the magnitudes and directions of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?

Write the vectors $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$ as column vectors in terms of α and β.

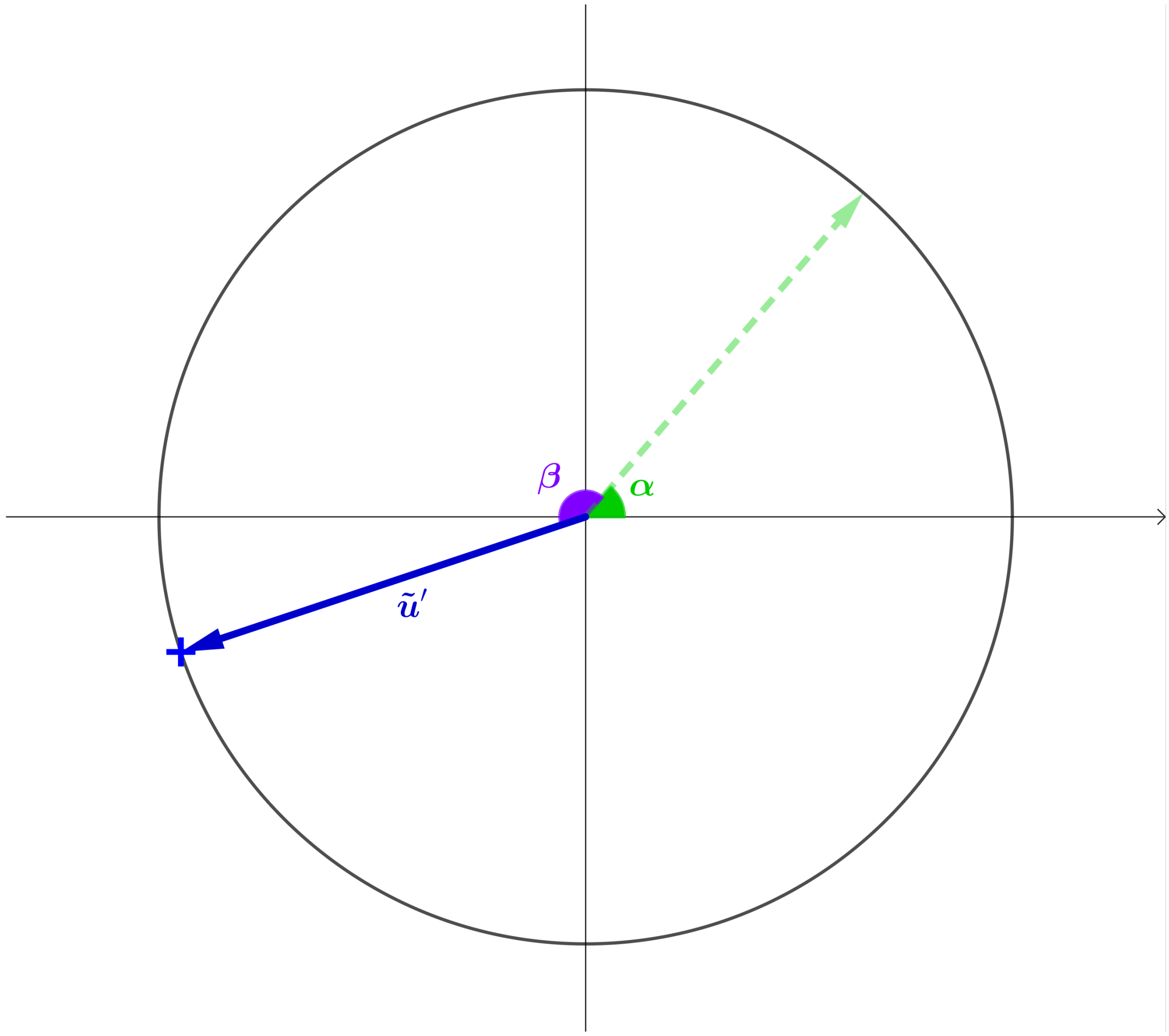


What is $\tilde{\boldsymbol{u}}'$ in terms of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?

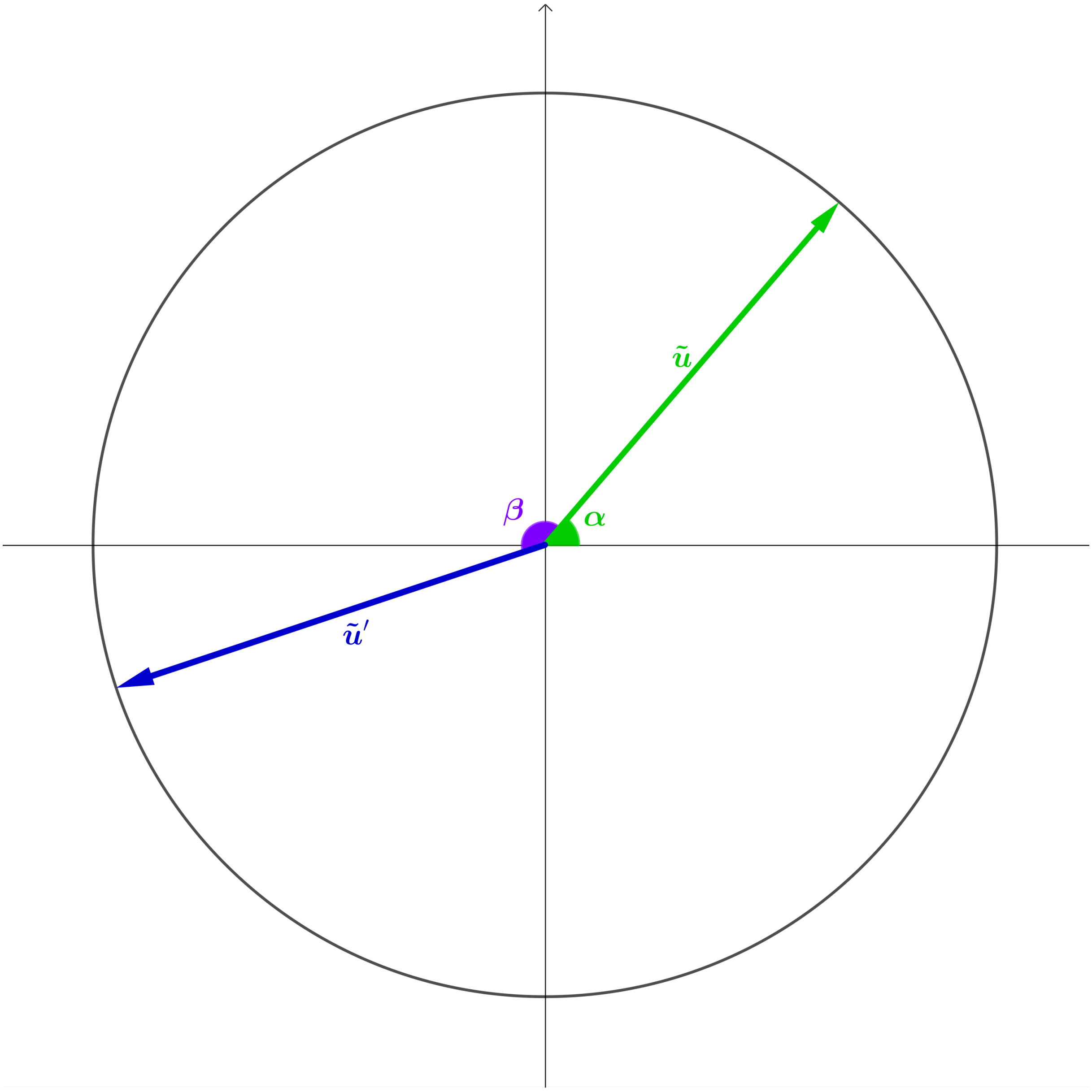


Use this to write $\tilde{\boldsymbol{u}}'$ in terms of α and β.

What are the coordinates of the blue cross?



Use the two expressions for $\tilde{\boldsymbol{u}}'$ together to find new expressions for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$.



Use these results to find $\tan(\alpha + \beta)$ in terms of $\tan \alpha$ and $\tan \beta$.

Use these results to find sin, cos, and tan of $\alpha - \beta$.

Use these results to find $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$.

Use $\cos^2 \alpha + \sin^2 \alpha = 1$ to find two different formulas for $\cos 2\alpha$

Find $\sin 75^\circ$, $\cos 75^\circ$, and $\tan 75^\circ$.

Find $\sin 15^\circ$, $\cos 15^\circ$, and $\tan 15^\circ$.

Use the formula $\cos 2\theta = 2\cos^2\theta - 1$ to find $\cos 15°$.

Compare the two expressions you now have for $\cos 15°$.

Find $\int \sin^2 x \, \mathrm{d}x$

If $\sin\theta = \dfrac{2}{5}$ (θ is acute) and $\cos\varphi = -\dfrac{3}{4}$ (φ is obtuse)

find, **without using your calculator**:

$\cos\theta$

$\tan 2\theta$

$\tan\theta$

$\cos\dfrac{\theta}{2}$

$\cos 2\theta$

$\sin\dfrac{\theta}{2}$

$\sin 2\theta$

$\tan \varphi$

$\tan \dfrac{\theta}{2}$

$\cos 2\varphi$

$\sin 2\varphi$

$\sin \varphi$

$\tan 2\varphi$

$\tan \dfrac{\varphi}{2}$

$\cos \dfrac{\varphi}{2}$

$\sin \dfrac{\varphi}{2}$

$\cos(\theta + \varphi)$

$\cos(\theta - \varphi)$

$\sin(\theta + \varphi)$

$\sin(\theta - \varphi)$

$\tan(\theta + \varphi)$

$\tan(\theta - \varphi)$

$\cos(\theta + \varphi)$

$\cos(\theta - \varphi)$

$\sin(\theta + \varphi)$

$\sin(\theta - \varphi)$

$\tan(\theta + \varphi)$

$\tan(\theta - \varphi)$

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Circular functions 7

Transforming and adding circular functions

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90° - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions

$\sin x + \cos x = \sqrt{2}\sin(x + 45°)$ and so on

Differentiating circular functions — radians, and tangents to graphs

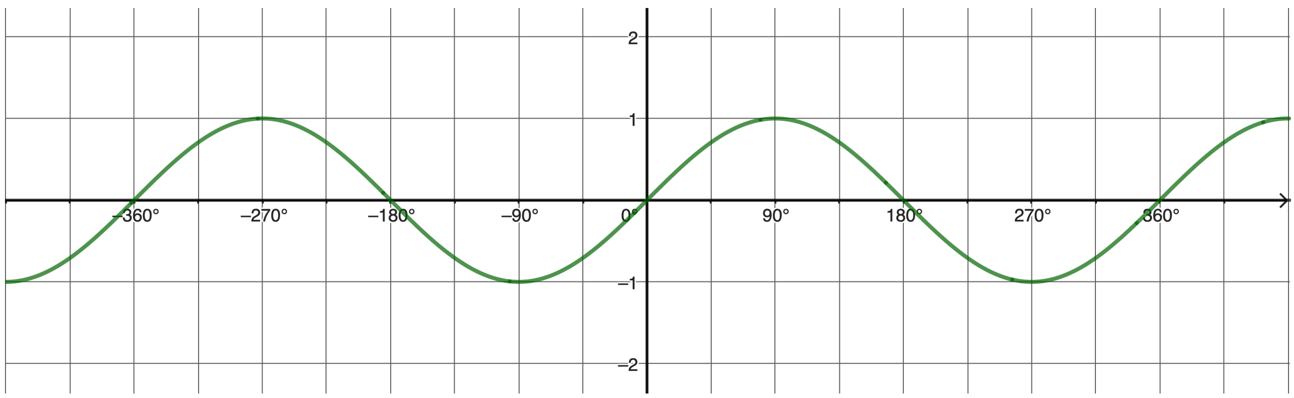
Integrating circular functions — areas

Inverses of circular functions — $\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Here is the graph $y = \sin x$.

Translate the graph left by $45°$.

What is the equation of your new graph?

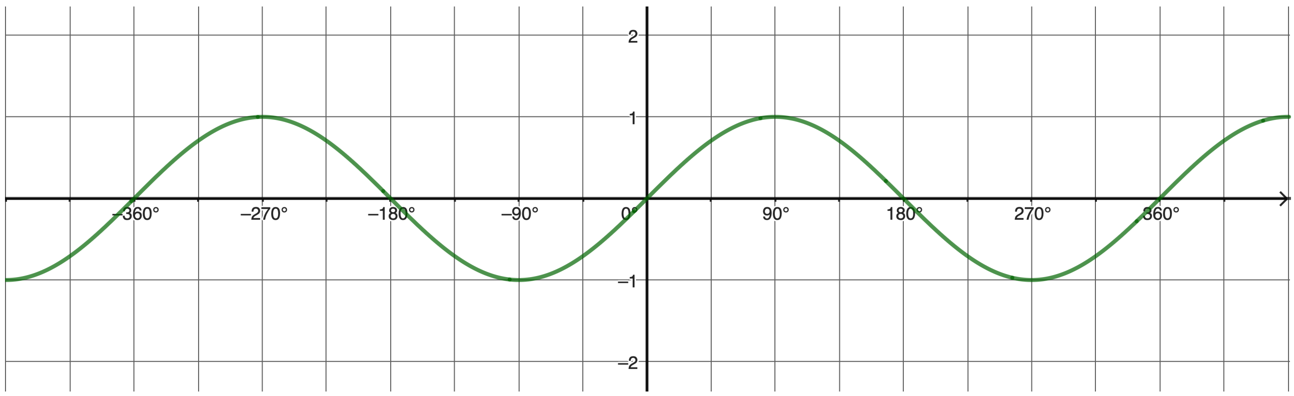


Now stretch the new graph parallel to the y axis scale factor $\sqrt{2}$.

What is the equation of your latest graph?

Now use a compound angle formula to expand $\sqrt{2}\sin\left(x+45^\circ\right)$

Draw the graph $y = \sin x + \cos x$

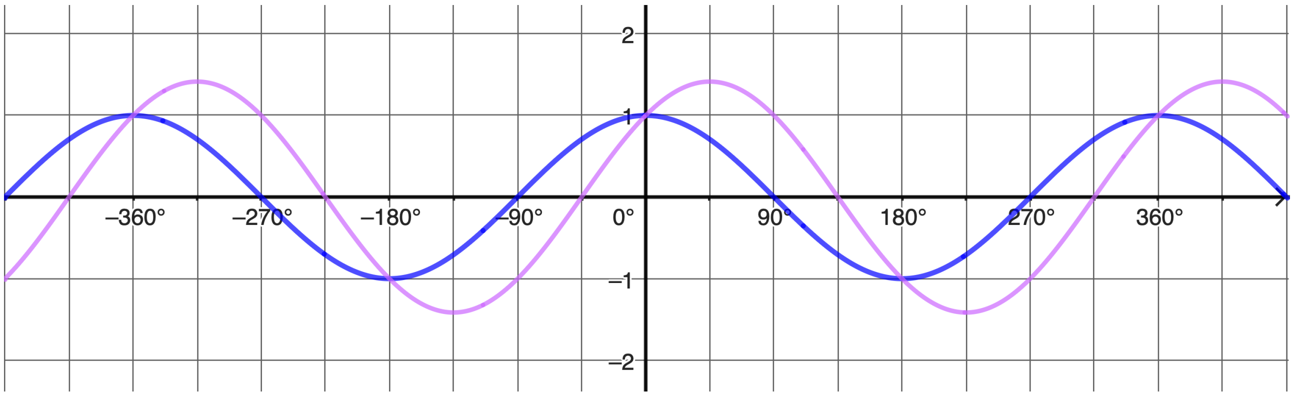


We can achieve the same graph

$y = \sin x + \cos x$

by applying transformations to the graph $y = \cos x$. Here are the two graphs together

What transformations will turn the blue ($y = \cos x$) into the pink ($y = \sin x + \cos x$)?



Expand $\sqrt{2}\cos\left(x - 45^\circ\right)$

What do you notice about the graphs

$$y = \sqrt{2}\sin\left(x + 45^\circ\right)$$

$$y = \sqrt{2}\cos\left(x - 45^\circ\right)$$

$$y = \sin x + \cos x$$

What transformations of $y = \sin x$ or $y = \cos x$ will result in the graphs

$$y = \sqrt{2}\,\sin(x + 405^\circ)$$

$$y = \sqrt{2}\,\cos(x - 405^\circ)$$

$$y = -\sqrt{2}\,\sin(x + 225^\circ)$$

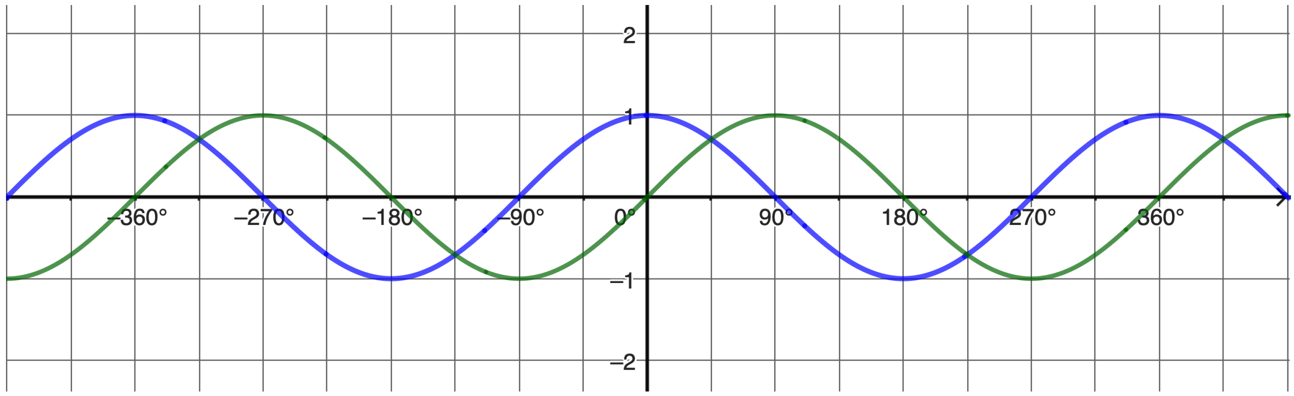
$$y = -\sqrt{2}\,\cos(x - 225^\circ)$$

What is the difference between these graphs and the graph $y = \sin x + \cos x$?

We can also look at this from the point of view of a molecule being moved around by the two waves $y = \sin x$ and $y = \cos x$.

Draw the graph obtained by “adding up” the two graphs.

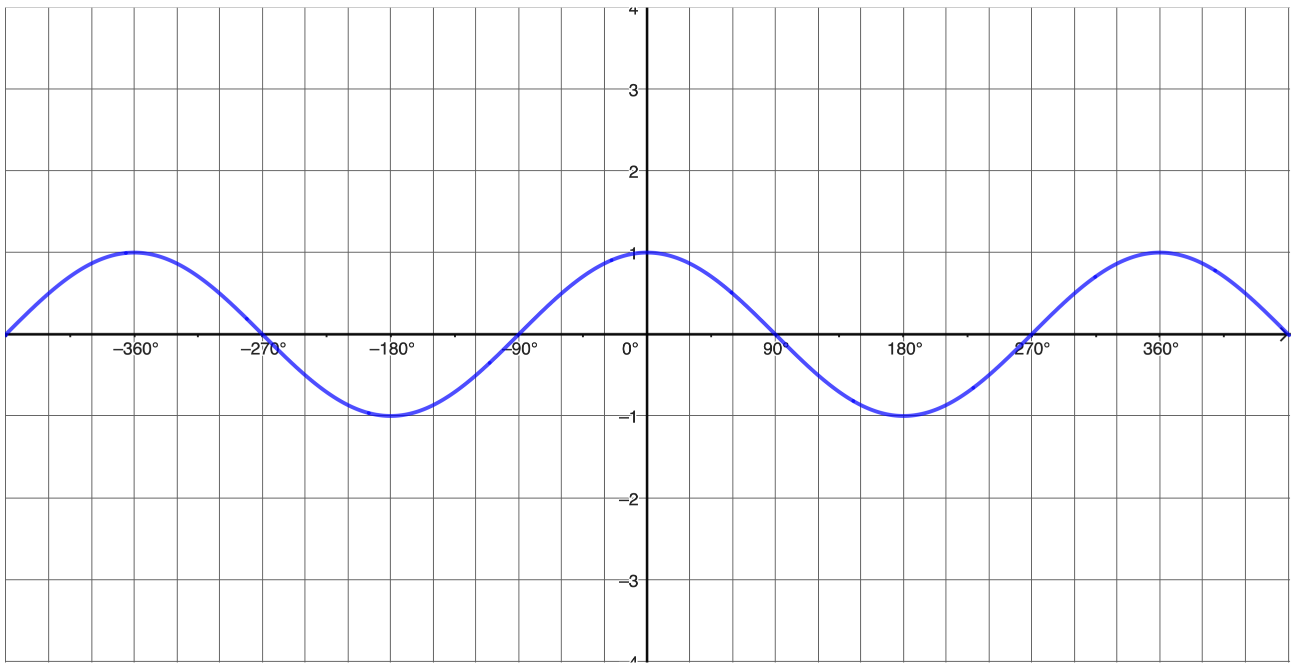
What is the equation of your new graph?



Now for a slightly trickier example.

What transformations take the graph $y = \cos x$ to the graph

$y = 2\sqrt{3}\cos(x + 30°)$?



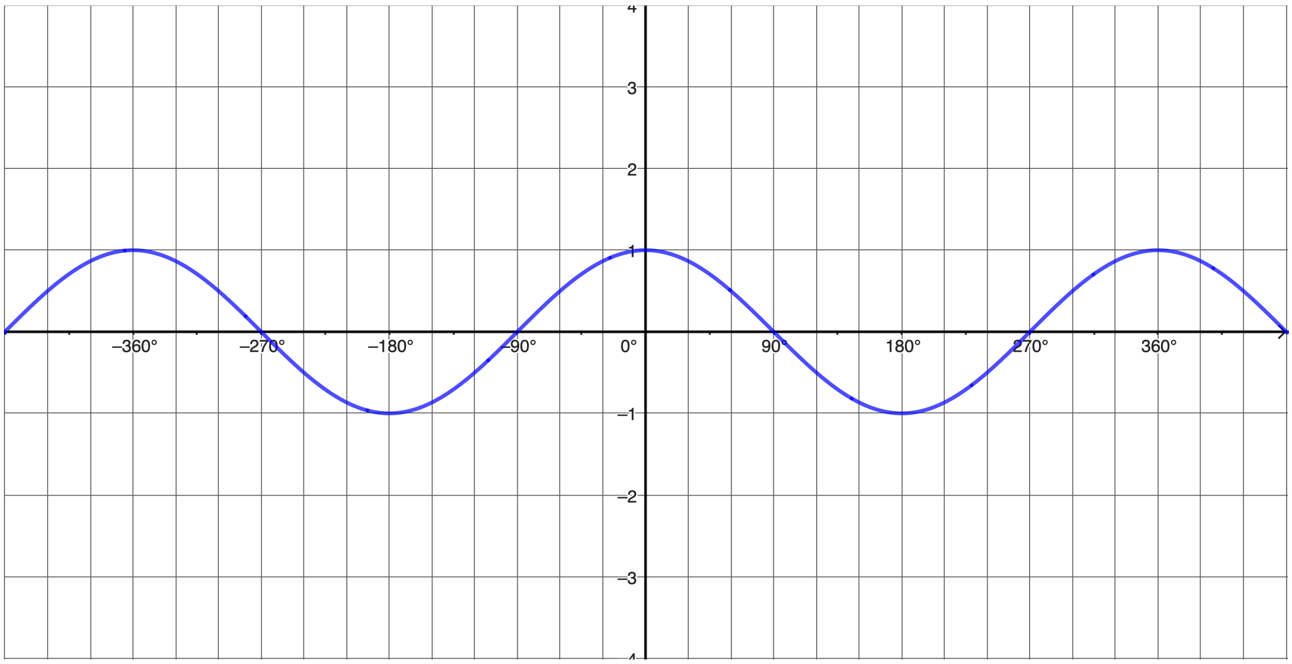
On the same axes, draw the graphs

$y = \cos(x + 30°)$ and $y = 2\sqrt{3}\cos(x + 30°)$.

Expand the brackets in $y = 2\sqrt{3}\cos(x + 30°)$.

Draw the graph

$$y = 3\cos x - \sqrt{3}\sin x$$



Next, let's take a similar but very slightly different example: the wave $y = 3\sin x$ meets the wave $y = \sqrt{3}\cos x$

Firstly, expand the brackets in $y = R\sin(x + \alpha)$.

If $R\sin(x + \alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

Use these results to find $\tan \alpha$.

Hence find the smallest positive value of α.

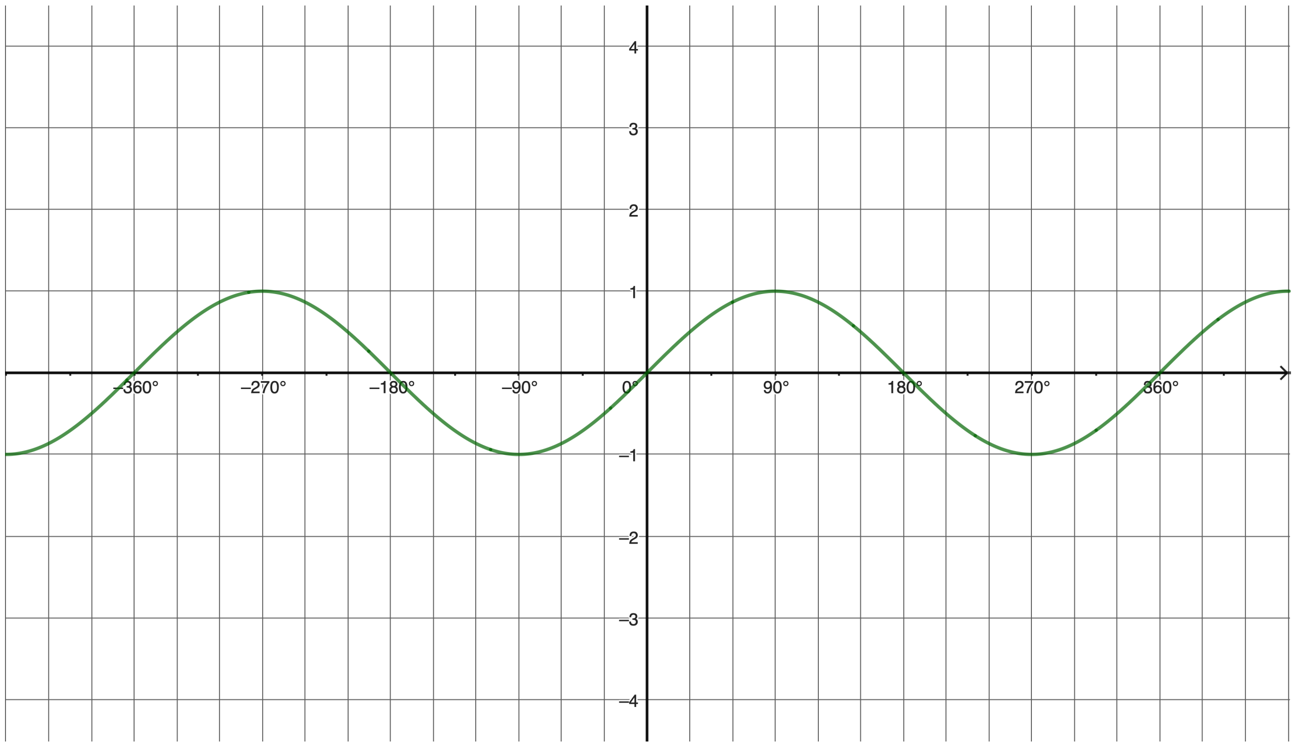
Find $(R \sin \alpha)^2 + (R \cos \alpha)^2$.

Hence find R.

Use these results to find the transformations that take the graph $y = \sin x$ to the graph

$$y = 3\sin x + \sqrt{3}\cos x$$

and draw this graph.



We could also do it this way.

If $R\cos(x-\alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

Use these results to find $\tan\alpha$.

Hence find the smallest positive value of α.

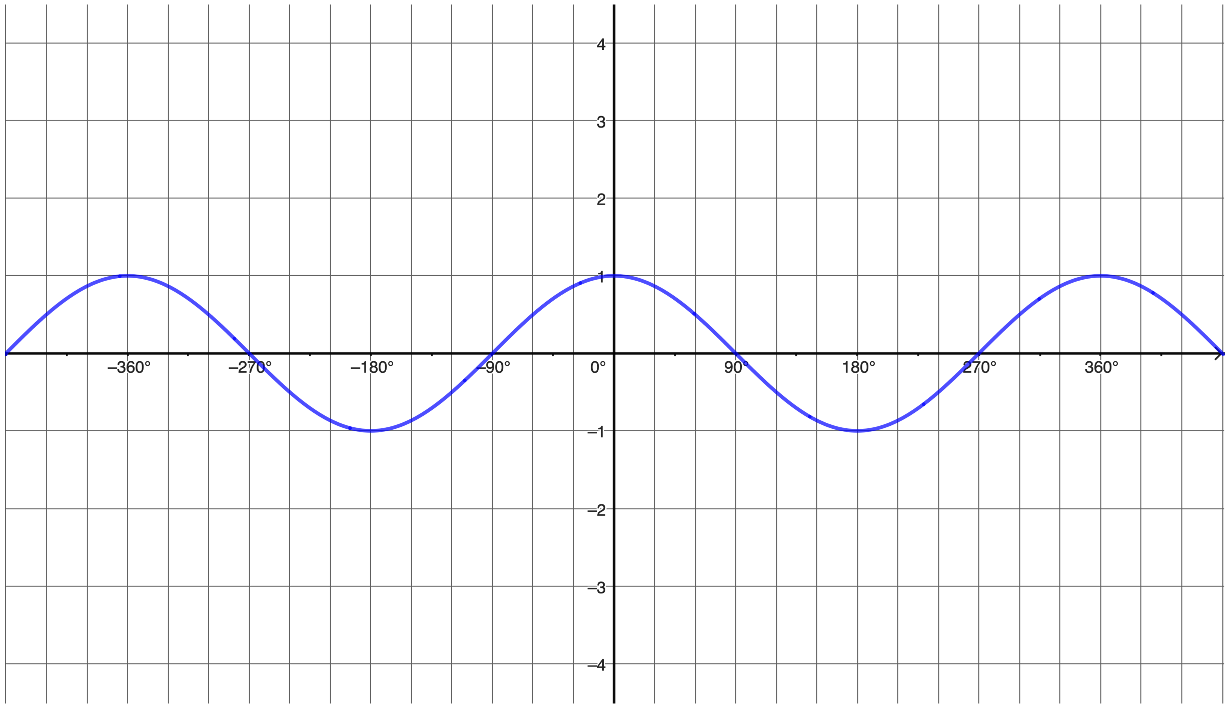
Find $(R\sin\alpha)^2 + (R\cos\alpha)^2$.

Hence find R.

Use these results to find the transformations that take the graph $y = \cos x$ to the graph

$$y = 3\sin x + \sqrt{3}\cos x$$

and draw this graph.



Any two waves $y = A\sin x$ and $y = B\cos x$ can be combined (either adding or subtracting) by these methods. Sometimes it's easier to transform the graph $y = \sin x$ to get $y = A\sin x \pm B\cos x$. Other times, starting with $y = \cos x$ is better.

Experiment with the combinations

$y = 5\sin x + 2\cos x$

$y = 5\sin x - 2\cos x$

$y = 2\cos x - 5\sin x$

using each of the following forms:

$R\sin(x + \alpha)$

$R\sin(x - \alpha)$

$R\cos(x + \alpha)$

$R\cos(x - \alpha)$

and taking $\tan 22°$ to be $\frac{2}{5}$.

Which R, α forms work best for each of the combinations?

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Circular functions 8

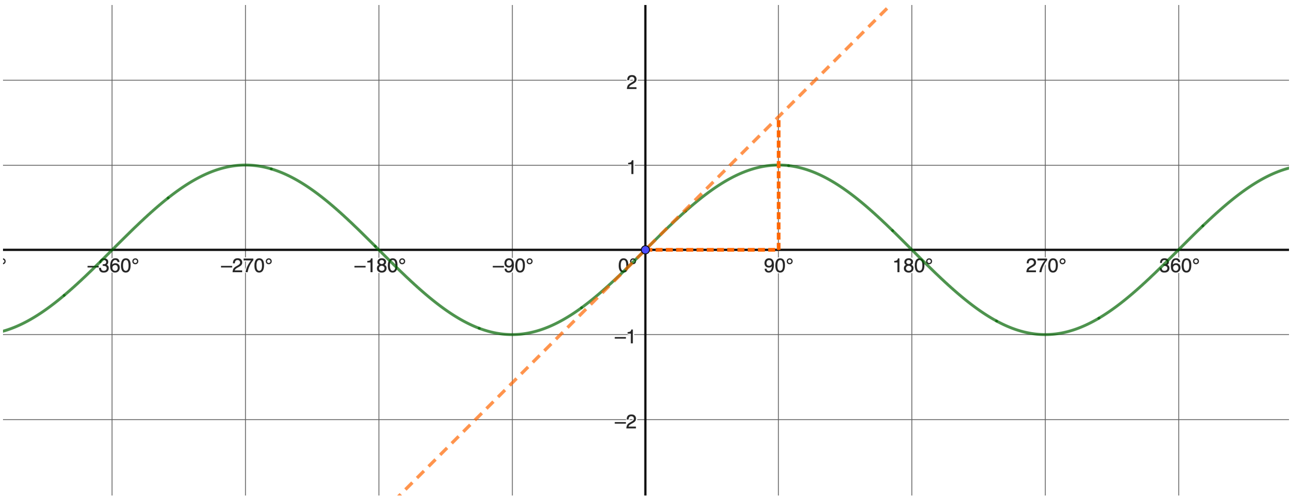
Differentials of circular functions

teacher version

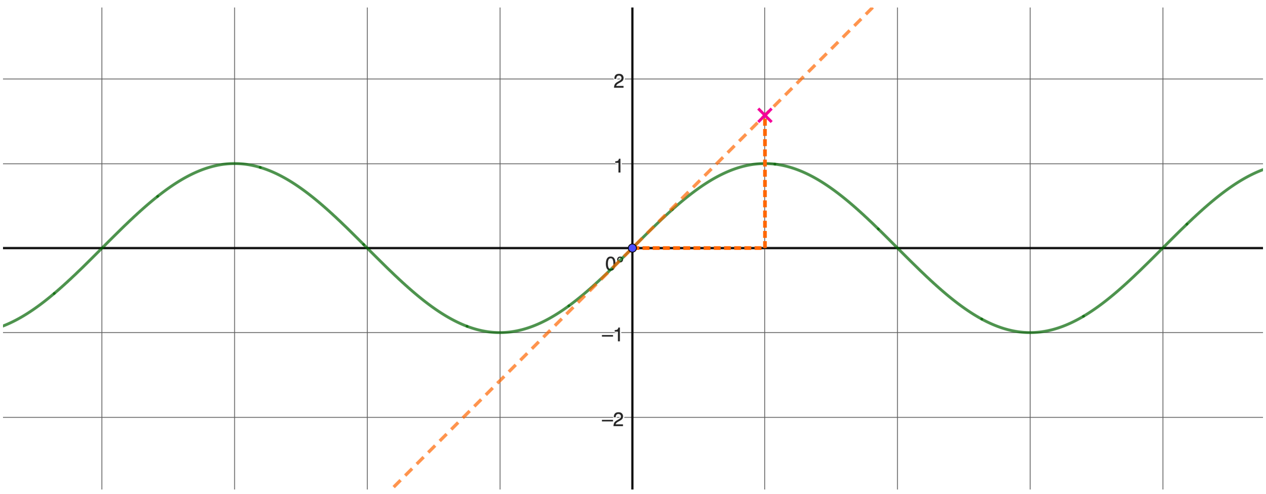
Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90° - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A+B) = \sin A\cos B + \cos A\sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45°\right)$ and so on
Differentiating circular functions	**radians, and tangents to graphs**
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

What (approximately) is the gradient of this tangent?

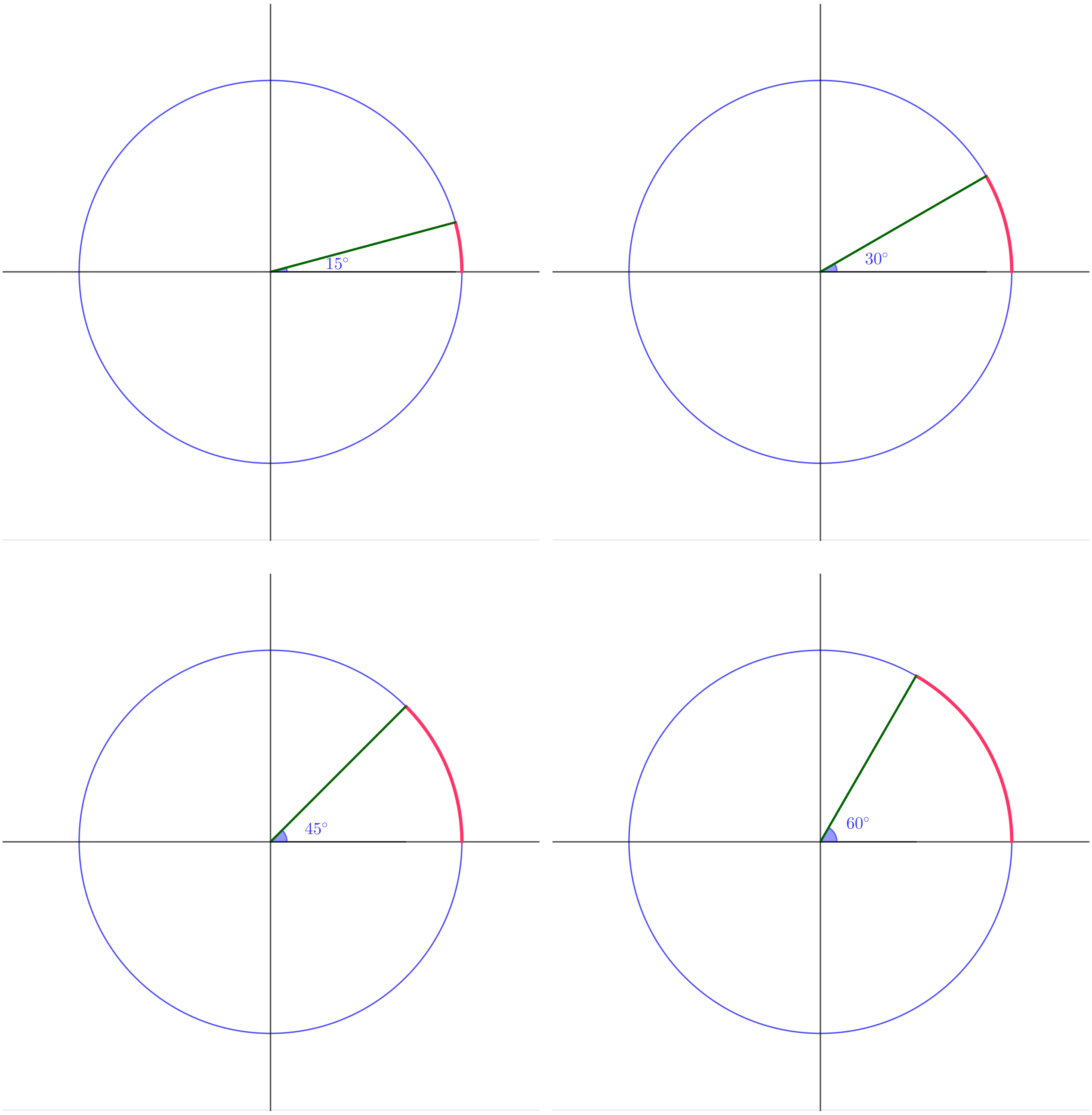


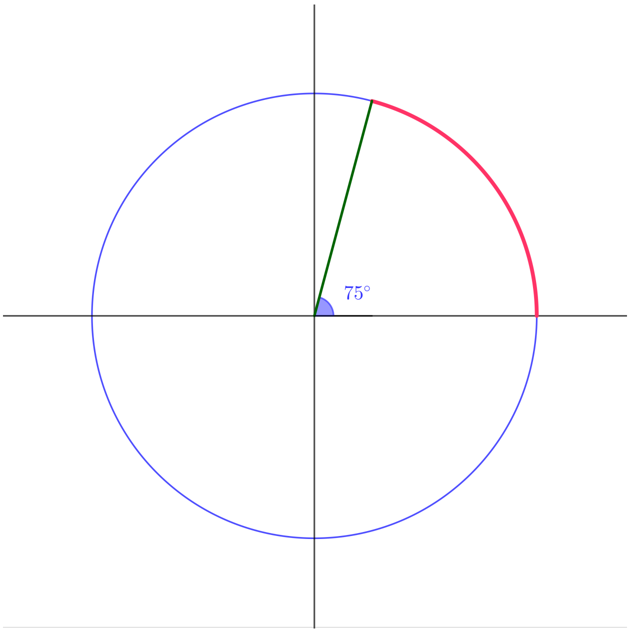
The gradient may look a bit like 1, but is it? Here is a new version of the graph without a scale on the x axis. If the gradient of this tangent is to be 1, what (approximately) would the x coordinate of the pink cross have to be?

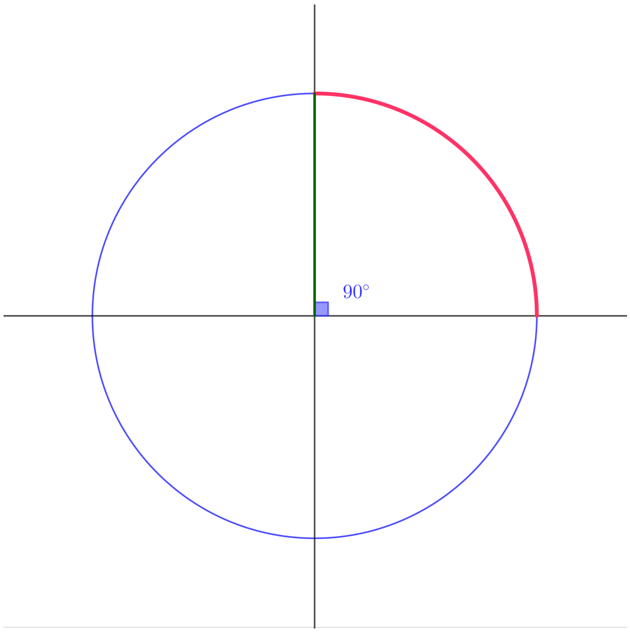


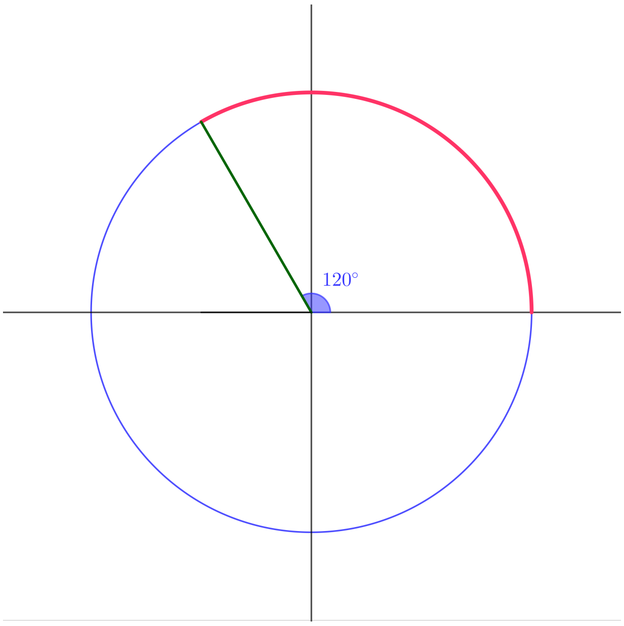
Now, we will try to find this x coordinate exactly. That is to say, we will find units for the x axis that makes this gradient 1. In theory, we can differentiate the circular functions without doing this, but everything works out so much more easily if we do, and that's the way it's done the world over. To do this, we need to go back to the unit circle.

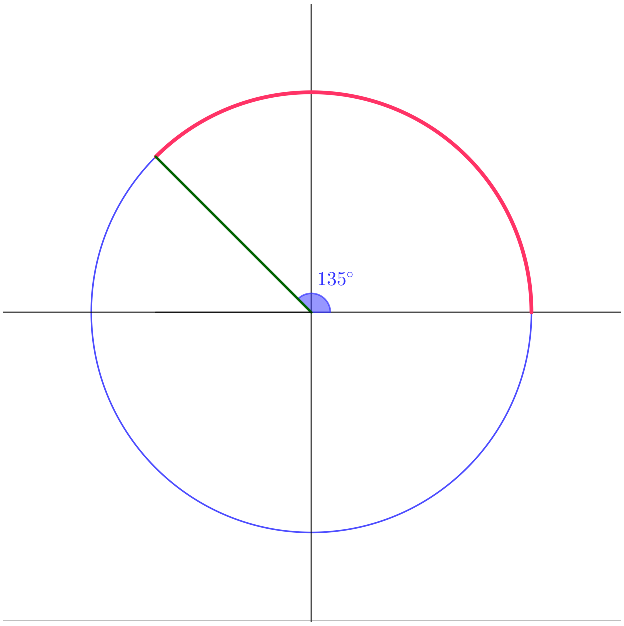
First of all, find the pink arc length on each of these circles with radius 1:

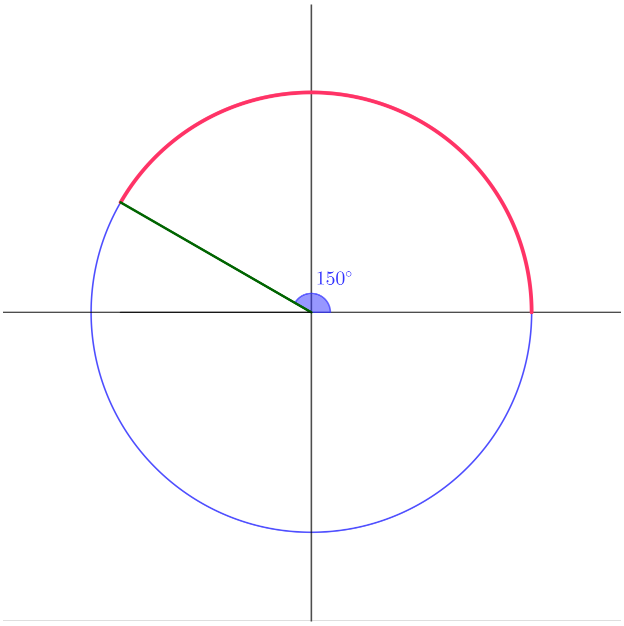
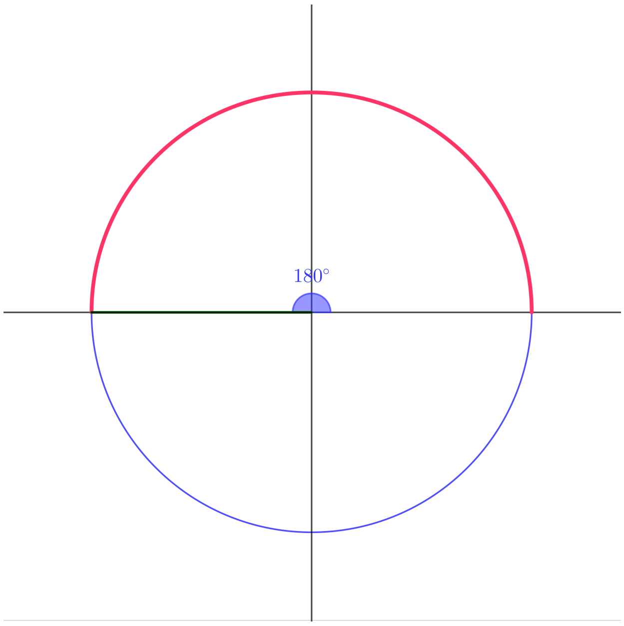
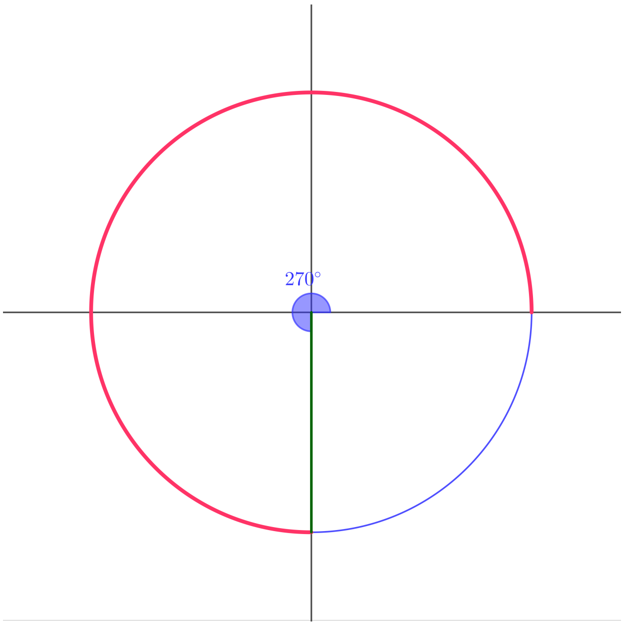
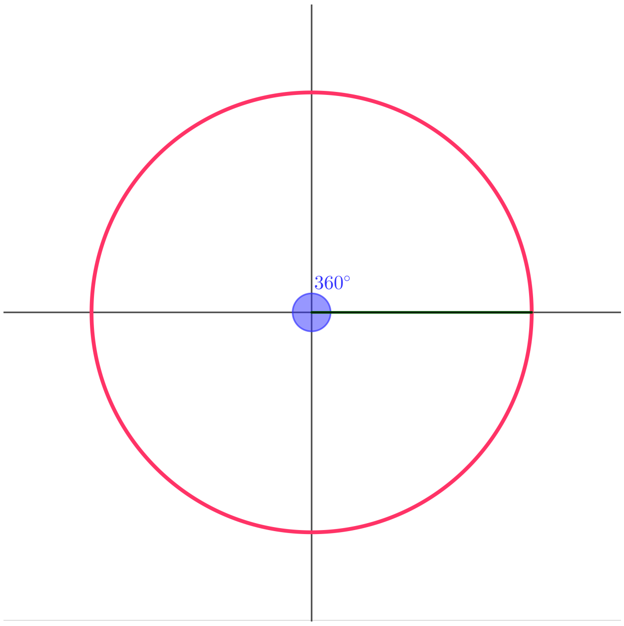



75°


90°

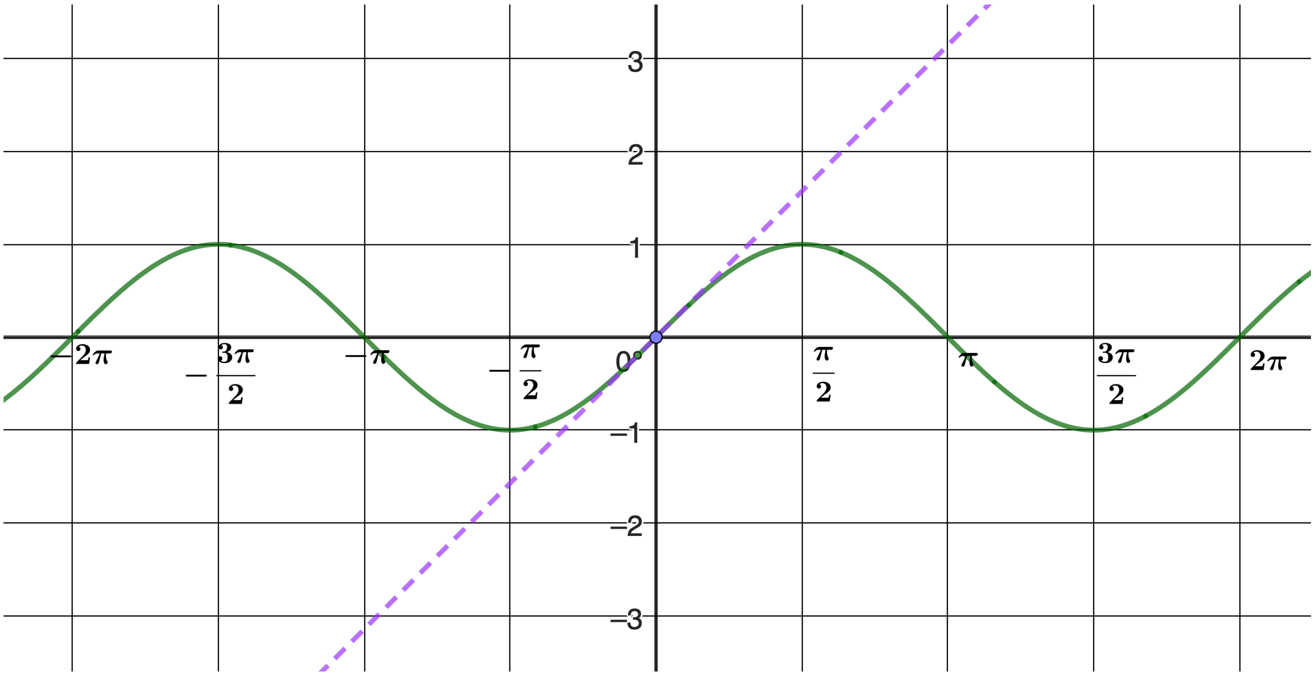

120°


135°


150°

180°

270°

360°

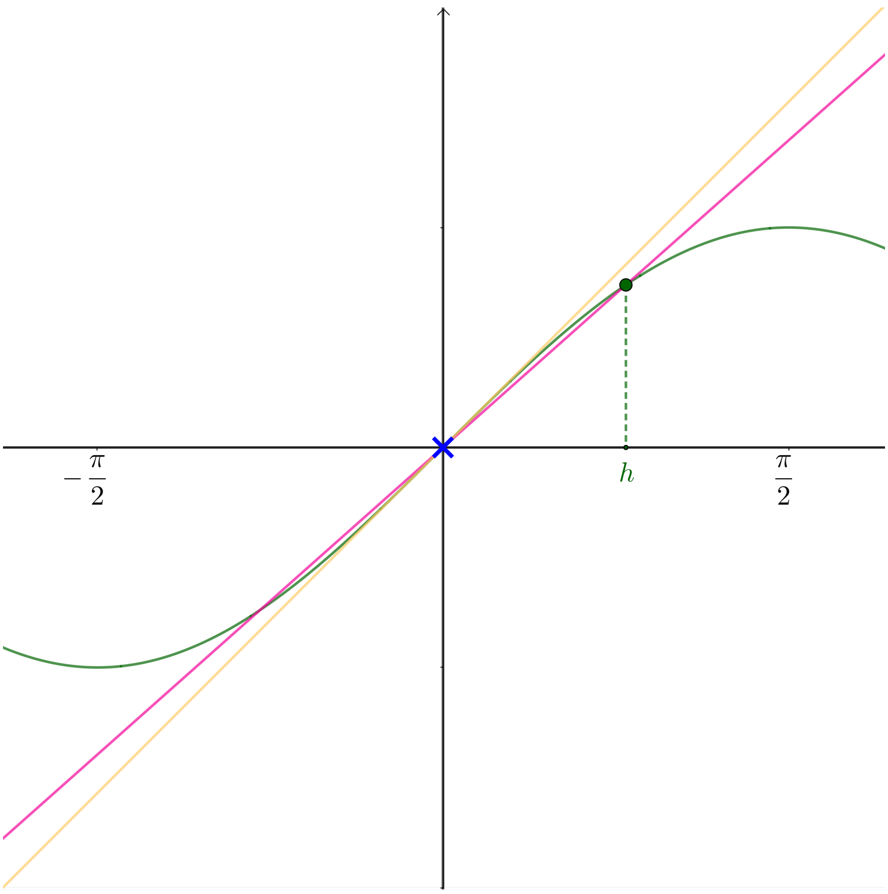
Using $\pi = 3.14159\ldots$, what is $\dfrac{\pi}{2}$ as a decimal?

What does the gradient of the tangent look like now?



To figure out whether this is the right scale to make the gradient of the tangent equal to 1, we will use the idea of the tangent as a limit.

First, what is the gradient of the pink line?

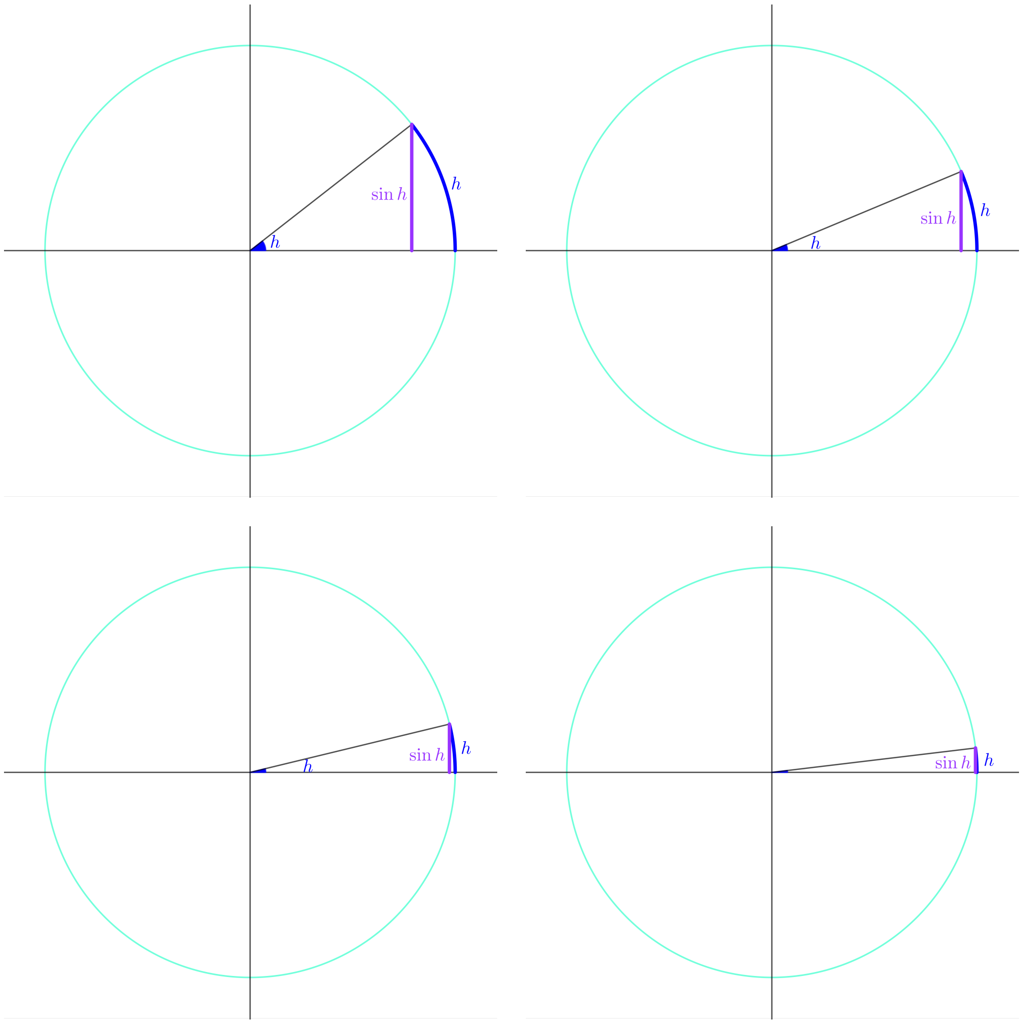


What will happen to the pink line as h gets increasingly small?

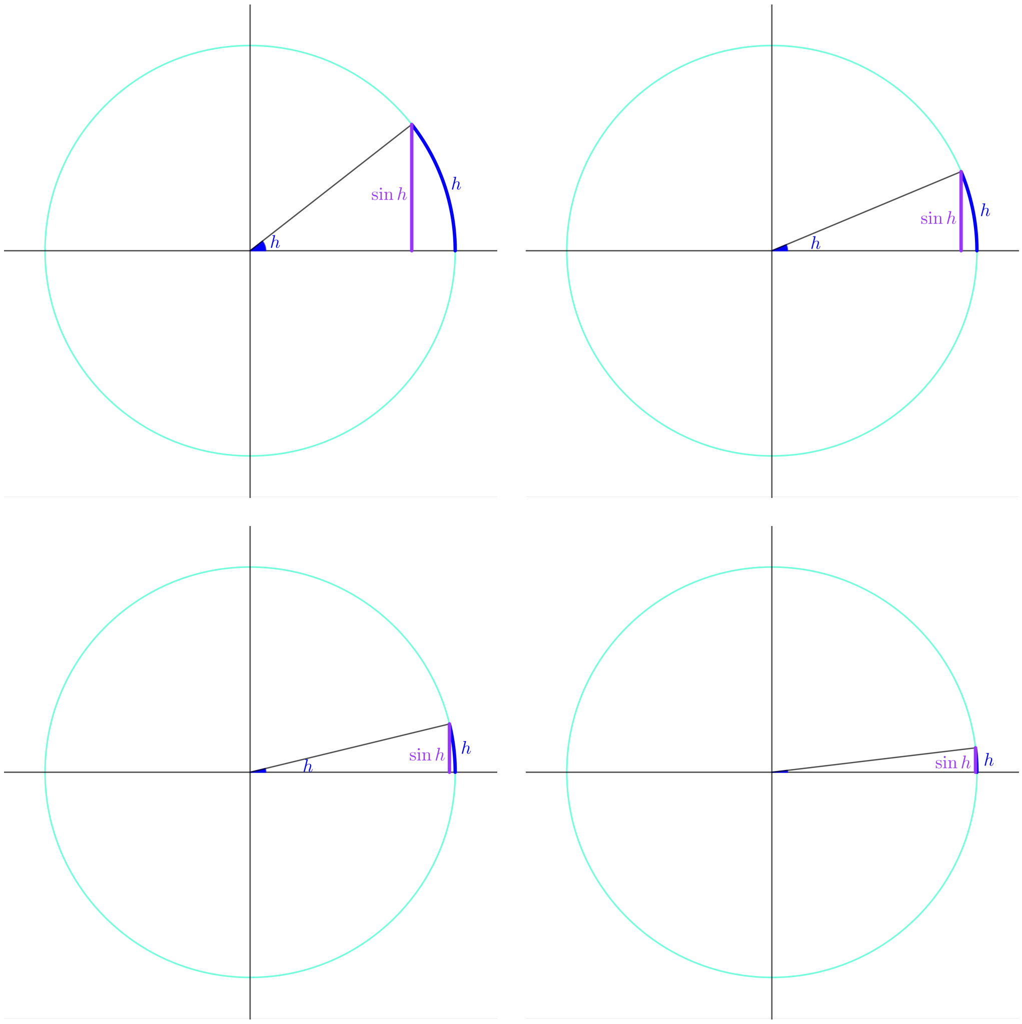
What does this tell us about the gradient of the tangent?

Look at this sequence of diagrams. Why have I used the same letter, h, for both the angles and the arc lengths?

What do you notice about the relationship between the angle size h and the ratio of the lengths of the blue arc and the purple segment?

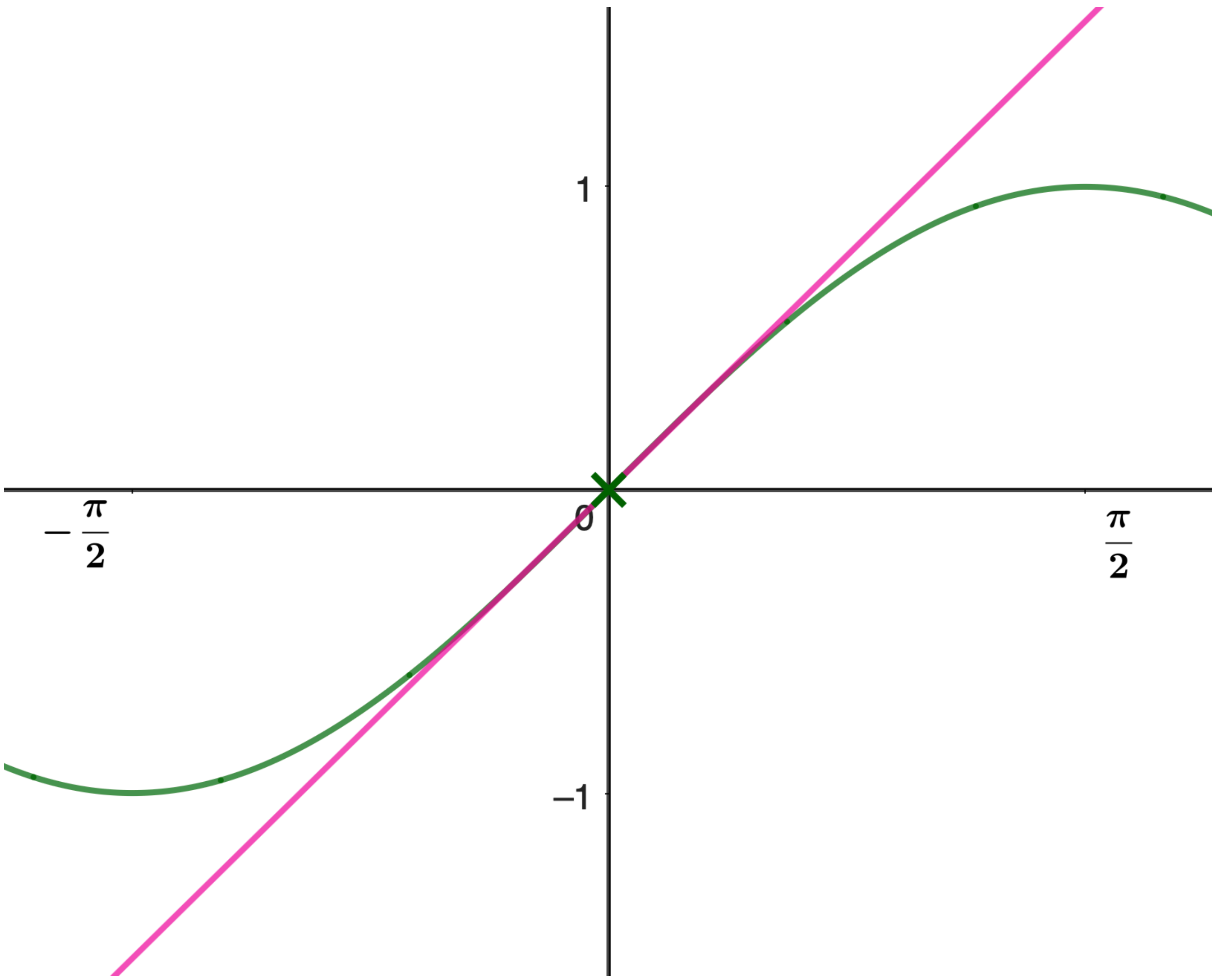


What does this tell you about the ratio $\dfrac{\sin h}{h}$ as $h \to 0$?



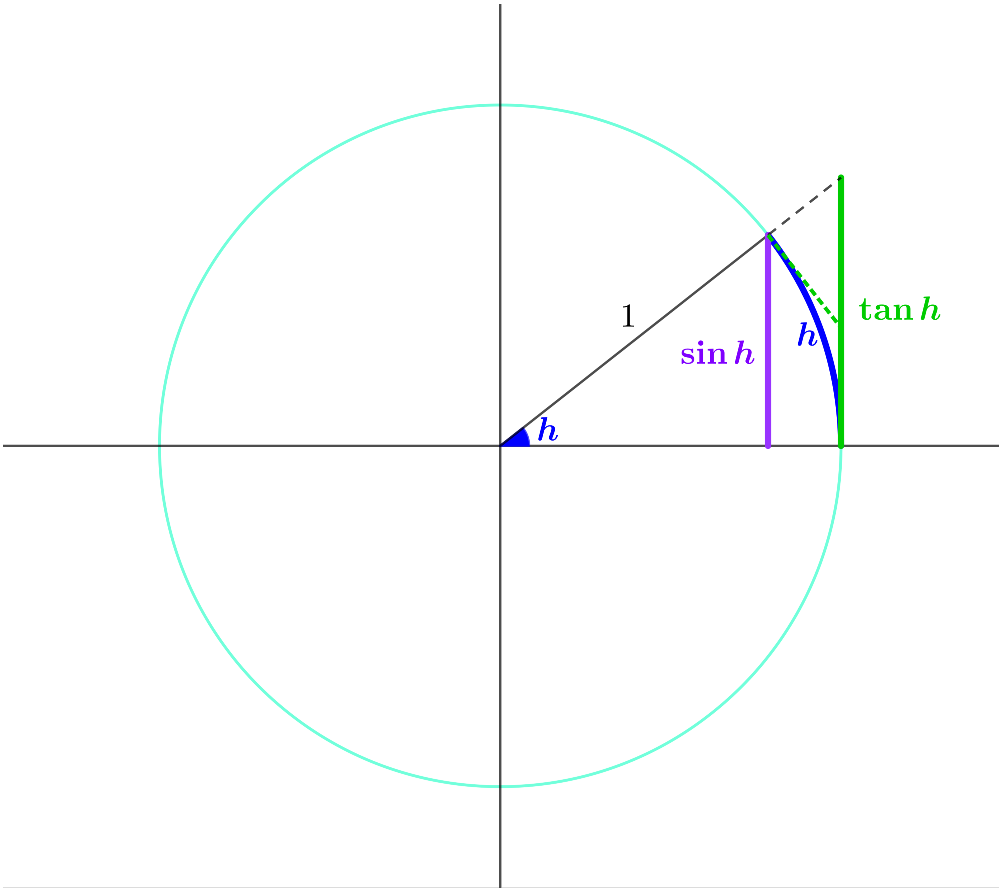
sin h
h
h
sin h
h
h
sin h
h
h
sin h
h

What is the gradient of the tangent to the curve $y = \sin x$ at the origin?



That gives you a sense of why $\lim_{h \to 0} \dfrac{\sin h}{h} = 1$, but it is not quite the whole story.

Here is a more complete "proof", in case you are interested.



Write down an inequality involving the pink and green line segments and the blue arc.

Use this to find lower and upper bounds for $\dfrac{h}{\sin h}$.

Now find lower and upper bounds for $\dfrac{\sin h}{h}$.

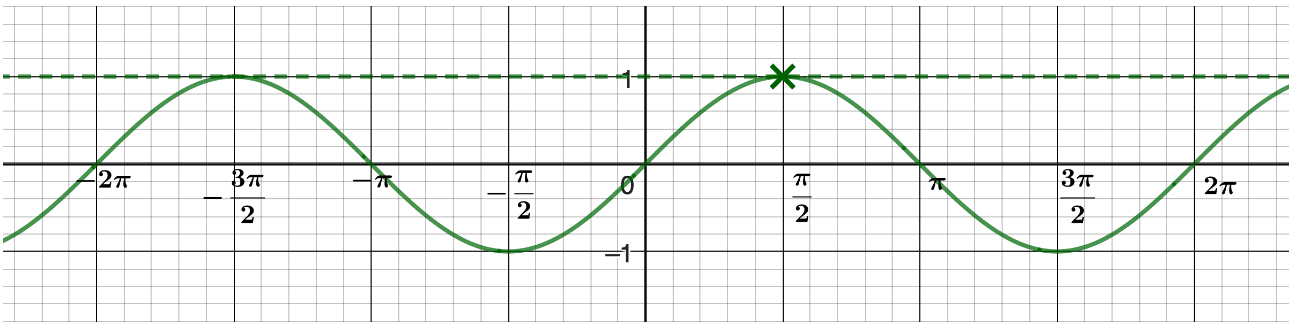
What is $\lim\limits_{h \to 0} \cos h$?

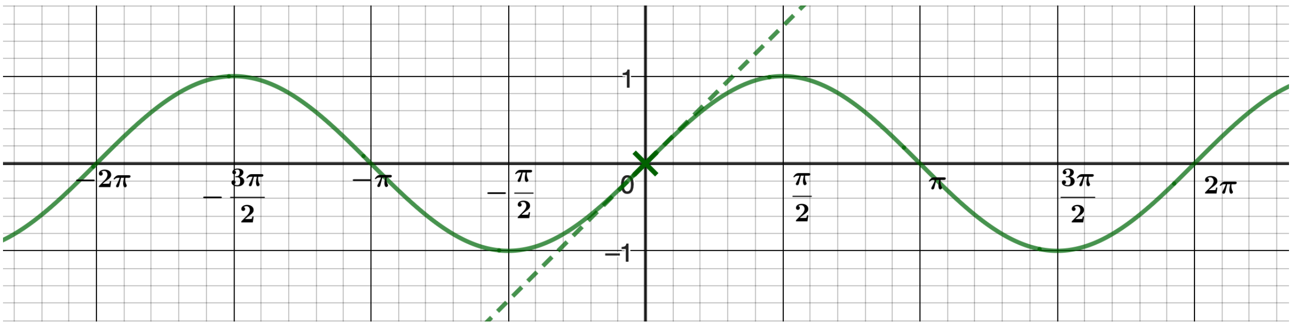
Use this to find $\lim\limits_{h \to 0} \dfrac{\sin h}{h}$.

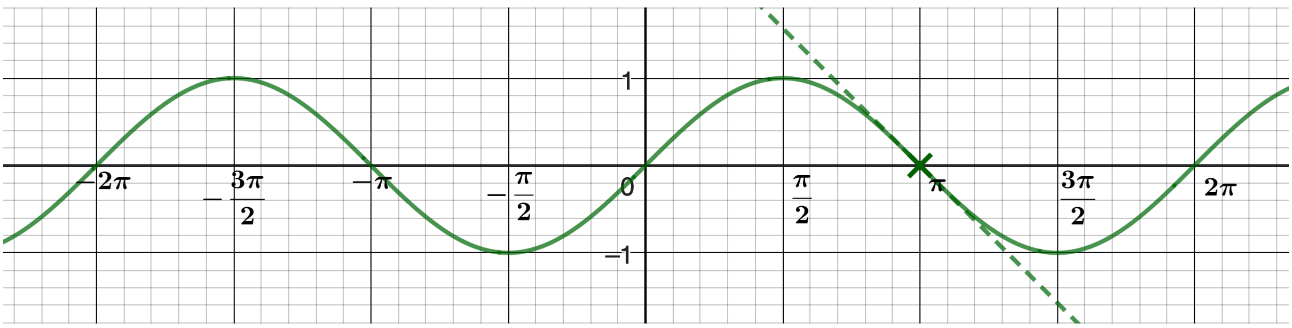
Now we know that the gradient of the tangent to the graph $y = \sin x$ at the origin is 1 (when we use radians as our unit of angles).

Next, we will think about the gradient of the curve at other points.

To start with, what are the gradients of these three tangents?



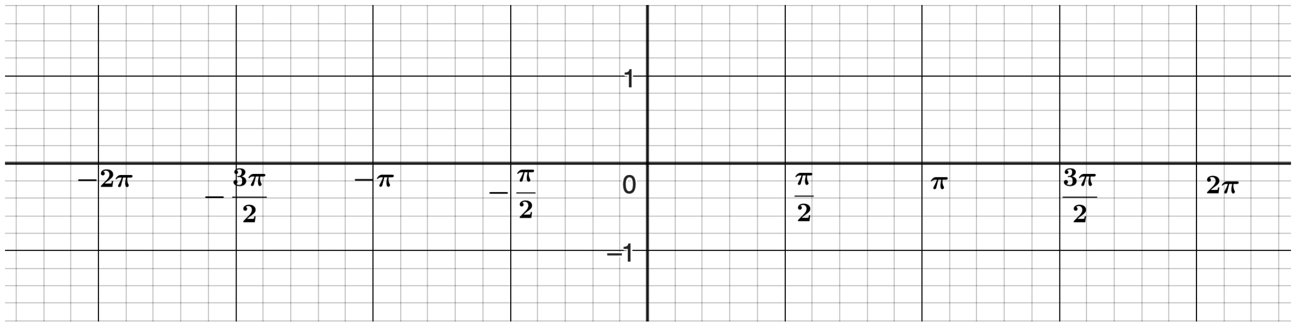




Use the graph to fill in this table:

x	**gradient of tangent**
0	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	
$-\frac{\pi}{2}$	
$-\pi$	
$-\frac{3\pi}{2}$	
-2π	

Now mark the values from the table on these axes with the left-hand column on the x axis and the right-hand column on the y axis.



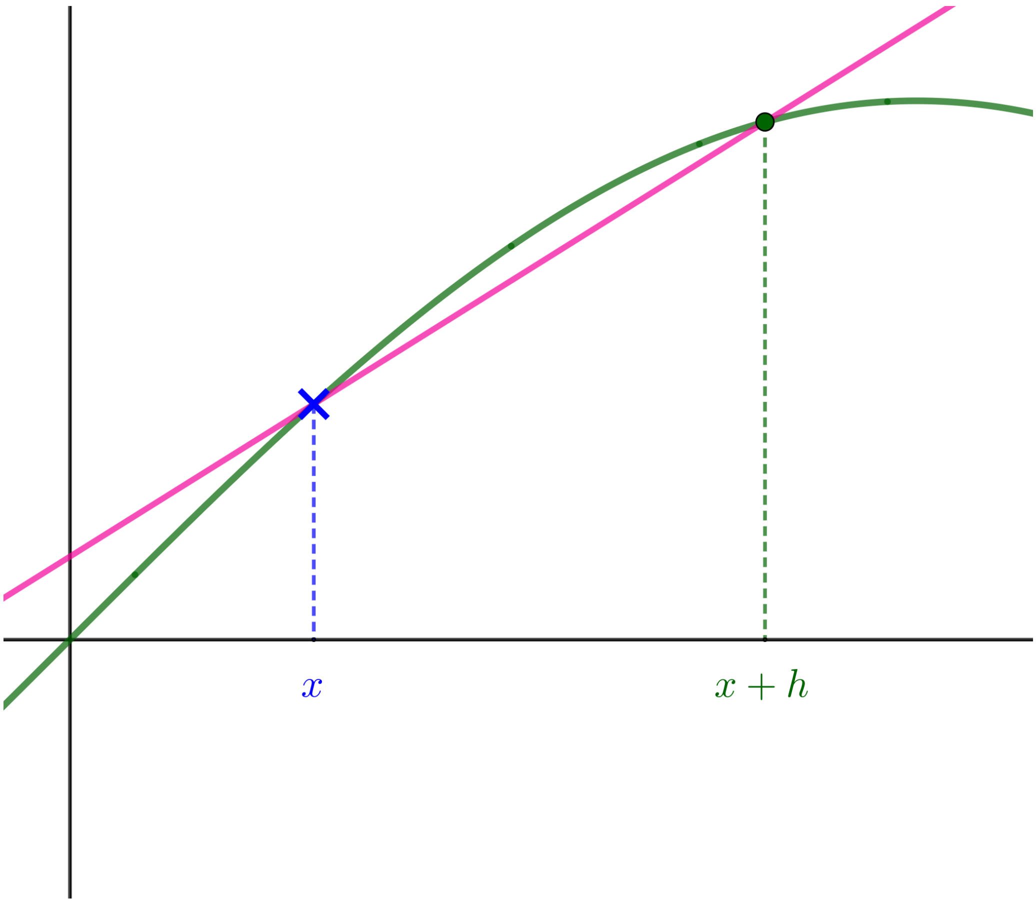
What does this graph look like?

What happens to the gradient between these points? Use this idea to draw the whole curve representing the gradient.

Now we know that the gradient of the sin graph looks rather like cos, and we are in a position to see that the differential of sin really is cos.

What is the gradient of the pink line in terms of x and h ?

What happens to the pink line as h gets increasingly small?



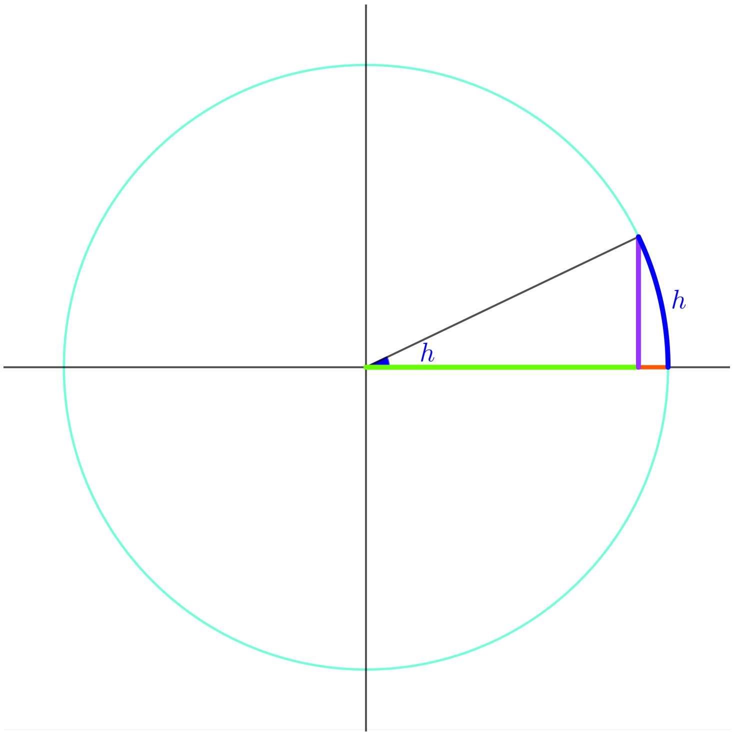
Use this to write the gradient of the tangent as a limit.

By using a compound angle formula and then rearranging, express this gradient as a multiple of $\cos x$ minus a multiple of $\sin x$.

Simplify this using the fact that $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$.

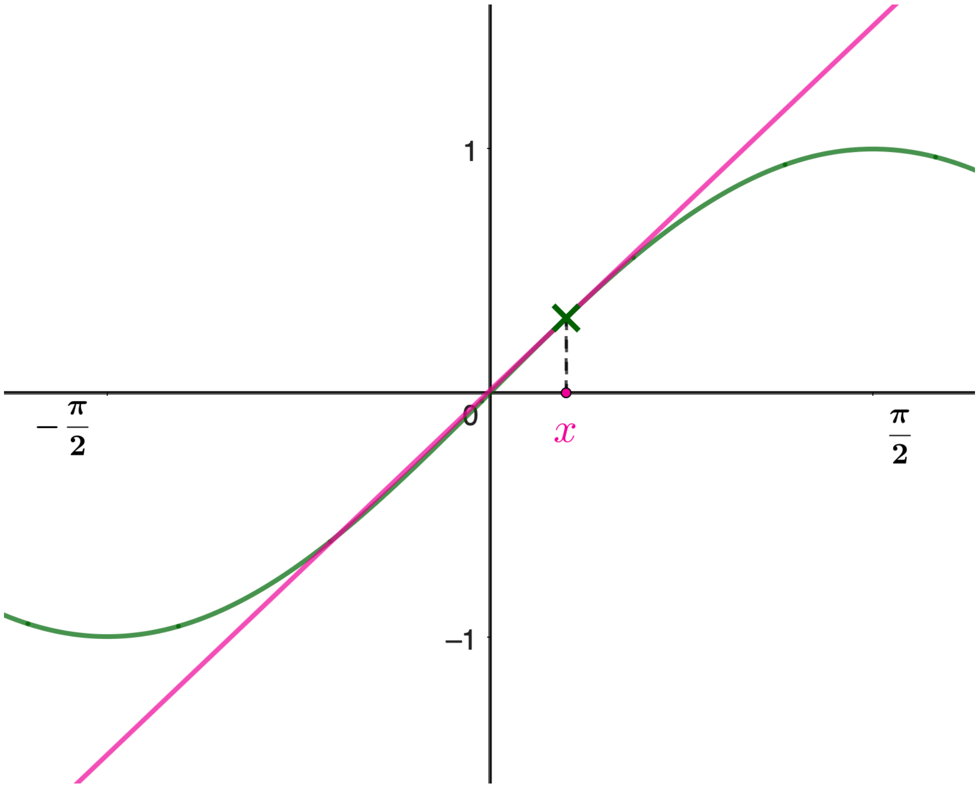
What is the length of the orange line segment?

What happens to the ratio of the orange line segment to the blue arc as h gets increasingly small?



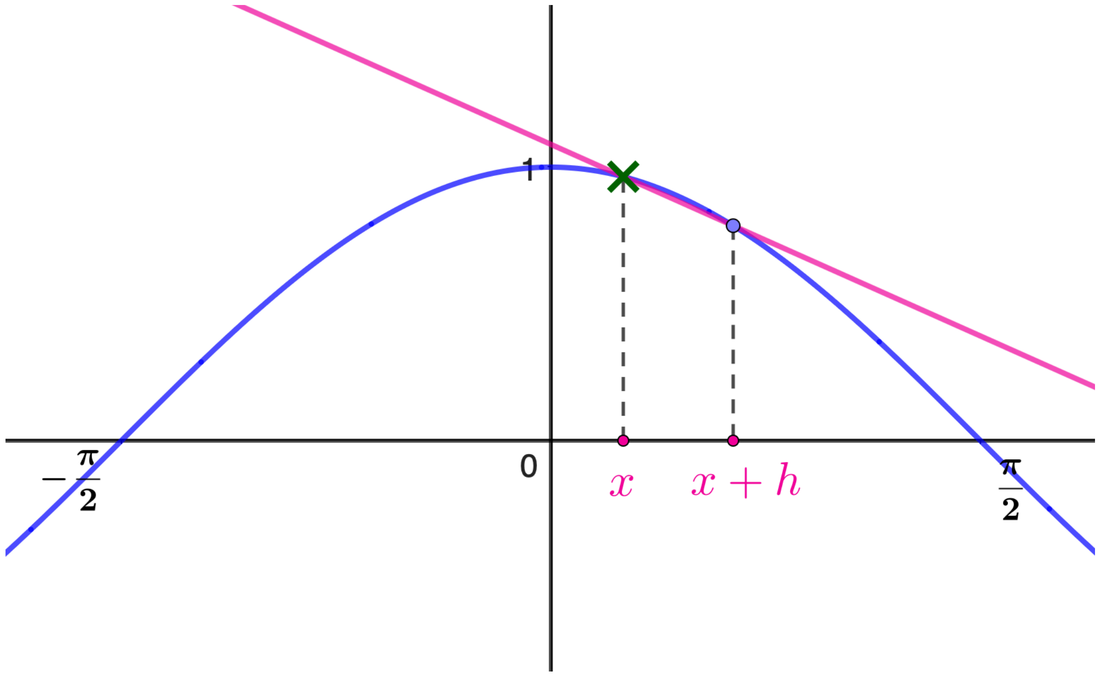
What does this tell you about $\lim_{h \to 0} \dfrac{1 - \cos h}{h}$?

Use this limit to find the gradient of the tangent to the curve $y = \sin x$ at the green cross.

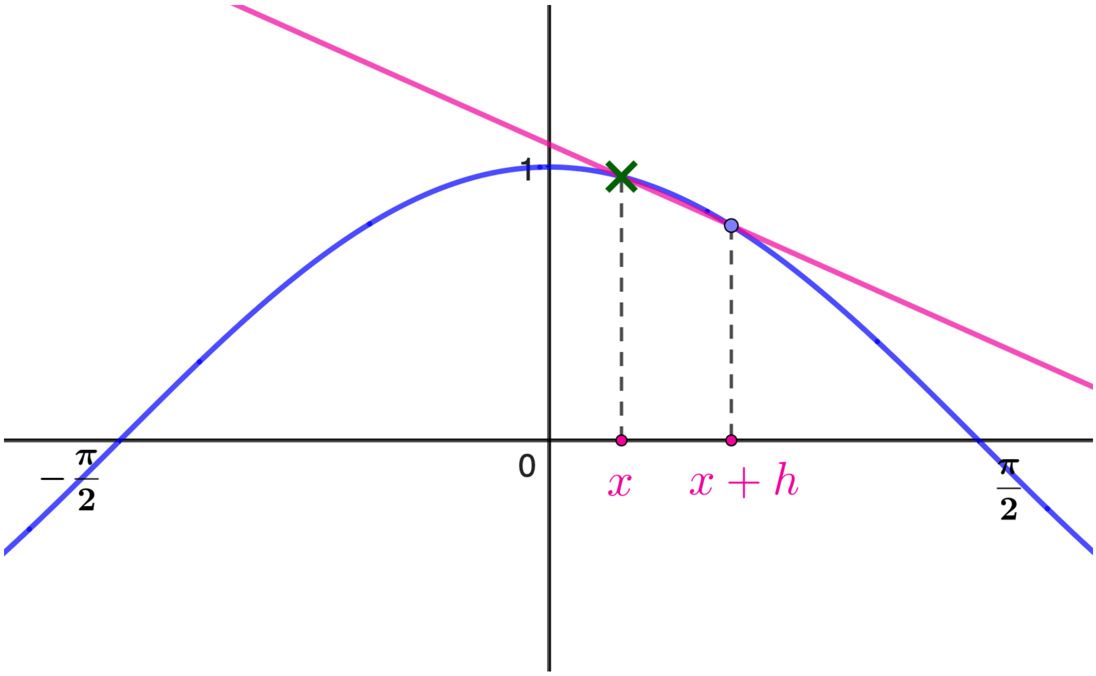


If $f(x) = \sin x$, what is $f'(x)$?

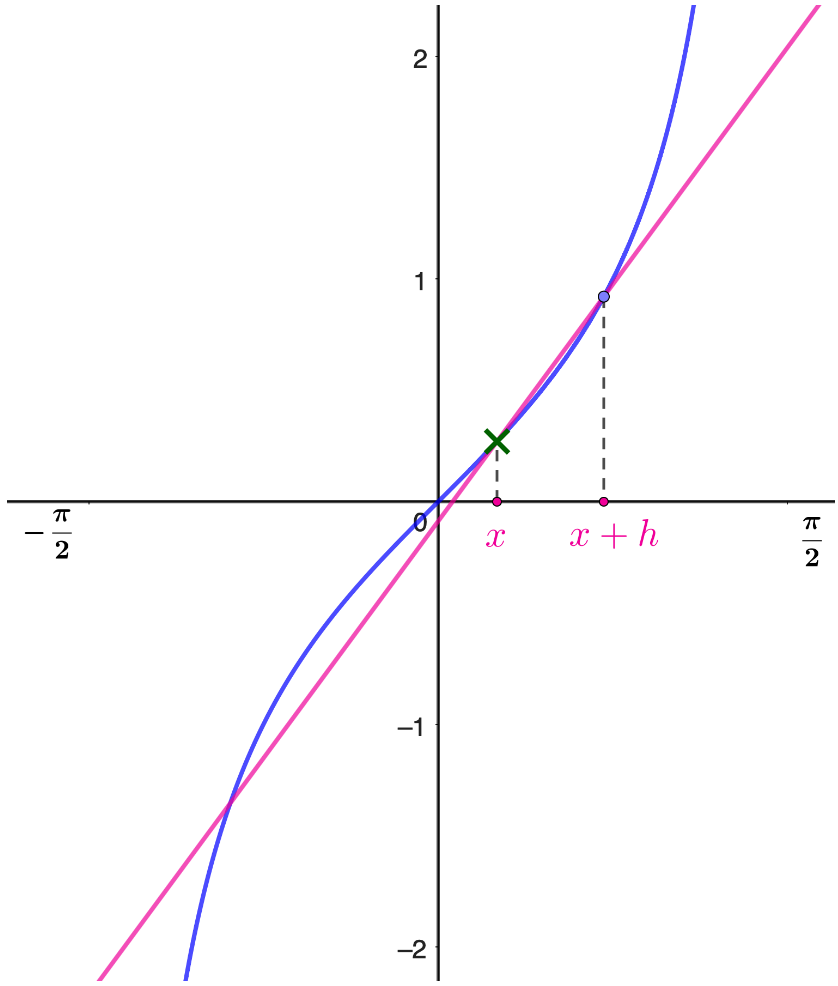
Now we know how to differentiate sine, we can tackle other circular functions.



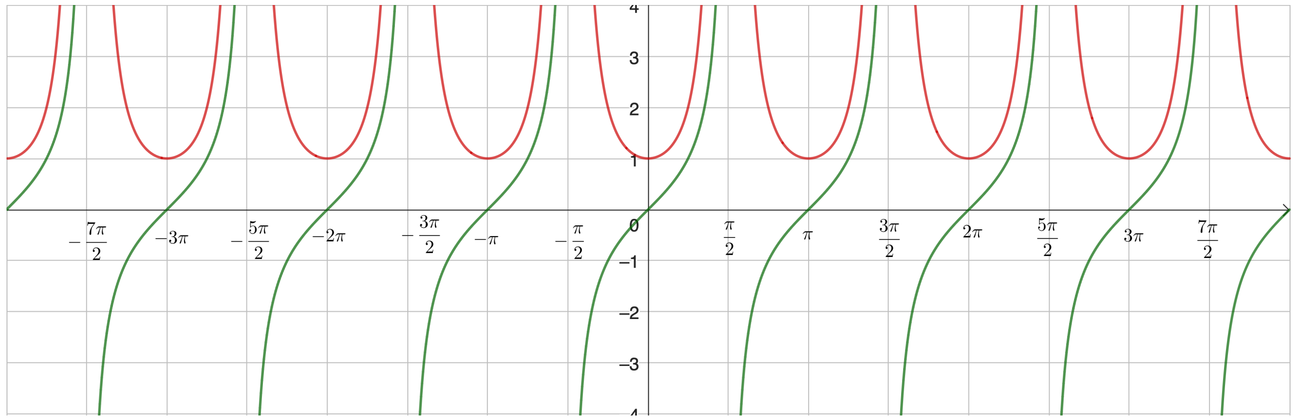
Use the same ideas to find the gradient of the tangent to the curve $y = \cos x$ at the green cross.


1
0
$-\frac{\pi}{2}$
x
$x+h$
$\frac{\pi}{2}$

What is the gradient of the tangent to the curve $y = \tan x$ at the green cross?

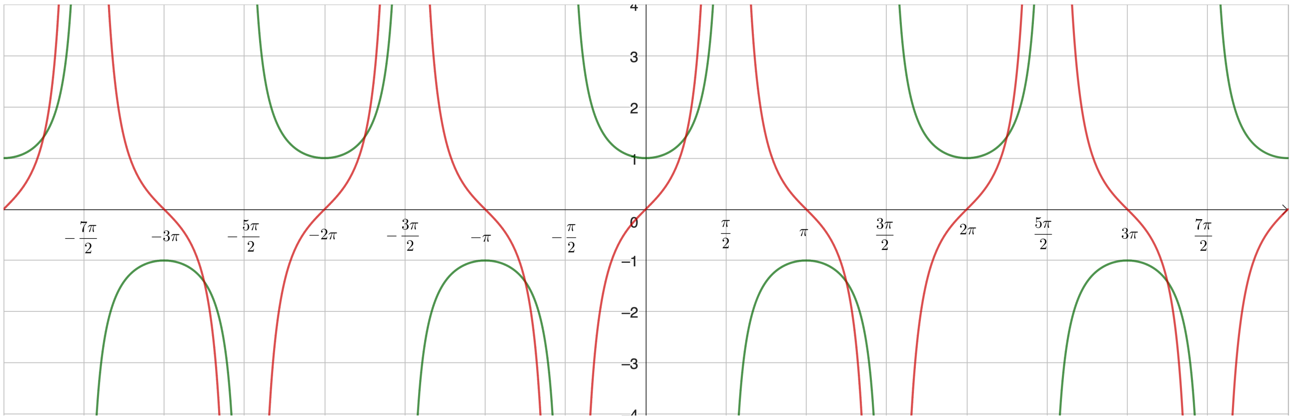


Here is the graph of tan x along with the graph of its differential. How do the two curves relate to each other?



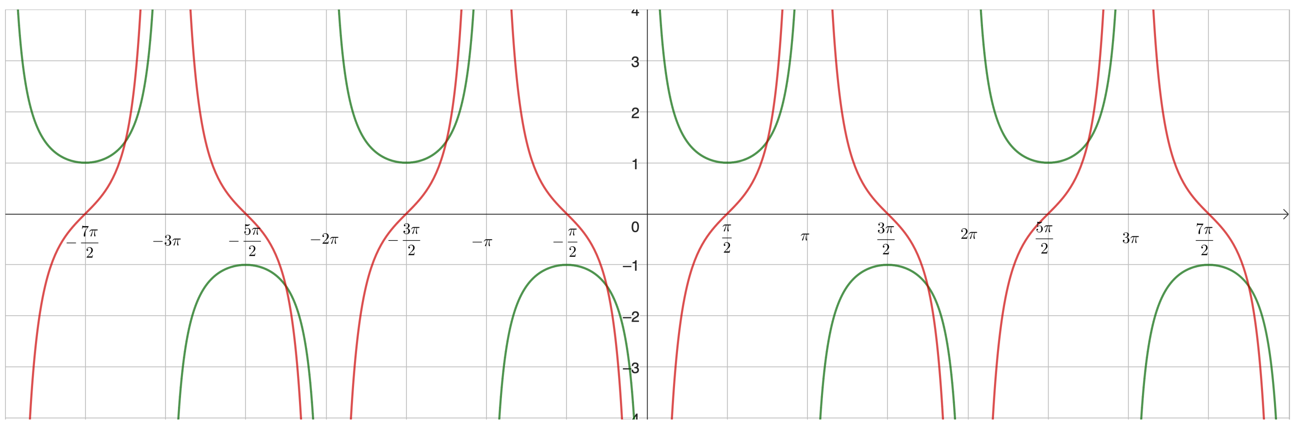
Differentiate $f(x) = \sec x$

Here is the graph of $\sec x$ along with the graph of its differential. How do the two curves relate to each other?



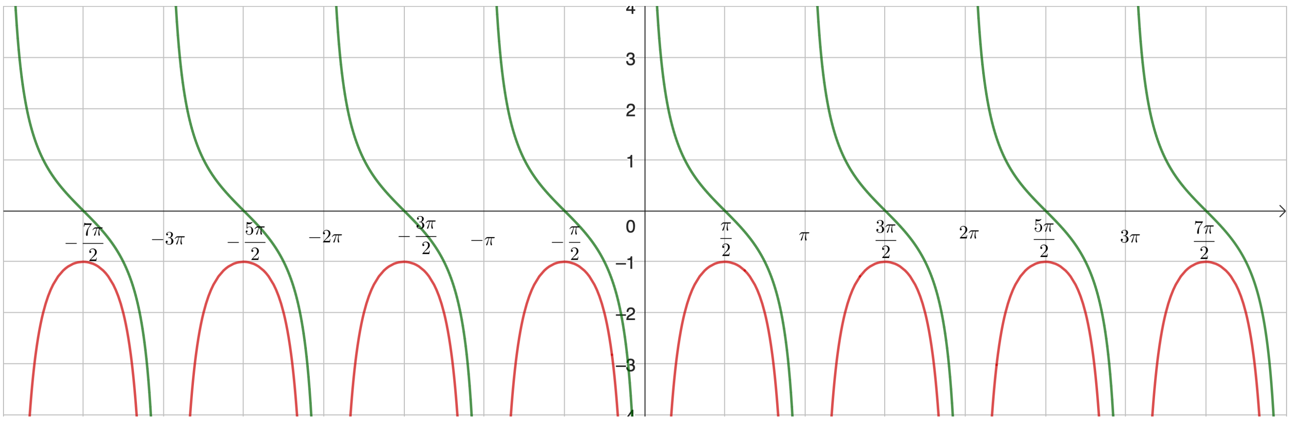
Differentiate $f(x) = \text{cosec}\, x$

Here is the graph of cosec$x$ along with the graph of its differential. How do the two curves relate to each other?



Differentiate $f(x) = \cot x$

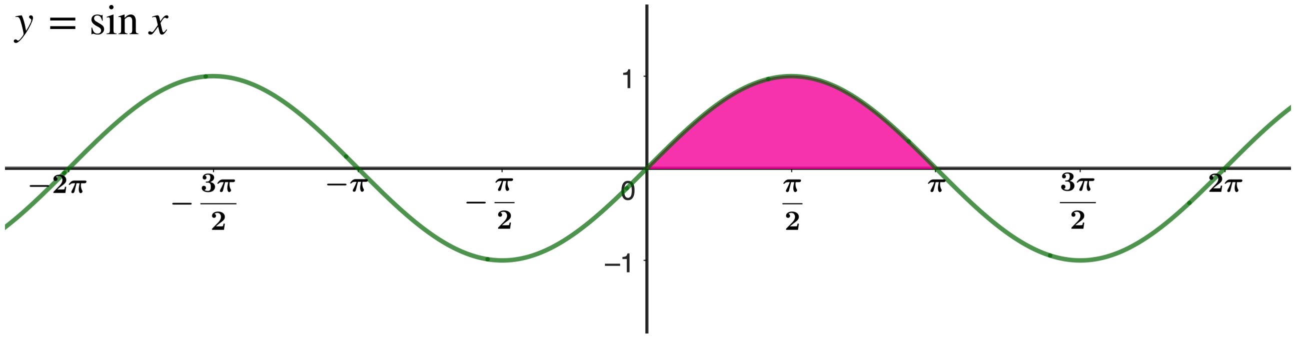
Here is the graph of $\cot x$ along with the graph of its differential. How do the two curves relate to each other?



Integrals of circular functions

Find these areas

$y = \sin x$



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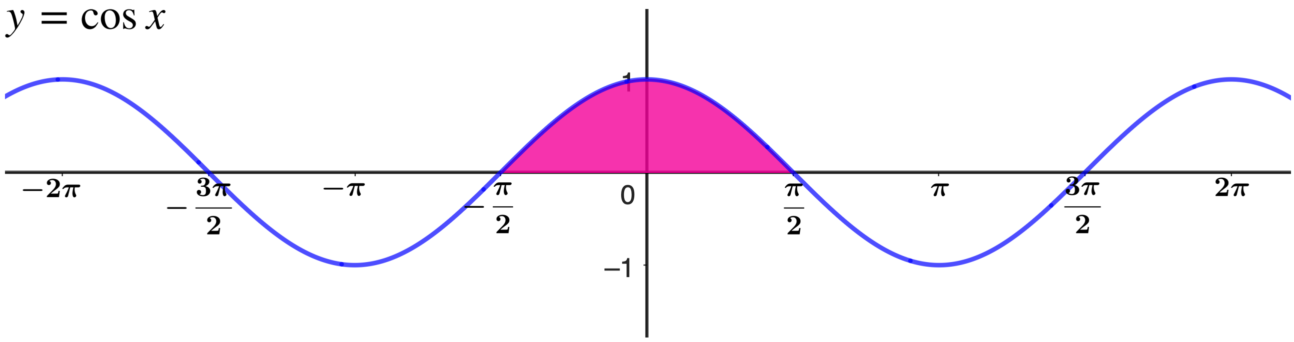
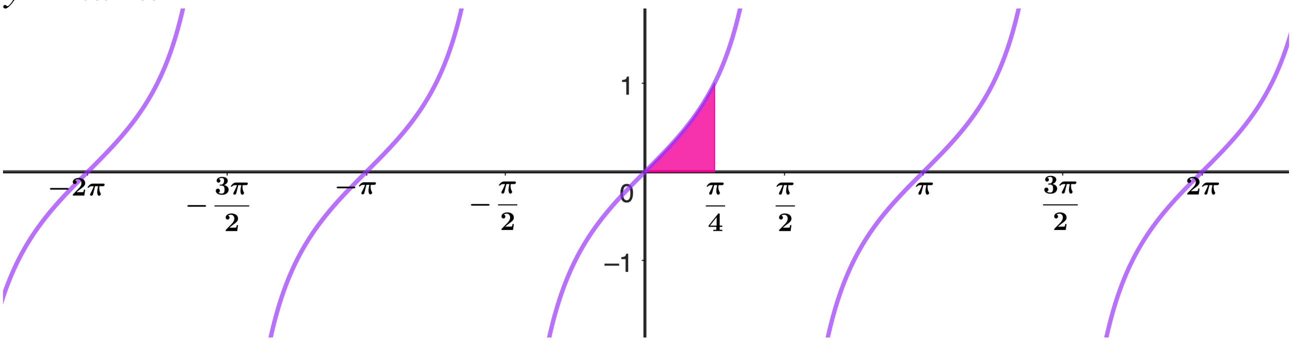
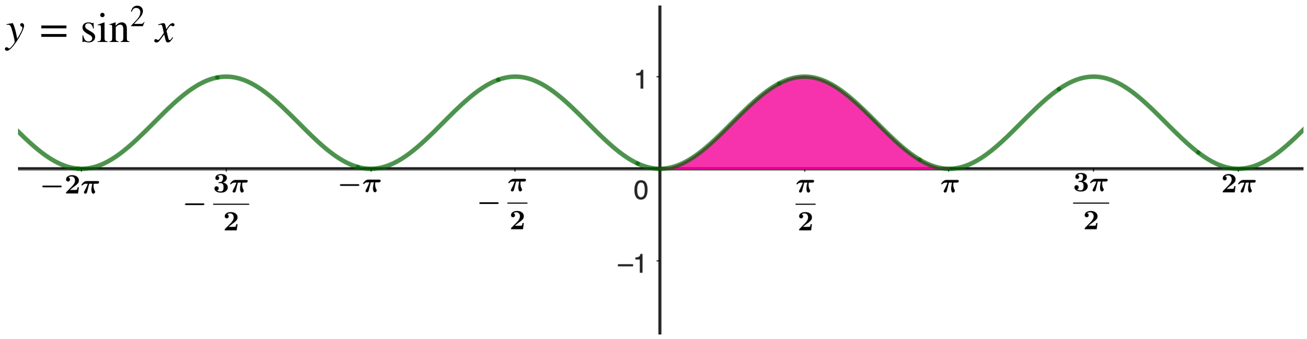
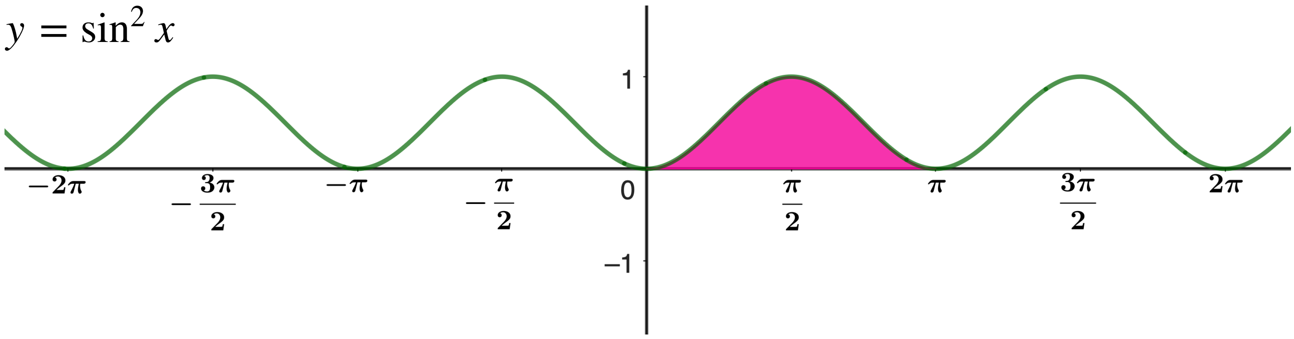

$y = \cos x$
1
0
−1
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

$y = \tan x$



1
−1
0
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


$y = \sin^2 x$
1
0
−1
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π



$$u = \sin x \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = \cos x \qquad v = -\cos x$$

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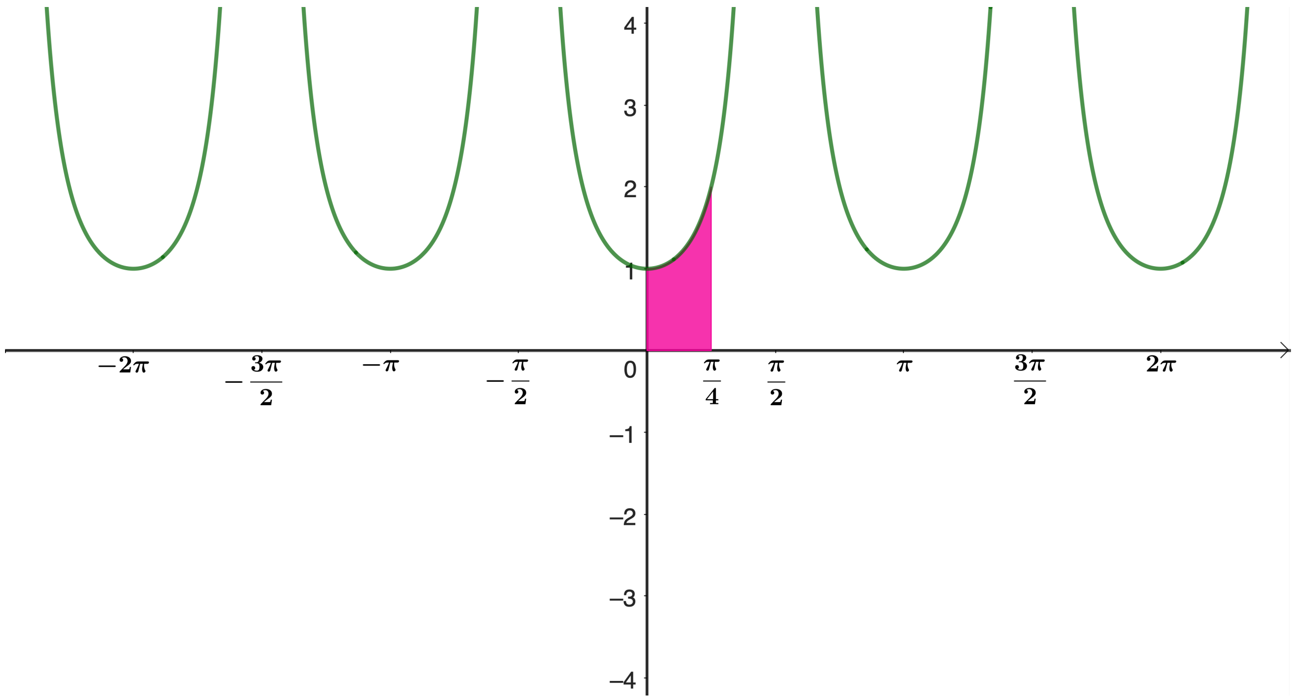
$y = \sec^2 x$



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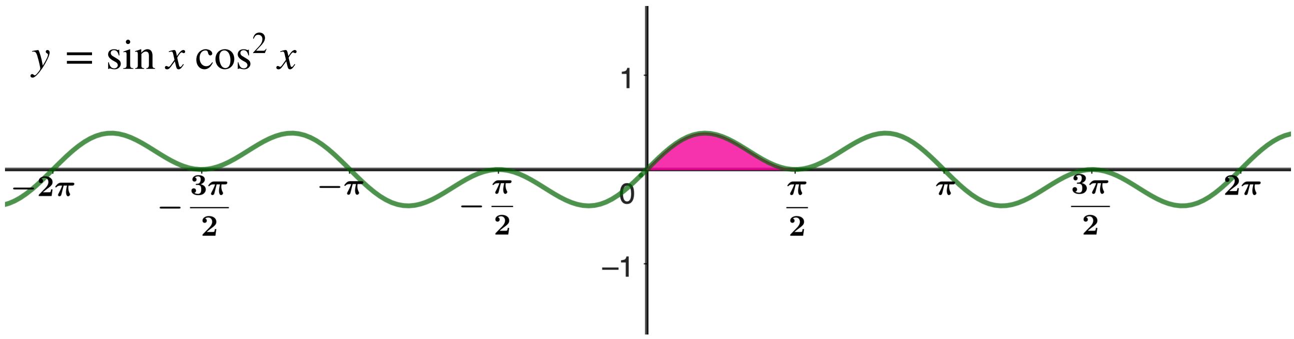
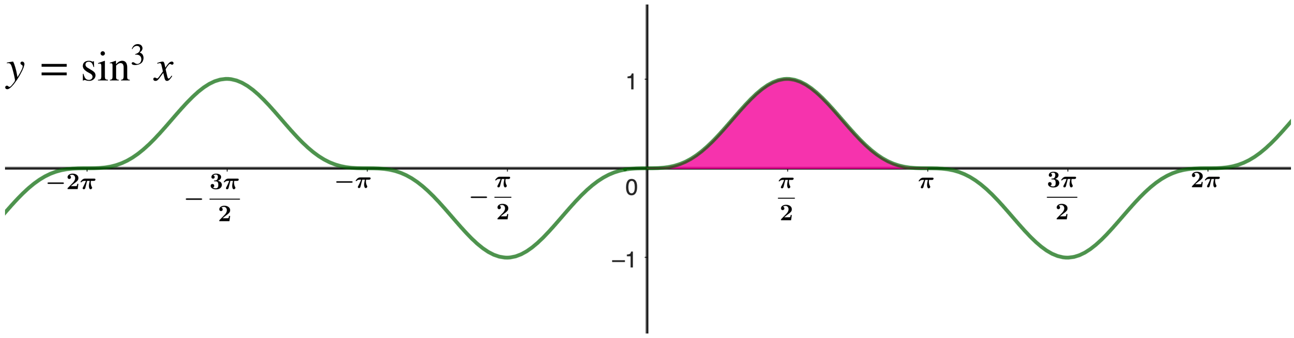
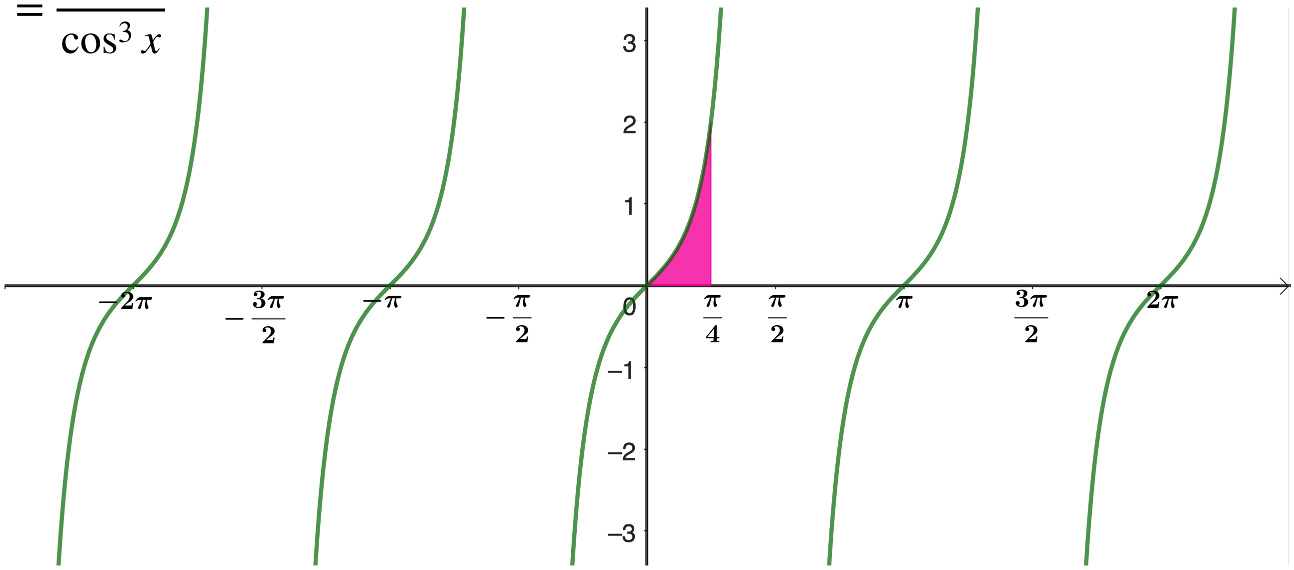


$y = \sin x \cos^2 x$
1
0
−1
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


$y = \sin^3 x$
1
0
−1
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

$$y = \frac{\sin x}{\cos^3 x}$$

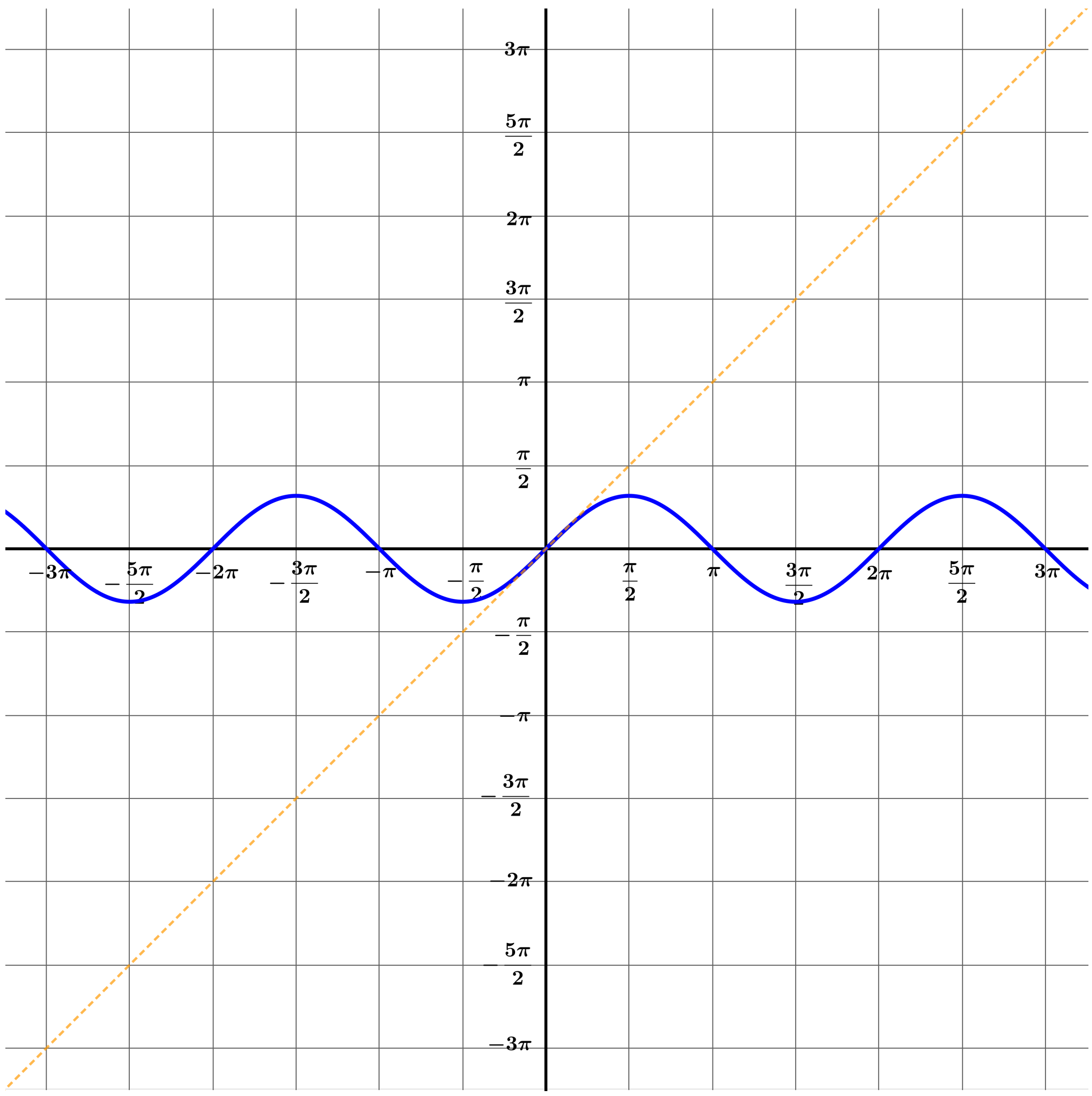


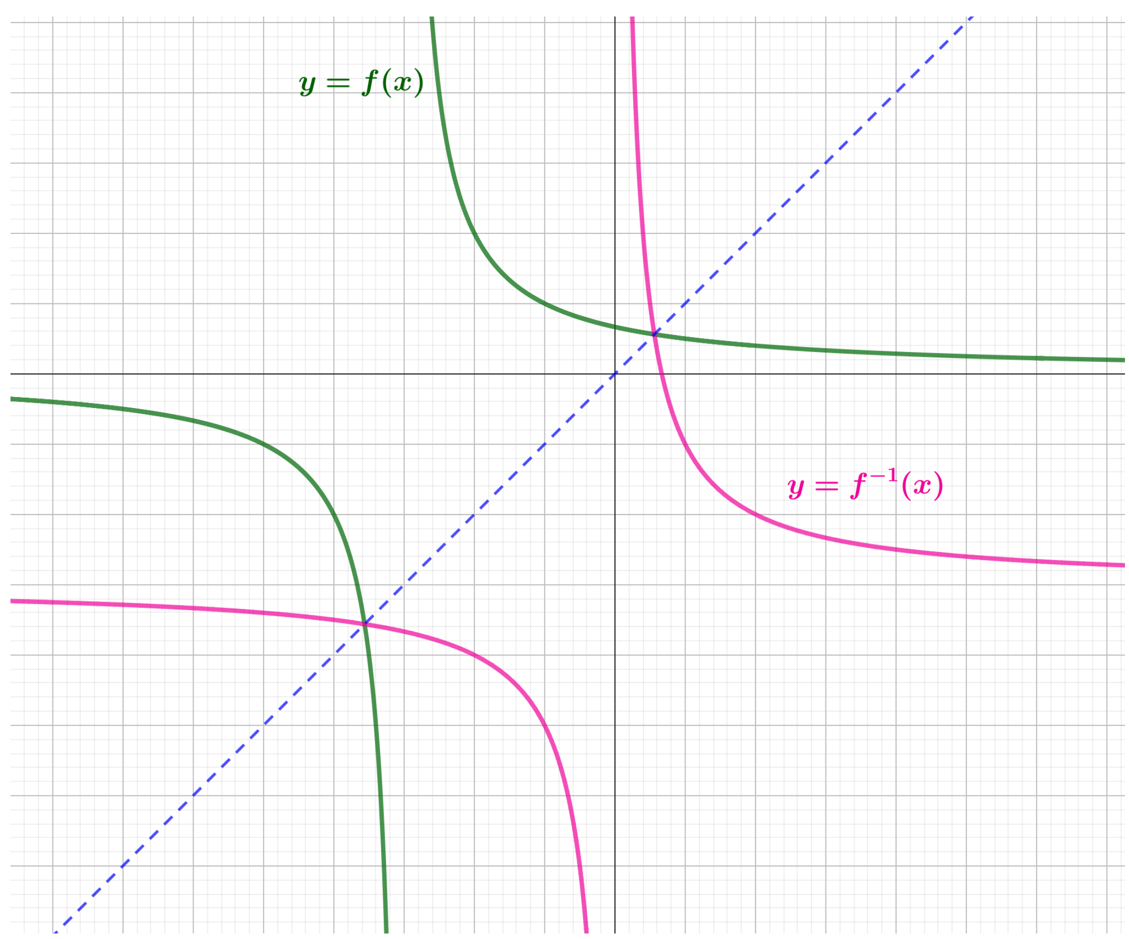
Here is an integral that is quite a bit more of a challenge:

$$\int \sec x \, \mathrm{d}x$$

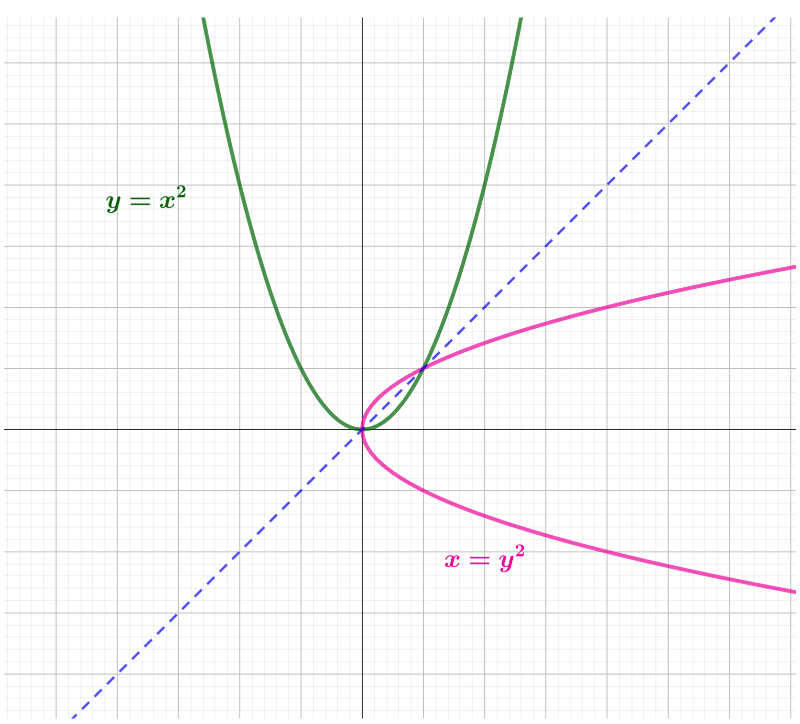
Inverse circular functions

Draw the graph $x = \sin y$.

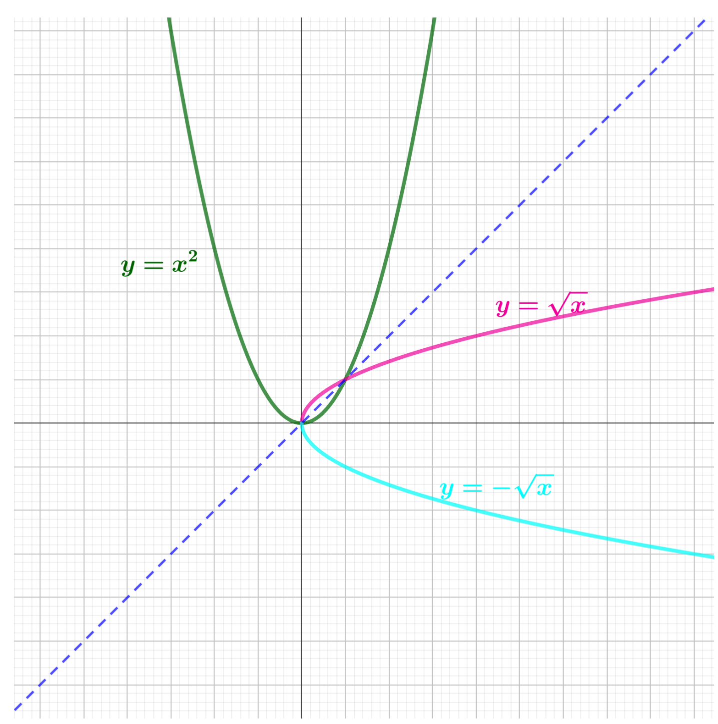




$y = f(x)$
$y = f^{-1}(x)$

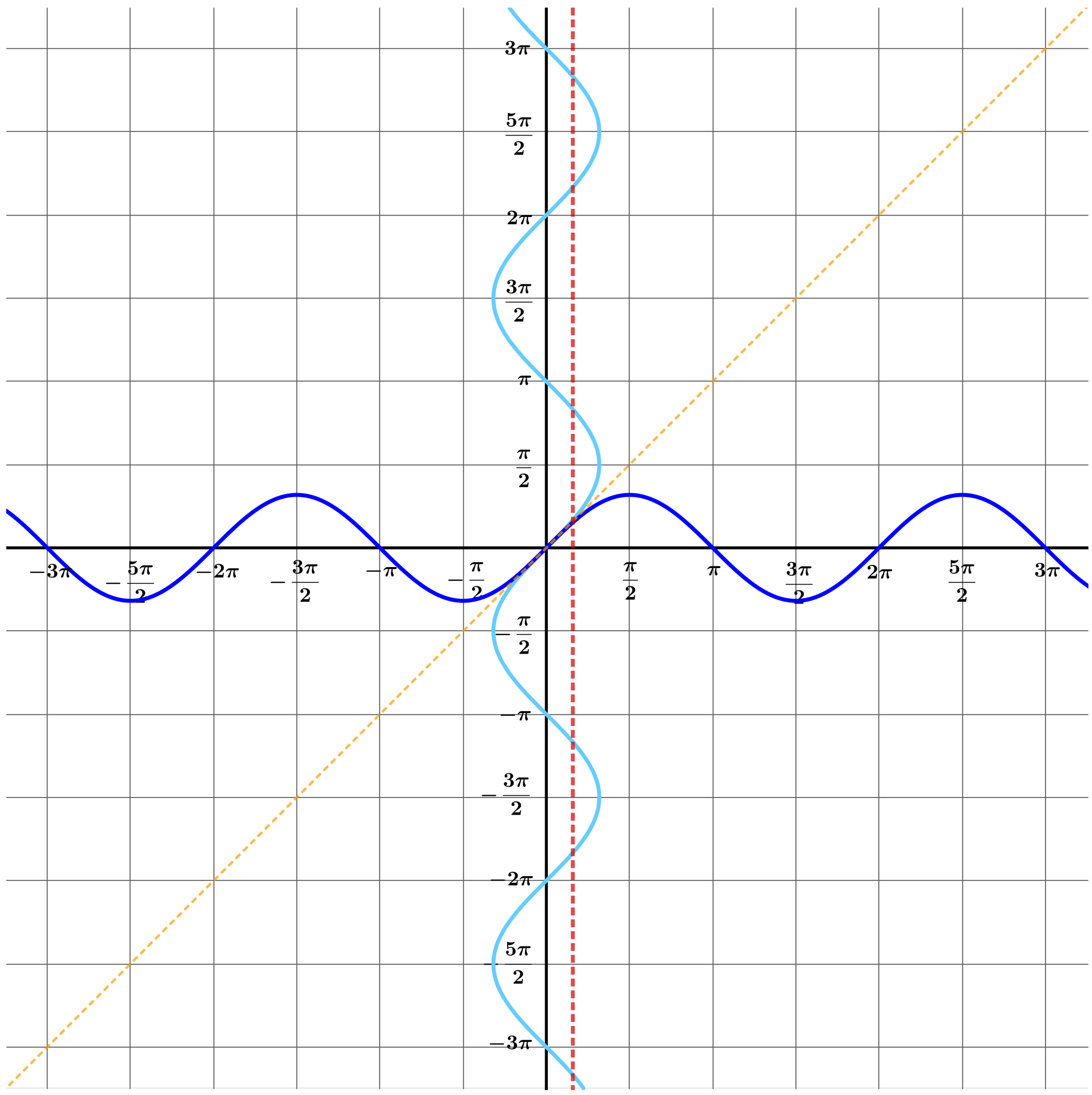


$y = x^2$
$x = y^2$



$y = x^2$
$y = \sqrt{x}$
$y = -\sqrt{x}$

Solve the equation $\sin y = \dfrac{1}{2}$.

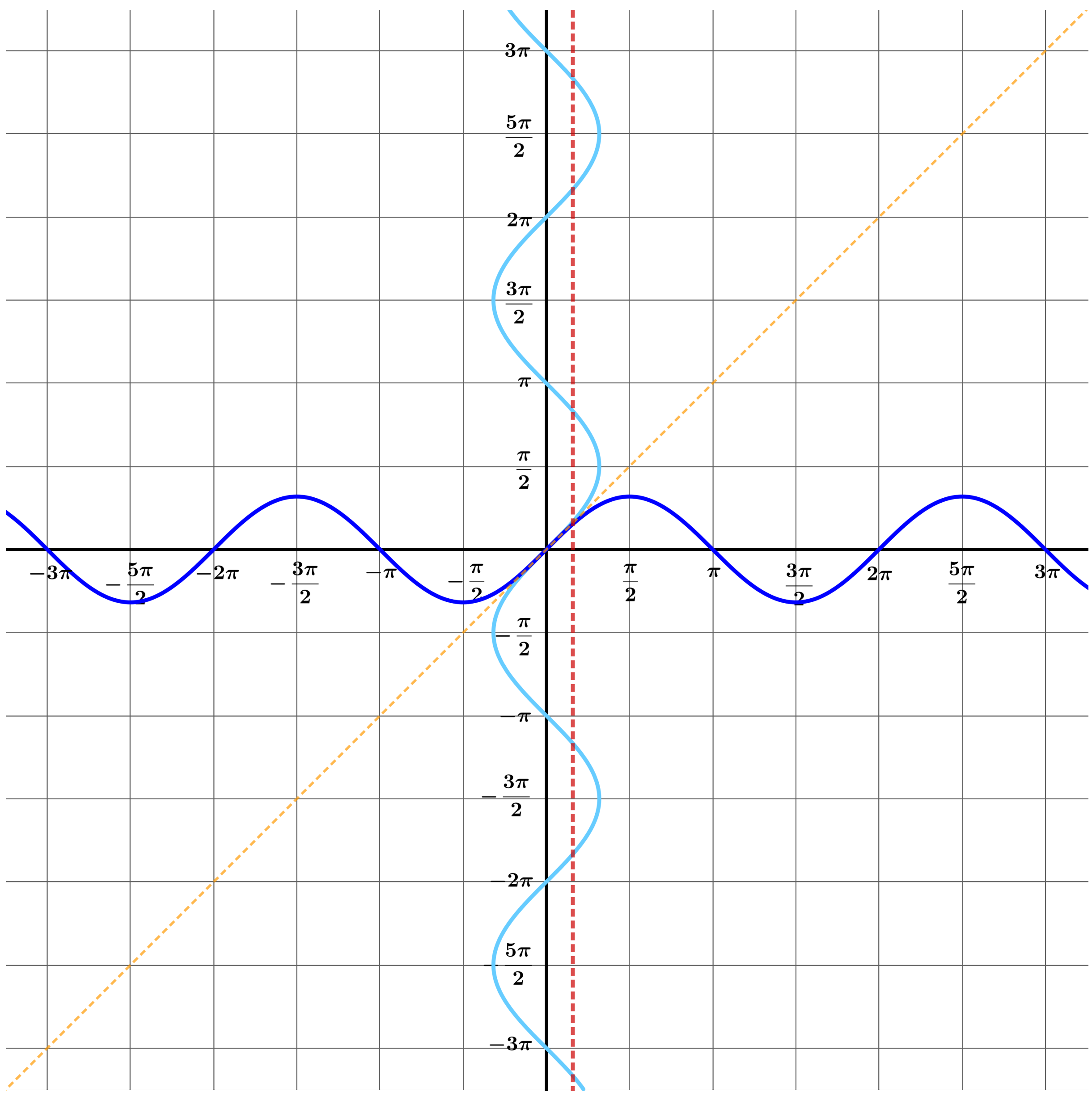


How does your answer relate to this graph?

How many values do you want for $\sin^{-1}\dfrac{1}{2}$?

How many times do you want the line $x = \dfrac{1}{2}$ to intersect with the graph $y = \sin^{-1} x$?

How can you adapt the graph $x = \sin y$ to ensure that any vertical line between $x = -1$ and $x = 1$ intersects it exactly once?



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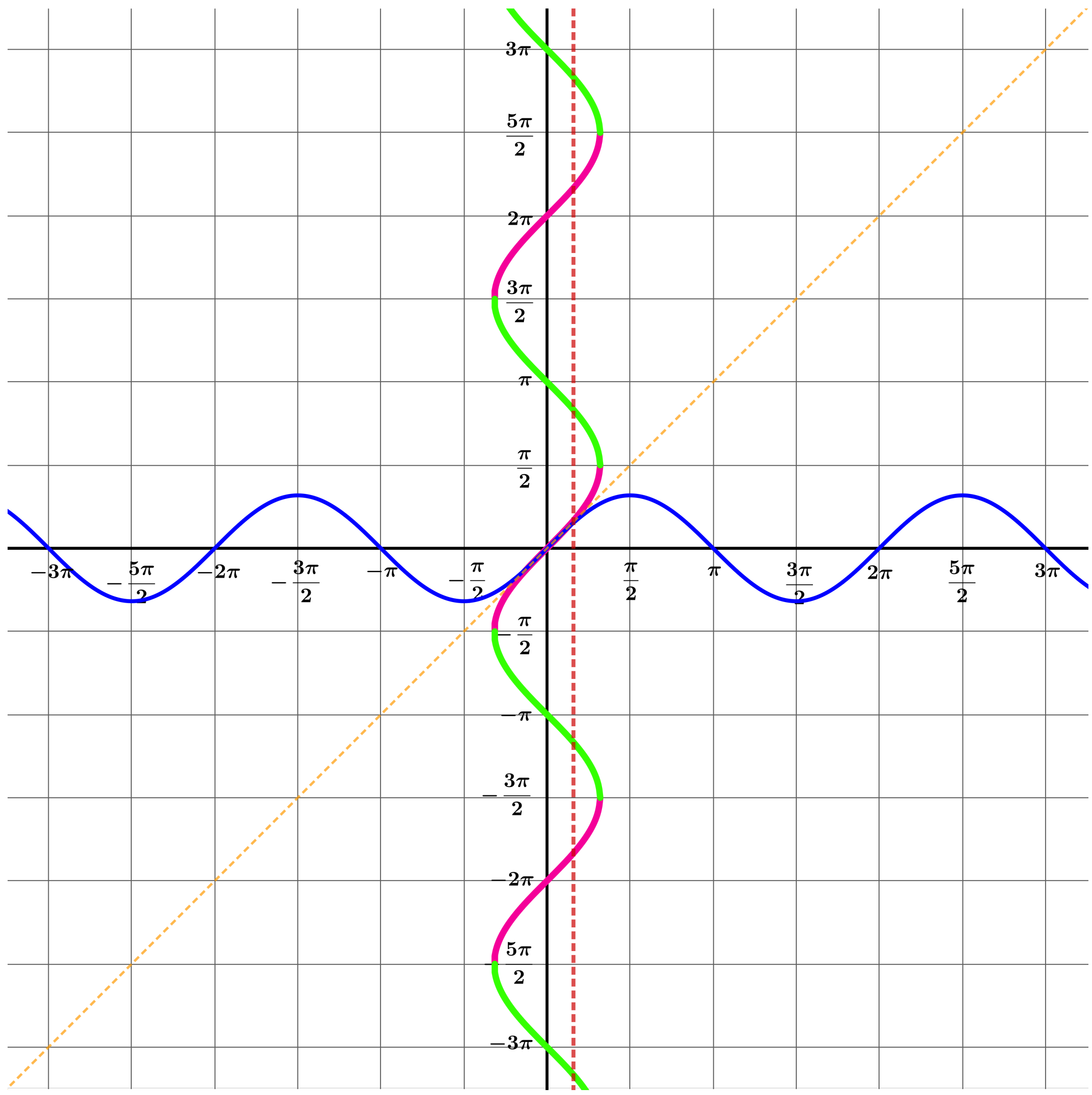
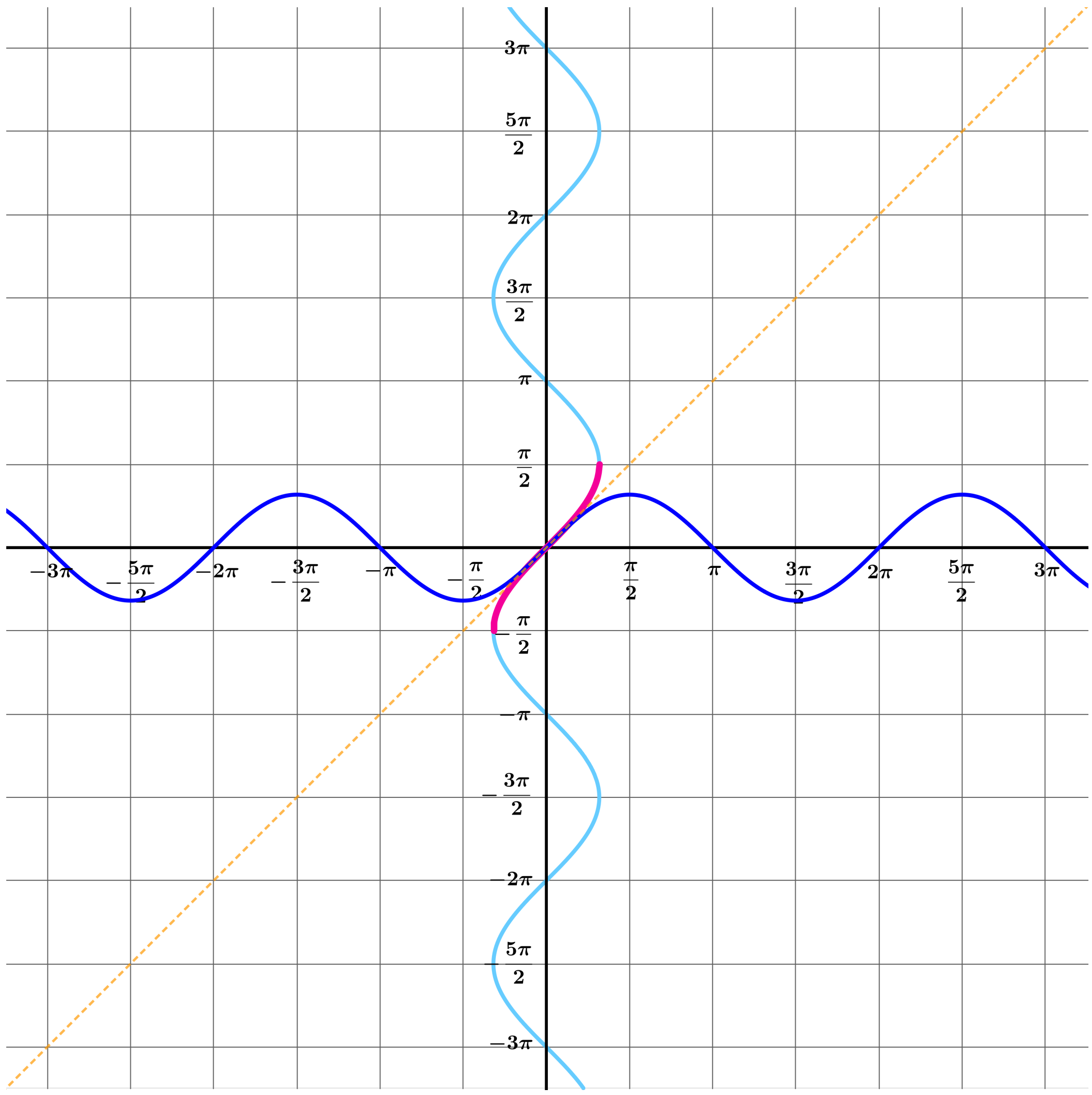
Which of these pink or green segments on the curve would correspond to a possible definition of the function $\sin^{-1} x$?

Which would be the most sensible to choose?

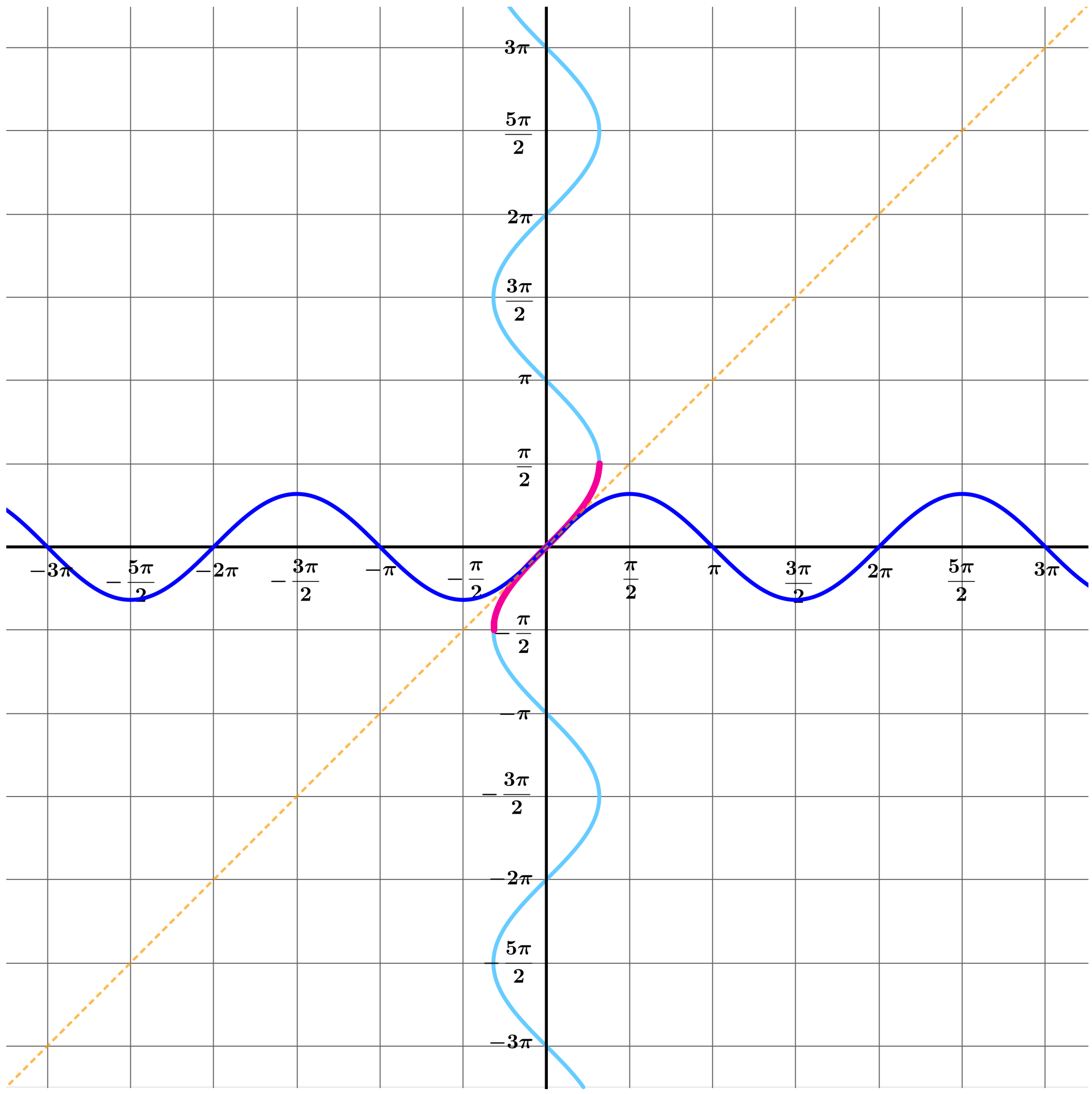


Here it is: the graph $y = \sin^{-1} x$.

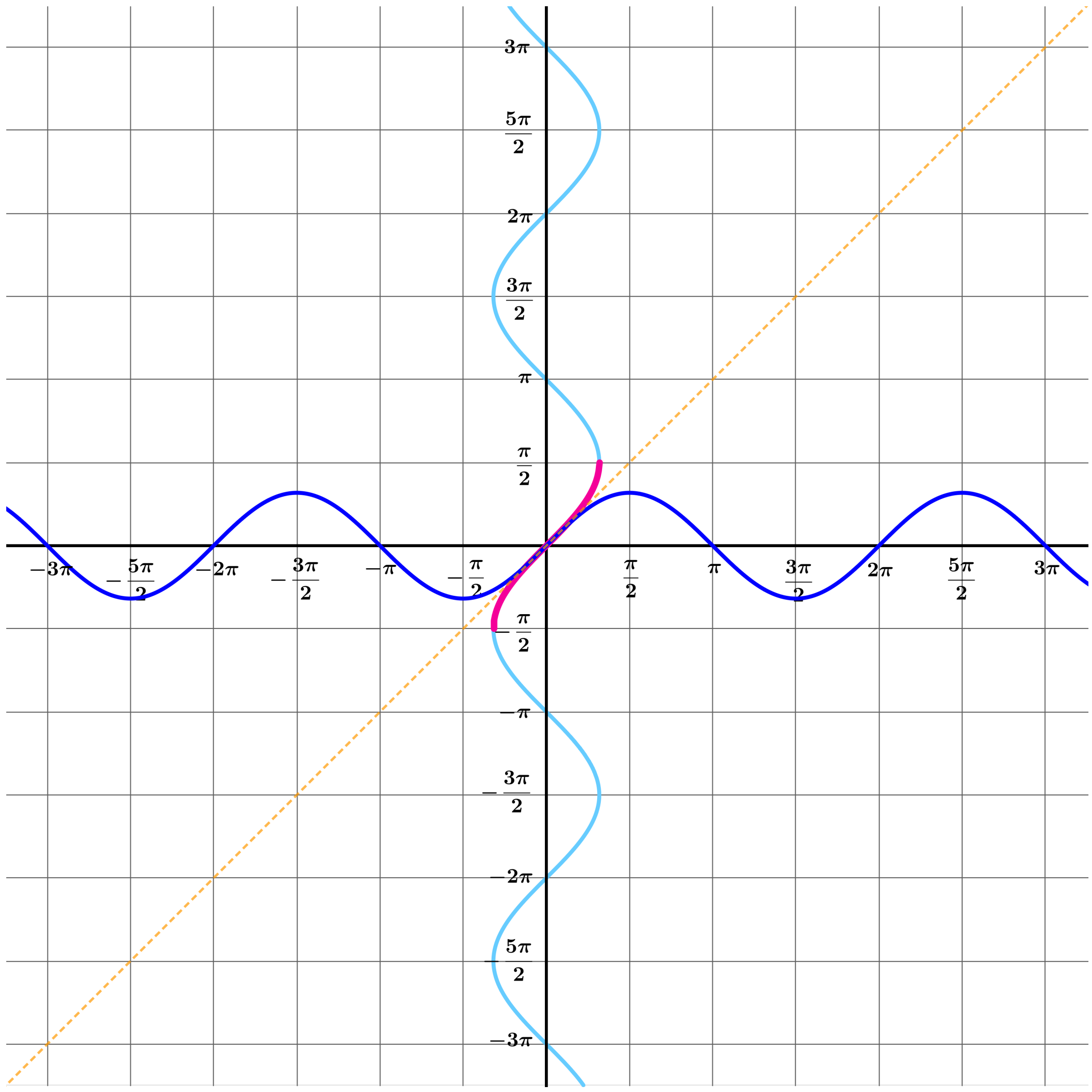


What are its domain and range?

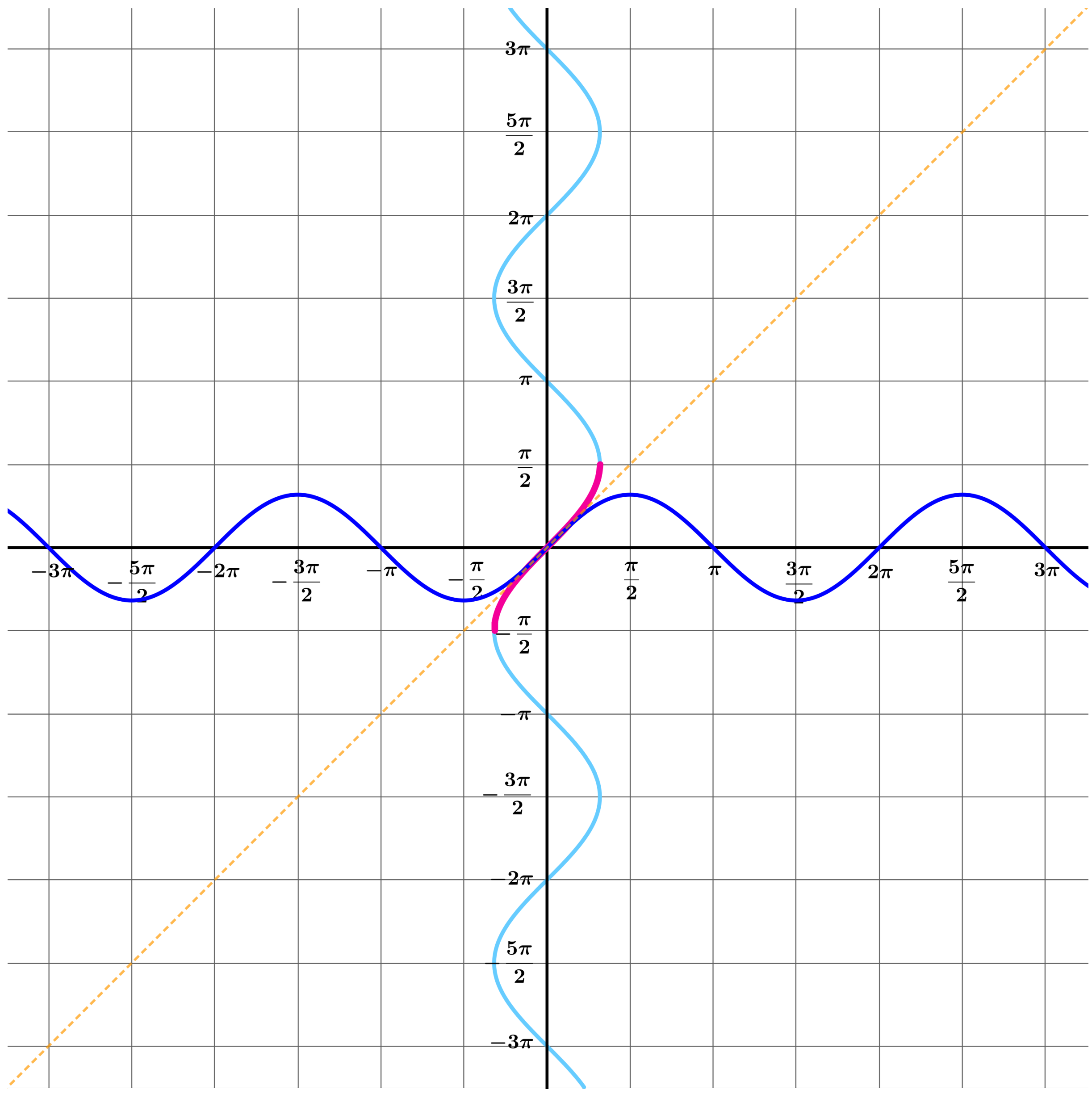
What are $\sin\left(\sin^{-1}\dfrac{1}{2}\right)$ and $\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$?



What are $\sin\left(\sin^{-1}(-1)\right)$ and $\sin^{-1}\left(\sin\dfrac{3\pi}{2}\right)$?



What are $\sin\left(\sin^{-1} x\right)$ and $\sin^{-1}\left(\sin x\right)$?



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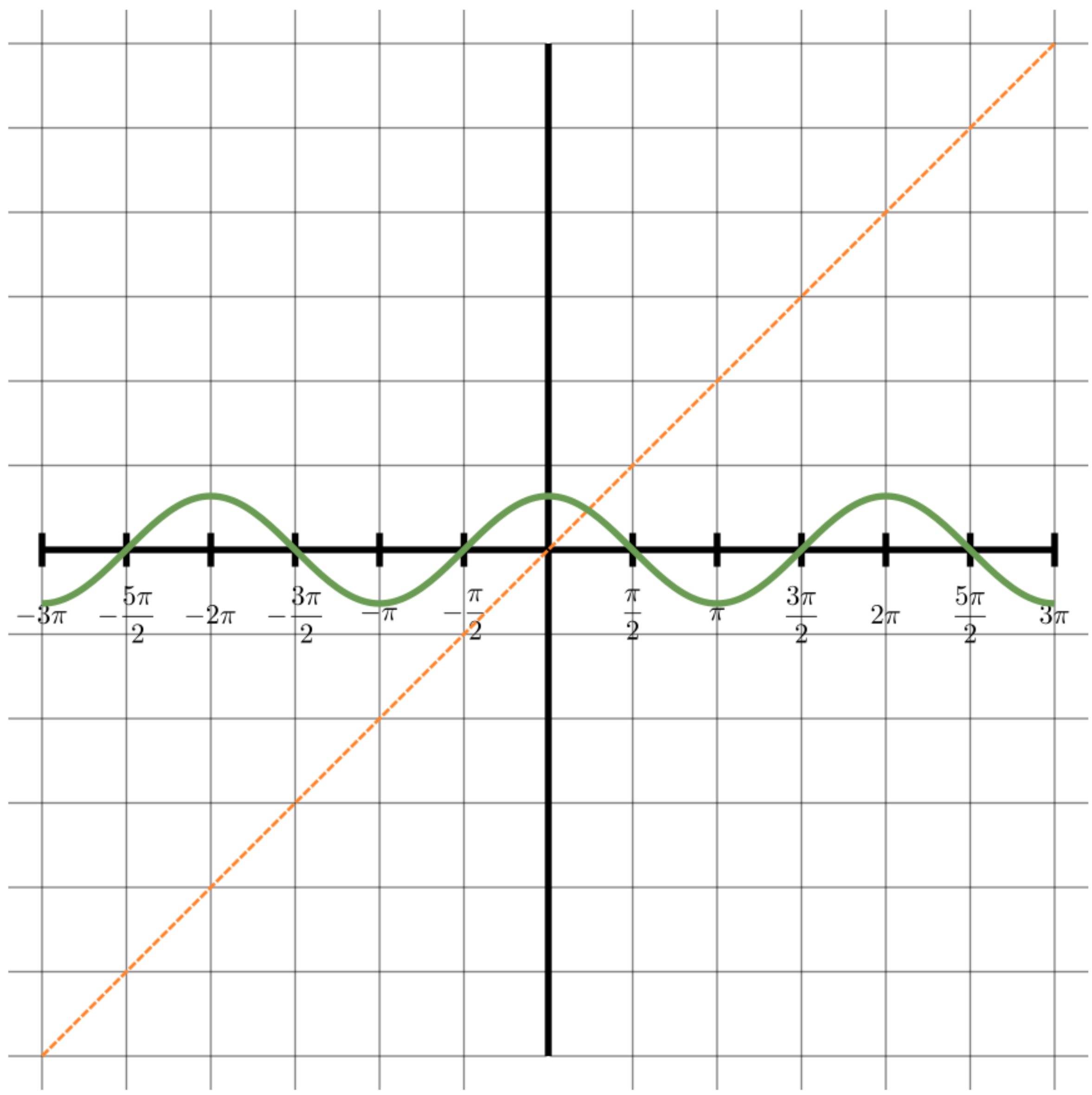
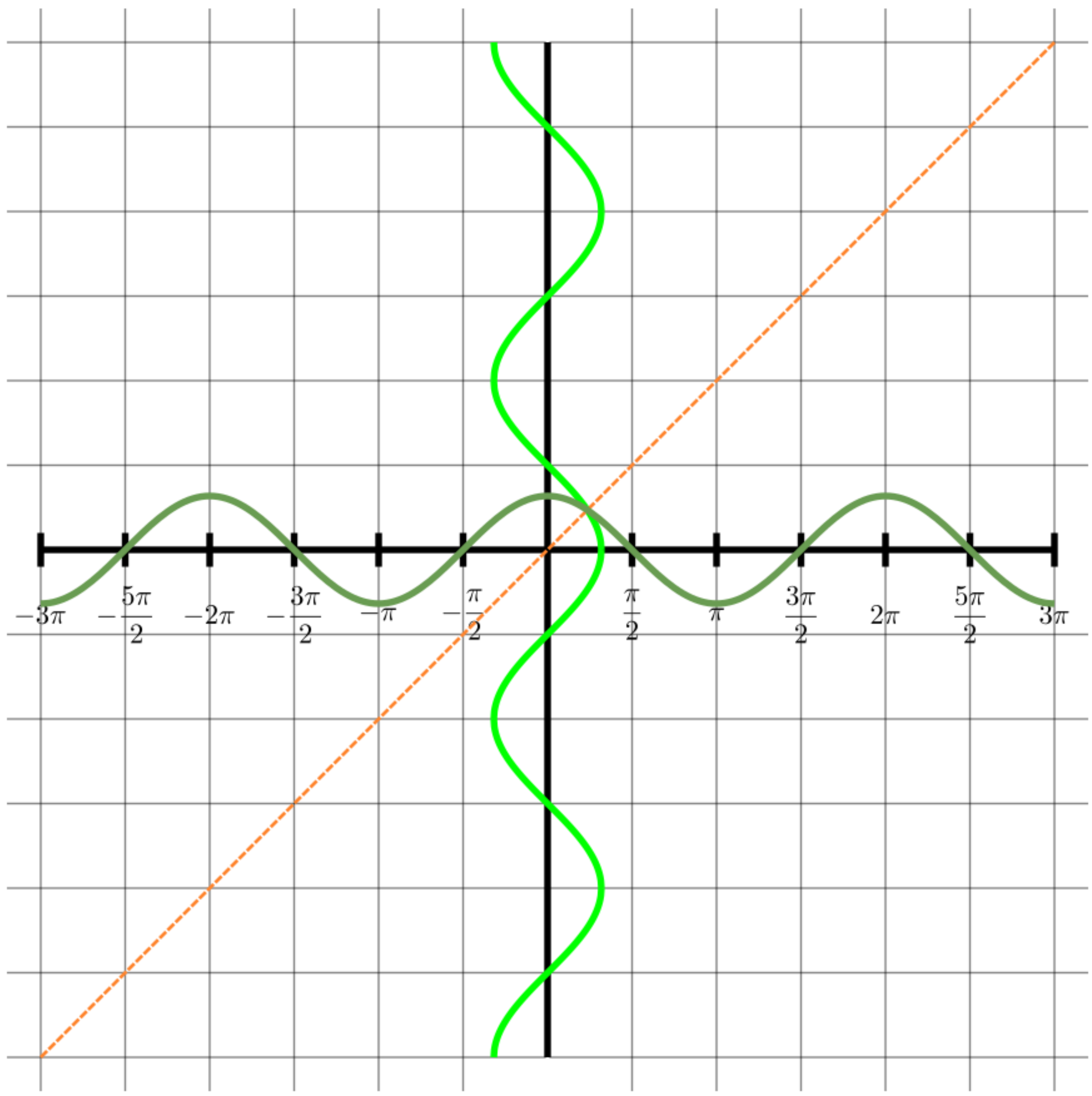
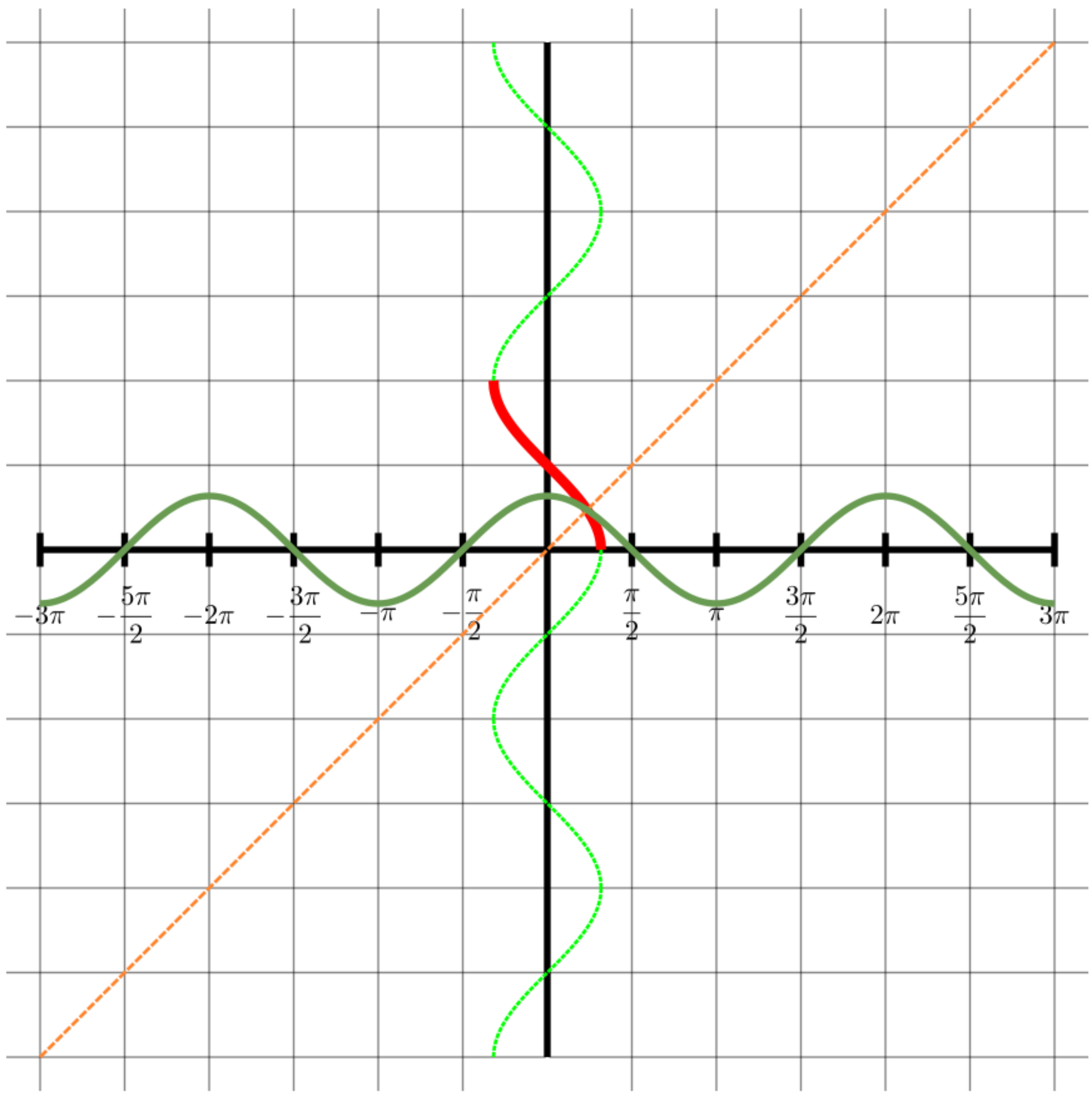
Draw the graph $x = \cos y$.



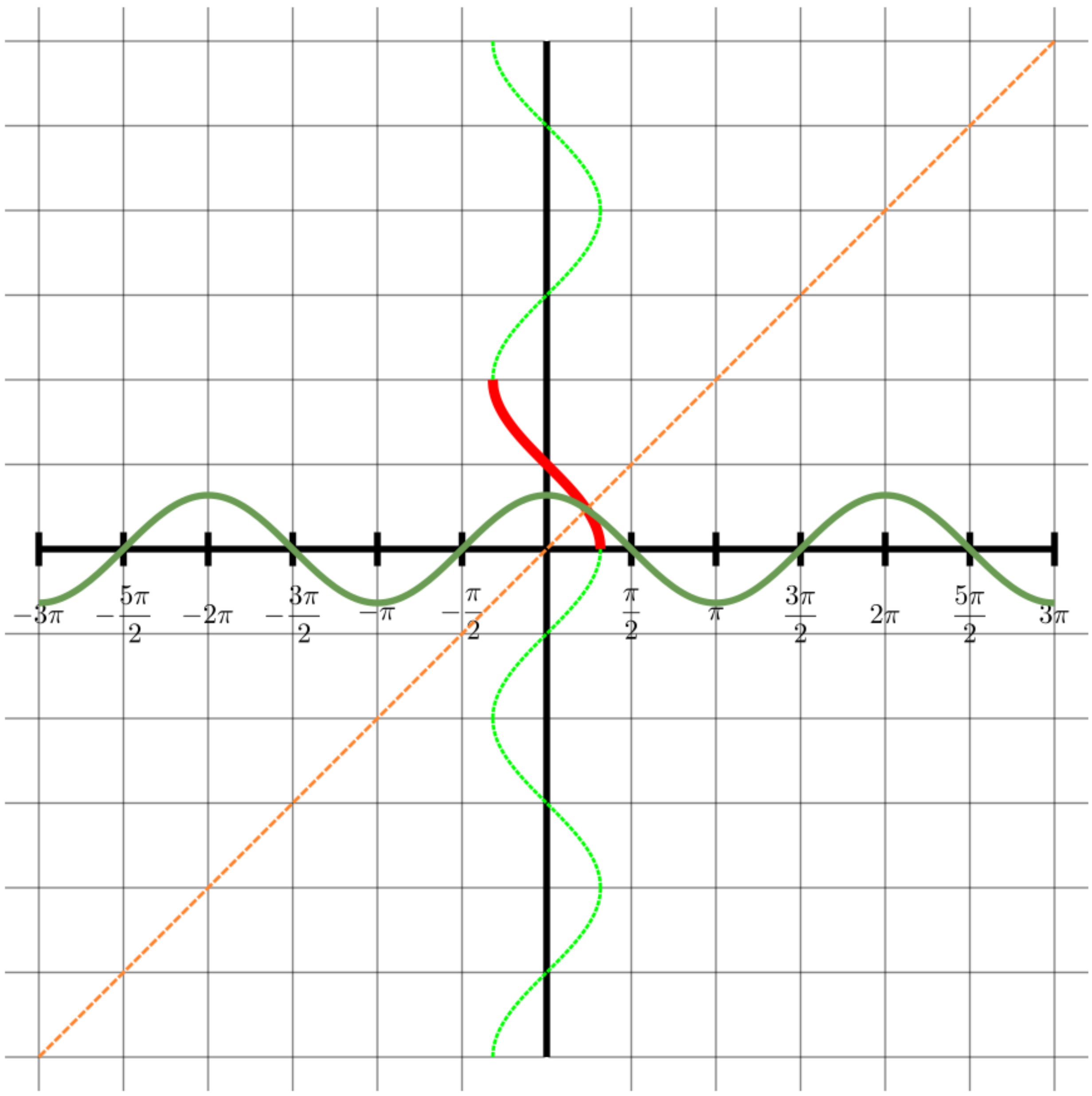
On the same axes, draw the graph $y = \cos^{-1} x$.



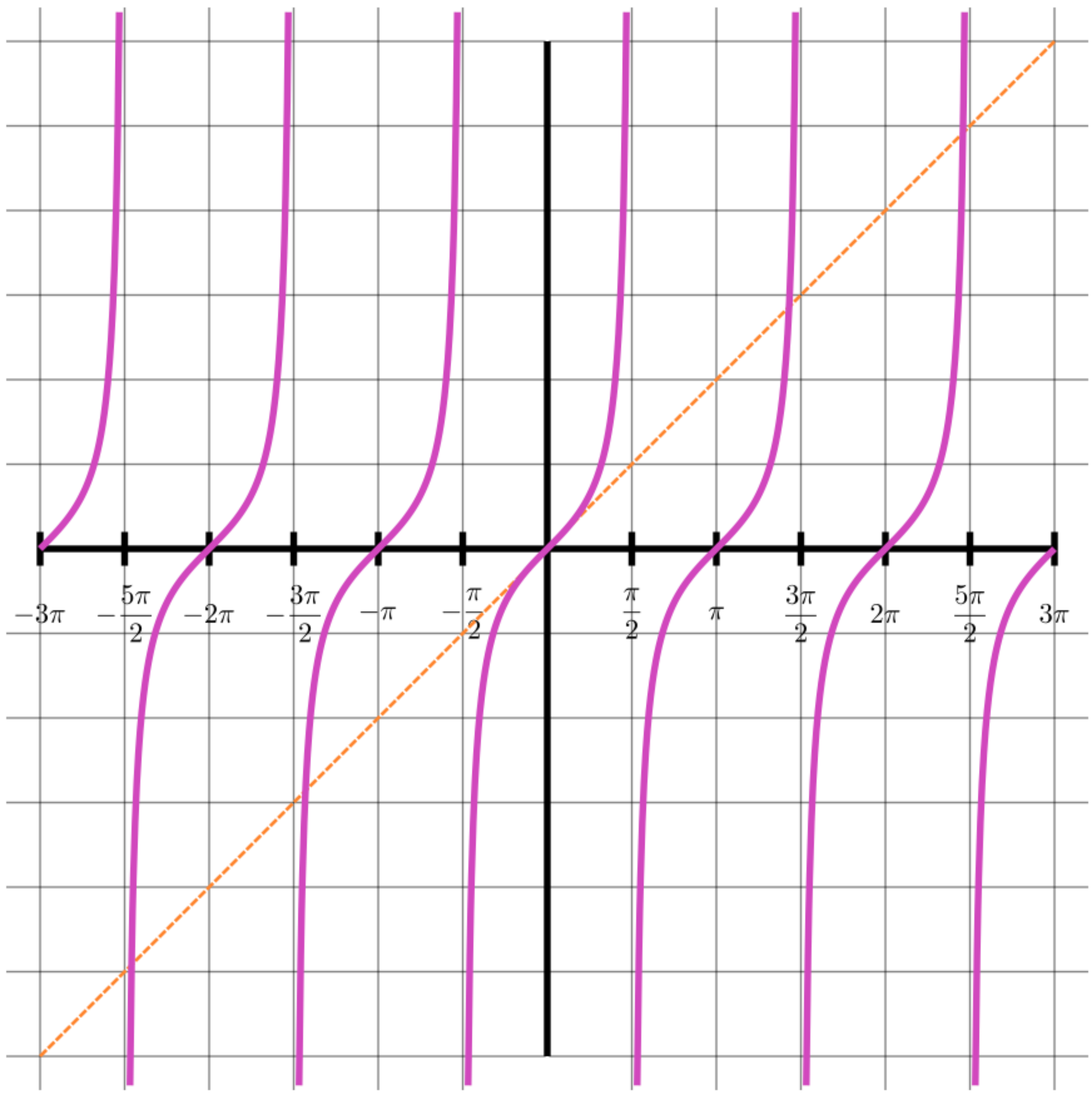
What are the domain and range of the function $f(x) = \cos^{-1} x$



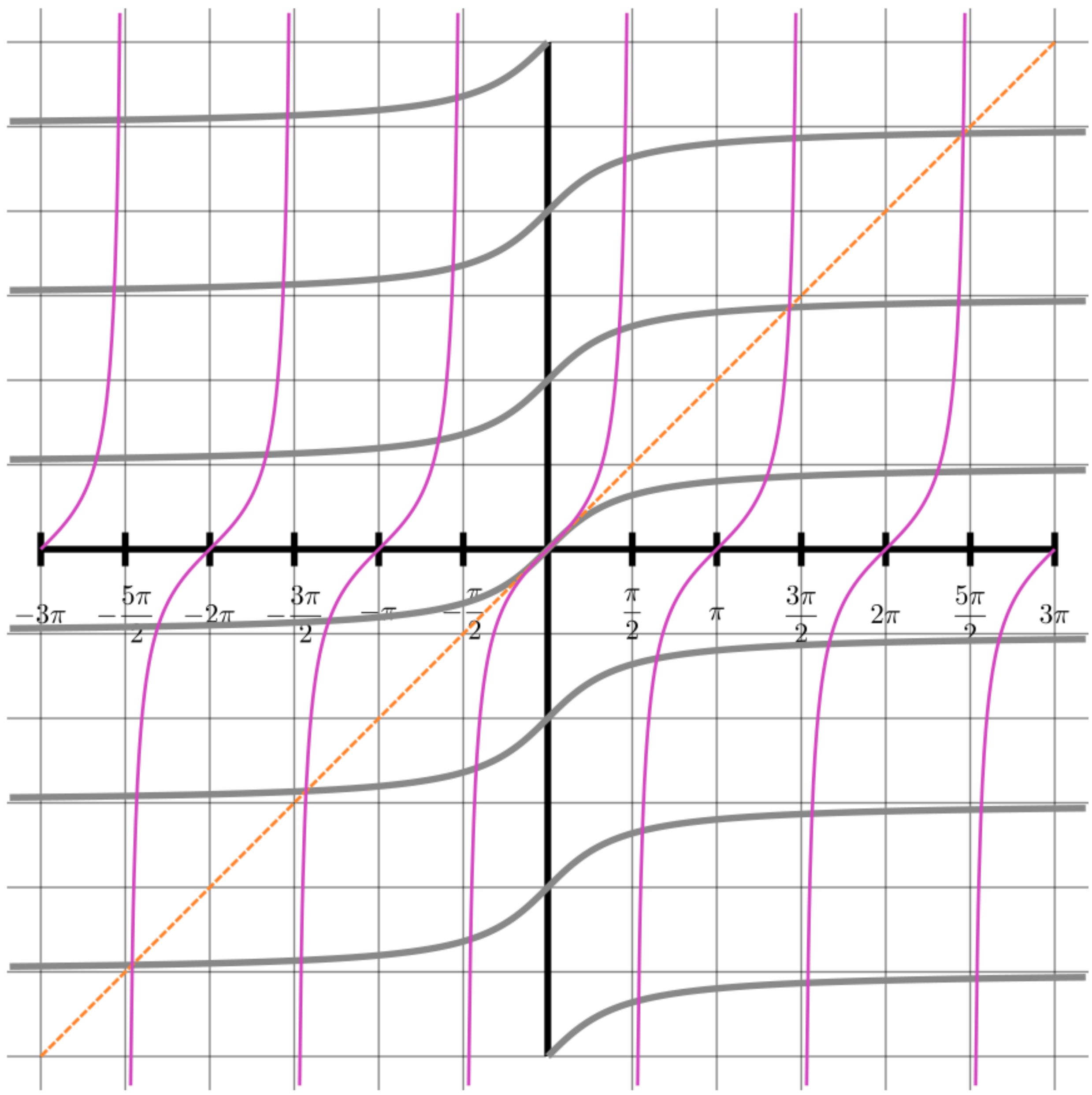
What are $\cos\left(\cos^{-1} x\right)$ and $\cos^{-1}(\cos x)$?



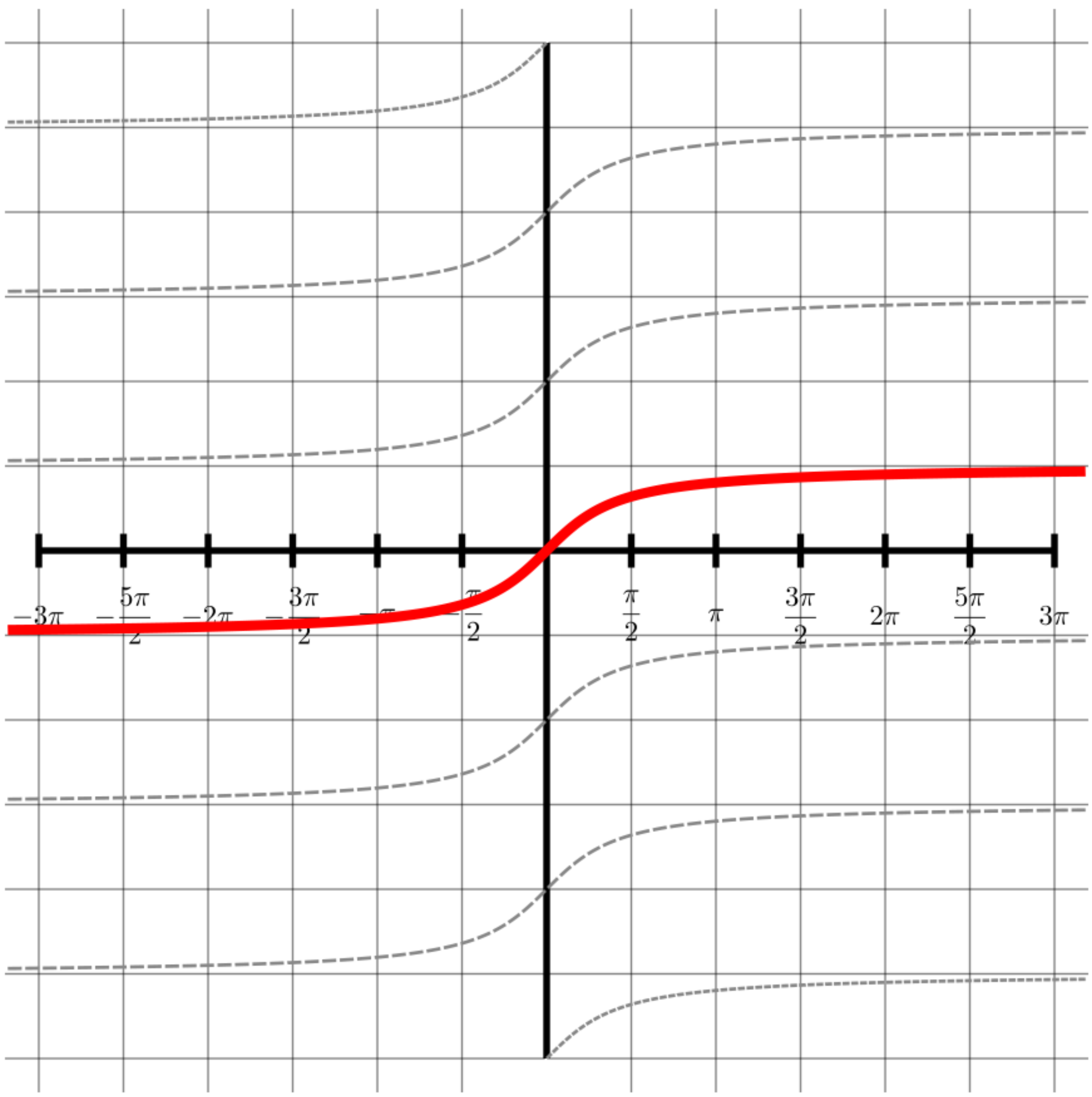
Draw the graph $x = \tan y$.



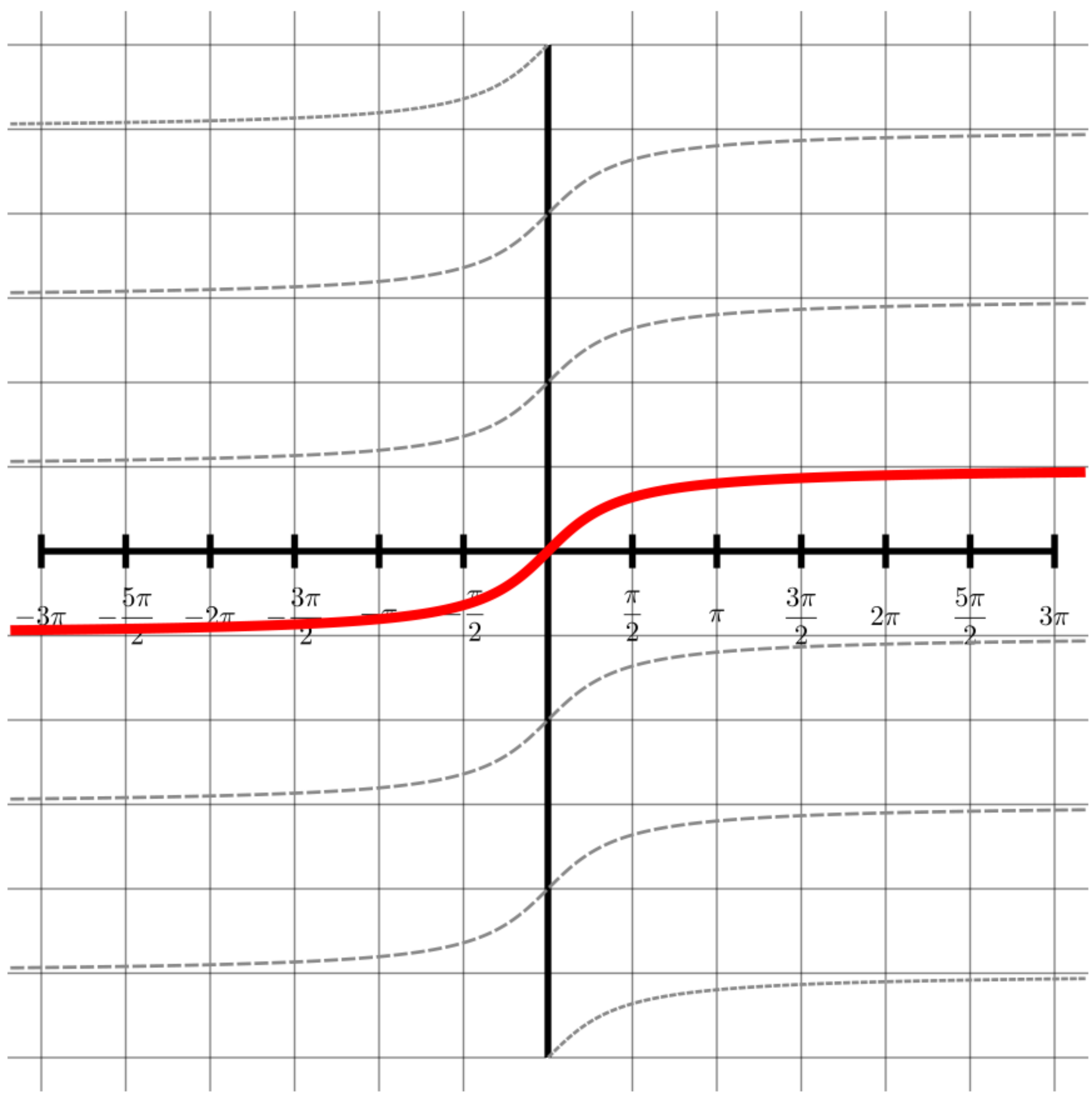
On the same axes, draw the graph $y = \tan^{-1} x$.



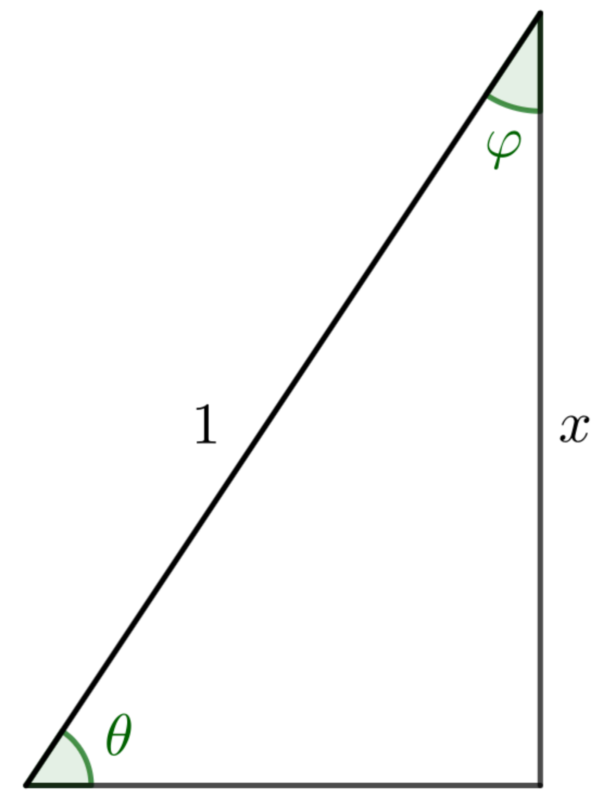
What are the domain and range of the function $f(x) = \tan^{-1} x$

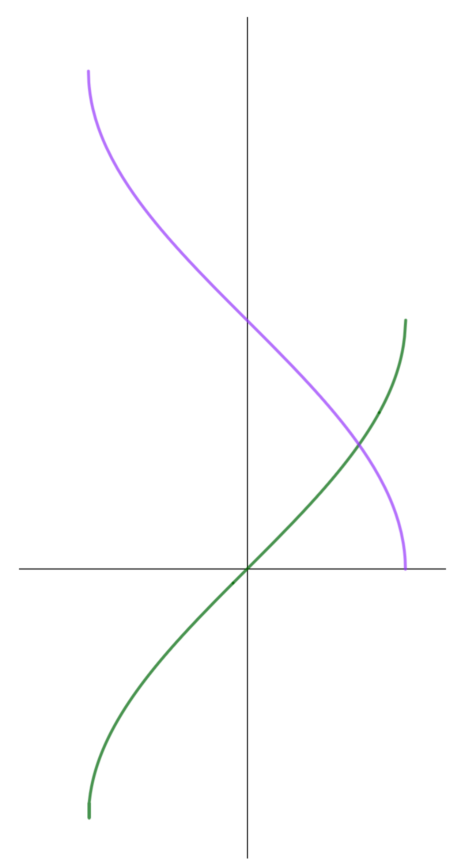


What are $\tan\left(\tan^{-1} x\right)$ and $\tan^{-1}(\tan x)$?



What is $\sin^{-1} x + \cos^{-1} x$?





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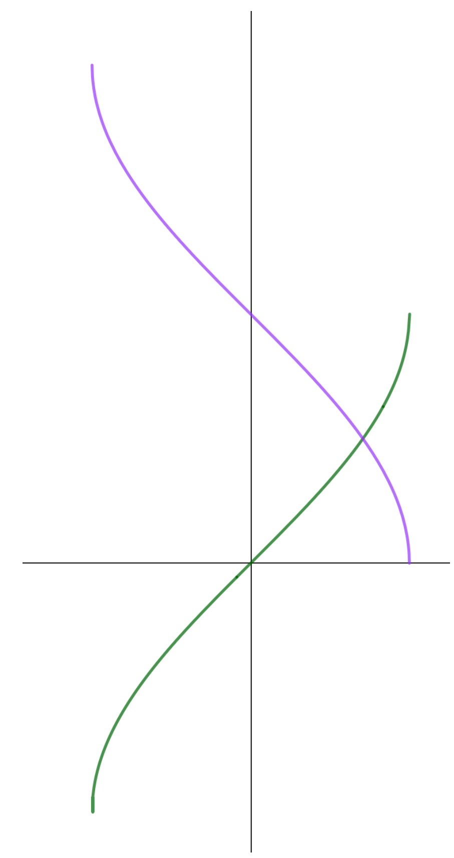
Here are the graphs $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

Where do they cross?



radical maths

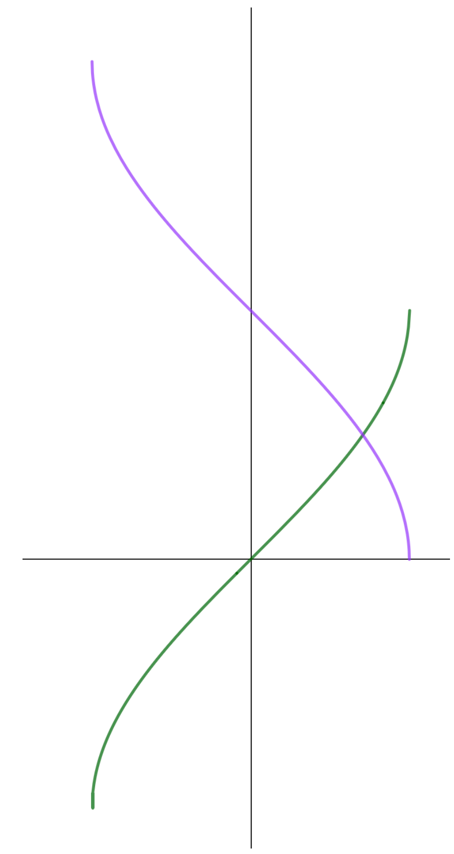
What are

$\sin^{-1} 0 + \cos^{-1} 0$

$\sin^{-1} 1 + \cos^{-1} 1$

$\sin^{-1}(-1) + \cos^{-1}(-1)$

$$\sin^{-1} \frac{\sqrt{2}}{2} + \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



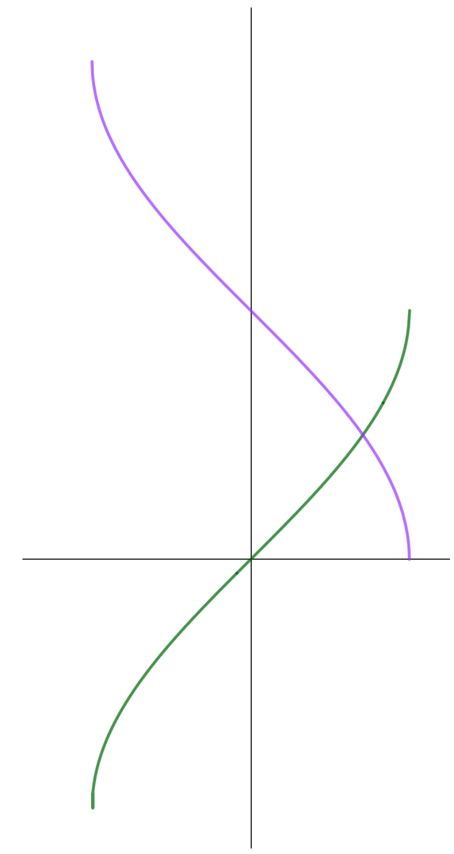
Draw any vertical line that intersects both of the curves.

What is the average of the y coordinates of the intersection points?

What is the sum of the y coordinates of the intersection points?

What is

$\sin^{-1} x + \cos^{-1} x$?



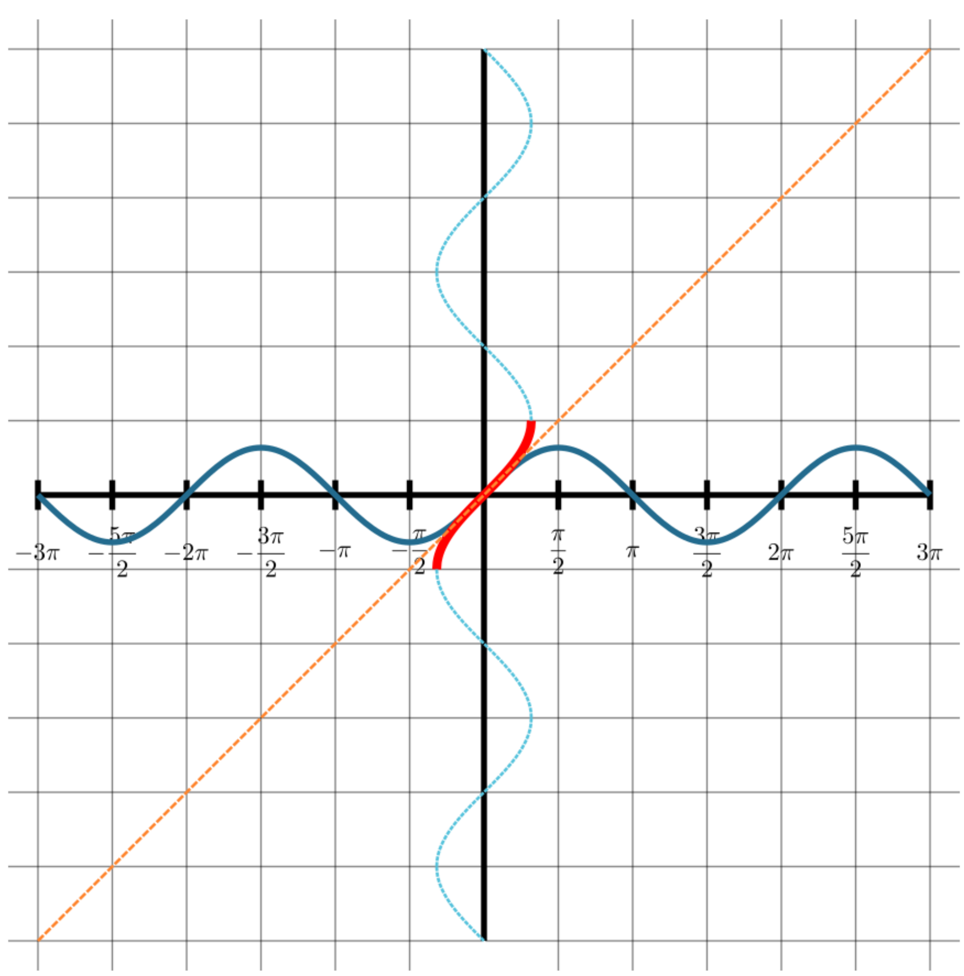
differentials of inverse circular functions

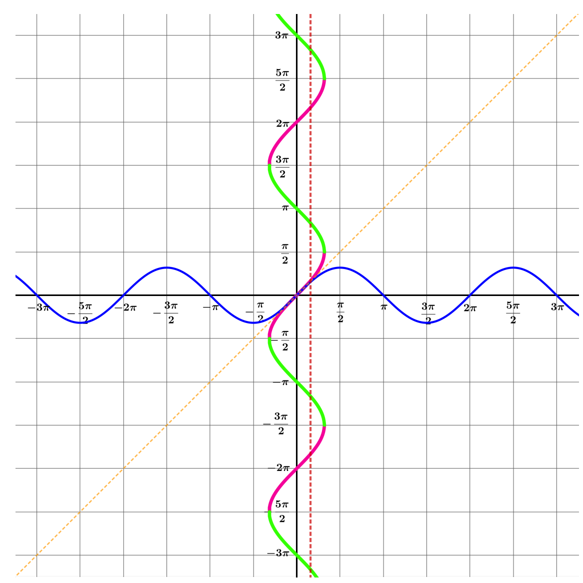
If $x = \sin y$, what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y ?

Use this to what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of x ?

Use this to what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x ?

What is $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x$?



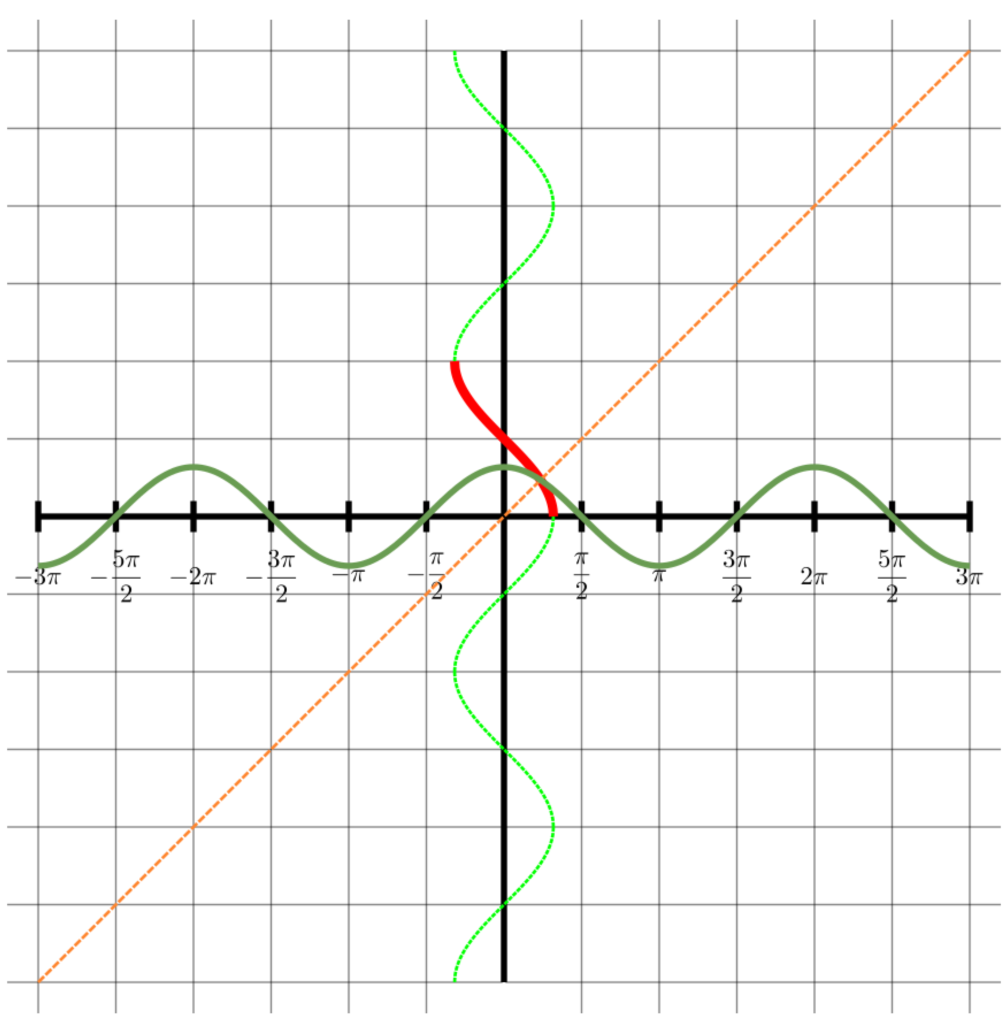

3π
5π/2
2π
3π/2
π
π/2
−3π
−5π/2
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
5π/2
3π
−π/2
−π
−3π/2
−2π
−5π/2
−3π

If $x = \cos y$, what is $\dfrac{dx}{dy}$ in terms of y ?

Use this to what is $\dfrac{dx}{dy}$ in terms of x ?

Use this to what is $\dfrac{dy}{dx}$ in terms of x ?

What is $\dfrac{d}{dx} \cos^{-1} x$?

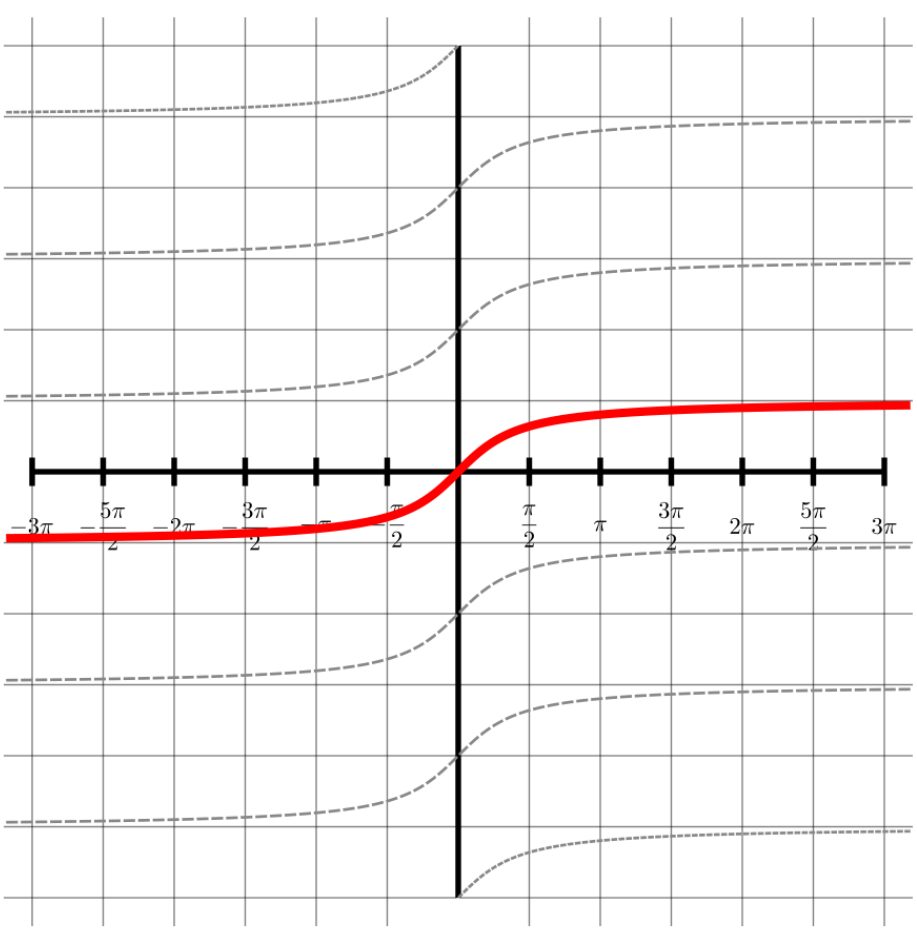


If $x = \tan y$, what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y ?

Use this to what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of x ?

Use this to what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x ?

What is $\dfrac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x$?



integrals using inverse circular functions

Use the substitution $x = \sin u$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

If $y = \sin^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

If $y = -\sin^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

Use the substitution $u = \sin^{-1} x$ for this integral:

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

Use the substitution $x = \cos u$ for this integral:

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

If $y = \cos^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

If $y = -\cos^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

Use the substitution $u = \cos^{-1} x$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x$$

Show that $\sin^{-1} x + c$ and $-\cos^{-1} x + c$ are equivalent solutions for the integral

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x$$

Use the substitution $x = \tan u$ for this integral:

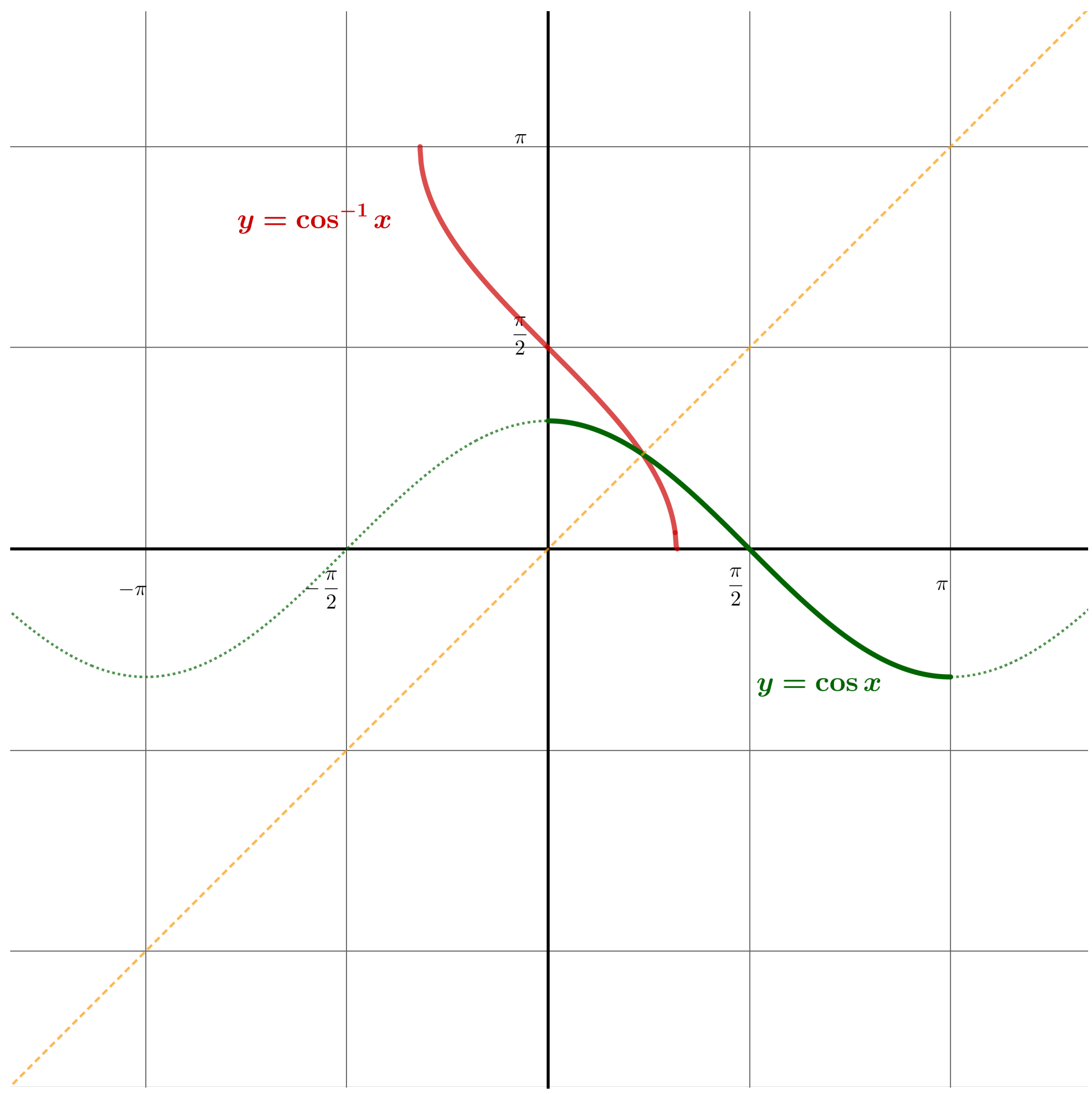
$$\int \frac{1}{1+x^2}\,\mathrm{d}x$$

Use the substitution $u = \tan^{-1} x$ for this integral:

$$\int \frac{1}{1+x^2}\,\mathrm{d}x$$

inverse circular functions: extension

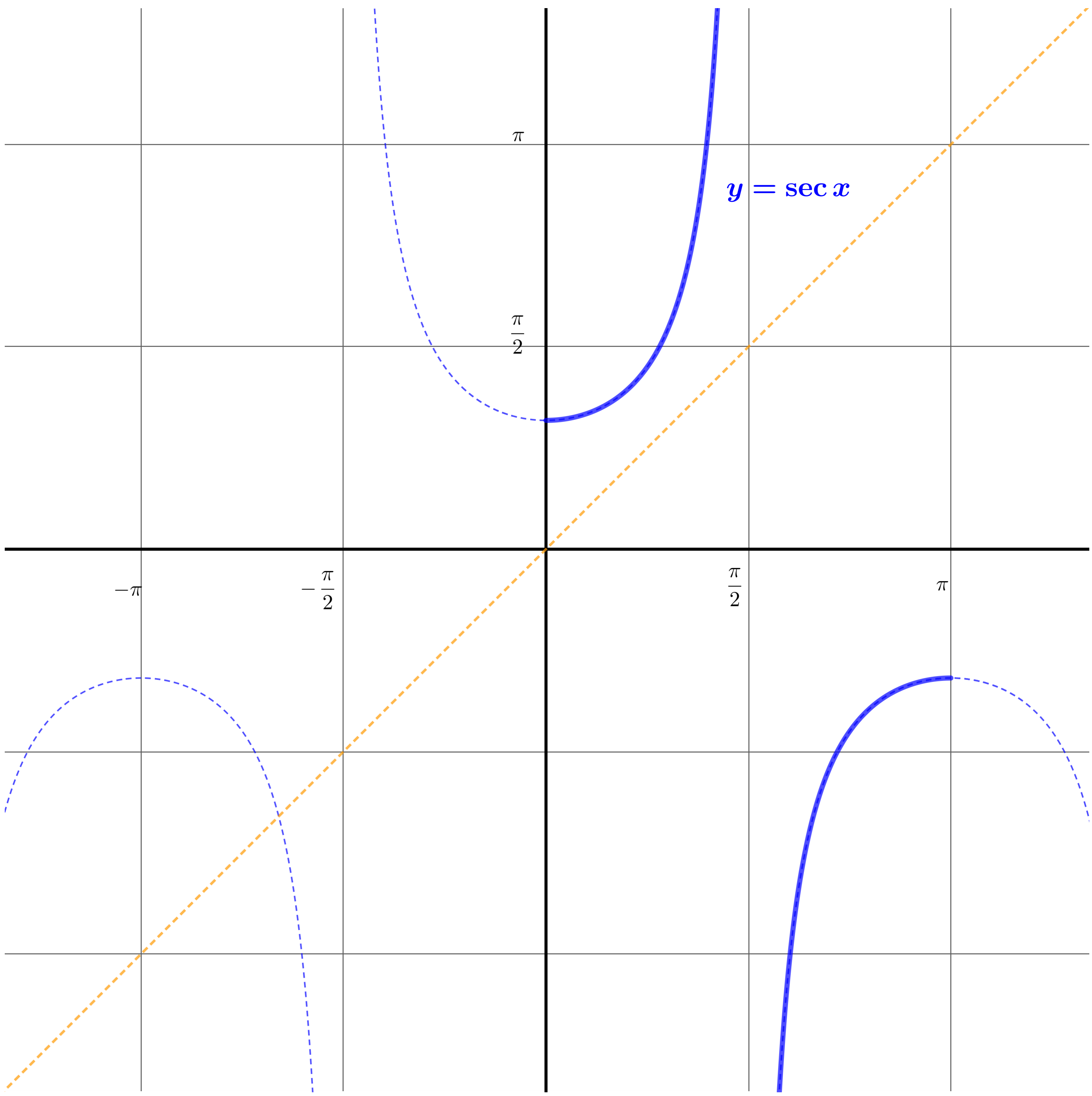
What are the domain and range of $f(x) = \cos^{-1} x$?



What are the domain and range of $f(x) = \sec^{-1} x$?

Here is the graph $y = \sec x$ over the domain
$$\left\{ x : 0 \le x \le \pi,\ x \ne \frac{\pi}{2} \right\}$$

Draw the graph $y = \sec^{-1} x$.

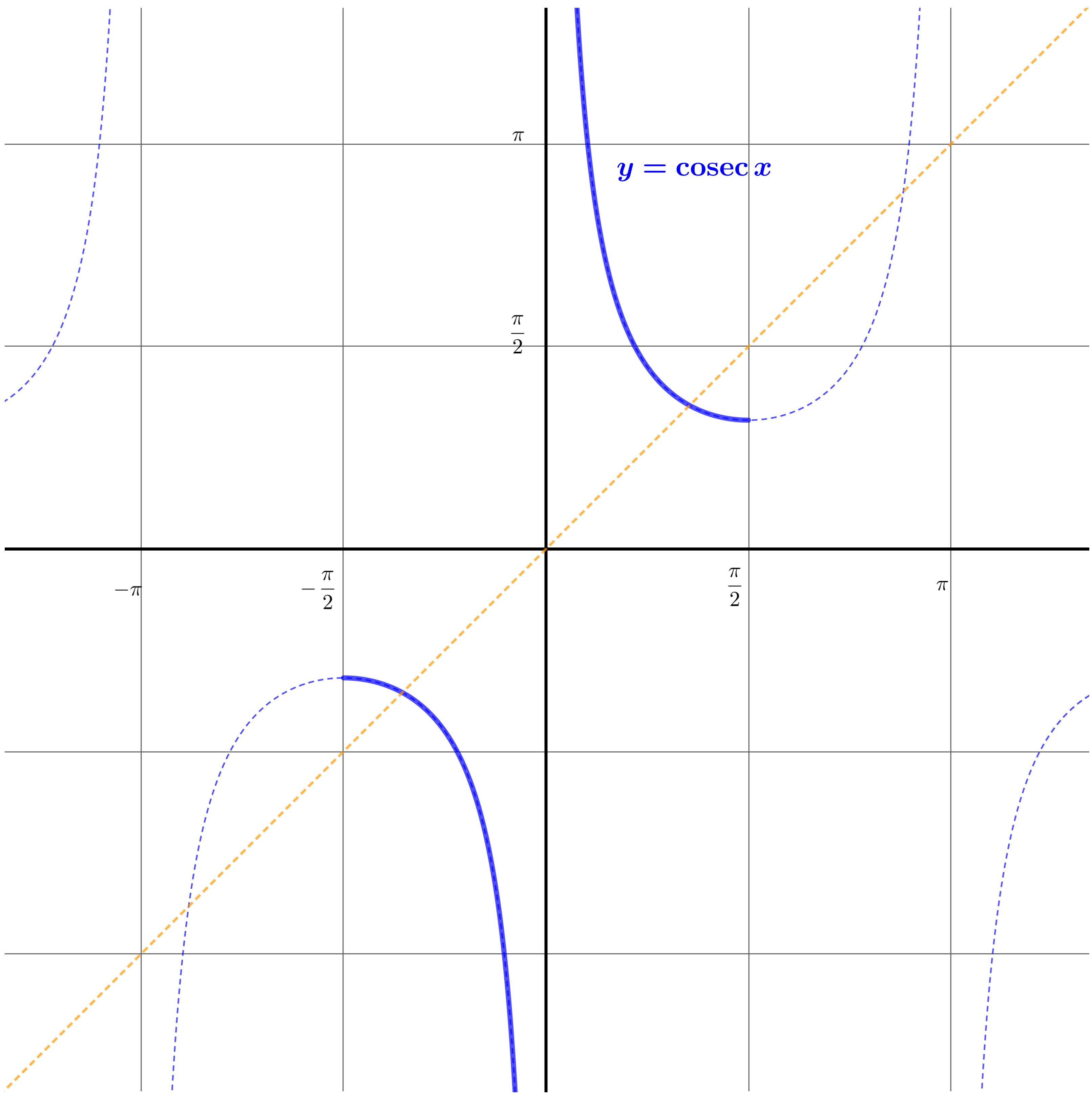


radical maths

Here is the graph $y = \operatorname{cosec} x$ over the domain

$$\left\{ x : -\frac{\pi}{2} \le x \le \frac{\pi}{2},\, x \ne 0 \right\}$$

Draw the graph $y = \operatorname{cosec}^{-1} x$.

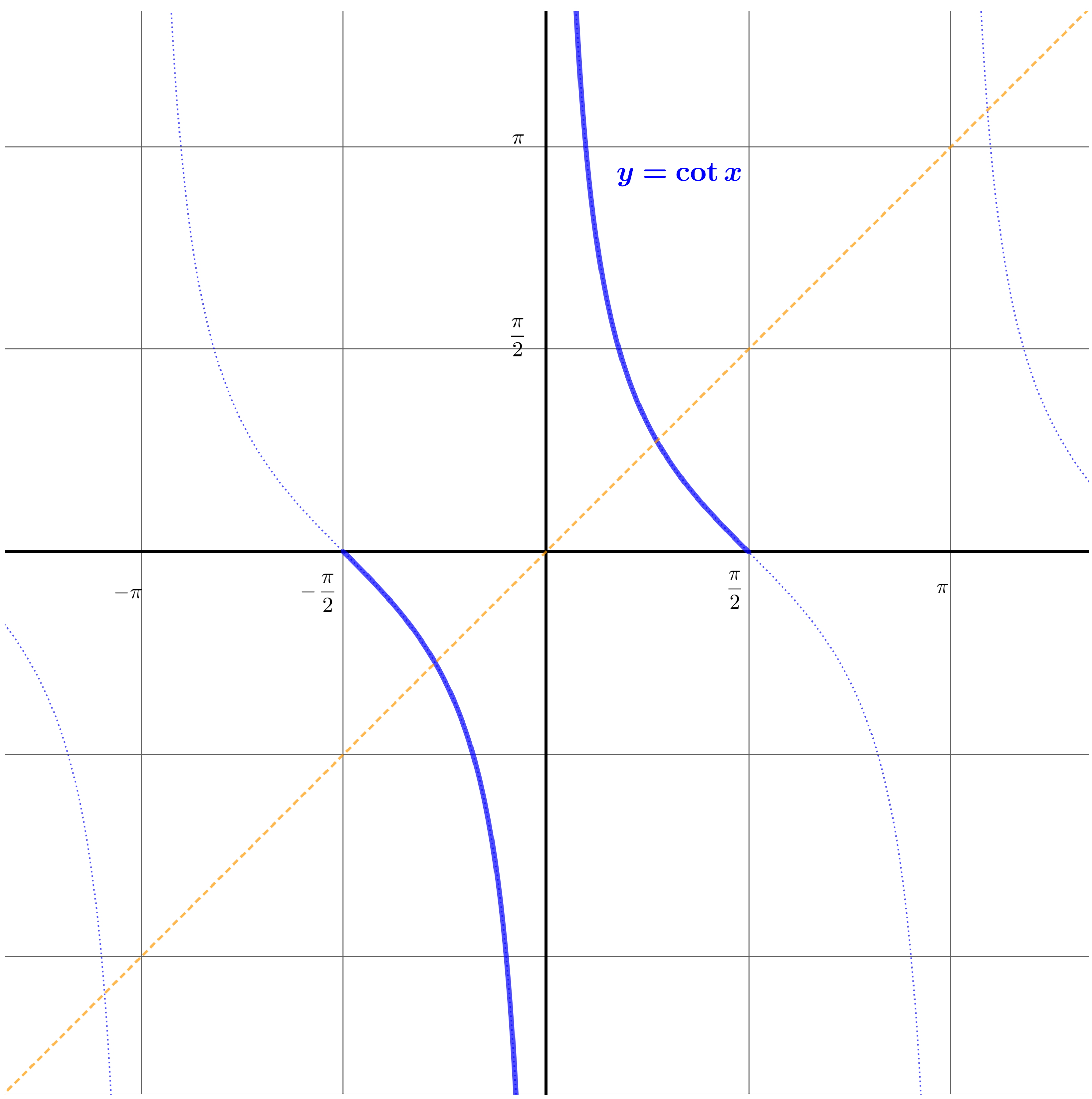


radical maths

Here is the graph $y = \cot x$ over the domain

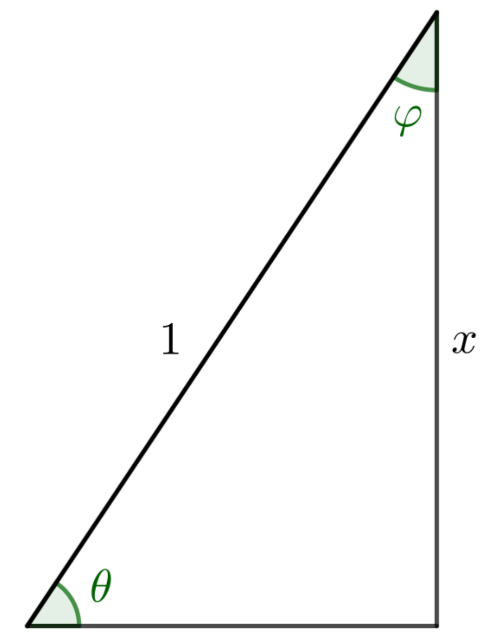
$$\left\{x : -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0\right\}$$

Draw the graph $y = \cot^{-1} x$.

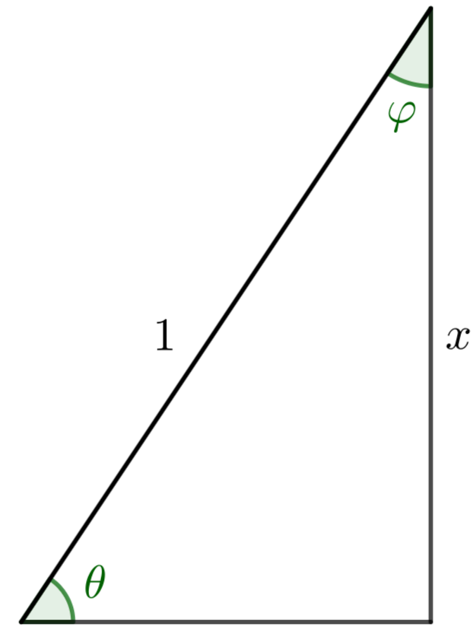


radical maths

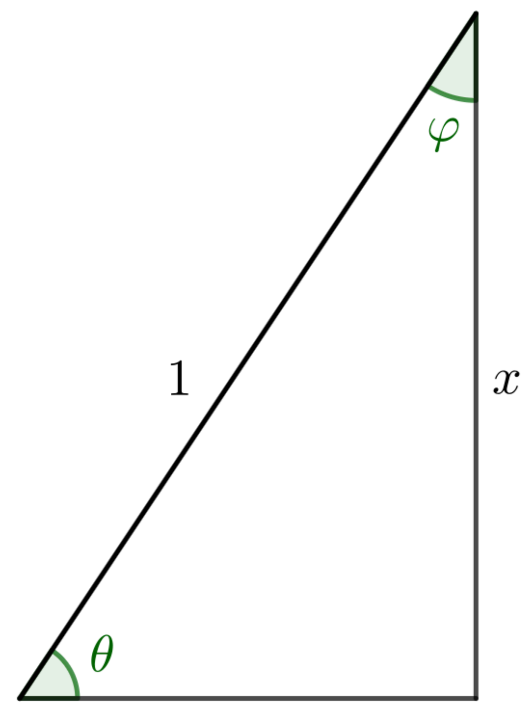
What is $\cos\left(\sin^{-1} x\right)$?



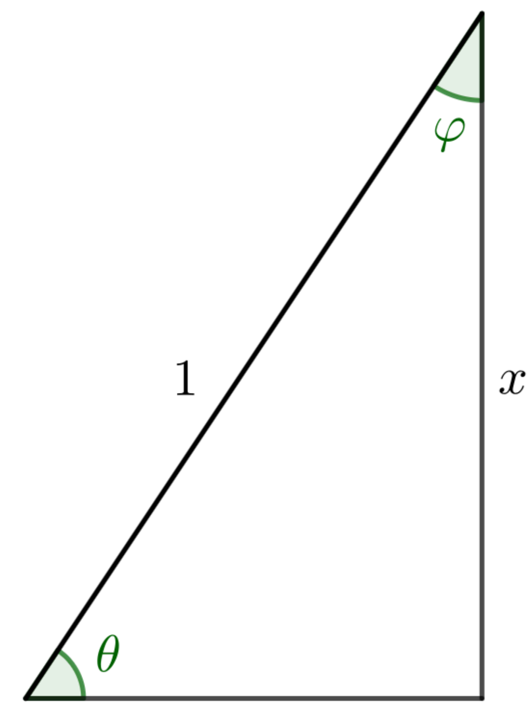
What is $\cos^{-1}\left(\sin\theta\right)$?



What are $\sin\left(\cos^{-1} x\right)$ and $\sin^{-1}(\cos \varphi)$?

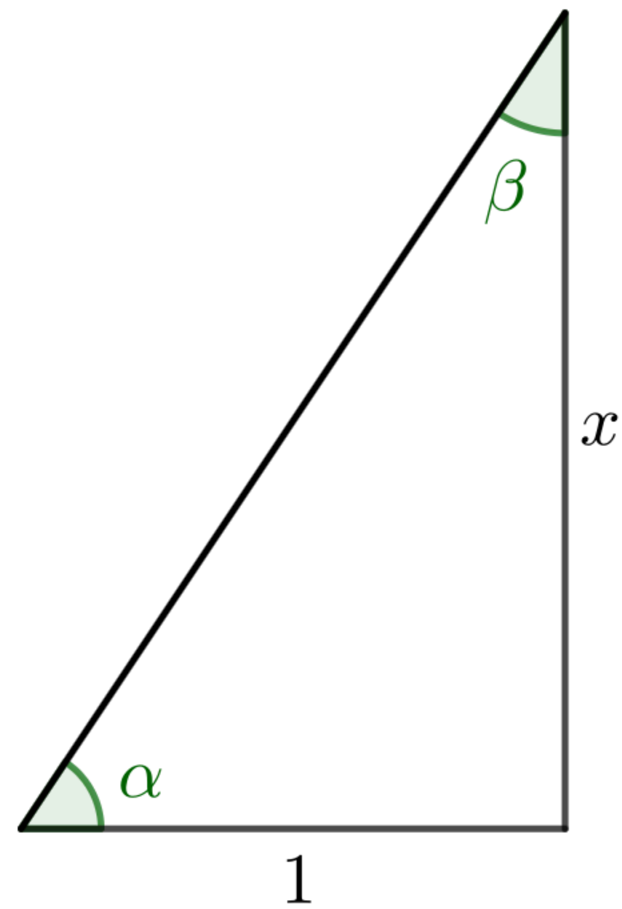


What are $\tan\left(\sin^{-1} x\right)$ and $\tan\left(\cos^{-1} x\right)$?



radical maths

What are $\sin\left(\tan^{-1} x\right)$ and $\cos\left(\tan^{-1} x\right)$?



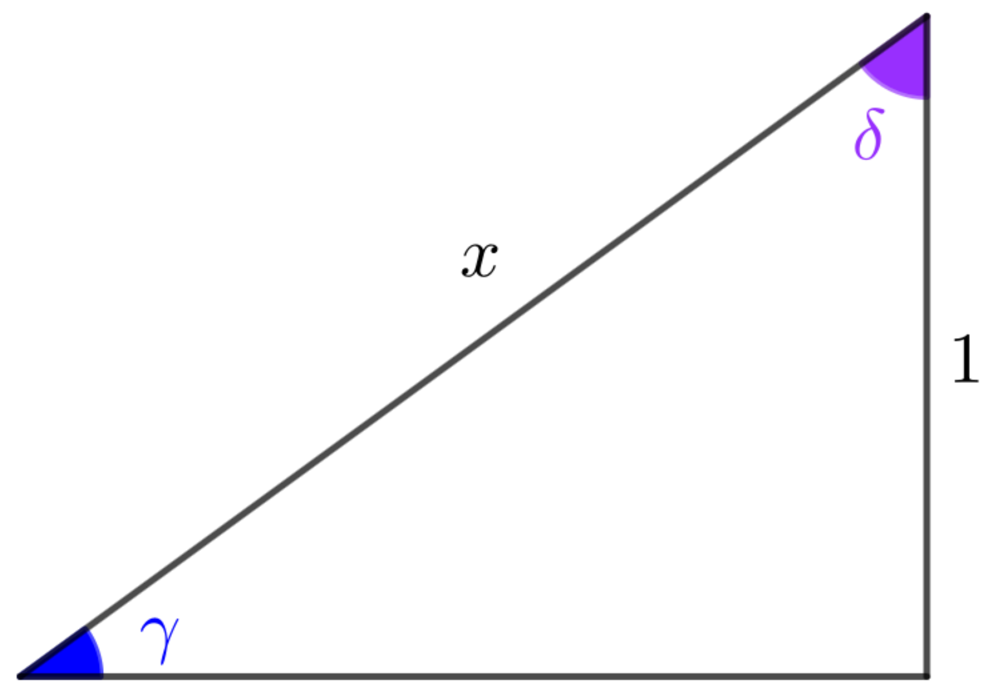
Simplify

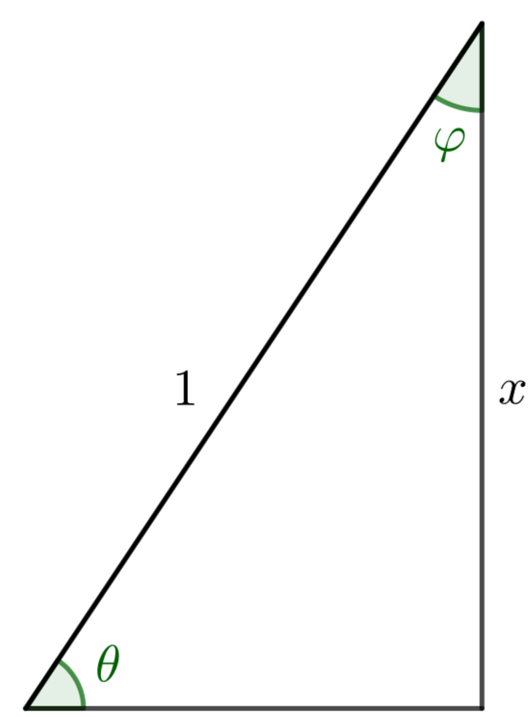
$\sin\left(\operatorname{cosec}^{-1} x\right)$

$\operatorname{cosec}\left(\sin^{-1} x\right)$

$\cos\left(\sec^{-1} x\right)$

$\sec\left(\cos^{-1} x\right)$

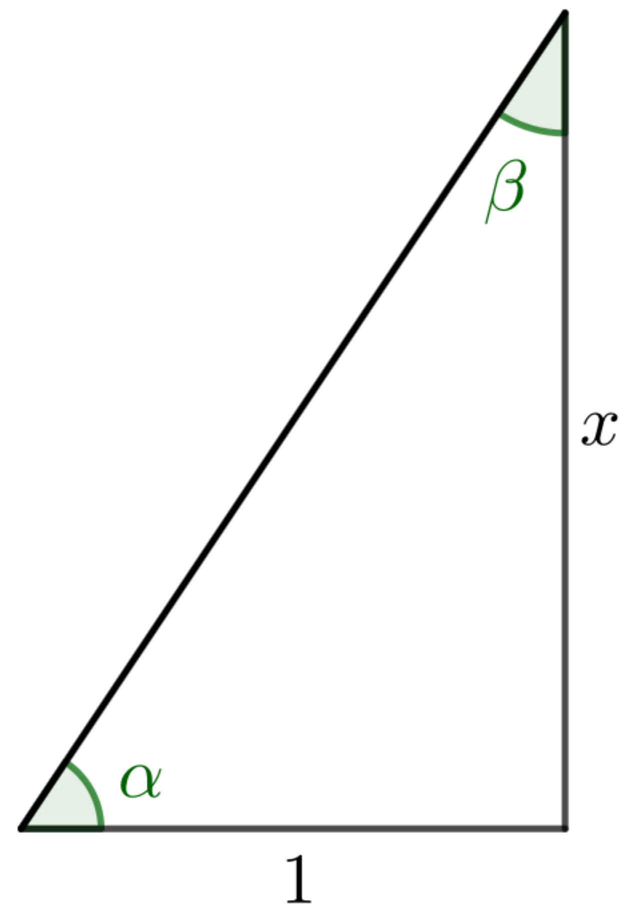




Simplify

$\tan\left(\cot^{-1} x\right)$

$\cot\left(\tan^{-1} x\right)$



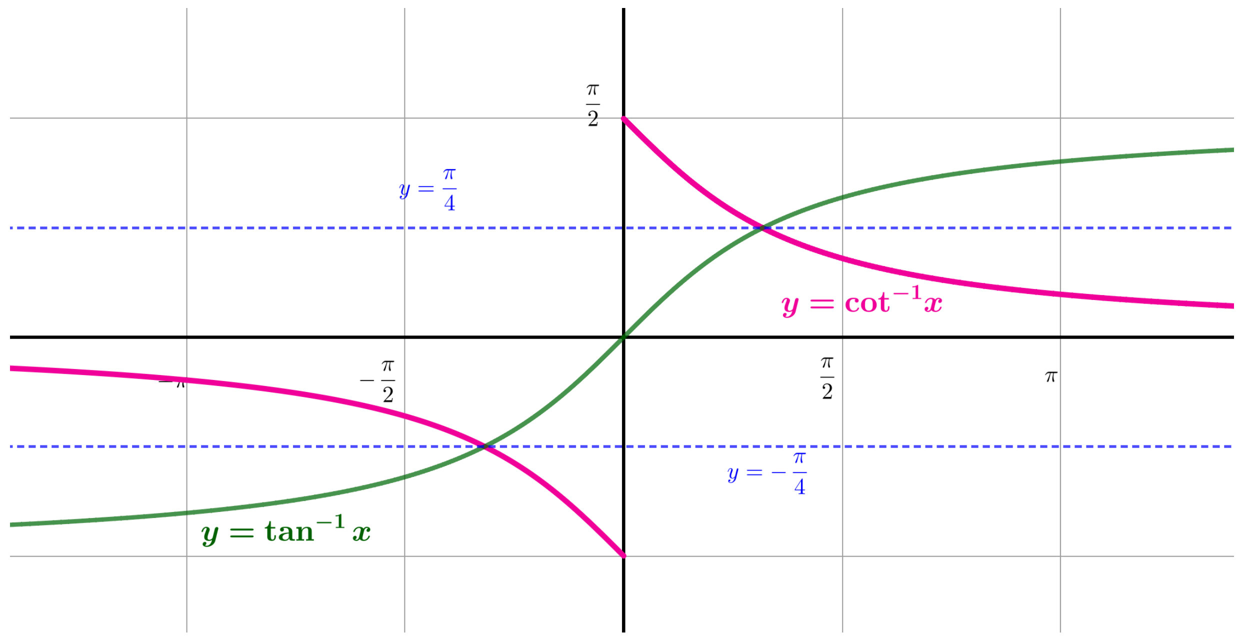
radical maths

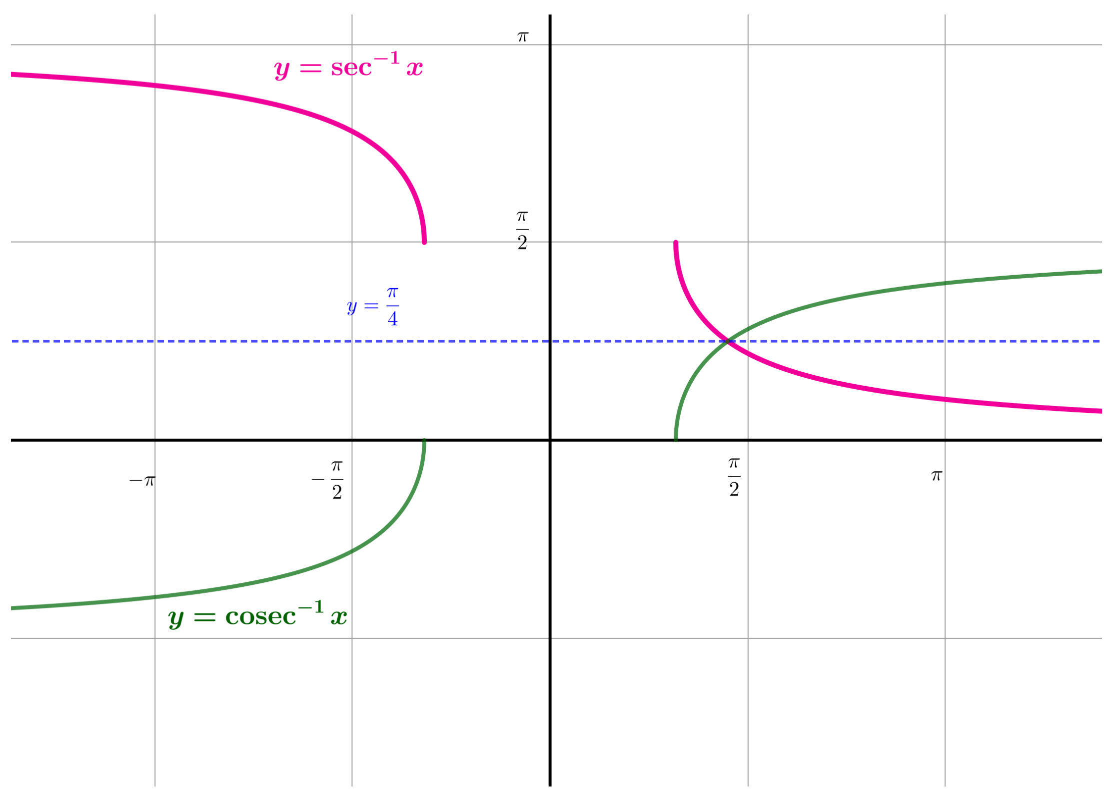
What are

$\cot^{-1} x + \tan^{-1} x$

and

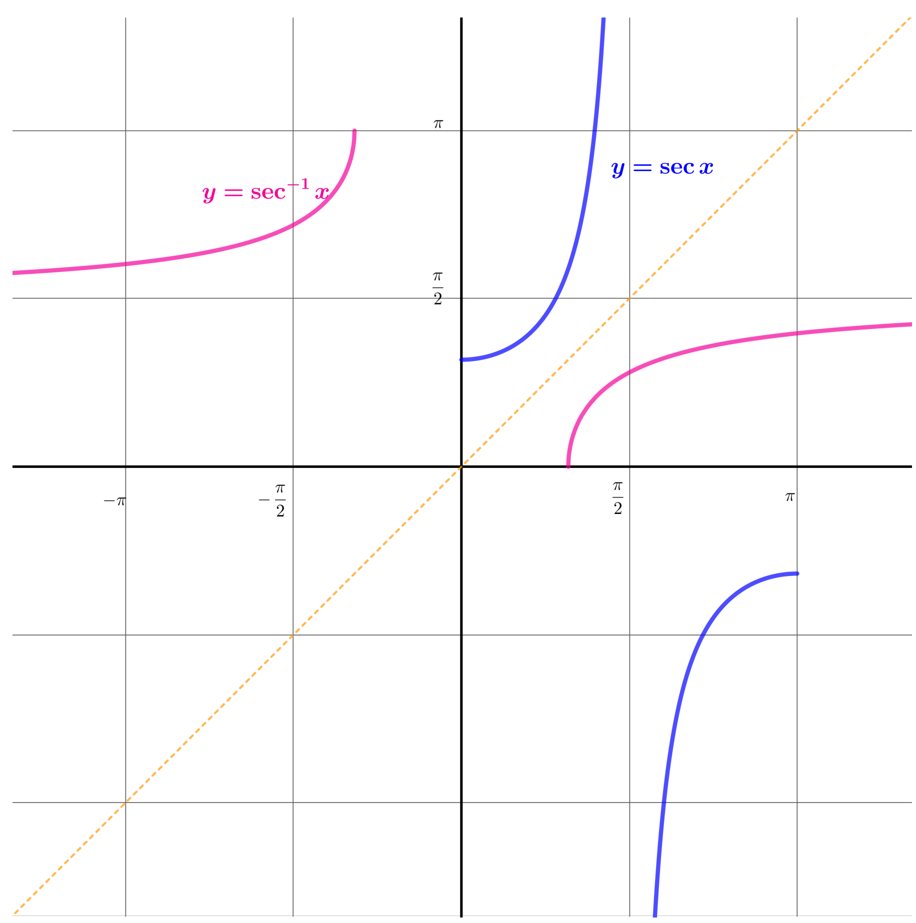
$\sec^{-1} x + \operatorname{cosec}^{-1} x$?





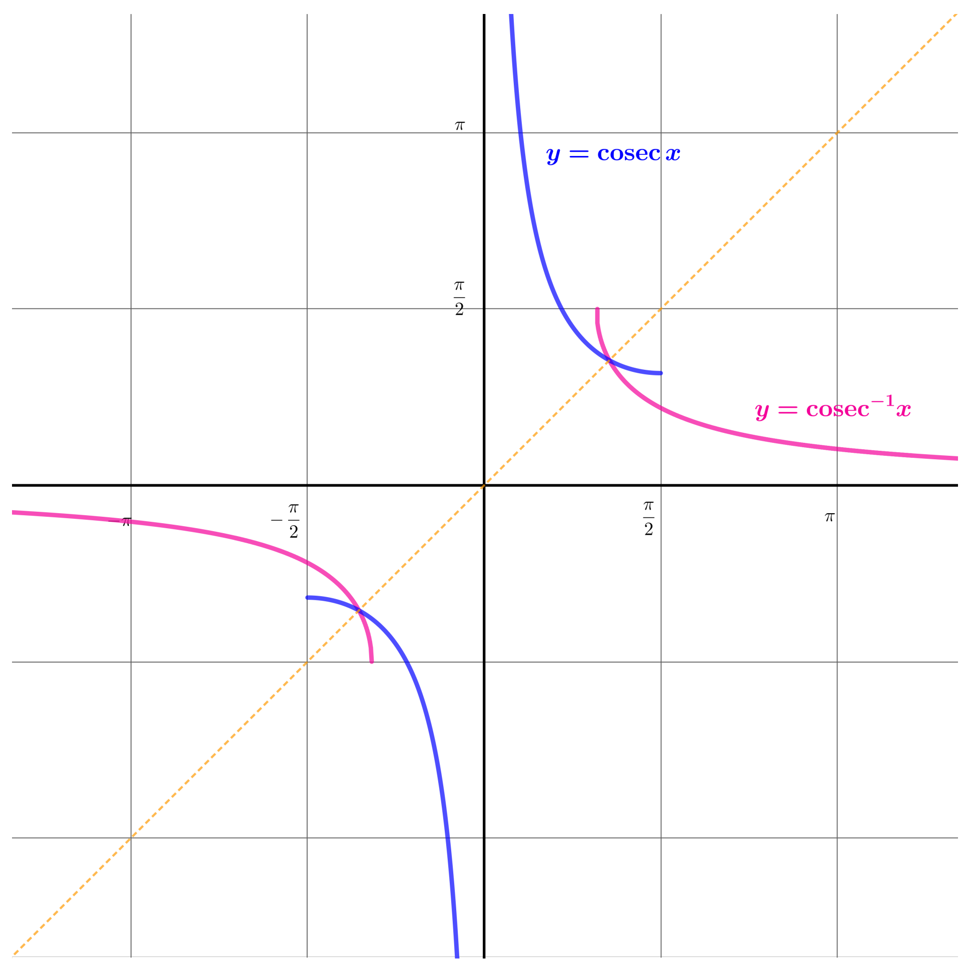
more differentials of inverse circular functions

What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\sec^{-1} x$?



radical maths

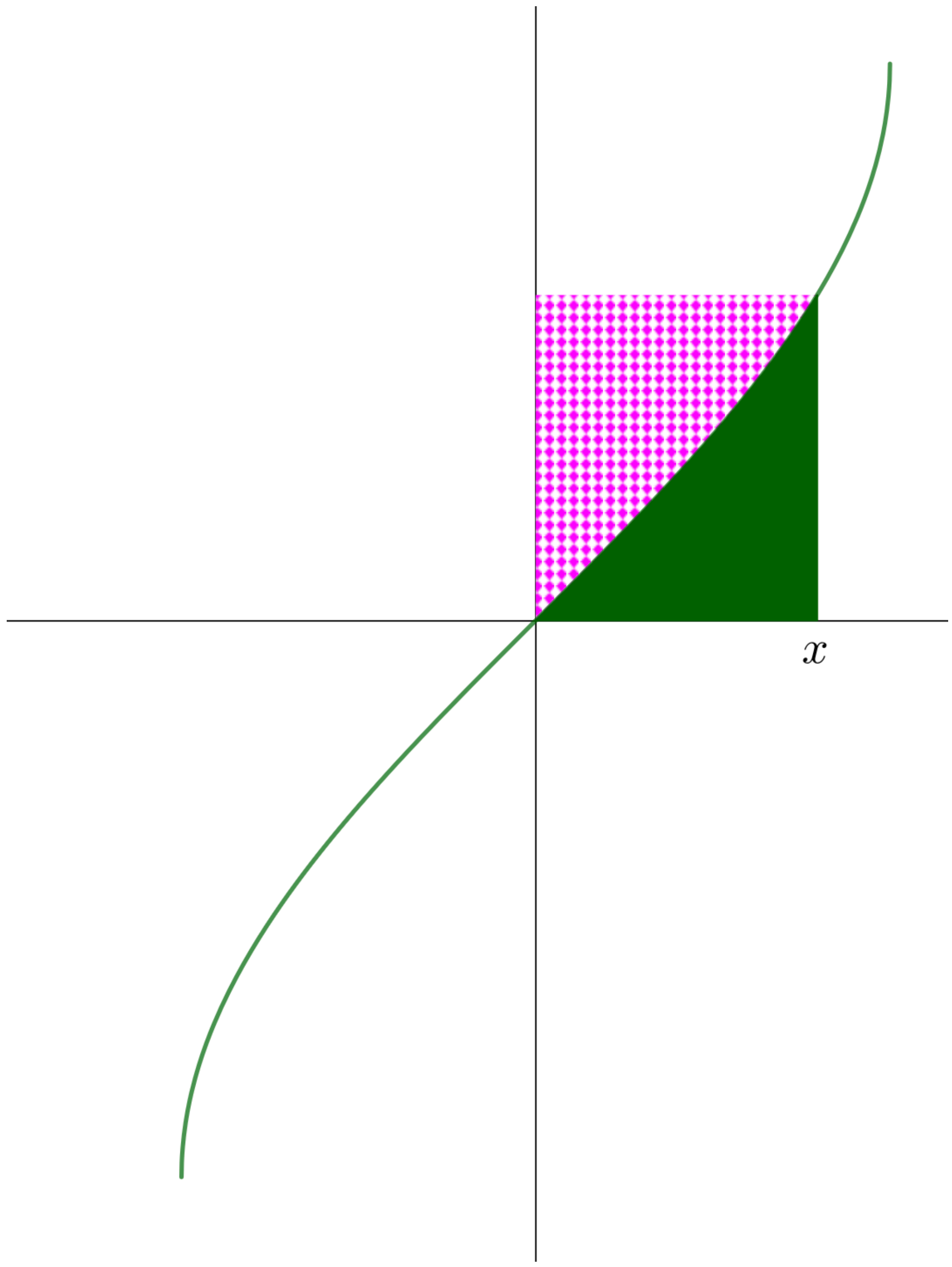
What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec}^{-1} x$?



What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x$?

radical maths

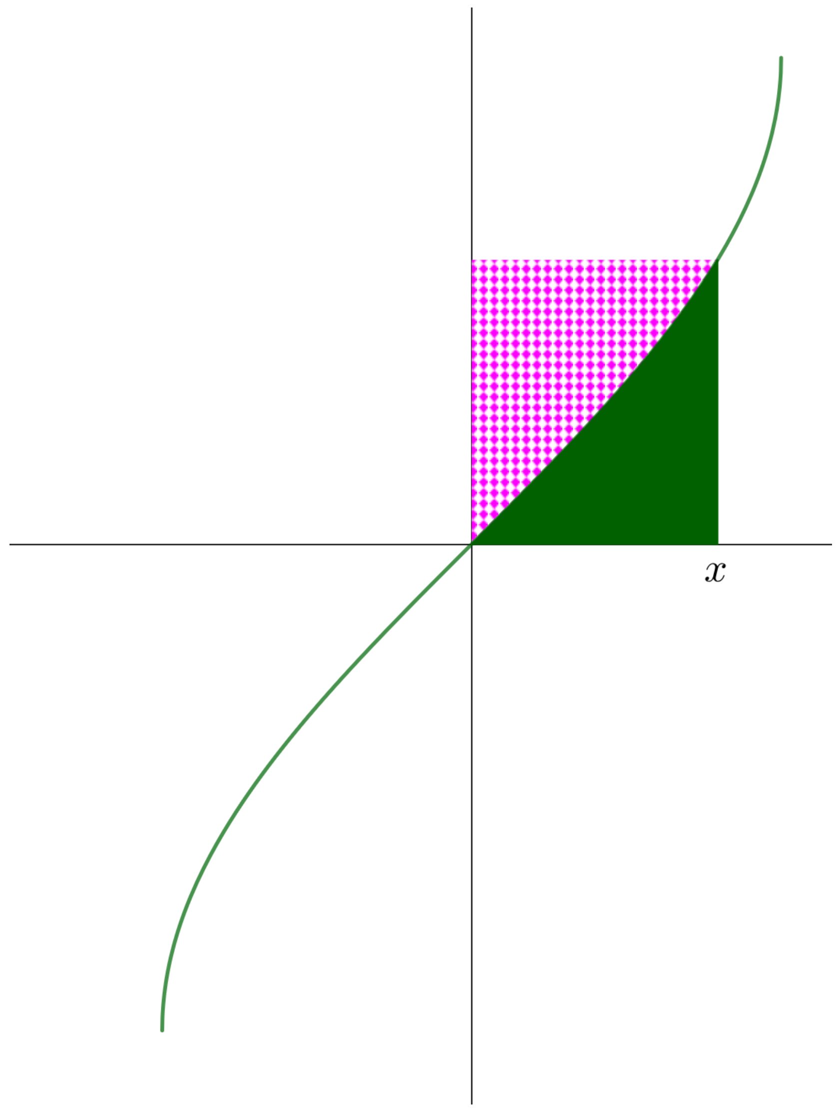
integrals of inverse circular functions



What is the area of the whole shaded rectangle?

Write the two shaded areas as integrals.

What is the area of the pink (chequered) area?



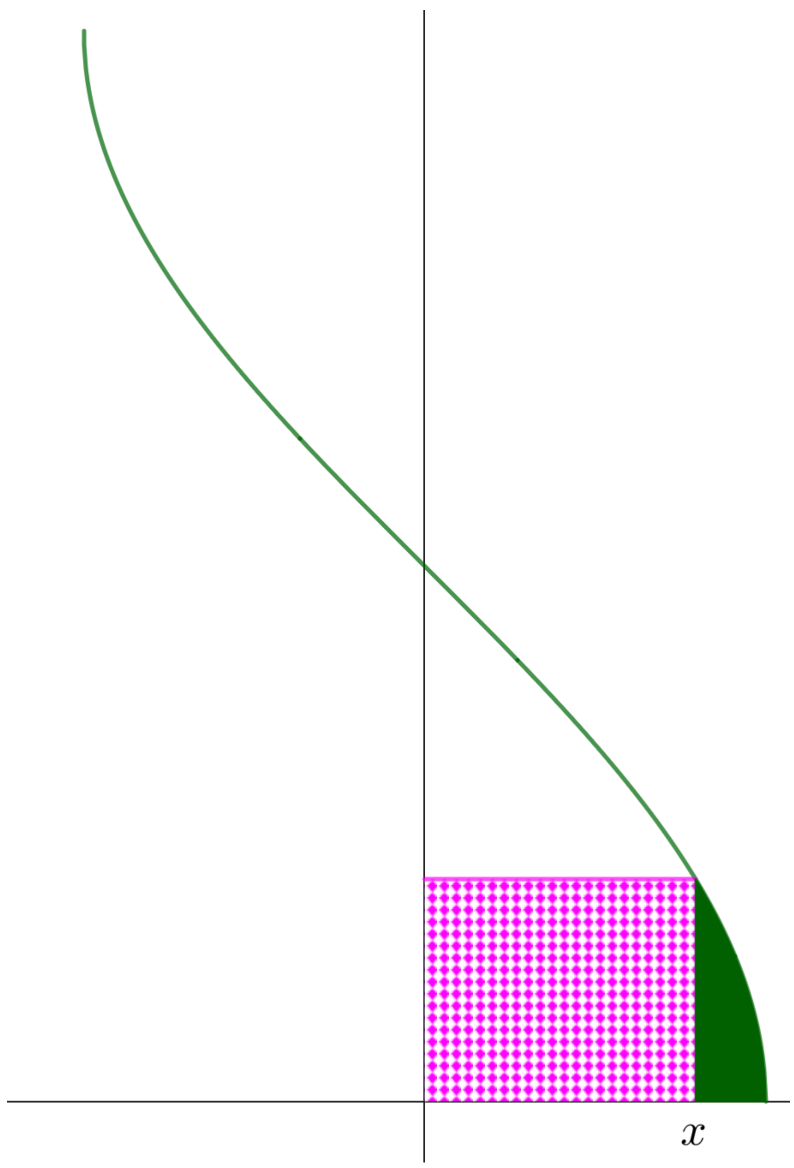
Use these results to find

$$\int_0^x \sin^{-1} x \, \mathrm{d}x$$

and

$$\int \sin^{-1} x \, \mathrm{d}x$$

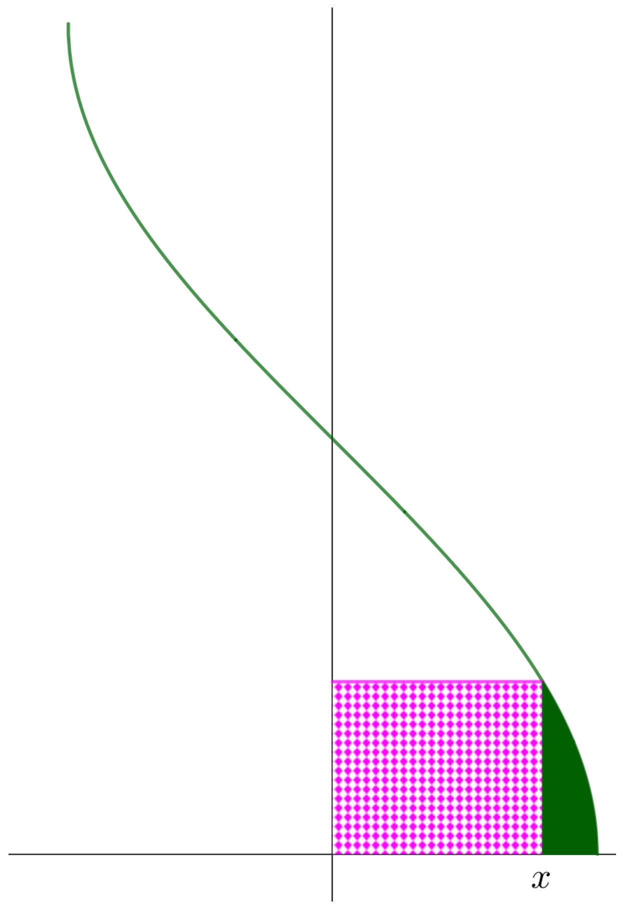
Here is the graph $y = \cos^{-1} x$.



What is the area of the pink (chequered) rectangle?

Write the entire shaded area as an integral.

Write the green (solid) shaded area as an integral.



Use these results to find

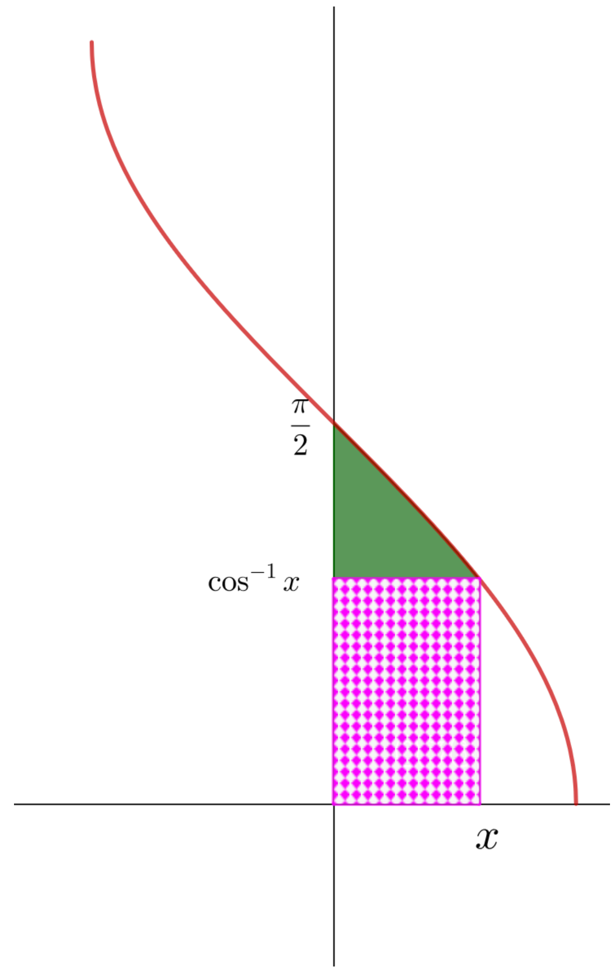
$$\int_x^1 \cos^{-1} x \, \mathrm{d}x$$

and

$$\int \cos^{-1} x \, \mathrm{d}x$$

radical maths

Here is the graph $y = \cos^{-1} x$.



Use these results to find

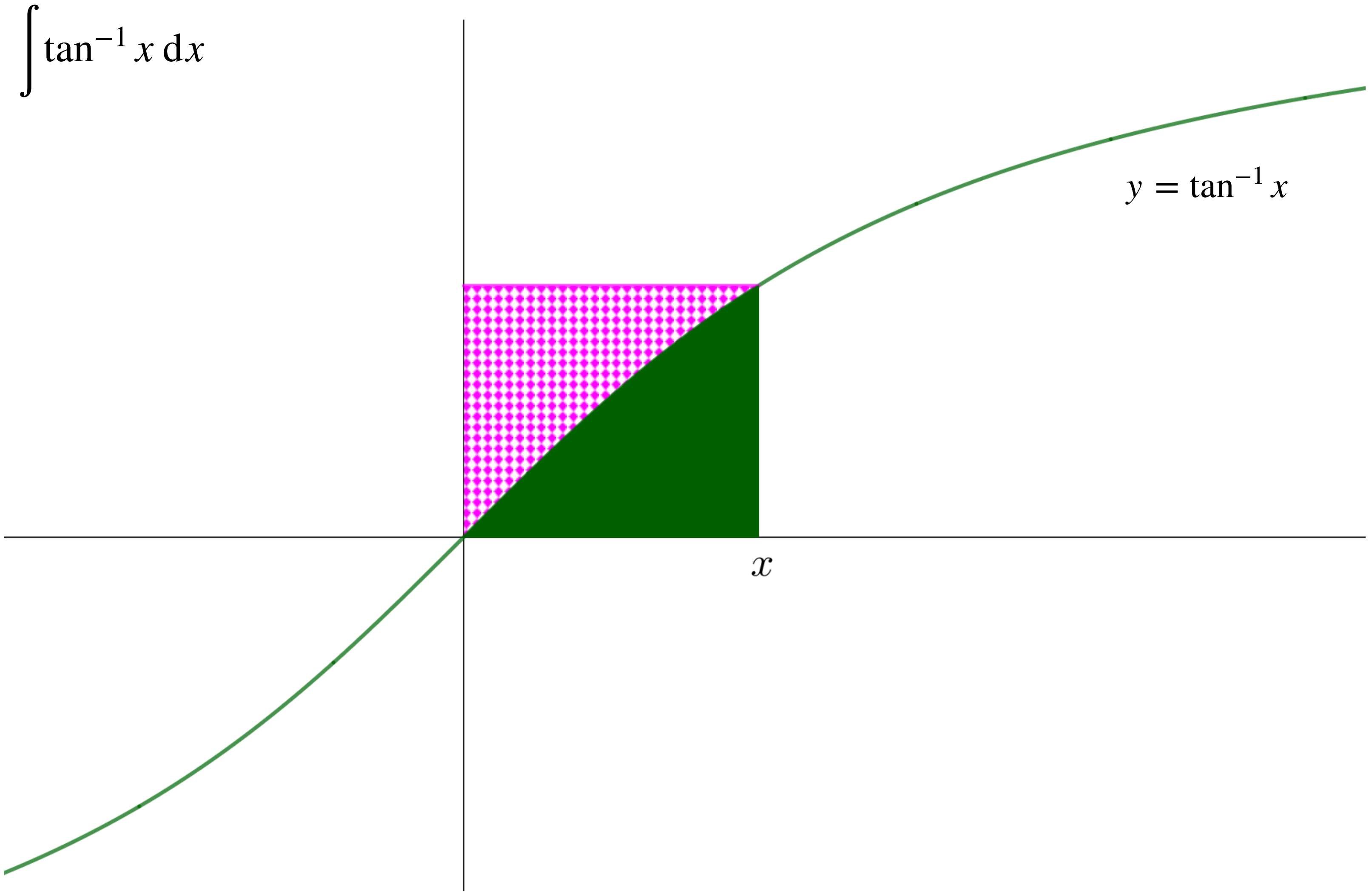
$$\int_0^x \cos^{-1} x \, \mathrm{d}x$$

and

$$\int \cos^{-1} x \, \mathrm{d}x$$

Use a similar strategy to find

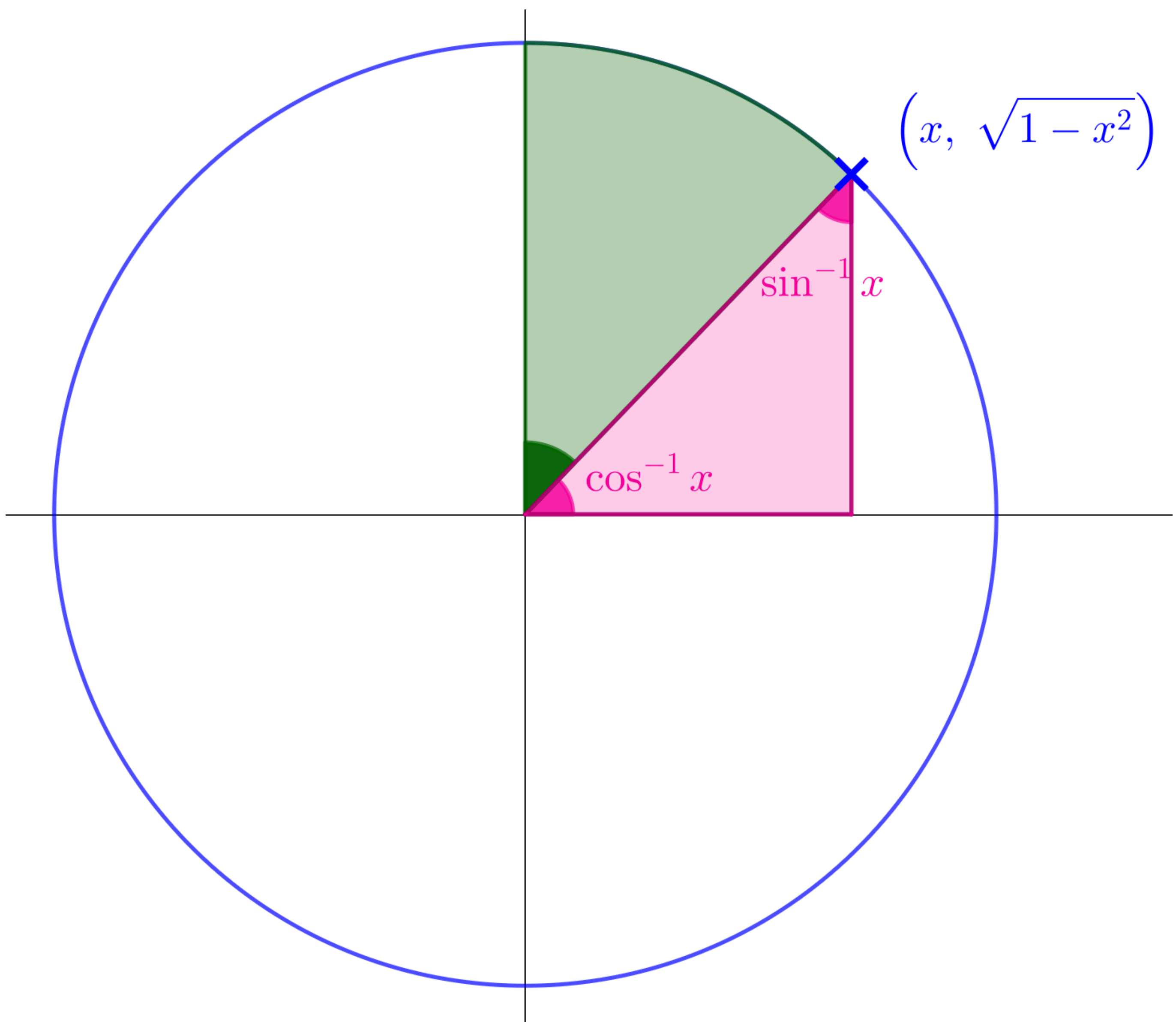
$$\int \tan^{-1} x \, \mathrm{d}x$$



Another integral using inverse circular functions

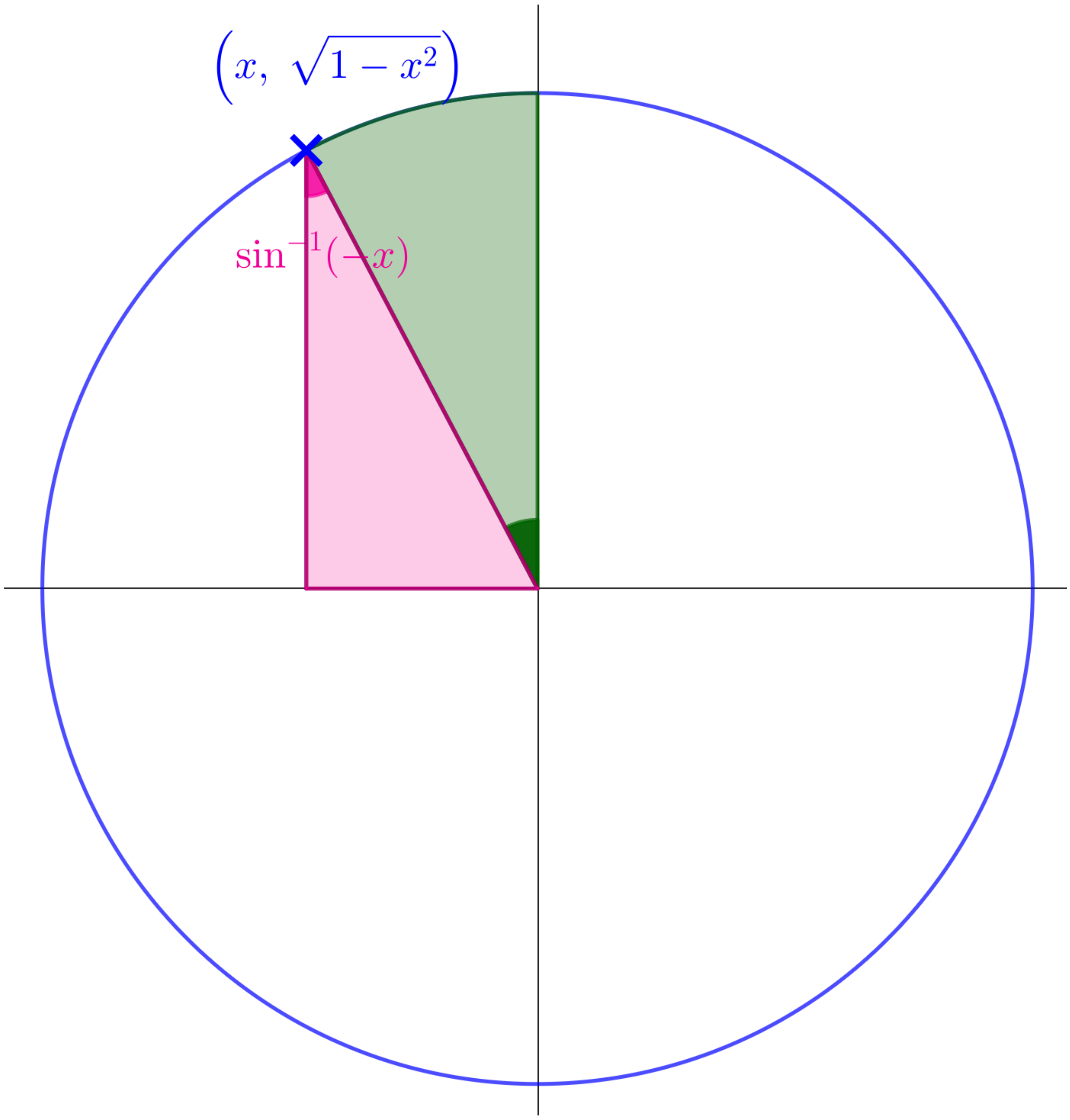
By finding the two shaded areas, find

$\int_0^x \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



$x < 0$, by finding the two shaded areas, find

$\int_x^0 \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



Use the substitutions $u = \sin x$ and $u = \cos x$ to find $\int \sqrt{1 - x^2}\, \mathrm{d}x$