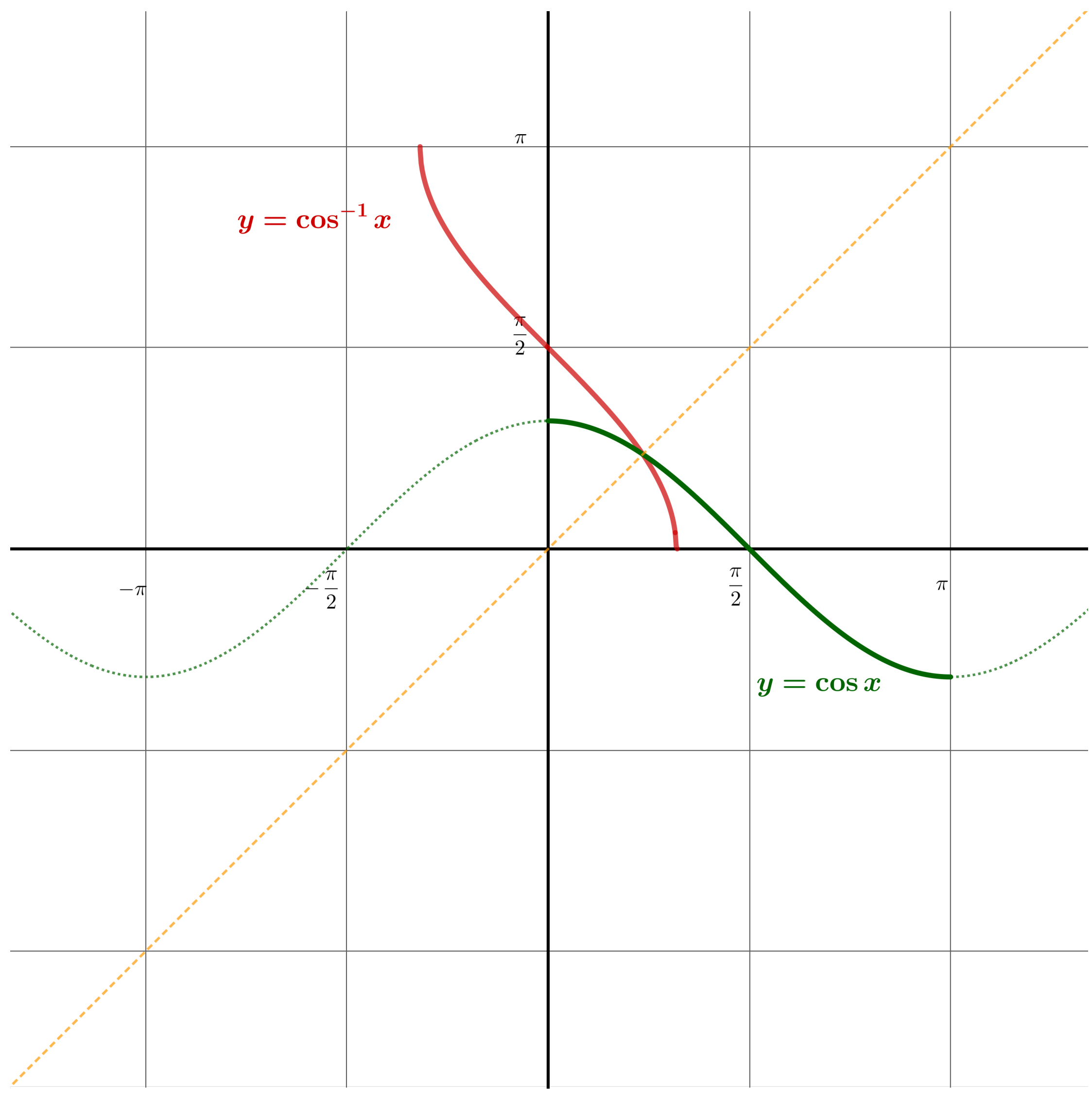
inverse circular functions: extension

What are the domain and range of $f(x) = \cos^{-1} x$?

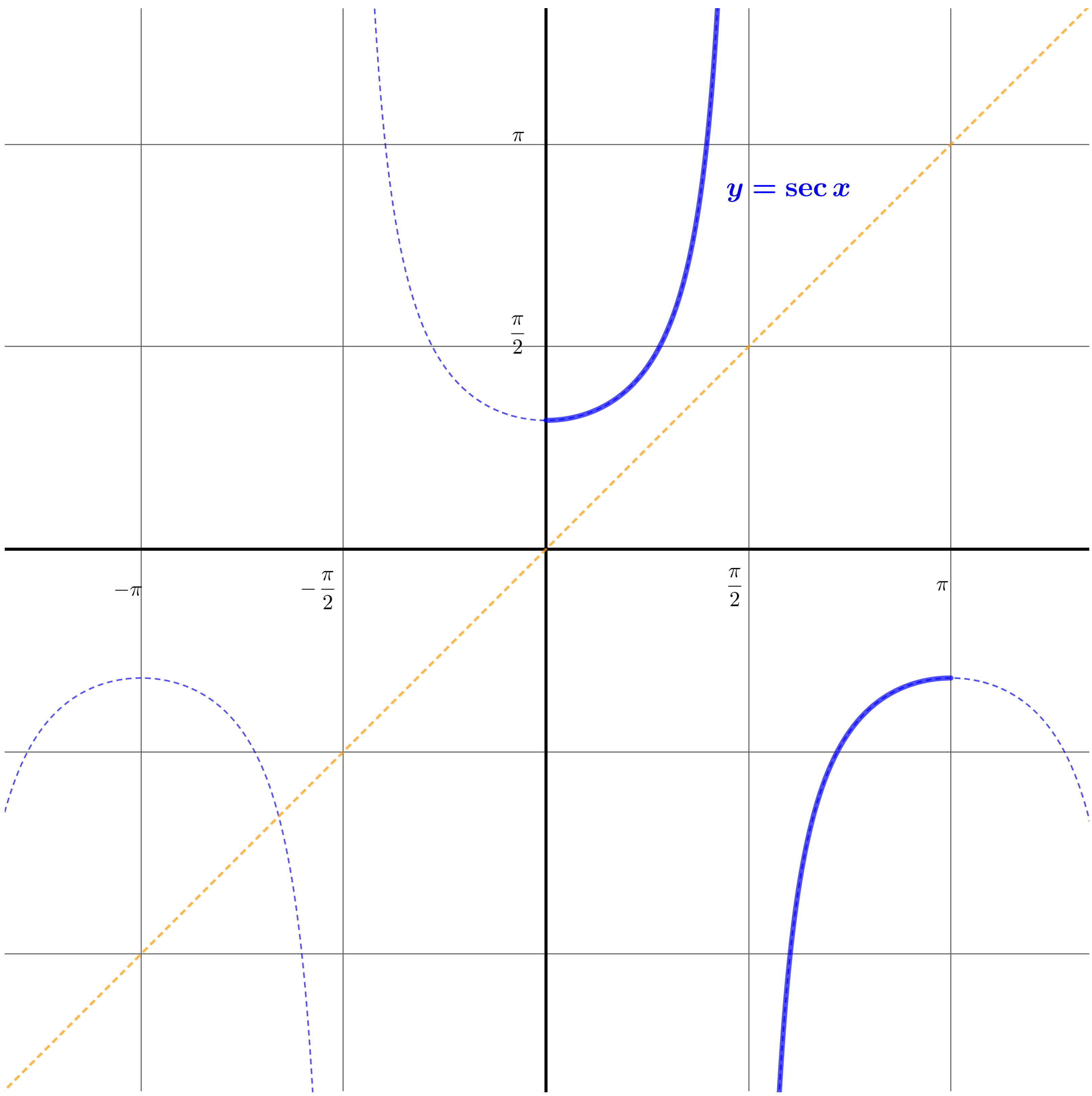


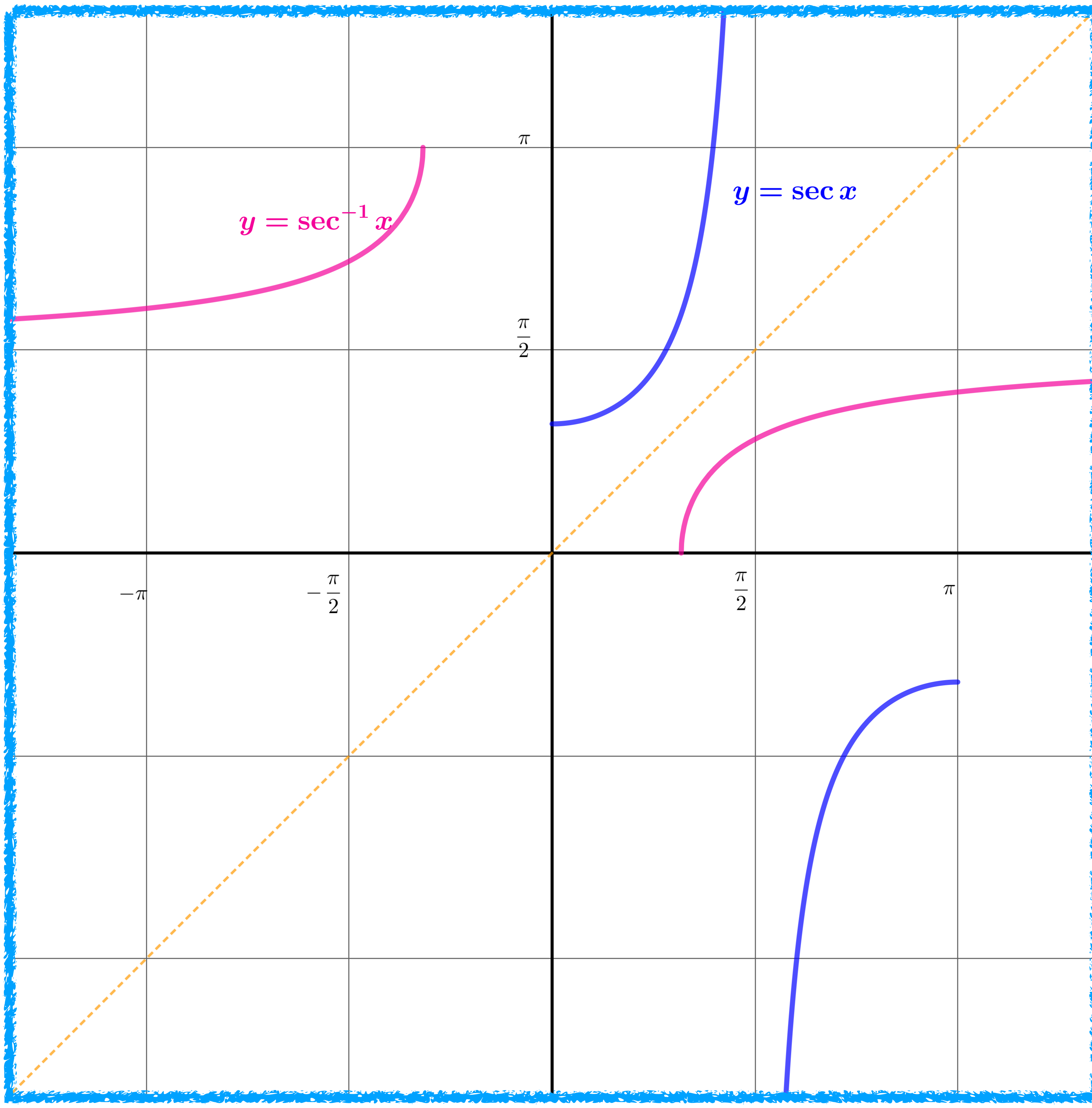
What are the domain and range of $f(x) = \sec^{-1} x$?

radical maths

Here is the graph $y = \sec x$ over the domain $\left\{ x : 0 \leq x \leq \pi,\, x \neq \dfrac{\pi}{2} \right\}$

Draw the graph $y = \sec^{-1} x$.



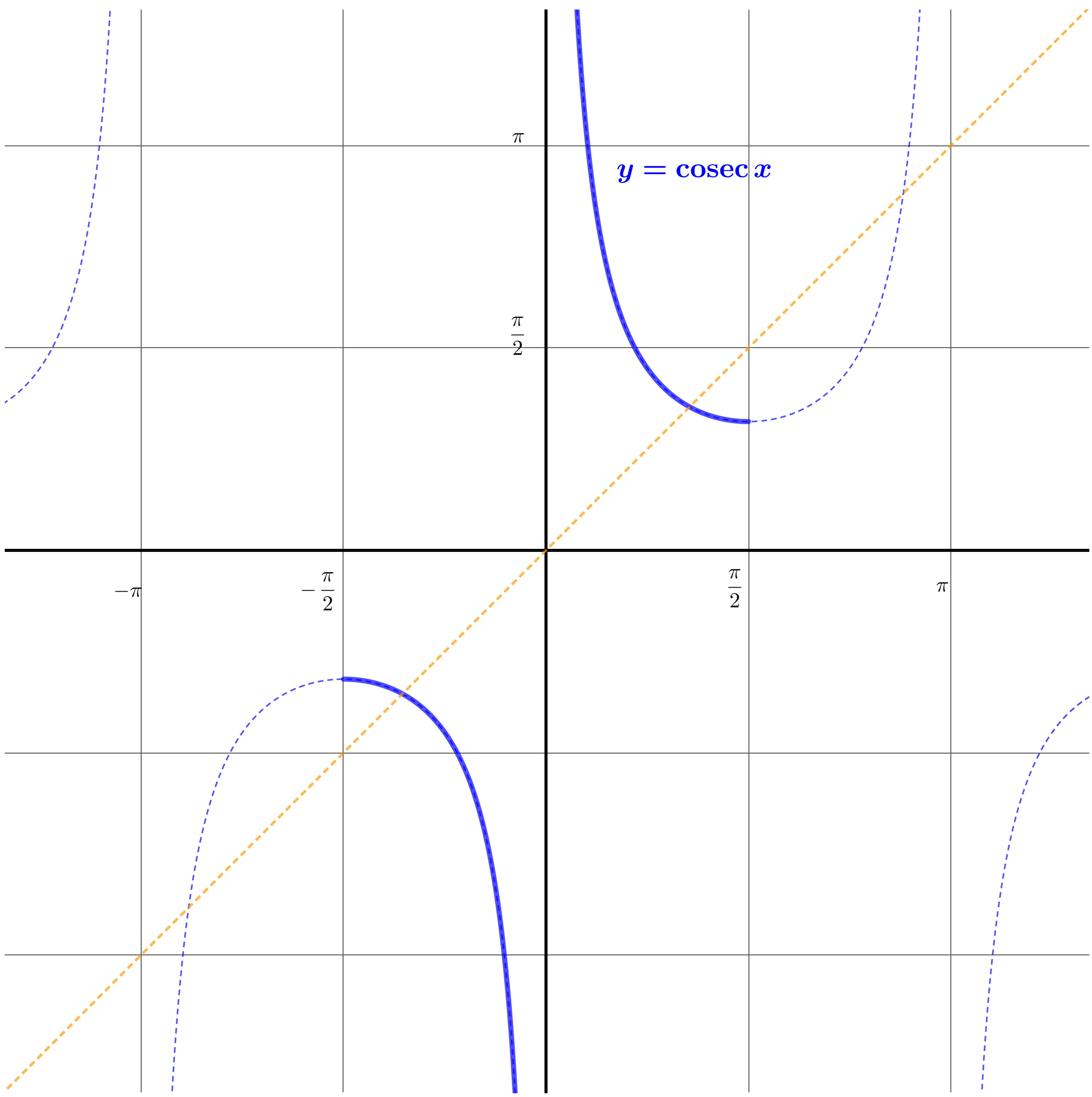


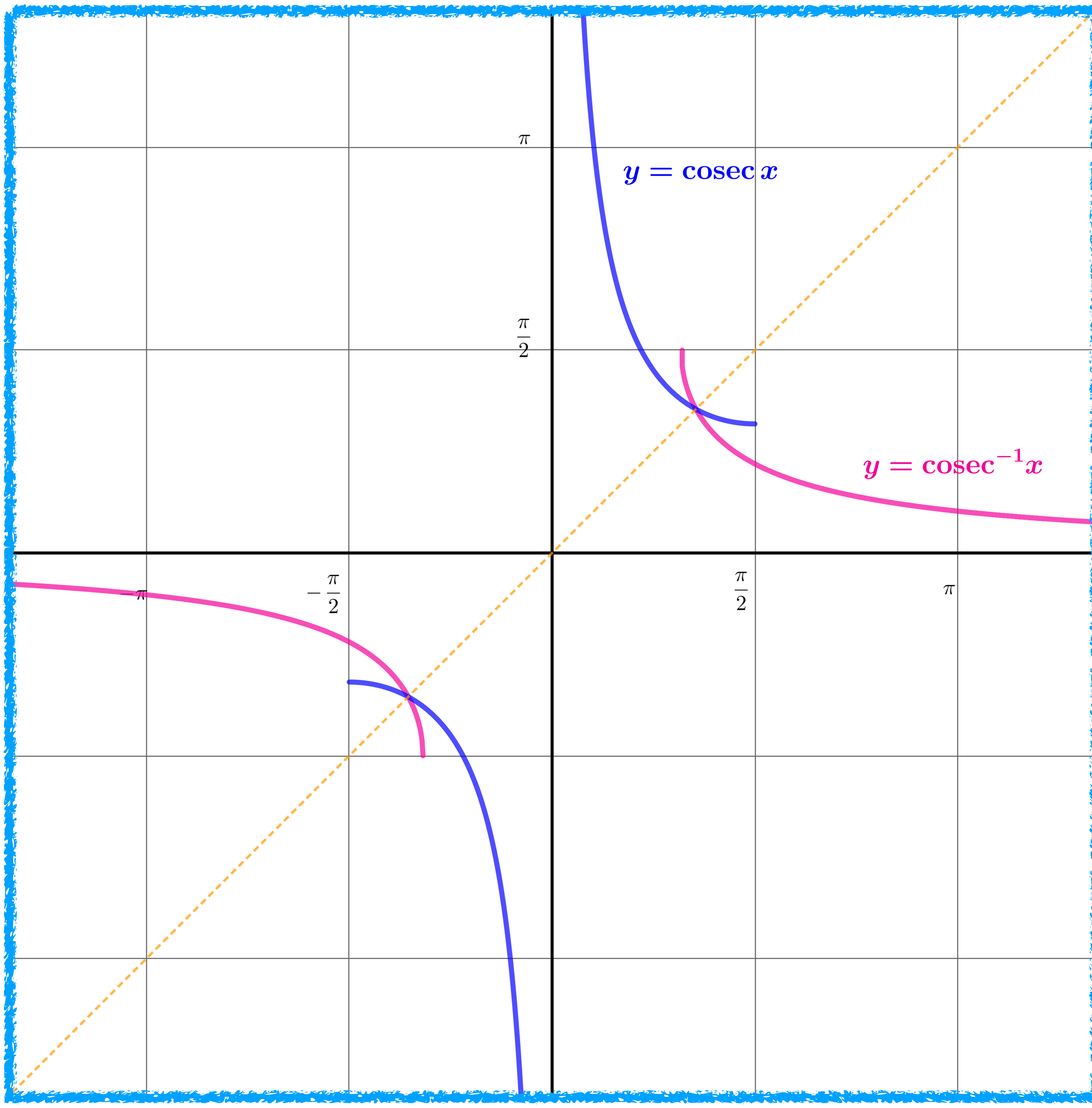
radical
maths

Here is the graph $y = \operatorname{cosec} x$ over the domain

$$\left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0\right\}$$

Draw the graph $y = \operatorname{cosec}^{-1} x$.



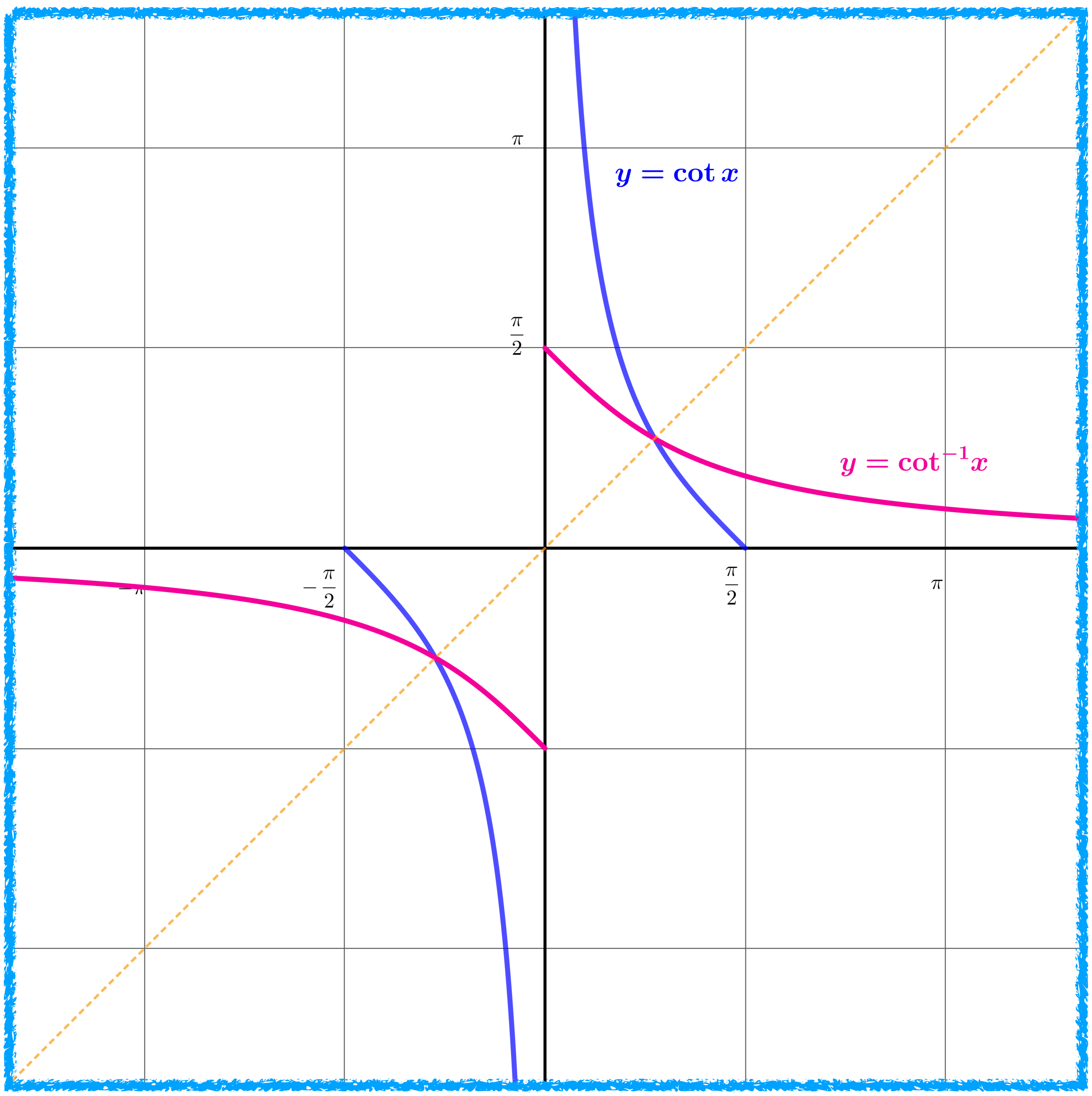


radical maths

Here is the graph $y = \cot x$ over the domain

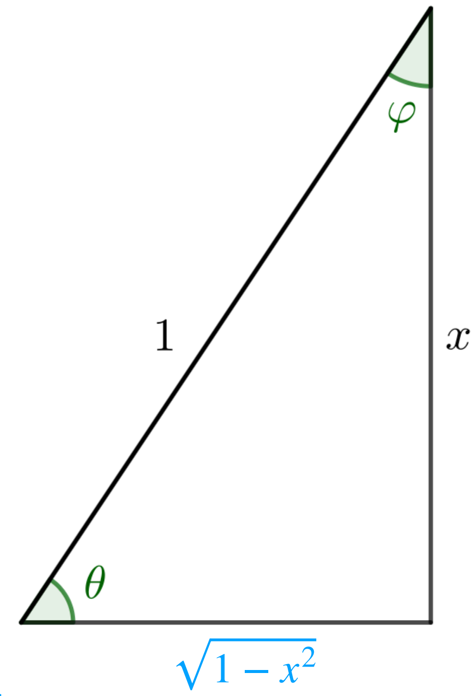
$$\left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0\right\}$$

Draw the graph $y = \cot^{-1} x$.



π
$\frac{\pi}{2}$
$y = \cot x$
$y = \cot^{-1} x$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π

What is $\cos\left(\sin^{-1} x\right)$?



From the diagram:

$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1-x^2}$.

You may decide that this is good enough!

However, if $-1 \leq x \leq 0$, we can no longer use the triangle. Whether or not you choose to delve into this with your students will be question of judgement, and probably you will decide against. But just in case:

There are a couple of ways of dealing with the negative values of x. Both are a little subtle.

First

If $x < 0$, look at the graph of $\sin^{-1} x$: it's symmetrical, so the value of $\sin^{-1} x$ must be the negative of the value of $\sin^{-1}|x|$. But $-\dfrac{\pi}{2} \leq \sin^{-1} x \leq \dfrac{\pi}{2}$.

This means that $\cos\left(\sin^{-1} x\right)$ is always positive, so

$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ for positive and negative values of x.

Second

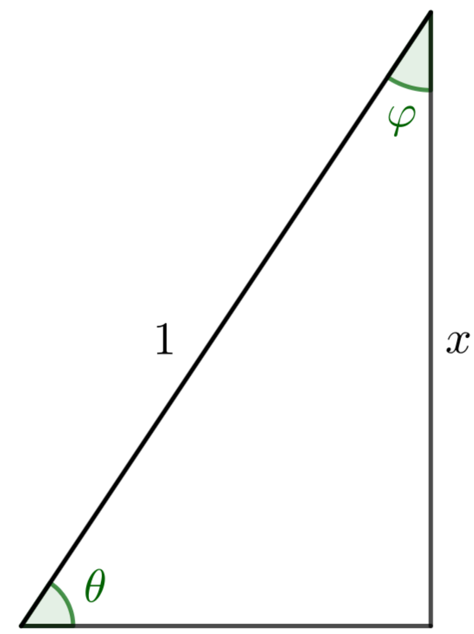
$$
\begin{aligned}
&\cos^2 A = 1 - \sin^2 A \\
\Rightarrow\ &\cos^2\left(\sin^{-1} x\right) = 1 - \sin^2\left(\sin^{-1} x\right) \\
&\qquad\qquad\qquad = \left(1 - \sin\left(\sin^{-1} x\right)\right)^2 \\
&\qquad\qquad\qquad = 1 - x^2 \\
\Rightarrow\ &\cos\left(\sin^{-1} x\right) = \pm\sqrt{1-x^2}
\end{aligned}
$$

But $-\dfrac{\pi}{2} \leq \sin^{-1} x \leq \dfrac{\pi}{2}$, so $\cos\left(\sin^{-1} x\right) > 0$

So for positive and negative x,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}.$$

What is $\cos^{-1}\left(\sin\theta\right)$?



From the diagram:

$\cos^{-1}\left(\sin\theta\right) = \cos^{-1} x = \varphi = \dfrac{\pi}{2} - \theta.$

You may decide that this is good enough!

However, if θ is outside the range $\left[0, \dfrac{\pi}{2}\right]$, we can no longer use the triangle.

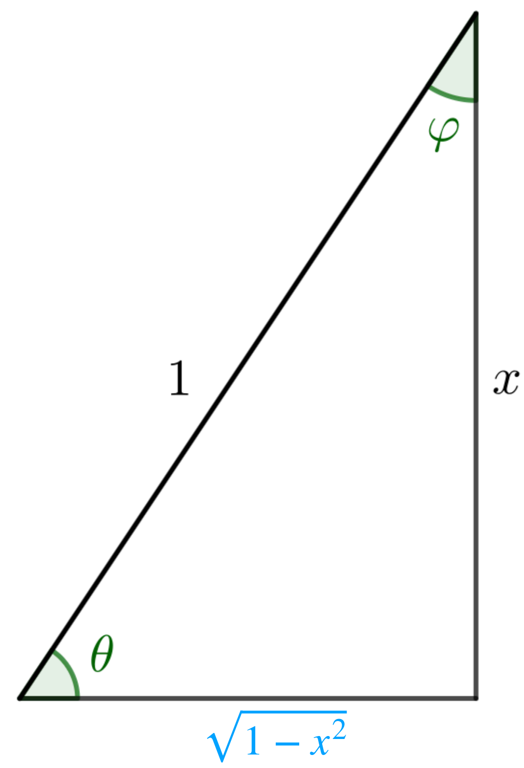
The range of $\cos^{-1}$ is $\{y : 0 \leq y \leq \pi\}$, so we have to add or subtract multiples of 2π to $\dfrac{\pi}{2} - \theta$ to bring it into this range.

$\cos^{-1}\left(\sin\theta\right) = \cos^{-1}\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\pi}{2} - \theta + 2n\pi$

where $n \in \mathbb{Z}$ is chosen to give a result in the range $\left[0, \pi\right]$.

radical maths

What are $\sin\left(\cos^{-1} x\right)$ and $\sin^{-1}(\cos\varphi)$?



From the diagram:

$\sin\left(\cos^{-1} x\right) = \sin\varphi = \sqrt{1-x^2}$, so this is true when x is positive.

Again, you will probably decide that this is enough. But just in case (again):

When x is negative, things are a little bit trickier with graphs. As before,

$$\begin{aligned}
&\sin^2 A = 1 - \cos^2 A \\
\Rightarrow\ &\sin^2\left(\cos^{-1} x\right) = 1 - \cos^2\left(\cos^{-1} x\right) \\
&\qquad = \left(1 - \cos\left(\cos^{-1} x\right)\right)^2 \\
&\qquad = 1 - x^2 \\
\Rightarrow\ &\sin\left(\cos^{-1} x\right) = \pm\sqrt{1-x^2}
\end{aligned}$$

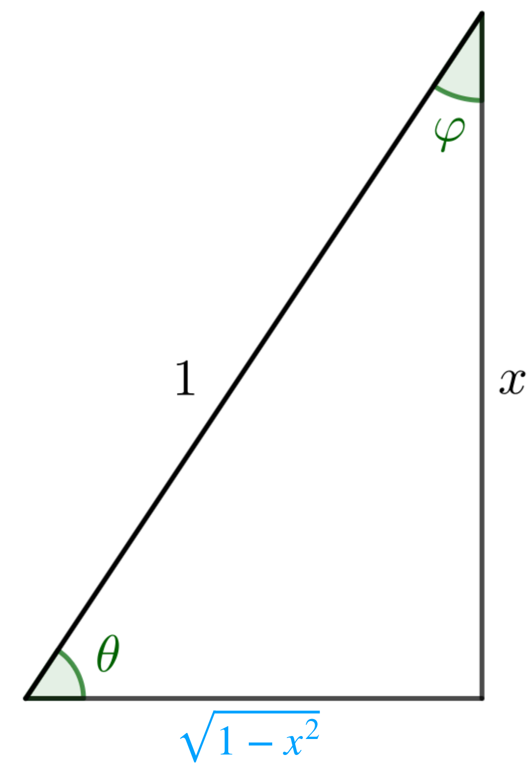
But $0 \le \cos^{-1} x \le \pi$, so $\sin\left(\cos^{-1} x\right) > 0$

So for positive and negative x,

$$\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}.$$

radical maths

What are $\tan\left(\sin^{-1}x\right)$ and $\tan\left(\cos^{-1}x\right)$?



From the diagram:

$$\tan\left(\sin^{-1}x\right) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\tan\left(\cos^{-1}x\right) = \tan\varphi = \frac{\sqrt{1-x^2}}{x}$$

For negative values of x:

$$\sin^{-1}x < 0 \Rightarrow \tan\left(\sin^{-1}x\right) < 0 \text{ and } \frac{x}{\sqrt{1-x^2}} < 0$$

$$\frac{\pi}{2} < \cos^{-1}x \leq \pi \Rightarrow \tan\left(\cos^{-1}x\right) < 0 \text{ and } \frac{\sqrt{1-x^2}}{x} < 0$$

so the results are still true.

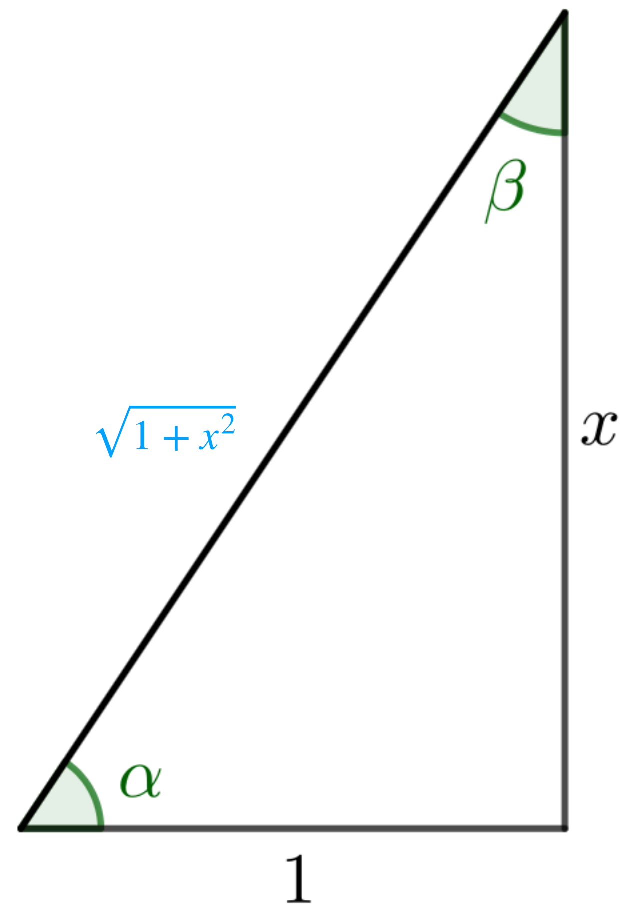
Or:

$$\tan\left(\sin^{-1}x\right) = \frac{\sin\left(\sin^{-1}x\right)}{\cos\left(\sin^{-1}x\right)} = \frac{x}{\sqrt{1-x^2}}$$

$$\tan\left(\cos^{-1}x\right) = \frac{\sin\left(\cos^{-1}x\right)}{\cos\left(\cos^{-1}x\right)} = \frac{\sqrt{1-x^2}}{x}$$

radical maths

What are $\sin\left(\tan^{-1}x\right)$ and $\cos\left(\tan^{-1}x\right)$?



$$\sin\left(\tan^{-1}x\right) = \sin\alpha = \frac{x}{\sqrt{1+x^2}}$$

When $x > 0$,

$0 < \tan^{-1}x < \frac{\pi}{2} \Rightarrow \sin\left(\tan^{-1}x\right) > 0$

When $x < 0$,

$-\frac{\pi}{2} < \tan^{-1}x < 0 \Rightarrow \sin\left(\tan^{-1}x\right) < 0$

$$\cos\left(\tan^{-1}x\right) = \cos\alpha = \frac{1}{\sqrt{1+x^2}}$$

Since $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$, we know that $\cos\left(\tan^{-1}x\right) > 0$.

$$\begin{aligned} 1 + x^2 = 1 + \tan^2\left(\tan^{-1}x\right) &= \sec^2\left(\tan^{-1}x\right) \\ &= \frac{1}{\cos^2\left(\tan^{-1}x\right)} \\ \Rightarrow \cos\left(\tan^{-1}x\right) &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

$$\begin{aligned} 1 + \frac{1}{x^2} = 1 + \cot^2\left(\tan^{-1}x\right) &= \operatorname{cosec}^2\left(\tan^{-1}x\right) \\ &= \frac{1}{\sin^2\left(\tan^{-1}x\right)} \\ \Rightarrow \sin\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+\frac{1}{x^2}}} &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

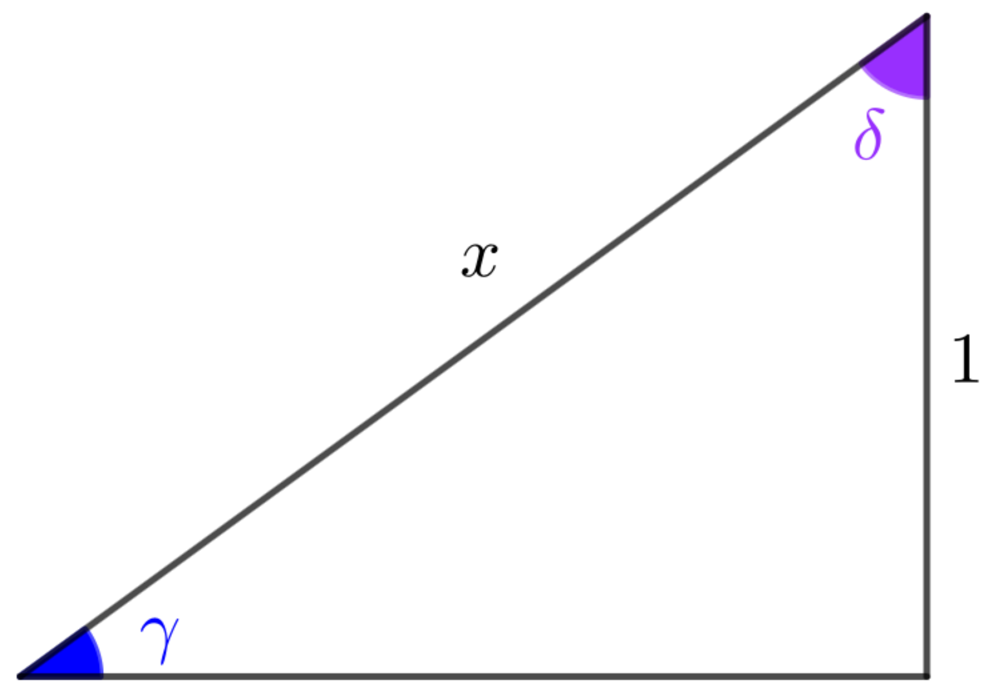
Simplify

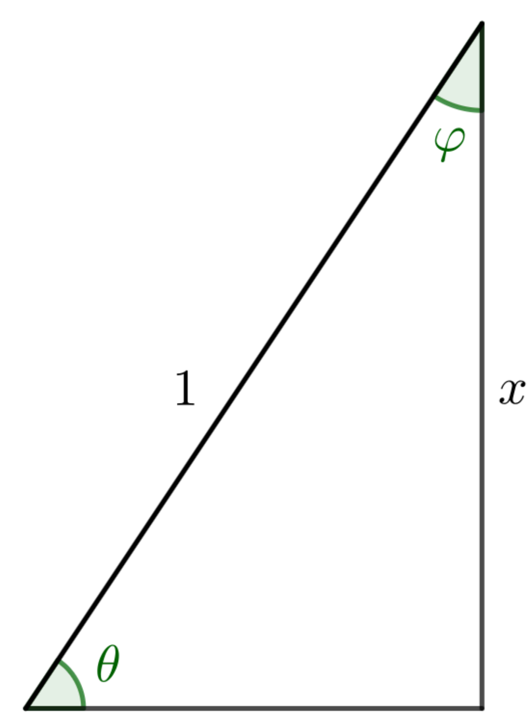
$\sin\left(\operatorname{cosec}^{-1} x\right)$

$\operatorname{cosec}\left(\sin^{-1} x\right)$

$\cos\left(\sec^{-1} x\right)$

$\sec\left(\cos^{-1} x\right)$





$$\sin\left(\operatorname{cosec}^{-1} x\right) = \sin\gamma = \frac{1}{x}$$

$$\operatorname{cosec}\left(\sin^{-1} x\right) = \operatorname{cosec}\theta = \frac{1}{x}$$

$$\cos\left(\sec^{-1} x\right) = \cos\delta = \frac{1}{x}$$

$$\sec\left(\cos^{-1} x\right) = \sec\varphi = \frac{1}{x}$$

So

$$\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$$

$$\sec^{-1} x = \cos^{-1}\frac{1}{x}$$

These are clearly true when $x > 0$. What about when x is negative?

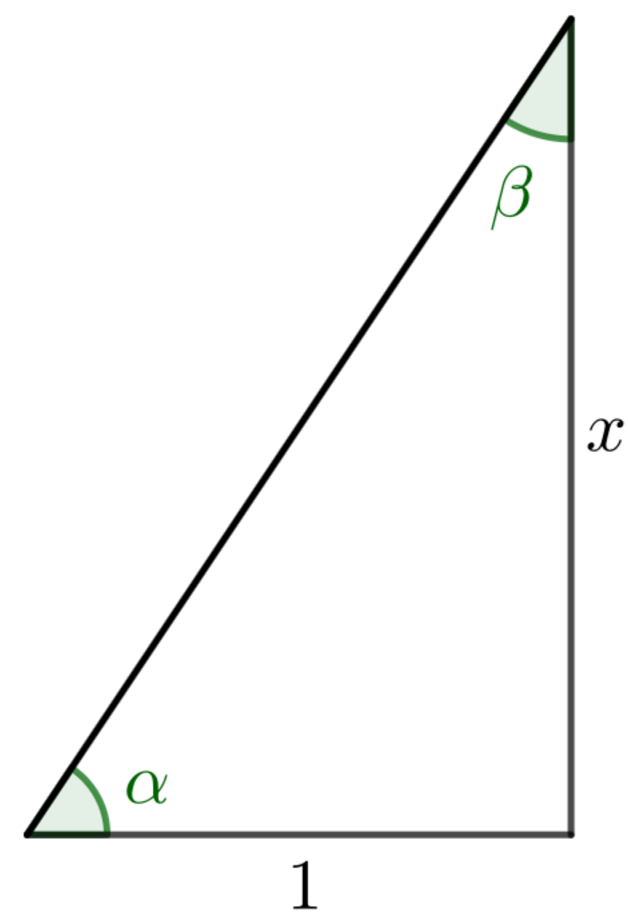
$$\sec\left(\cos^{-1}\frac{1}{x}\right) = \frac{1}{\cos\left(\cos^{-1}\frac{1}{x}\right)} = x$$

is true for all x, and similarly for sin and cosec.

Simplify

$\tan\left(\cot^{-1} x\right)$

$\cot\left(\tan^{-1} x\right)$



$\tan\left(\cot^{-1} x\right) = \tan\beta = \dfrac{1}{x}$

$\cot\left(\tan^{-1} x\right) = \cot\alpha = \dfrac{1}{x}$

So

$\cot^{-1} x = \tan^{-1}\dfrac{1}{x}$

These are clearly true when $x > 0$. What about when x is negative?

$\cot\left(\tan^{-1}\dfrac{1}{x}\right) = \dfrac{1}{\tan\left(\tan^{-1}\frac{1}{x}\right)} = x$

is true for all x.

What are

$\cot^{-1} x + \tan^{-1} x$

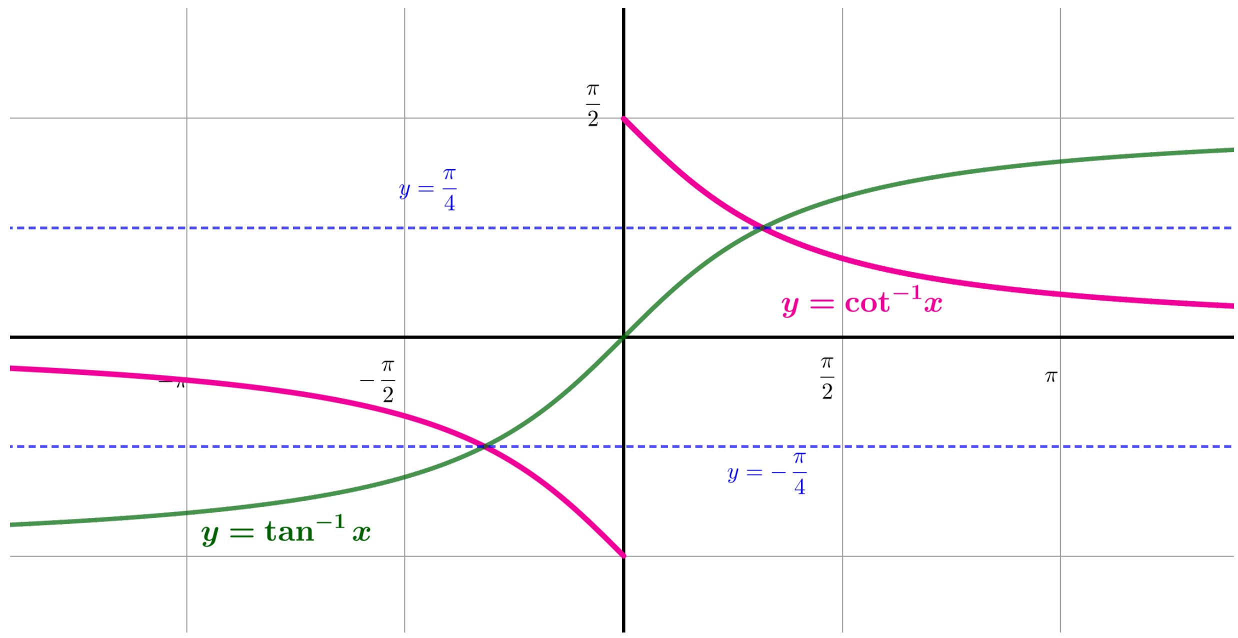
and

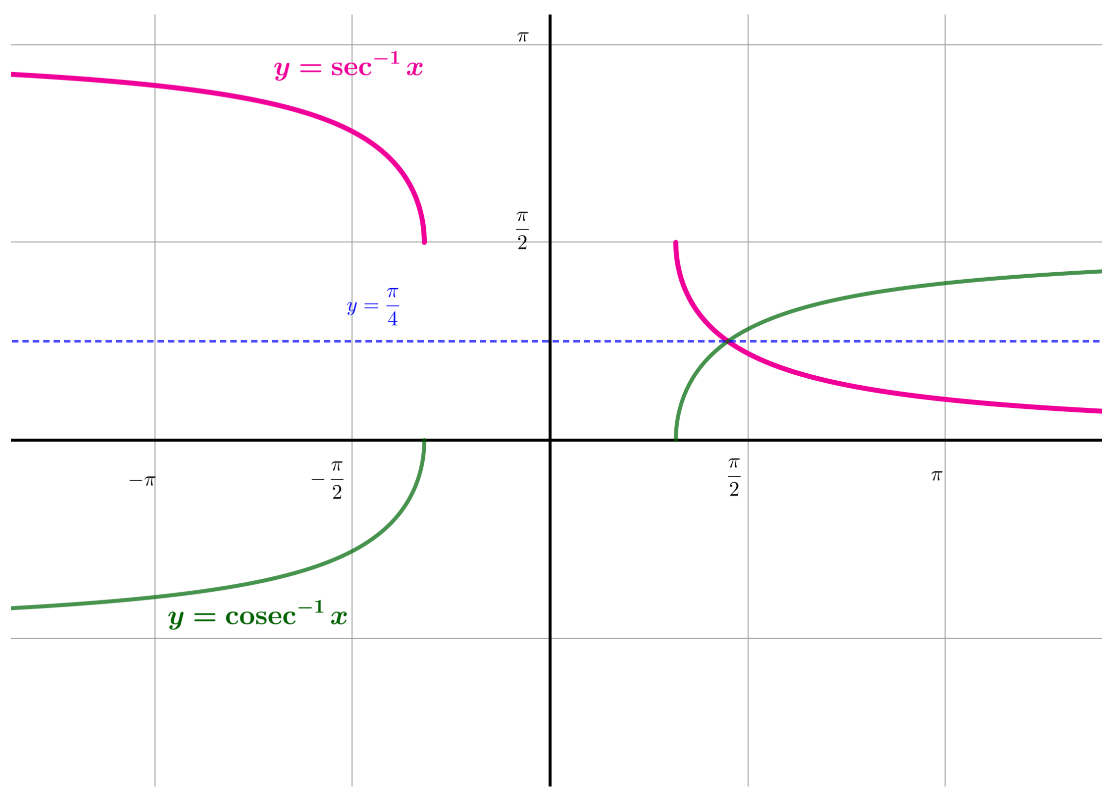
$\sec^{-1} x + \operatorname{cosec}^{-1} x$?

Symmetry:

$\cot^{-1} x + \tan^{-1} x = \pm \dfrac{\pi}{2}$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$





more differentials of inverse circular functions

What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x$?

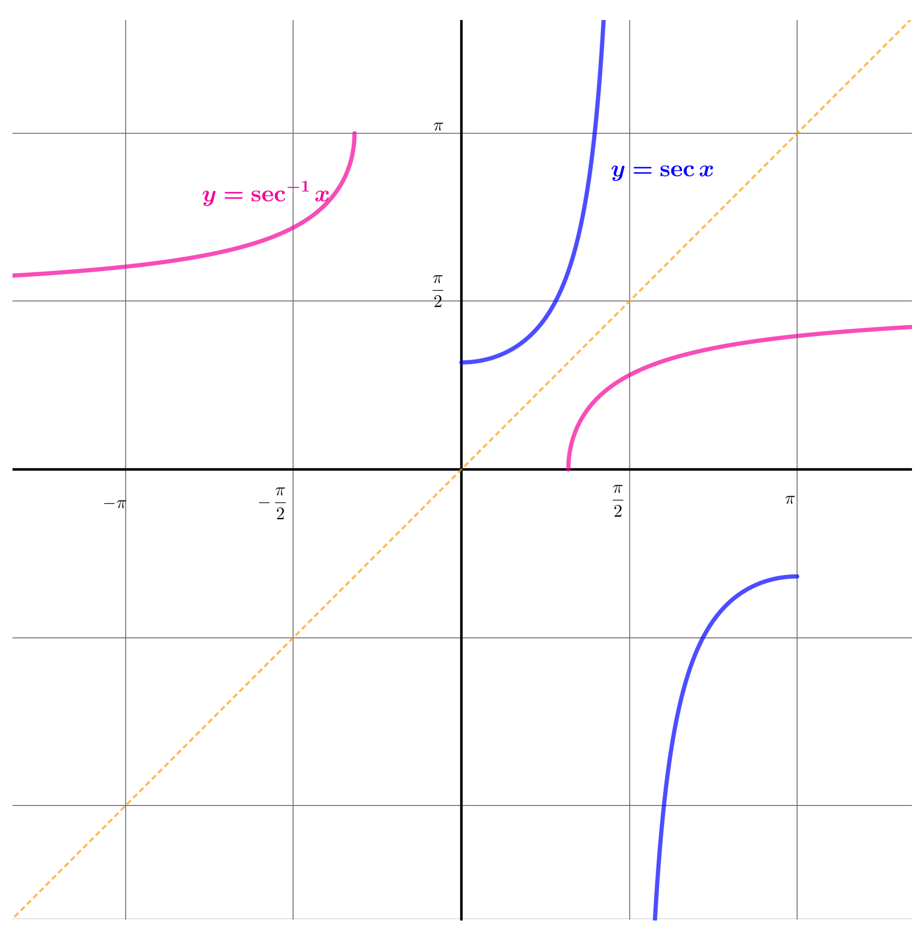
$$\begin{aligned}
& y = \sec^{-1}x \\
\Rightarrow\ & x = \sec y \\
\Rightarrow\ & \frac{\mathrm{d}x}{\mathrm{d}y} = \sec y \tan y = \pm\, x\sqrt{x^2-1} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

$$\begin{aligned}
& y = \sec^{-1}x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

From the graph, gradient is always positive.

But the integral on the right is negative when x is negative. What went wrong?

$\sqrt{a^2b} = |a|b$, not ab



So

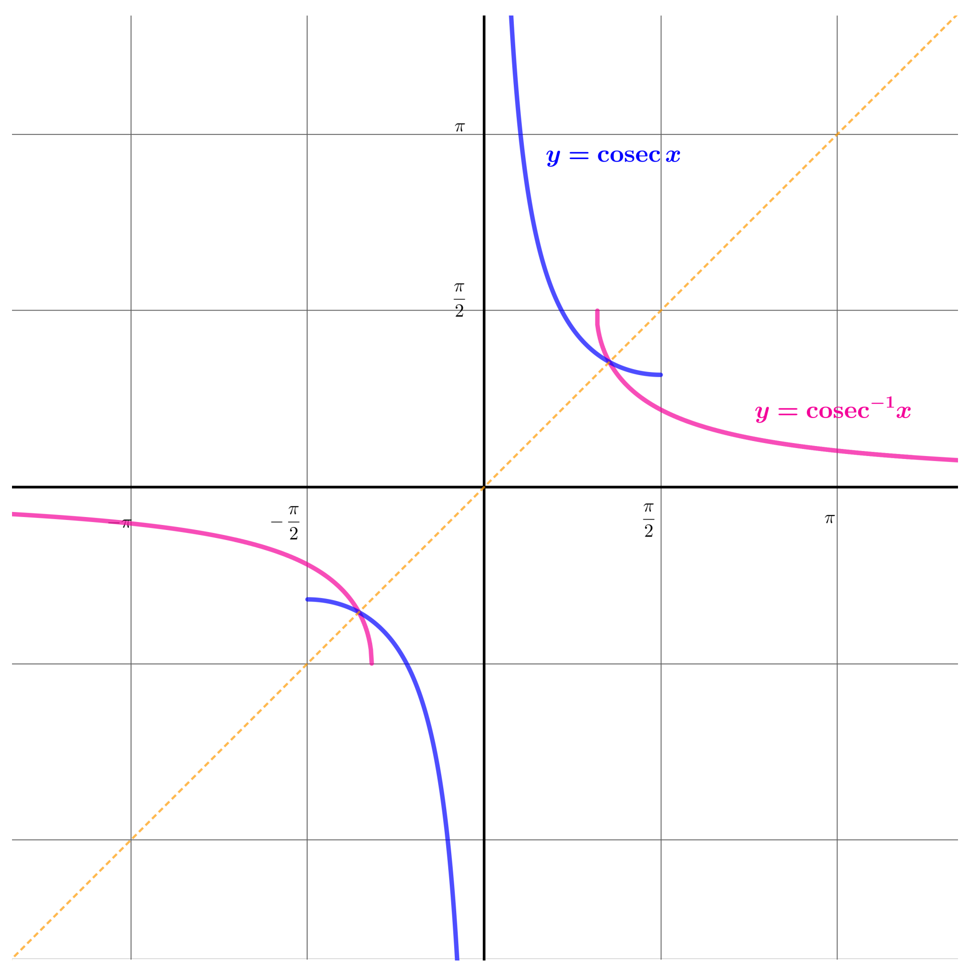
$$\begin{aligned}
& y = \sec^{-1}x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
& = \frac{1}{|x|\,|x|}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

radical maths

What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec}^{-1} x$?

$$
\begin{aligned}
& y = \operatorname{cosec}^{-1} x \\
\Rightarrow\ & x = \operatorname{cosec} y \\
\Rightarrow\ & \frac{\mathrm{d}x}{\mathrm{d}y} = -\operatorname{cosec} y \cot y = \mp x\sqrt{x^2 - 1} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \mp \frac{1}{x\sqrt{x^2 - 1}}
\end{aligned}
$$

$$
\begin{aligned}
& y = \operatorname{cosec}^{-1} x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}\frac{-1}{\sqrt{1 - \frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{|x|\sqrt{x^2 - 1}}
\end{aligned}
$$

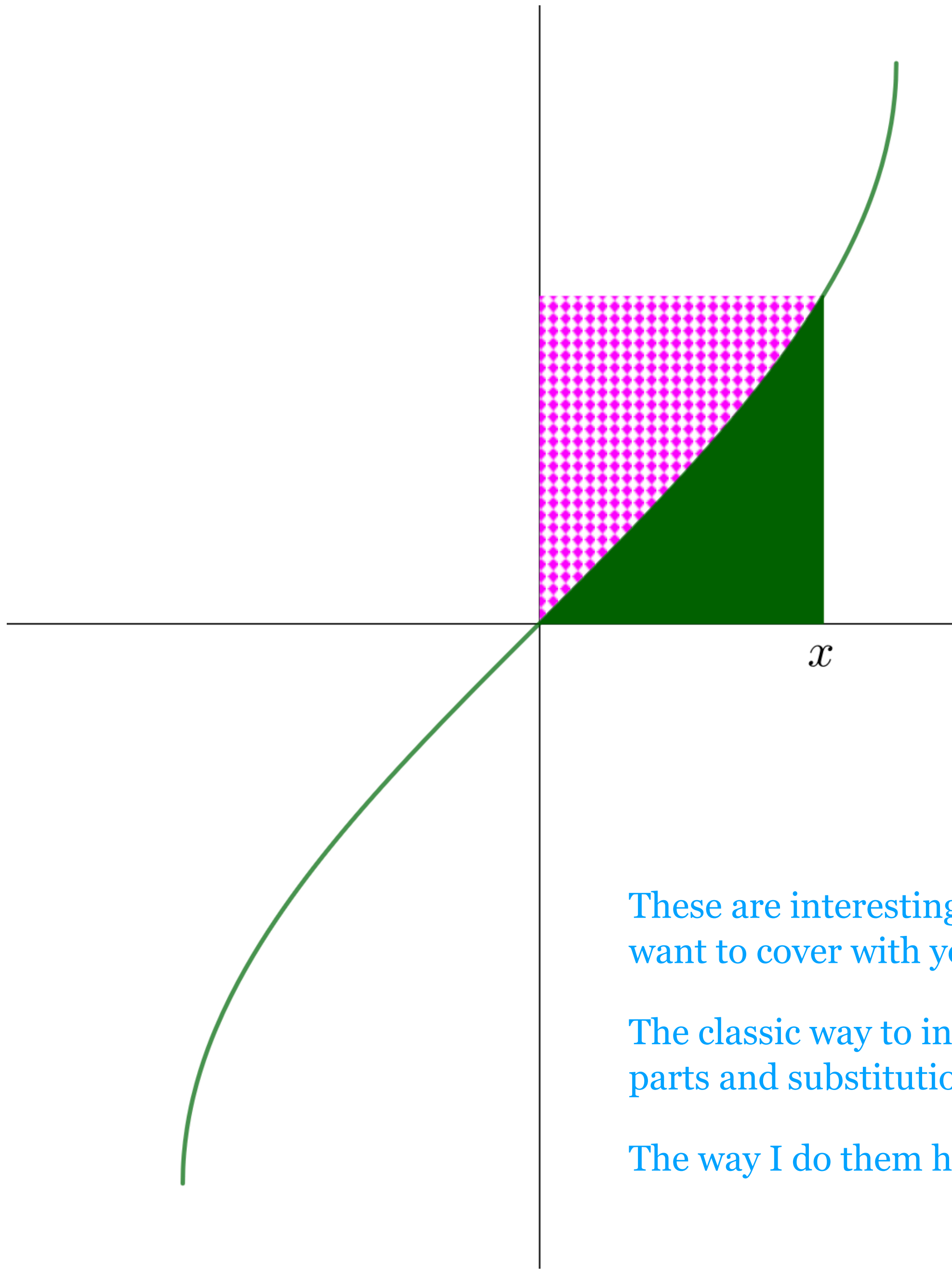


What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x$?

$$y = \cot^{-1} x$$
$$\Rightarrow x = \cot y$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-\sin^2 y - \cos^2 y}{\sin^2 y}$$
$$= -\operatorname{cosec}^2 y$$
$$= -1 - \cot^2 y$$
$$= -1 - x^2$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}$$

$$y = \cot^{-1} x$$
$$\Rightarrow y = \tan^{-1} \frac{1}{x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2} \times \frac{1}{1+\frac{1}{x^2}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}$$

integrals of inverse circular functions



These are interesting, but may well be more than you either need or want to cover with your students.

The classic way to integrate these functions is with a combination of parts and substitution (in either order).

The way I do them here is rather whimsical, but still instructive.

What is the area of the whole shaded rectangle?

The coordinates of the top right-hand vertex of the rectangle are $(x, \sin^{-1} x)$, so the rectangle area is $x \sin^{-1} x$.

Write the two shaded areas as integrals.

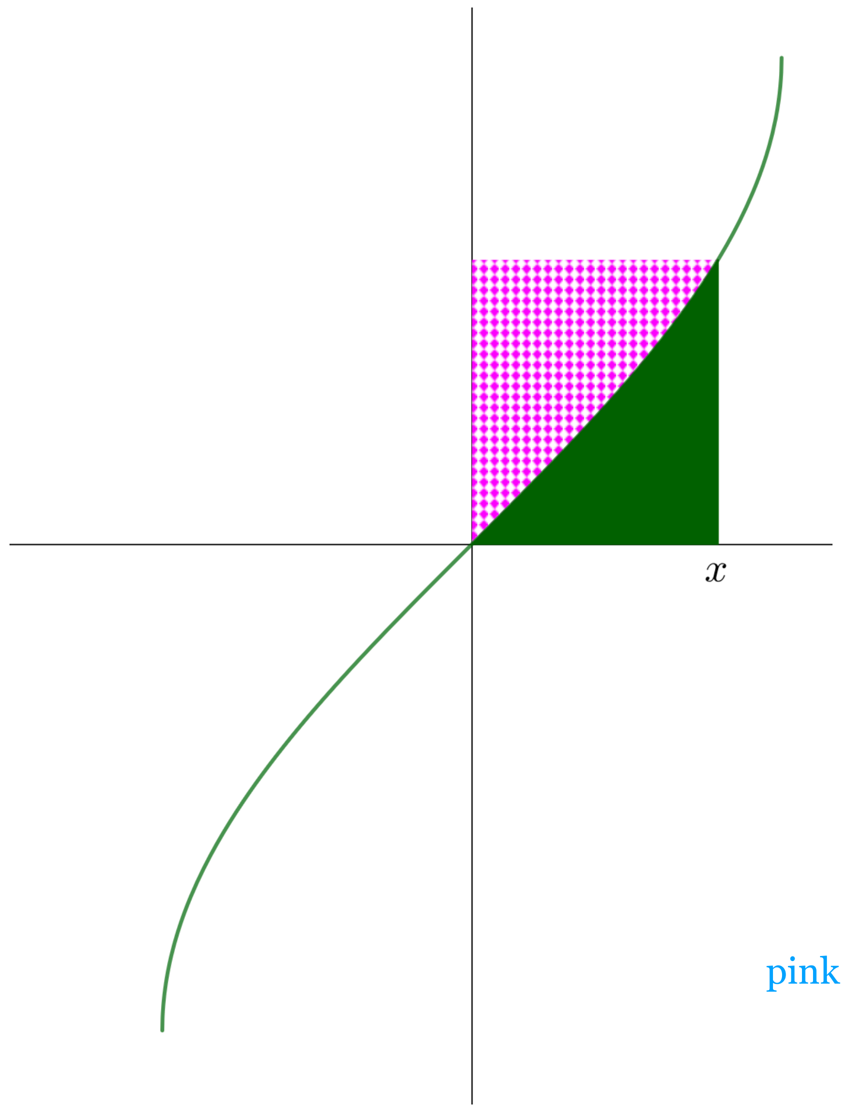
$$\text{green (solid) area} = \int_0^x \sin^{-1} x \, \mathrm{d}x$$

$$\text{pink (chequered) area} = \int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y$$

What is the area of the pink (chequered) area?

$$\begin{aligned}\text{pink (chequered) area} &= \int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y \\ &= [-\cos y]_0^{\sin^{-1} x} \\ &= -\cos \sin^{-1} x - -1 \\ &= 1 - \sqrt{1 - x^2}\end{aligned}$$

radical maths



$$\text{pink (chequered) area} = \int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y$$
$$= \left[-\cos y\right]_0^{\sin^{-1} x}$$
$$= -\cos \sin^{-1} x - -1$$
$$= 1 - \sqrt{1 - x^2}$$

Use these results to find

$$\int_0^x \sin^{-1} x \, \mathrm{d}x$$

$$= \text{green (solid) area} = \text{rectangle} - \text{pink (chequered) area}$$
$$= x \sin^{-1} x - 1 + \sqrt{1 - x^2}$$

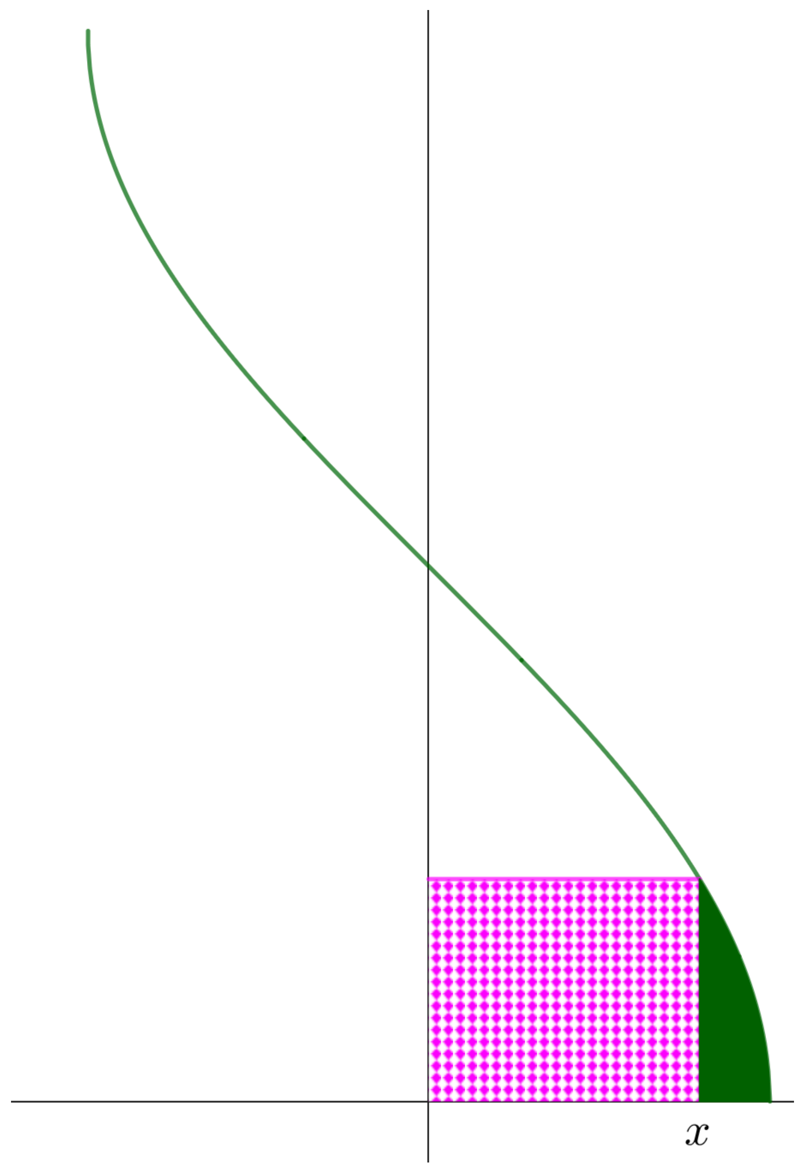
and

$$\int \sin^{-1} x \, \mathrm{d}x$$

$$= x \sin^{-1} x - 1 + \sqrt{1 - x^2} + \text{constant}$$
$$= x \sin^{-1} x + \sqrt{1 - x^2} + c$$

remember: 1 less than a constant is just another constant

Here is the graph $y = \cos^{-1} x$.



What is the area of the pink (chequered) rectangle?

The coordinates of the top right-hand vertex of the rectangle are $\left(x, \cos^{-1} x\right)$, so the rectangle area is $x \cos^{-1} x$.

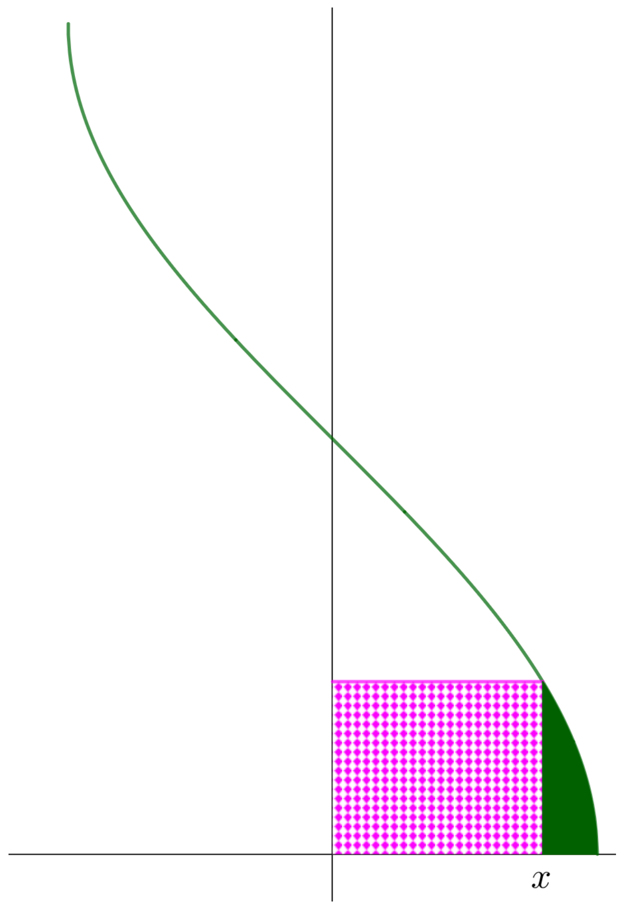
Write the entire shaded area as an integral.

$$\text{entire shaded area } = \int_0^{\cos^{-1} x} \cos y \, \mathrm{d}y$$

Write the green (solid) shaded area as an integral.

$$\text{green (solid) area } = \int_x^1 \cos^{-1} x \, \mathrm{d}x$$

radical maths



$$\begin{aligned}\text{entire shaded area} &= \int_{-\cos^{-1}x}^{0} \cos y \, \mathrm{d}y \\ &= \left[\sin y\right]_{-\cos^{-1}x}^{0} \\ &= \sin \cos^{-1} x \\ &= \sqrt{1-x^2}\end{aligned}$$

Use these results to find

$$\int_x^1 \cos^{-1} x \, \mathrm{d}x$$

$$= \text{green (solid) area} = \text{entire shaded area} - \text{pink (chequered) area}$$

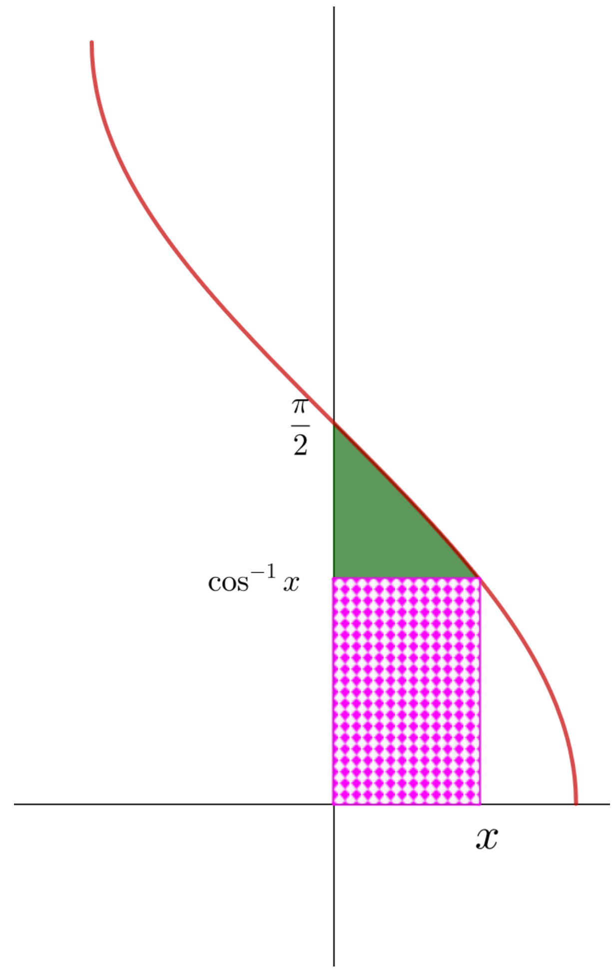
$$= \sqrt{1-x^2} - x\cos^{-1} x$$

and

$$\int \cos^{-1} x \, \mathrm{d}x$$

$$\begin{aligned} I(x) &= \int \cos^{-1} x \, \mathrm{d}x \\ \Rightarrow \int_x^1 \cos^{-1} x \, \mathrm{d}x &= \left[I\right]_x^1 \\ &= I(1) - I(x) \\ &= 1 - I(x) \\ \Rightarrow I(x) &= 1 + x\cos^{-1} x - \sqrt{1-x^2} + c \\ &= x\cos^{-1} x - \sqrt{1-x^2} + c' \end{aligned}$$

Here is the graph $y = \cos^{-1} x$.



Use these results to find

$$\int_0^x \cos^{-1} x \, \mathrm{d}x$$

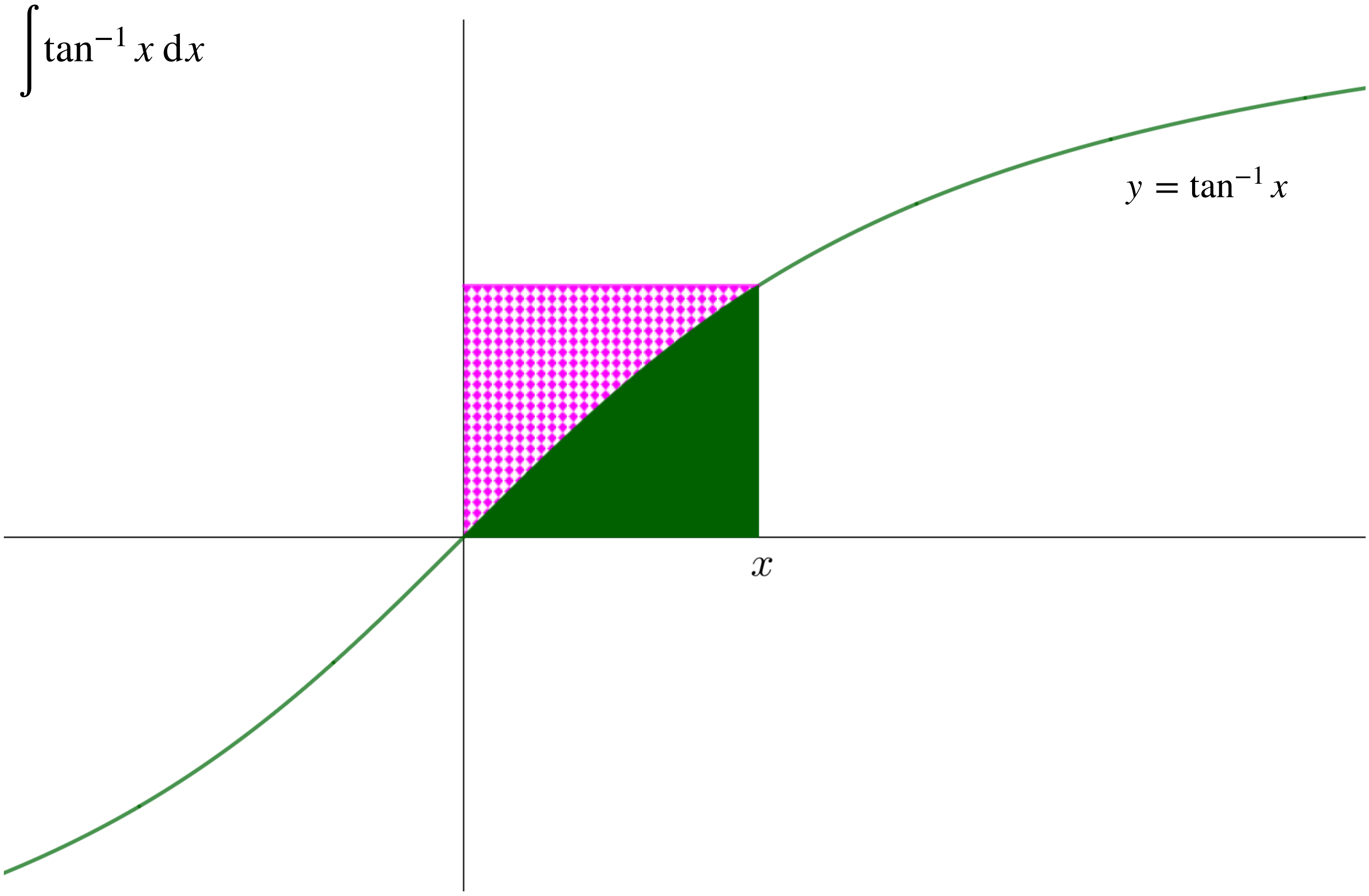
$$\begin{aligned} &= \text{green (solid) area} + \text{pink (chequered) area} \\ &= \int_0^{\frac{\pi}{2} - \cos^{-1} x} \sin x \, \mathrm{d}x + x \cos^{-1} x \\ &= \int_0^{\sin^{-1} x} \sin x \, \mathrm{d}x + x \cos^{-1} x \\ &= -\left[\cos x\right]_0^{\sin^{-1} x} + x \cos^{-1} x \\ &= -\cos(\sin^{-1} x) + 1 + x \cos^{-1} x \\ &= x \cos^{-1} x - \sqrt{1 - x^2} + 1 \end{aligned}$$

and

$$\int \cos^{-1} x \, \mathrm{d}x \qquad = x \cos^{-1} x - \sqrt{1 - x^2} + c$$

Use a similar strategy to find

$$\int \tan^{-1} x \, \mathrm{d}x$$



$$\int \tan^{-1} x \, \mathrm{d}x = x \tan^{-1} x - \int_0^{\tan^{-1} x} \tan y \, \mathrm{d}y$$

$$\begin{aligned}\int \tan^{-1} x \, \mathrm{d}x &= x \tan^{-1} x + \left[\ln \cos y\right]_0^{\tan^{-1} x} \\ &= x \tan^{-1} x + \ln \left(\cos \left(\tan^{-1} x\right)\right) \\ &= x \tan^{-1} x + \ln \frac{1}{\sqrt{1+x^2}} \\ &= x \tan^{-1} x - \frac{1}{2} \ln \sqrt{1+x^2}\end{aligned}$$

$$\int \sin^{-1} x \, \mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad v = x$$

$$u = \sin^{-1} x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}$$

$$\int \sin^{-1} x \, \mathrm{d}x = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

$$= x \sin^{-1} x + \sqrt{1 - x^2}$$

where the final integral is done either by substitution or by "intuition".

$$\int \cos^{-1} x \, \mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad v = x$$

$$u = \cos^{-1} x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{1 - x^2}}$$

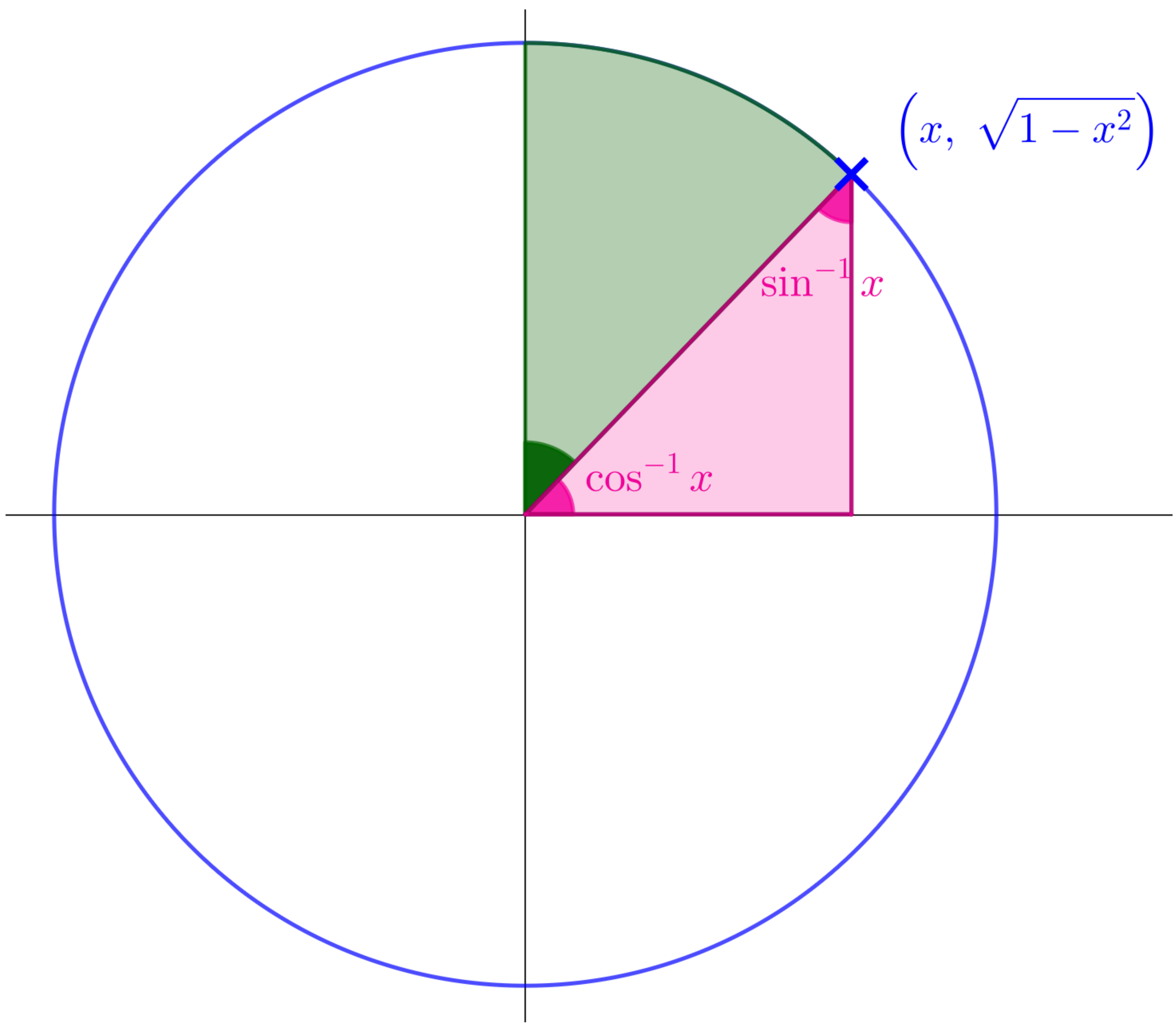
$$\int \cos^{-1} x \, \mathrm{d}x = x \cos^{-1} x + \int \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

$$= x \cos^{-1} x - \sqrt{1 - x^2}$$

Another integral using inverse circular functions

By finding the two shaded areas, find

$\int_0^x \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



$$\int_0^x \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}$$

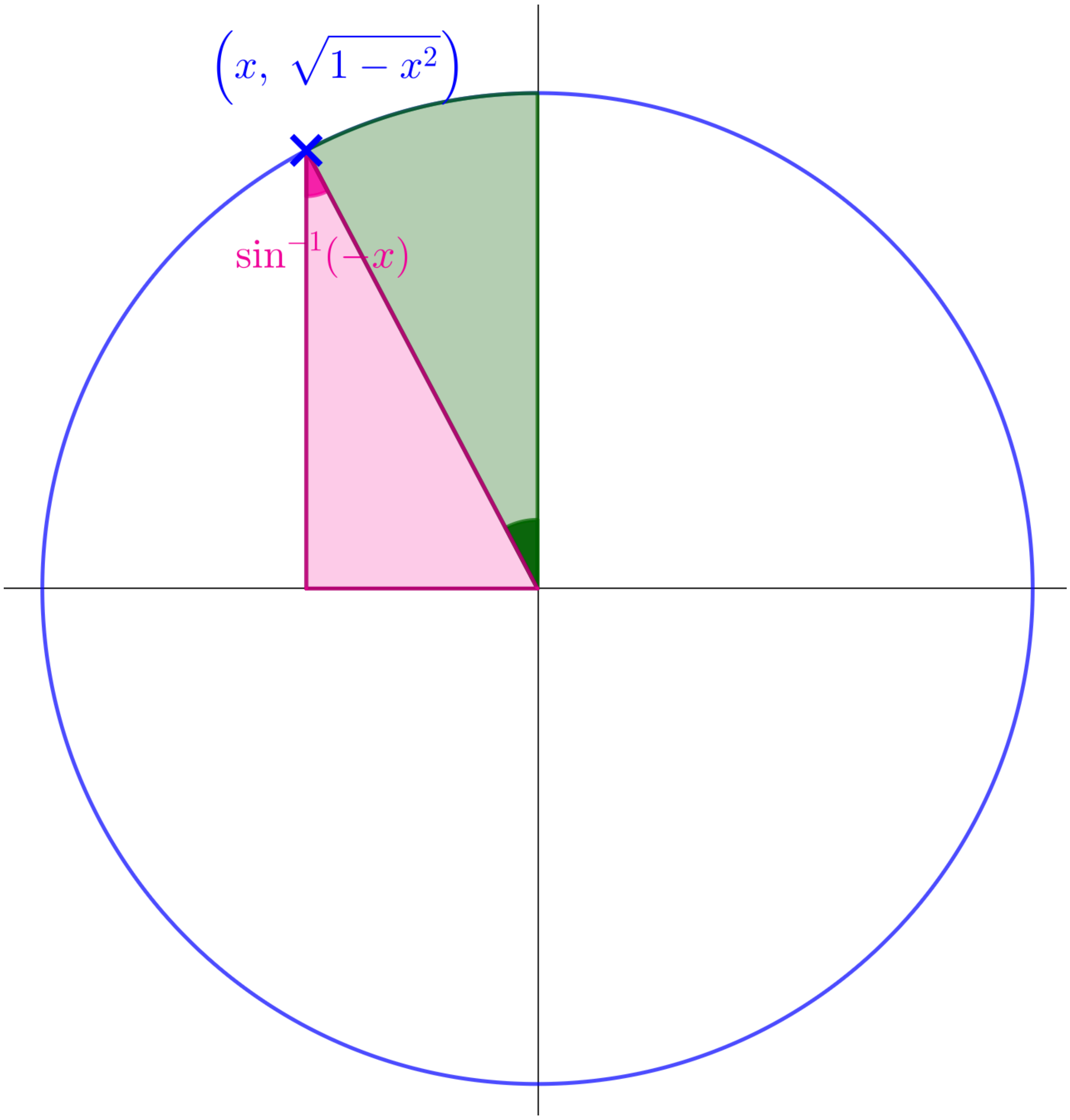
because the green angle is $\sin^{-1} x$ and the area of a segment is $\frac{1}{2}r^2\theta$

and

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + c$$

$x < 0$, by finding the two shaded areas, find

$\int_x^0 \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



$$\begin{aligned}
\int_x^0 \sqrt{1-x^2}\,\mathrm{d}x &= \left[\frac{1}{2}\sin^{-1}(-x) + \frac{1}{2}(-x)\sqrt{1-x^2}\right]_x^0 \\
&= \left[-\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2}\right]_x^0 \\
&= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}
\end{aligned}$$

and

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + c$$

Use the substitutions $u = \sin x$ and $u = \cos x$ to find $\int \sqrt{1 - x^2}\,\mathrm{d}x$

$$
\begin{aligned}
\int \sqrt{1 - x^2}\,\mathrm{d}x &= \int \sqrt{1 - x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}x \\
&= \int \cos^2 u\,\mathrm{d}u \\
&= \frac{1}{2}\int \cos 2u + 1\,\mathrm{d}u \\
&= \frac{1}{4}\sin 2u + \frac{1}{2}u + c \\
&= \frac{1}{2}\sin u \cos u + \frac{1}{2}u + c \\
&= \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x + c
\end{aligned}
$$

$$
\begin{aligned}
\int \sqrt{1 - x^2}\,\mathrm{d}x &= \int \sqrt{1 - x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}x \\
&= -\int \sin^2 u\,\mathrm{d}u \\
&= \frac{1}{2}\int \cos 2u - 1\,\mathrm{d}u \\
&= \frac{1}{4}\sin 2u \frac{1}{2}u + c \\
&= \frac{1}{2}\sin u \cos u + \frac{1}{2}u + c \\
&= \frac{1}{2}x\sqrt{1 - x^2} - \frac{1}{2}\cos^{-1} x + c
\end{aligned}
$$