radical maths

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for insight

Circular functions 1

Defining the circular functions

sin, cos, tan and the unit circle

teacher version

Circular functions

Defining the circular functions	**sin, cos, tan and the unit circle**
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90° - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A+B) = \sin A\cos B + \cos A\sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45°\right)$ and so on
Differentiating circular functions	radians, and tangents to graphs
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Defining the circular functions

My idea is to start with the trigonometric functions that my students already know, namely, ratios of sides in right-angled triangles, and to give them the chance to see for themselves how to extend these functions to include all real numbers in their domains. The purpose here is to begin a long process of divorcing the trigonometric functions from right-angled triangles, and ultimately from angles altogether.

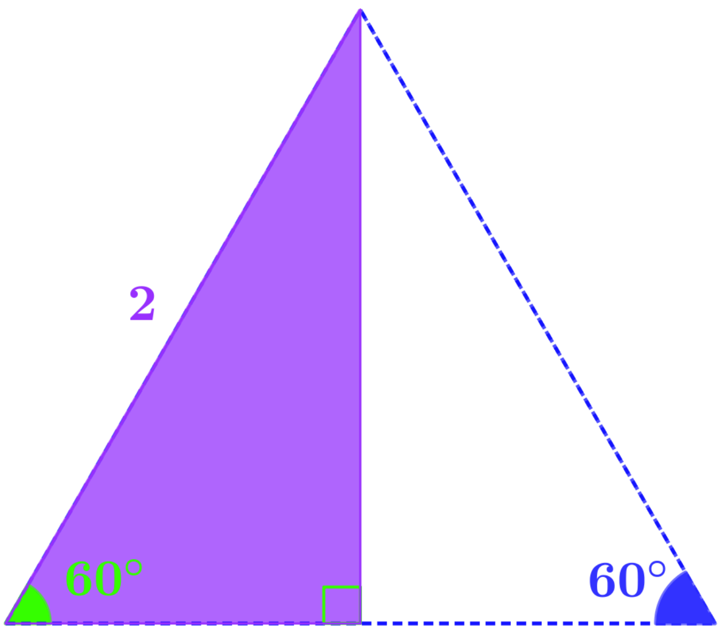
At the same time, I want to decouple the functions from calculators; that is, to remind students that the trigonometric functions are more than just something that your calculator gives you. To do this, I want my students to know the easy exact values of trigonometric functions. I'm not entirely sure why I think this important, but it has This has to do with embedding the image of the unit circle in the minds' eyes of my students, and with continually reinforcing the interpretation of the functions in terms of the unit circle, again in opposition to an understanding of the functions as something that the calculator tells you. This then means that the solution of trigonometric equations is built on a firm foundation.

Any text in blue and any diagrams with a blue border are only for you, and do not appear on the student version.

For you as a teacher, you can, if you feel attracted to the idea, use them to hand more responsibility for progress to your students, encouraging them always to think for themselves rather than to outsource their thinking to you, to their calculator, to the mark-scheme, or to artificial intelligence. This takes a tremendous amount of patience on your part: the patience to give them time to figure something out for themselves either individually or, more likely, by collaborating with each other.

These sheets are not in any way a complete course: they provide a context within which to develop insight and understanding by encouraging self-reliance and curiosity. They will not replace more traditional resources; in particular, you will need both routine exercises and more challenging sets of problems both for fluency and for creative problem-solving.

Use this diagram to find sin, cos, and tan of 60° and 30°.



This is a perfect introduction to the idea that your students can work together to figure things out without your direct help. Collaboration is the key here: they help each other along both with ideas and with motivation. They can achieve so much more together than they can alone.

They have to figure out

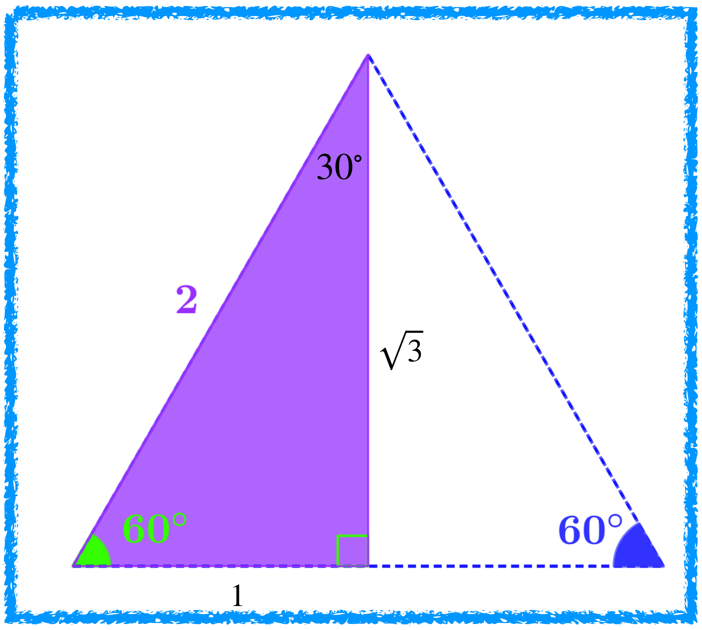
- that they need to know the sides of the right-angled triangle
- that the large triangle is equilateral and is divided exactly into two halves
- that they can use Pythagoras to find the height
- that they can then use the ratios that they already know to find the trig functions

and these are all steps they can take without the help of their teacher.

There is always a balance to find when deciding how much information to put in the question or diagram. I could have put the third 60˚ angle in the diagram, for example, or included the fact that the triangle is equilateral in the “question”. I could have left out the shading of the right-angled triangle and the vertical line segment, or even have left out the side-length of the equilateral triangle. The problem would still have been solvable, but this version seems to me to strike a good balance for a wide range of classes.

The temptation for the teacher will always be to help out, to give a hint, to answer a request for help, or to show them how to do it. Resisting this temptation takes considerable patience, sometimes in the face of strongly expressed frustration. But by giving your students the space to try out ideas, you allow them to gain greatly in terms of confidence in their own abilities, and to experience the satisfaction of having figured out something significant.

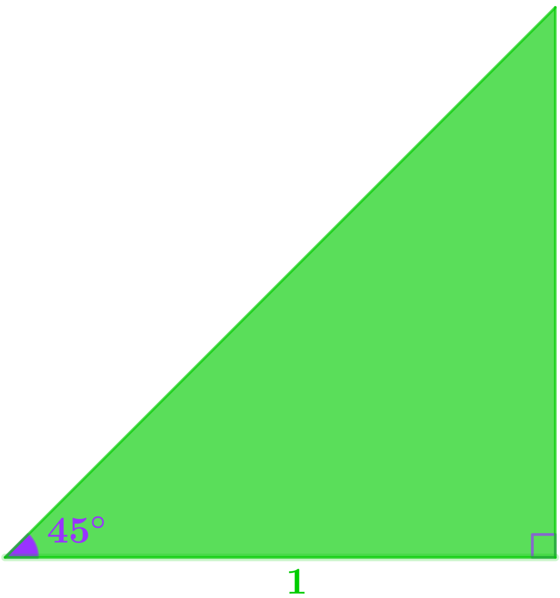
It is tremendously important that they tackle this page, and indeed most if not all of this lesson, without a calculator. I want them to be comfortable with exact values (rather than decimal approximations) and with fractions that include square roots. It is also important to me that they understand why, for example, $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, and are able to use both forms later in calculations if necessary.



$\cos 60^\circ = \frac{1}{2}$ $\quad \sin 60^\circ = \frac{\sqrt{3}}{2}$ $\quad \tan 60^\circ = \sqrt{3}$

$\sin 30^\circ = \frac{1}{2}$ $\quad \cos 30^\circ = \frac{\sqrt{3}}{2}$ $\quad \tan 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

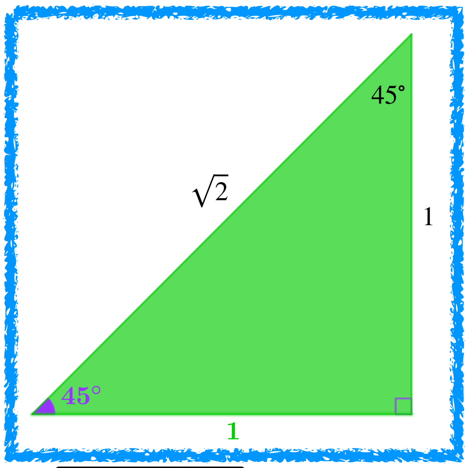
Use this diagram to find sin, cos, and tan of 45°.



Here, your students will need to see that:

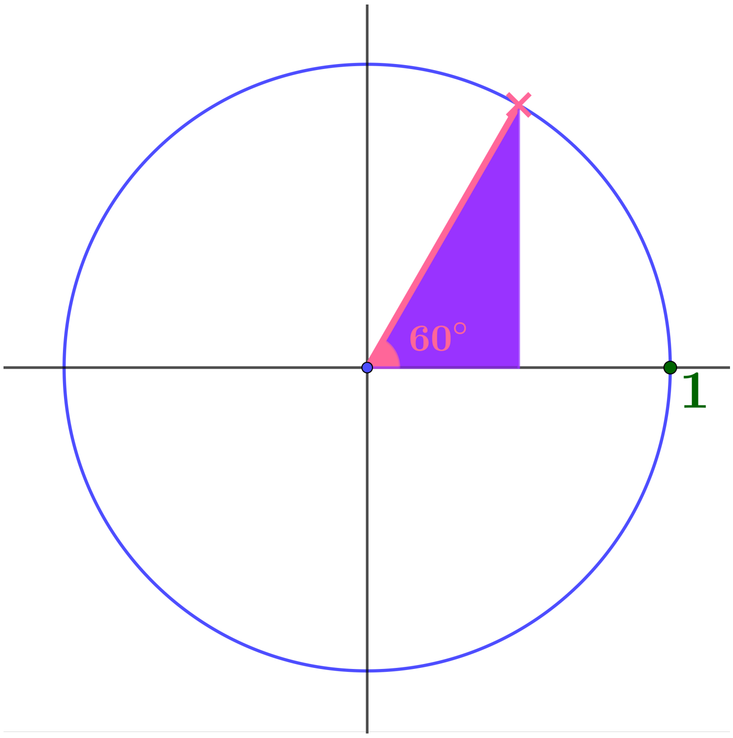
- the top angle is also 45°
- the triangle is isosceles
- the vertical side of the triangle is also 1
- they can use Pythagoras to find the hypotenuse
- they can use the trig ratios that they already know
- they can rationalise the denominator

These are all things that they can figure out for themselves, or at least by working with others: there is really no need for you to help at all!



$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \tan 45^\circ = 1$$

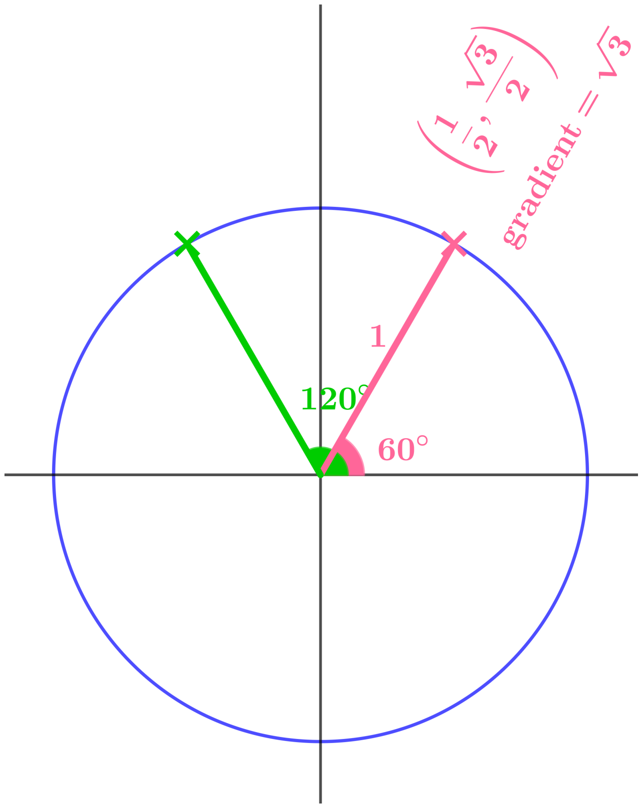
What are the coordinates of the pink point and the gradient of the pink radius?



Putting what they have just done into the context of a unit circle is the first step towards generalising the trigonometric ratios in right-angled triangles to circular functions of any real number (whether or not that number corresponds to an angle). It might take a while before your students figure out that they are dealing with exactly the same problem as before, but they will, if given the time, the space, the encouragement, and the confidence to do so.

Some might have the coordinates and the gradient as $\left(\cos 60^{\circ},\ \sin 60^{\circ}\right)$ and $\tan 60^{\circ}$, others as $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\sqrt{3}$, but in discussion, it is really important to be sure that all your students understand both ways of writing the coordinates and gradient as this is the clue to extending the domains of the circular (trigonometric) functions.

What are the coordinates of the green point and the gradient of the green radius?



Again, resist the temptation to help by pointing out the symmetry and what this might mean for the coordinates and gradient. They can do this without you, and will benefit far more from doing so.

Suggest values for sin, cos, and tan of 120°.

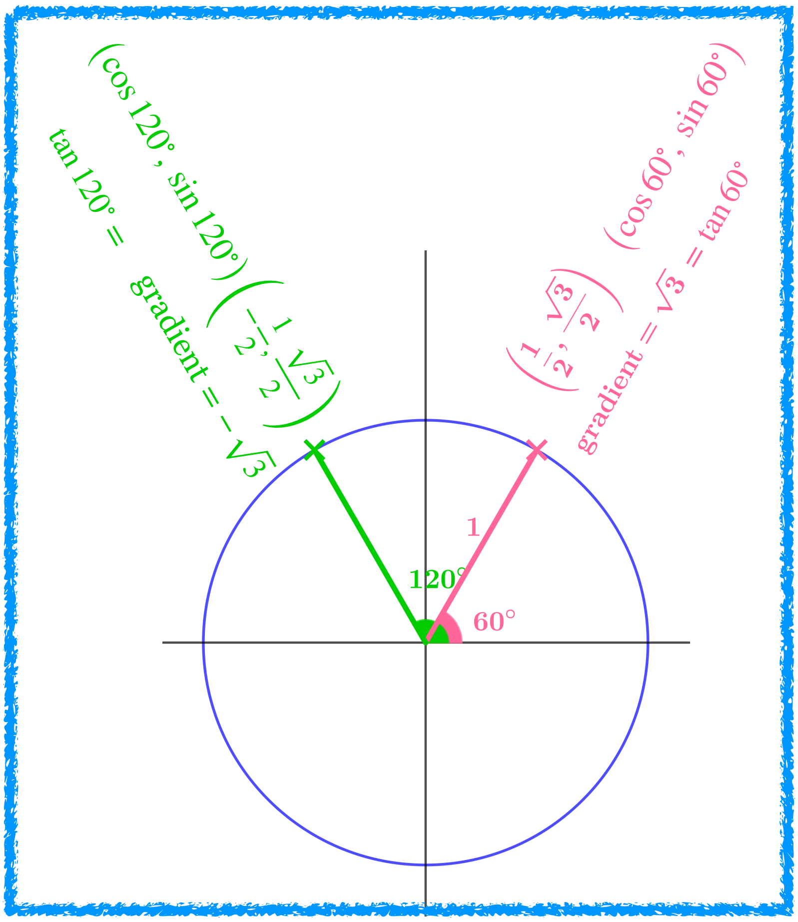
This is where they begin to see the relationship between coordinates and gradients on the one hand, and the circular functions sin, cos, and tan on the other.

The key word here is symmetry. It is the symmetry of the diagram that allows us to see right away that the coordinates of the pink point are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and that the gradient is $-\sqrt{3}$.

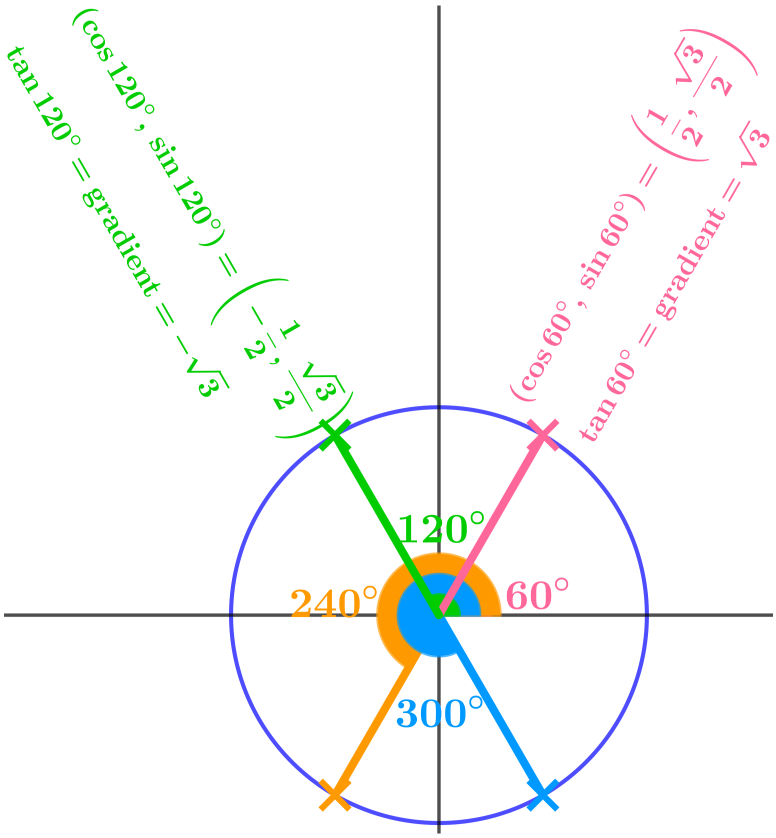
The sequence of questions is designed to suggest that cos is the x coordinate, sin is the y coordinate, and tan is the gradient; suggest so strongly, in fact, that this conclusion appears more or less inevitable. What it achieves is giving students an idea of why this way of defining cos, sin, and tan agrees with their earlier understanding of the functions for angles between 0° and 90°. and hence why assigning the value $-\frac{1}{2}$ to cos 120°, for example, is reasonable.

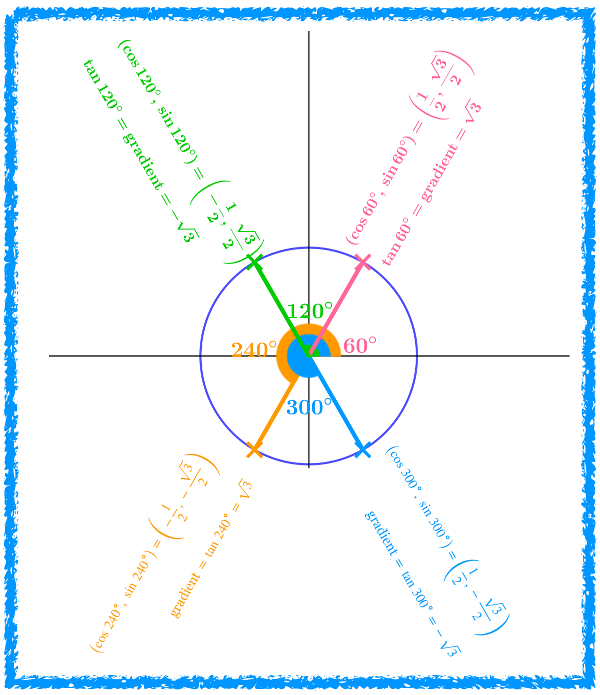
Again, following your class discussion of this page, your students should have a diagram that has both the exact values of the coordinates and gradient, and the values in terms of cos, sin, and tan.

$$\cos 120^\circ = -\frac{1}{2} \quad \sin 120^\circ = \frac{\sqrt{3}}{2} \quad \tan 120^\circ = -\sqrt{3}$$



Suggest values for sin, cos, and tan of 240° and 300°.

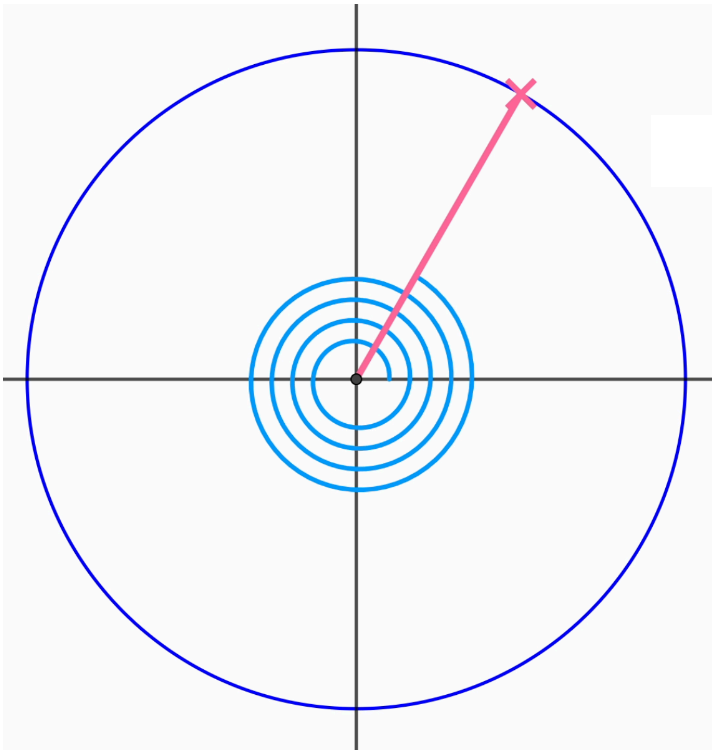




Now we can build on this to extend our definitions to angles between 0° and 360°, but your students have to figure out each step: what are the coordinates and gradients, and what are the angles, all using symmetry? This way, they become more familiar with the unit circle diagram and with its value for defining trigonometric functions.

This is probably a good moment to think about the name: trigon is a three-sided shape, and "metry" is to do with measurement, so "trigonometry" means "measurement of triangles". Here, however, we are trying to decouple the functions from the triangles, so the name "circular functions" makes far more sense. Consider introducing this new name for the functions at this point, though with an awareness that most resources will continue to use the term "trigonometric functions". I suppose we could justify this by understanding the term to mean "functions related to the measurement of triangles".

What are sin, cos, tan of 420°, 780°, 1140°, 1500° ?



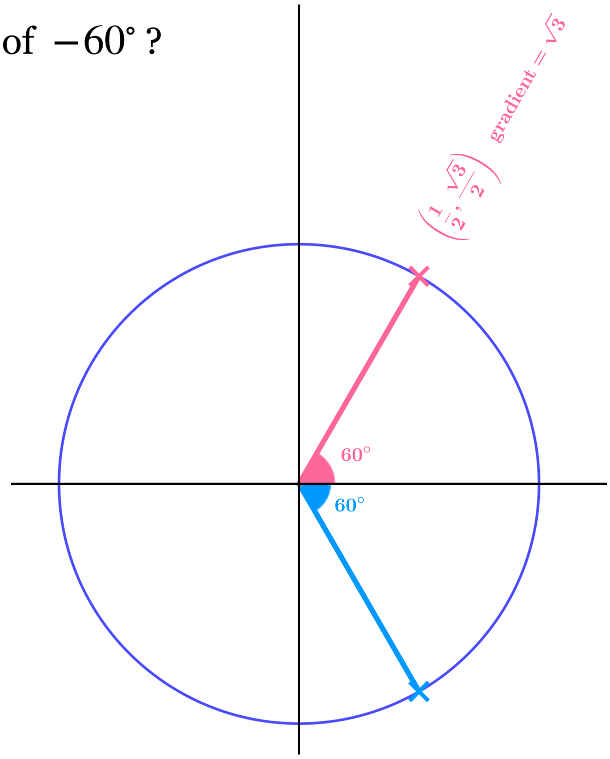
What about angles bigger than 360° and negative angles?

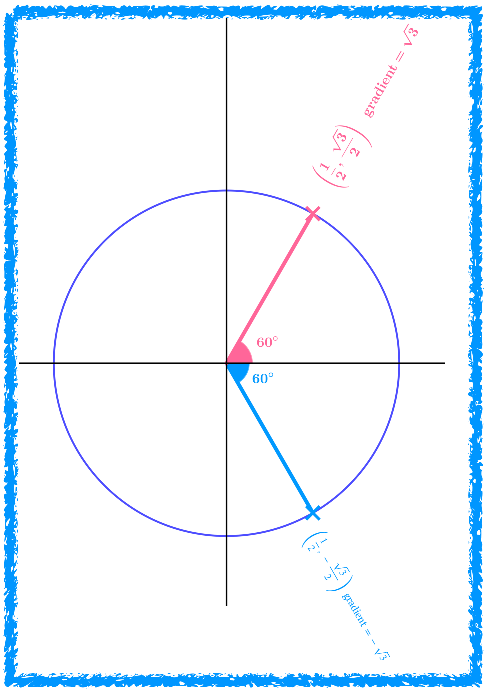
These are strange ideas: how can an angle be negative, and how can an angle be bigger than 360°? The answer is that they can't really, so here we are beginning to uncouple the functions from the idea of angles altogether. Ultimately, this will lead to definitions that have nothing to do with triangles or circles, and graphs will become more useful as tools for thinking about them. For now, though, we can extend our ideas of angles to include these strange animals using the unit circle.

Here, students are led to the extension of the circular functions to these "non-existent" angles just as they were at the start to angles bigger than right-angles.

Each of these "angles" has the same sin, cos, and tan as 60°.

Suggest values for sin, cos, tan of -60° ?





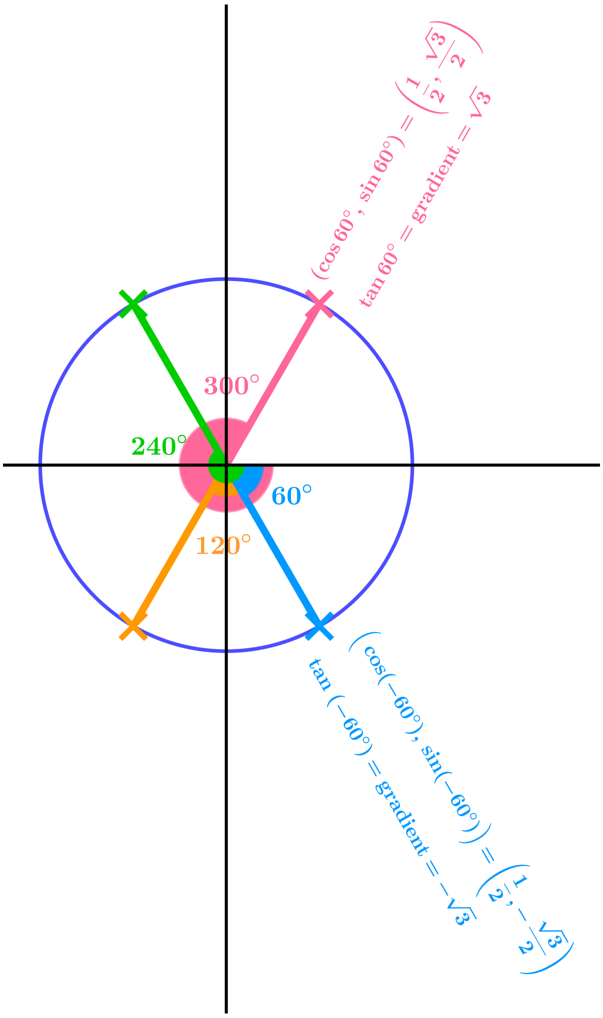
$$\cos(-60^\circ) = \frac{1}{2} \quad \sin(-60^\circ) = -\frac{\sqrt{3}}{2} \quad \tan(-60^\circ) = -\sqrt{3}$$

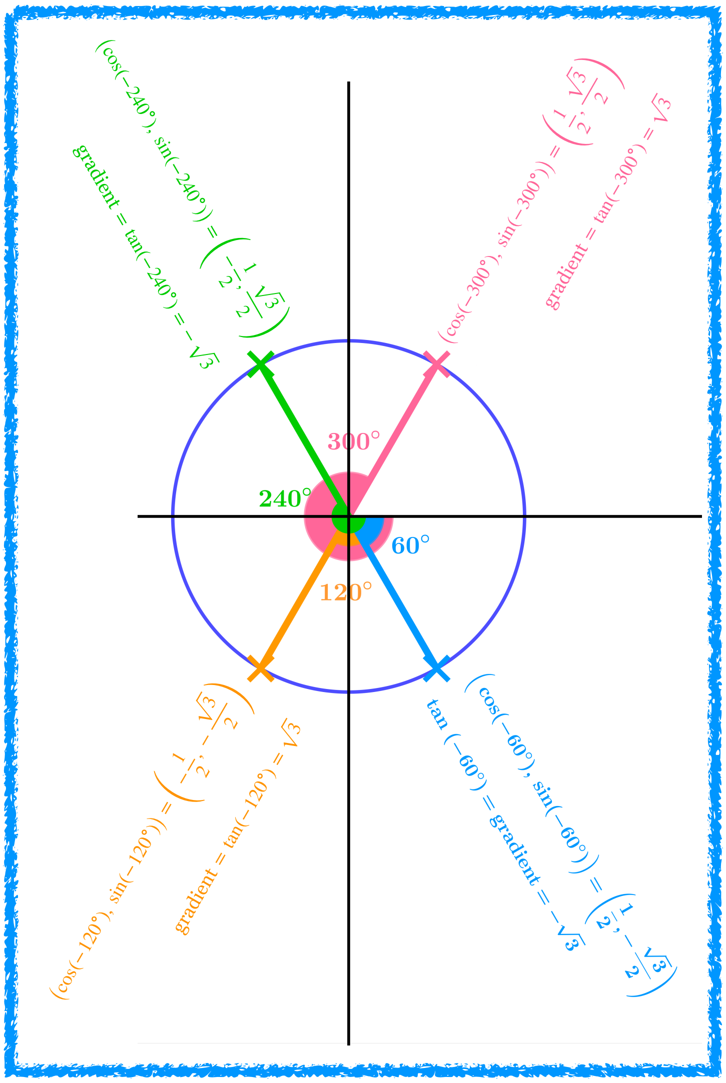
So far, all angles have been measured anticlockwise from the positive x axis. This has been so obvious from the start that it may well not even have been mentioned up to this point.

The time has come to make this more explicit and to discuss the appropriateness of using angles measured clockwise from the positive x axis as representative of negative angles.

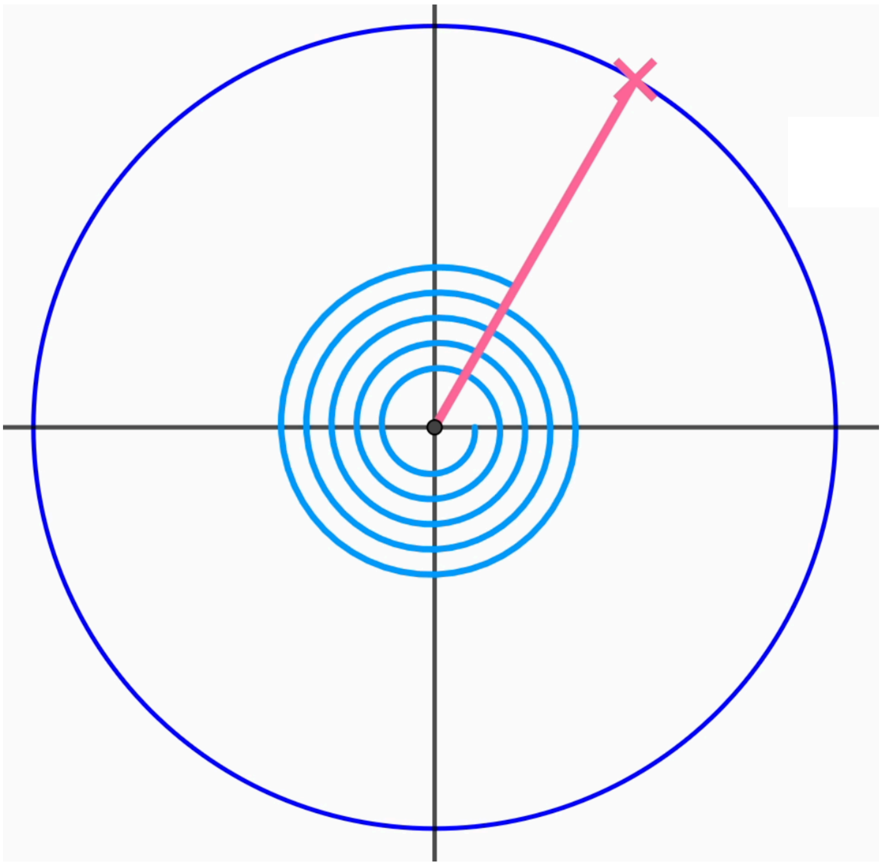
This always raises the question of labelling on the diagram: should we label the blue angle here 60° or -60°? There is no hard and fast rule here. I prefer to label the diagram with a positive number, because, for example here, the angle really is 60° in the sense that your students have always understood it. But then I use -60° in my working.

Suggest values for sin, cos, tan of -120° -240° -300° .





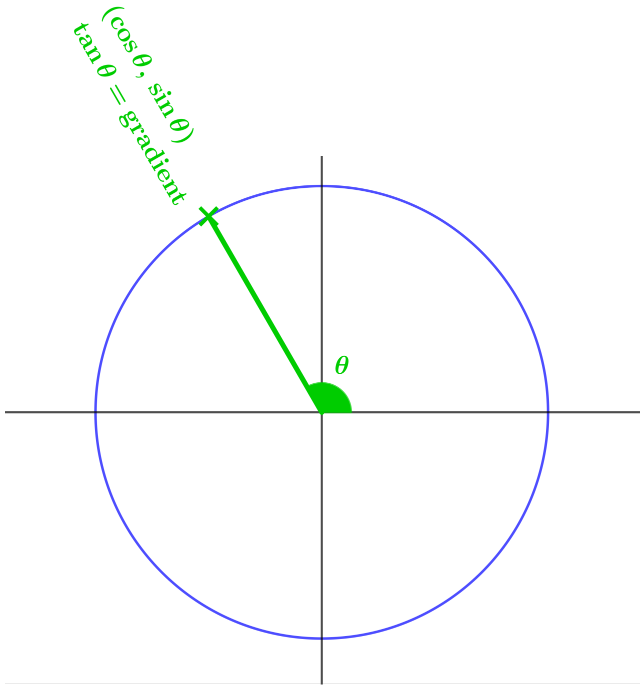
Suggest values for sin, cos, tan of $-660^\circ, -1020^\circ\ -1380^\circ\ -1740^\circ$.

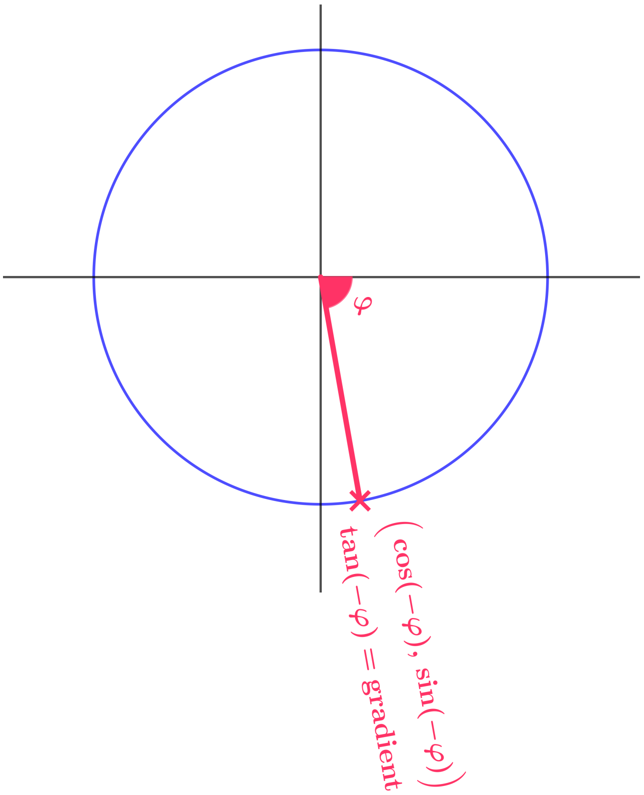


Each of these “angles” has the same sin, cos, and tan as -60°.

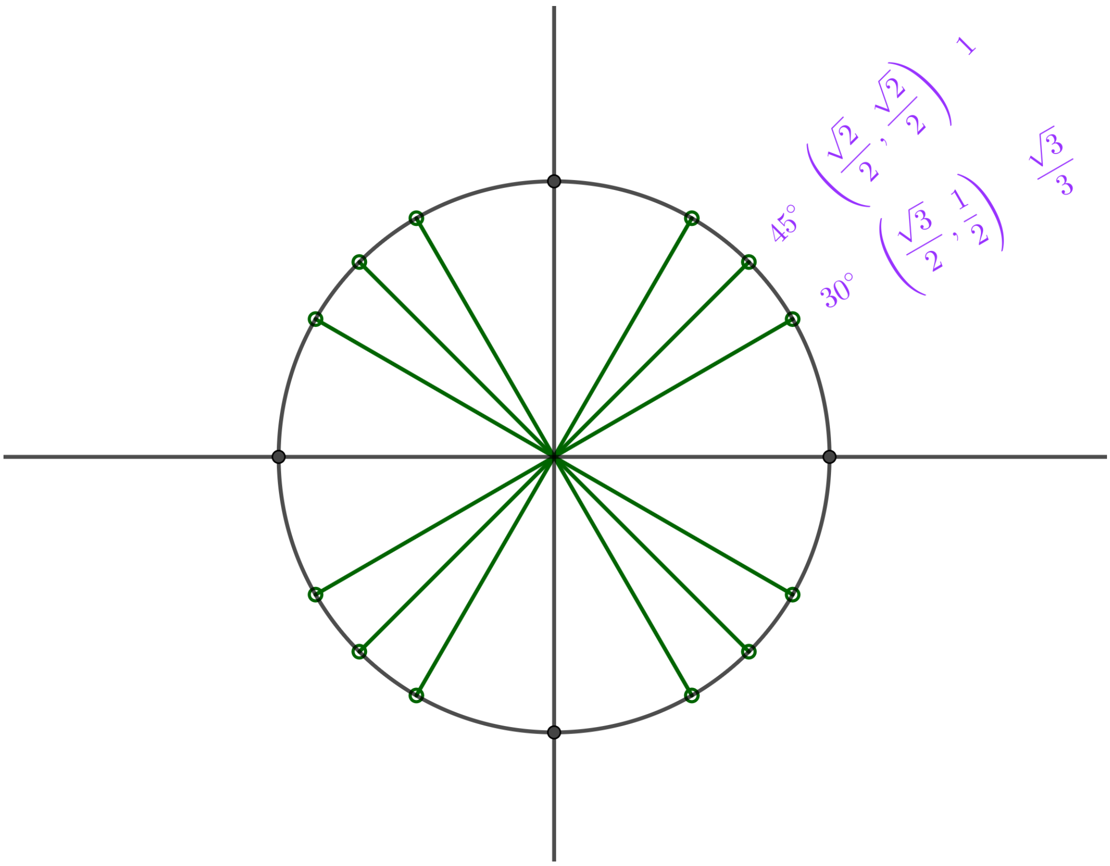
These diagrams essentially form definitions of sin, cos, and tan of any number (angle).

There are other ways to define these functions using, for example, graphs or power series, but this one makes intuitive sense and forms a reliable foundation for what is to come.



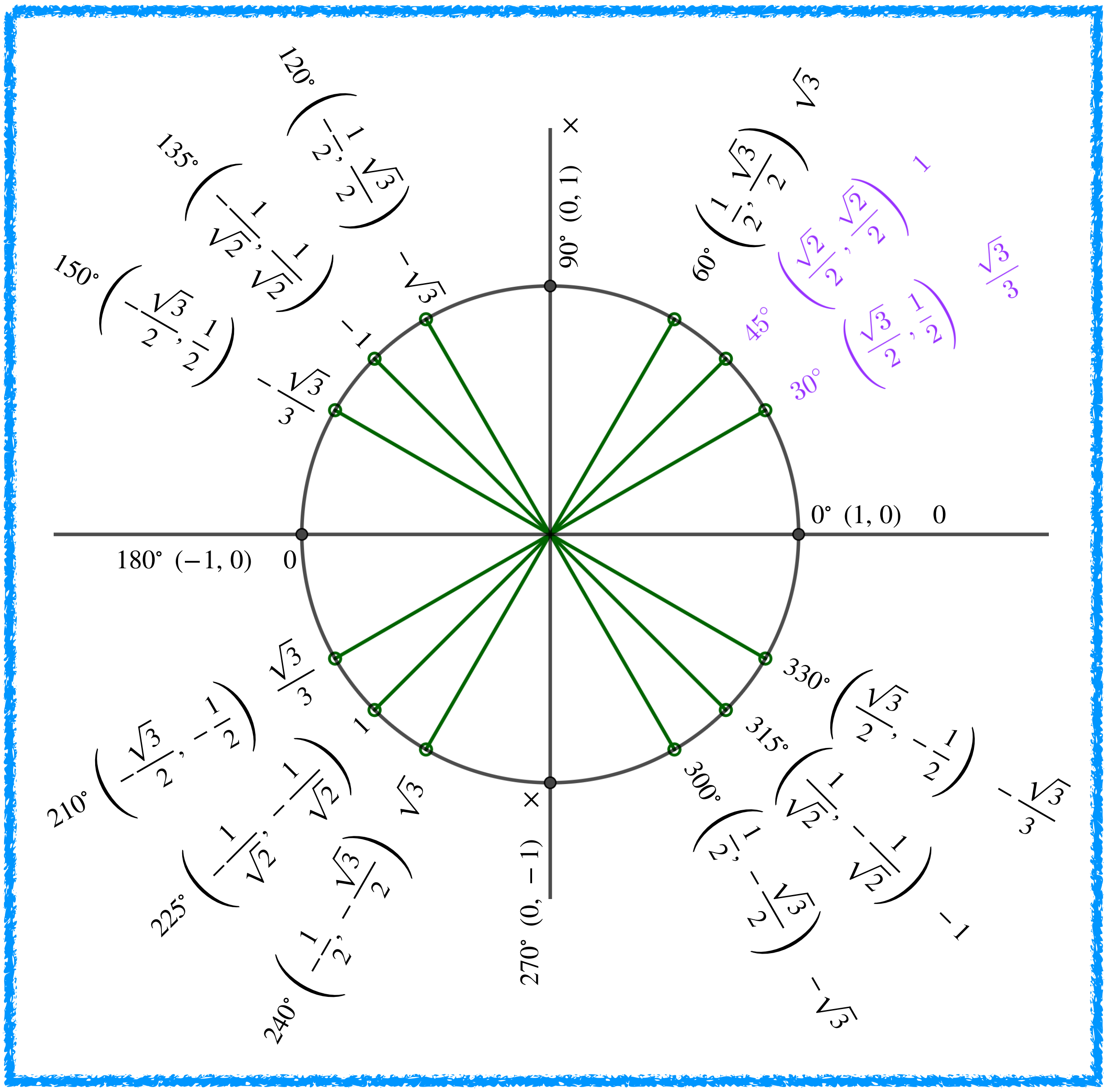


Complete this diagram.



This diagram is absolutely fundamental to all understanding of the circular functions, and I want my students to know it intimately. To this end, I ask them to fill in a blank copy at the start of several lessons; I even set a timing goal. If they can do this fluently, they have a really strong understanding of what the circular functions are all about.

It's handy to know these without the calculator, but ultimately, knowing this diagram is about insight, not about practicality. Every time you use a calculator for one of these, you erode your understanding of the meaning of the circular functions.



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Circular functions 2

Solving circular functions equations

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — **like $\sin\theta = 0.4$**

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions — $\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

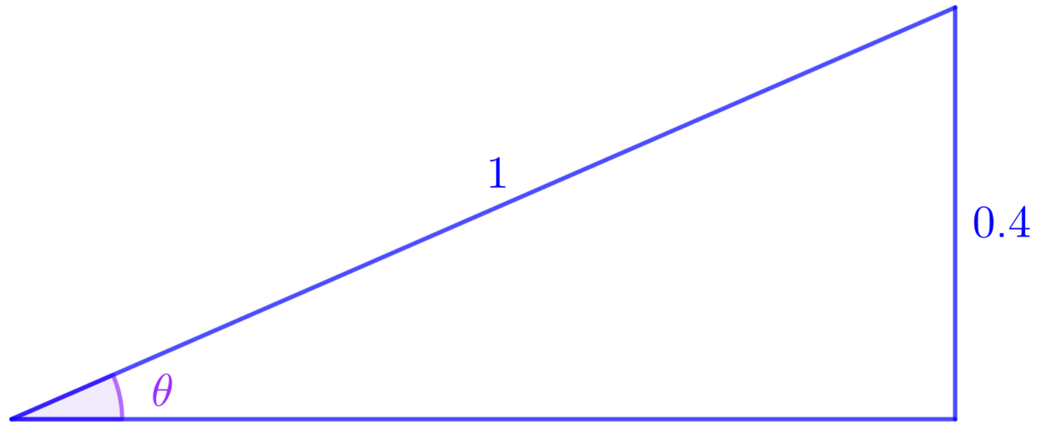
Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Solving equations with circular functions

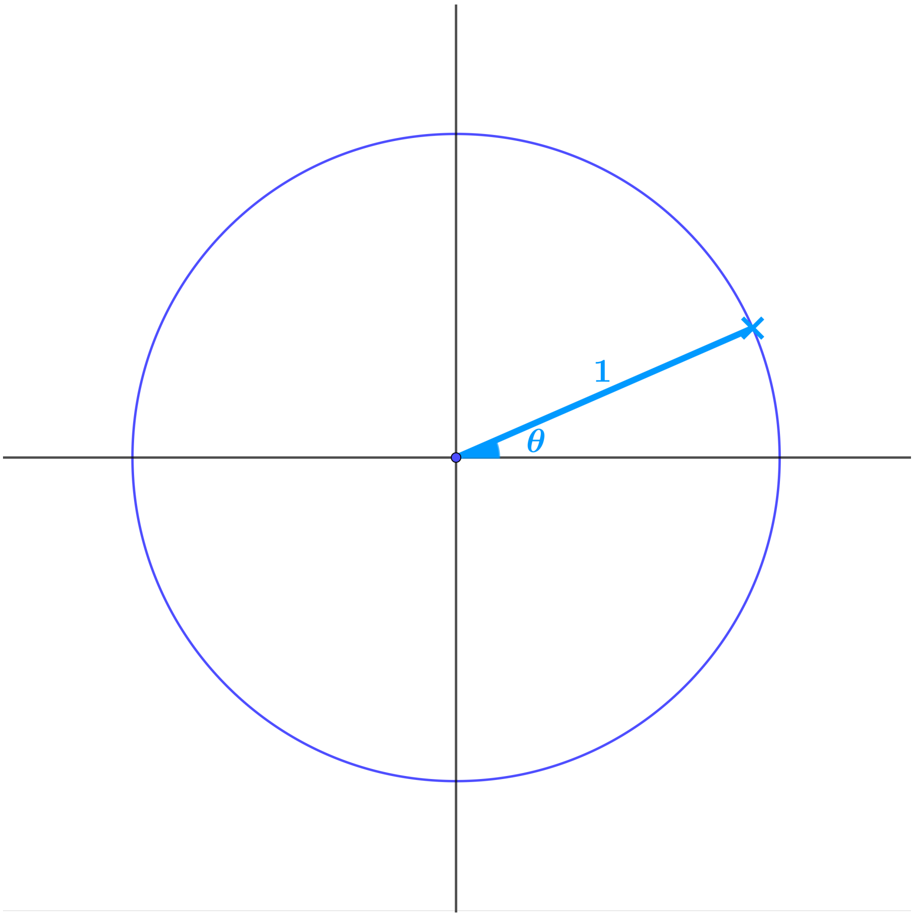
Use your calculator to find the angle θ.



There are an infinite number of solutions to an equation like $\sin\theta = 0.4$, and this is only the smallest positive solution. It's worth spending the time on this sheet, as this is the next step in the process of divorcing the circular functions from triangles. Trigonometry means "measurement of three-sided shapes", but sin, cos, tan and so on are really functions rather than ratios of sides of "trigons", and their power is far greater than that needed for the measurement of three-sided shapes.

Finding other solutions is a question of symmetry, and you can either use this unit circle to do your visualisation or you can use graphs. I am a great believer in the advantages of the unit circle over graphs, and that is the approach I take here.

If the y coordinate of the blue point is 0.4, find the angle θ.

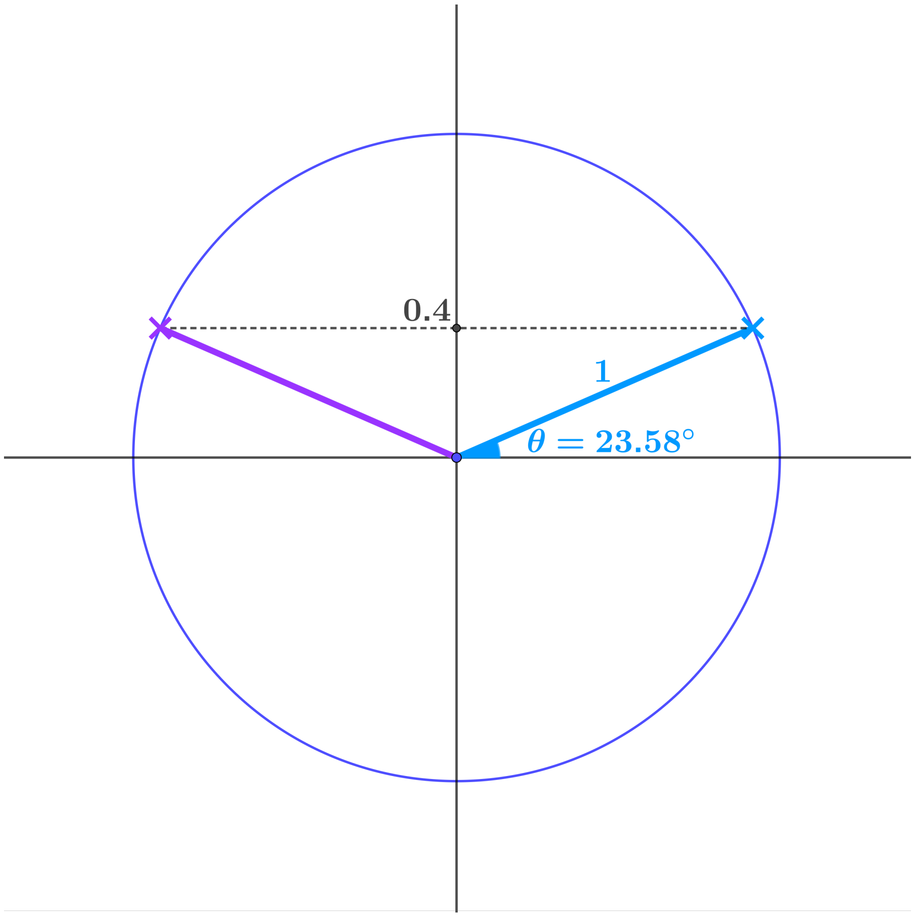


What other point on the circle has the same y coordinate as the blue point?

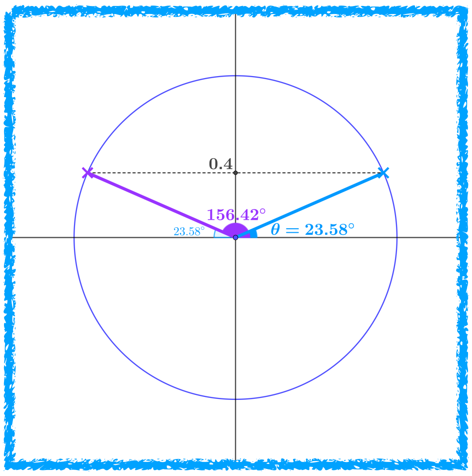
If you have always taught your students to use graphs to find solutions to equations like this, I strongly recommend that you give this way a go. It has many advantages: for example, the unit circle is easy to sketch when solving a problem, and easy to visualise without sketching. It's easy to remember that the coordinates of a point are $(\cos\theta, \sin\theta)$ and that the gradient of the radius is $\tan\theta$. On the other hand, for graphs, you have to remember quite a bit of detail about three different graphs, and then it's much harder to use symmetry to read off the solutions.

You might find it takes some getting used to, but I don't think you will ever look back!

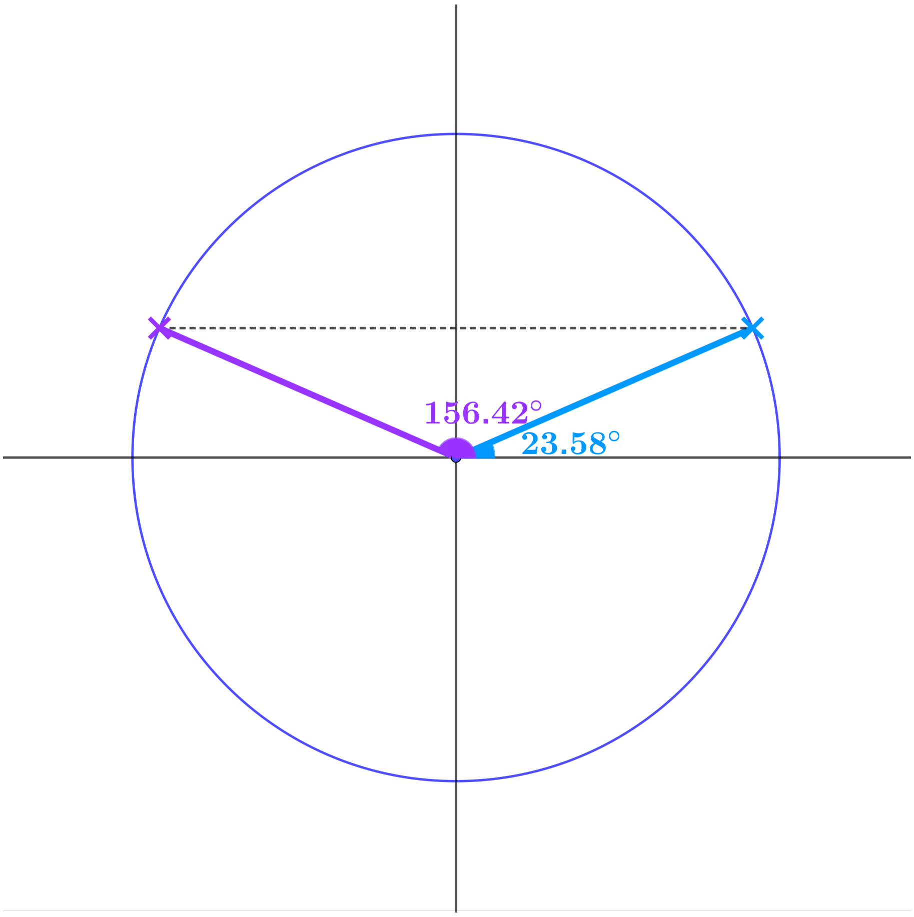
What other positive angle between 0° and 360° is a solution of $\sin\theta = 0.4$?



Symmetry is the key (as it is with graphs), so
$180 - 23.58 = 156.42°$ is another solution of $\sin\theta = 0.4$

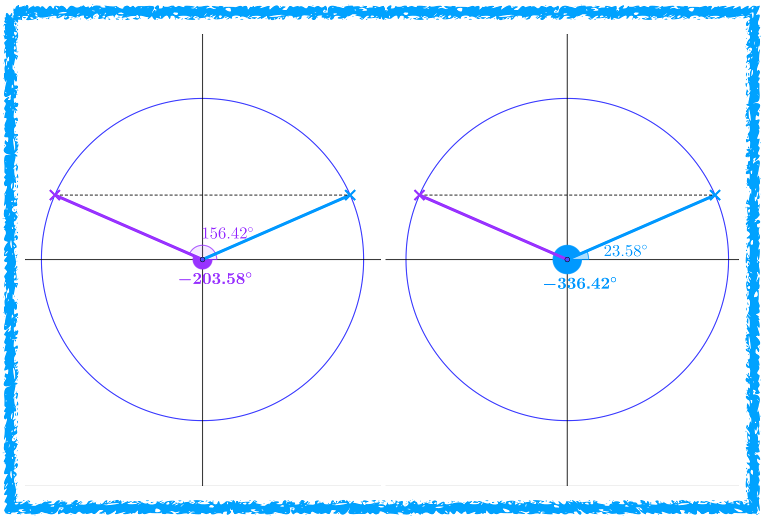


What negative angles between $-360°$ and $0°$ are solutions of $\sin\theta = 0.4$?

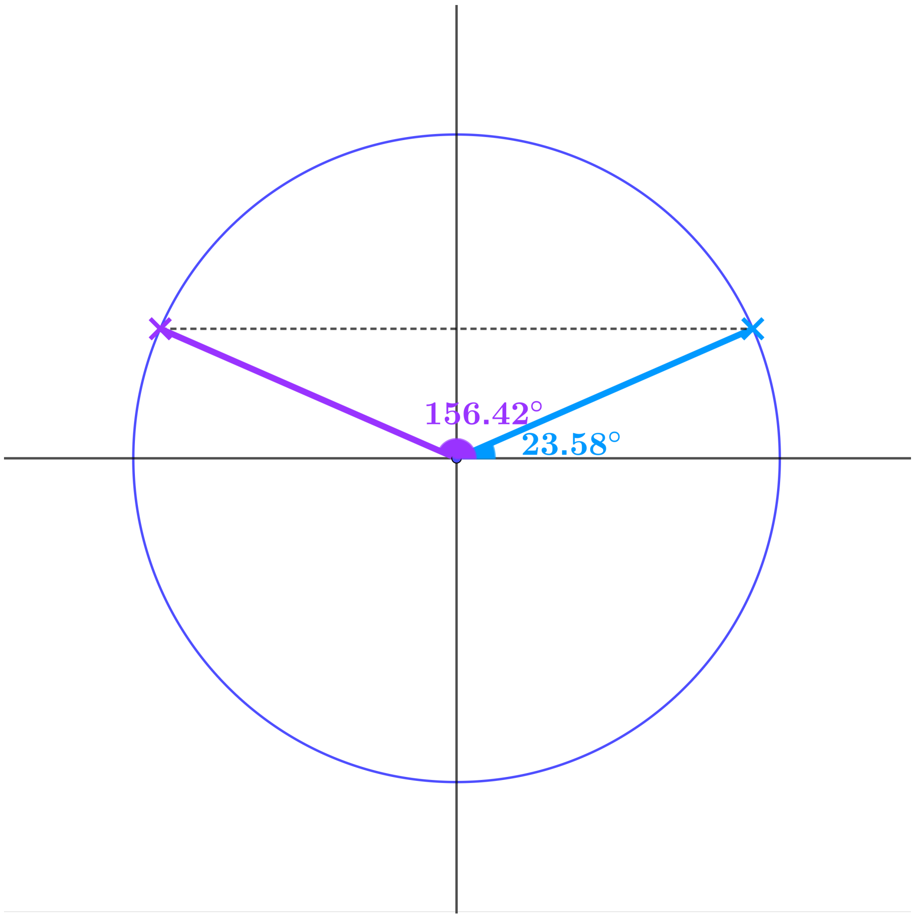


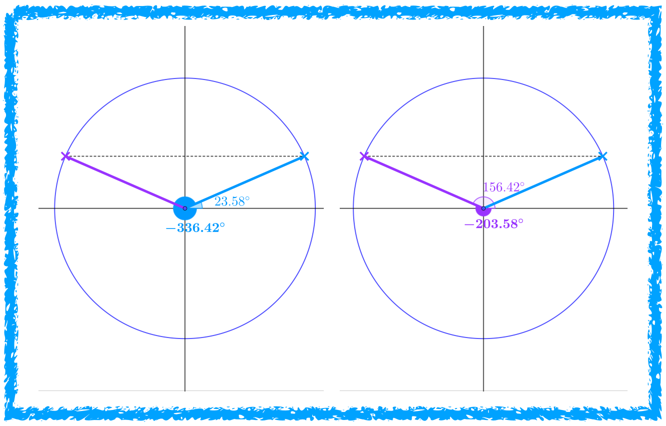
To find the size of the angles, subtract the positive angles from 360°.

But they are measured clockwise from the positive x axis, so they are negative.



Solve the equation $\sin\theta = 0.4$





$\theta = 23.58°, 156.42°, 383.58°, 516.42° \ \ldots$

$-203.58°, \ -336.42°, \ -563.58°, \ -696.42° \ \ldots$

Keep adding or subtracting 360° as often as you like.

If α is any solution of the equation $\sin\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

$180 - \alpha$

$\alpha + 360$

$\alpha - 360$

The point of asking this question is to get to some easy rules for solving sin, cos, and tan equations that don't even need the unit circle, let alone graphs.

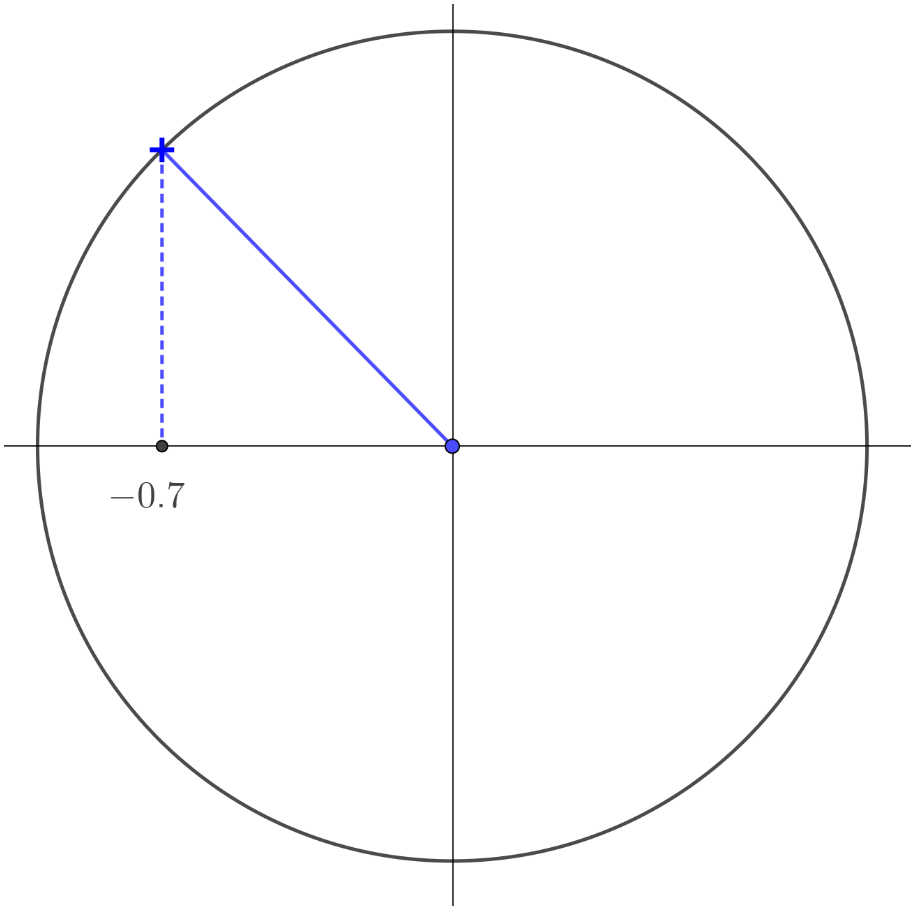
Here is the first easy rule to remember:

- for sin, take your answer from 180.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

Later, we will see that $180 + \alpha$ works for tan and $-\alpha$ works for cos.

The last two options, $\alpha \pm 360$, work for sin, cos, and tan.

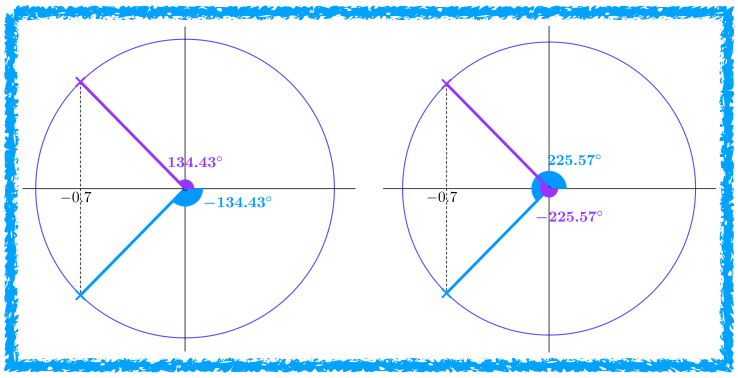
Adapt the previous method to solve the equation $\cos\theta = -0.7$.



This may seem like a bit leap, but it's just the same method, using x instead of y coordinates. Give your students plenty of time to figure this out by working together, rather than pointing the way. If they have had that time, then your help later will make far more sense.

$\theta = 134.43^{\circ}, 225.57^{\circ}, 494.43^{\circ}, 585.57^{\circ}\ \ldots$

$-134.43^{\circ},\ -225.57^{\circ},\ -494.43^{\circ},\ -585.57^{\circ}\ \ldots$



If α is any solution of the equation $\cos\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

This is the second easy rule to remember:

- for cos, take the negative of your answer.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

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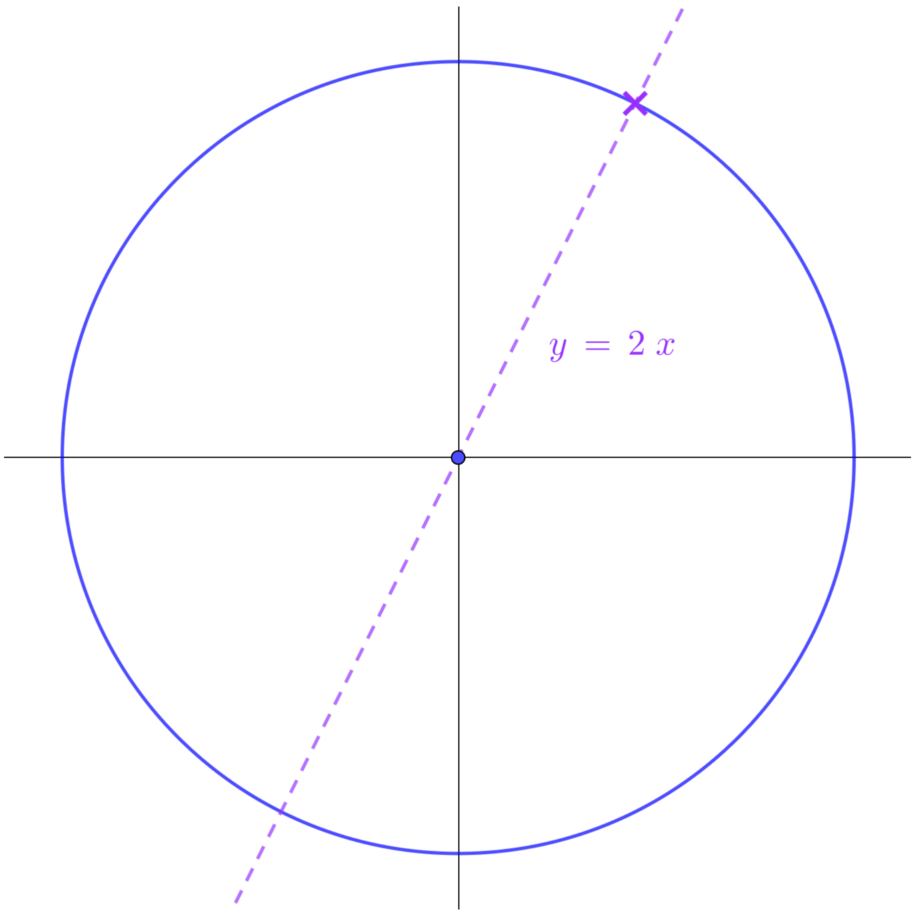
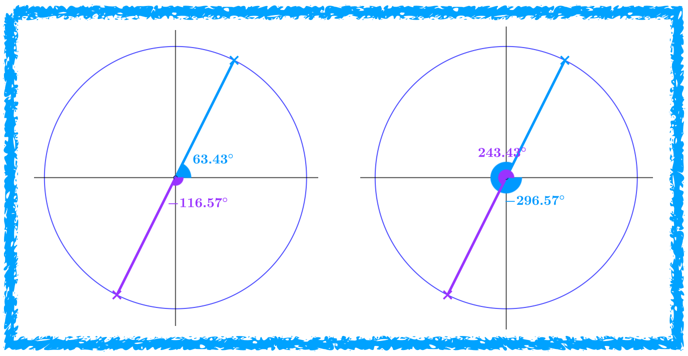
Use this diagram and a calculator to solve the equation

$$\tan\theta = 2$$



$\theta = 63.43°, 243.43°, 423.43°, 603.43° \ \ldots$

$\quad -116.57°, \ -296.57°, \ -476.57°, \ -656.57° \ \ldots$



If α is any solution of the equation $\tan\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

$180 + \alpha$

$\alpha + 360$

$\alpha - 360$

Here is the third easy rule to remember:

- for tan, add 180 to your answer.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

Now recap the three rules:

sin: find one solution, take it away from 180, and then repeatedly ± 360.

cos: find one solution, take it's negative, and then repeatedly ± 360.

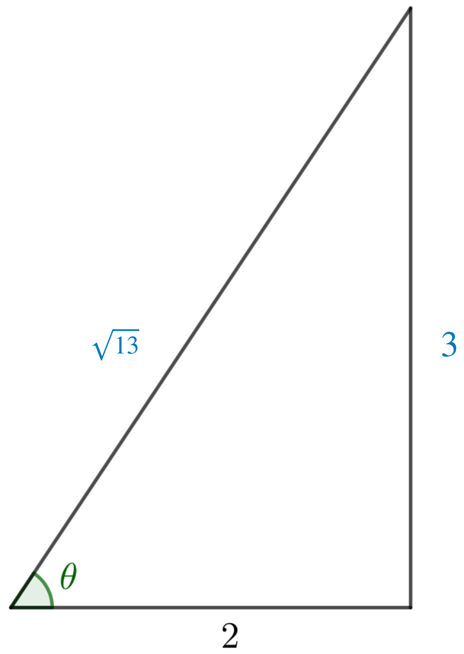
tan: find one solution, add it to 180, and then repeatedly ± 360.

At this point, your students will need plenty of basic practice. When they first encounter an equation like $\sin(2\theta) = 0.4$, they will probably find $\sin^{-1} 0.4$, then divide this by two, and then use the $\pm 180 \pm 360$ rule to find what they think are the other solutions.

I like to let them do this once, use their calculator to find sine of one of their non-solutions, see that it is not 0.4 and ask them to figure out what went wrong.

They will still need a bit of prompting to get the right method, which is to find all the possible values of 2θ and then divide these all by 2.

If $\tan\theta = \dfrac{3}{2}$, find $\sin\theta$ and $\cos\theta$.



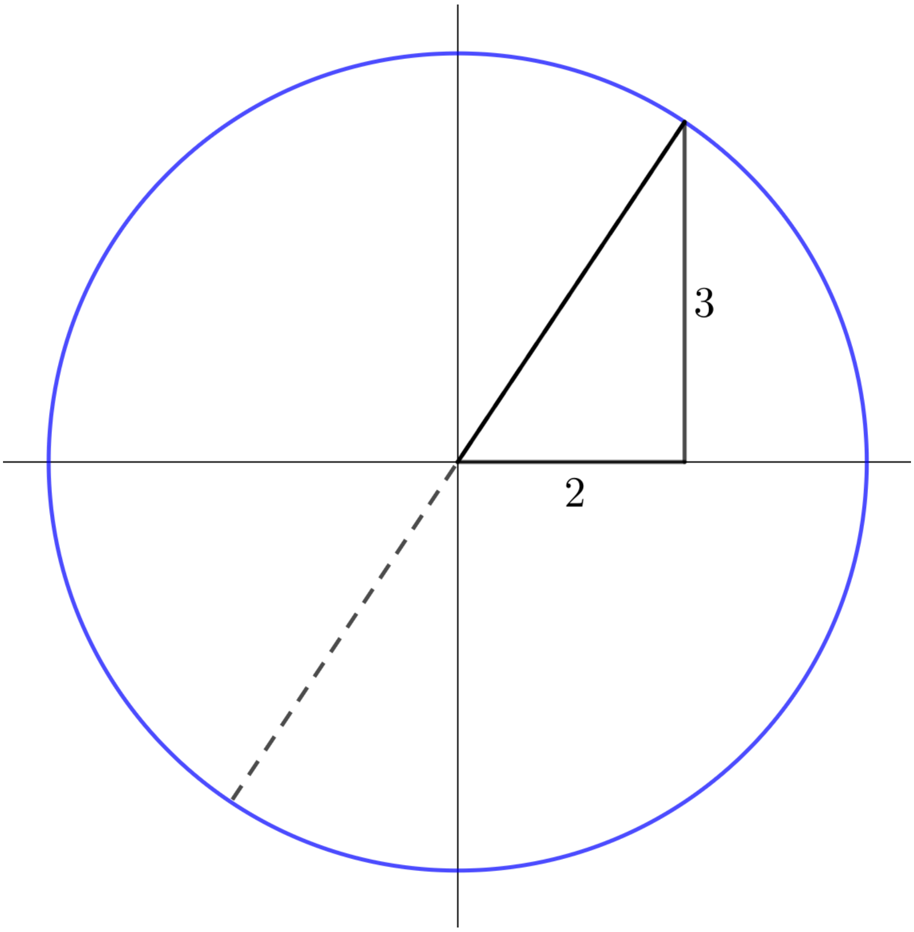
This is a right-angled triangle question.

$\sin\theta = \dfrac{3}{\sqrt{13}} \qquad \cos\theta = \dfrac{2}{\sqrt{13}}$

Here is a very useful non-calculator technique that tends to get overlooked. Of course, you can use the calculator to find θ and then find sin and cos. This, however, teaches you nothing about the circular functions, how they work, and how they relate to each other.

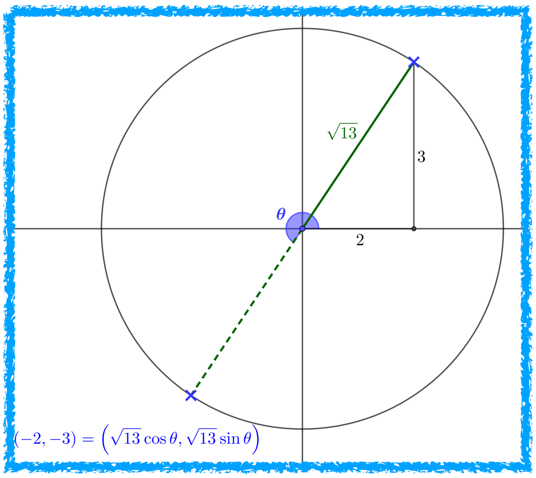
So don't let you students use their calculators for these!

If $\tan\theta = \dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.



Here, I am using a circle that no longer has radius 1. However, all the same considerations from earlier apply. In this case, the whole unit-circle diagram has been enlarged by scale factor $\sqrt{13}$.

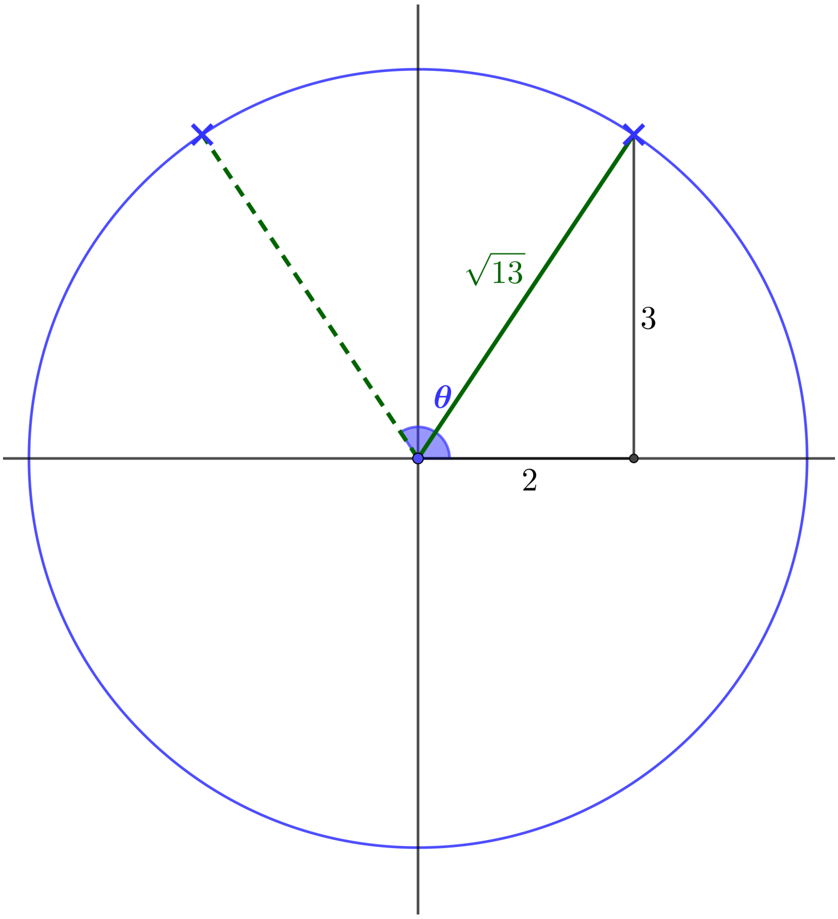
We could redraw the diagram with radius 1, but it's really more trouble than it's worth.



Here, $\tan\theta$ is still $\dfrac{3}{2}$, but

$$\sin\theta = -\frac{3}{\sqrt{13}} \qquad \cos\theta = -\frac{2}{\sqrt{13}}$$

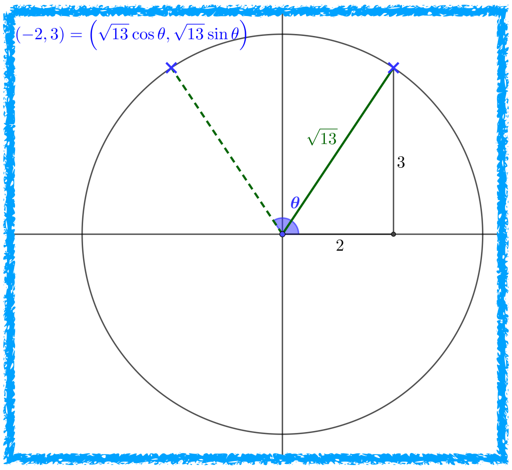
If $\tan\theta = -\dfrac{3}{2}$, and θ is obtuse, find $\sin\theta$ and $\cos\theta$.



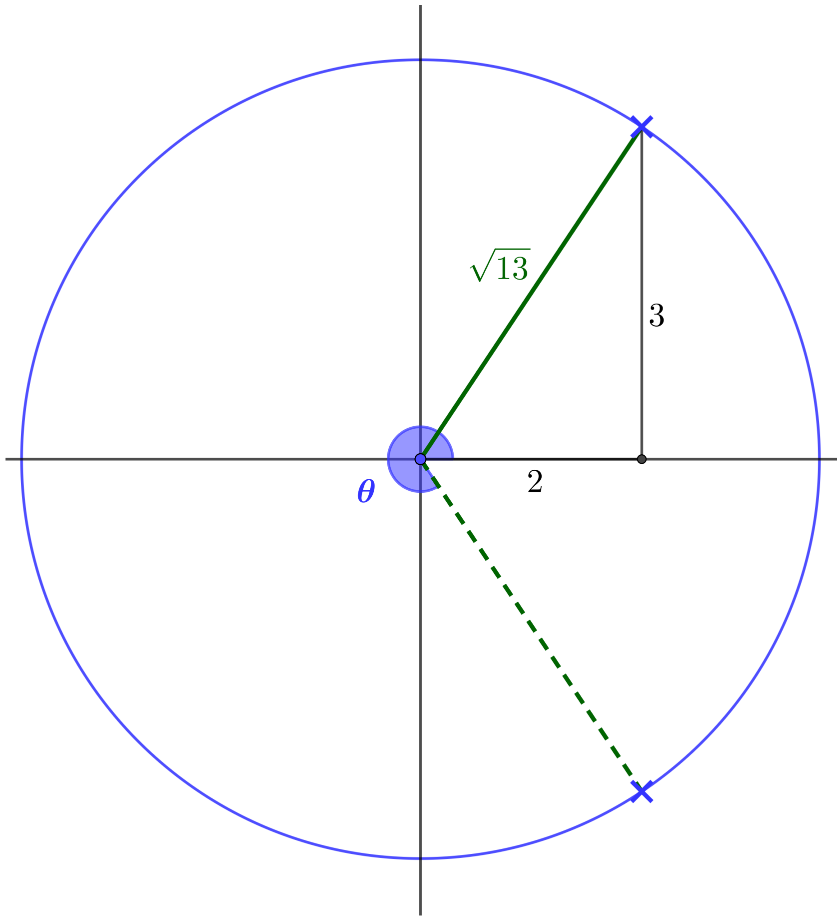
Now we need the point where the tan of the angle is the same size as before, but negative.

Here, $\tan\theta = -\dfrac{3}{2}$ and

$\sin\theta = \dfrac{3}{\sqrt{13}}$ $\qquad$ $\cos\theta = -\dfrac{2}{\sqrt{13}}$

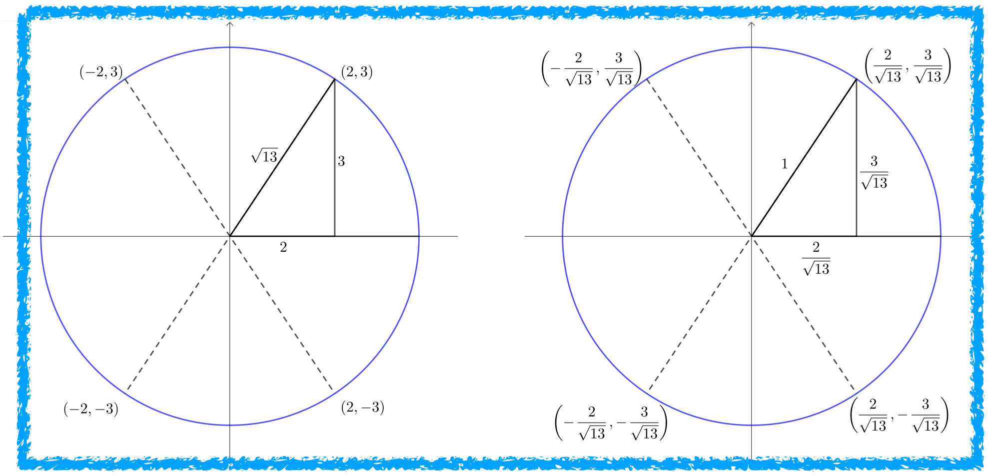


If $\tan\theta = -\dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.

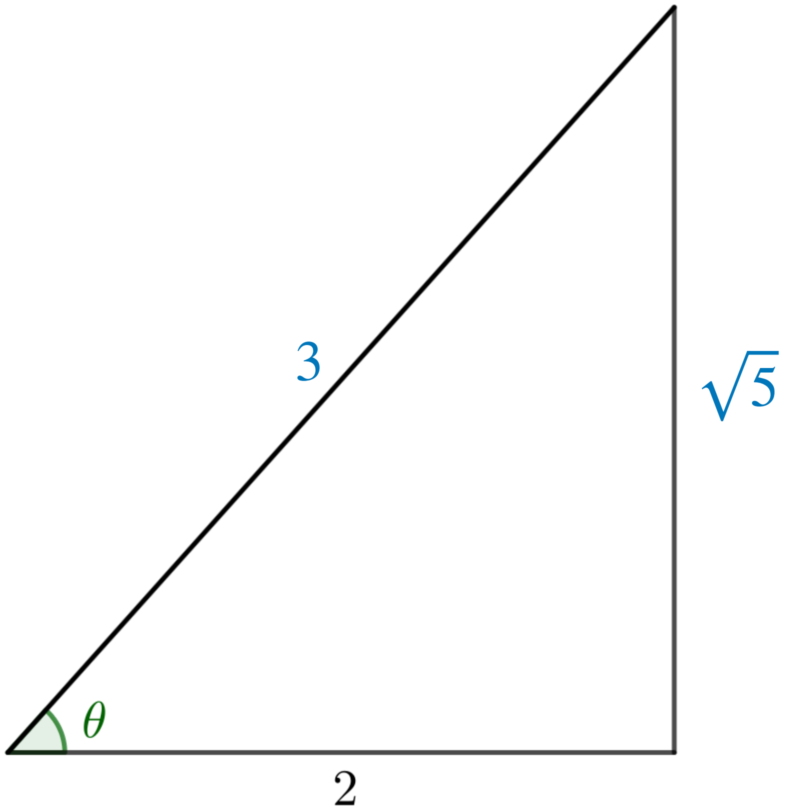


Here,

$\tan\theta = -\dfrac{3}{2}$ and $\sin\theta = -\dfrac{3}{\sqrt{13}}$ $\qquad \cos\theta = \dfrac{2}{\sqrt{13}}$



If $\cos\theta = \dfrac{2}{3}$, find $\tan\theta$ and $\sin\theta$.



$\tan\theta = \dfrac{\sqrt{5}}{2}$ $\sin\theta = \dfrac{\sqrt{5}}{3}$

Find $\tan\theta$ and $\sin\theta$ when:

$$\cos\theta = \frac{2}{3}, \text{ and } \theta \text{ is between } 270^\circ \text{ and } 360^\circ$$

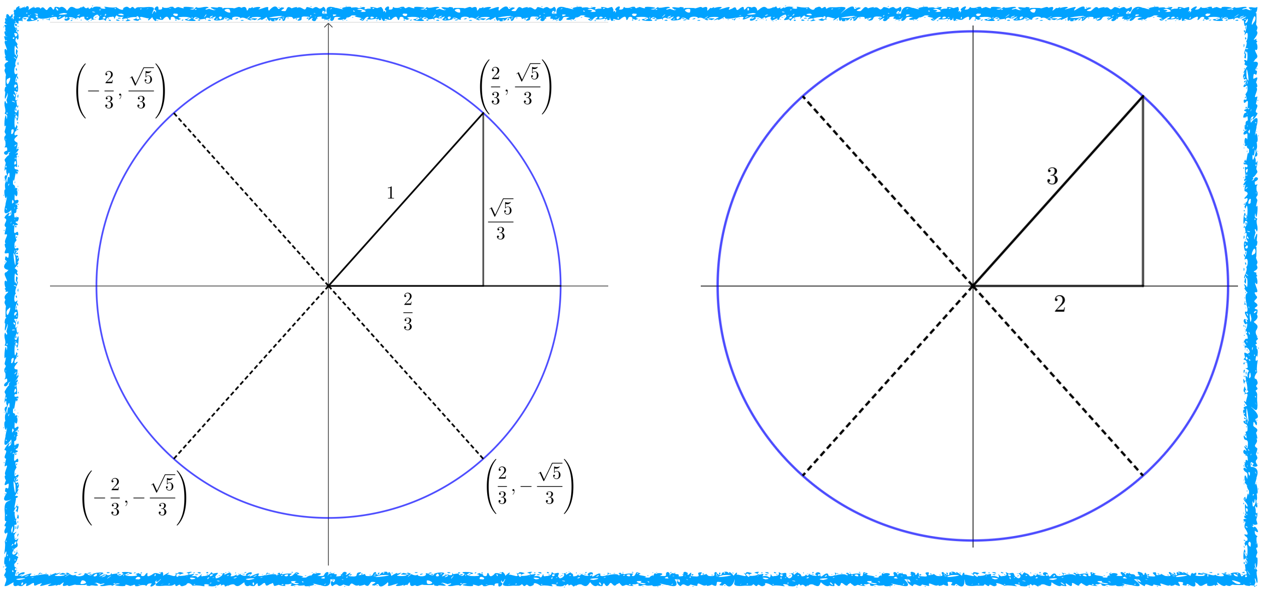
$$\tan\theta = -\frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$

$$\cos\theta = -\frac{2}{3}, \text{ and } \theta \text{ is between } 180^\circ \text{ and } 270^\circ$$

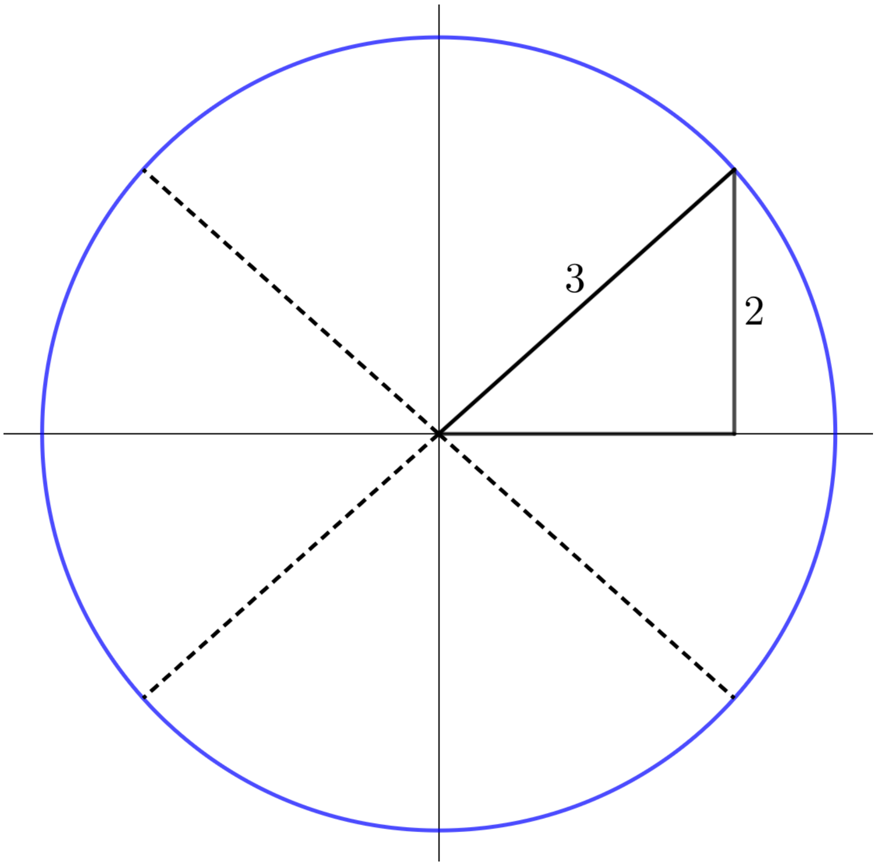
$$\tan\theta = \frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$

$$\cos\theta = -\frac{2}{3}, \text{ and } \theta \text{ is obtuse.}$$

$$\tan\theta = \frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$



Find $\tan\theta$ and $\cos\theta$ when $\sin\theta = \pm\frac{2}{3}$ for the various possible values of θ.



$0 < \theta < 90^\circ$ $\quad \sin\theta = \frac{2}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$ $\quad \cos\theta = \frac{\sqrt{5}}{3}$

$90^\circ < \theta < 180^\circ$ $\quad \sin\theta = \frac{2}{3}$ $\quad \tan\theta = -\frac{2}{\sqrt{5}}$ $\quad \cos\theta = -\frac{\sqrt{5}}{3}$

$180^\circ < \theta < 270^\circ$ $\quad \sin\theta = -\frac{2}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$ $\quad \cos\theta = -\frac{\sqrt{5}}{3}$

$270^\circ < \theta < 360^\circ$ $\quad \sin\theta = -\frac{2}{3}$ $\quad \tan\theta = -\frac{2}{\sqrt{5}}$ $\quad \cos\theta = \frac{\sqrt{5}}{3}$

radical maths

for independence
for confidence
for creativity
for insight

Circular functions 3

Graphs of circular functions

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — **graphs $y = \cos x$ and the like**

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions — $\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

The main difficulty with the unit circle approach to circular functions is that the axes on the graph with the unit circle are not the same as the axes on the graphs of the functions. For example, with $y = \sin x$, the angle on the circle becomes the x coordinate on the graph, but the y coordinate on the circle is still the y coordinate on the graph, whereas on the graph $y = \cos x$, it is the x coordinate on the circle that becomes the y coordinate on the graph. I have tried so many ways to introduce the graphs over the years, most of which do the job but leave understanding just a little hazy. The questions here are designed to demonstrate to my students just how the circle and the graphs are related; or look at it another way, how to generate the graphs from the unit circle definitions. This seems to me rather more worthwhile than making a table of values with (or without) a calculator, as it shows just why the curve is the shape that it is.

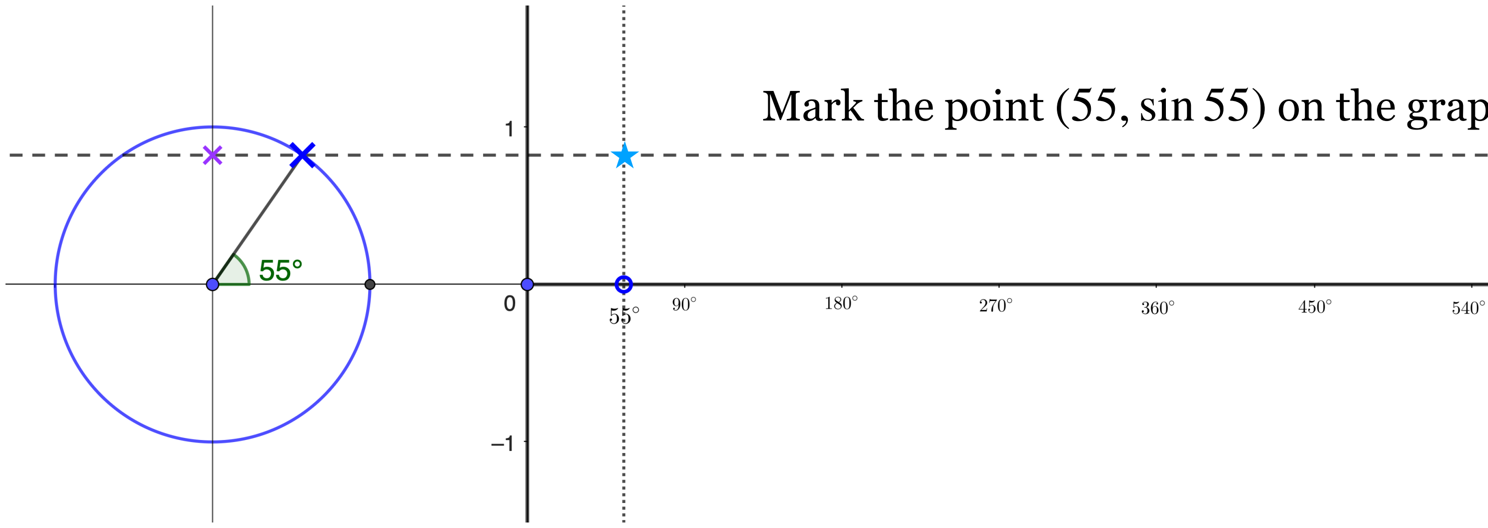
Without some kind of exercise along these lines, it's quite possible to leave school either without having seen the unit circle definitions of sin, cos, and tan, or without understanding the shape of the curve, or both. This makes working with these functions a skill without any understanding, almost magic.

Of course this relationship is far easier to demonstrate with dynamic geometry or animation, and this is a valuable approach. Even this, though, still shows students something that they can nod their heads to without gaining an embodied (that is, the strongest possible) sense of why things are the way they are.

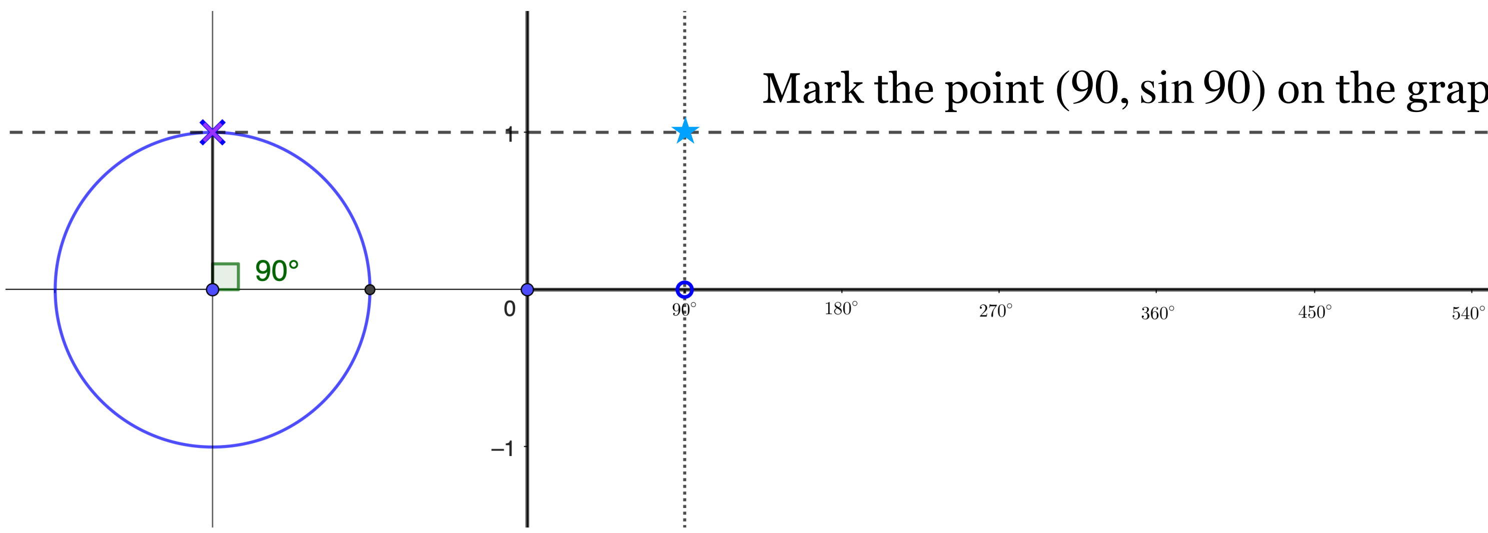
In the first sequence, your students may well get the principle right away or they may need a little time to understand the diagrams and what the question is asking. Give them space to work this out, or challenge them to work it out for themselves, rather than pointing it out.

In the cos sequence, it's a bit harder, but ask them to think about the relationship between the lengths of the pink lines on the circles and the lengths of the pink lines on the graphs. Soon enough, someone will spot what's going on!

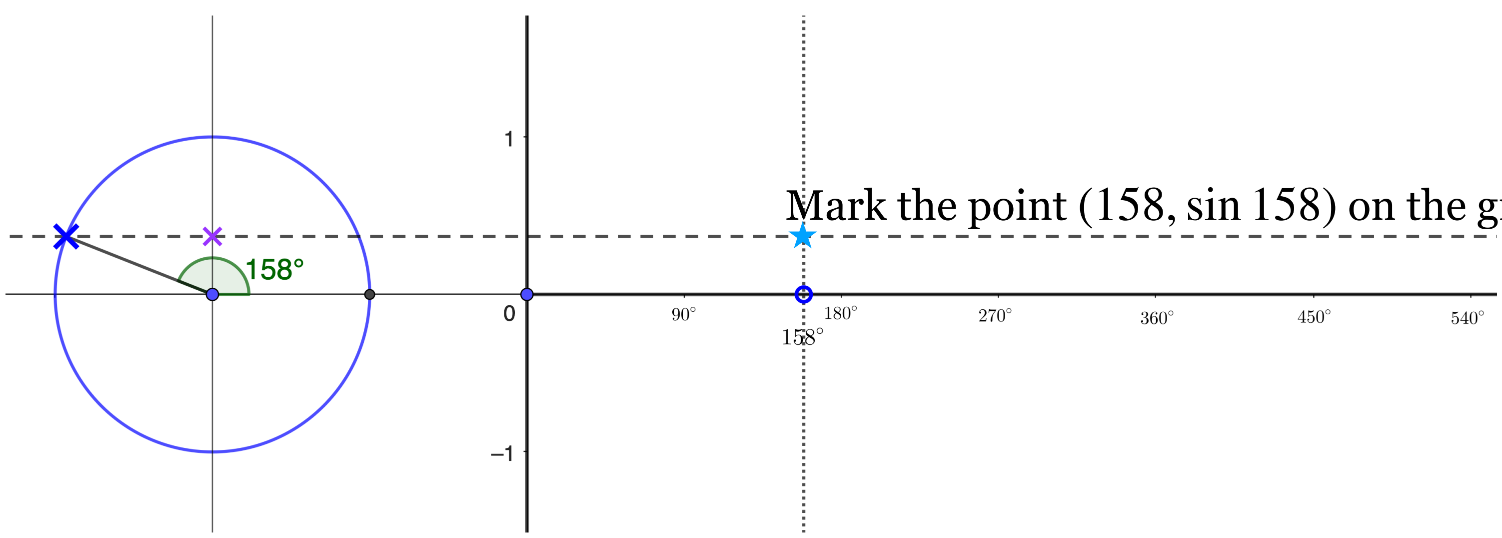
In the tan sequence, spotting the relationship between the pink lines is much harder and you may well need to lead them more actively toward understanding that length of the pink line on the graph represents the gradient of the pink line on the unit circle.



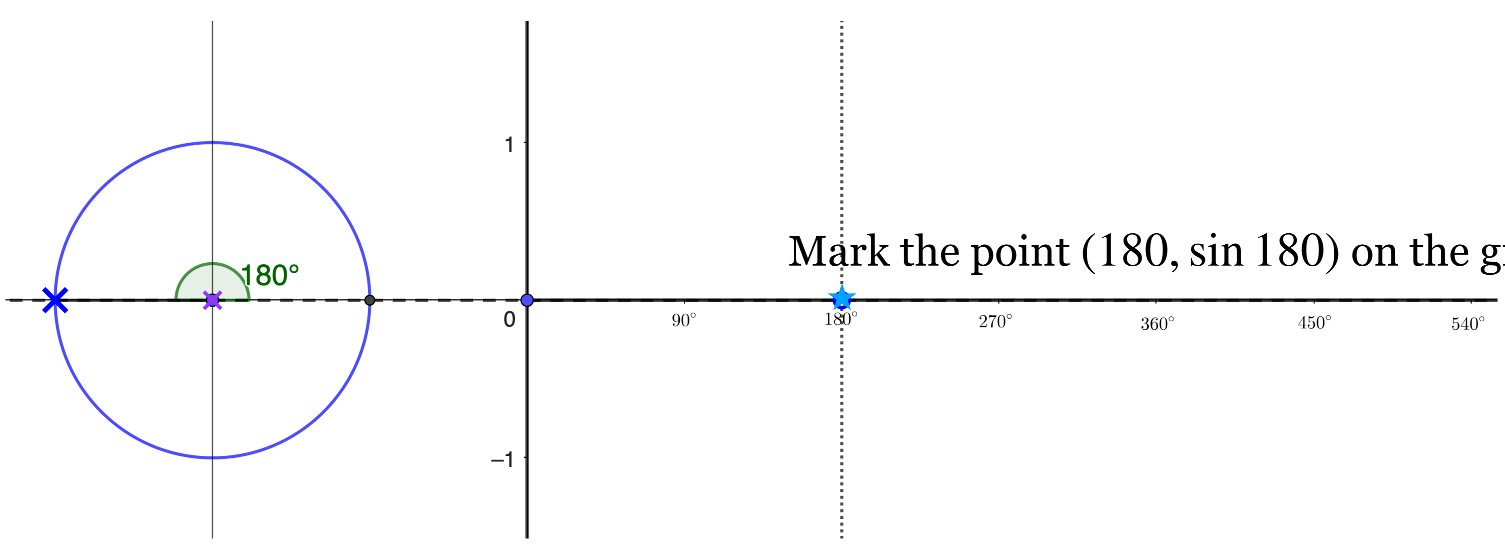
Mark the point (55, sin 55) on the graph



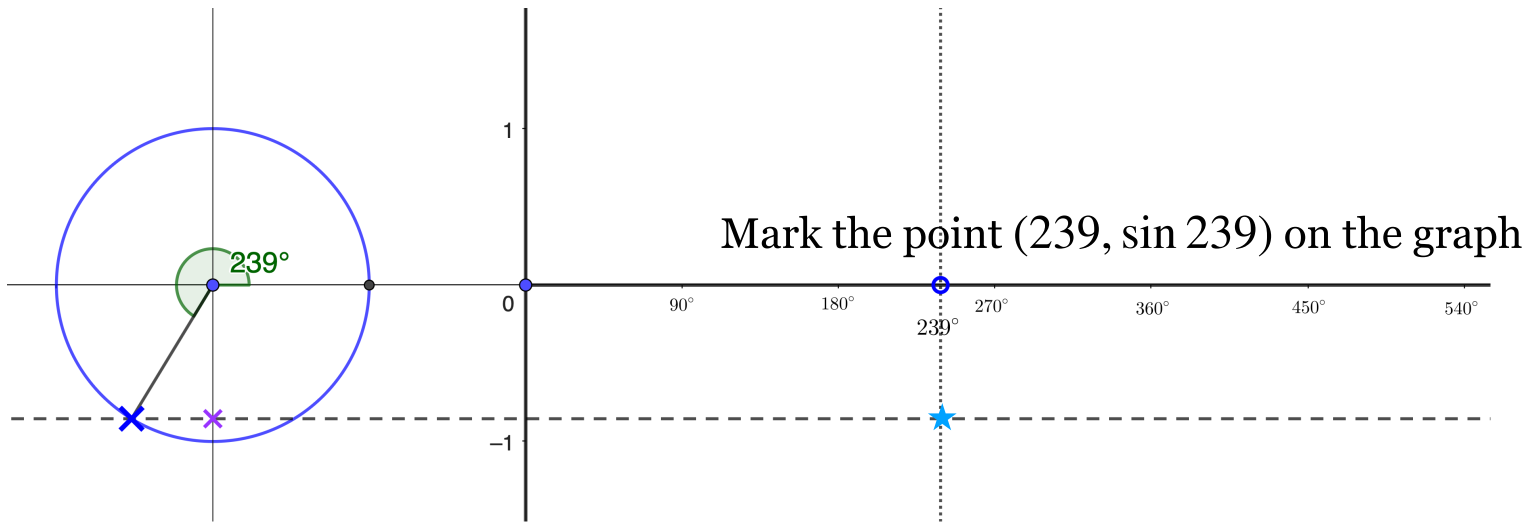
Mark the point (90, sin 90) on the graph



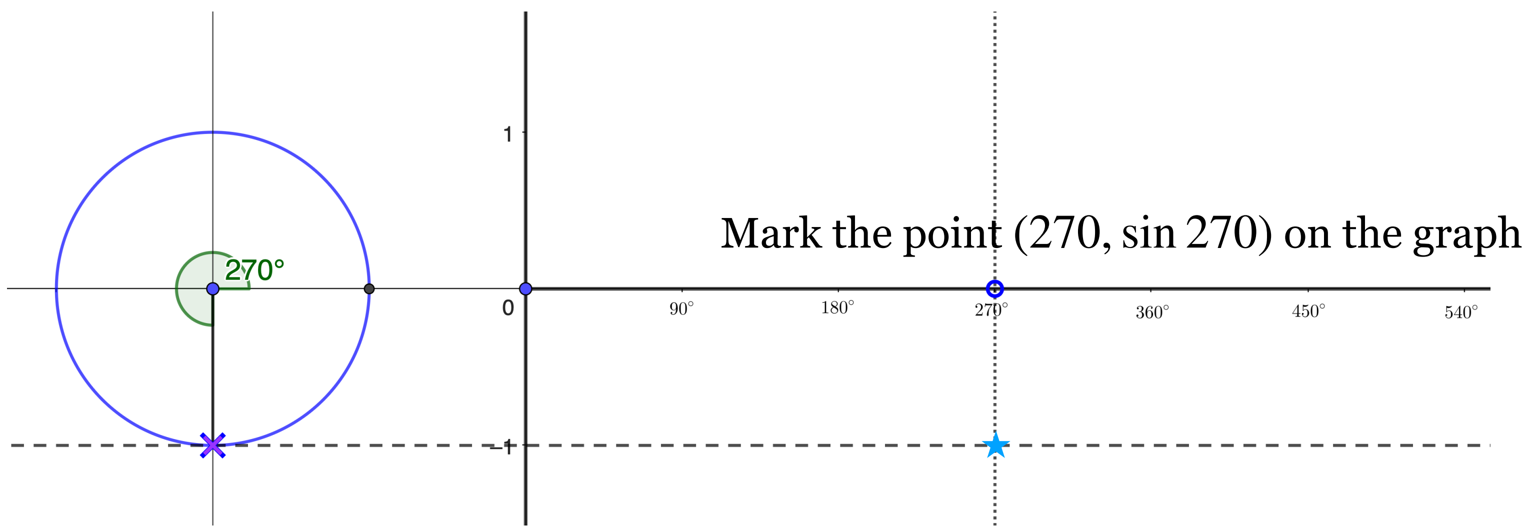
Mark the point (158, sin 158) on the graph



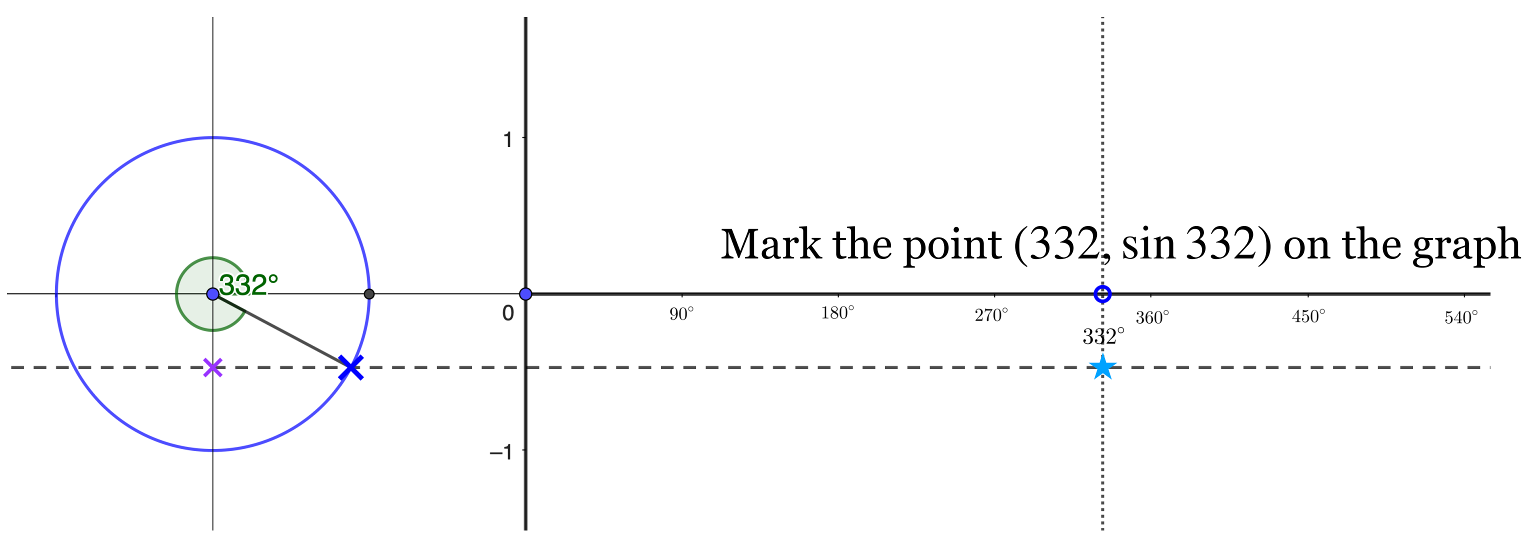
Mark the point (180, sin 180) on the graph



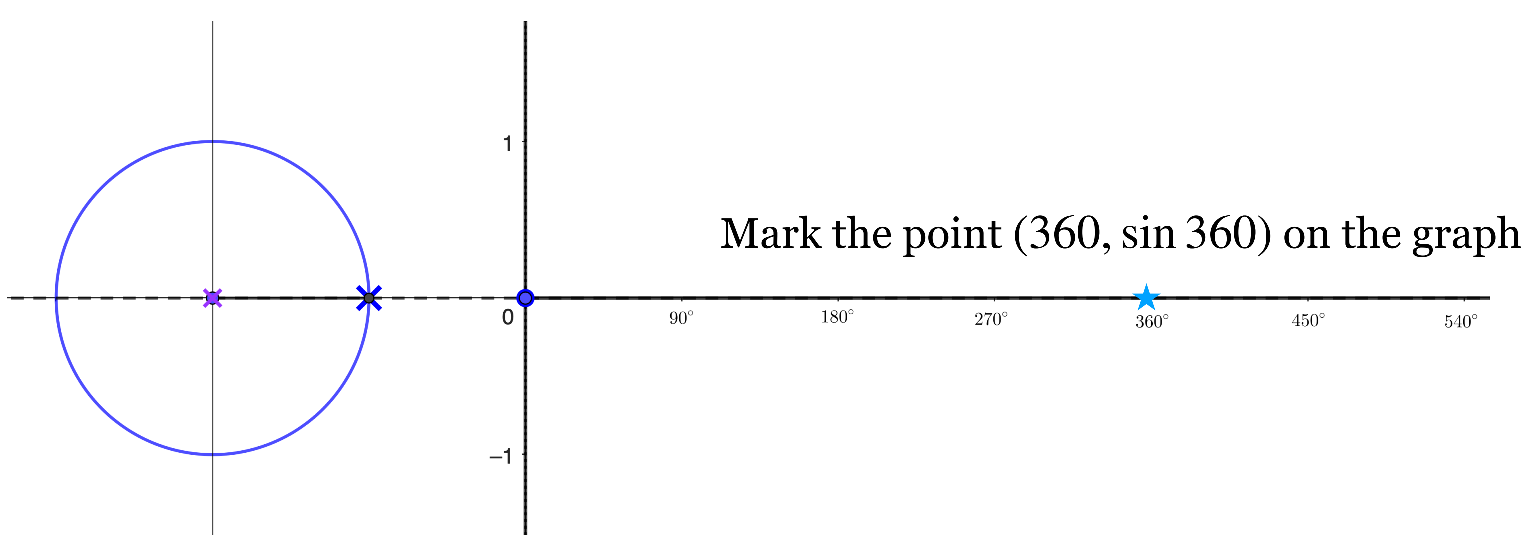
Mark the point (239, sin 239) on the graph
239°
1
0
–1
90°
180°
270°
360°
450°
540°



Mark the point (270, sin 270) on the graph
270°
1
0
–1
90°
180°
270°
360°
450°
540°

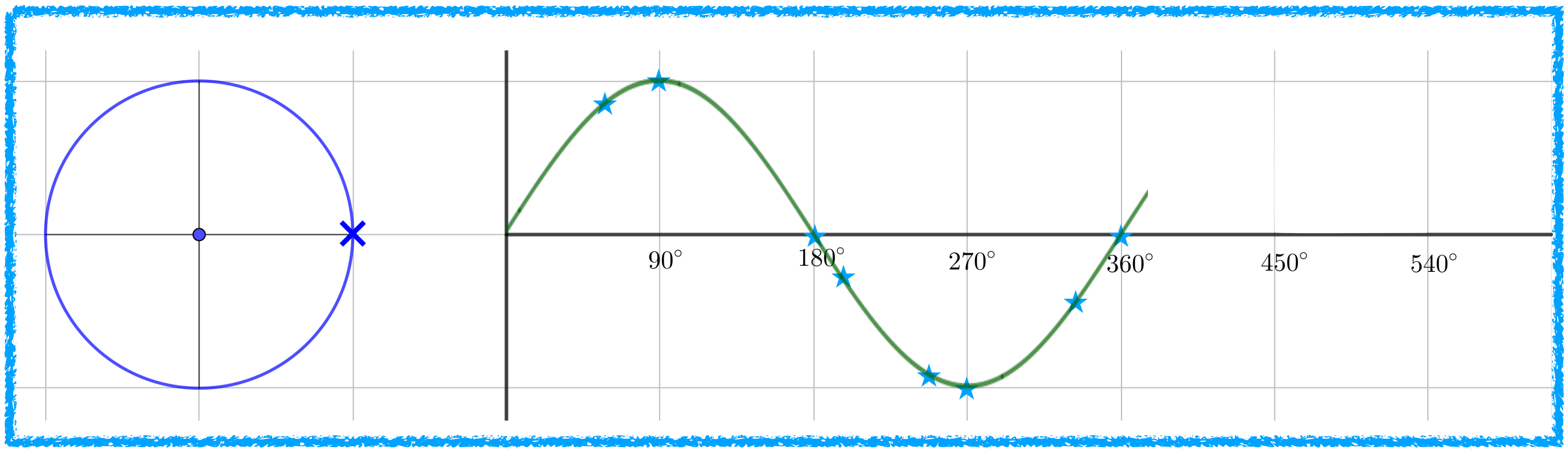


Mark the point (332, sin 332) on the graph
332°
1
0
–1
90°
180°
270°
360°
450°
540°

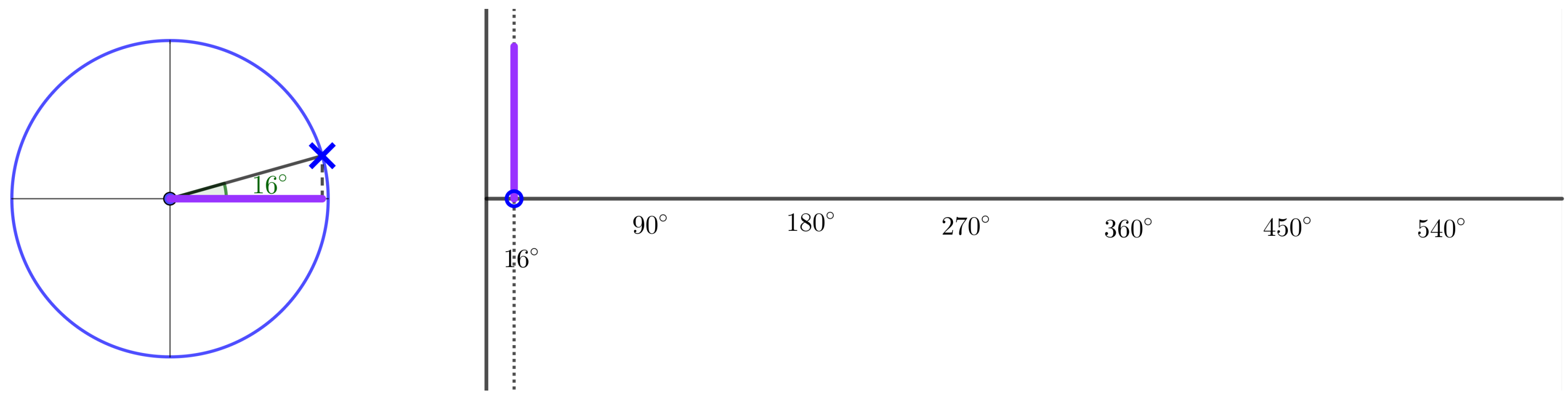


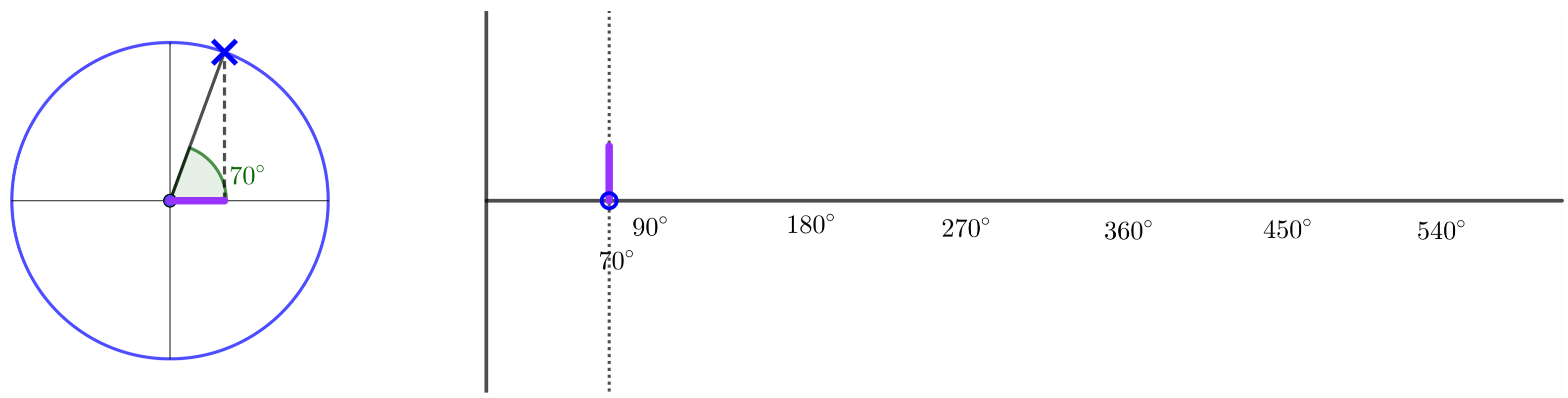
Mark the point (360, sin 360) on the graph
1
0
–1
90°
180°
270°
360°
450°
540°

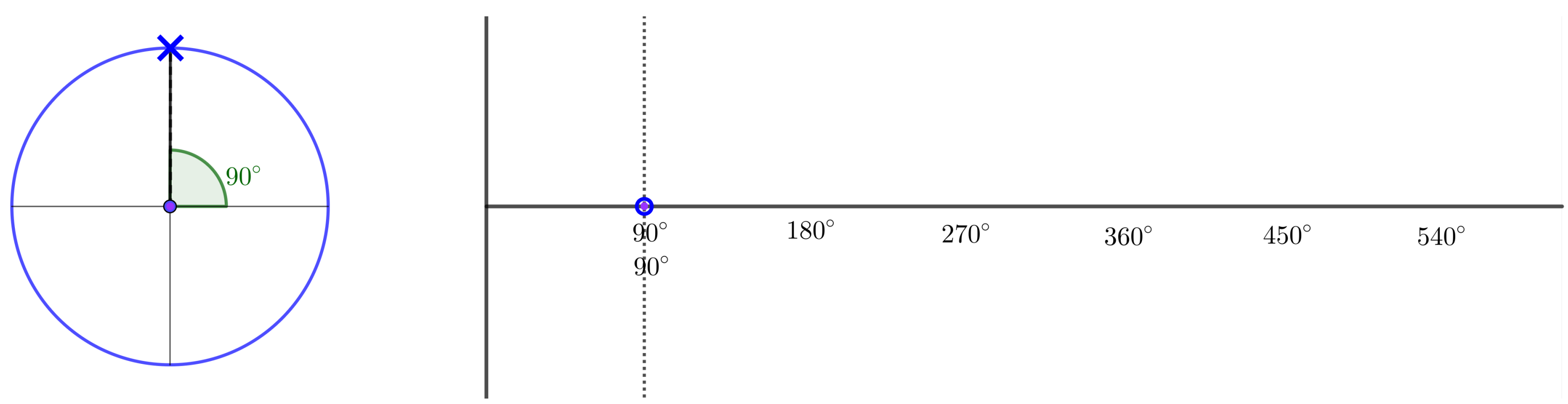
Use these points as a guide to draw the graph $y = \sin x$.

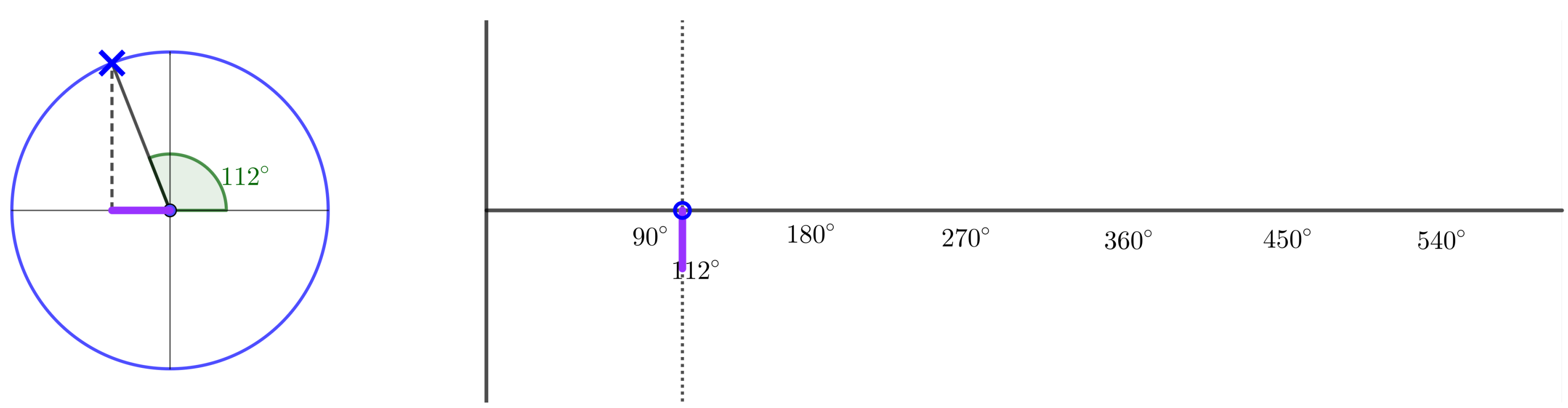


Look at the next sequence of images, and think about the relationship between the purple line segment on the left and the purple line segment on the right.

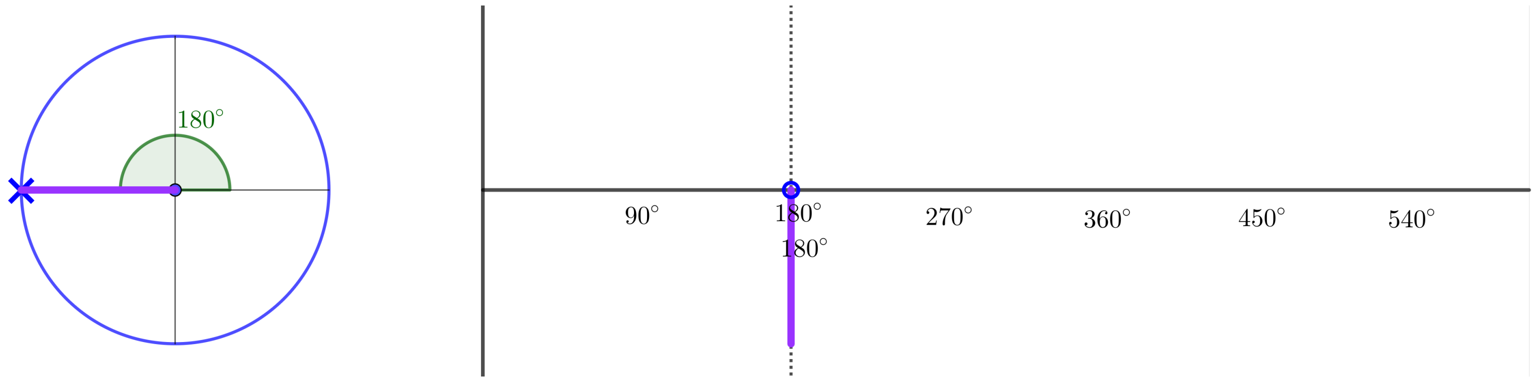


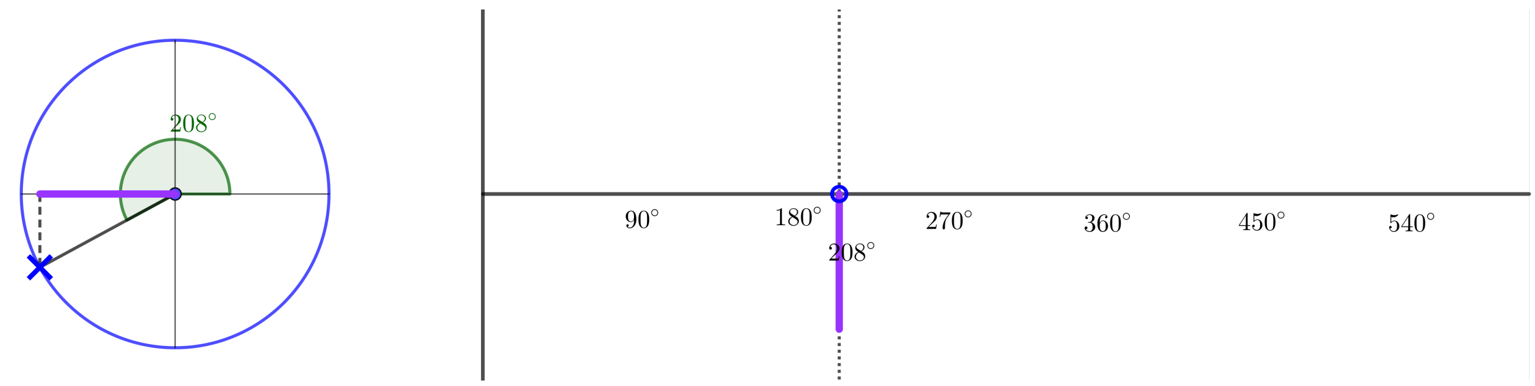


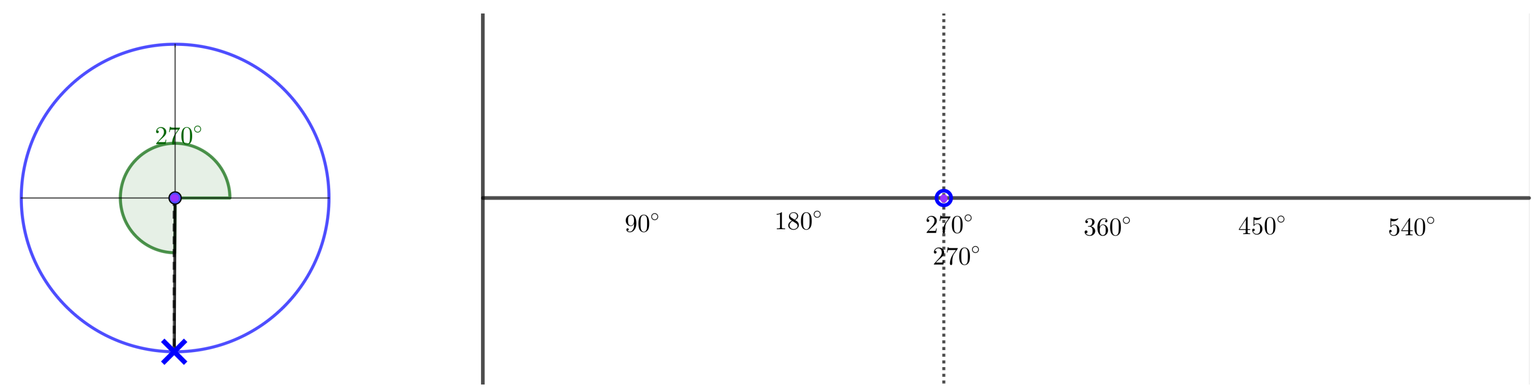


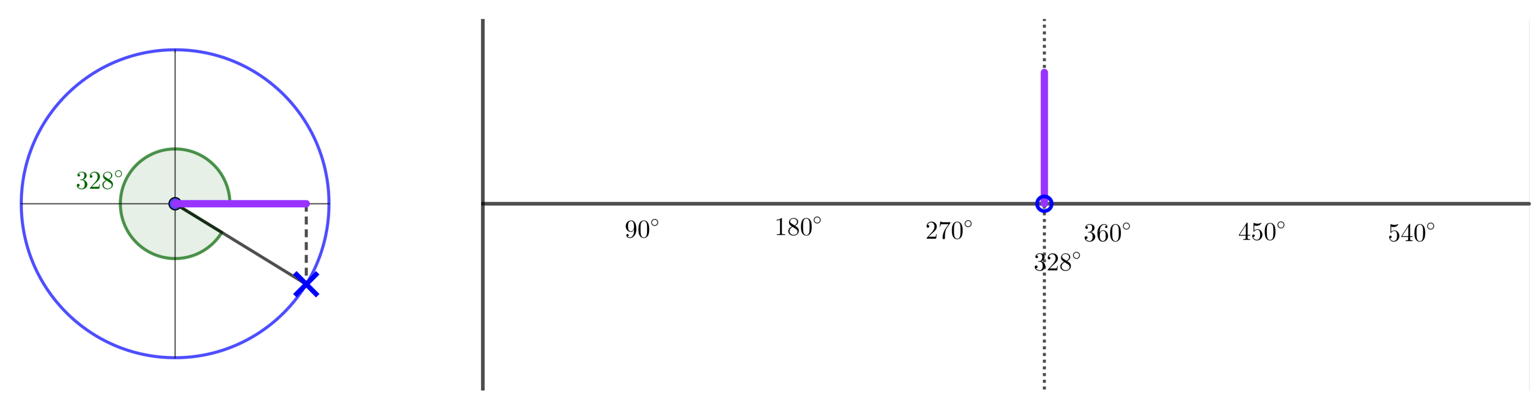


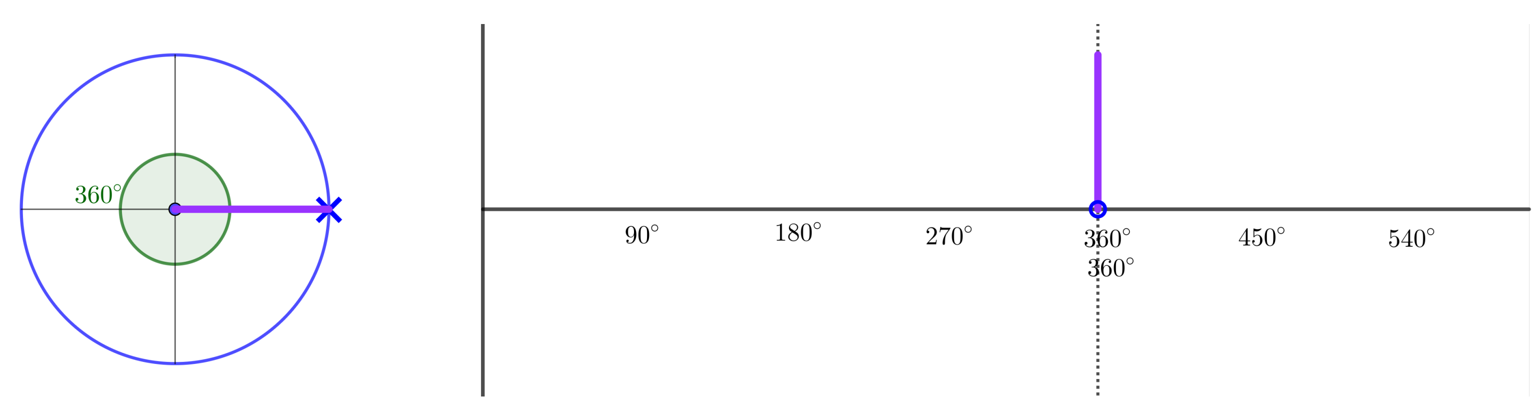
Demonstrating how the x coordinates on the unit circle become the y coordinates on the graph is a bit trickier. The signed length of the segment on the left is the same as the signed length of that on the right.


180°
90°
180°
180°
270°
360°
450°
540°

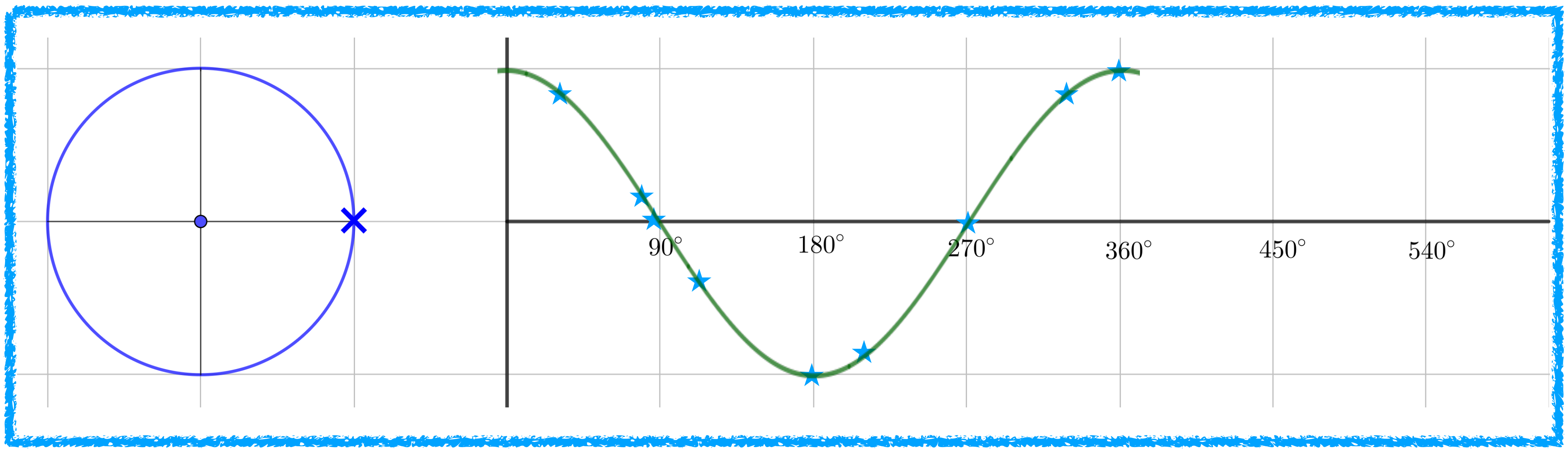

208°
90°
180°
208°
270°
360°
450°
540°


270°
90°
180°
270°
270°
360°
450°
540°

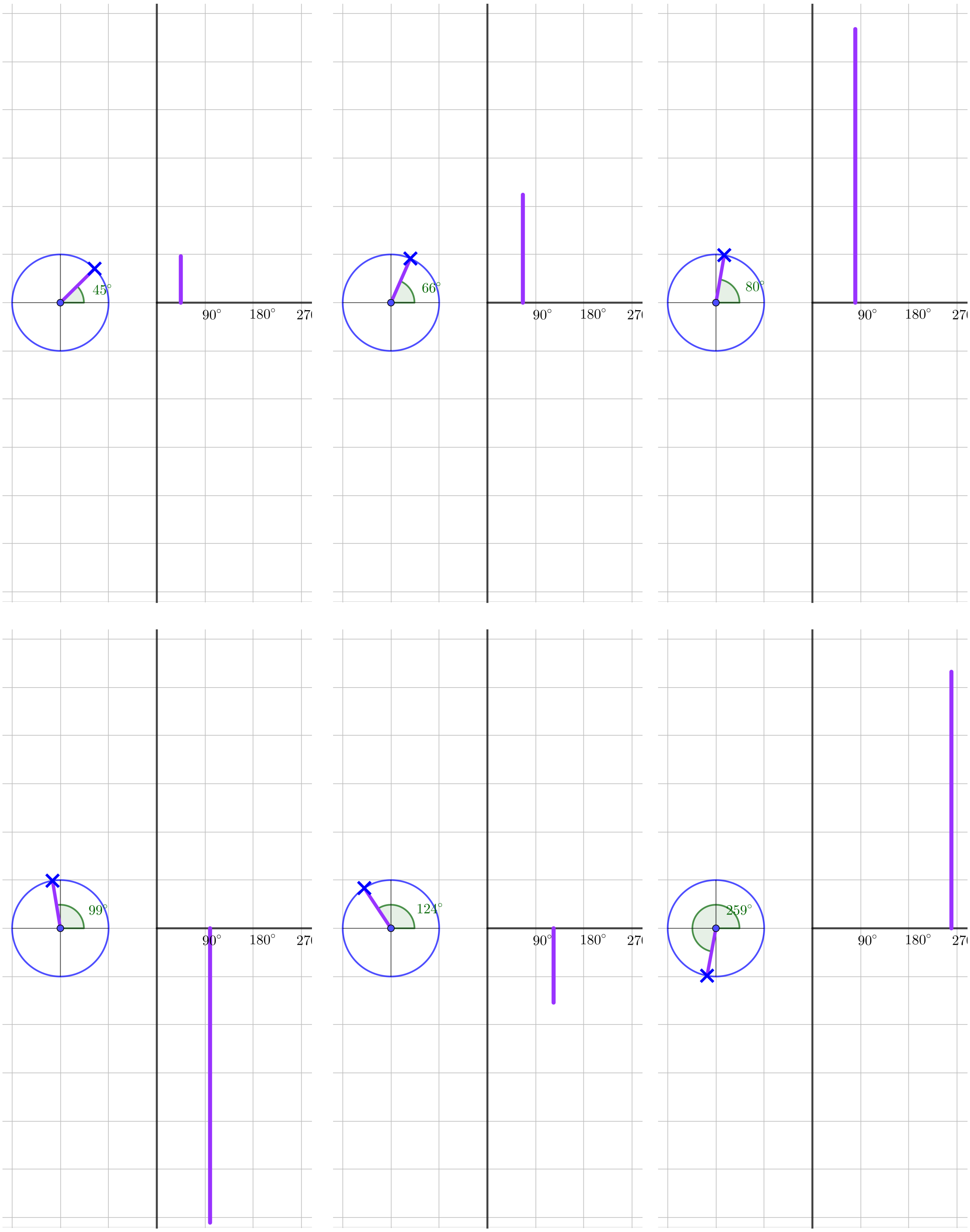

328°
90°
180°
270°
328°
360°
450°
540°


360°
90°
180°
270°
360°
360°
450°
540°

Use these points as a guide to draw the graph $y = \cos x$.



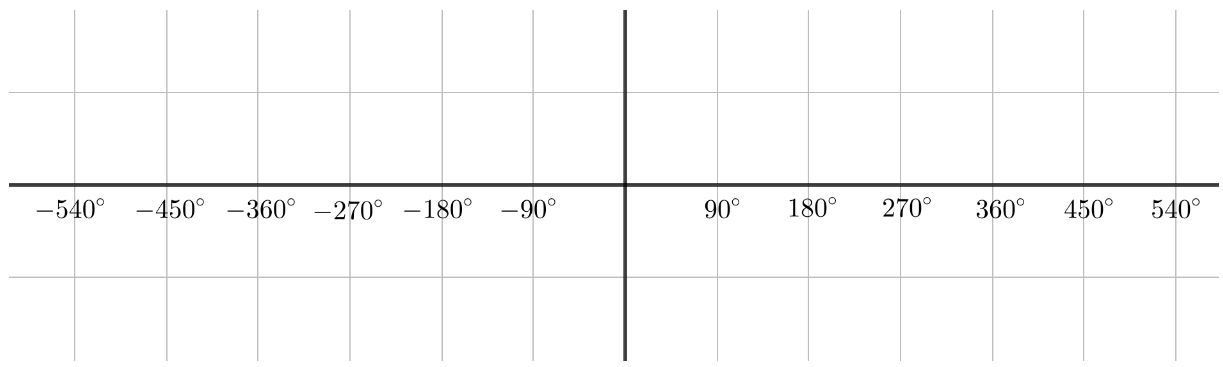
Look at this sequence of images, and describe the relationship between the purple line segment on the left and the purple line segment on the right.

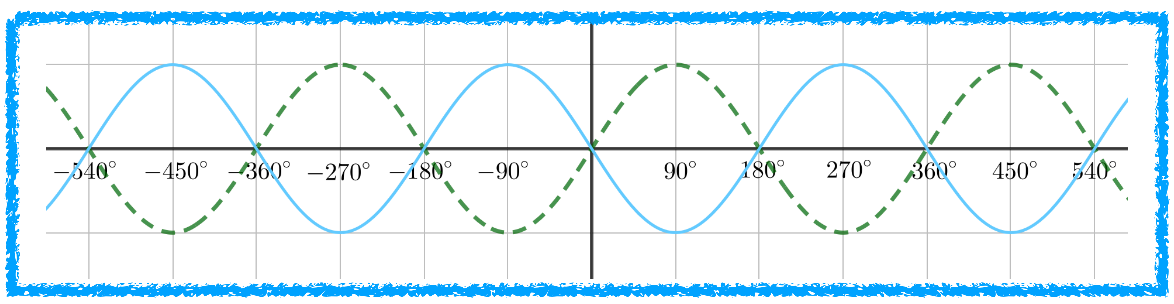


Relationships between circular functions

Draw the following graphs:

$y = -\sin x$

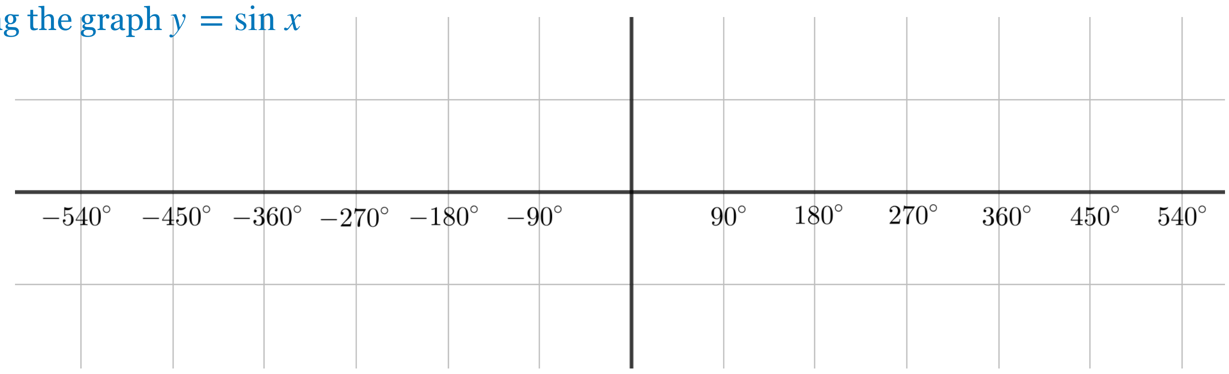




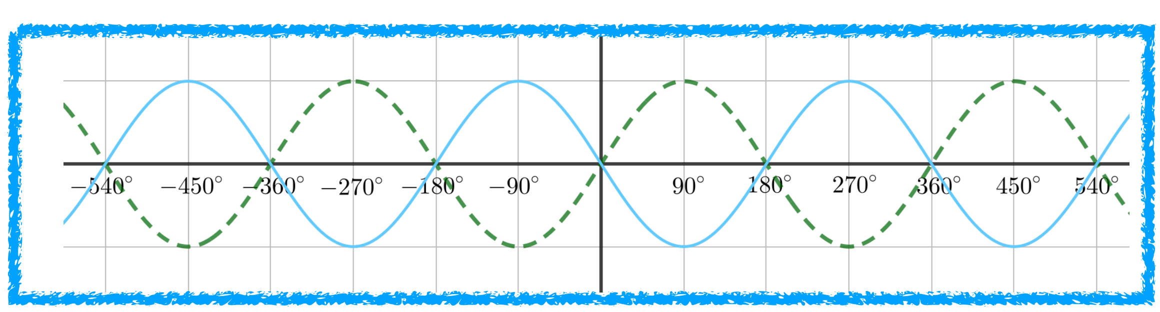
Reflect sin graph in the x axis.

It's easiest if they start by drawing the graph $y = \sin x$

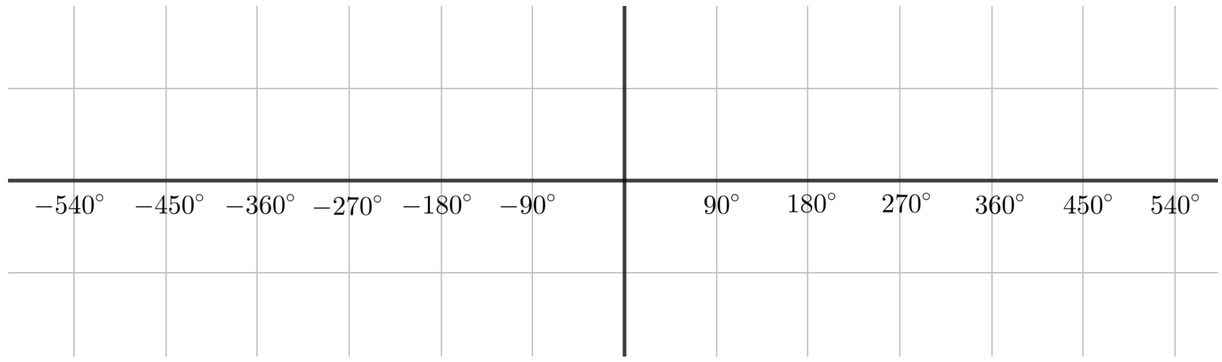
$y = \sin(-x)$

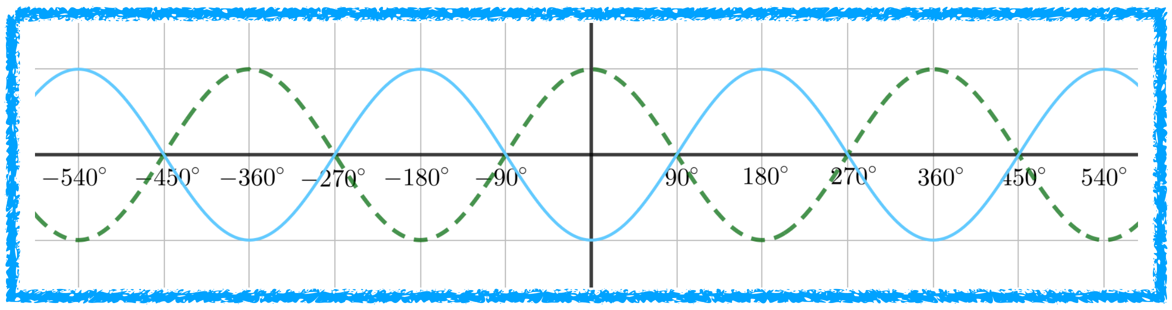


Reflect sin graph in the y axis.



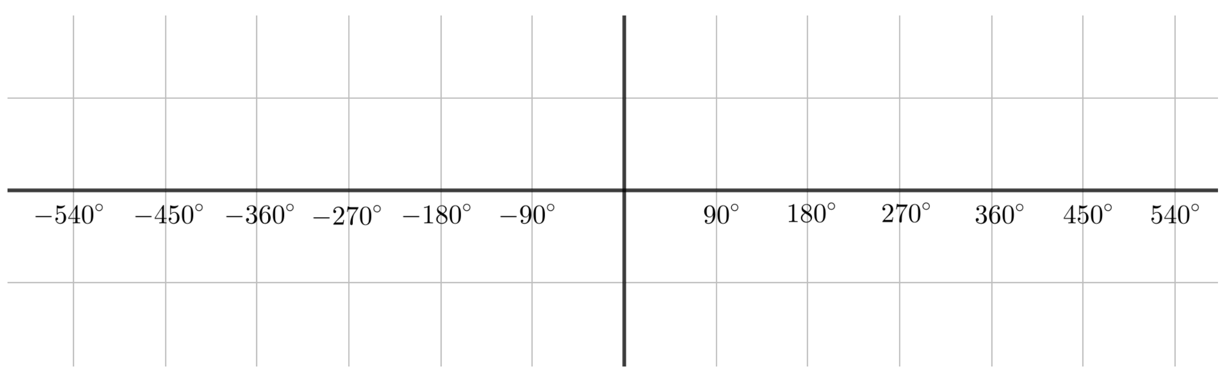
$y = -\cos x$

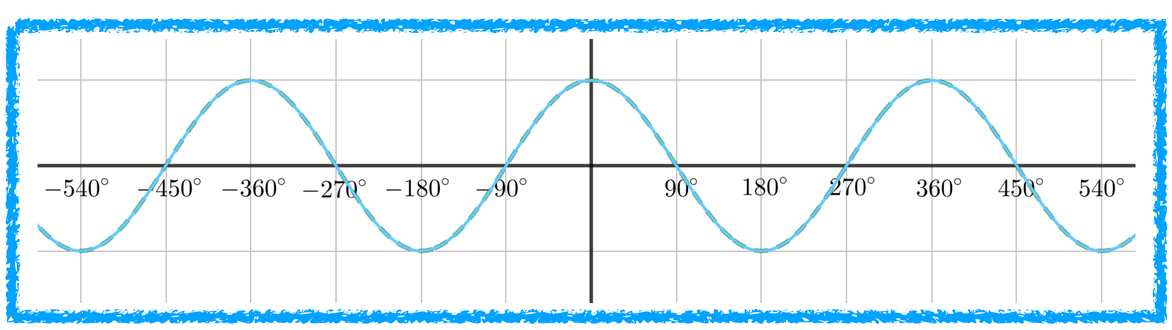




Reflect cos graph in the *x* axis.

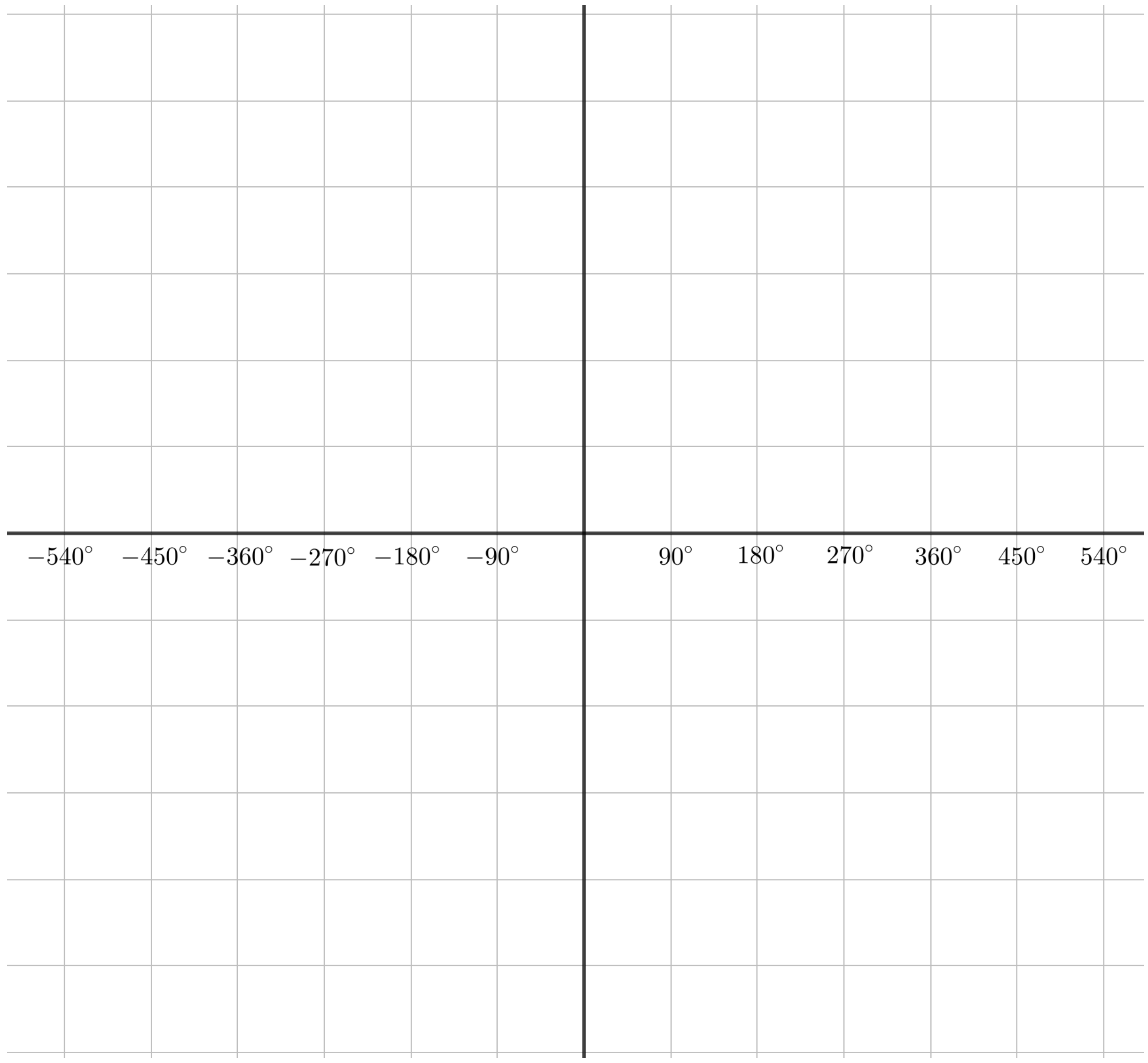
$y = \cos(-x)$





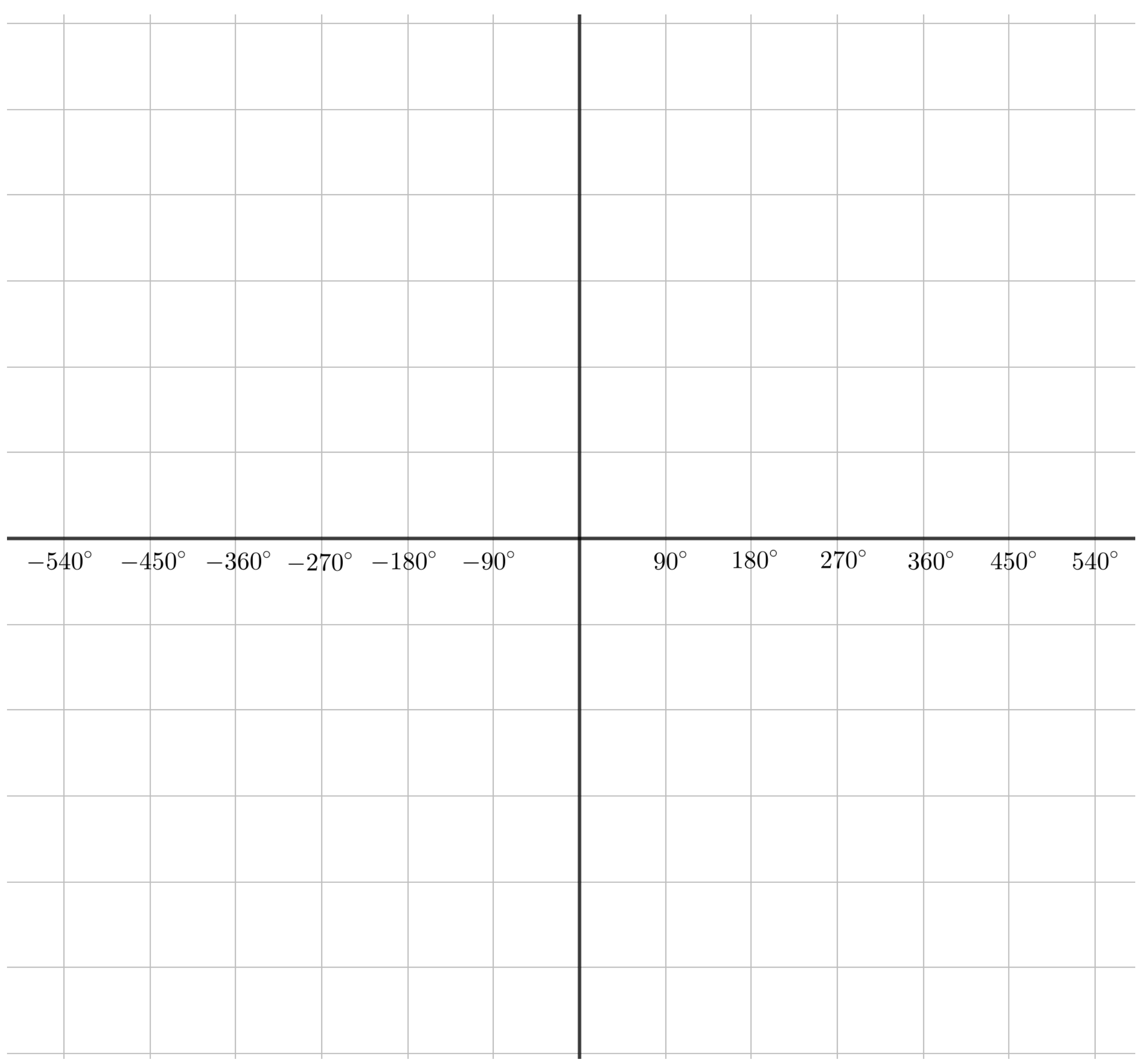
Reflect cos graph in the *y* axis.

$y = -\tan x$

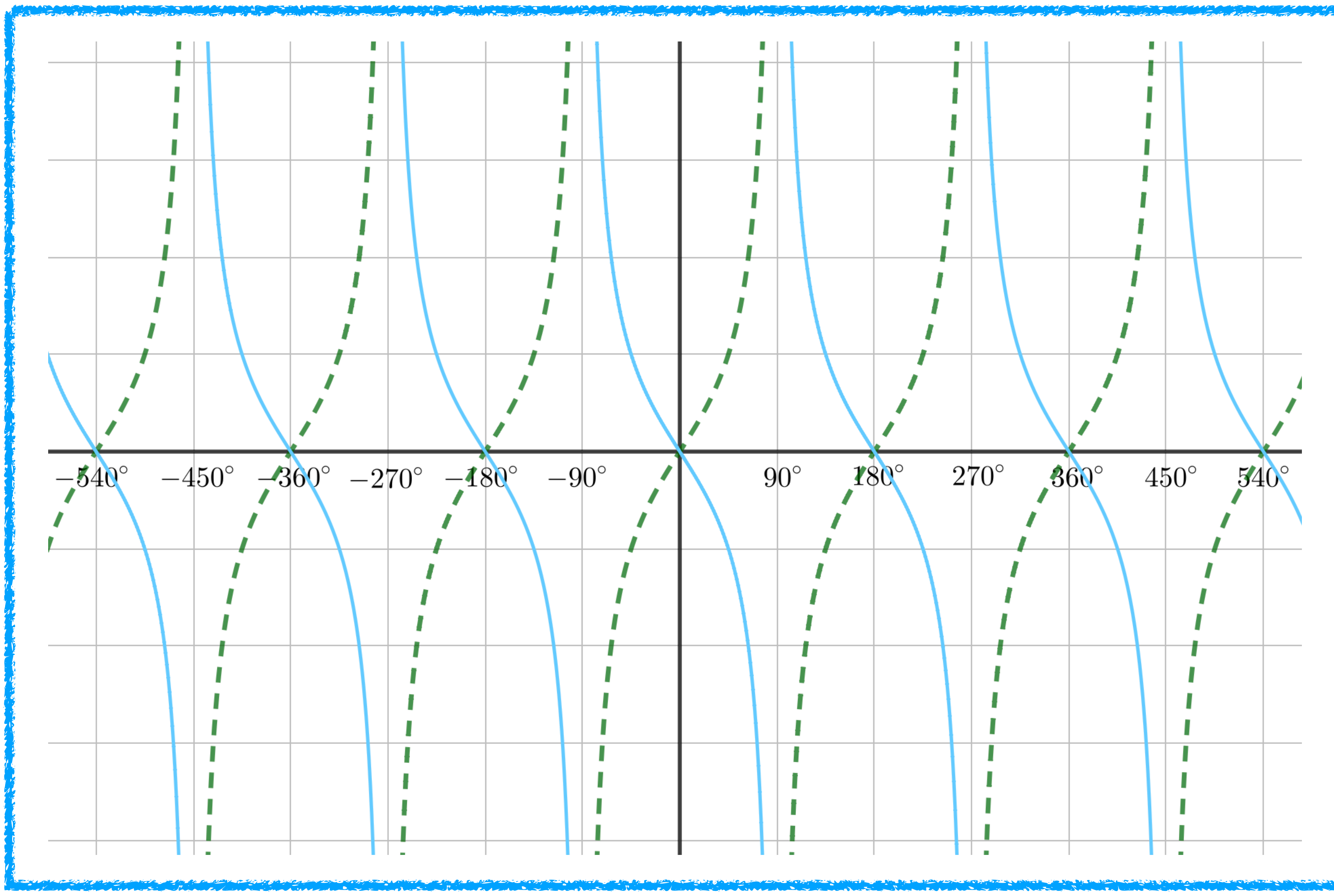


−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

$y = \tan(-x)$



−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°



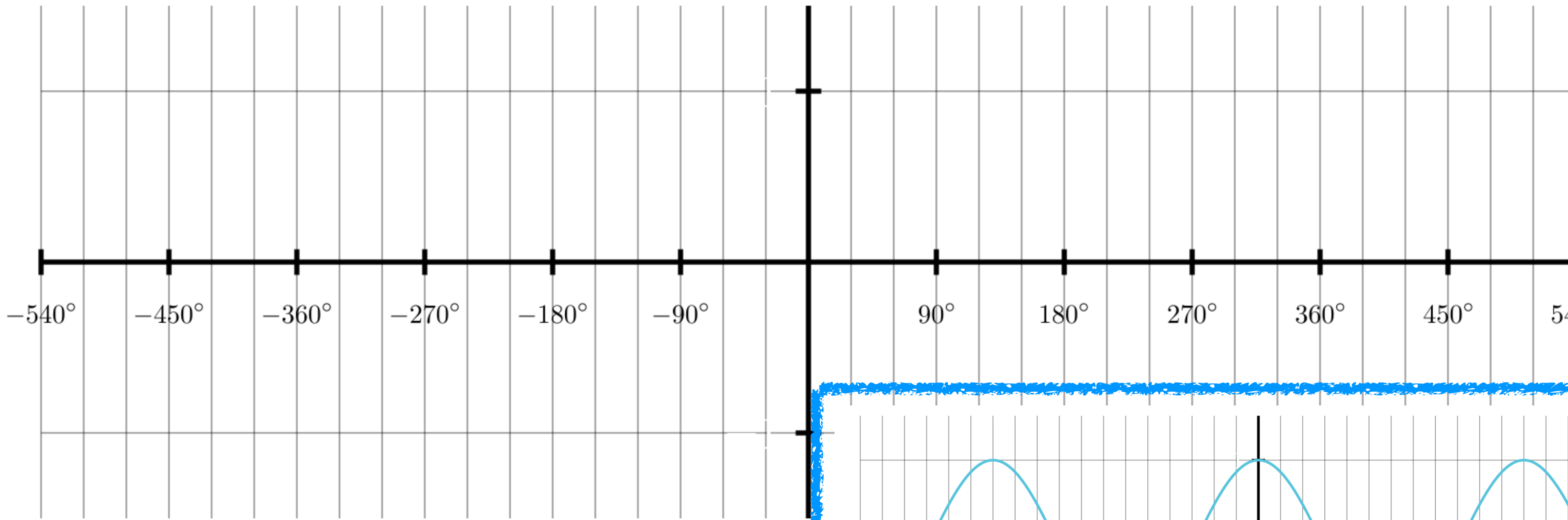
Reflecting the tan graph in the x axis and reflecting the tan graph in the y axis result in the same graph.

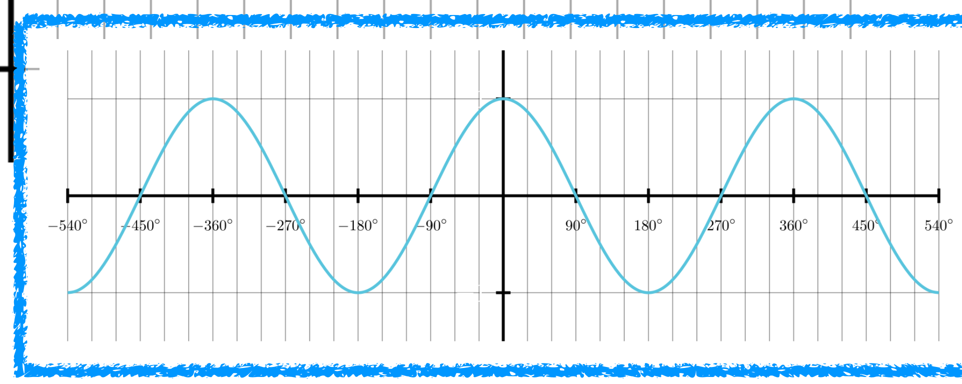
The identities

$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$

are fundamental bites of subject knowledge that your students should have at their fingertips. This last sequence of graphs reinforces these facts, and it's worth discussing them again and how they relate to the unit circle.

$y = \sin(90^\circ - x)$



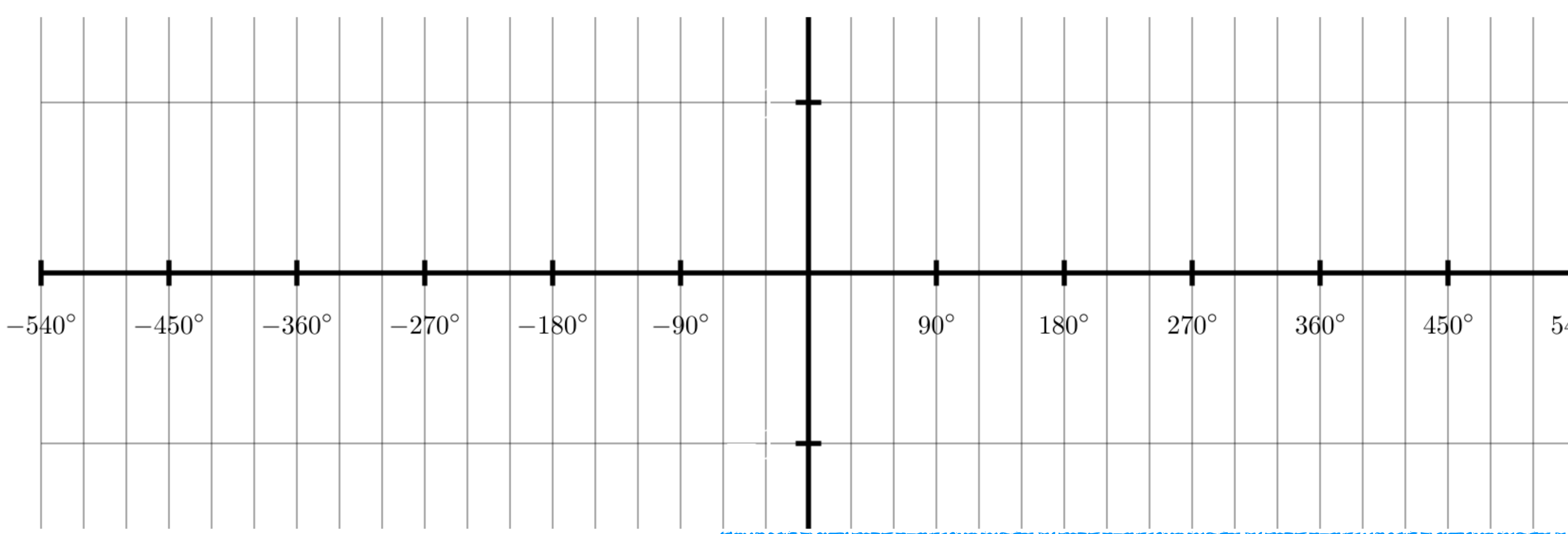


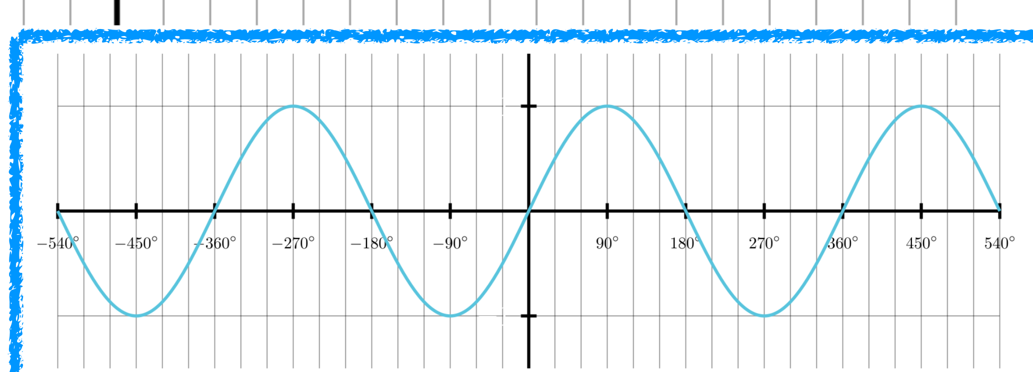
The easiest way to see this is to put some values into the function such as $0, 90, -90\ldots$

You could, in theory, use compound angle formulas, but that is not really in the spirit of this worksheet.

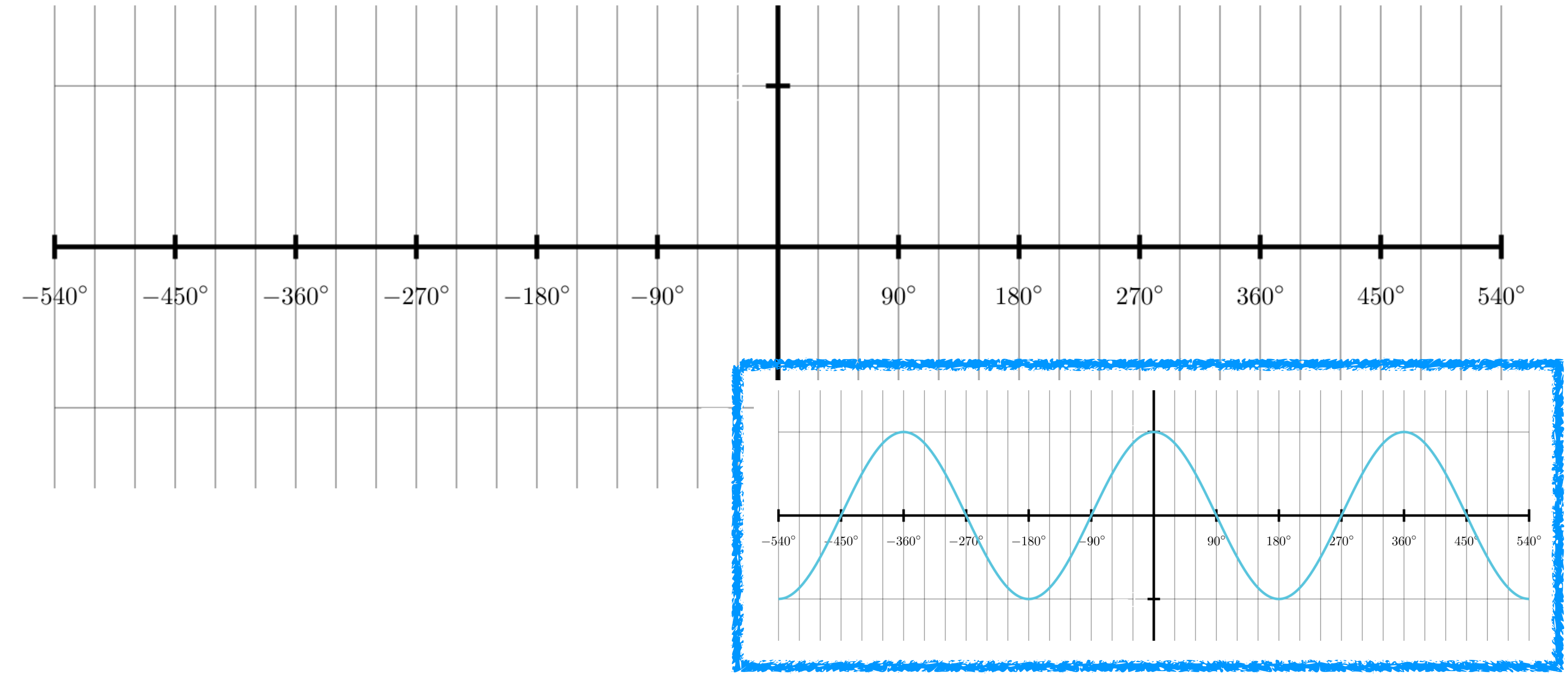
You could, in theory, use transformations, but this example is a bit tricky and would take you off course.

$y = \cos(90^\circ - x)$





$y = \sin(90^\circ + x)$

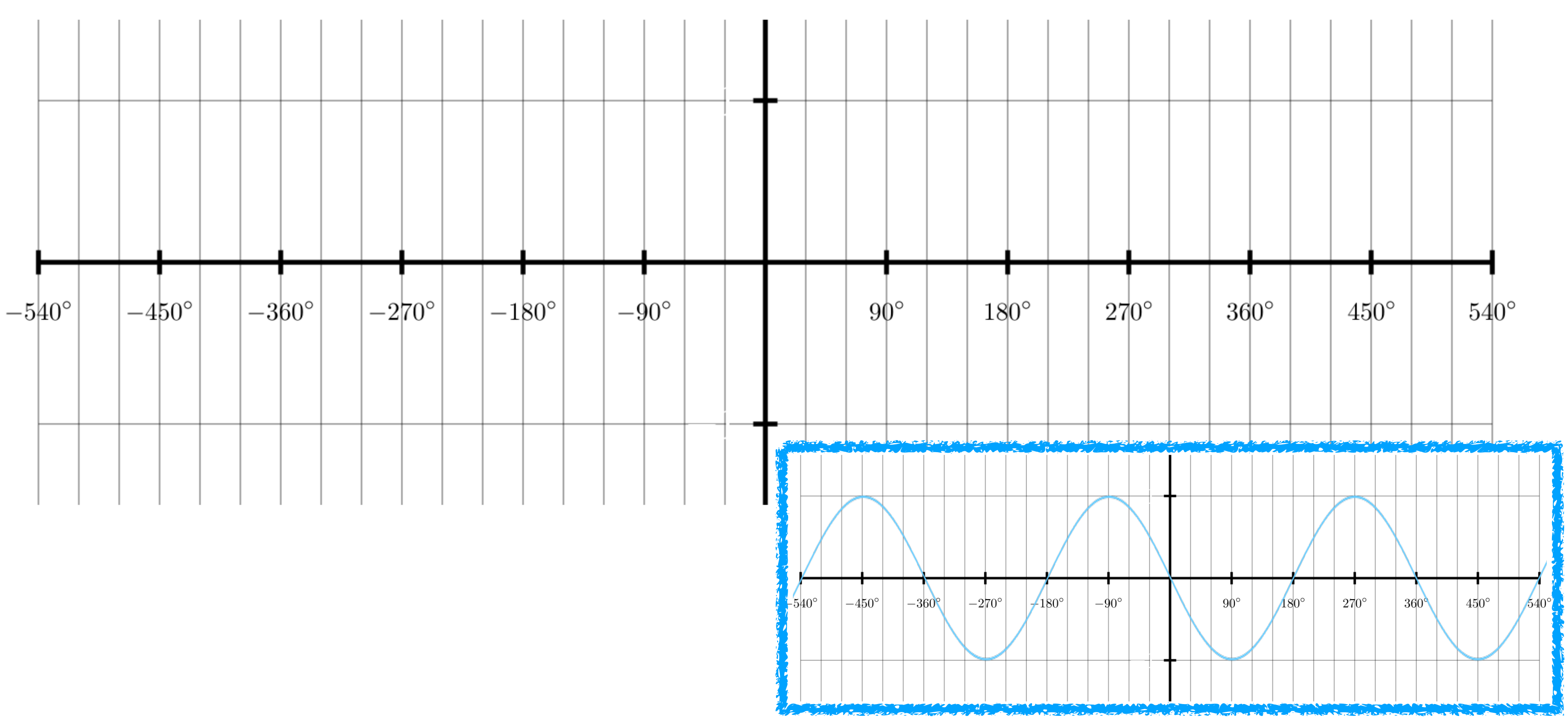


The easiest way to see this is to put some values into the function such as 0, 90, $-90\ldots$

You could, in theory, use compound angle formulas, but that is not really in the spirit of this worksheet.

You could, in theory, use transformations, but this example is a bit tricky and would take you off course.

$y = \cos(90^\circ + x)$



Use these graphs to find some relationships between sin and cos.

$\sin(90° - x) = \sin(90° + x) = \cos x$ $\quad$ $\cos(90° - x) = \cos(x - 90°) = \sin x$ $\quad$ $\cos(90° + x) = \sin(x - 90°) = -\sin x$

$\sin(-x) = -\sin x$ $\quad$ $\cos(-x) = \cos x$ $\quad$ $\tan(-x) = -\tan x$

Show that

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

whenever θ is not a multiple of 90°.

If you were to ask your students how to prove this identity before you have discussed it, you might see something like this:

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

which is true, so the original identity is true.

We all know what they mean, and in fact, if only they had written

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\iff \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\iff \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

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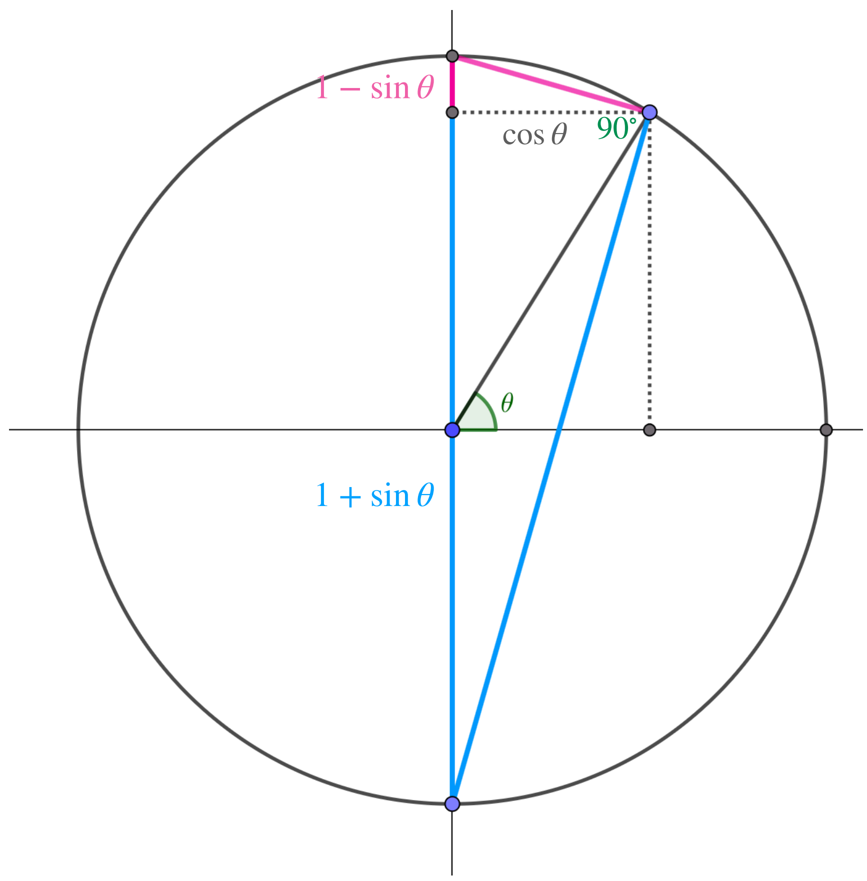
Here's a lovely geometrical proof, at least for angles between 0 and 90°.



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We can read it as "is equivalent to", meaning "is equal to for every value of θ.

In fact, it is not quite true, because when the bottom of either fraction is 0, the fraction is not defined.

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Make life a bit easier by writing $s = \sin\theta \quad c = \cos\theta$.

Now we need to show that

$$\frac{s - c + 1}{s + c - 1} = \frac{1 + s}{c}$$

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$$\begin{aligned}\frac{s - c + 1}{s + c - 1} \times \frac{c}{1 + s} &= \frac{sc - c^2 + c}{s + c - 1 + s^2 + sc - s} \\ &= \frac{sc - 1 + s^2 + c}{c - 1 + sc + s^2} \\ &= 1\end{aligned}$$

which seems perfectly adequate to me, because $x \times \dfrac{1}{y} = 1 \Rightarrow x = y$.

Alternatively:

$$\begin{aligned}\frac{1 + s - c}{c} &= \frac{1 + s}{c} - 1 \\ &= \frac{c}{1 - s} - 1 \\ &= \frac{c - 1 + s}{1 - s} \\ \Rightarrow \frac{1 + s - c}{c - 1 + s} &= \frac{c}{1 - s} = \frac{1 + s}{c}\end{aligned}$$

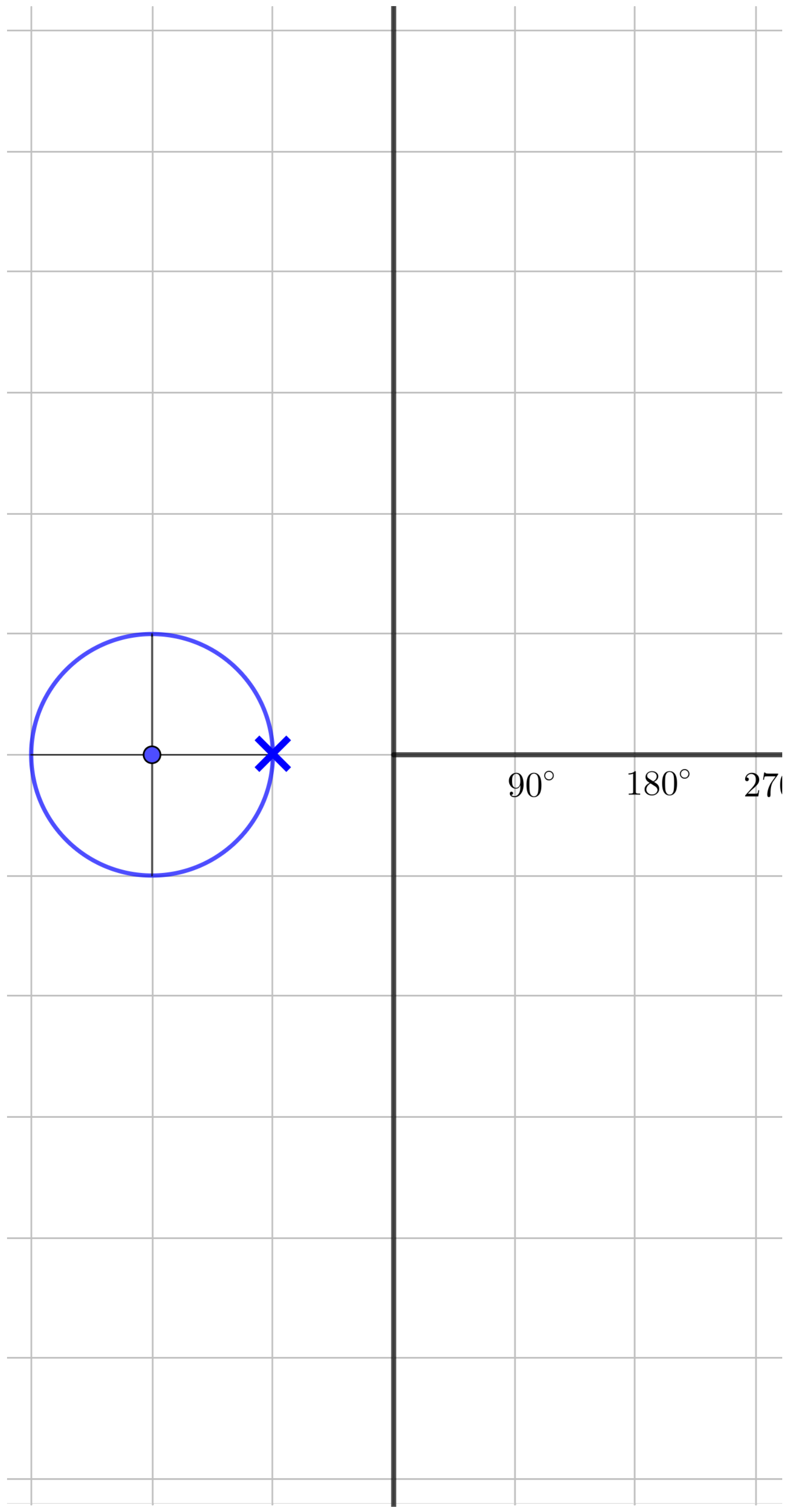
or

$$\begin{aligned}\frac{1 + s - c}{1 + s} &= 1 - \frac{c}{1 + s} \\ &= 1 - \frac{1 - s}{c} \\ &= \frac{c - 1 + s}{c} \\ \Rightarrow \frac{1 + s - c}{c - 1 + s} &= \frac{1 + s}{c}\end{aligned}$$

I could turn this proof into something that the obsessive examiner would approve of, but I wouldn't bother. No one will object to this proof outside the context of a school examination.

By the way, whether you use the equivalence sign $\equiv$ is really a matter of personal style. It distinguishes between an equation to solve ($=$) and an identity to prove ($\equiv$), but I tend to rely on the context to make the difference and stick to $=$, otherwise I am certain to be inconsistent sooner or later.

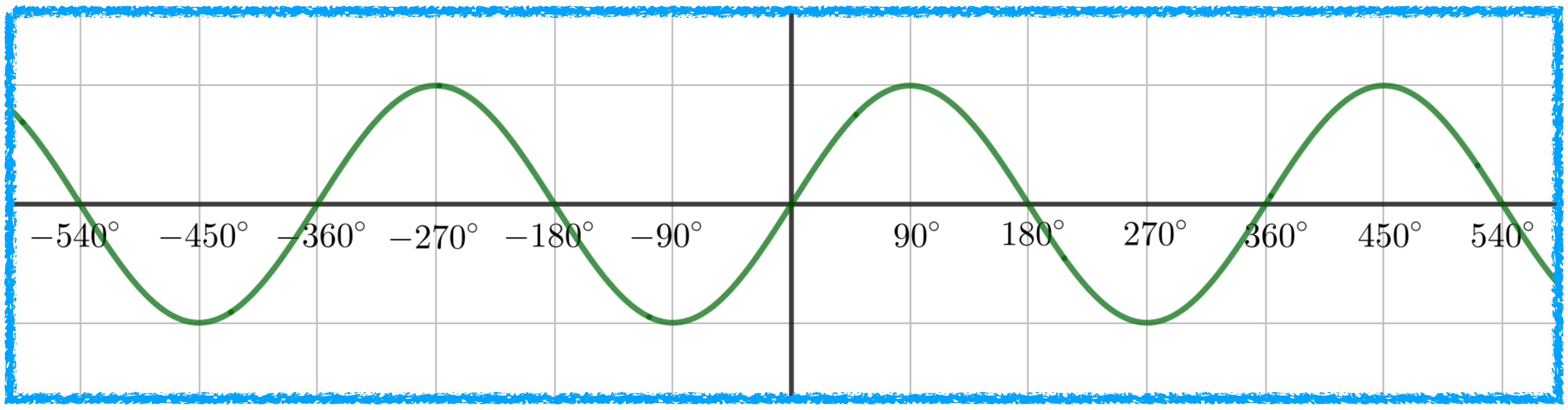
Use this idea to draw the graph $y = \tan x$.



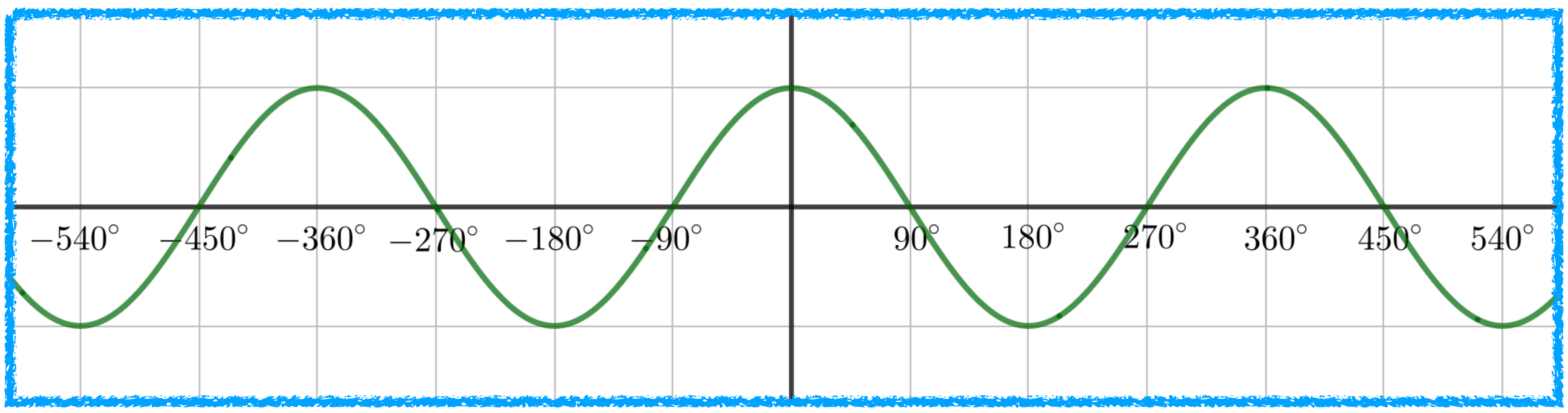
In the last sequence of diagrams, it is the gradient of the segment on the left that determines the signed length of the segment on the right. This should clarify the reason for the behaviour around the asymptotes: as the angle gets closer to 90 from below, the gradient gets large without limit. As the angle gets closer to 90 from above, the gradient gets increasingly large in the negative direction.

The value of this section is in understanding the relationship between the graph and the gradient of the segment in the unit circle.

Draw the graph $y = \sin x$.

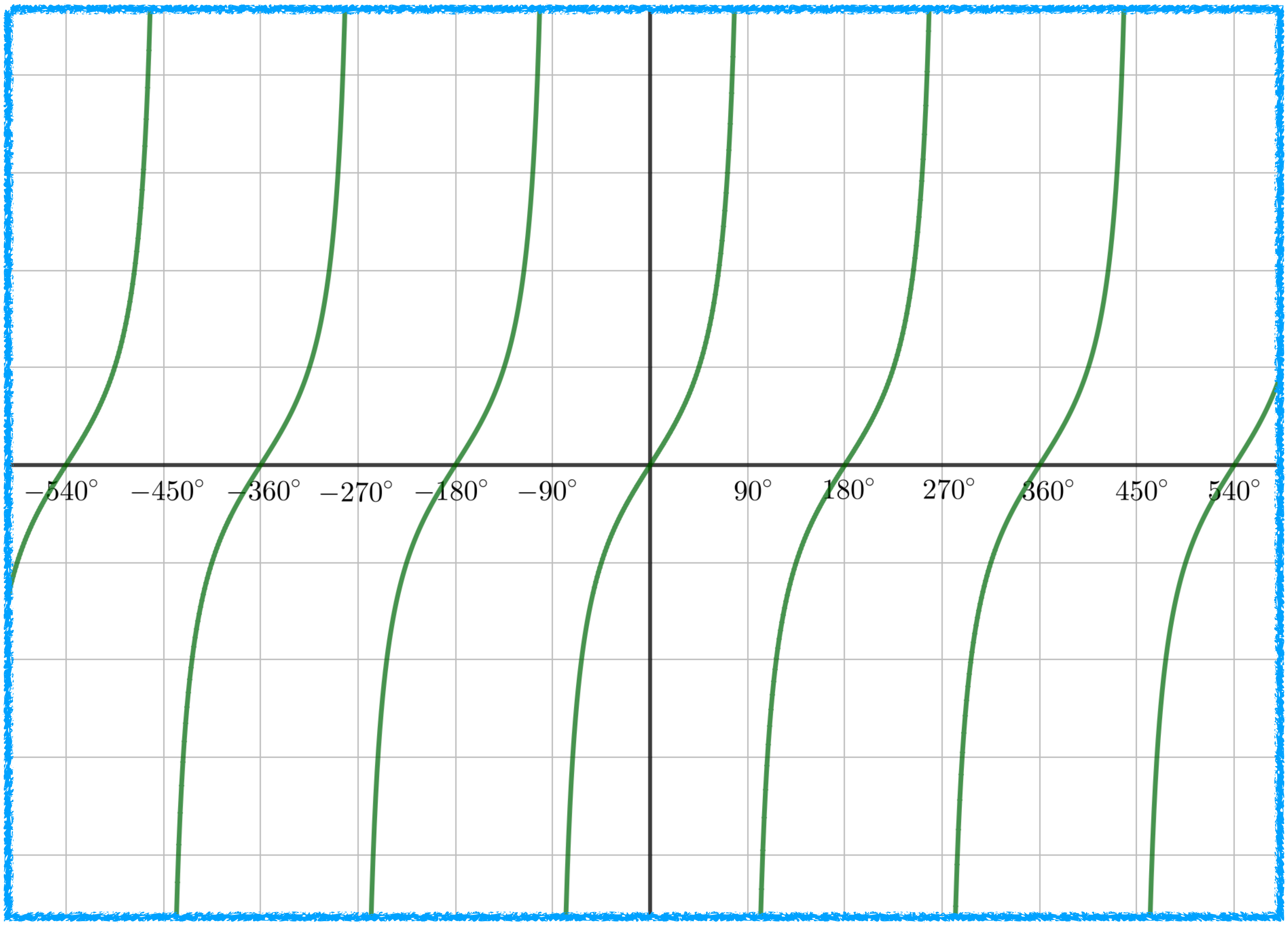


Draw the graph $y = \cos x$.

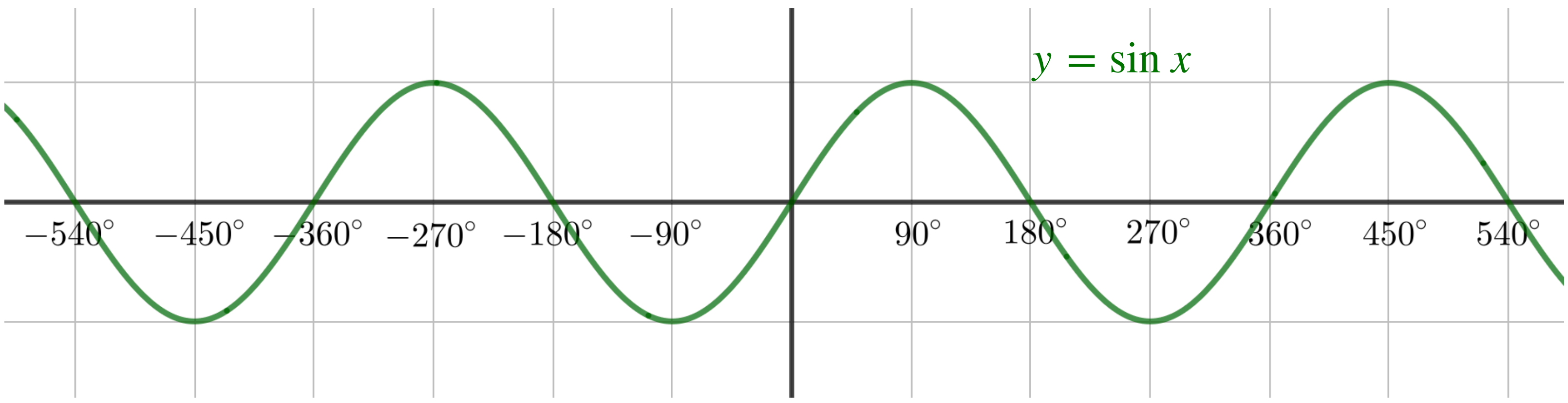


So far, we have only drawn the graphs for positive values of x, so here we add in the negative values by continuing the pattern, checking that we are correct by looking at the unit circle from time to time.

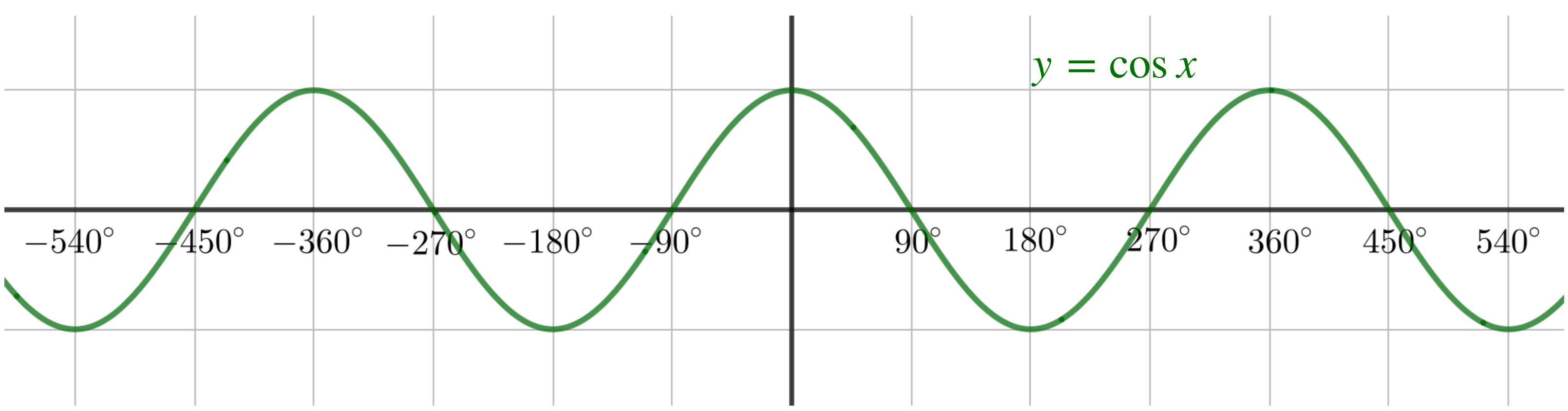
Draw the graph $y = \tan x$.



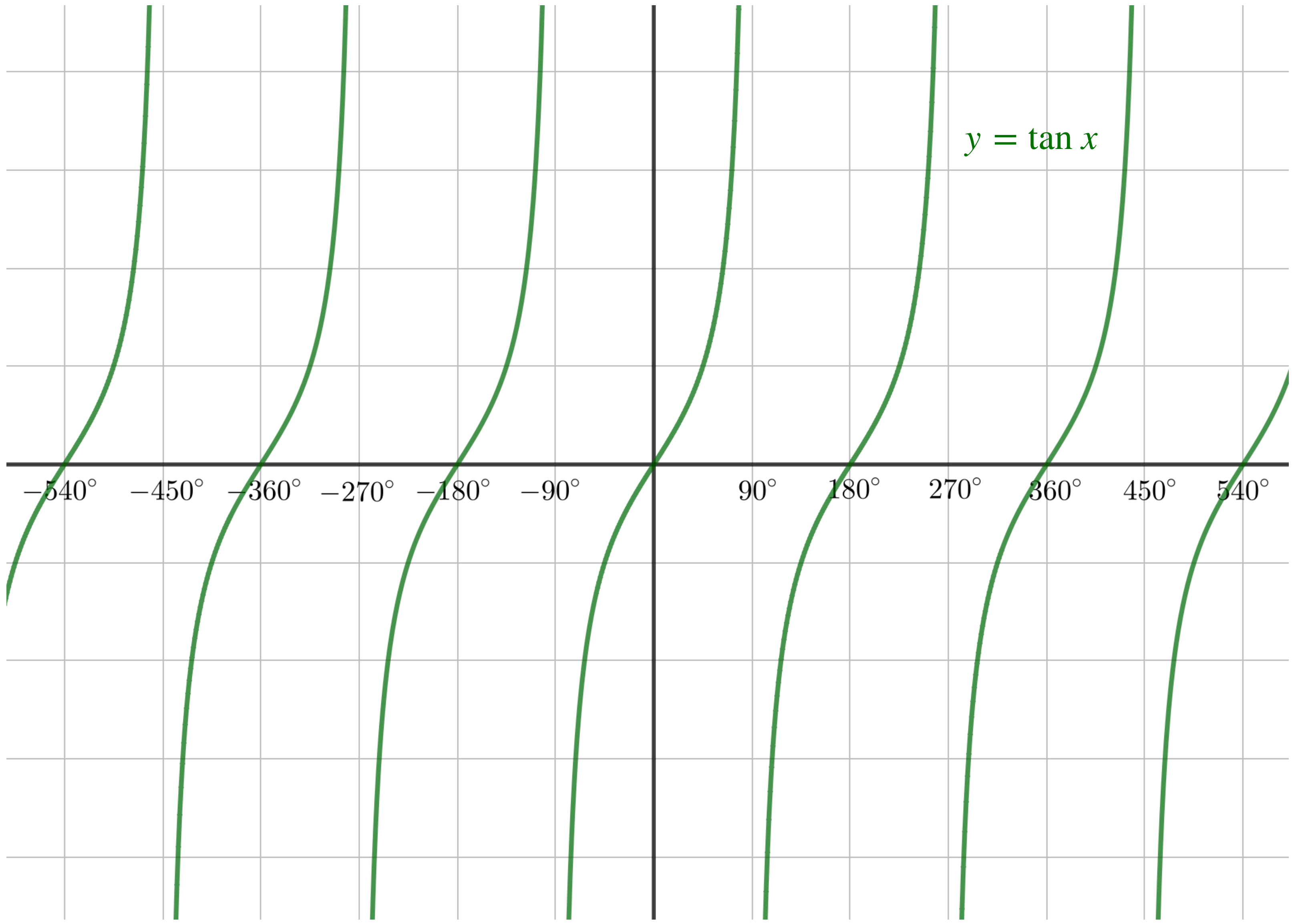
Describe the symmetry of each of these graphs.



Reflect in both axes or in the origin. Or reflect in x axis and the line $x = 180n$ for any integer n.
Or reflect in any points $(180n, 0)$



Reflect in the y axis or in the line $x = 180n$ for any integer n

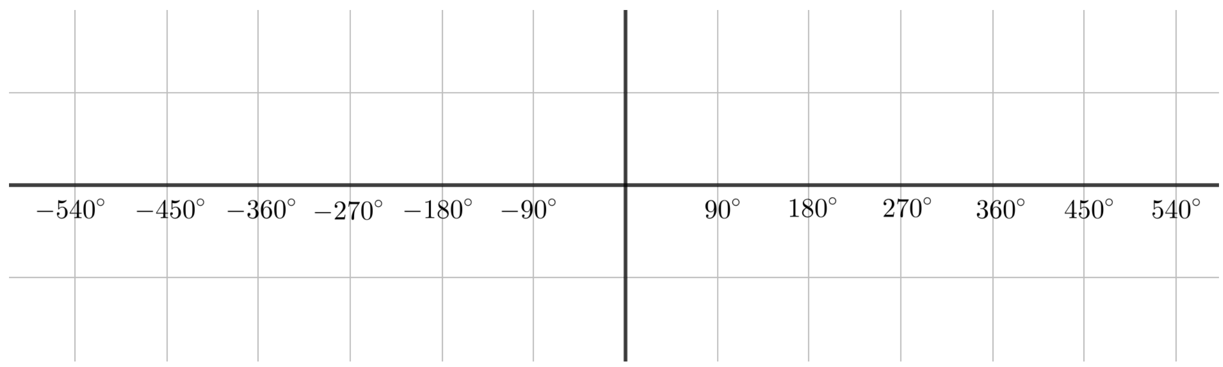


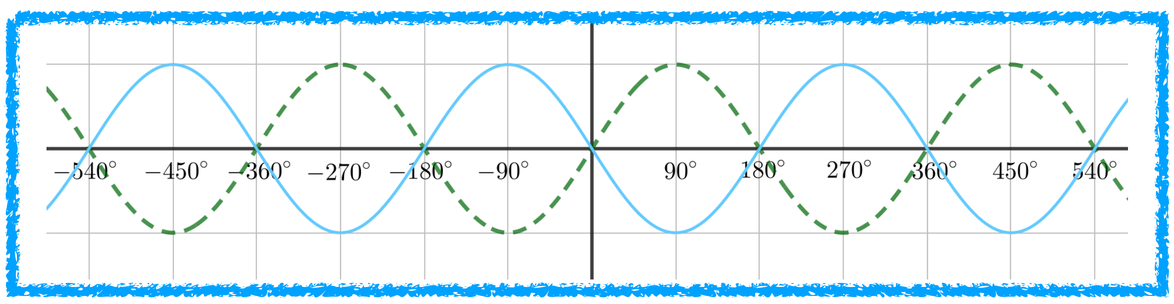
Reflect in both axes or in the origin. Or reflect in x axis and the line $x = 180n$ for any integer n. Or reflect in any points $(180n, 0)$

Relationships between circular functions

Draw the following graphs:

$y = -\sin x$

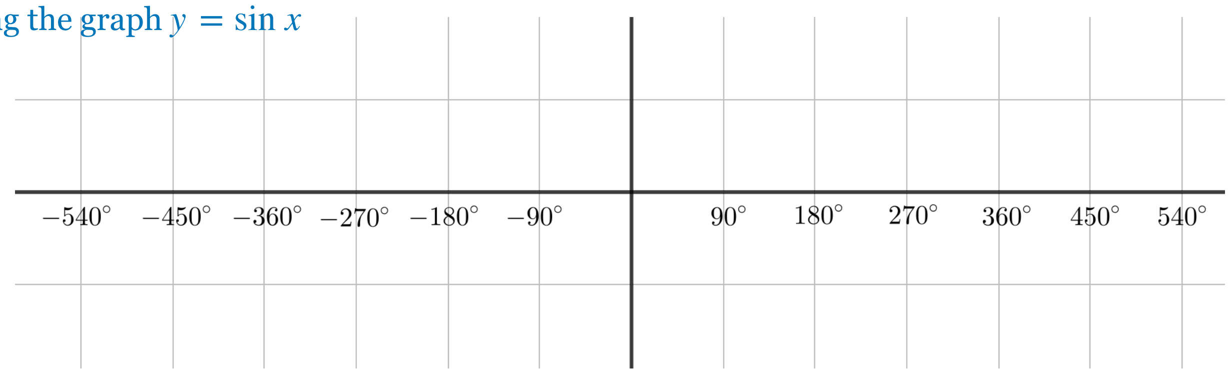




Reflect sin graph in the x axis.

It's easiest if they start by drawing the graph $y = \sin x$

$y = \sin(-x)$



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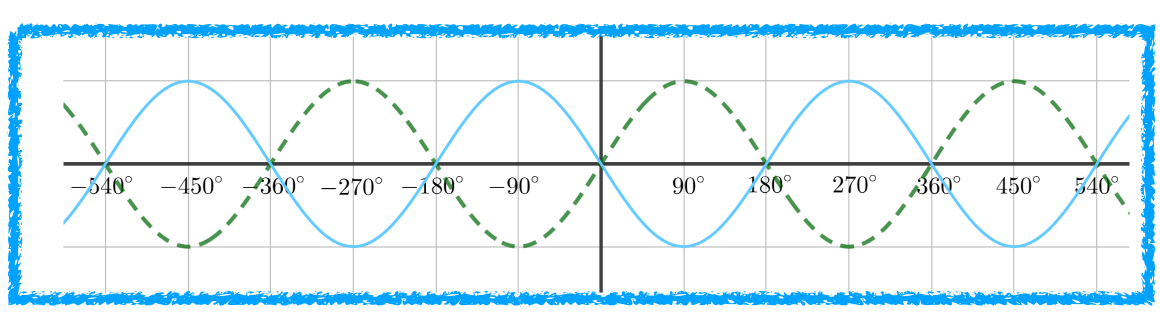

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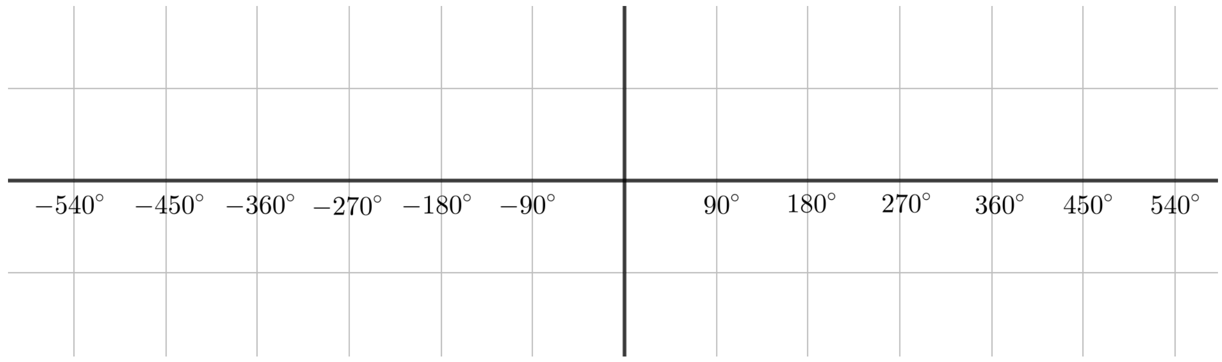
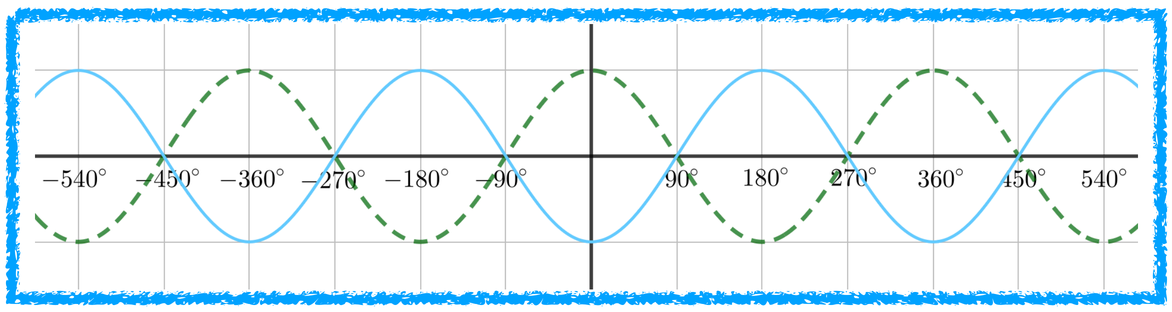
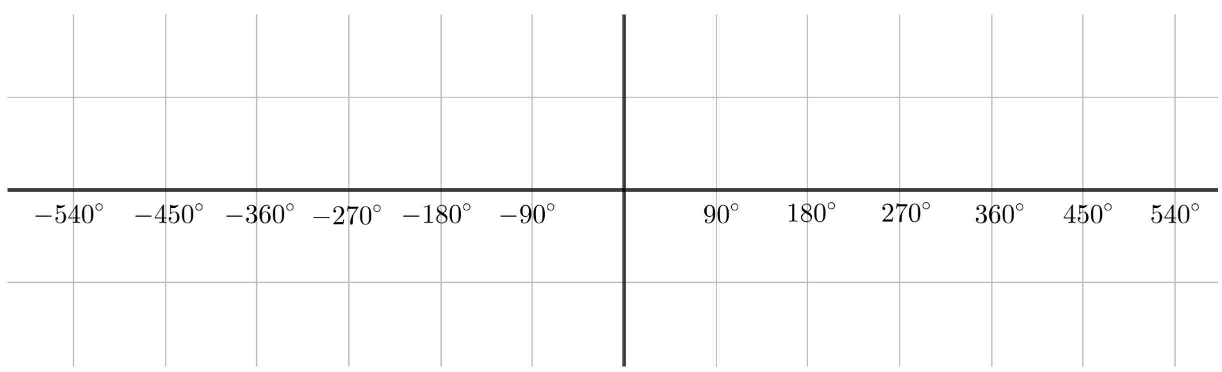
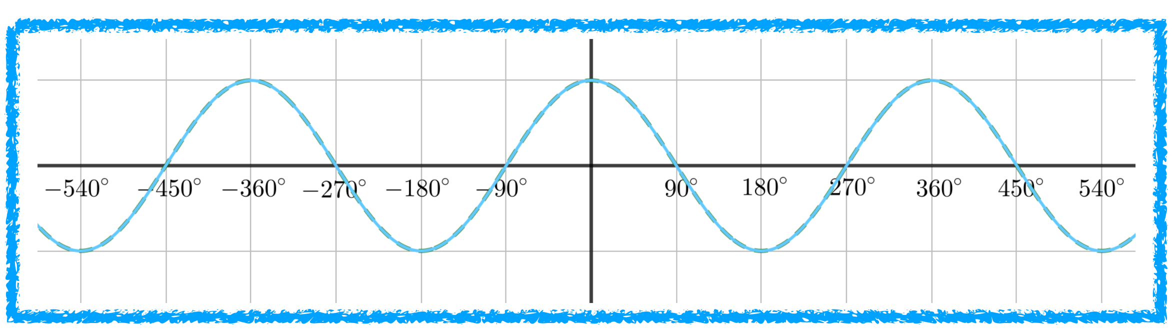




Reflect cos graph in the x axis.

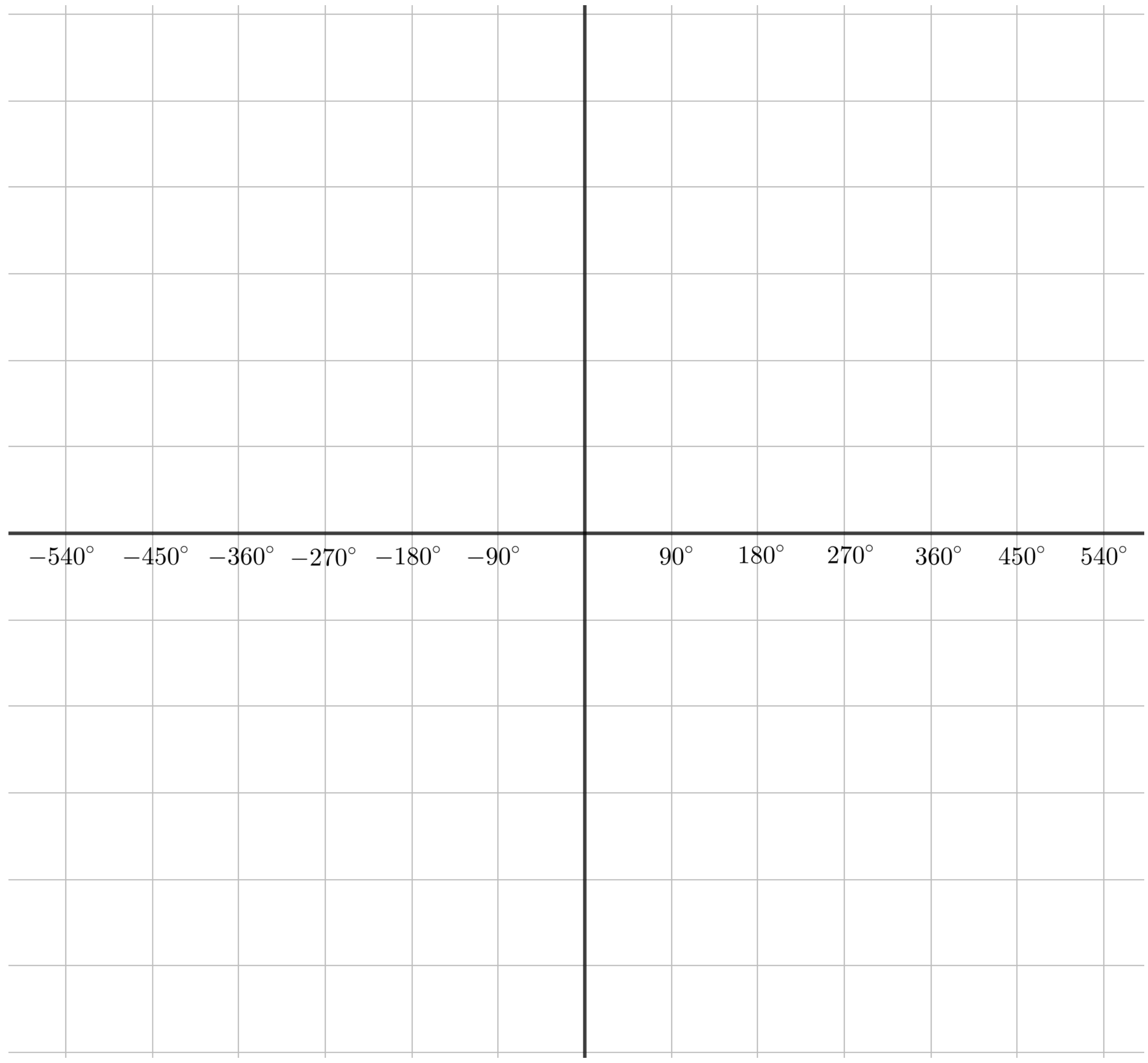
$y = \cos(-x)$





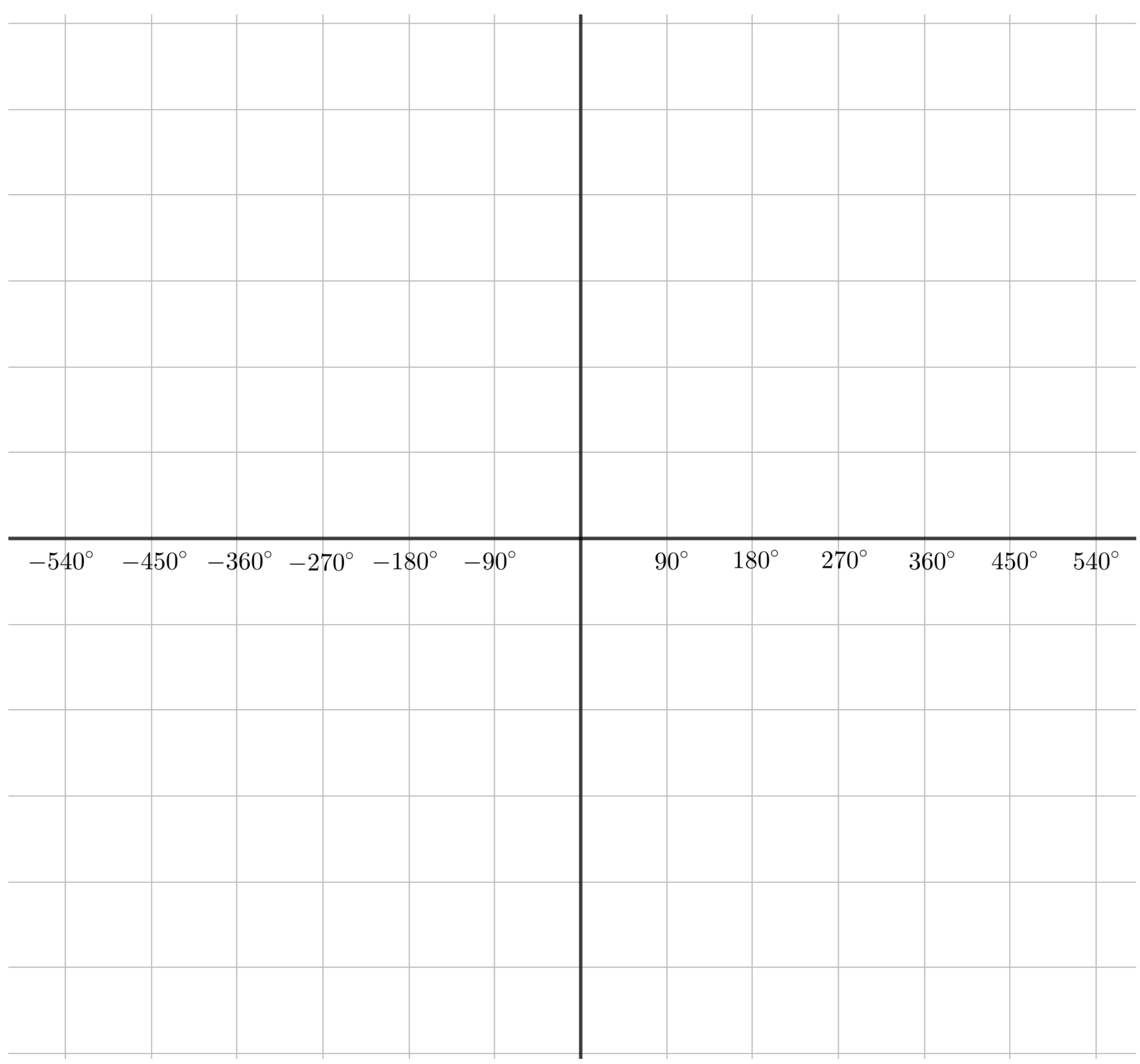
Reflect cos graph in the y axis.

$y = -\tan x$



−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

$y = \tan(-x)$



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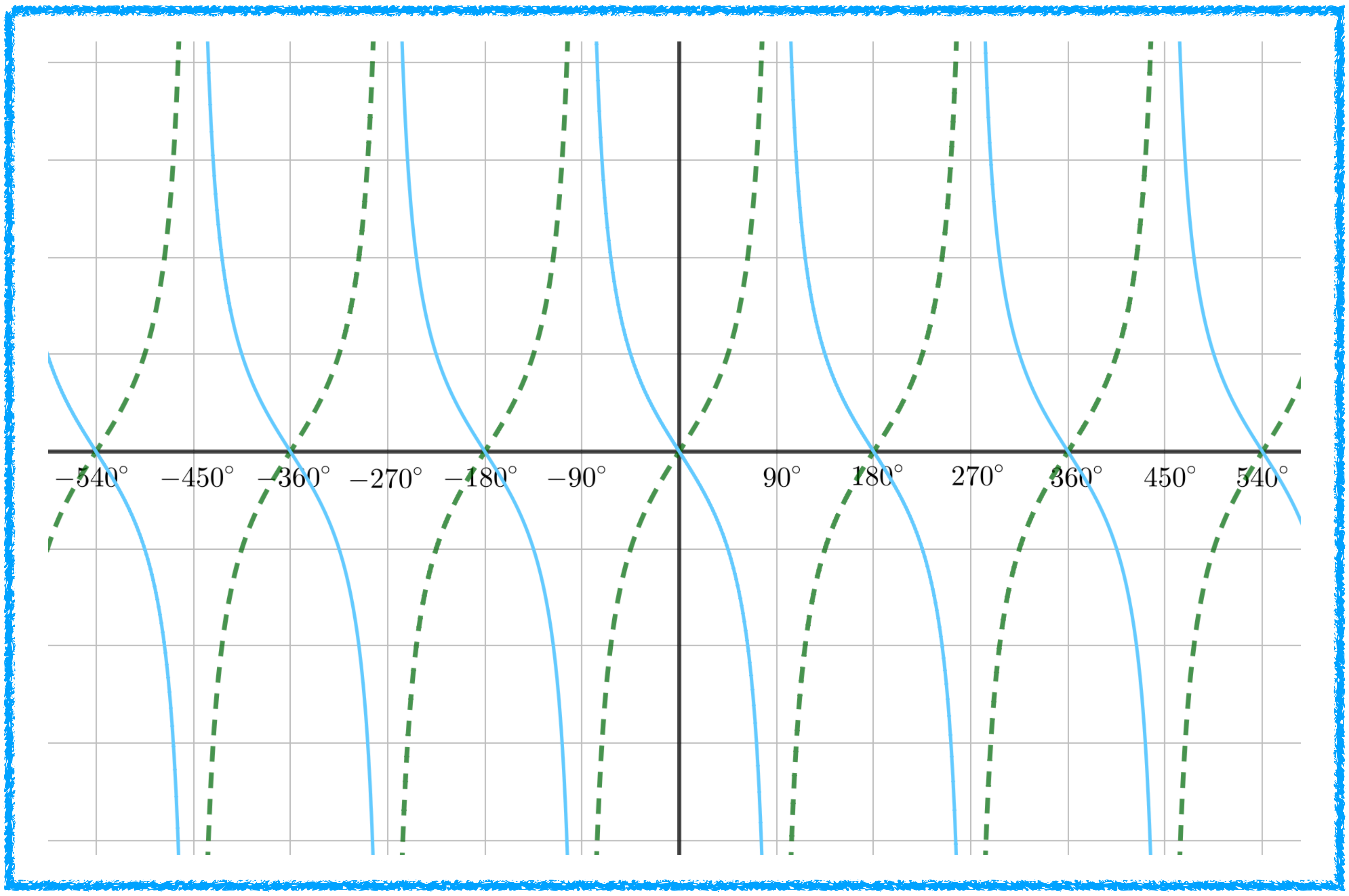


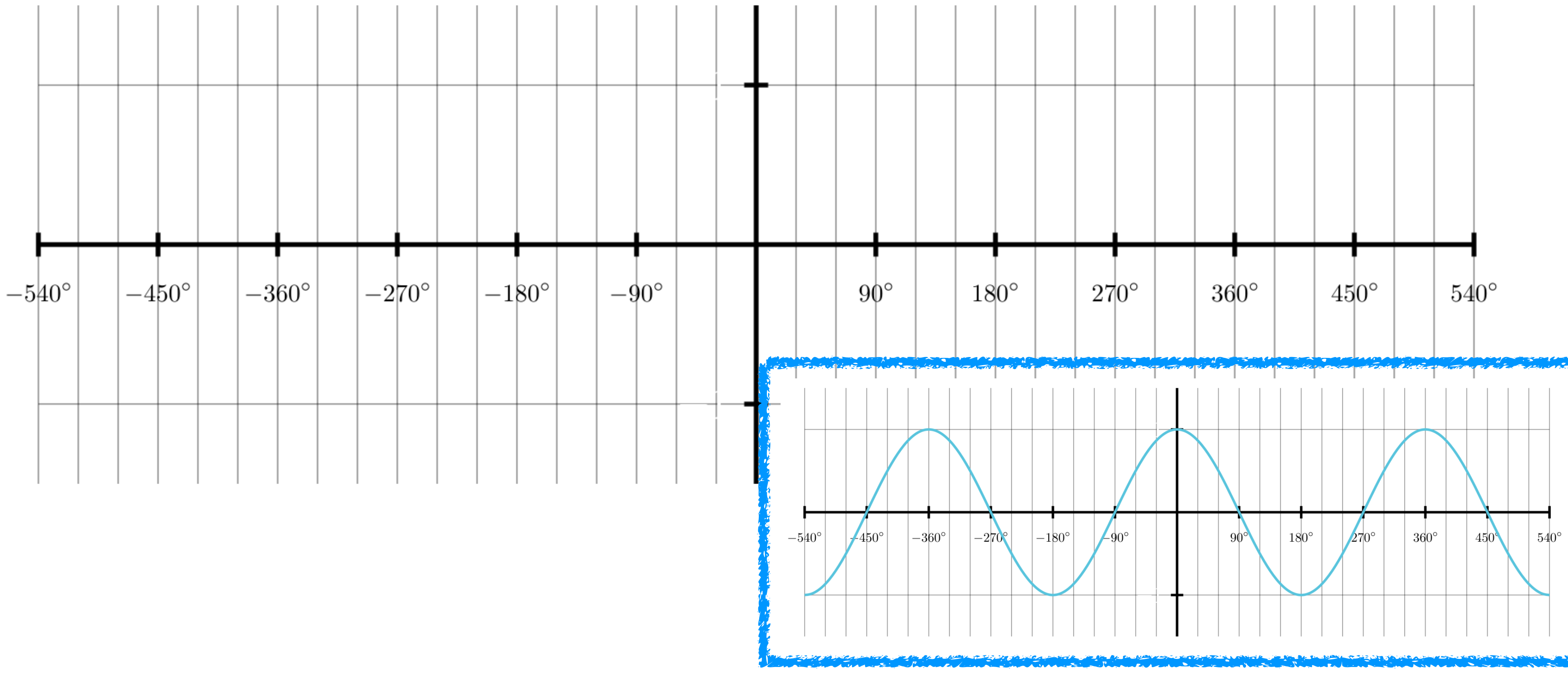
Reflecting the tan graph in the x axis and reflecting the tan graph in the y axis result in the same graph.

The identities

$\sin(-x) = -\sin x$ $\quad\quad$ $\cos(-x) = \cos x$ $\quad\quad$ $\tan(-x) = -\tan x$

are fundamental bites of subject knowledge that your students should have at their fingertips. This last sequence of graphs reinforces these facts, and it's worth discussing them again and how they relate to the unit circle.

$y = \sin(90^\circ - x)$

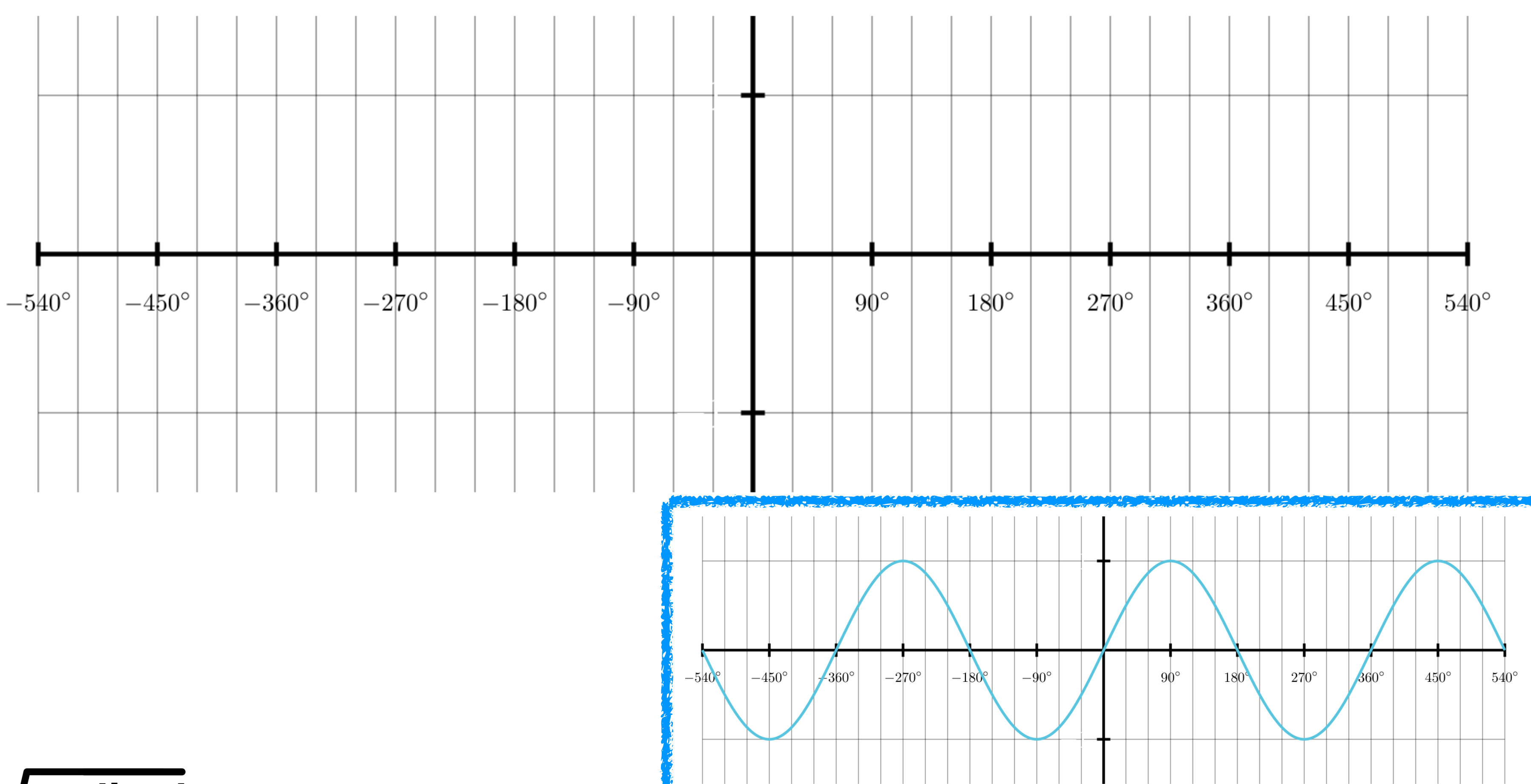


The easiest way to see this is to put some values into the function such as $0,\ 90,\ -90\ldots$

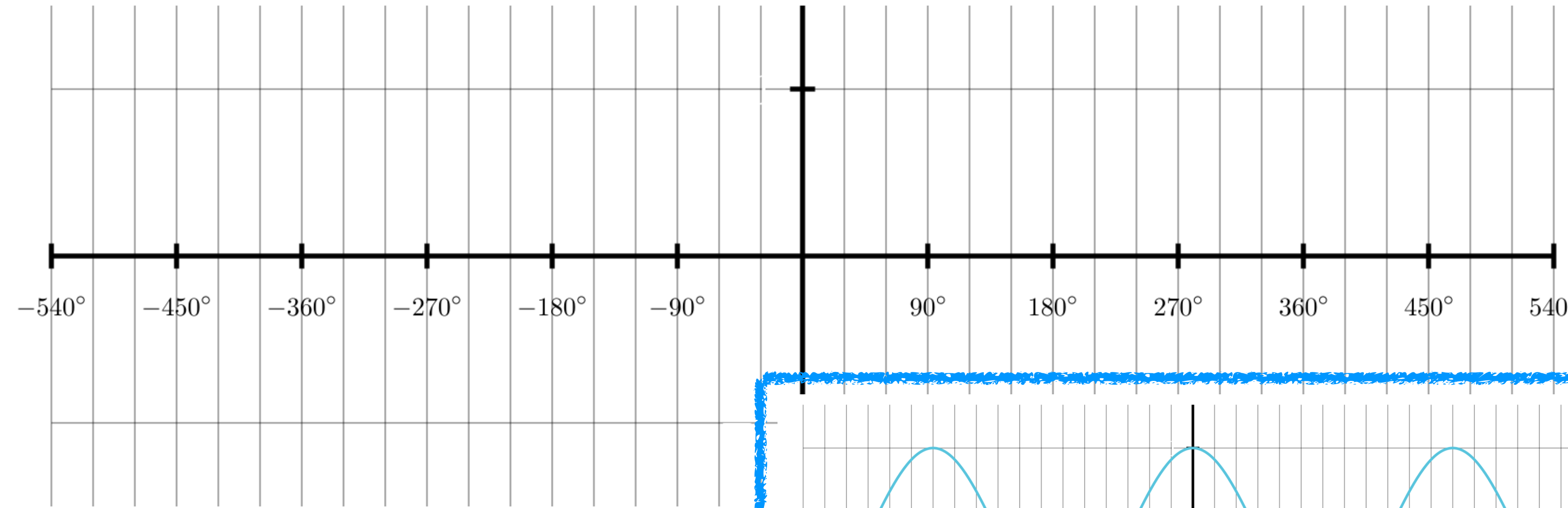
You could, in theory, use compound angle formulas, but that is not really in the spirit of this worksheet.

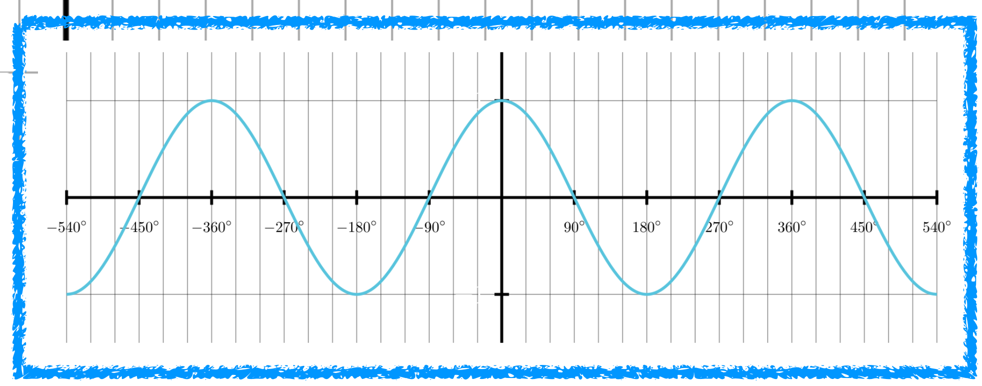
You could, in theory, use transformations, but this example is a bit tricky and would take you off course.

$y = \cos(90^\circ - x)$



$y = \sin(90^\circ + x)$





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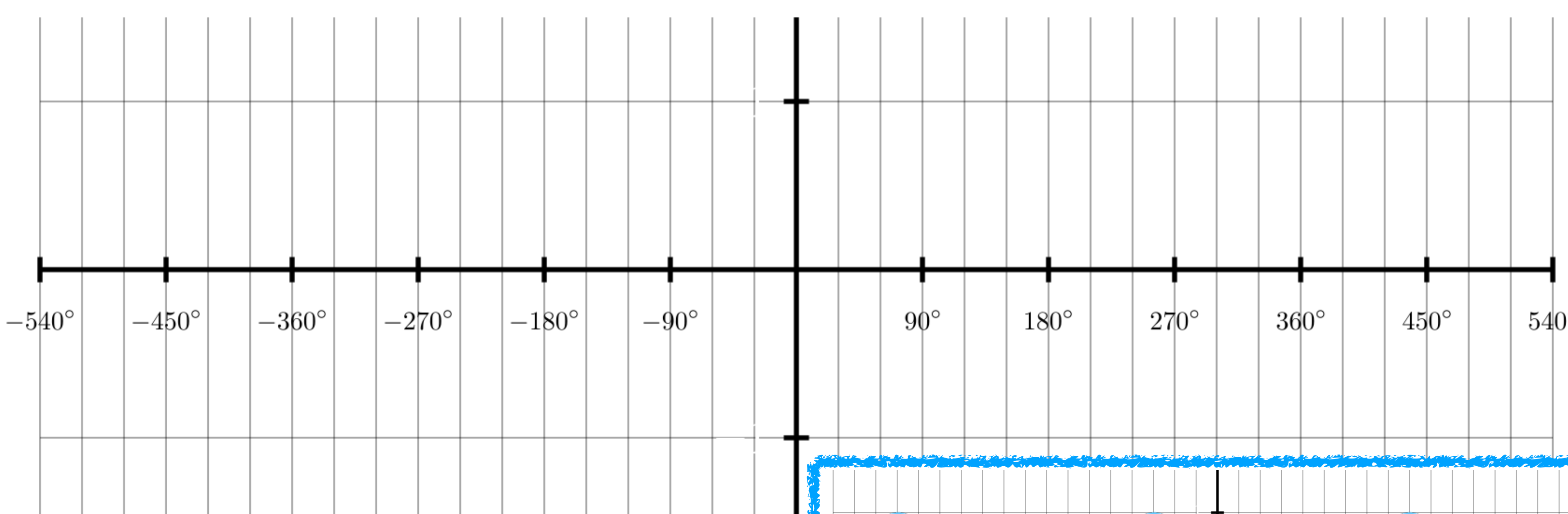

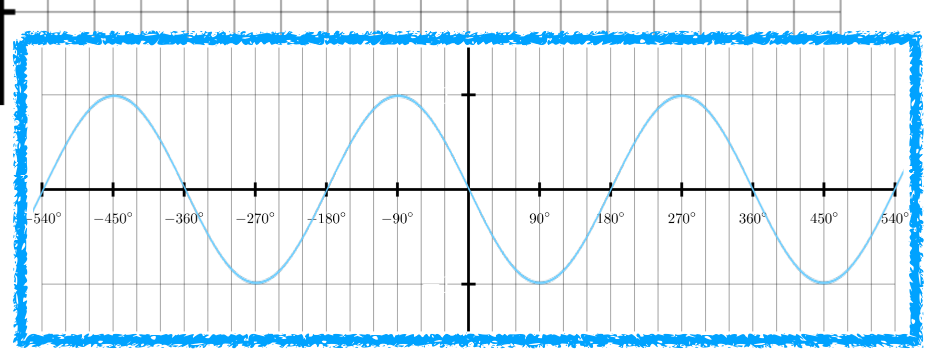

Use these graphs to find some relationships between sin and cos.

$\sin(90° - x) = \sin(90° + x) = \cos x$ $\cos(90° - x) = \cos(x - 90°) = \sin x$ $\cos(90° + x) = \sin(x - 90°) = -\sin x$

$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$

Show that

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

whenever θ is not a multiple of 90°.

If you were to ask your students how to prove this identity before you have discussed it, you might see something like this:

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

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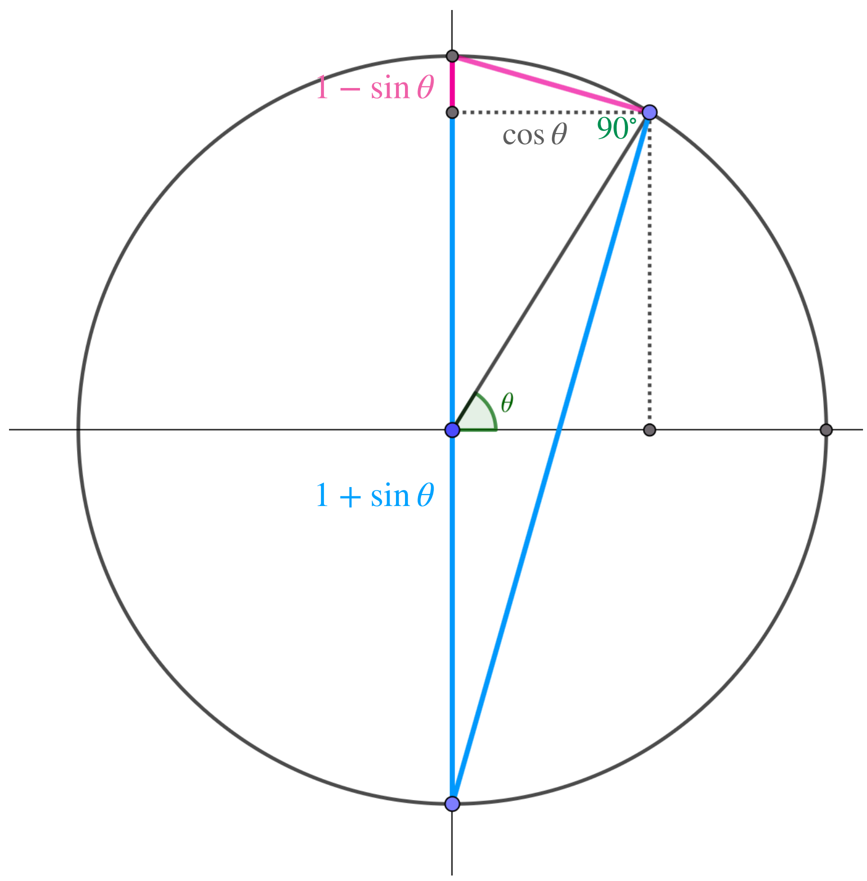

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Alternatively:

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\frac{1 + s - c}{c} &= \frac{1 + s}{c} - 1 \\
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&= \frac{c - 1 + s}{1 - s} \\
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\end{aligned}$$

or

$$\begin{aligned}
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I could turn this proof into something that the obsessive examiner would approve of, but I wouldn't bother. No one will object to this proof outside the context of a school examination.

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Reciprocal circular functions

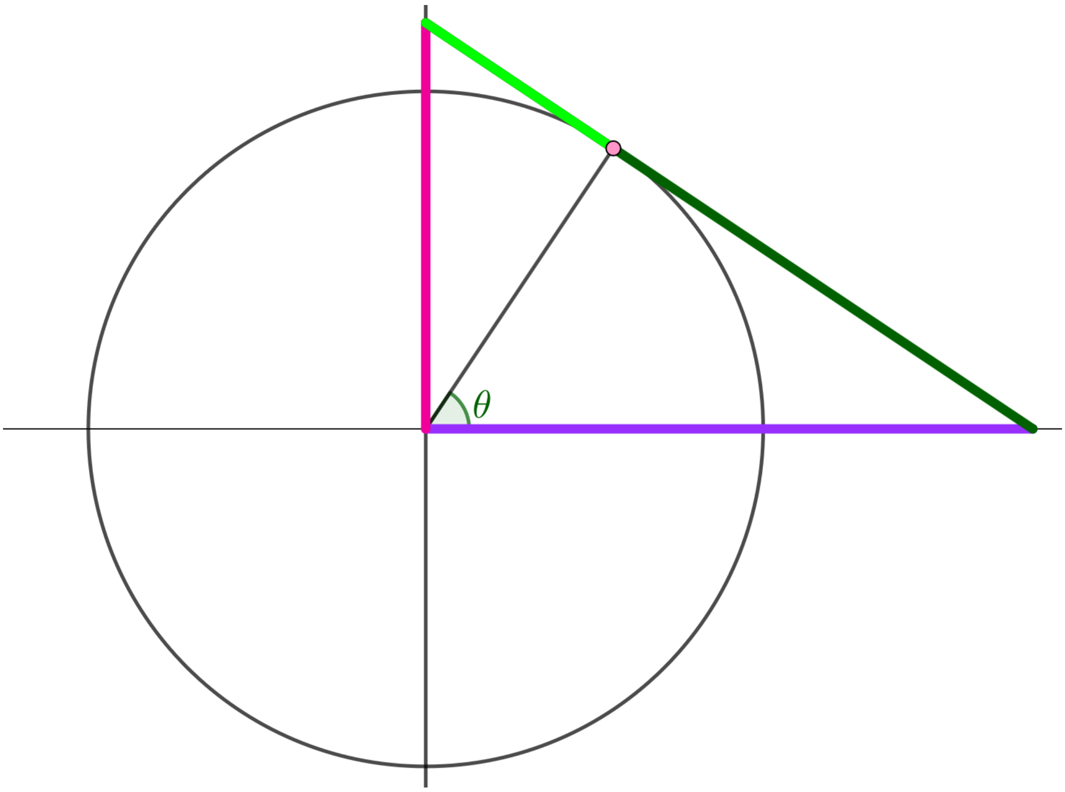
sec, cosec (or csc) and cotan (or cot) get far less attention than sin, cos, and tan, but they also have interesting interpretations on the unit circle, which I introduce using this diagram.

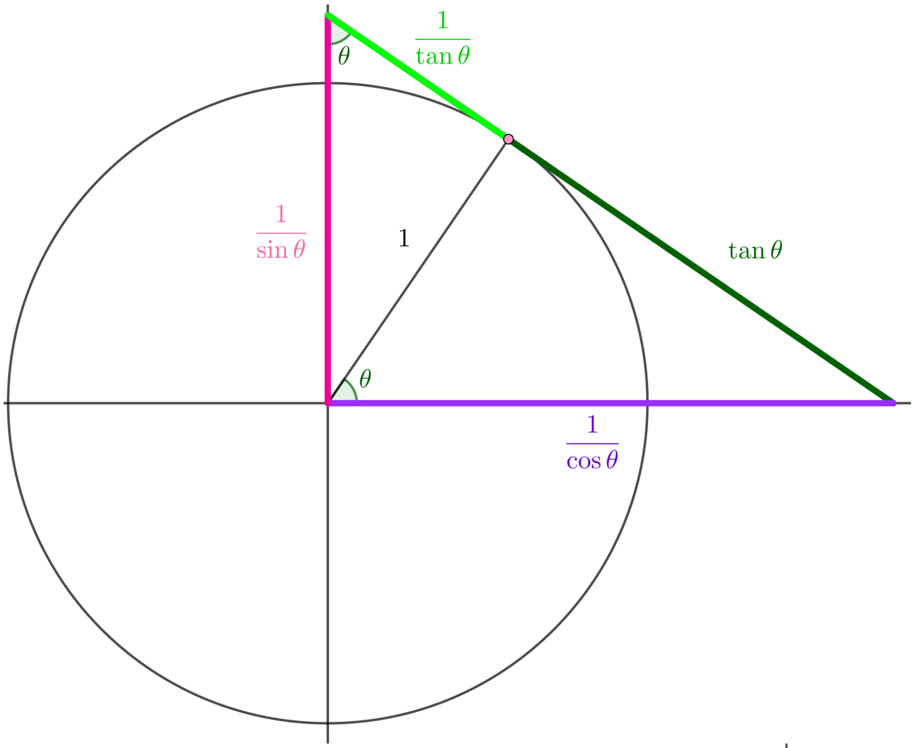
If you were to look at the diagram on the next page and say to me: “what’s the point of a diagram like this? Why not just tell them what sec, cosec, and cotan are?” I would say: “you are quite right, there is no real point, go ahead and miss it out.” I don’t really believe this, though. It all helps to build up a really strong idea of the circular functions, and it’s just plain interesting.

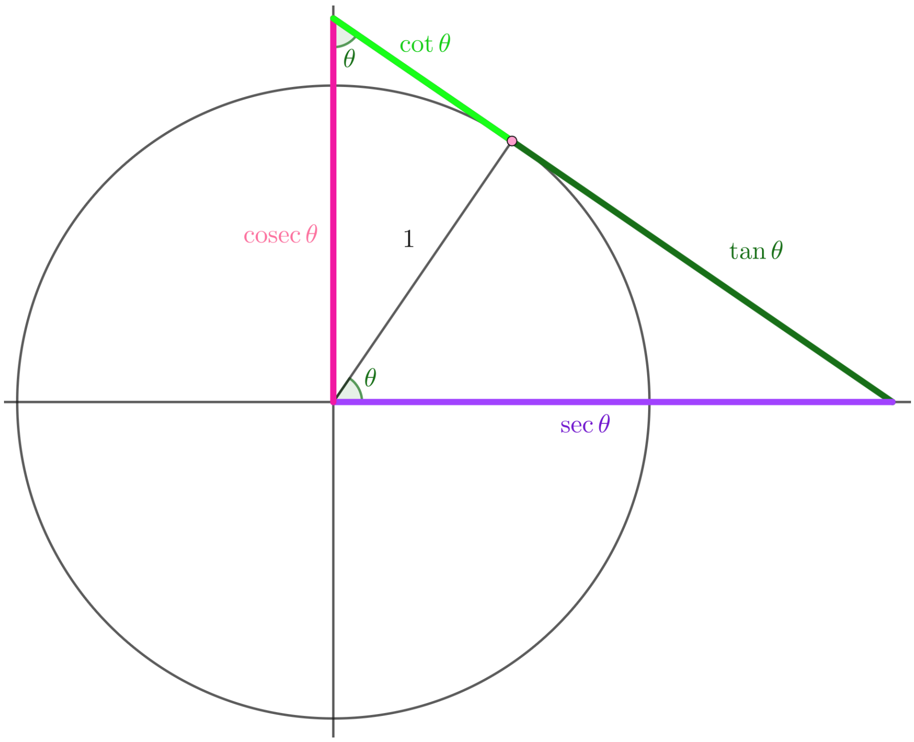
We tend to sketch their graphs by starting from sin cos and tan and sketching their inverses, but here too we can relate the functions to the unit circle. And here again, you might ask what the point is, and I would give the same answer.

This is a circle radius 1.

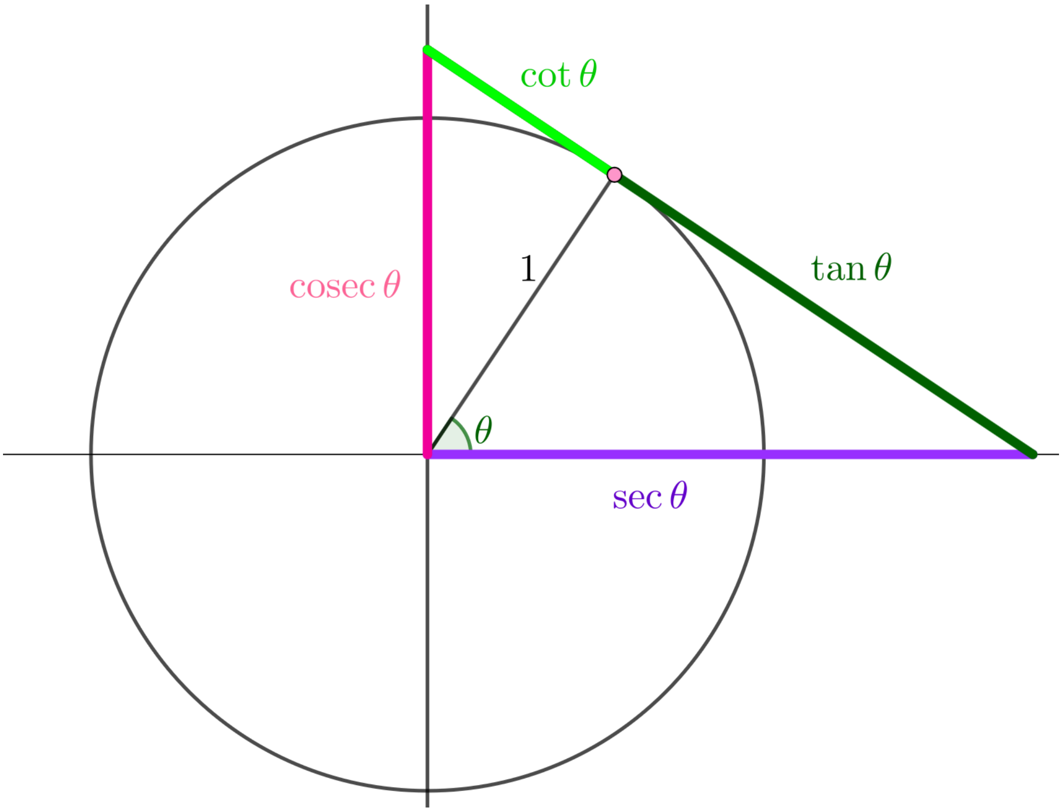
What are the lengths of the highlighted segments?







I suggest starting with the top diagram, which is what your students' work should look like.
Use this to introduce the new names in the right-hand triangle.



There are three right-angled triangles here. What would Pythagoras say about each of them?

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\operatorname{cosec}^2\theta + \sec^2\theta = (\tan\theta + \cot\theta)^2$

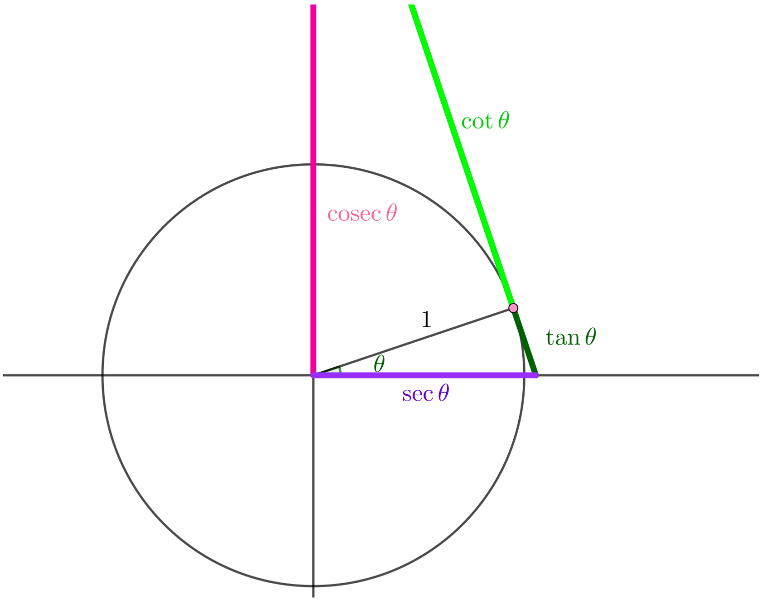
The first two of these can easily be derived from the identity

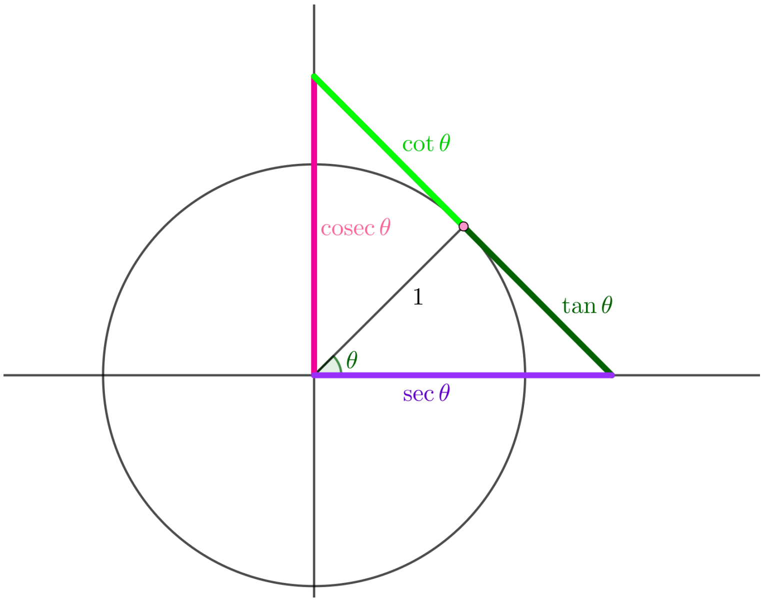
$\cos^2\theta + \sin^2\theta = 1$

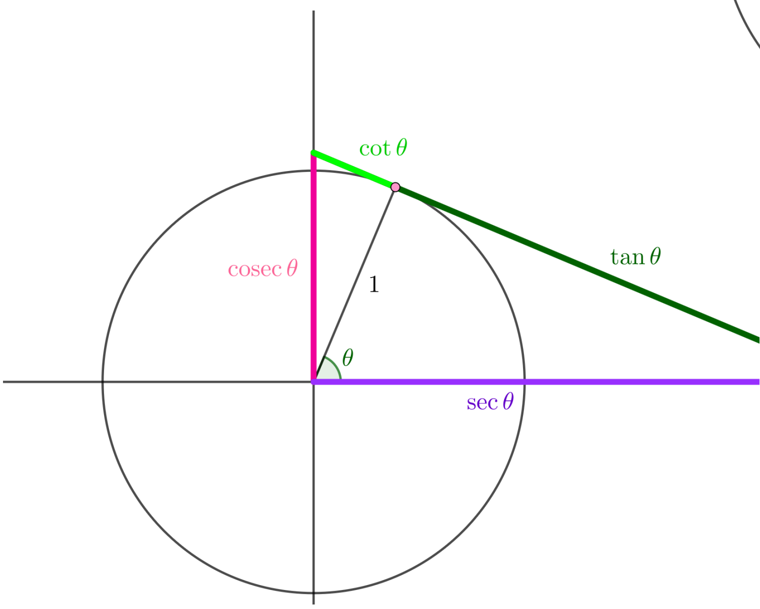
by dividing through either by $\cos^2\theta$ or $\sin^2\theta$.

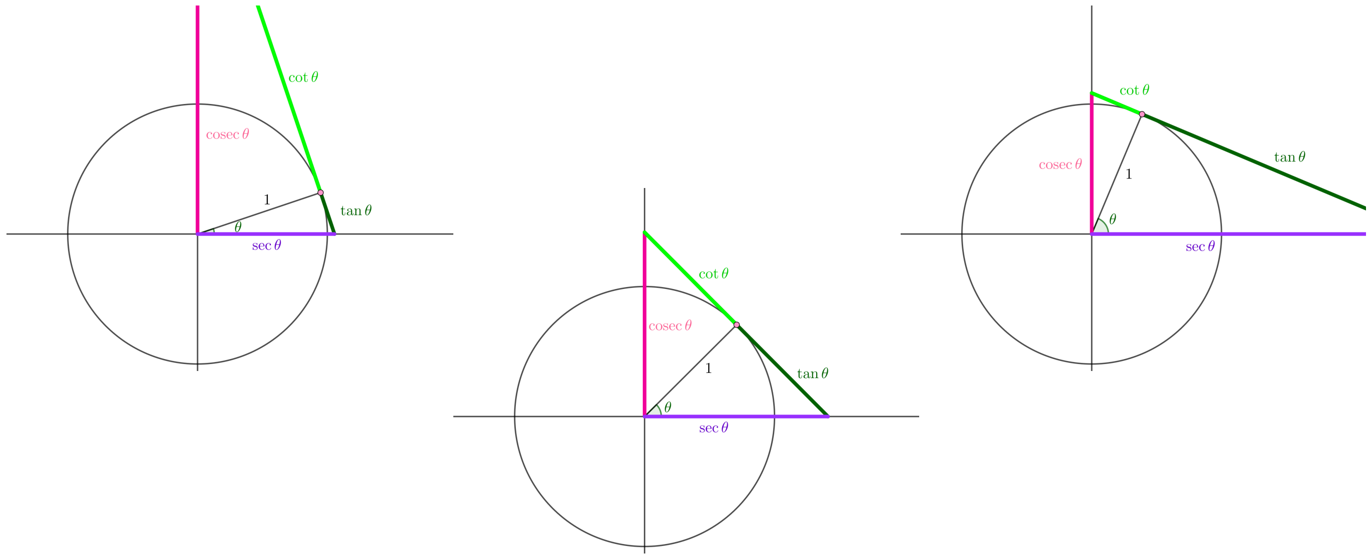
The last is a bit exotic: its not one that have ever taught or bothered to learn.

Use this sequence of diagrams as a guide to help you answer the questions on the next page:









What happens to $\sec\theta$ as $\theta \to 0$? $\quad$ $\sec\theta \to 1$ as $\theta \to 0$?

What is the minimum value of $\sec\theta$? $\quad$ minimum value of $\sec\theta = 1$?

What happens to $\sec\theta$ as $\theta \to 90^\circ$? $\quad$ $\sec\theta \to \infty$ as $\theta \to 90^\circ$?

What happens to $\operatorname{cosec}\theta$ as $\theta \to 0$? $\quad$ $\operatorname{cosec}\theta \to \infty$ as $\theta \to 0$?

What is the minimum value of $\operatorname{cosec}\theta$? $\quad$ minimum value of $\operatorname{cosec}\theta = 1$?

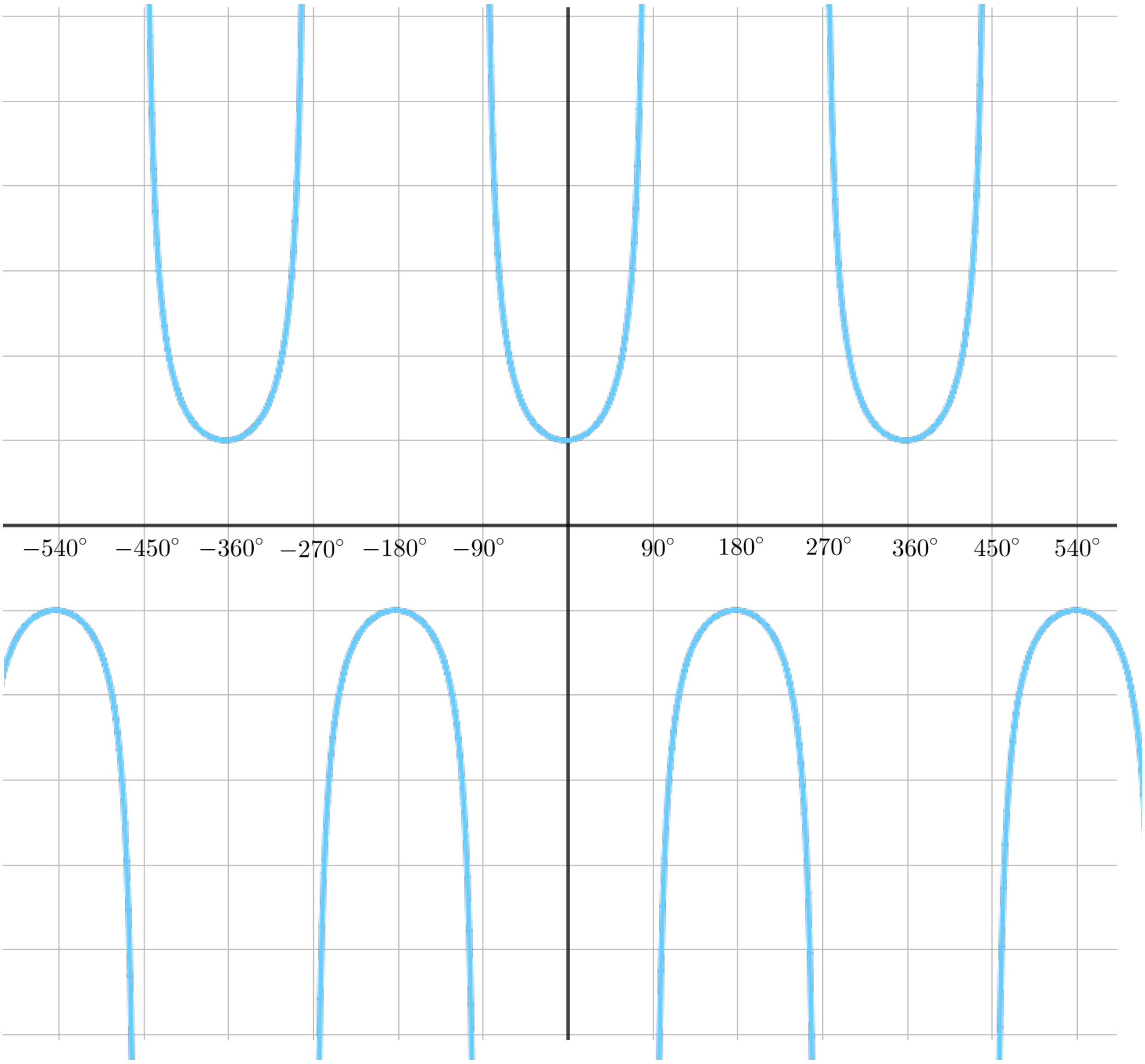
What happens to $\operatorname{cosec}\theta$ as $\theta \to 90^\circ$? $\quad$ $\operatorname{cosec}\theta \to 1$ as $\theta \to 90^\circ$?

What happens to $\cot\theta$ as $\theta \to 0$? $\quad$ $\cot\theta \to \infty$ as $\theta \to 0$?

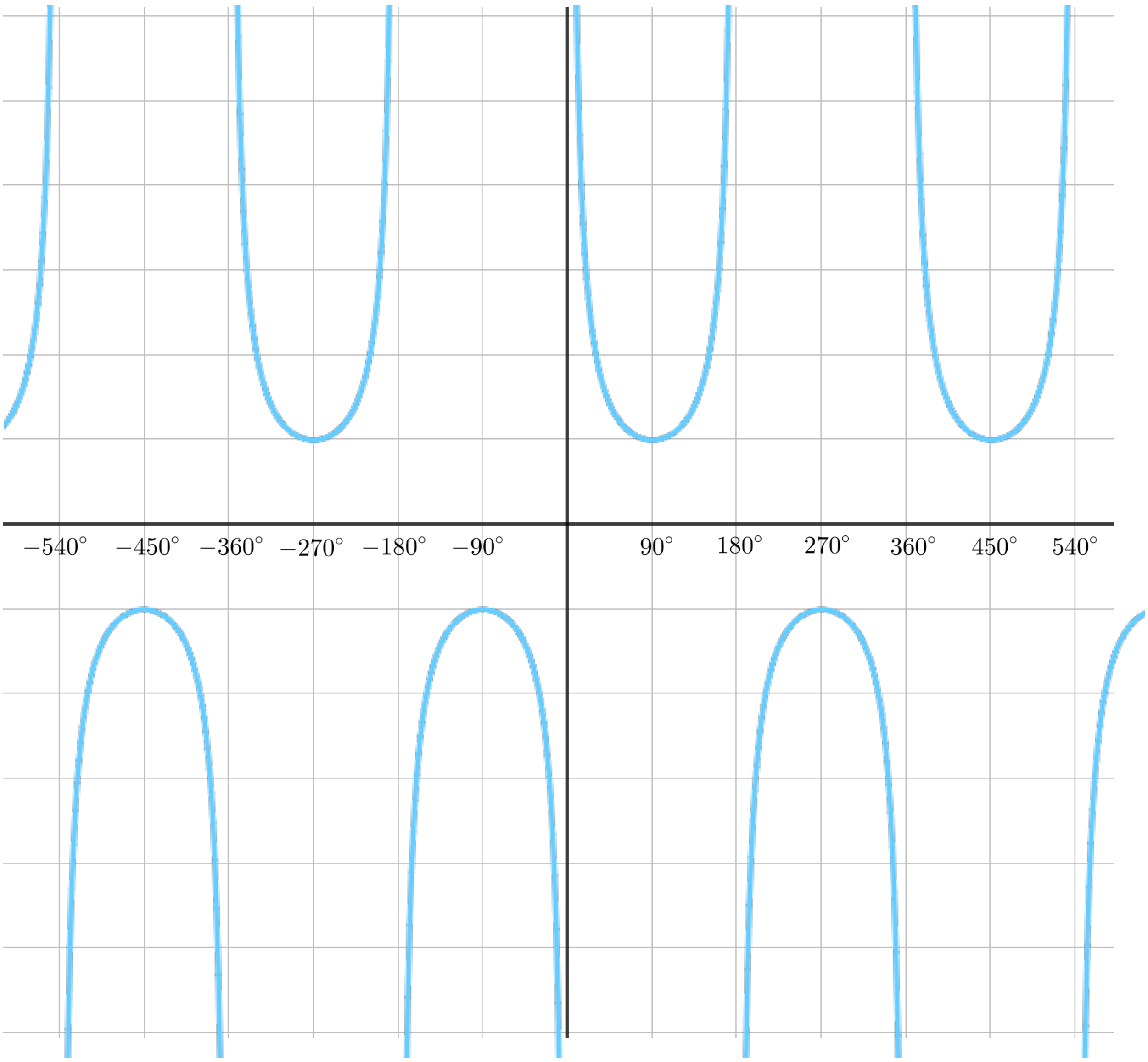
What is $\cot 45^\circ$? $\quad$ minimum value of $\cot 45^\circ = 1$?

What happens to $\cot\theta$ as $\theta \to 90^\circ$? $\quad$ $\cot\theta \to 0$ as $\theta \to 90^\circ$?

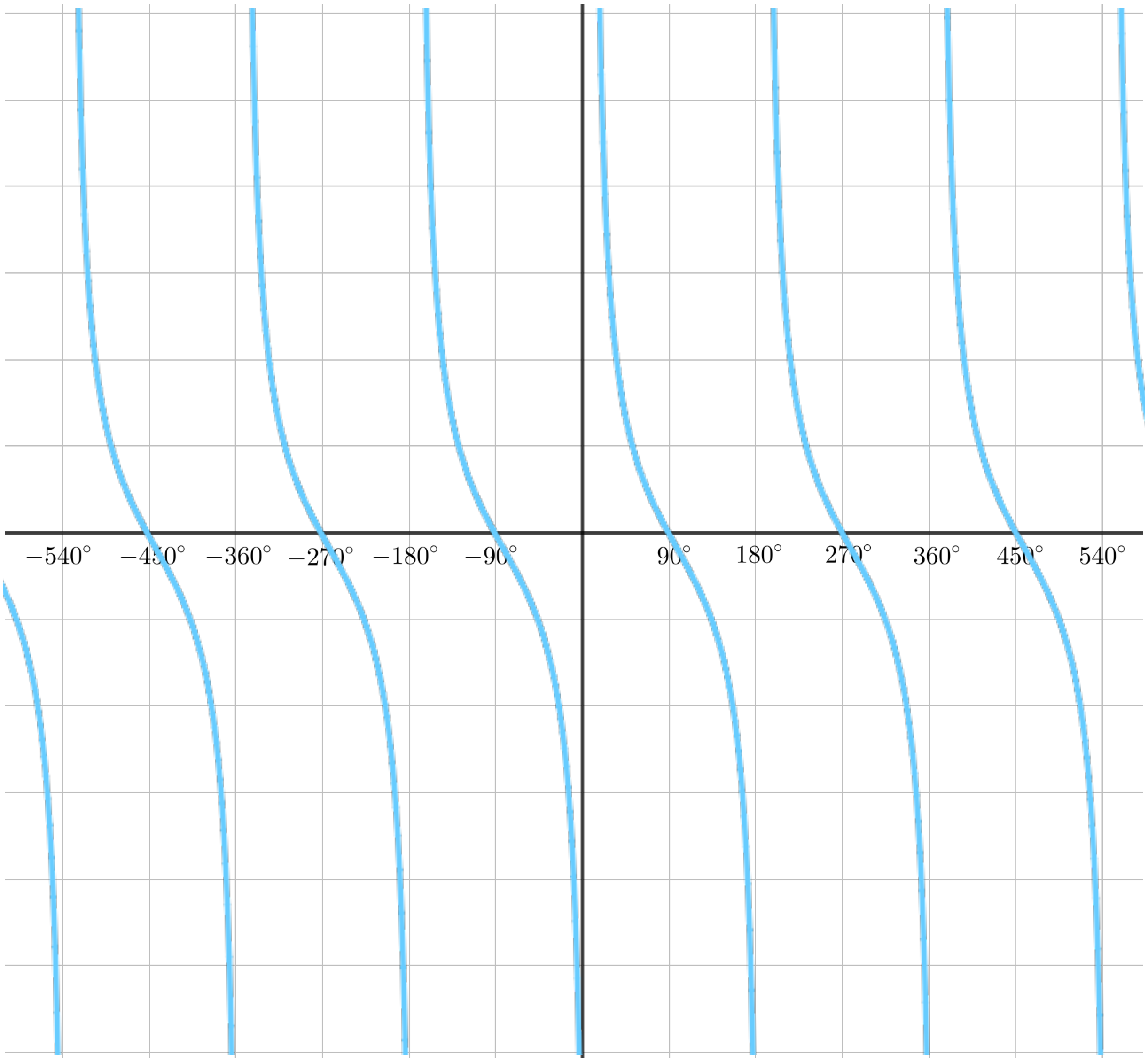
Draw the graph $y = \sec x$.



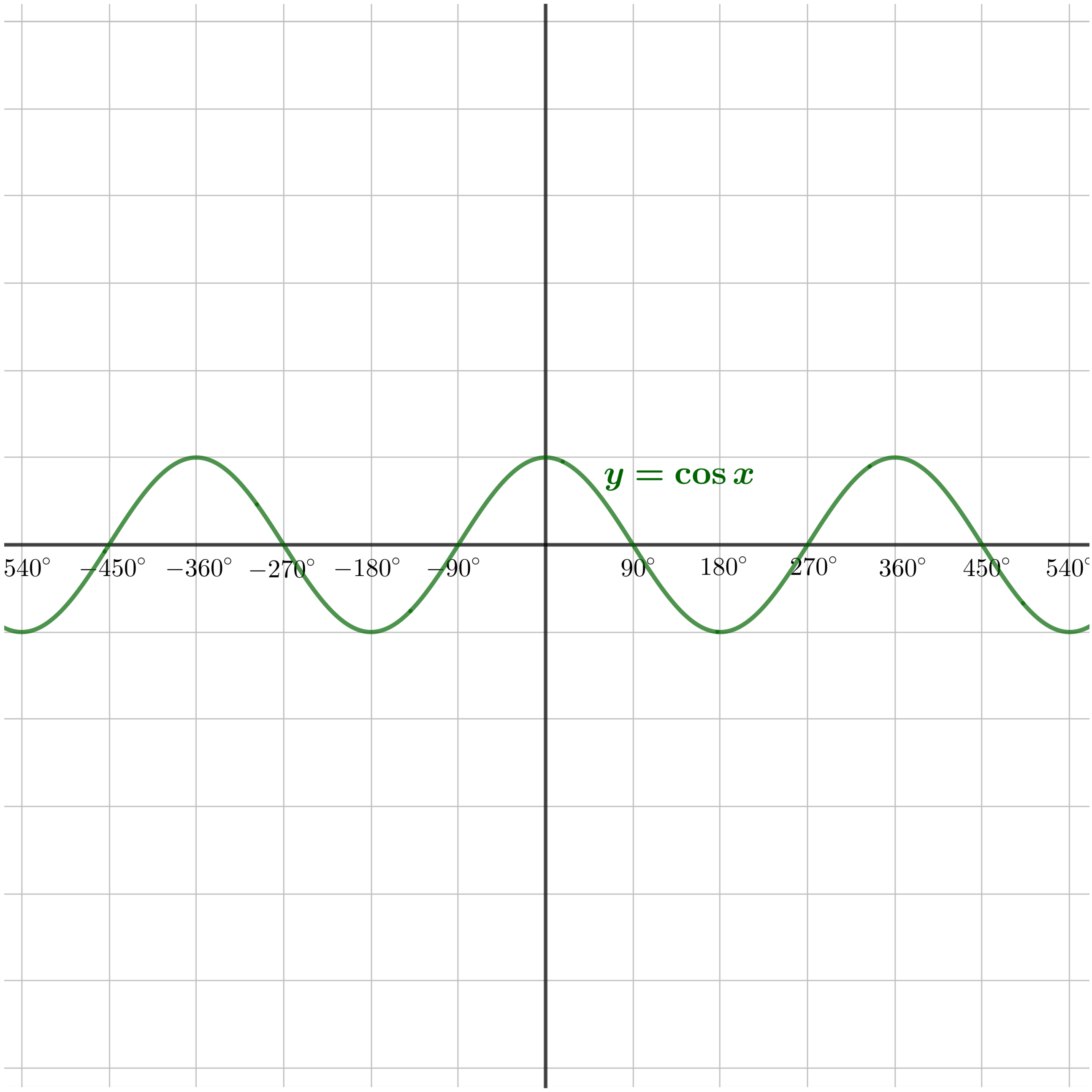
Draw the graph $y = \text{cosec } x$.

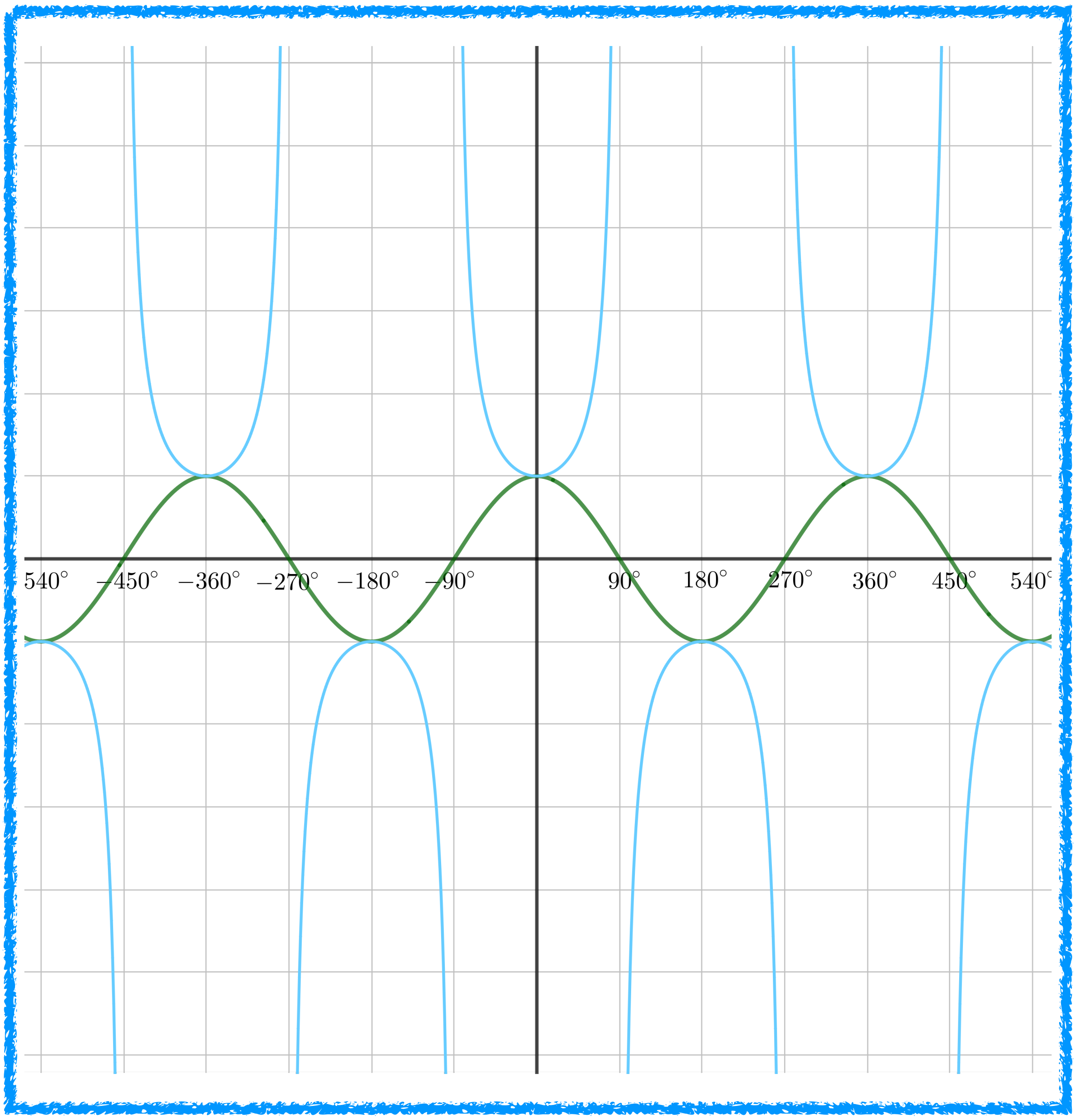


Draw the graph $y = \cot x$.



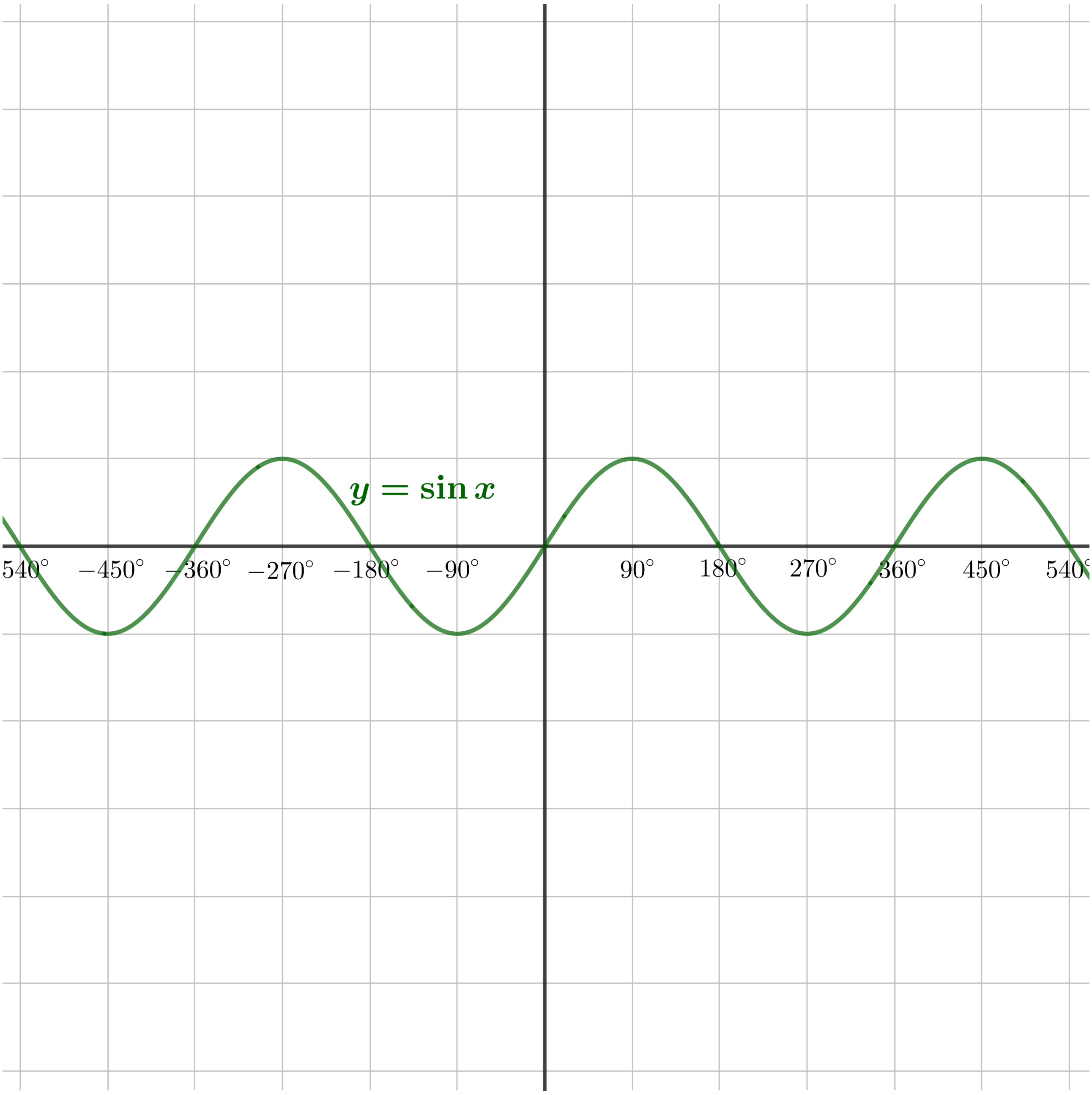
Draw the graph $y = \sec x$ again and describe its relationship to the graph $y = \cos x$.

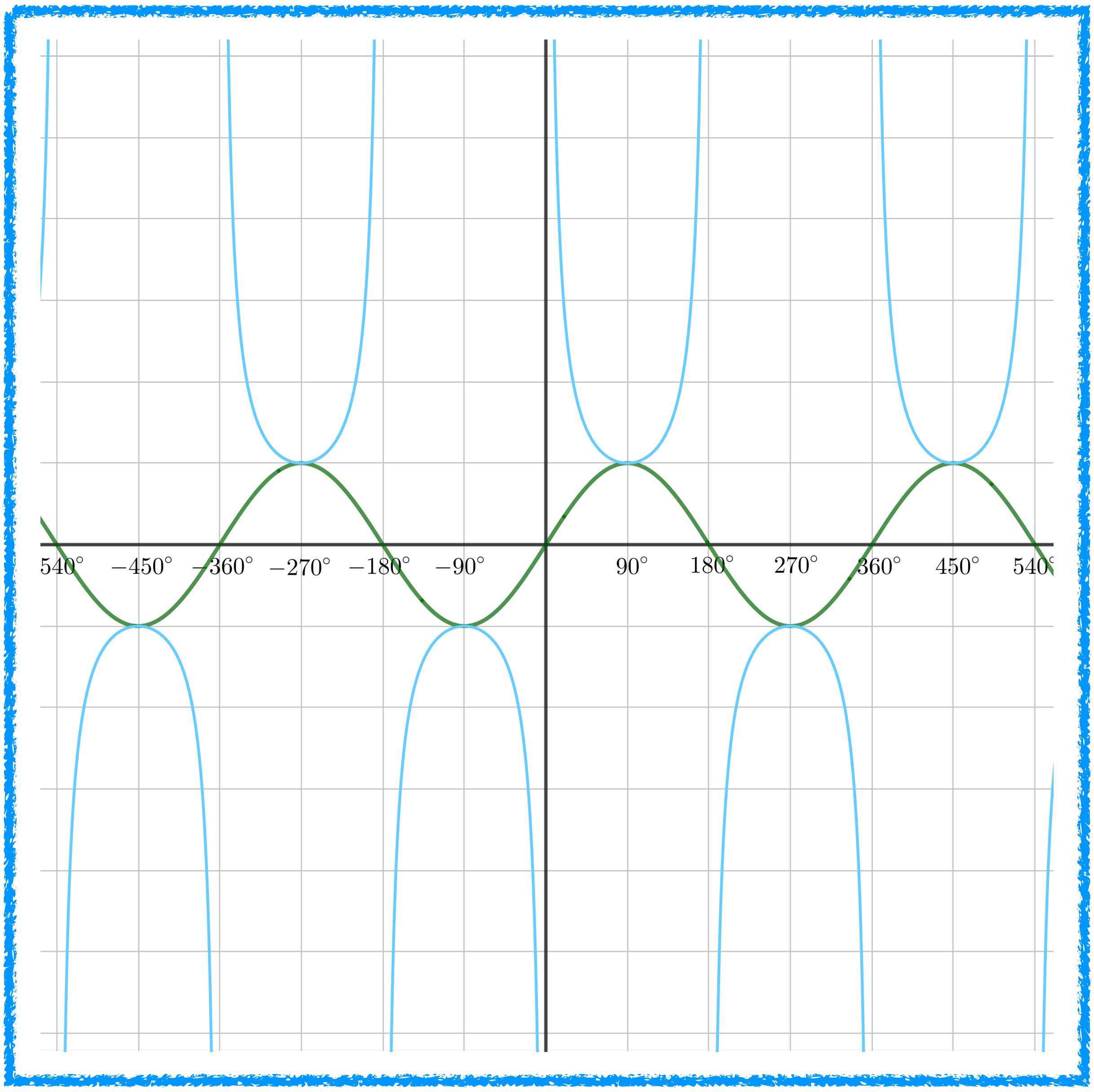




One graph is just the reciprocal of the other.

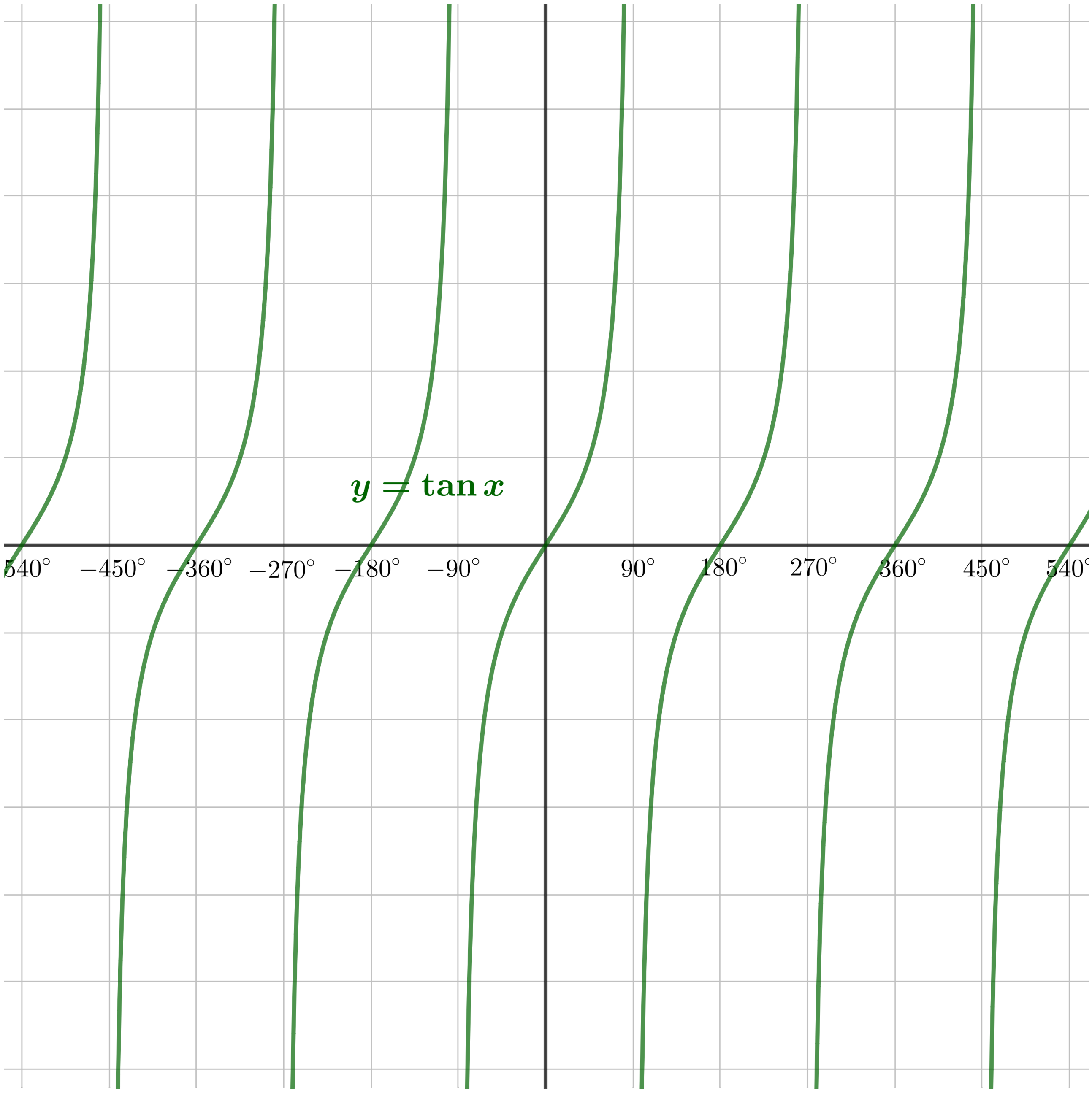
Draw the graph $y = \operatorname{cosec} x$ again and describe its relationship to the graph $y = \sin x$.

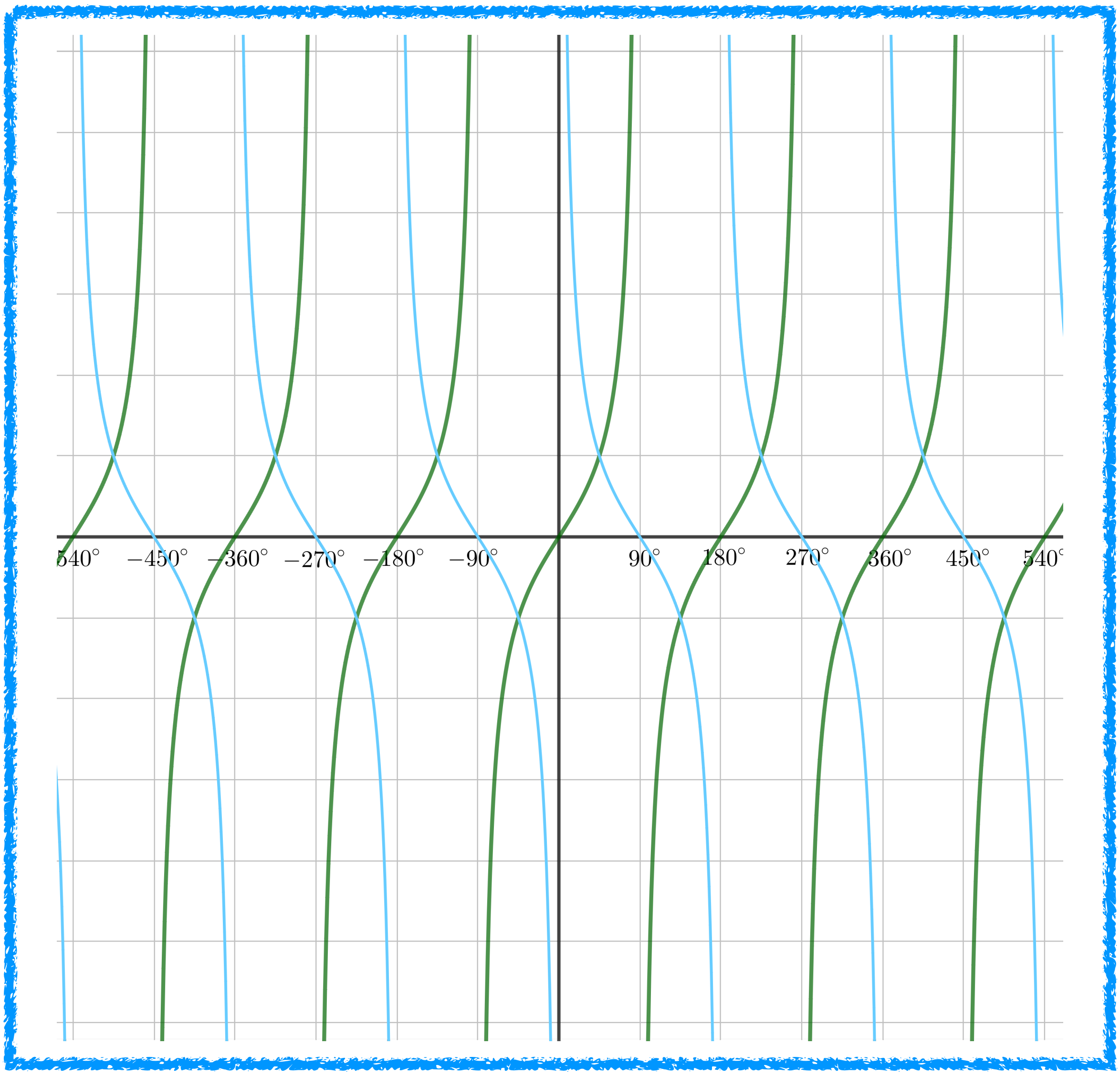




−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

Draw the graph $y = \cot x$ again and describe its relationship to the graph $y = \tan x$.





One graph is the reciprocal of the other, but there is more going on:

> they cross at odd multiples of 45°
>
> they are reflections of each other in, for example, $x = 45°$

Show that

$$\frac{1}{\cot\theta} + \cot\theta = \sec\theta\,\mathrm{cosec}\theta$$

whenever θ is not a multiple of 90°.

Proving identities involving these reciprocal circular functions can usually be approached in two ways:

- convert the sec, cosec, and cotan to sin, cos, and tan, and then prove as before, or
- use the identities

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\mathrm{cosec}\,\theta}{\sec\theta},\ \ 1+\tan^2\theta = \sec^2\theta,\ \text{and}\ 1+\cot^2\theta = \mathrm{cosec}^2\,\theta$$

We have already done this example earlier by the first method. For the second, try this:

$$\begin{aligned}\frac{1}{\cot\theta} + \cot\theta &= \frac{1+\cot^2\theta}{\cot\theta}\\ &= \frac{\mathrm{cosec}^2\,\theta}{\cot\theta}\\ &= \mathrm{cosec}^2\,\theta \times \frac{\sec\theta}{\mathrm{cosec}\,\theta}\\ &= \sec\theta\,\mathrm{cosec}\,\theta\end{aligned}$$

radical maths

for independence
for confidence
for creativity
for insight

Circular functions 6

$\sin(A + B)$ etc

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions — $\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x$, $\cos^{-1} x$, $\cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

We all know formulas for $(a+b)^2$ or e^{a+b} and so on, but what about the circular functions? Are there formulas for $\sin(\alpha+\beta)$ and the other circular functions? Yes there are, and they are of fundamental importance in more or less all applications of mathematics to the physical world.

There are so many ways to derive the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. Everyone has their favourite, it seems, and will argue passionately for its supremacy. I find nearly all of them a bit unsatisfactory for one or more of the following reasons:

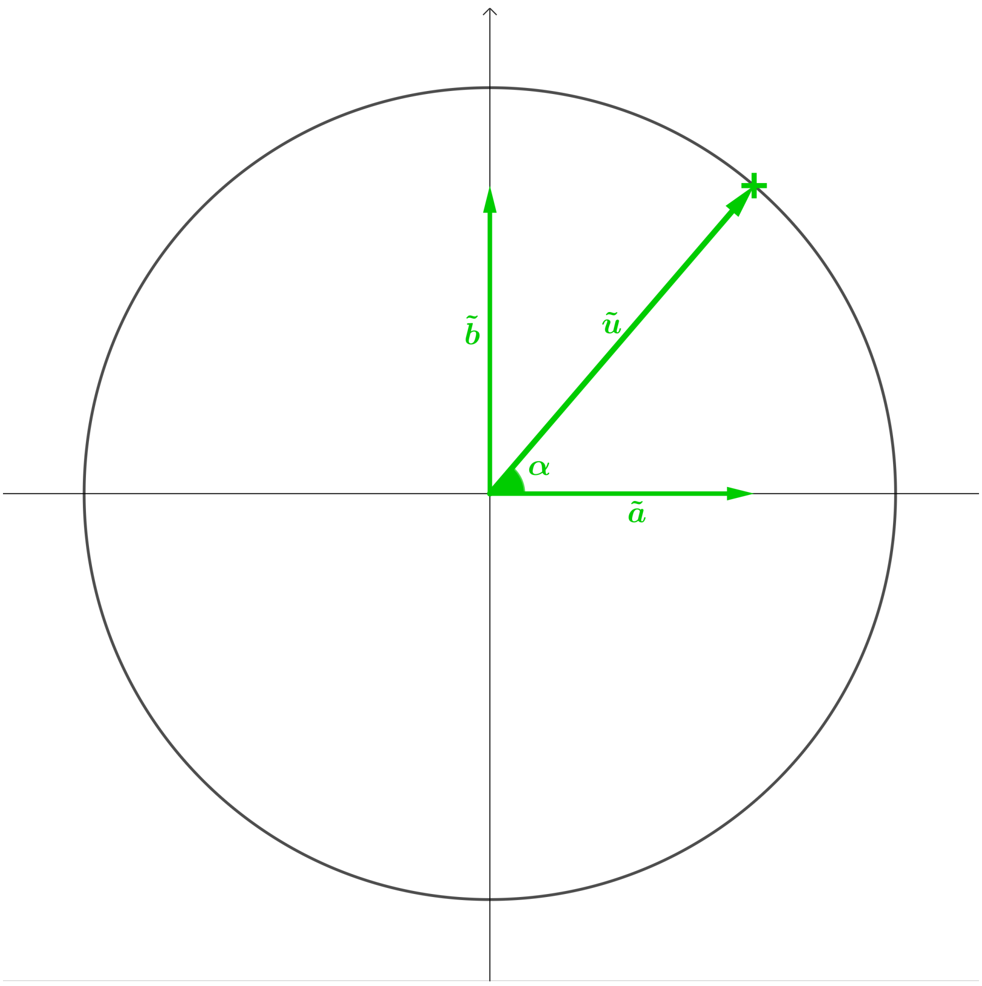
- they only cover cases when $\alpha < 90^\circ, \quad \beta < 90^\circ,$ and perhaps even $\alpha+\beta < 90^\circ$
- they are a bit contrived, often involving either lengths or areas of right-angled triangles in a rather complicated and unilluminating way.

Since I am taking my students on a journey that splits the circular functions away from right-angled triangles and attaches them instead to unit circles, I prefer to rely on the idea of rotations. This method does not rely on the angles being acute—it is quite general (at least, I claim that it is, but perhaps someone will show me why it isn't quite as general as I would like it to be). The downside is that it requires some knowledge of vectors. Even though it doesn't involve very much, it may be prudent, for some classes, to have a quick look at how vectors work before tackling this sheet.

Here is a circle radius 1.

What are the coordinates of the green cross?

Write each of the vectors as column vectors in terms of α.

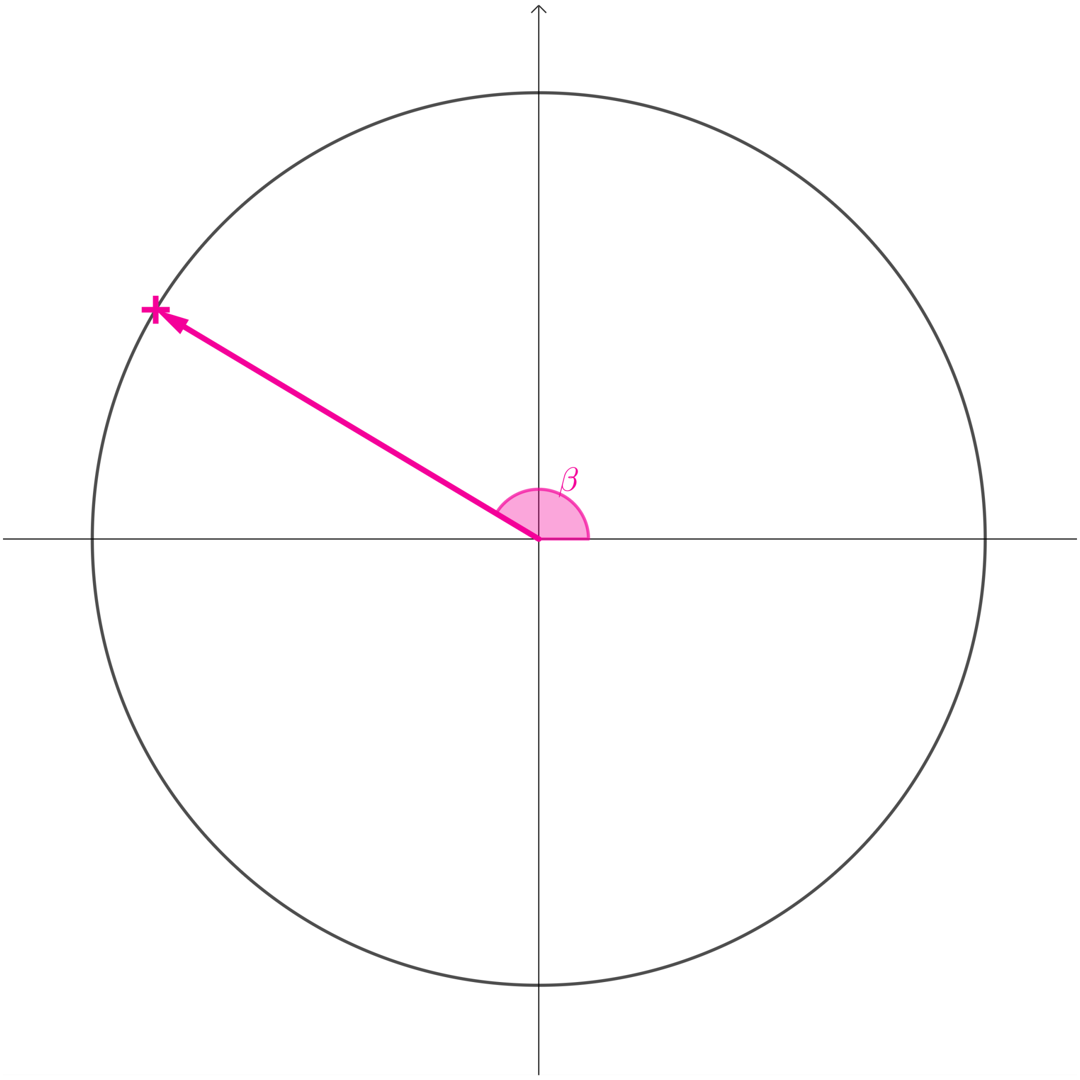


The green cross is at $(\cos\alpha,\ \sin\alpha)$

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}} = \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

What are the coordinates of the red cross?

Write the red vector as column vector in terms of β.

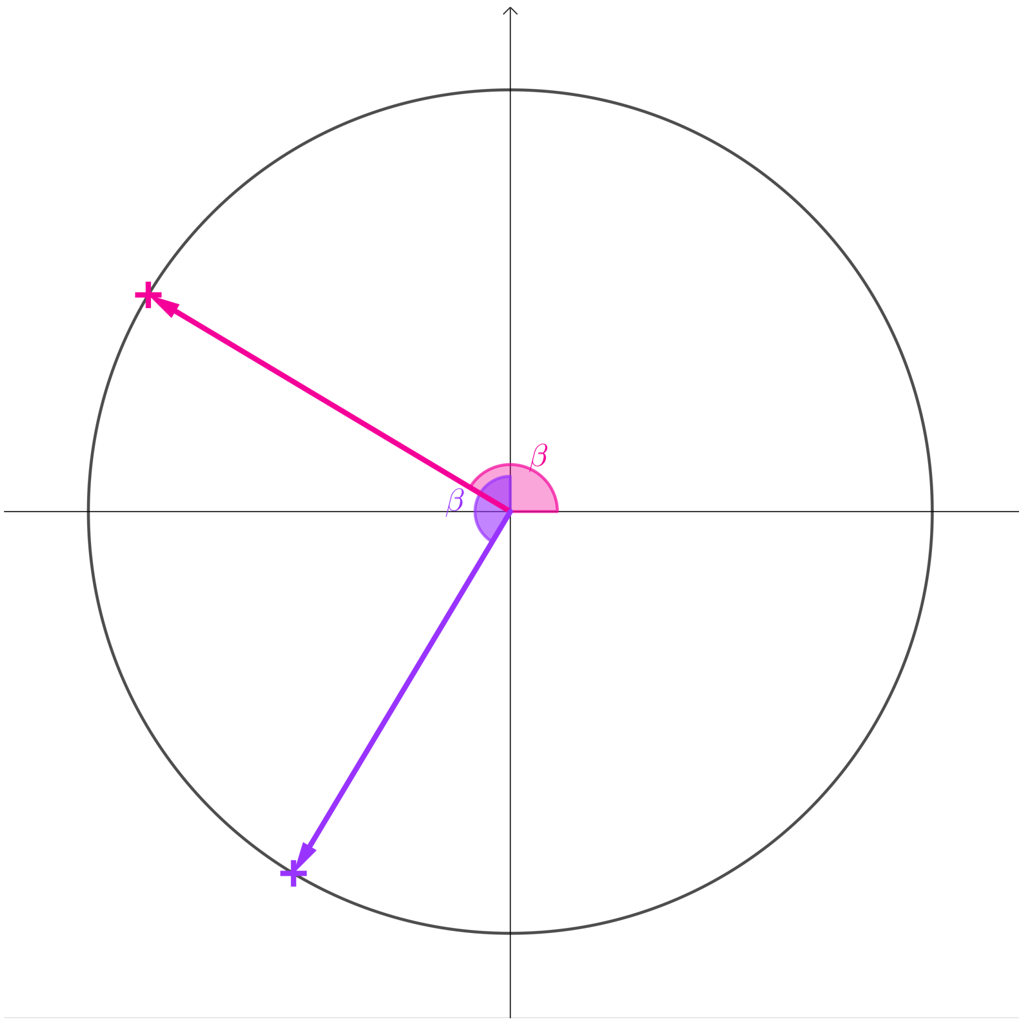


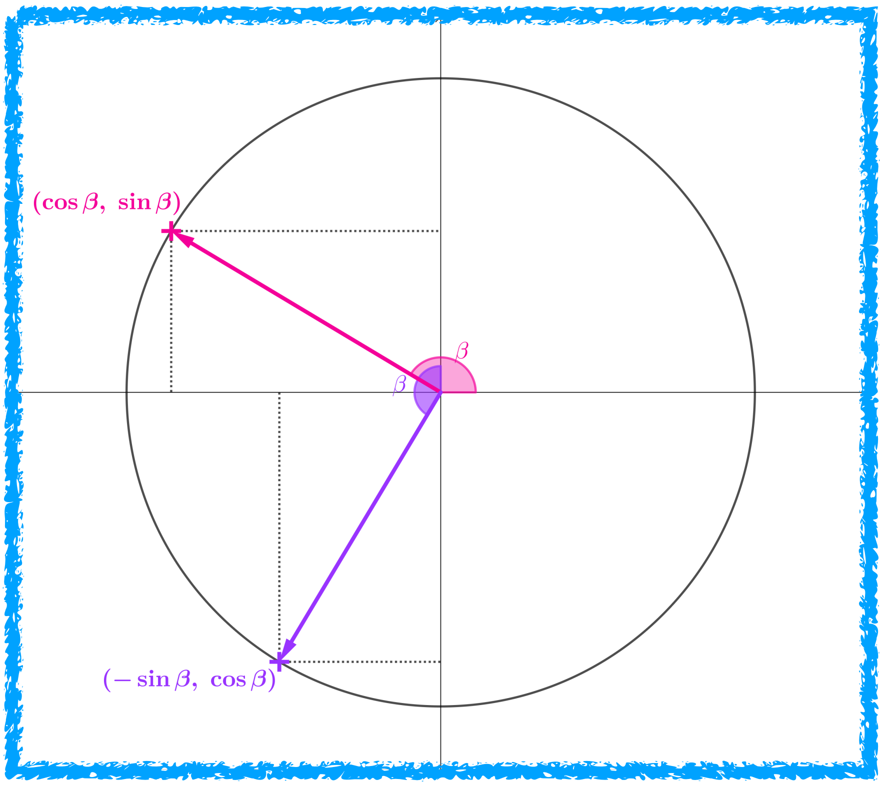
The pink cross is at $(\cos\beta,\ \sin\beta)$. This is just the definition of cos and sine as the x and y coordinates on the unit circle.

The pink vector is $\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$.

What are the coordinates of the darker cross?

Write the purple vector as column vector in terms of β.





The purple cross is at $(-\sin\beta,\ \cos\beta)$.

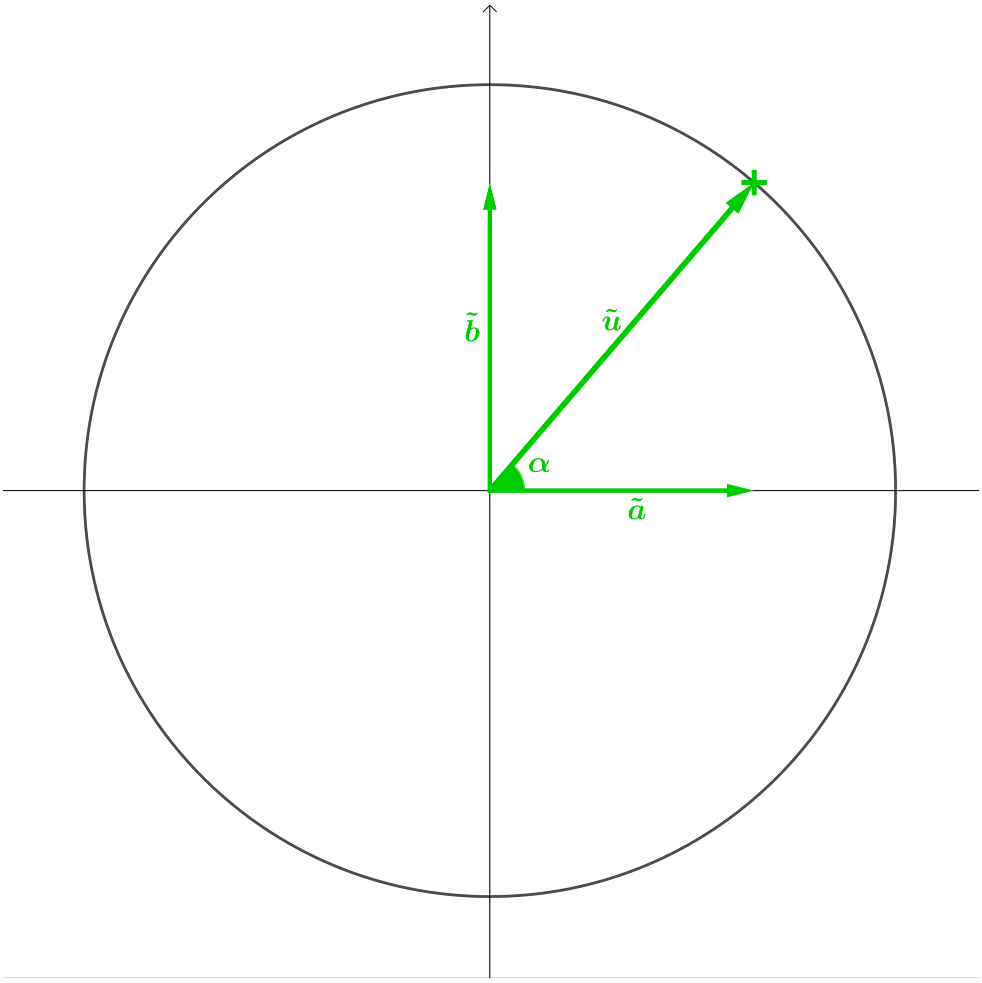
The purple vector is $\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$

The sizes of the x and y components of the purple vector are the sizes of the y and x components respectively of the pink vector.

Notice that, in this case, $\sin\beta > 0$ and $\cos\beta < 0$, and both components of the rotated purple vector are negative. It might be worth spending a bit of time showing that these signs work out correctly no matter what the size of β.

Quick reminder:

Write each of the vectors as column vectors in terms of α.

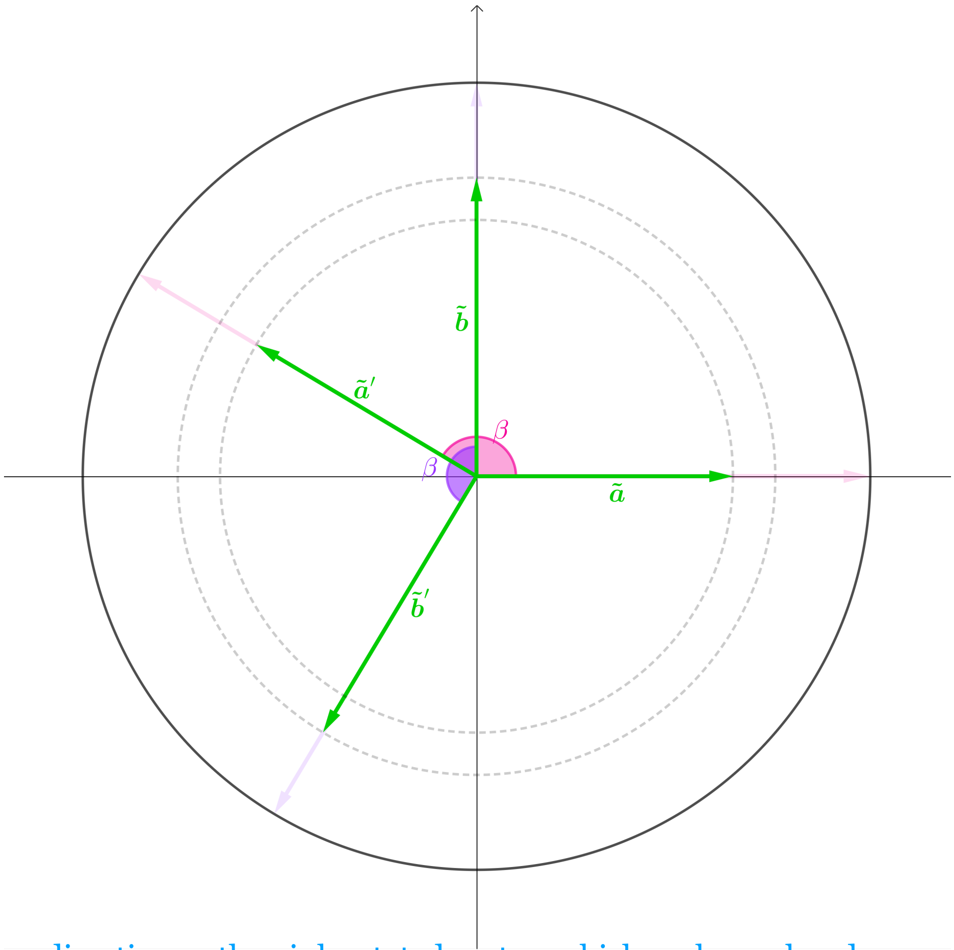


The green cross is at $(\cos\alpha,\ \sin\alpha)$

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}} = \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

What are the magnitudes and directions of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?

Write the vectors $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$ as column vectors in terms of α and β.



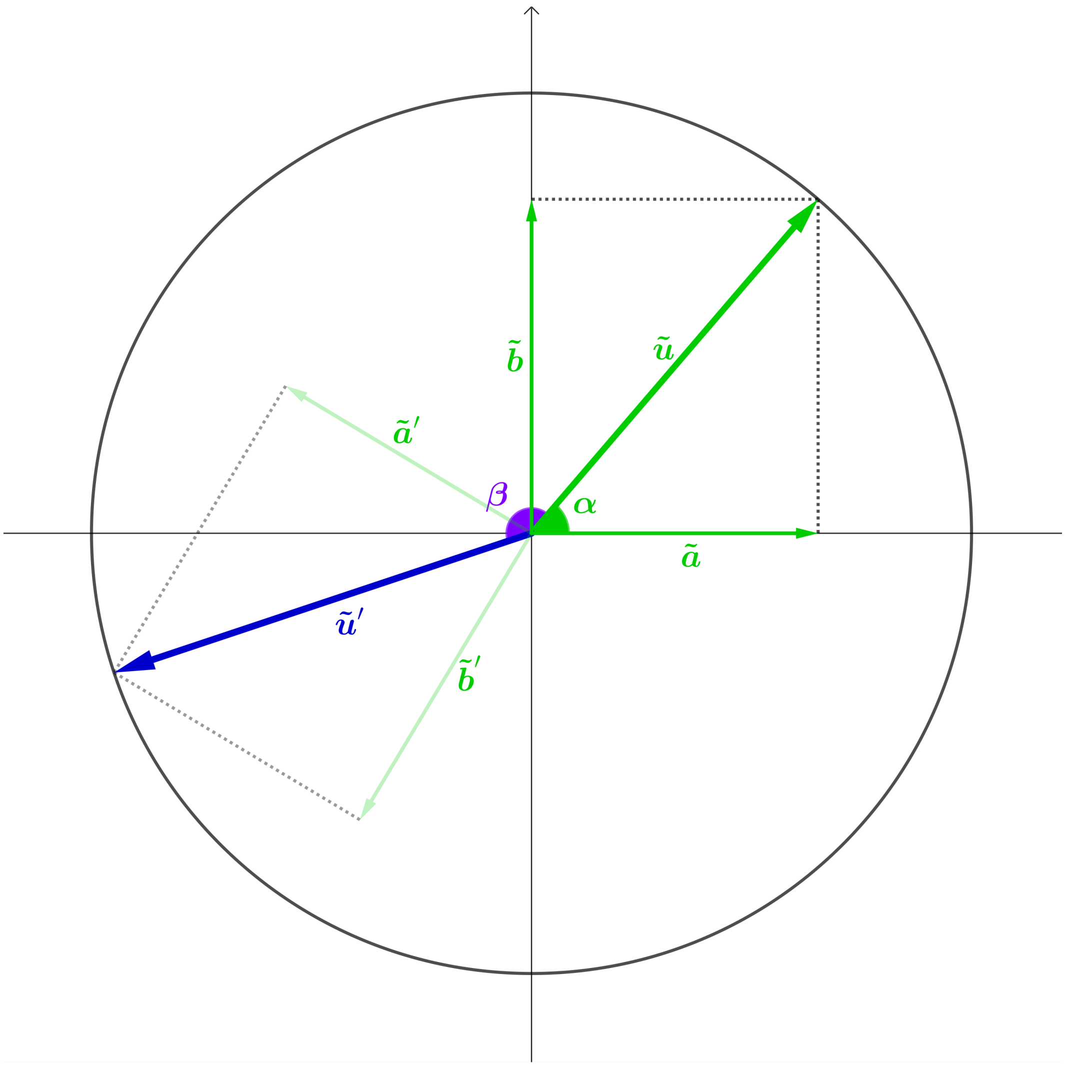
$\tilde{\boldsymbol{a}}'$ has the same direction as the pink rotated vector, which we have already seen is $\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$, and it has the same magnitude as $\tilde{\boldsymbol{a}}$, which is $\cos\alpha$. So

$$|\tilde{\boldsymbol{a}}'| = |\tilde{\boldsymbol{a}}| \Rightarrow \tilde{\boldsymbol{a}}' = \cos\alpha \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$$

and similarly,

$$|\tilde{\boldsymbol{b}}'| = |\tilde{\boldsymbol{b}}| \Rightarrow \tilde{\boldsymbol{b}}' = \sin\alpha \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$$

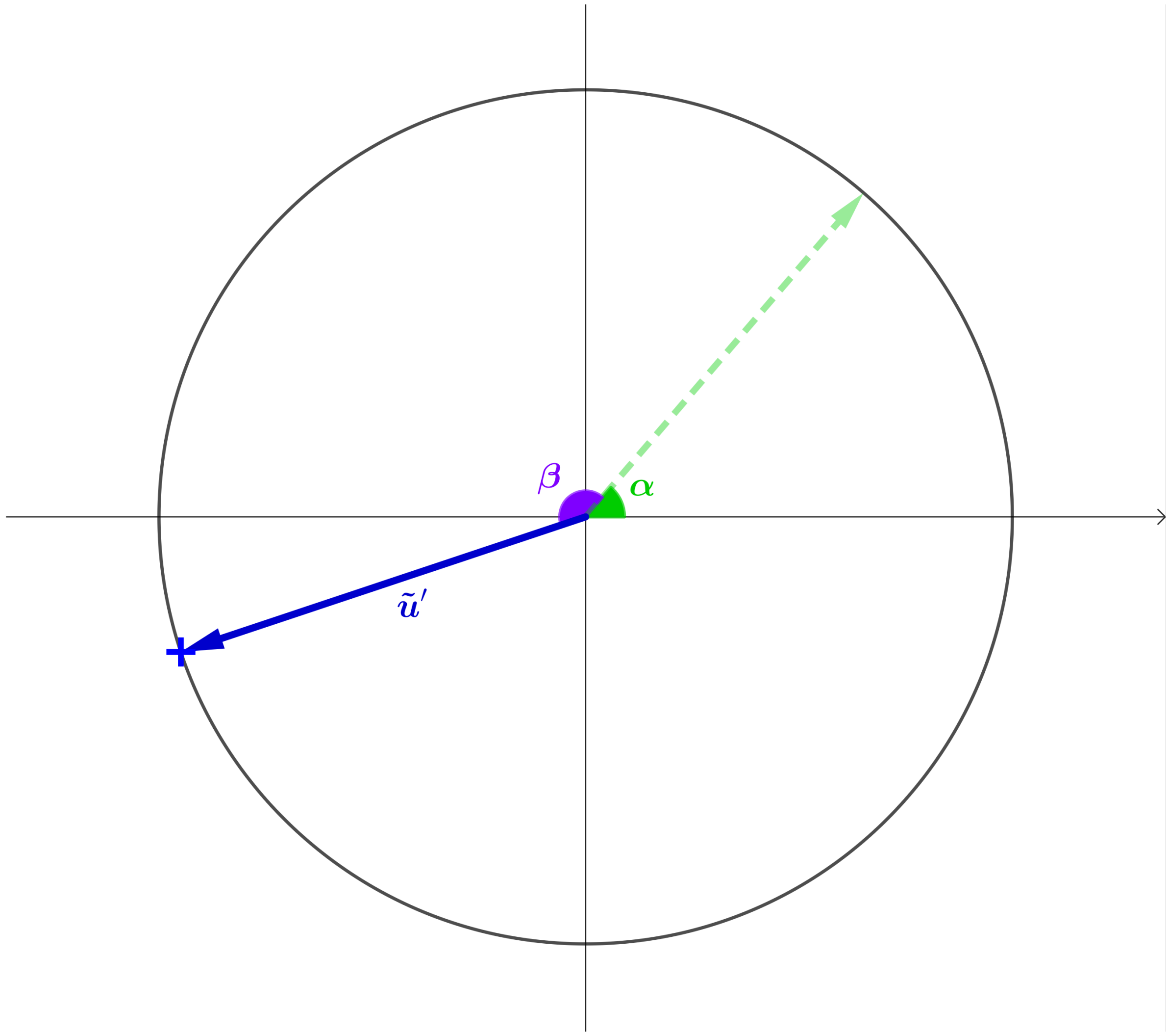
What is $\tilde{\boldsymbol{u}}'$ in terms of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?



Use this to write $\tilde{\boldsymbol{u}}'$ in terms of α and β.

$$\begin{aligned}\tilde{\boldsymbol{u}}' &= \tilde{\boldsymbol{a}}' + \tilde{\boldsymbol{b}}' \\ &= \cos\alpha\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix} + \sin\alpha\begin{pmatrix}-\sin\beta\\ \cos\beta\end{pmatrix} \\ &= \begin{pmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \cos\alpha\sin\beta + \sin\alpha\cos\beta\end{pmatrix}\end{aligned}$$

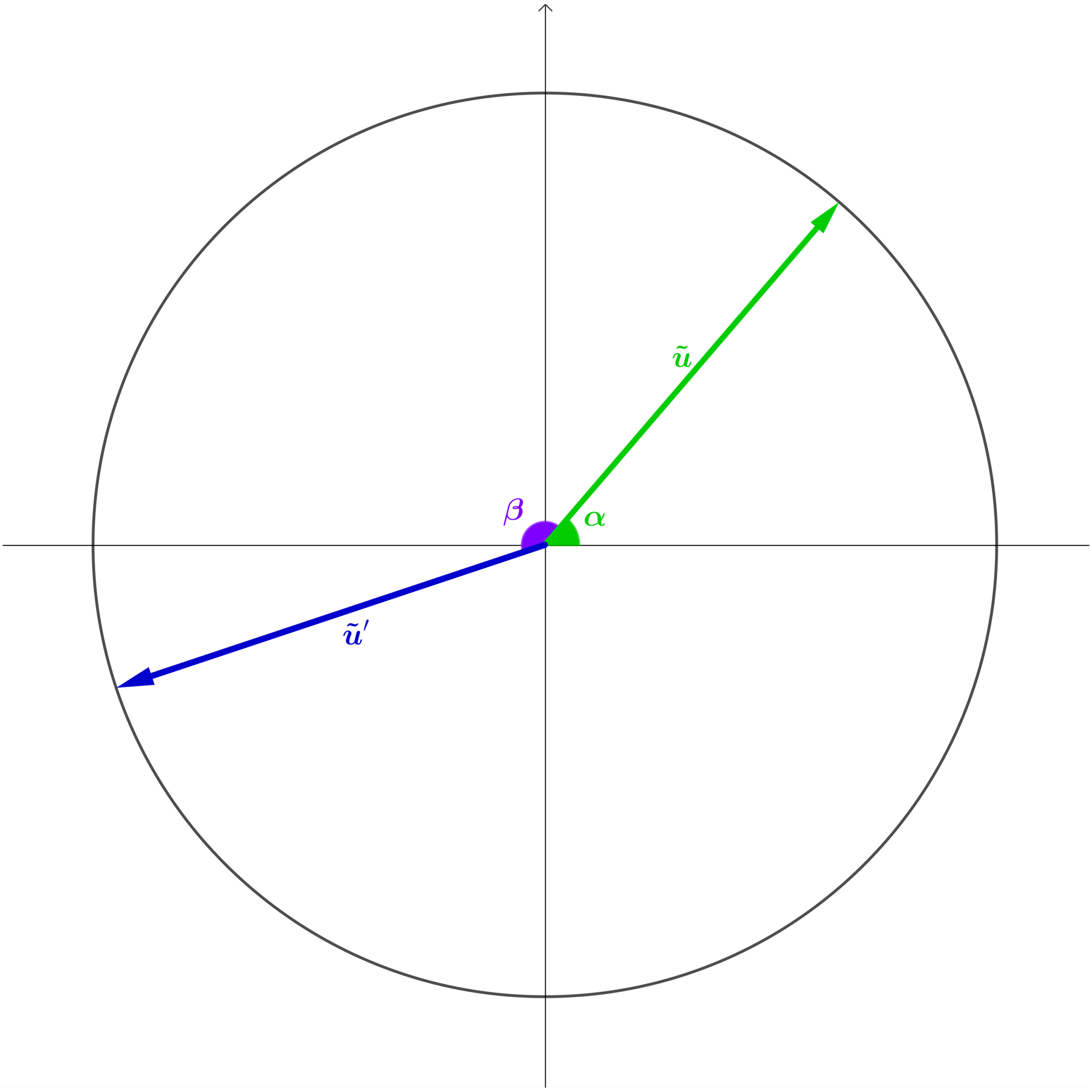
What are the coordinates of the blue cross?



The coordinates of the blue point are $\big(\cos(\alpha+\beta),\ \sin(\alpha+\beta)\big)$. Again, this is just the definition of cos and sine. So:

$$\tilde{\boldsymbol{u}}' = \begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix}$$

Use the two expressions for $\tilde{\boldsymbol{u}}'$ together to find new expressions for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.



$$\tilde{\boldsymbol{u}}' = \begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta \end{pmatrix}$$

which gives the familiar formulas right away.

Use these results to find $\tan(\alpha + \beta)$ in terms of $\tan \alpha$ and $\tan \beta$.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
&= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}
\end{aligned}$$

Use these results to find sin, cos, and tan of $\alpha - \beta$.

Remember

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

so

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + -\beta) \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

In the same way,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Use these results to find $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$.

$$\begin{aligned} \sin 2\alpha &= \sin(\alpha + \alpha) \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ &= 2\sin\alpha\cos\alpha \end{aligned}$$

$$\begin{aligned} \cos 2\alpha &= \cos(\alpha + \alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \end{aligned}$$

$$\begin{aligned} \tan 2\alpha &= \tan(\alpha + \alpha) \\ &= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha} \\ &= \frac{2\tan\alpha}{1 - \tan^2\alpha} \end{aligned}$$

Use $\cos^2\alpha + \sin^2\alpha = 1$ to find two different formulas for $\cos 2\alpha$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1 - \cos^2\alpha) \\ &= 2\cos^2\alpha - 1\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= (1 - \sin^2\alpha) - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Find $\sin 75^\circ$, $\cos 75^\circ$, and $\tan 75^\circ$.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ)\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Find $\sin 15°$, $\cos 15°$, and $\tan 15°$.

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° \cos 30° - \cos 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin 15° &= \cos(90° - 15°)\\ &= \cos 75°\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

or

$$\begin{aligned}\cos 15° &= \sin(90° - 15°)\\ &= \sin 75°\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\tan 15° &= \tan(45° - 30°)\\ &= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\end{aligned}$$

$$\begin{aligned}\tan 15° &= \frac{1}{\tan(90° - 15°)}\\ &= \frac{1}{\tan 75°}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\end{aligned}$$

Use the formula $\cos 2\theta = 2\cos^2\theta - 1$ to find $\cos 15^\circ$.

$$\cos 30^\circ = 2\cos^2 15^\circ - 1$$

$$\Rightarrow \cos^2 15^\circ = \frac{\cos 30^\circ + 1}{2}$$

$$= \frac{\frac{\sqrt{3}}{2} + 1}{2}$$

$$= \frac{\sqrt{3} + 2}{4}$$

$$\Rightarrow \cos 15^\circ = \frac{\sqrt{\sqrt{3} + 2}}{2}$$

Compare the two expressions you now have for $\cos 15^\circ$.

$$\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2 = \frac{8 + 2\sqrt{12}}{16}$$

$$= \frac{8 + 4\sqrt{3}}{16}$$

$$= \frac{2 + \sqrt{3}}{4}$$

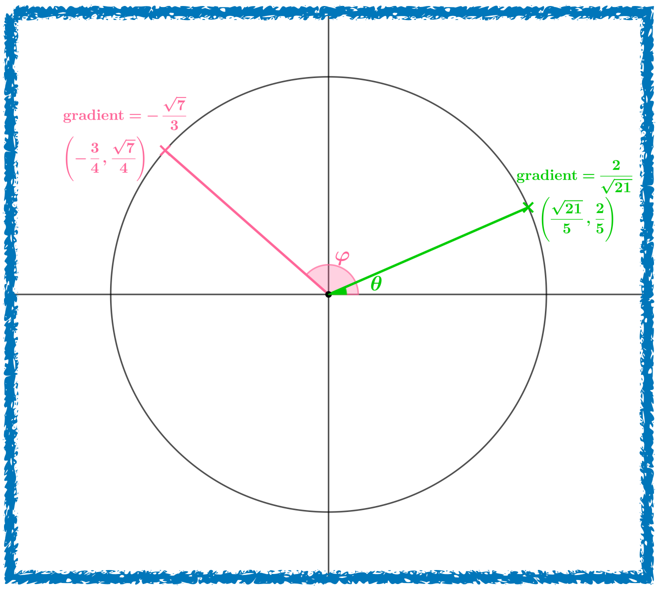
so they are the same.

Find $\int \sin^2 x \, \mathrm{d}x$

$$\begin{aligned}\int \sin^2 x \, \mathrm{d}x &= \int \frac{1-\cos 2x}{2} \, \mathrm{d}x \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + c\end{aligned}$$

If $\sin\theta = \dfrac{2}{5}$ (θ is acute) and $\cos\varphi = -\dfrac{3}{4}$ (φ is obtuse)

find, **without using your calculator**:



If $\sin\theta = \frac{2}{5}$ (θ is acute) and $\cos\varphi = -\frac{3}{4}$ (φ is obtuse)

find, **without using your calculator**:

$\cos\theta$

$\cos^2\theta = 1 - \sin^2\theta$ and

$\cos\theta > 0 \Rightarrow \cos\theta = \frac{\sqrt{21}}{5}$

$\tan 2\theta$

$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{4\sqrt{21}}{17}$

$\tan\theta$

$\tan\theta = \text{gradient} = \frac{2}{\sqrt{21}}$

$\cos\frac{\theta}{2}$

$\cos\frac{\theta}{2} > 0$

$\Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{5+\sqrt{21}}{10}}$

$\cos 2\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \frac{21}{25} - \frac{4}{25} = \frac{17}{25}$$

$\sin\frac{\theta}{2}$

$\sin\frac{\theta}{2} > 0$

$\Rightarrow \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{5-\sqrt{21}}{10}}$

$\sin 2\theta$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{2}{5}\frac{\sqrt{21}}{5} = \frac{4\sqrt{21}}{25}$$

$\tan\varphi$

$$\tan\varphi = \text{gradient} = -\frac{\sqrt{7}}{3}$$

$\tan\dfrac{\theta}{2}$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \sqrt{\frac{5-\sqrt{21}}{5+\sqrt{21}}}$$

$\cos 2\varphi$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$
$$= \frac{9}{16} - \frac{7}{16} = \frac{1}{8}$$

$\sin\varphi$

$\sin^2\varphi = 1 - \cos^2\varphi$ and

$$\sin\varphi > 0 \Rightarrow \sin\varphi = \frac{\sqrt{7}}{4}$$

$\sin 2\varphi$

$$\sin 2\varphi = 2\sin\varphi\cos\varphi$$
$$= 2 \times \frac{\sqrt{7}}{4}\frac{-3}{4} = -\frac{3\sqrt{7}}{8}$$

$\tan 2\varphi$

$\tan\dfrac{\varphi}{2}$

$$\tan 2\varphi = \frac{\sin 2\varphi}{\cos 2\varphi} = -3\sqrt{7}$$

$$\tan\frac{\varphi}{2} = \frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} = \sqrt{7}$$

$\cos\dfrac{\varphi}{2}$

$$\cos\frac{\varphi}{2} > 0 \Rightarrow \cos\frac{\varphi}{2} = \sqrt{\frac{1+\cos\varphi}{2}} = \frac{\sqrt{2}}{4}$$

$\sin\dfrac{\varphi}{2}$

$$\sin\frac{\varphi}{2} > 0 \Rightarrow \sin\frac{\varphi}{2} = \sqrt{\frac{1-\cos\varphi}{2}} = \frac{\sqrt{14}}{4}$$

$$\cos(\theta+\varphi)$$

$$\cos(\theta+\varphi)=\cos\theta\cos\varphi-\sin\theta\sin\varphi$$

$$=-\frac{\sqrt{21}}{5}\frac{3}{4}-\frac{2}{5}\frac{\sqrt{7}}{4}$$

$$=-\frac{\sqrt{7}\left(3\sqrt{3}+2\right)}{20}$$

$$\cos(\theta-\varphi)$$

$$\cos(\theta-\varphi)=\cos\theta\cos\varphi+\sin\theta\sin\varphi$$

$$=-\frac{\sqrt{21}}{5}\frac{3}{4}+\frac{2}{5}\frac{\sqrt{7}}{4}$$

$$=\frac{\sqrt{7}\left(2-3\sqrt{3}\right)}{20}$$

$$\sin(\theta+\varphi)$$

$$\sin(\theta+\varphi)=\sin\theta\cos\varphi+\cos\theta\sin\varphi$$

$$=-\frac{2}{5}\frac{3}{4}+\frac{\sqrt{21}}{5}\frac{\sqrt{7}}{4}$$

$$=\frac{7\sqrt{3}-6}{20}$$

$$\sin(\theta-\varphi)$$

$$\sin(\theta-\varphi)=\sin\theta\cos\varphi-\cos\theta\sin\varphi$$

$$=-\frac{2}{5}\frac{3}{4}-\frac{\sqrt{21}}{5}\frac{\sqrt{7}}{4}$$

$$=-\frac{7\sqrt{3}+6}{20}$$

$$\tan(\theta+\varphi)$$

$$\tan(\theta+\varphi)=\frac{\sin(\theta+\varphi)}{\cos(\theta+\varphi)}$$

$$=\frac{6-7\sqrt{3}}{\sqrt{7}\left(3\sqrt{3}+2\right)}$$

$$\tan(\theta-\varphi)$$

$$\tan(\theta-\varphi)=\frac{\sin(\theta-\varphi)}{\cos(\theta-\varphi)}$$

$$=\frac{6+7\sqrt{3}}{\sqrt{7}\left(3\sqrt{3}-2\right)}$$

In the last two, you could rationalise the denominator, but I don't think you gain much, if anything.

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Circular functions 7

Transforming and adding circular functions

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions

$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

What happens when two perfectly regular waveforms interact this each other? Many wonderful things that can be analysed and explained with some rather complicated equations. Here, we will start to think about the simplest examples: when the wavelengths of the two waves are the same, and when they are out of phase by a quarter of a wavelength. Even slightly harder examples will have to wait for university, but they will rely on this first step.

Here, we will investigate what happens when a wave whose equation is something like $y = 2\sin x$ meets a wave whose equation is something like $y = 5\cos x$. We will discover that the combination is another sine wave, and we can easily figure out its amplitude and the difference between its phase and that of the original sine wave.

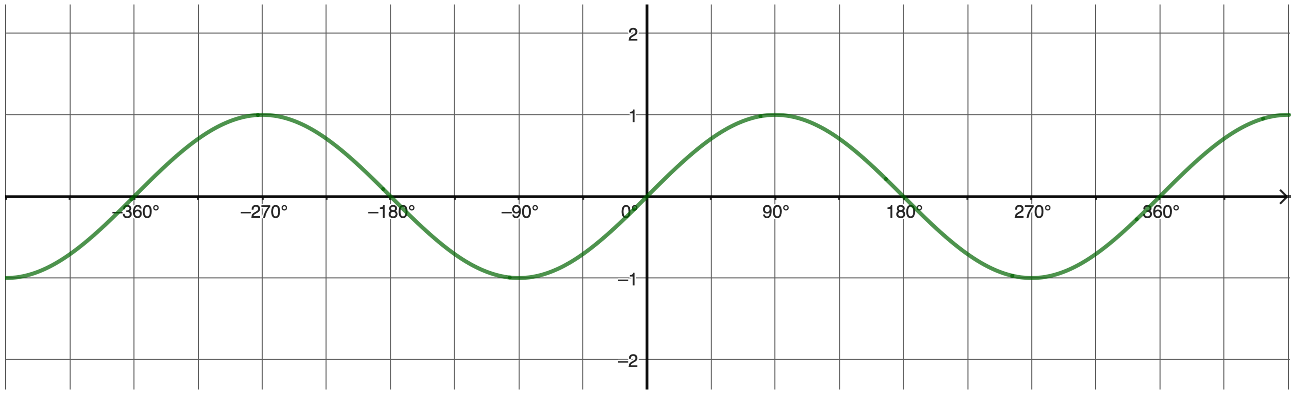
My approach is highly graphical: I explore graph transformations first and then see how these help with the waves superimposition problem.

Investigating functions like $f(x) = 2\sin x + 5\cos x$ through graphs and transformations reinforces the association of circular functions with graphs, and gives an opportunity to explore the relationship between transformations and compound angle formulas.

Here is the graph $y = \sin x$.

Translate the graph left by $45°$.

What is the equation of your new graph?



Now stretch the new graph parallel to the y axis scale factor $\sqrt{2}$.

What is the equation of your latest graph?

This first task will raise the question: why add 45° to x when the graph is translated to the left. The easy way to address this is to look at the point on the translated graph when $x = -45°$:

$$x = -45° \Rightarrow x + 45° = 0 \Rightarrow \sin\left(x + 45°\right) = 0$$

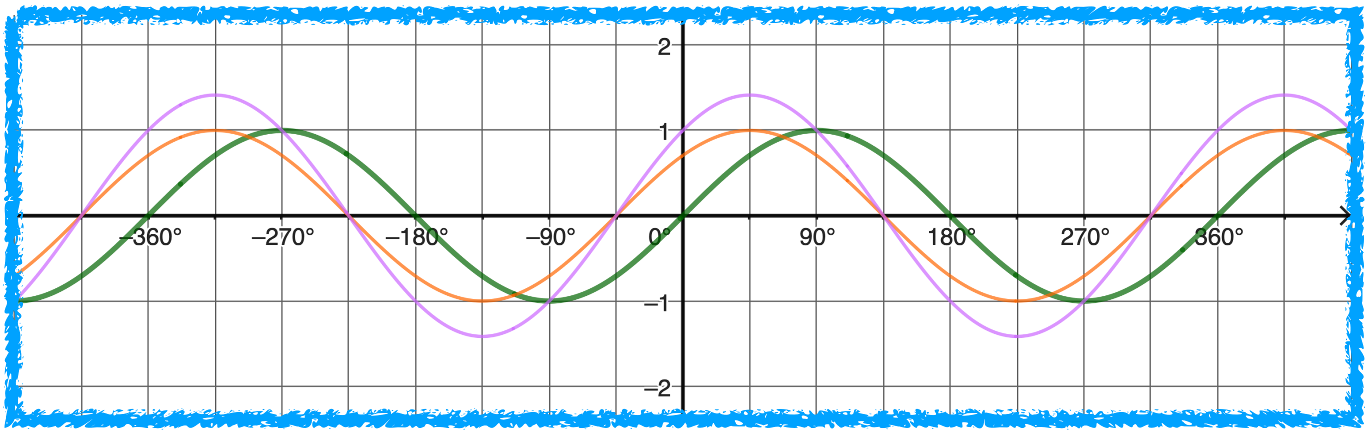
So the first "new"graph is

$$y = \sin\left(x + 45°\right)$$

and the second "new" graph is

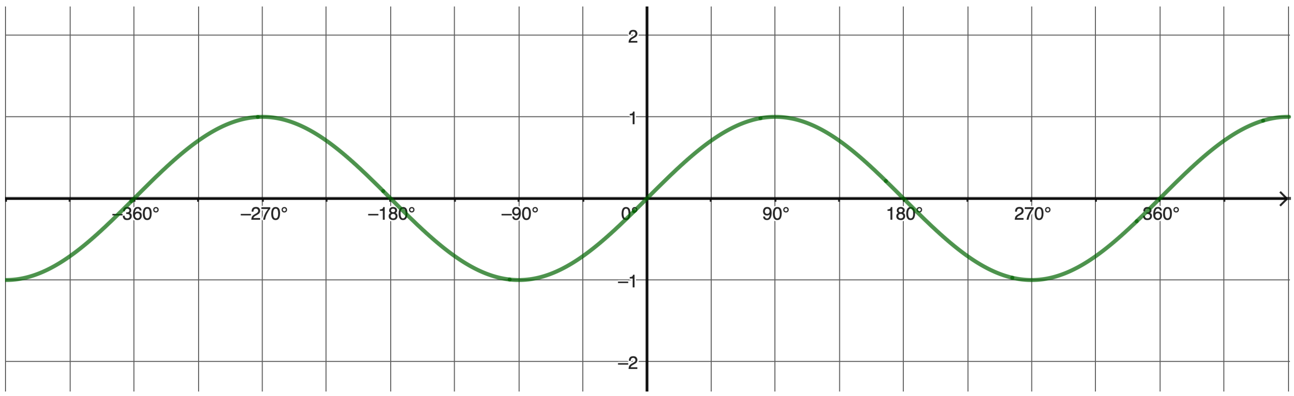
$$y = \sqrt{2}\sin\left(x + 45°\right)$$

Take care when stretching: the points where the graph crosses the x axis do not move! So the pink and orange curves on this diagram intersect on the x axis.



Now use a compound angle formula to expand $\sqrt{2}\sin\left(x + 45^\circ\right)$

Draw the graph $y = \sin x + \cos x$



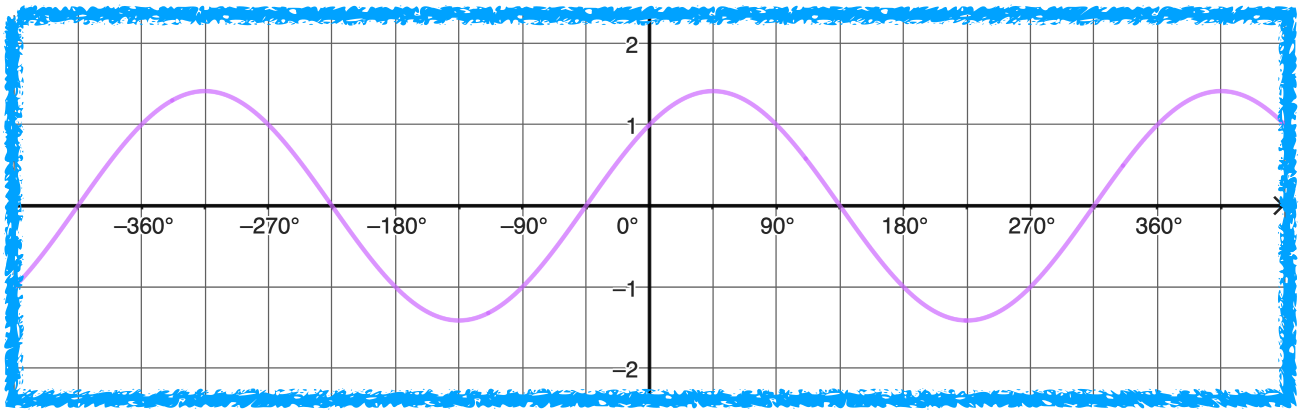
$$\sqrt{2}\sin(x+45^\circ) = \sqrt{2}(\sin x\cos 45^\circ + \cos x\sin 45^\circ)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sin x + \cos x$$

Don't let your students use their calculators for sin and cos of 45°!

Use the fact that $\sin x + \cos x = \sqrt{2}\sin(x+45^\circ)$

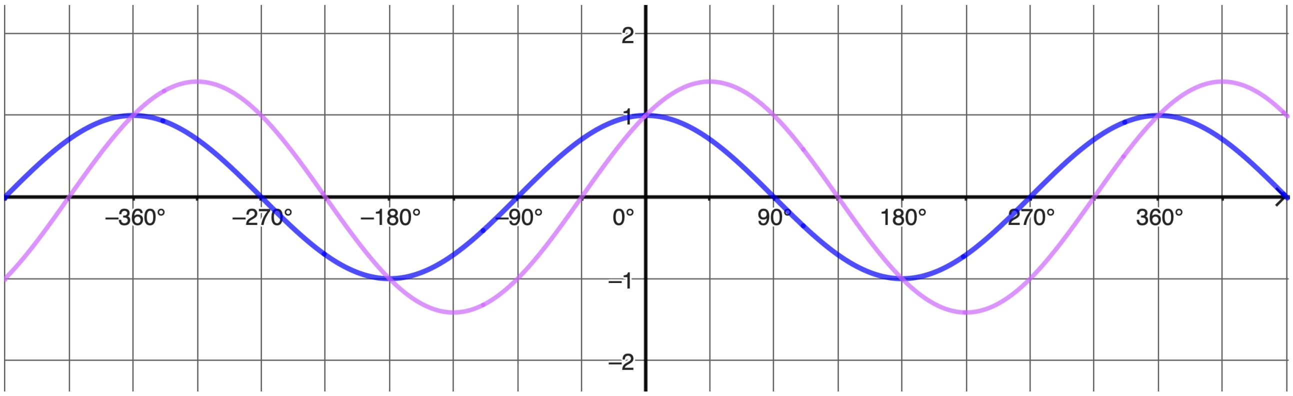


We can achieve the same graph

$y = \sin x + \cos x$

by applying transformations to the graph $y = \cos x$. Here are the two graphs together

What transformations will turn the blue ($y = \cos x$) into the pink ($y = \sin x + \cos x$)?



Here, students explore the transformations idea a bit more deeply by starting with $y = \cos x$ and figuring out how to get from this to the red graph $y = \sin x + \cos x$.

Translate right by 45° and stretch parallel to the y axis scale factor $\sqrt{2}$ to give the equation

$$y = \sqrt{2}\cos(x - 45^\circ)$$

You could also translate right by $45^\circ + 360n^\circ$ and stretch parallel to the y axis scale factor $\sqrt{2}$.

You could even translate right by $45^\circ + (2n+1)180^\circ$ and stretch parallel to the y axis scale factor $-\sqrt{2}$.

The video looks at these alternatives a bit more closely.

Ultimately, to keep things definite, we will always choose a positive value of R and then the smallest possible value of α.

Expand $\sqrt{2}\cos\left(x - 45^\circ\right)$

$$\sqrt{2}\cos\left(x - 45^\circ\right) = \sqrt{2}\left(\cos x \cos 45^\circ + \sin x \sin 45^\circ\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \cos x + \sin x$$

Don’t let your students use their calculators for sin and cos of 45°!

What do you notice about the graphs

$$y = \sqrt{2}\sin\left(x + 45^\circ\right)$$

$$y = \sqrt{2}\cos\left(x - 45^\circ\right)$$

$$y = \sin x + \cos x$$

At this point, they have discovered that

$$\sqrt{2}\sin\left(x+45^\circ\right)=\sqrt{2}\cos\left(x-45^\circ\right)=\sin x+\cos x$$

They could have reached this point far more easily simply by using the compound angle formulas, but, by looking closely at graphs, their understanding will have increased so much more.

What transformations of $y = \sin x$ or $y = \cos x$ will result in the graphs

$$y = \sqrt{2}\sin(x + 405^\circ)$$

$$y = \sqrt{2}\cos(x - 405^\circ)$$

$$y = -\sqrt{2}\sin(x + 225^\circ)$$

$$y = -\sqrt{2}\cos(x - 225^\circ)$$

What is the difference between these graphs and the graph $y = \sin x + \cos x$?

Expanding the brackets in each of these shows that they are also the graph $y = \sin x + \cos x$

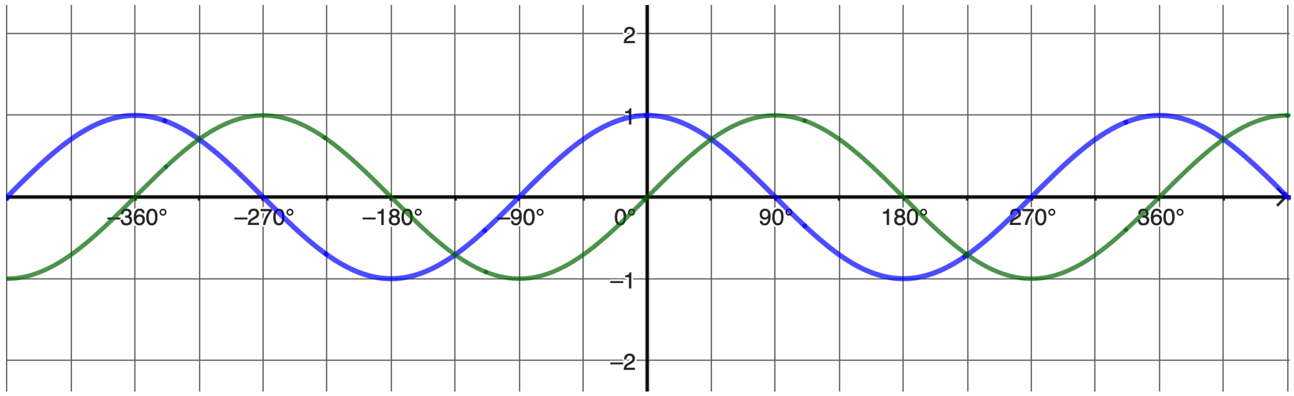
Translating left or right by $45^\circ + 180n^\circ$ and stretching parallel to the y axis scale factor $\sqrt{2}$ when n is even and $-\sqrt{2}$ when n is odd will always yield the same curve, and you can see a demonstration of this on the video.

Ultimately, to keep things definite and straightforward, we will always choose the positive value of R and then the smallest positive value of α.

We can also look at this from the point of view of a molecule being moved around by the two waves $y = \sin x$ and $y = \cos x$.

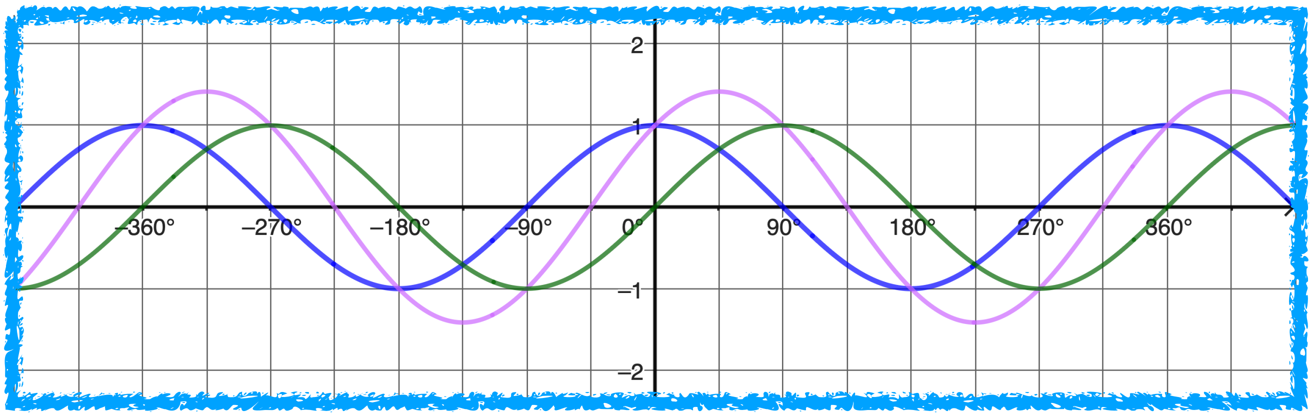
Draw the graph obtained by “adding up” the two graphs.

What is the equation of your new graph?



By “adding”, I mean, for each x value, adding the two y values from the green and blue graphs to get a new y value.

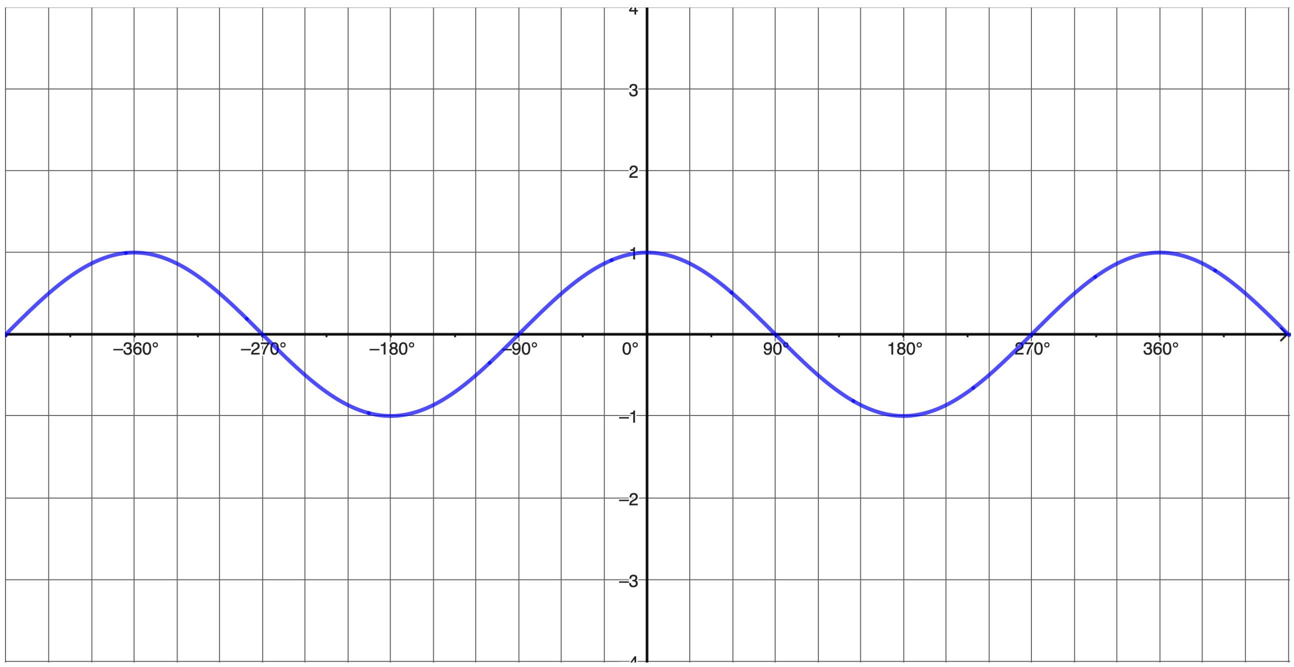
This will generate the graph $y = \sin x + \cos x$ in a more direct way than by transformations—probably not be a curriculum skill, but it does offer another angle on understanding.



Now for a slightly trickier example.

What transformations take the graph $y = \cos x$ to the graph

$y = 2\sqrt{3}\cos(x + 30°)$?



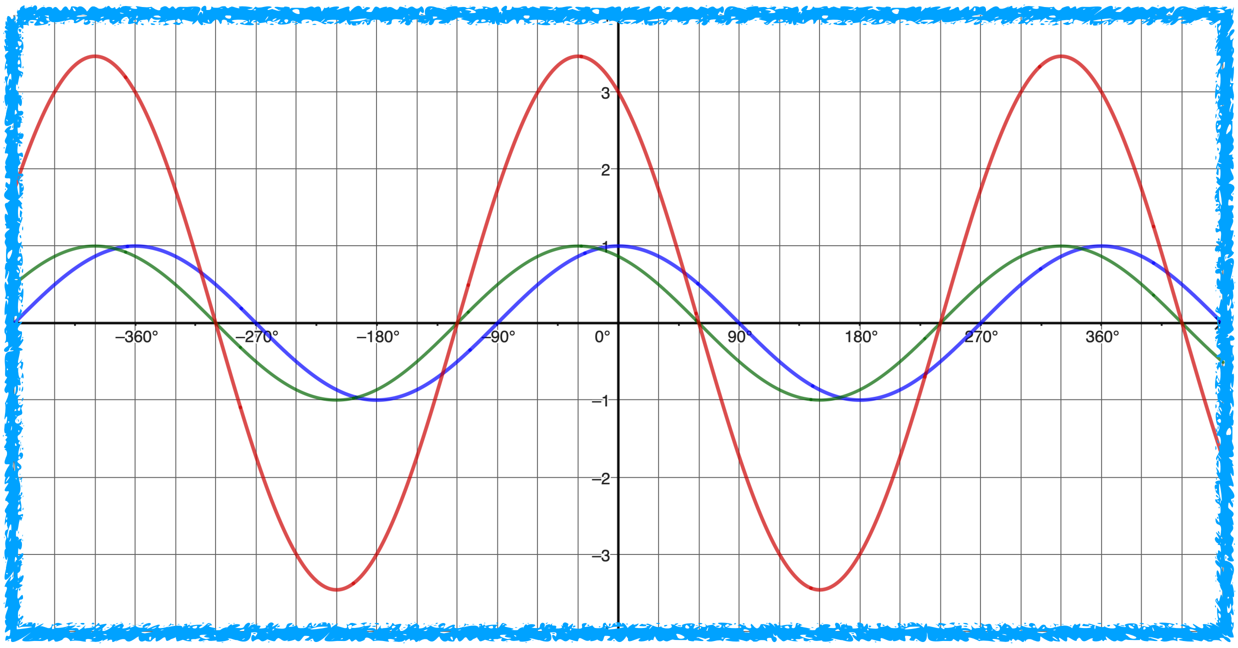
On the same axes, draw the graphs

$y = \cos(x + 30°)$ and $y = 2\sqrt{3}\cos(x + 30°)$.

A slightly harder example, but the principles are the same.

Translate left by 30° and stretch parallel to the y axis scale factor $2\sqrt{3}$.

You could also add or subtract multiples of π, changing the sign of the scale factor if necessary. However, for simplicity's sake, we will always choose the smallest translation and the positive scale factor.



Expand the brackets in $y = 2\sqrt{3}\cos(x + 30°)$.

$$2\sqrt{3}\cos(x+30°) = 2\sqrt{3}\left(\cos x \cos 30° - \sin x \sin 30°\right)$$

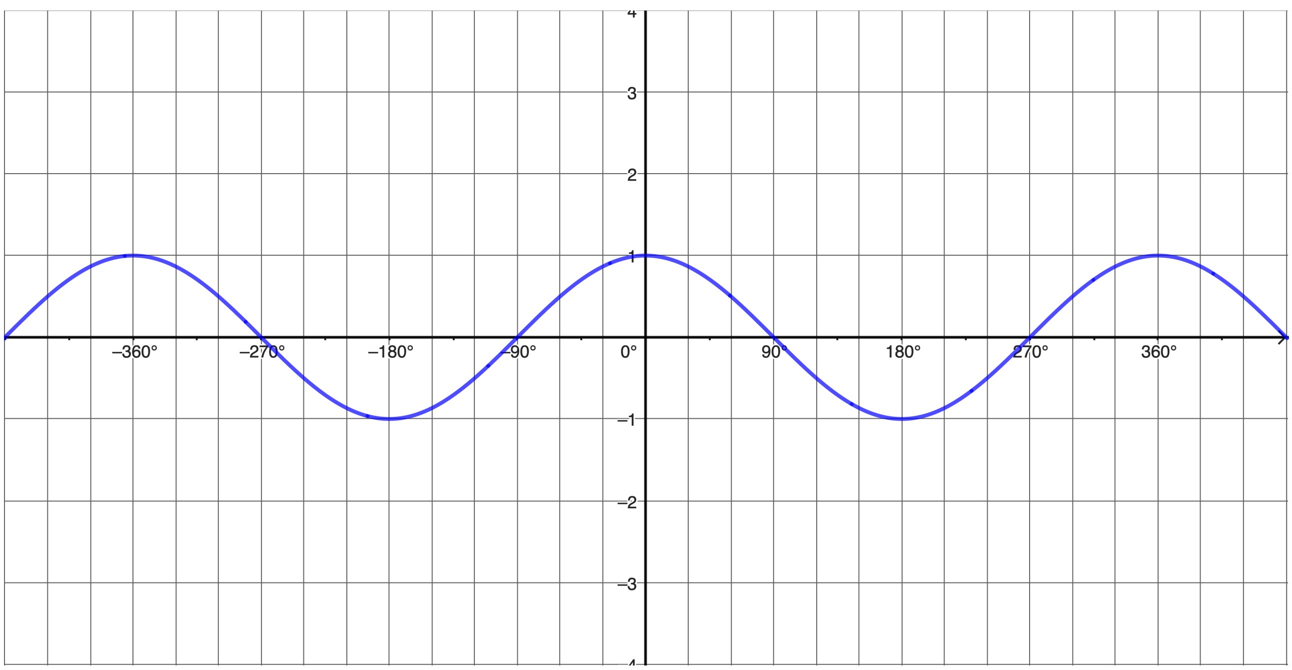
$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$= 3\cos x - \sqrt{3}\sin x$$

Don't let your students use their calculators for sin and cos of 30°!

Draw the graph

$$y = 3\cos x - \sqrt{3}\sin x$$

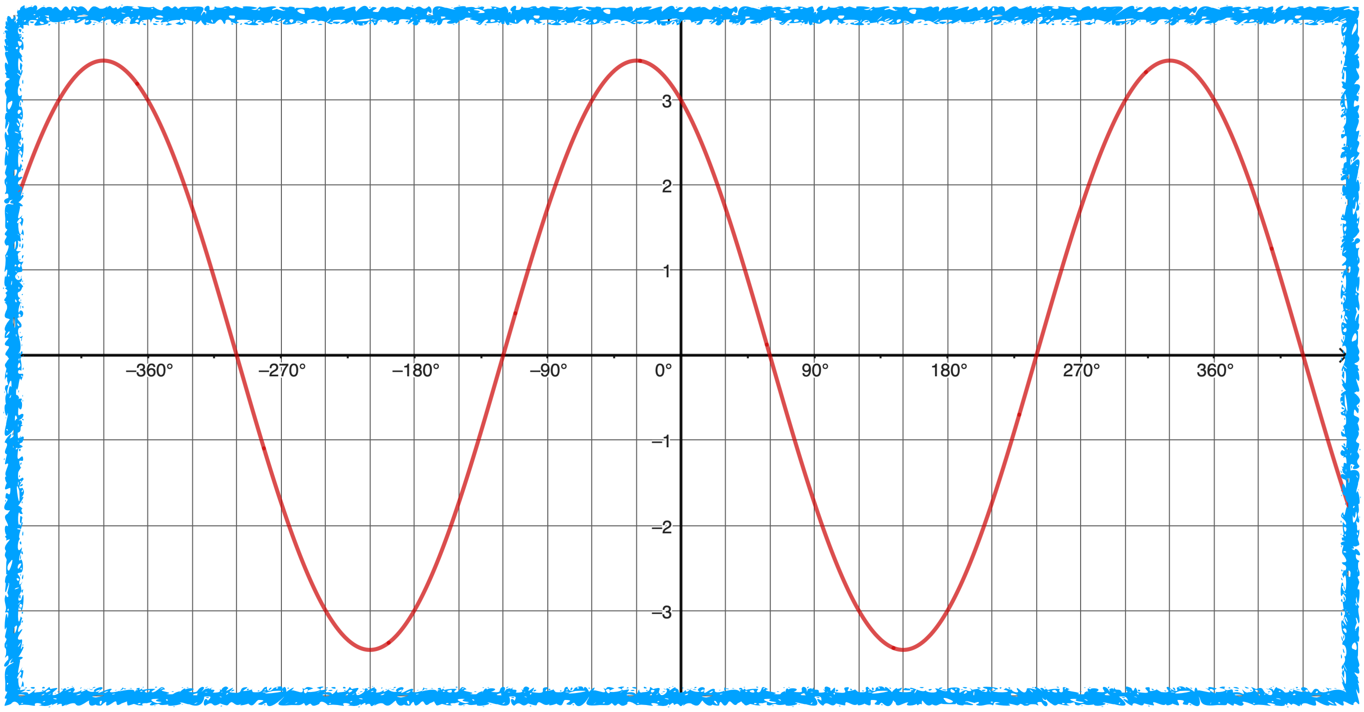


$$2\sqrt{3}\cos(x+30^\circ) = 2\sqrt{3}\left(\cos x\cos 30^\circ - \sin x\sin 30^\circ\right)$$

$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$= 3\cos x - \sqrt{3}\sin x$$

Don’t let your students use their calculators for sin and cos of 30°!



Next, let's take a similar but very slightly different example: the wave $y = 3\sin x$ meets the wave $y = \sqrt{3}\cos x$

Firstly, expand the brackets in $y = R\sin(x + \alpha)$.

$$y = R(\sin x\cos\alpha + \sin\alpha\cos x)$$

$$= (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$$

Obviously brackets are not needed, but they will help in a minute.

If $R\sin(x + \alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

$R\sin(x + \alpha) = (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$= 3\sin x + \sqrt{3}\cos x$

This must be true for every value of x, and the only way this can happen is if

$R\cos\alpha\sin x = 3\sin x \Rightarrow R\cos\alpha = 3$

and

$R\sin\alpha\cos x = \sqrt{3}\cos x \Rightarrow R\sin\alpha = \sqrt{3}$

Use these results to find $\tan\alpha$.

Hence find the smallest positive value of α.

Find $(R\sin\alpha)^2 + (R\cos\alpha)^2$.

Hence find R.

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3} \qquad \Rightarrow \tan\alpha = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \alpha = 30°$$

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = \left(\sqrt{3}\right)^2 + 3^2 = 12$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 12$$

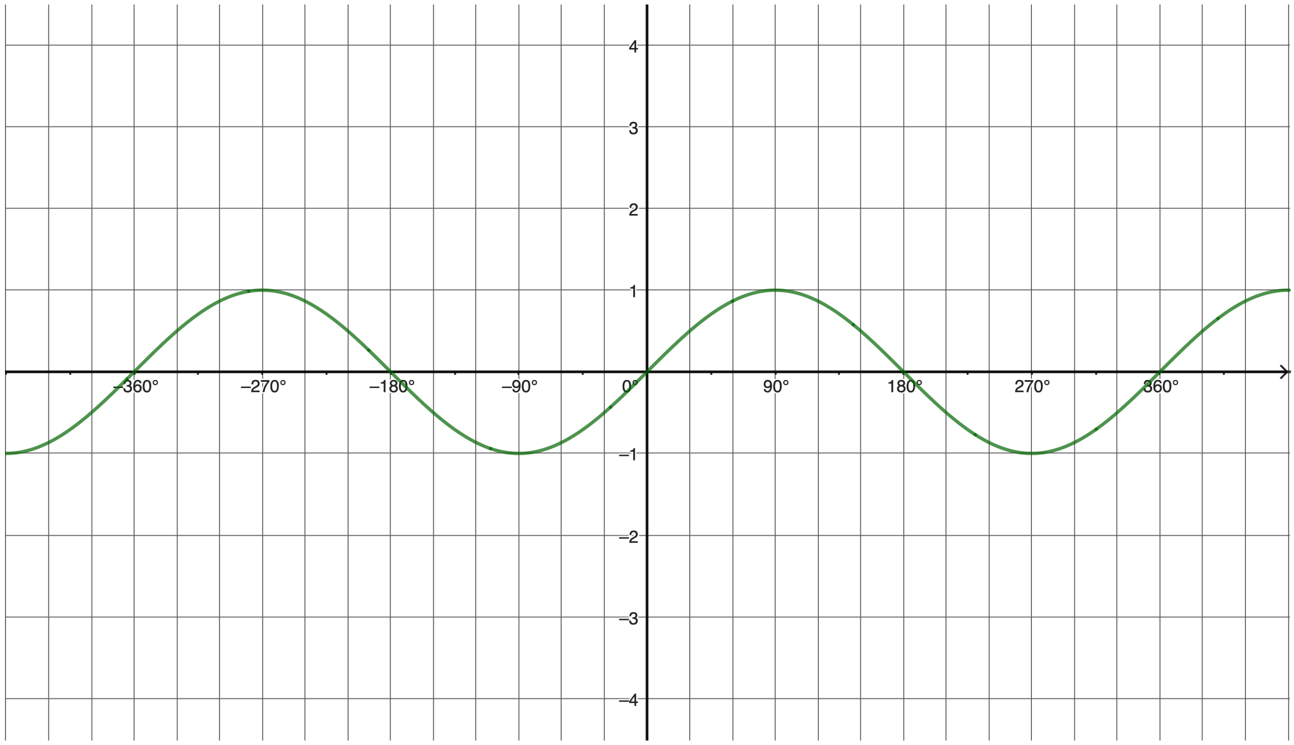
$$\Rightarrow R^2 = 12$$

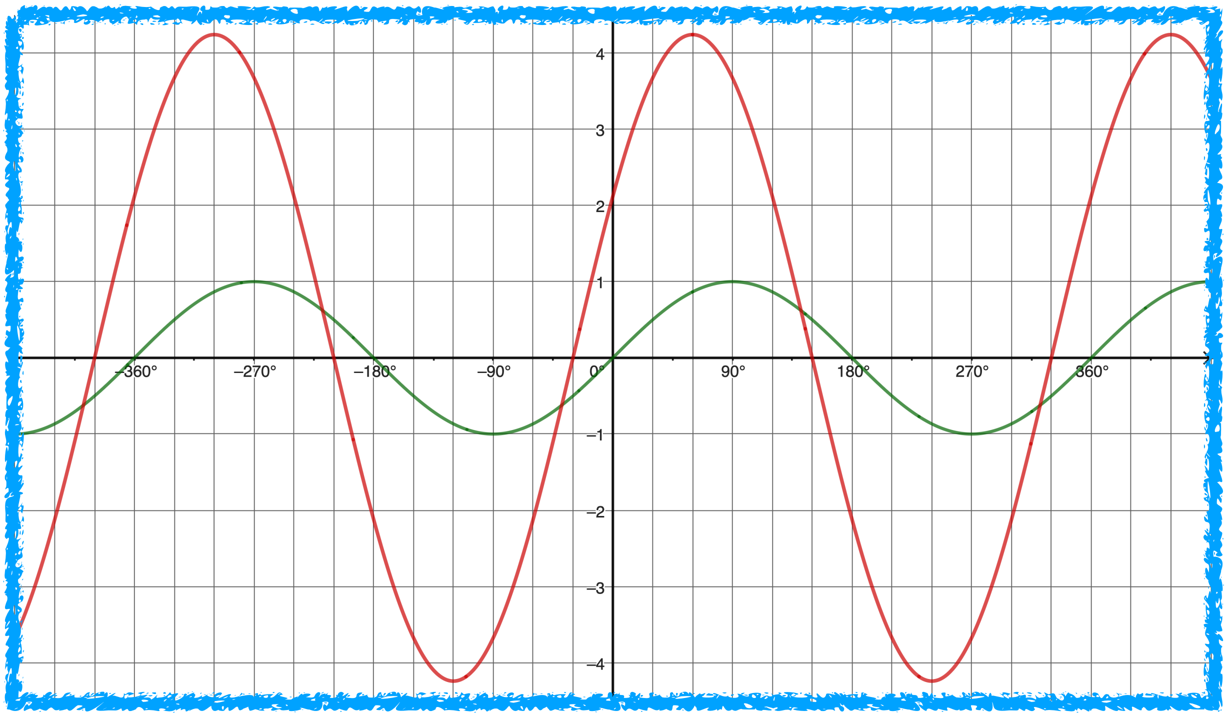
$$\Rightarrow R = 2\sqrt{3}$$

Use these results to find the transformations that take the graph $y = \sin x$ to the graph

$$y = 3\sin x + \sqrt{3}\cos x$$

and draw this graph.





$$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin(x + 30^\circ)$$

Translation by $\begin{pmatrix} -30^\circ \\ 0 \end{pmatrix}$ and stretch parallel to y axis scale factor $3\sqrt{2}$.

We could also do it this way.

If $R\cos(x-\alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

Use these results to find $\tan\alpha$.

Hence find the smallest positive value of α.

Find $(R\sin\alpha)^2 + (R\cos\alpha)^2$.

Hence find R.

$$R\cos(x-\alpha) = R(\cos x\cos\alpha + \sin\alpha\sin x)$$

$$= (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

$$R\sin\alpha = 3$$

$$R\cos\alpha = \sqrt{3}$$

$$\Rightarrow \frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{\sqrt{3}}$$

$$\Rightarrow \tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\Rightarrow \alpha = 60^\circ$$

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 3^2 + \left(\sqrt{3}\right)^2 = 12$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 12$$

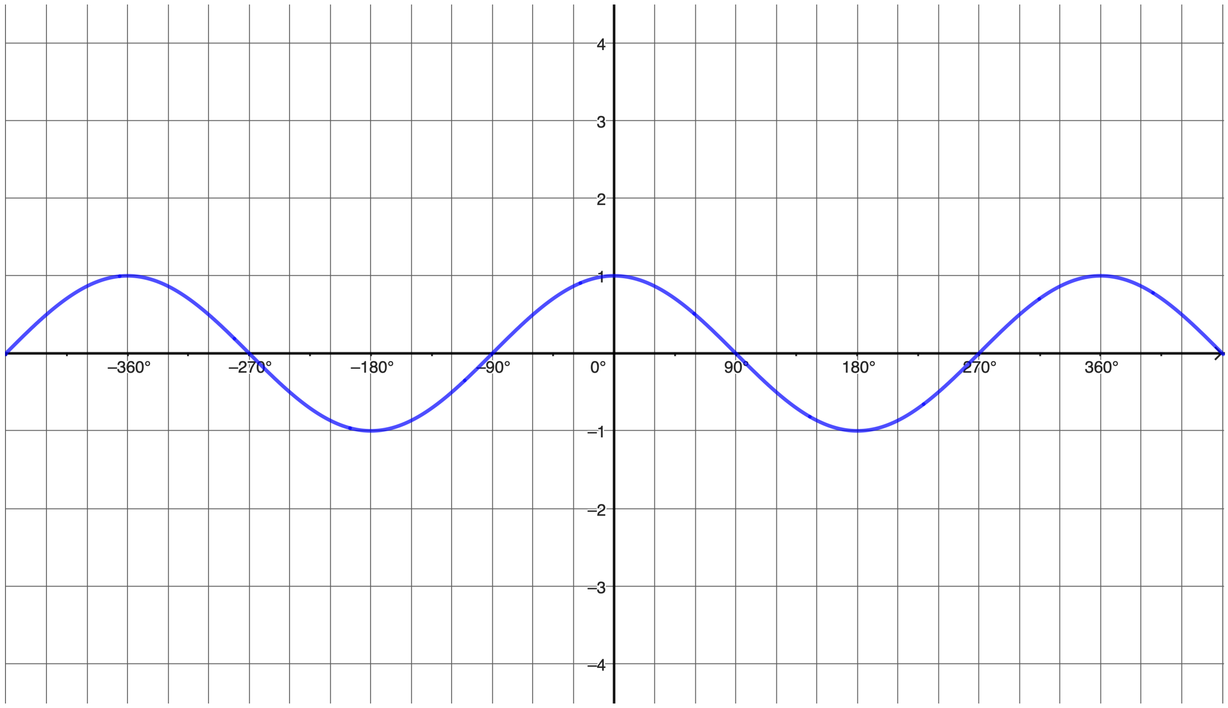
$$\Rightarrow R^2 = 12$$

$$\Rightarrow R = 2\sqrt{3}$$

Use these results to find the transformations that take the graph $y = \cos x$ to the graph

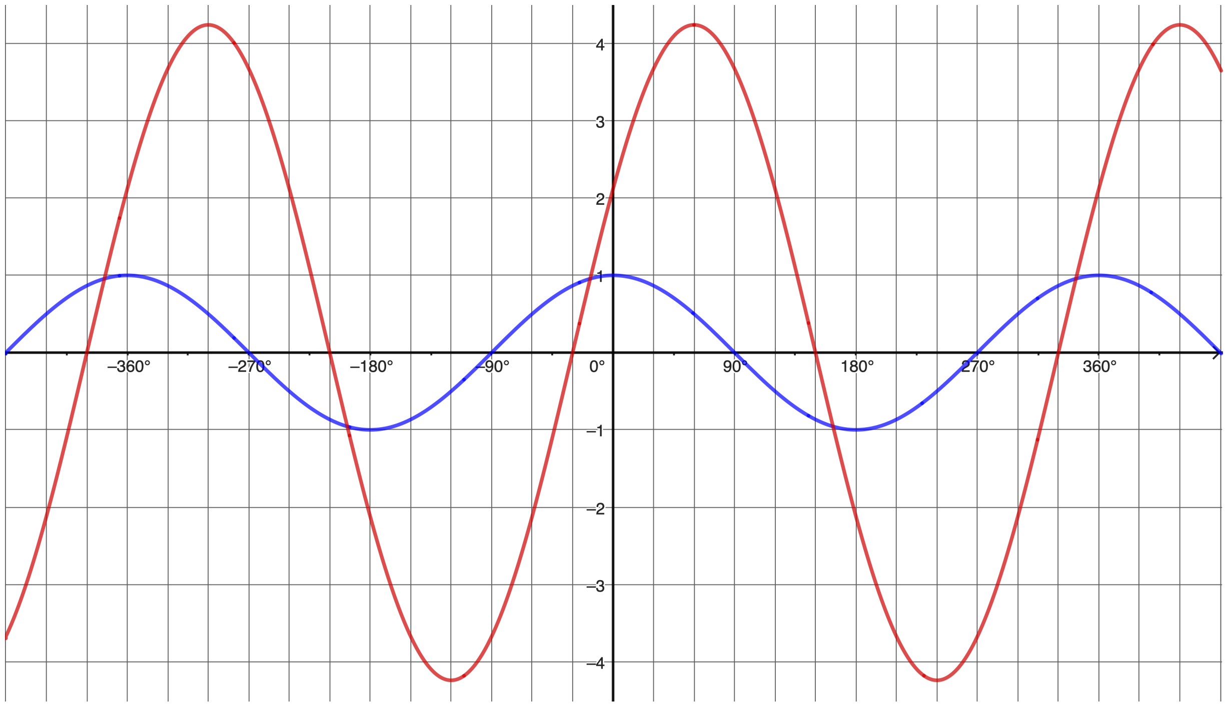
$$y = 3 \sin x + \sqrt{3} \cos x$$

and draw this graph.



$$3\sin x + \sqrt{3}\cos x = 3\sqrt{2}\cos\left(x - 60^\circ\right)$$

Translation by $\begin{pmatrix} 60^\circ \\ 3 \end{pmatrix}$ 0 and stretch parallel to y axis scale factor $3\sqrt{2}$.



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Any two waves $y = A\sin x$ and $y = B\cos x$ can be combined (either adding or subtracting) by these methods. Sometimes it's easier to transform the graph $y = \sin x$ to get $y = A\sin x \pm B\cos x$. Other times, starting with $y = \cos x$ is better.

Experiment with the combinations

$y = 5\sin x + 2\cos x$

$y = 5\sin x - 2\cos x$

$y = 2\cos x - 5\sin x$

using each of the following forms:

$R\sin(x + \alpha)$

$R\sin(x - \alpha)$

$R\cos(x + \alpha)$

$R\cos(x - \alpha)$

and taking $\tan 22°$ to be $\dfrac{2}{5}$.

Which R, α forms work best for each of the combinations?

$R \sin(x + \alpha)$ or $R \cos(x - \alpha) = 5 \sin x + 2 \cos x$

$R \sin(x - \alpha) = 5 \sin x - 2 \cos x$

$R \cos(x + \alpha) = 2 \cos x - 5 \sin x$

are the best: they all guarantee the possibility for finding the combination of a positive R with $0 \leq \alpha \leq 90^{\circ}$.

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Circular functions 8

Differentials of circular functions

teacher version

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90^\circ - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A+B) = \sin A\cos B + \cos A\sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on
Differentiating circular functions	**radians, and tangents to graphs**
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

My approach here is to begin by getting a sense of how the gradient of a sine graph works, and to see why the fact that the differential is cosine might make sense.

From this, students will see that, for the result to be true, the gradient of the sine graph at the origin must be 1. Then we move on to understanding what this means in terms of the limit as a line approaches the tangent at the origin.

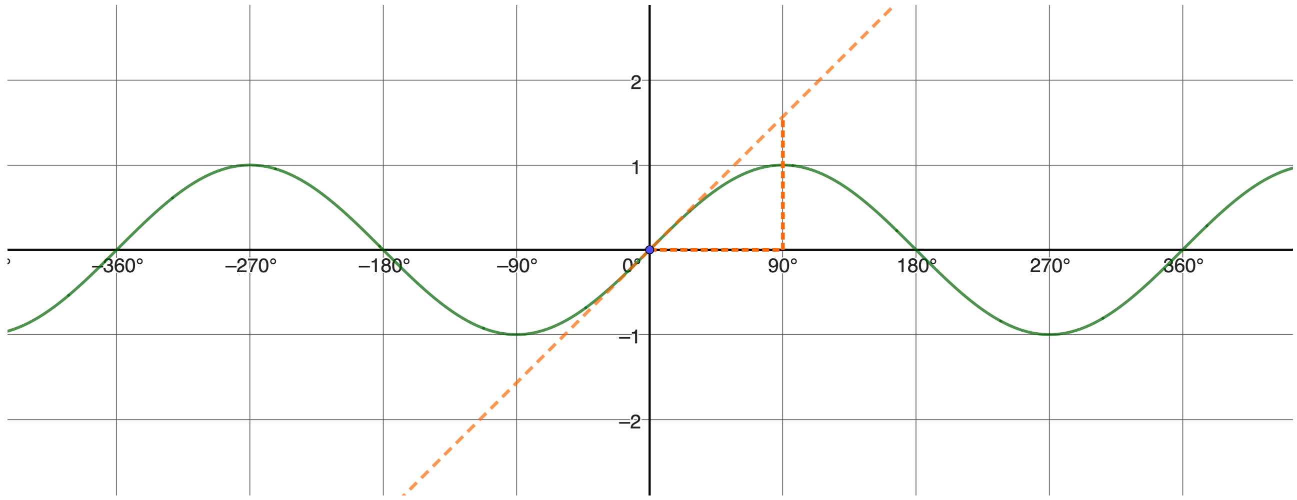
To see why this limit really is 1, we need to switch to a unit circle representation of sine and cosine. There are different degrees of rigour with which we can approach the proof, and how much detail you choose to explore will depend very much on the appetite and aptitude of your students.

The unit circle is the classic way to understand this limit, and a geometrical demonstration is certainly enough for now. However, some of your students may find this less satisfying than they would like, so I've included a more thorough version for your consideration.

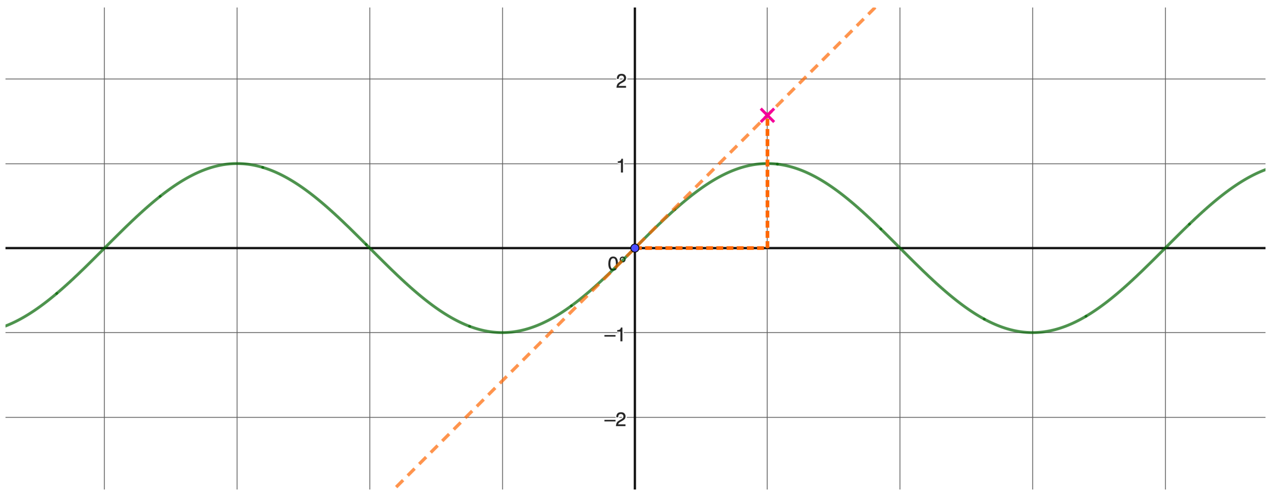
Once we know that the differential of sine is cosine, we can get all the other differentials of circular functions by using the chain, product, and quotient rules. For cosine, we can also use a "first principles" method as we did with sine; we can do this for other functions, too, but beyond sine and cosine, it's probably more trouble than it's worth. I've included a tan version in case you are interested.

The difficulty with capturing all this on a worksheet is that limiting processes really cry out for animations to bring them to life. I recommend using this worksheet in conjunction with my video version of the journey.

What (approximately) is the gradient of this tangent?



The gradient may look a bit like 1, but is it? Here is a new version of the graph without a scale on the x axis. If the gradient of this tangent is to be 1, what (approximately) would the x coordinate of the pink cross have to be?



Look at the orange triangle.

y-step is approximately 1.6.

x-step is 90

so

gradient $\approx \dfrac{1.6}{90}$

which is very small.

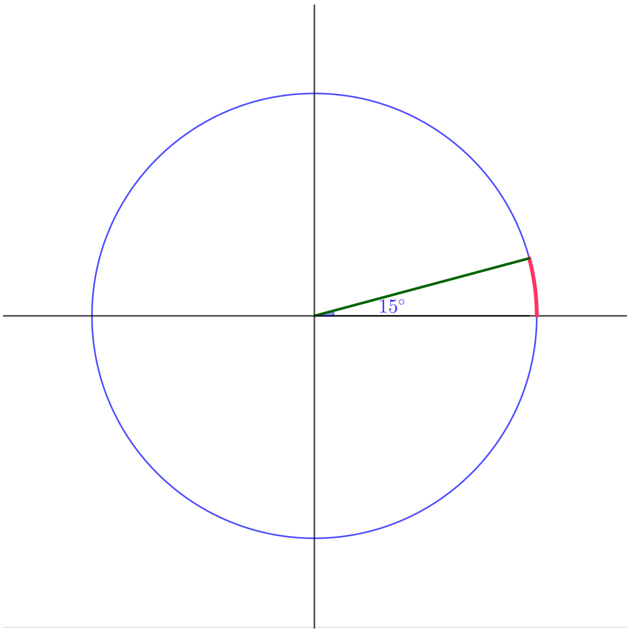
It looks much bigger than this, but this is because the axis scales are so different.

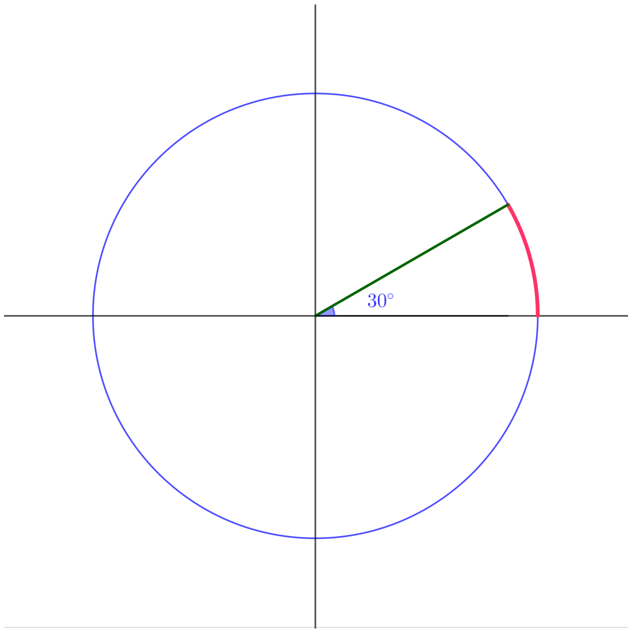
To make the gradient 1, the x coordinate of the pink cross would have to be approximately 1.6.

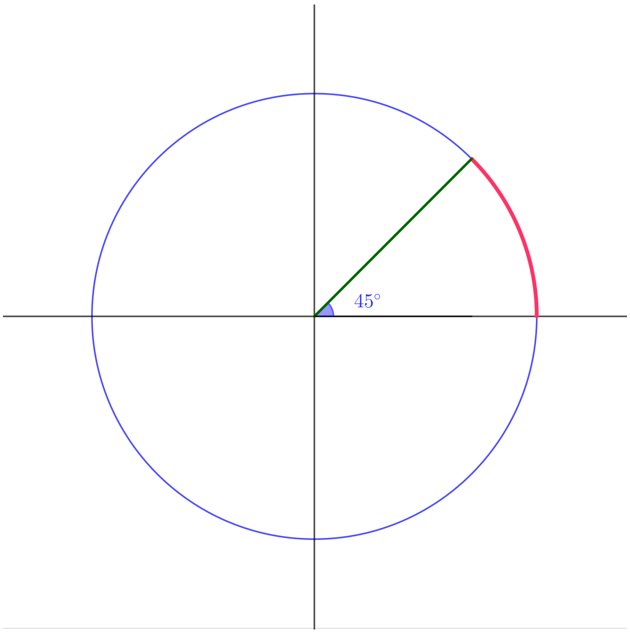
Now we will see how to find this value exactly.

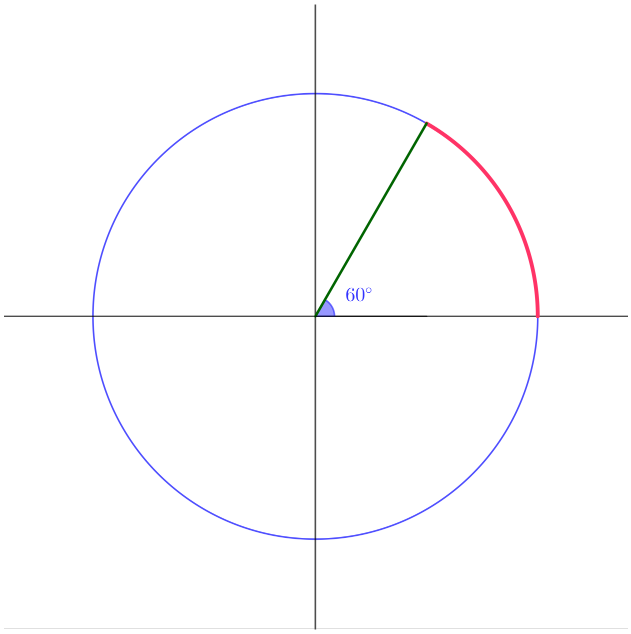
Now, we will try to find this x coordinate exactly. That is to say, we will find units for the x axis that makes this gradient 1. In theory, we can differentiate the circular functions without doing this, but everything works out so much more easily if we do, and that's the way it's done the world over. To do this, we need to go back to the unit circle.

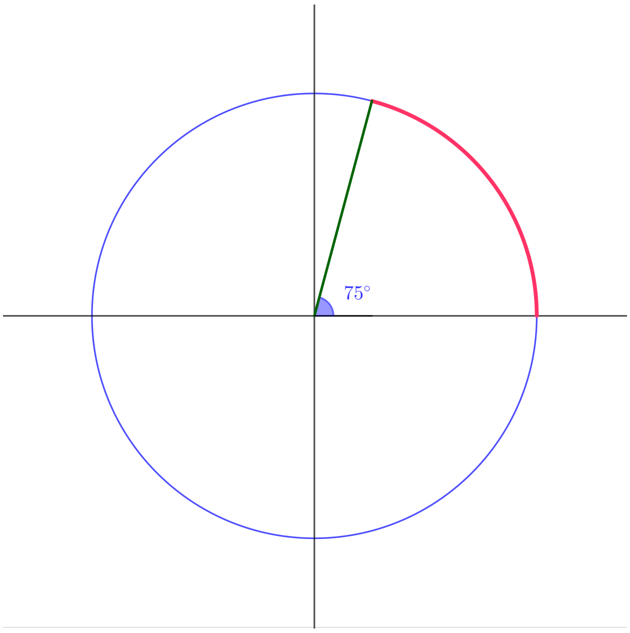
First of all, find the pink arc length on each of these circles with radius 1:

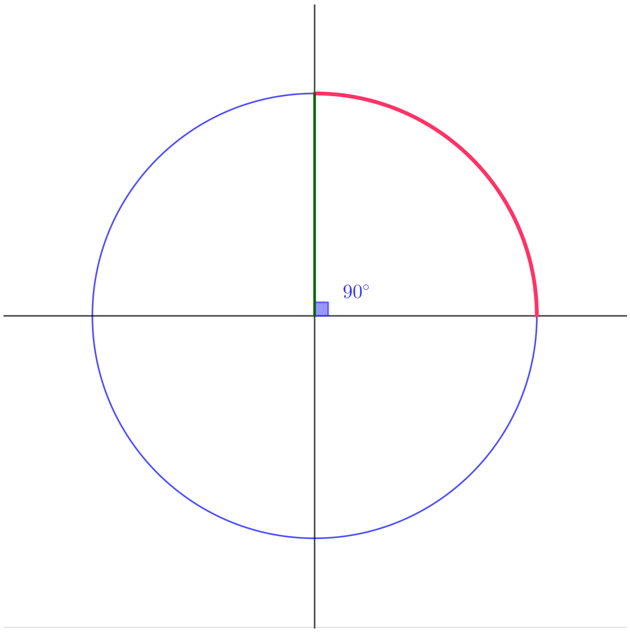


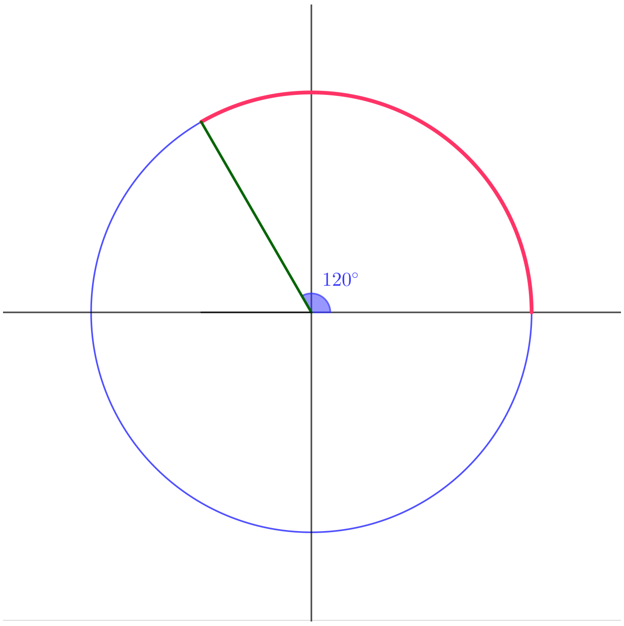


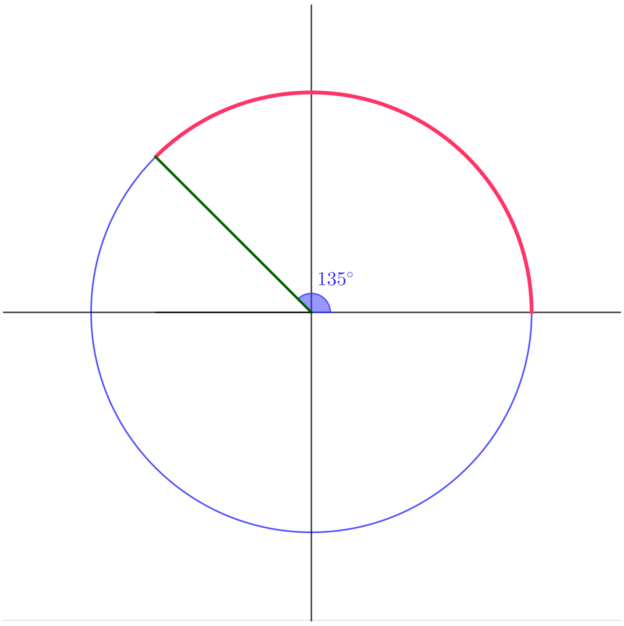


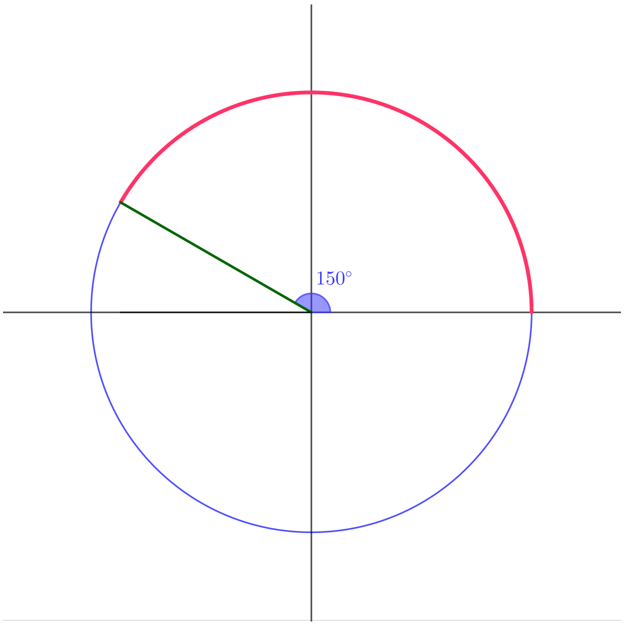


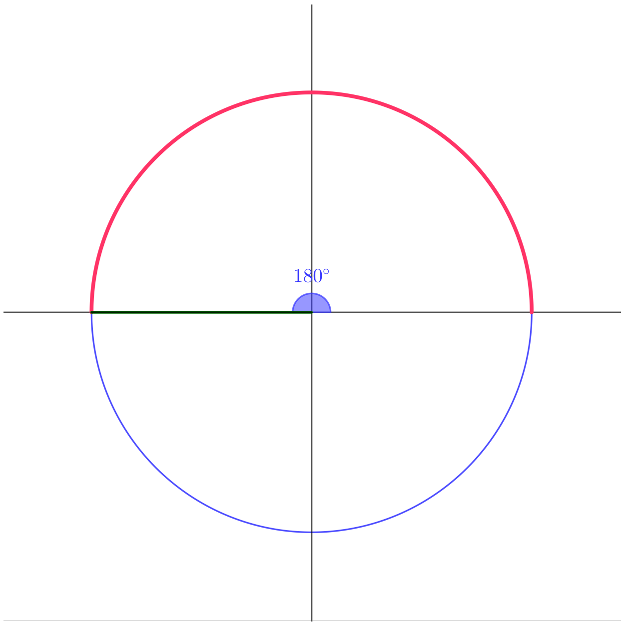

75°

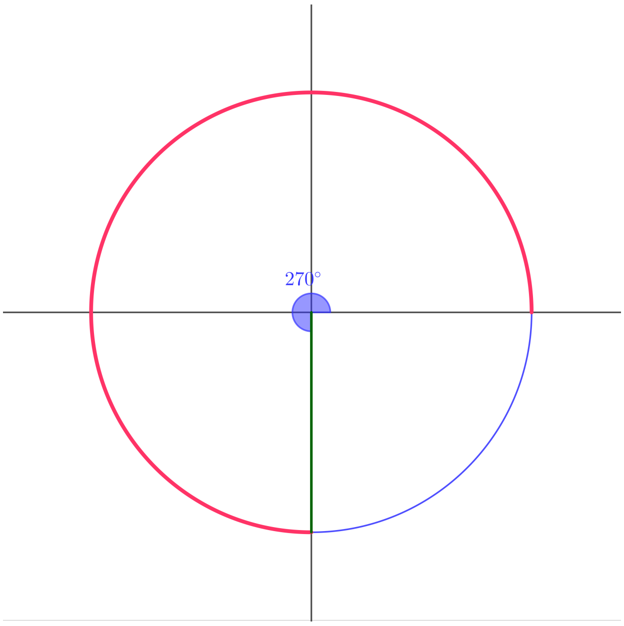

90°

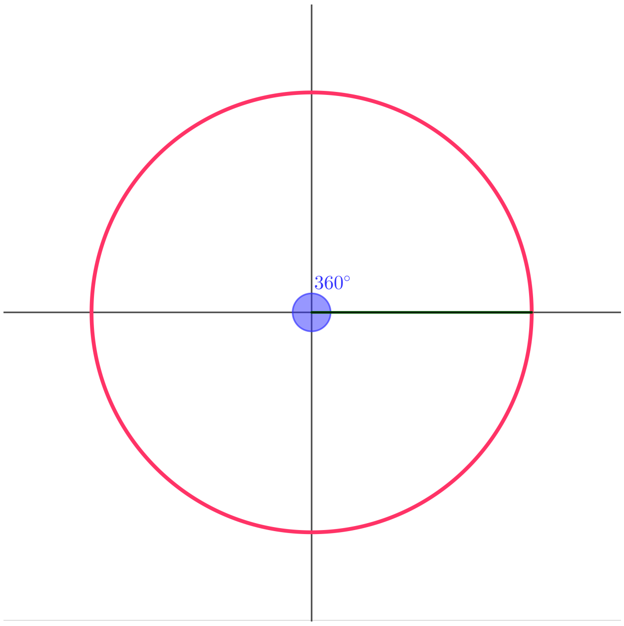

120°


135°


150°

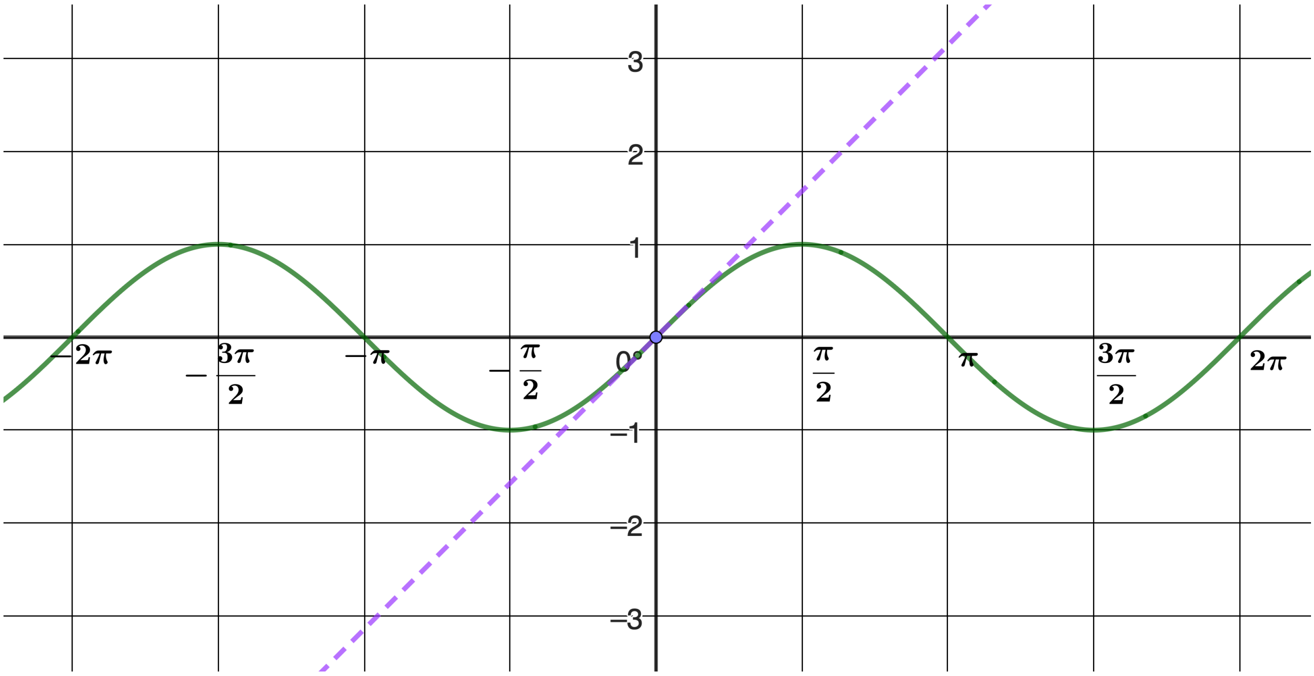

180°


270°


360°

Using $\pi = 3.14159\ldots$, what is $\dfrac{\pi}{2}$ as a decimal?

What does the gradient of the tangent look like now?



This is the place to introduce the idea of radians: instead of measuring angles in degrees, we will use the arc lengths on the unit circle as the measure of the angle itself.

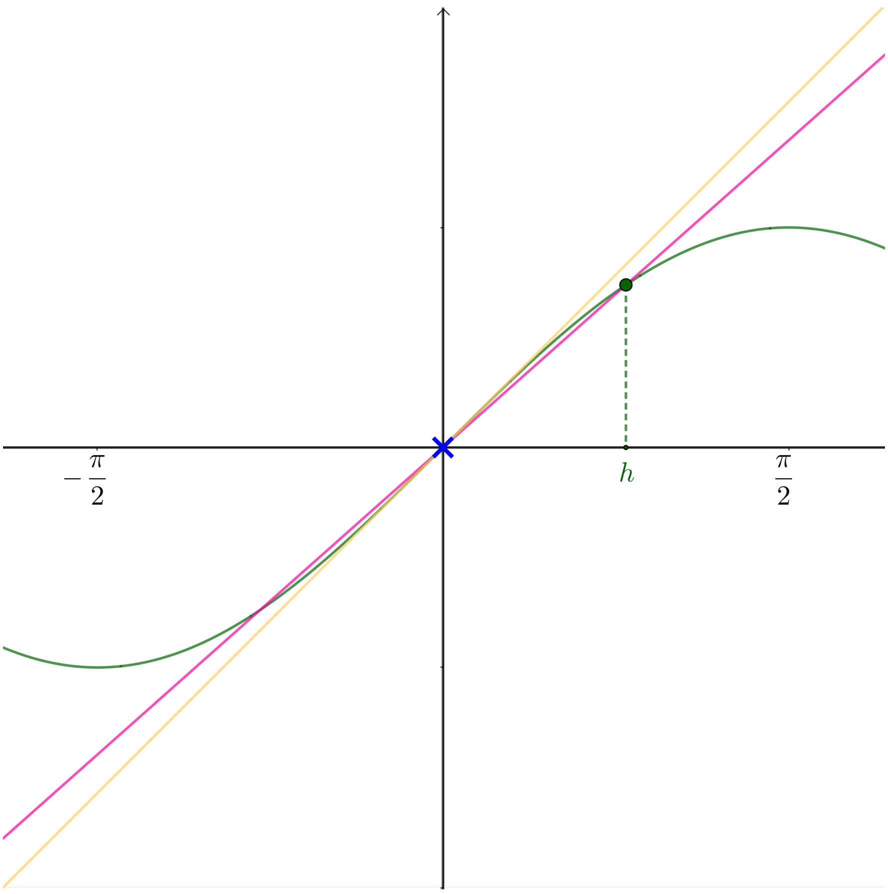
This means that we can still call the equation of the graph

$y = \sin x$

but the units of x are now radians rather than degrees.

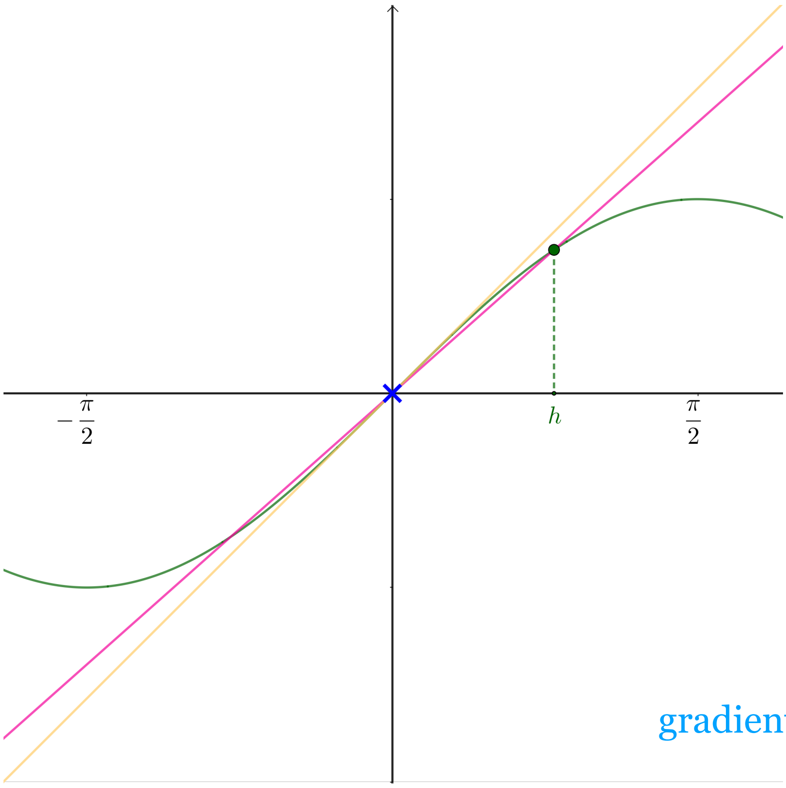
To figure out whether this is the right scale to make the gradient of the tangent equal to 1, we will use the idea of the tangent as a limit.

First, what is the gradient of the pink line?



What will happen to the pink line as h gets increasingly small?

What does this tell us about the gradient of the tangent?



$$\begin{aligned} \text{gradient} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{\sin h - \sin 0}{h} \\ &= \frac{\sin h}{h} \end{aligned}$$

Remember that the units of h is radians.

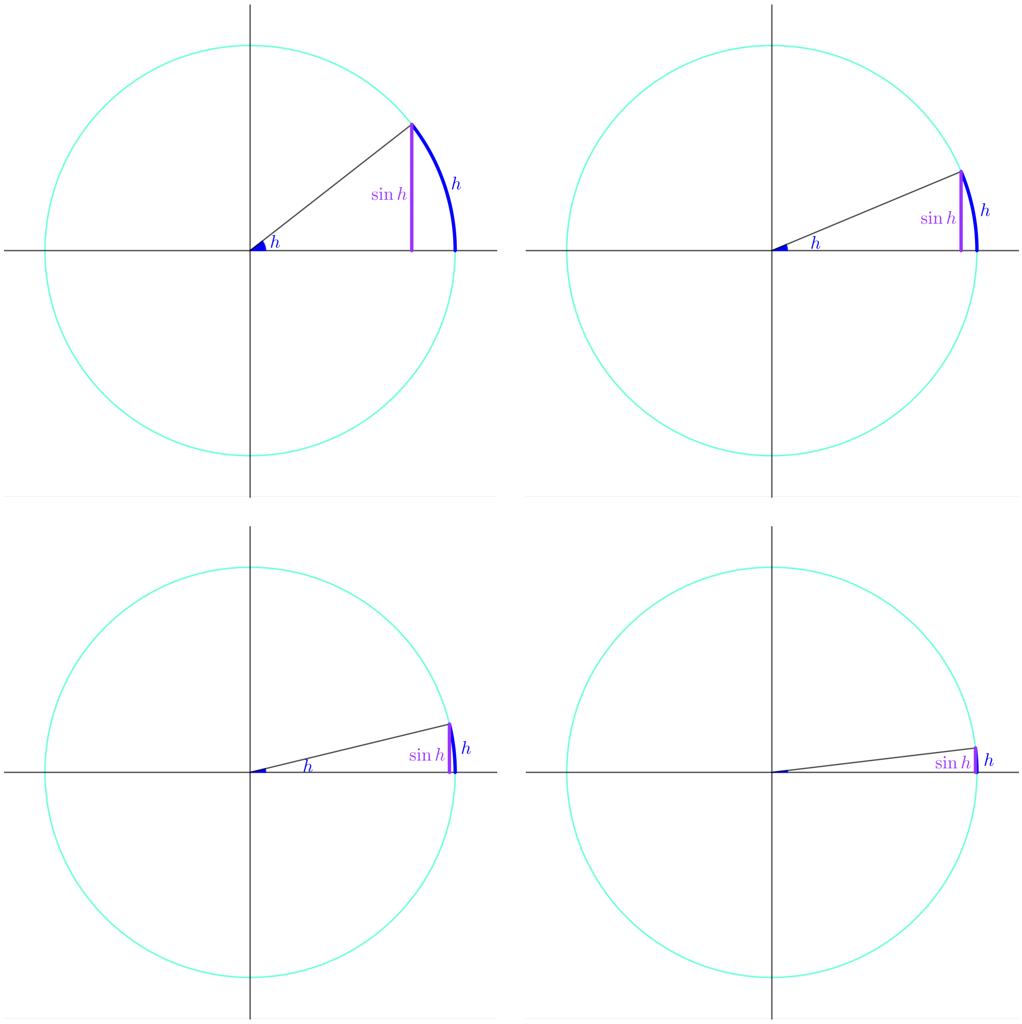
The pink line will get increasingly close to the tangent, and its gradient will get increasingly close to

$$\lim_{h \to 0} \frac{\sin h}{h}$$

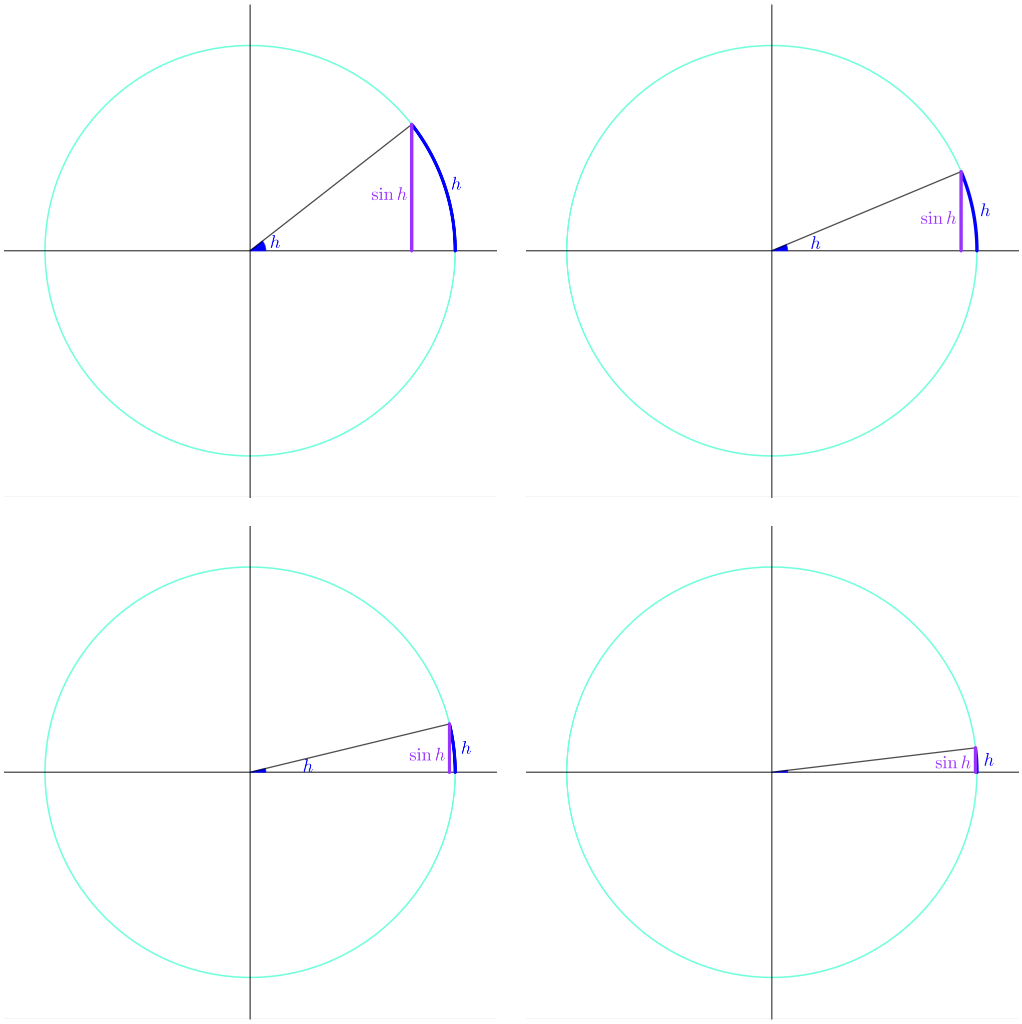
So the gradient of the tangent is this limit. We already know that it looks like 1, and that we would like it to be 1. Next, we will see why it really is 1. To do this, we must turn back yet again to the unit circle.

Look at this sequence of diagrams. Why have I used the same letter, h, for both the angles and the arc lengths?

What do you notice about the relationship between the angle size h and the ratio of the lengths of the blue arc and the purple segment?



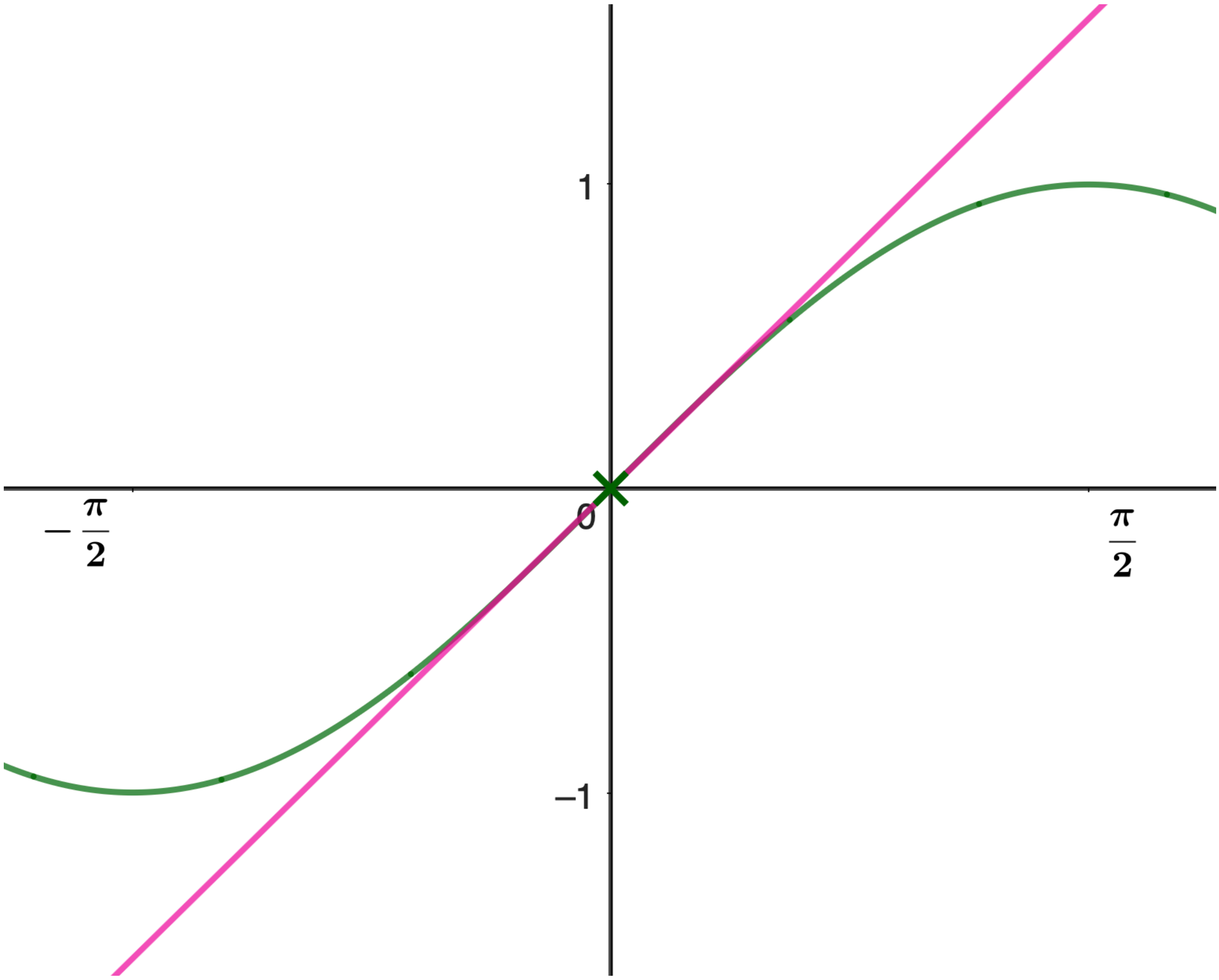
What does this tell you about the ratio $\dfrac{\sin h}{h}$ as $h \to 0$?



The smaller h gets, the closer the lengths are to each other. This is not quite the same as saying that the ratio gets closer to 1. We don't now yet if the arc length is always the same fraction (slightly over 1) of the mauve segment. But this is a subtle technical point.

It turns out that the ratio really does tend to 1, but I would be tempted not to alert your students to this little technicality unless they are going to attempt the more rigorous proof below.

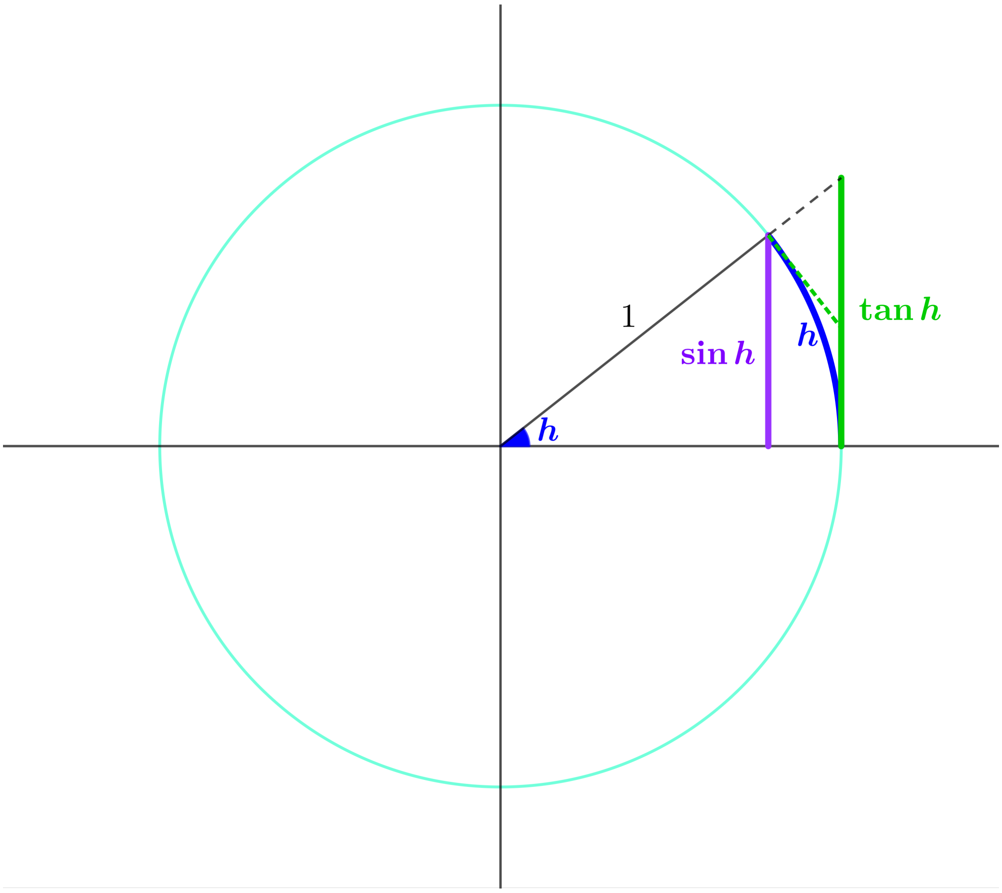
What is the gradient of the tangent to the curve $y = \sin x$ at the origin?



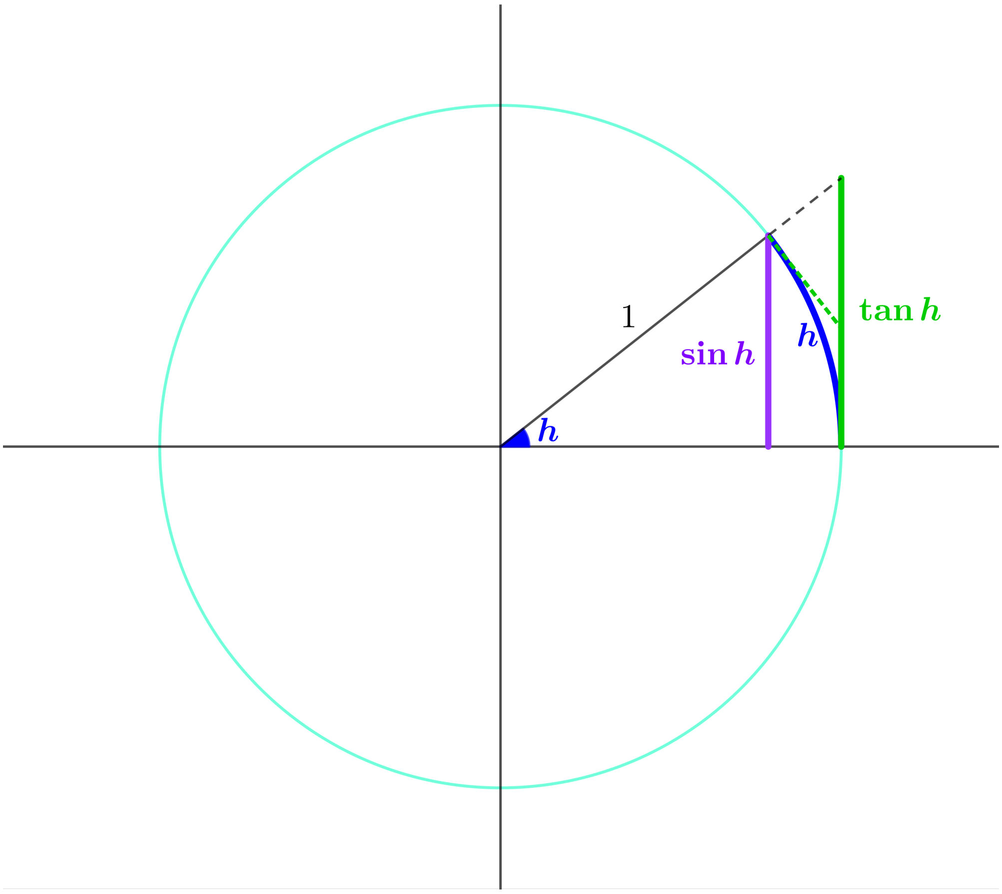
$$\text{gradient of tangent} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

That gives you a sense of why $\lim_{h \to 0} \dfrac{\sin h}{h} = 1$, but it is not quite the whole story.

Here is a more complete "proof", in case you are interested.



Write down an inequality involving the pink and green line segments and the blue arc.



Write down an inequality involving the pink and green line segments and the blue arc.

These two pages are really for your more able classes. They address the technical problem that I flagged up a couple of pages back.

It looks immediately as though $\sin h < h < \tan h$. Can we be sure that $\tan h$ really is greater than h?

Well, the little triangle at the top right is right angled, and the hypotenuse is half of $\tan h$. So replacing the top half of the tan segment with the green dotted segment makes a path that is definitely longer than the blue arc. This shows pretty convincingly that $\tan h > h$, and hence that

$$\sin h < h < \tan h$$

Use this to find lower and upper bounds for $\dfrac{h}{\sin h}$.

Now find lower and upper bounds for $\dfrac{\sin h}{h}$.

$$\begin{aligned} & \sin h < h < \tan h \\ \Rightarrow\ & \frac{\sin h}{\sin h} < \frac{h}{\sin h} < \frac{1}{\cos h} \\ \Rightarrow\ & 1 < \frac{h}{\sin h} < \frac{1}{\cos h} \\ \Rightarrow\ & \cos h < \frac{\sin h}{h} < 1 \end{aligned}$$

What is $\lim\limits_{h \to 0} \cos h$?

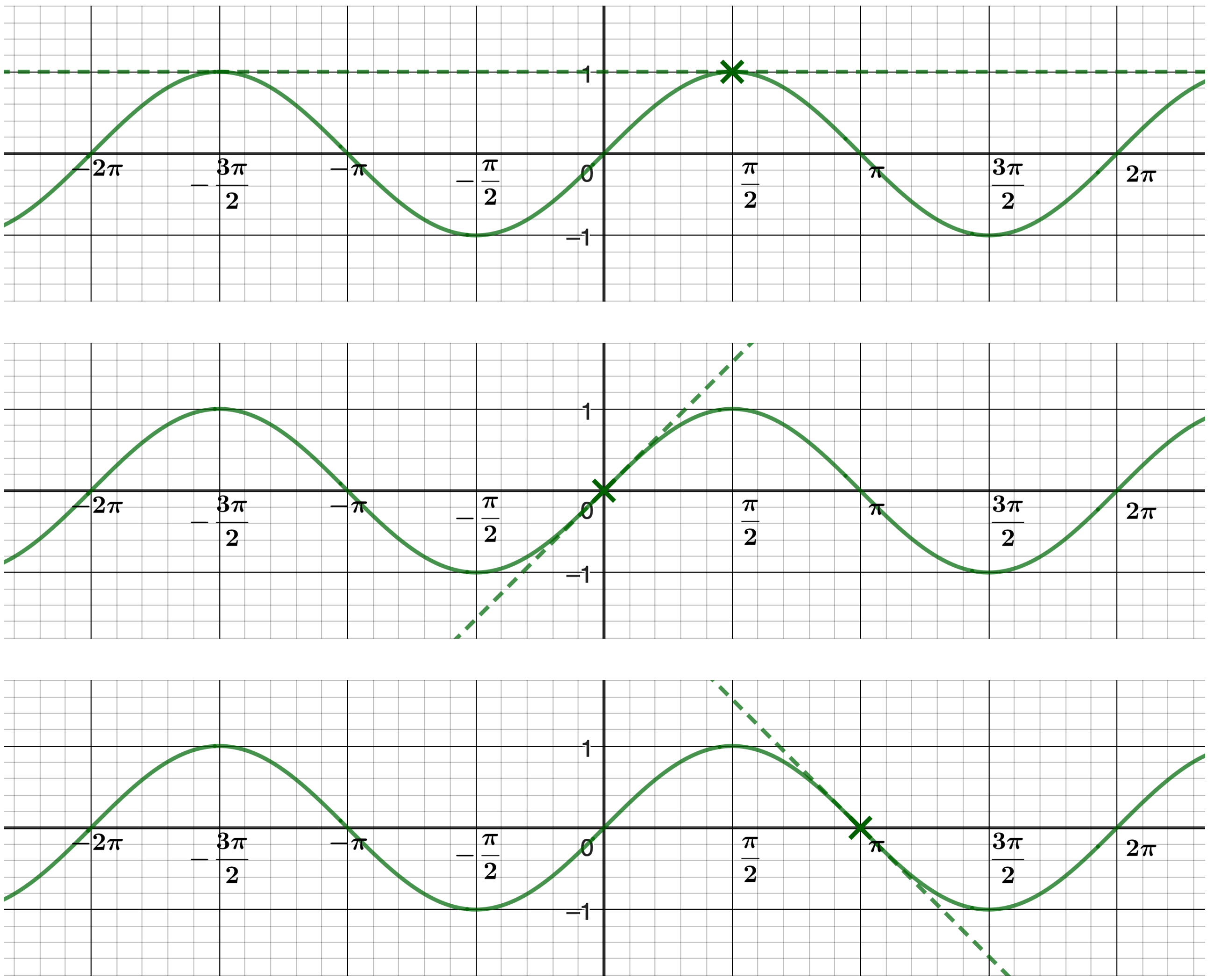
Use this to find $\lim\limits_{h \to 0} \dfrac{\sin h}{h}$.

$$\begin{aligned} & \cos h \to 1 \text{ as } h \to 0 \text{ and } \cos h < \frac{\sin h}{h} < 1 \\ \Rightarrow\ & 1 \le \lim_{h \to 0} \frac{\sin h}{h} \le 1 \\ \Rightarrow\ & \lim_{h \to 0} \frac{\sin h}{h} = 1 \end{aligned}$$

Now we know that the gradient of the tangent to the graph $y = \sin x$ at the origin is 1 (when we use radians as our unit of angles).

Next, we will think about the gradient of the curve at other points.

To start with, what are the gradients of these three tangents?

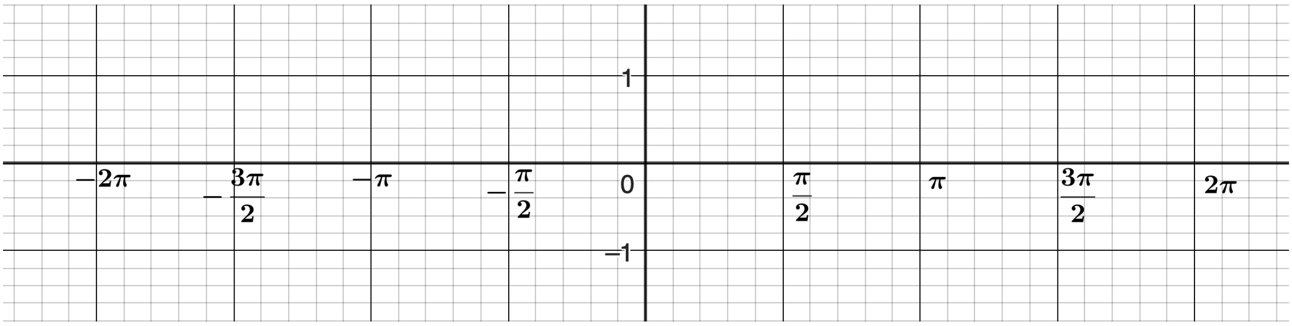


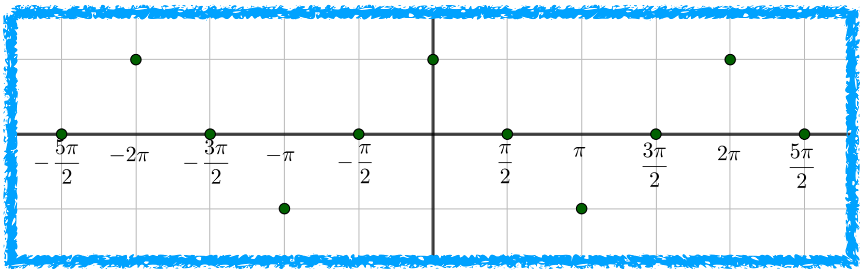
Symmetry tells us that the third of these has gradient -1.

Use the graph to fill in this table:

x	**gradient of tangent**
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1
$-\frac{\pi}{2}$	0
$-\pi$	-1
$-\frac{3\pi}{2}$	0
-2π	1

Now mark the values from the table on these axes with the left-hand column on the x axis and the right-hand column on the y axis.

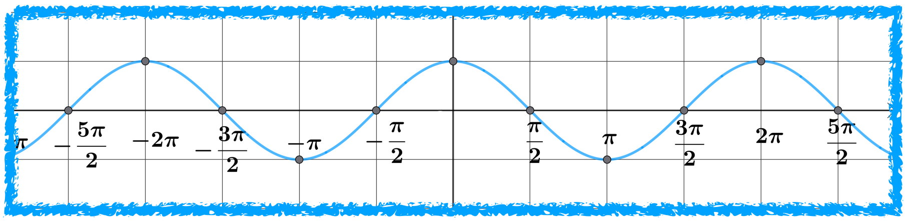




What does this graph look like?

This looks like the outline of a cos graph. The next section of this sheet is devoted to showing that it really is a cos graph.

What happens to the gradient between these points? Use this idea to draw the whole curve representing the gradient.

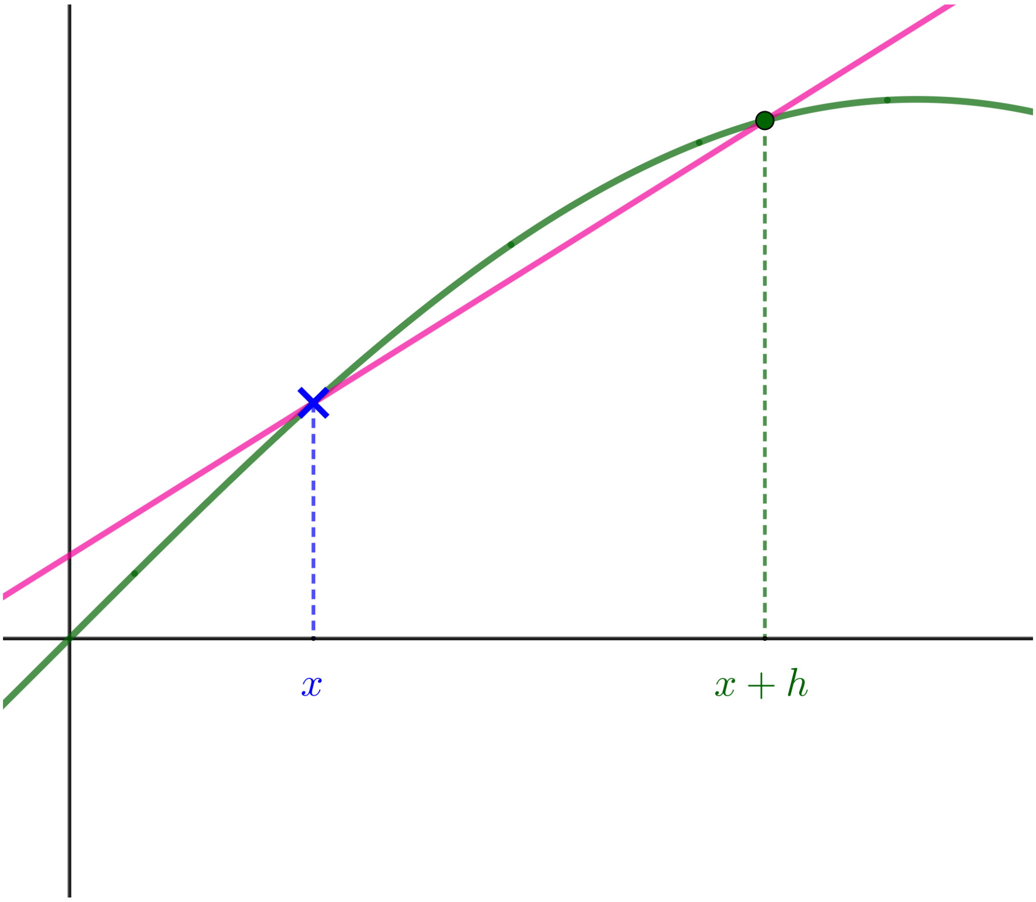


This now looks like the cos graph. Is it?

Now we know that the gradient of the sin graph looks rather like cos, and we are in a position to see that the differential of sin really is cos.

What is the gradient of the pink line in terms of x and h ?

What happens to the pink line as h gets increasingly small?



The gradient of the line is

$$\frac{\sin(x+h)-\sin x}{h}$$

and the line gets increasingly close to the tangent at the blue cross as h approaches 0.

Use this to write the gradient of the tangent as a limit.

By using a compound angle formula and then rearranging, express this gradient as a multiple of $\cos x$ minus a multiple of $\sin x$.

$$\frac{\sin(x+h)-\sin x}{h} = \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \cos x \frac{\sin h}{h} - \sin x \frac{1-\cos h}{h}$$

$$\Rightarrow \text{gradient of tangent} = \lim_{h \to 0}\left[\cos x \frac{\sin h}{h} - \sin x \frac{1-\cos h}{h}\right]$$

Simplify this using the fact that $\displaystyle\lim_{h \to 0} \frac{\sin h}{h} = 1$.

$$\lim_{h \to 0}\left[\cos x \frac{\sin h}{h} - \sin x \frac{1-\cos h}{h}\right] = \lim_{h \to 0} \cos x \frac{\sin h}{h} - \lim_{h \to 0} \sin x \frac{1-\cos h}{h}$$

$$\lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right] = \lim_{h\to 0}\cos x\frac{\sin h}{h} - \lim_{h\to 0}\sin x\frac{1-\cos h}{h}$$

because the limit of a sum (or difference) is the sum (or difference) of two limits. This is a rather technical point, and perhaps it is perfectly obvious, at least for well-behaved functions such as these. It doesn't need much time in class, if any, but it is something lurking in the background that will come up in a first term analysis course of a maths degree.

Now notice that, as far as h is concerned, $\cos x$ is constant, as is $\sin x$. This means that we can "take them out of the limit"--the limit of a multiple of a function is the same multiple of the limit of the function. Again, little or no time needed on this in class. Again, first term analysis.

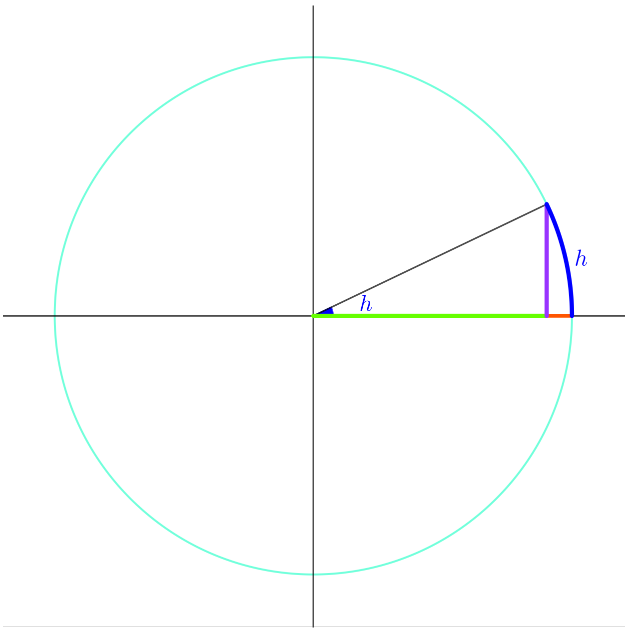
So

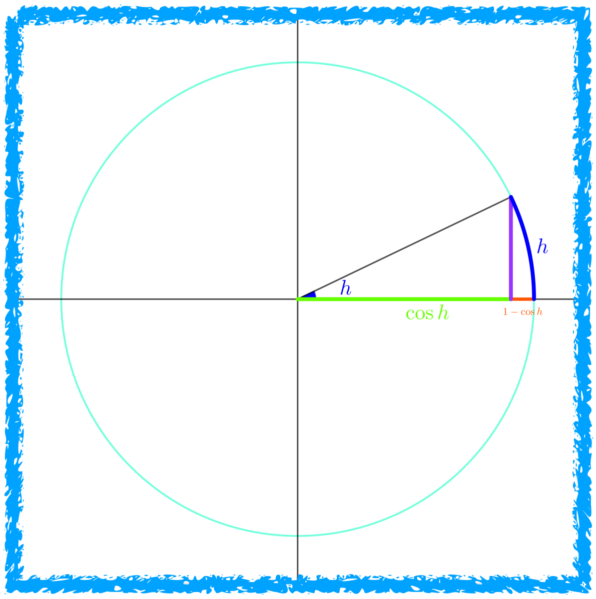
$$\begin{aligned}\text{gradient of tangent} &= \lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right]\\ &= \lim_{h\to 0}\cos x\frac{\sin h}{h} - \lim_{h\to 0}\sin x\frac{1-\cos h}{h}\\ &= \cos x\lim_{h\to 0}\frac{\sin h}{h} - \sin x\lim_{h\to 0}\frac{1-\cos h}{h}\end{aligned}$$

but we already know that $\lim_{h\to 0}\frac{\sin h}{h} = 1$.

What is the length of the orange line segment?

What happens to the ratio of the orange line segment to the blue arc as h gets increasingly small?





The orange segment is $1 - \cos h$.

What does this tell you about $\lim\limits_{h\to 0} \dfrac{1-\cos h}{h}$?

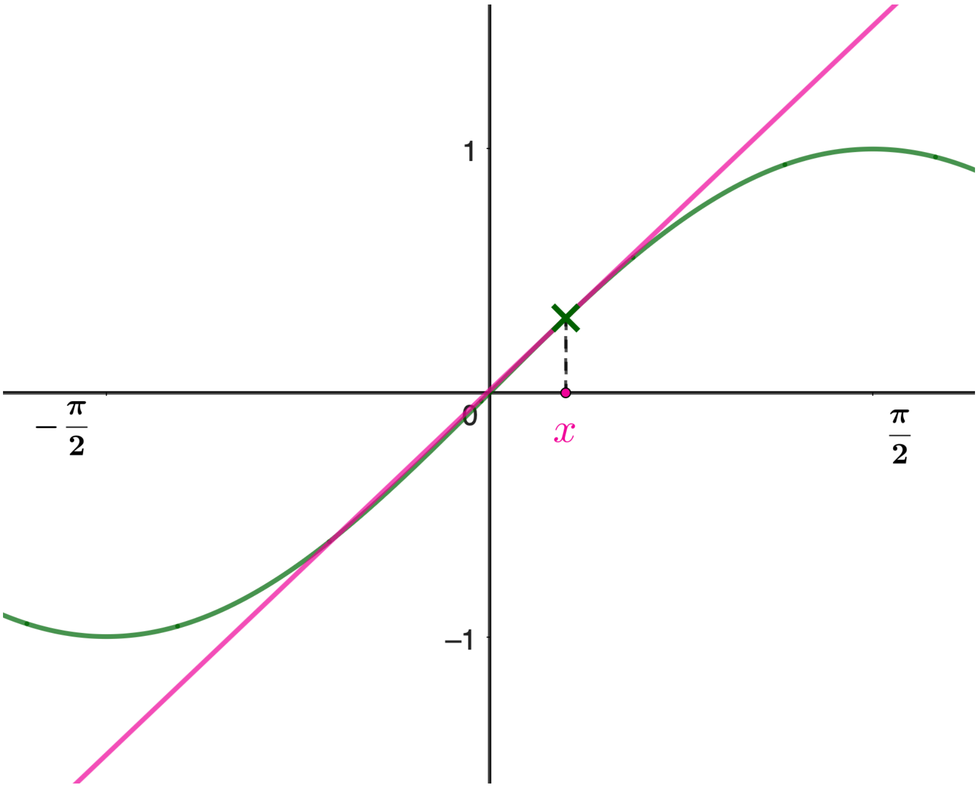
$$\lim_{h\to 0} \frac{1-\cos h}{h} = 0$$

It certainly looks as though this ratio tends to 0, and this is really enough for this stage in your students' mathematical career. However, as before, it's not the whole story, and you may want to lead them through a more rigorous proof such as this:

$$\begin{aligned}
\frac{1-\cos h}{h} &= \frac{1-\cos^2 h}{h(1+\cos h)} \\
&= \frac{\sin^2 h}{h(1+\cos h)} \\
&= \frac{\sin h}{h} \times \frac{\sin h}{1+\cos h}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{h\to 0} \frac{1-\cos h}{h} &= \lim_{h\to 0} \frac{\sin h}{h} \times \frac{\sin h}{1+\cos h} \\
&= 1 \times 0 \\
&= 1
\end{aligned}$$

Use this limit to find the gradient of the tangent to the curve $y = \sin x$ at the green cross.



$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\sin h}{h} - \sin x \lim_{h \to 0} \frac{1 - \cos h}{h}$$

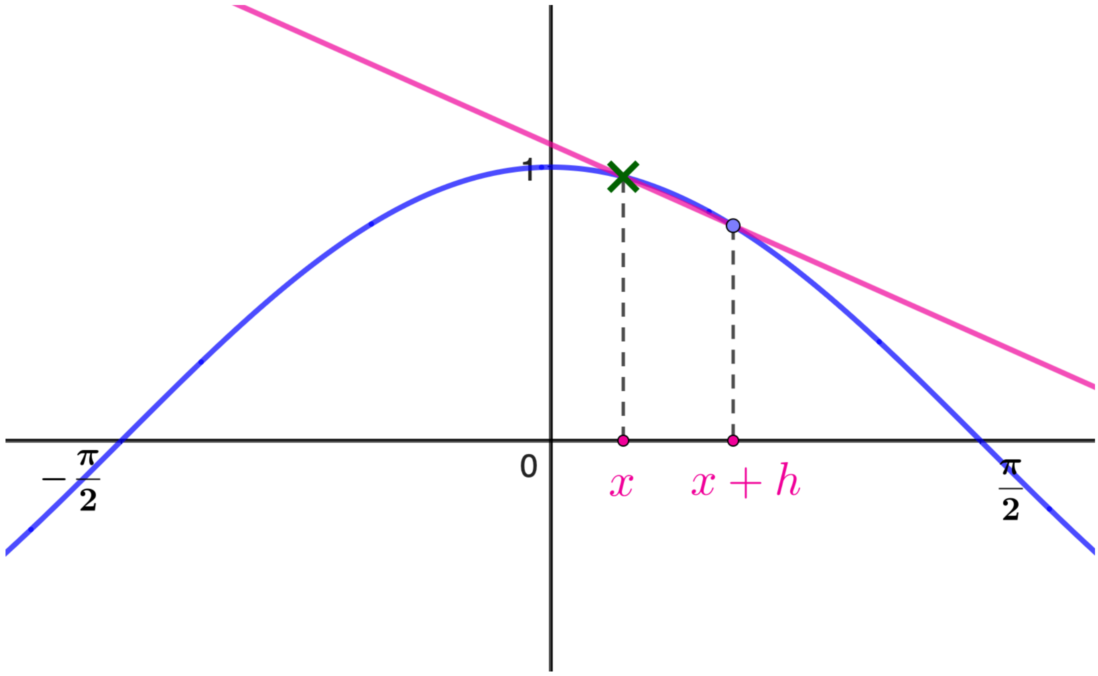
$$= \cos x \times 1 - \sin x \times 0$$

$$= \cos x$$

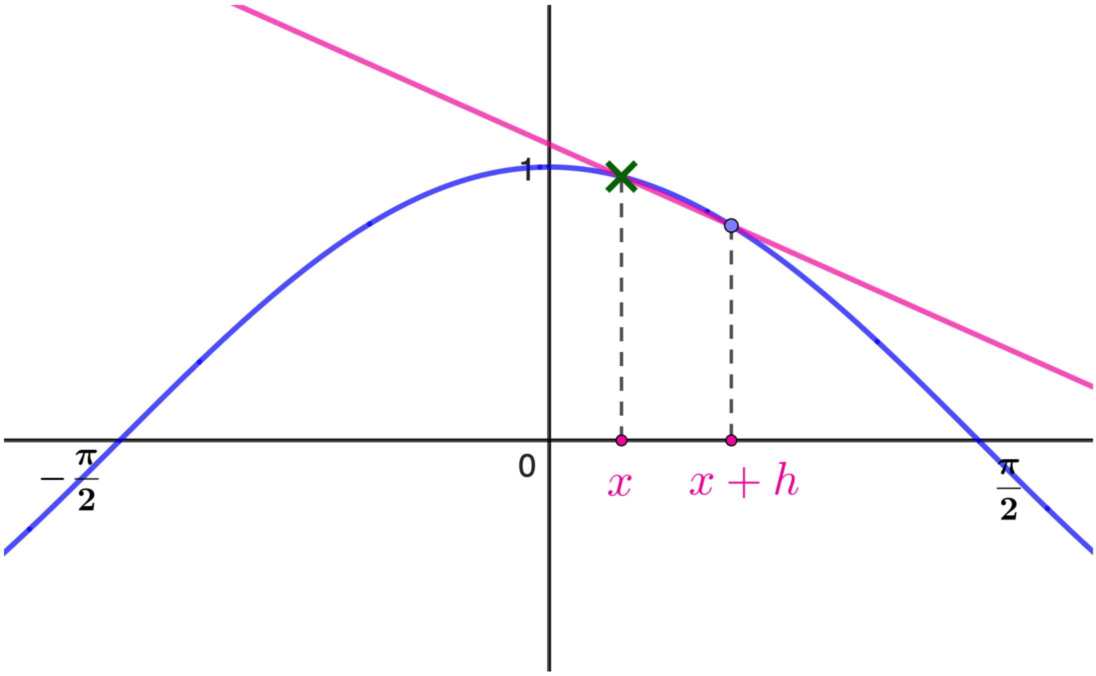
If $f(x) = \sin x$, what is $f'(x)$?

$$f'(x) = \cos x$$

Now we know how to differentiate sine, we can tackle other circular functions.



Use the same ideas to find the gradient of the tangent to the curve $y = \cos x$ at the green cross.



$$\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin h \sin x - \cos x}{h}$$

$$= -\sin x \lim_{h \to 0} \frac{\sin h}{h} - \cos x \lim_{h \to 0} \frac{1 - \cos h}{h}$$

$$= -\sin x$$

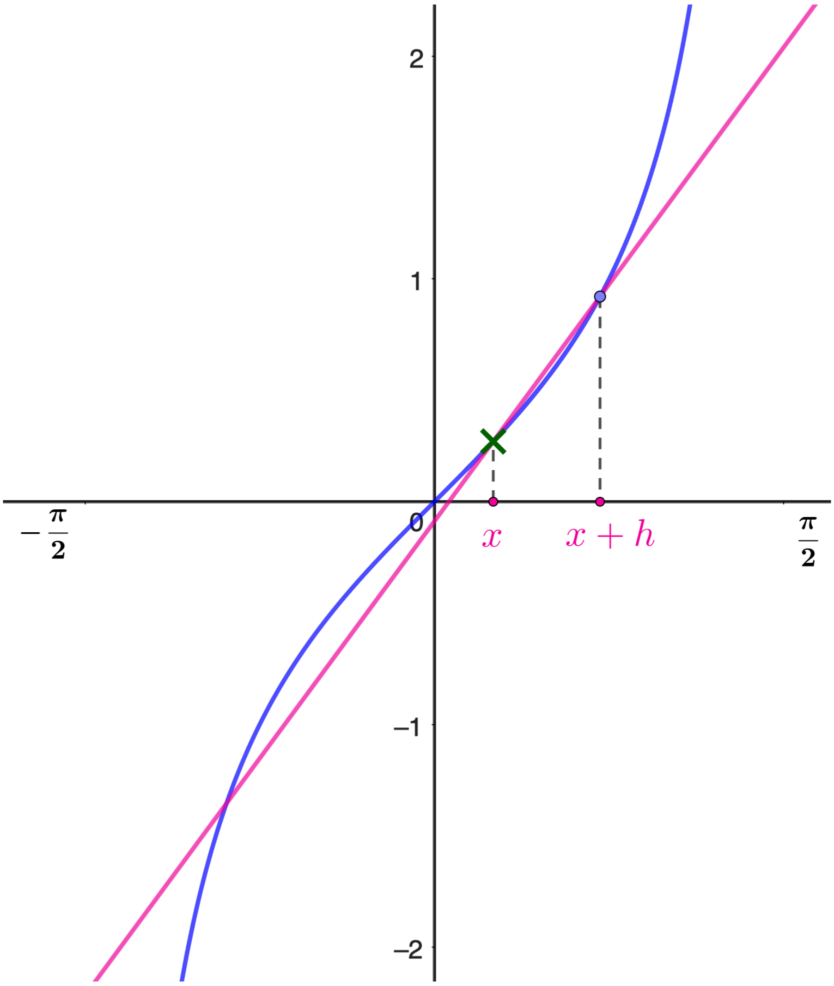
We could also find this differential by translating the cos graph to become the sin curve.

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\cos\left(\frac{\pi}{2} - x\right) \text{ by the chain rule}$$

$$= -\sin x$$

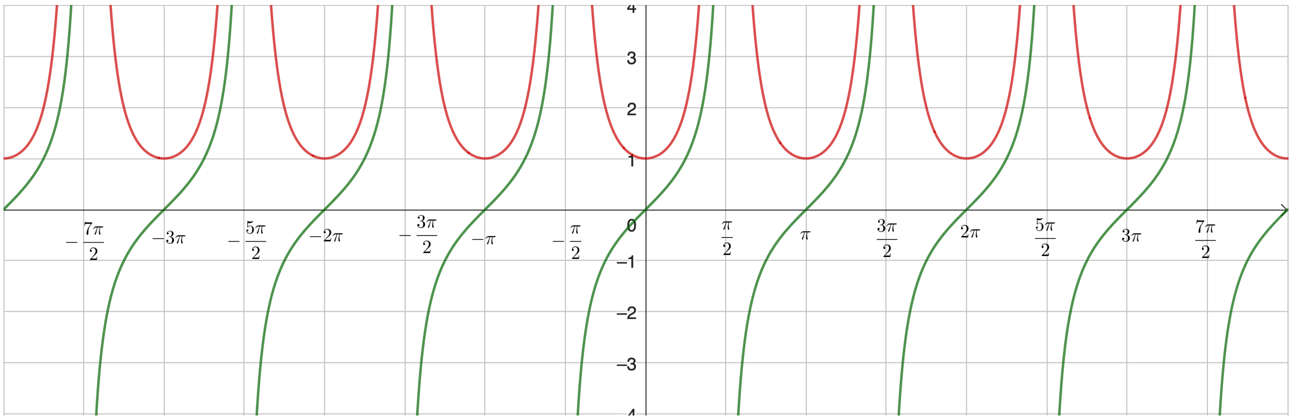
What is the gradient of the tangent to the curve $y = \tan x$ at the green cross?



Using the quotient rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Here is the graph of tan x along with the graph of its differential. How do the two curves relate to each other?



The point here is to see that the expressions for the differential do make some graphical sense. This differential isn't just slightly random expressions, but it does actually relate to something a bit more tangible! The same for the next few functions.

The tan curve (green) has points of inflection at the same time that the differential curve has local minima, and the differential curve tends to $\pm\infty$ when the gradient of the green curve tends to $\pm\infty$.

We can also find this differential by thinking about limits:

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}=\lim_{h\to 0}\frac{1}{h}\left(\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\tan x+\tan h-\tan x+\tan^2 x\tan h}{1-\tan x\tan h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\tan^2 x\tan h+\tan h}{1-\tan x\tan h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan h(1+\tan^2 x)}{h(1-\tan x\tan h)}\right)$$

$$=\lim_{h\to 0}\frac{\tan h}{h}\left(\frac{\sec^2 x}{1-\tan x\tan h}\right)$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\times\frac{1}{\cos h}\times\frac{\sec^2 x}{1-\tan x\tan h}$$

$$=\sec^2 x$$

because

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \qquad \lim_{h\to 0}\cos h=1 \qquad \lim_{h\to 0}\tan h=0$$

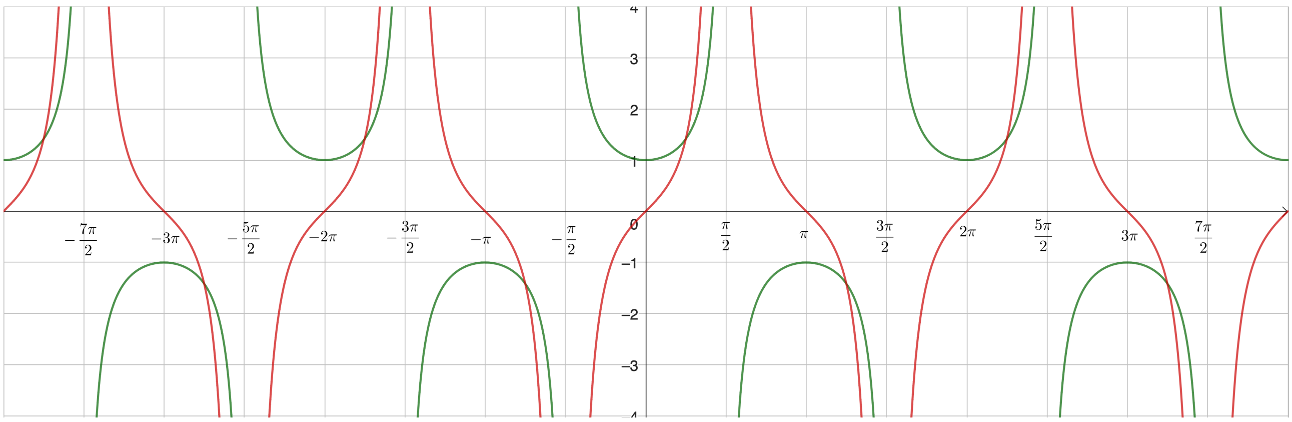
Differentiate $f(x) = \sec x$

$$y = \sec x = \frac{1}{\cos x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x = \sin x \sec^2 x$$

Here is the graph of $\sec x$ along with the graph of its differential. How do the two curves relate to each other?



The sec curve (green) has turning points at the same time that the differential curve crosses the x axis, and the differential curve tends to $\pm\infty$ when the gradient of the red curve tends to $\pm\infty$.

You might also like this implicit differentiation version, although your students will quite possibly not yet be ready for this:

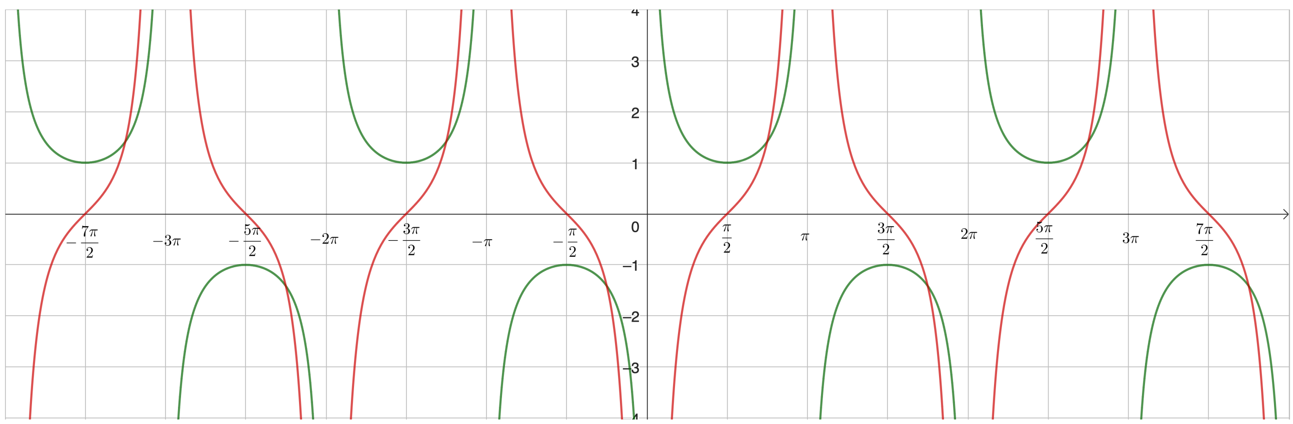
$$
\begin{aligned}
&y \cos x = 1 \\
\Rightarrow\ &-y \sin x + \frac{\mathrm{d}y}{\mathrm{d}x} \cos x = 0 \\
\Rightarrow\ &\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y \sin x}{\cos x} \\
&\qquad = \sec x \tan x
\end{aligned}
$$

Differentiate $f(x) = \operatorname{cosec} x$

$$y = \operatorname{cosec} x = \frac{1}{\sin x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x = -\cos x \operatorname{cosec}^2 x$$

Here is the graph of cosec$x$ along with the graph of its differential. How do the two curves relate to each other?



The cosec curve (green) has turning points at the same time that the differential curve crosses the x axis, and the differential curve tends to $\pm\infty$ when the gradient of the red curve tends to $\pm\infty$.

Differentiate $f(x) = \cot x$

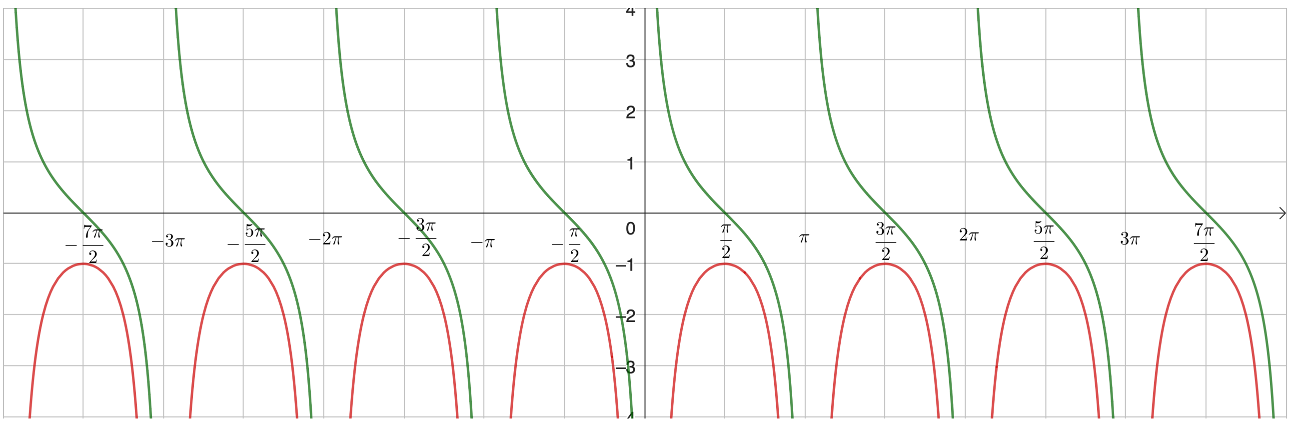
$$y = \cot x$$

$$\Rightarrow y = \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

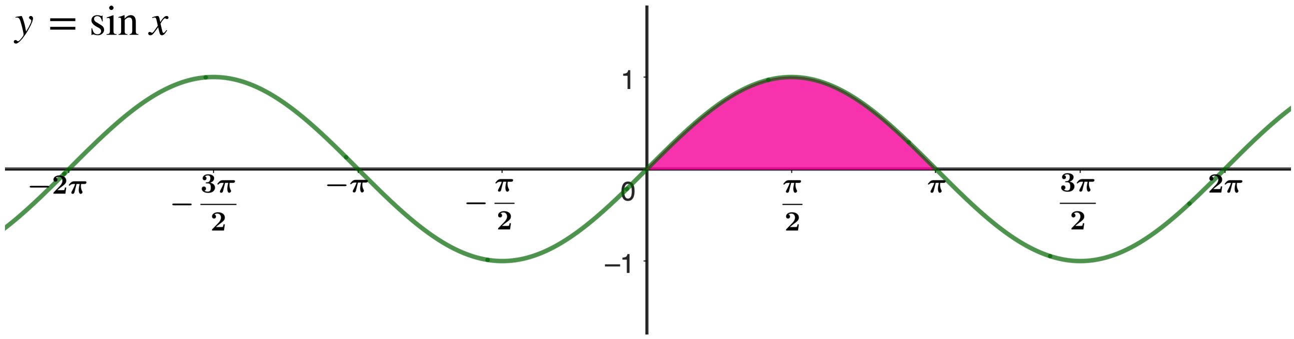
Here is the graph of $\cot x$ along with the graph of its differential. How do the two curves relate to each other?



The cot curve (green) has points of inflection at the same time that the differential curve has local maxima, and the differential curve tends to $\pm\infty$ when the gradient of the green curve tends to $\pm\infty$.

Integrals of circular functions

Find these areas



Since the differential of cos is $-$sin, it must be the case that the integral of sin is $-$cos.

$$\text{area} = \int_0^{\pi} \sin x \, \mathrm{d}x$$

$$= \Big[-\cos x\Big]_0^{\pi}$$

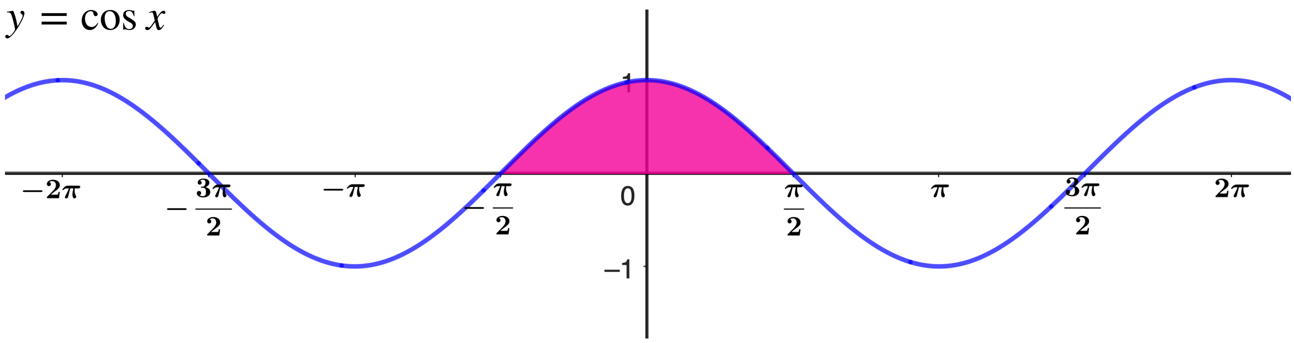
$$= -\cos\pi - (-\cos 0)$$

$$= 1 + 1 = 2$$

Please don't let your students use their calculators for any steps such as

- putting the integral into the calculator
- finding $\cos 0$ or $\cos\pi$

Of course they may be able to do so in an exam, but if they do so now, they will learn nothing.



No calculators!

$$\text{area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, \mathrm{d}x$$

$$= \Big[\sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)$$

$$= 1 + 1 = 2$$

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln|\cos x|$

Integrating $\tan x$ needs integration by substitution, with which your students may not yet be familiar. However, they will surely know the chain rule, so if necessary, this page will provide a fix for this.

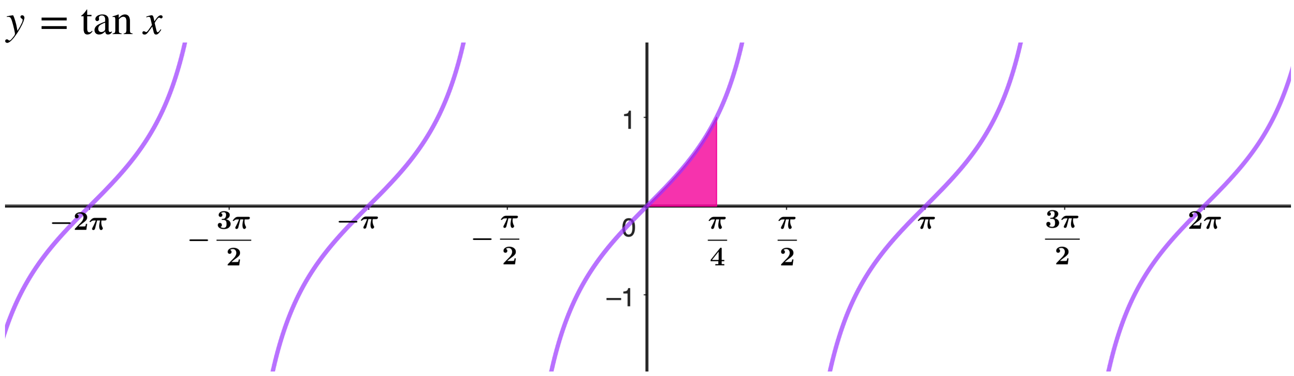
$w = \ln|\cos x| \qquad u = \cos x$

$w = \ln|u|$

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{u} = \frac{1}{\cos x} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\cos x}\sin x = -\tan x$$

Find this area:



For the integral of tan, either use the result on the previous page, or use the substitution $u = \cos x$. Or they might remember the rule for $\int \frac{f'}{f}$, but even so, it's a good idea to do the substitution to remind themselves why this rule works.

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x \qquad u = \cos x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$= -\int \frac{\sin x}{\cos x} \frac{1}{\sin x} \, \mathrm{d}u$$

$$= -\int \frac{1}{u} \, \mathrm{d}u$$

$$= \ln |u|$$

$$= -\ln |\cos x|$$

$$= \ln |\sec x|$$

Often, students will get the final version from the formula book without noticing that $-\ln|\cos x|$ and $\ln|\sec x|$ are the same thing, so it's a good moment to emphasise this.

For the area, we can use either form of the integral of tan, but please, no calculators, even for the manipulation of logs.

$$\text{area} = \int_0^{\frac{\pi}{4}} \tan x \, \mathrm{d}x$$

$$= \Big[-\ln(\cos x)\Big]_0^{\frac{\pi}{4}}$$

$$= -\ln\left(\cos\frac{\pi}{4}\right) - \big(-\ln(\cos 0)\big)$$

$$= -\ln\frac{1}{\sqrt{2}} + \ln 1$$

$$= \ln\sqrt{2}$$

$$= \frac{1}{2}\ln 2$$

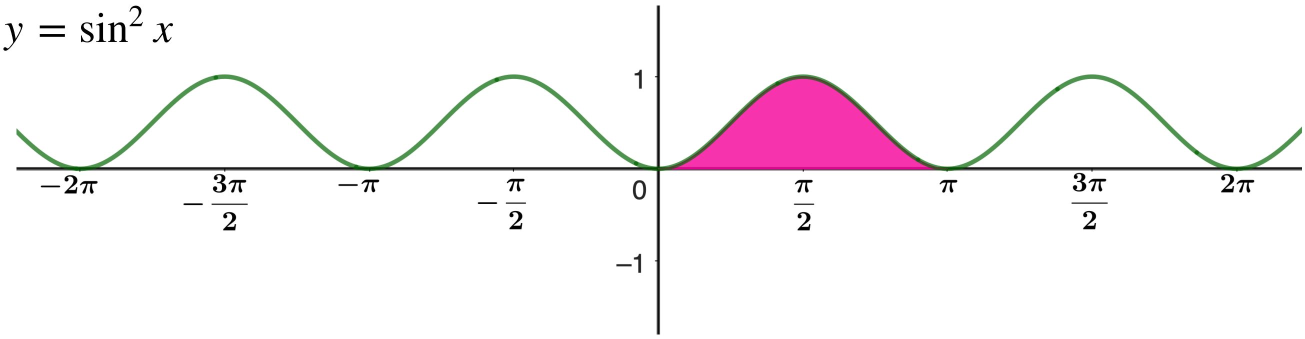
$$\text{area} = \int_0^{\frac{\pi}{4}} \tan x \, \mathrm{d}x$$

$$= \Big[\ln(\sec x)\Big]_0^{\frac{\pi}{4}}$$

$$= \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$$

$$= \ln\sqrt{2} - \ln 1$$

$$= \frac{1}{2}\ln 2$$



Compare your answer with $\int_0^{\pi} \sin x \, \mathrm{d}x$. Which is bigger? Explain your answer in terms of the graphs.

Here is the most straightforward method, but even once they have seen this idea and used it a few times, students often need reminding to use a double angle formula for $\cos 2x$ to integrate $\sin^2 x$ or $\cos^2 x$.

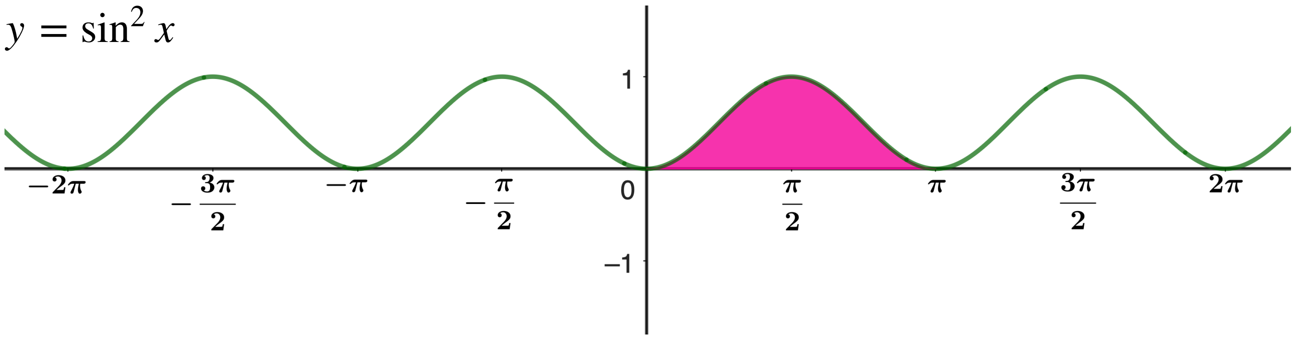
$$\text{area} = \int_0^{\pi} \sin^2 x \, \mathrm{d}x$$

$$= \int_0^{\pi} \frac{1 - \cos 2x}{2} \, \mathrm{d}x$$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

$\frac{\pi}{2} < 2$. When $0 < x < \pi$, $0 < \sin x < 1 \Rightarrow \sin^2 x < \sin x$, so the area on the $\sin^2$ graph must be less.



Probably the double angle method is the easiest, but, left to their own devices, your students might decide to try integrating by parts. You may or may not want to teach this method, but in either case, it's a good idea to support someone who sets off along these lines:

$$\int \sin^2 x \, \mathrm{d}x = -\sin x \cos x + \int \cos x \cos x \, \mathrm{d}x$$

$$= -\frac{1}{2}\sin 2x + \int \cos^2 x \, \mathrm{d}x$$

$$= -\frac{1}{2}\sin 2x + \int 1 - \sin^2 x \, \mathrm{d}x$$

$$= -\frac{1}{2}\sin 2x + x - \int \sin^2 x \, \mathrm{d}x$$

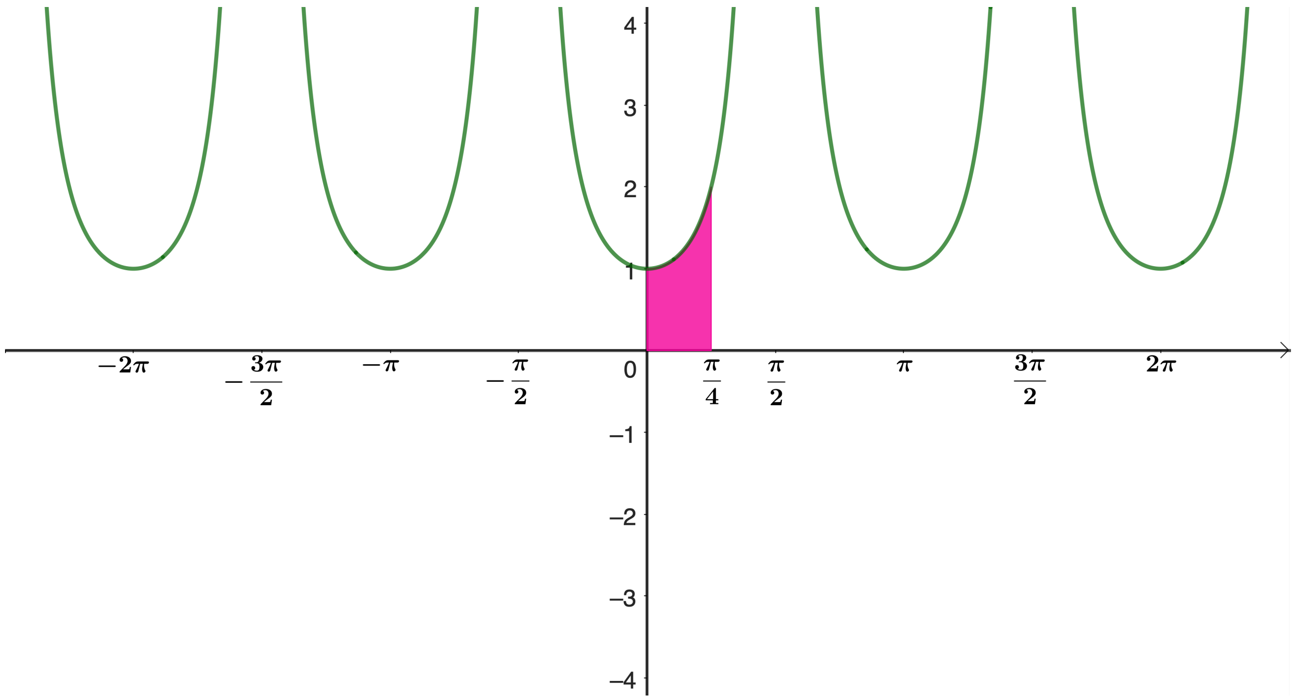
$$\Rightarrow 2\int \sin^2 x \, \mathrm{d}x = x - \frac{1}{2}\sin 2x$$

$$\Rightarrow \int \sin^2 x \, \mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$u = \sin x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \qquad v = -\cos x$$

$y = \sec^2 x$



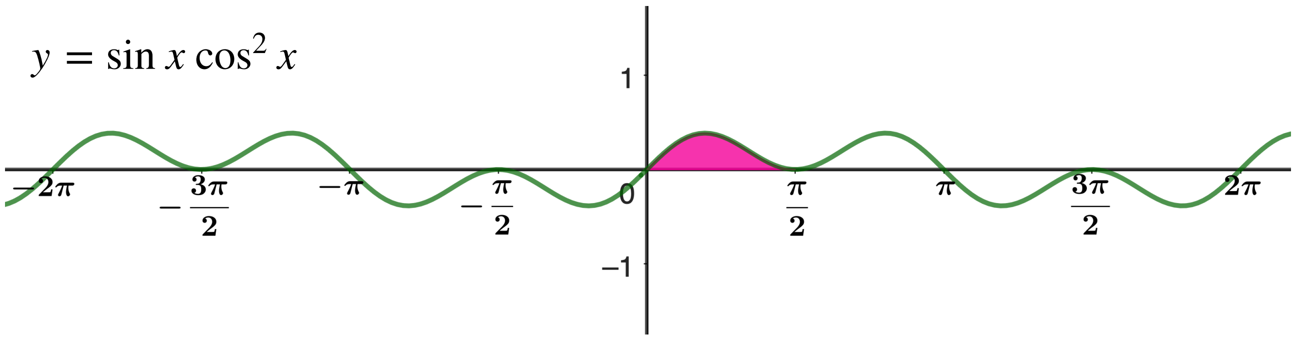
Remember that the differential of tan is $\sec^2$, so

$$\text{area} = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}} \quad .$$

$$= \tan \frac{\pi}{4} - \tan 0$$

$$= 1$$



Some integrals look terrifying but actually turn out to be pretty simple. Here, perhaps you can see straight away that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^3 x = -3\sin x\cos^2 x$$

so that the integral is $-\frac{1}{3}\cos^3 x$.

This kind of "integrating by looking" is always something to be on the alert for.

If not, there is always substitution:

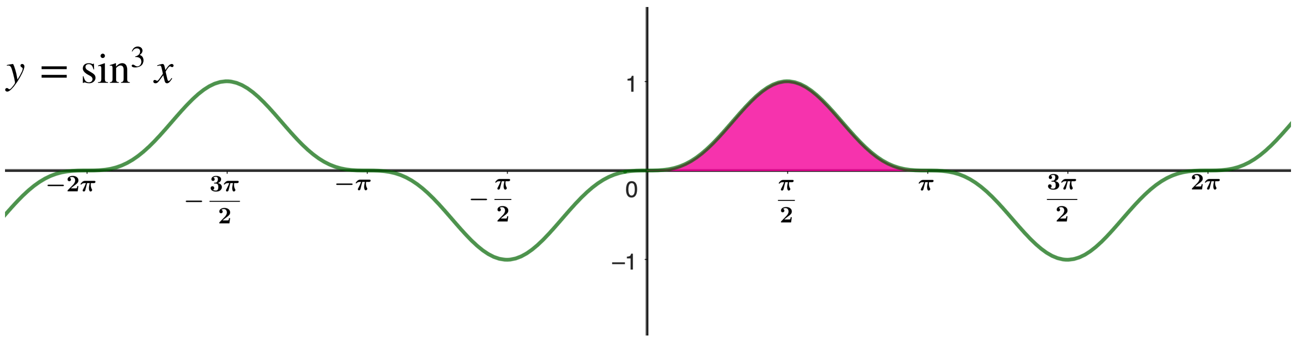
$$u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$\int_0^{\frac{\pi}{2}} \sin x\cos^2 x\,\mathrm{d}x = \int_1^0 \sin x\cos^2 x\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int_1^0 \frac{u^2\sin x}{\sin x}\,\mathrm{d}u$$

$$= \int_0^1 u^2\,\mathrm{d}u = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

Note the change of order in limits in the last line to change the sign.



Here, I've used two results from earlier in the sheet (but doubled one of them)

$$
\begin{aligned}
\int_0^{\pi} \sin^3 x \, \mathrm{d}x &= \int_0^{\pi} \sin x \left(1 - \cos^2 x\right) \, \mathrm{d}x \\
&= \int_0^{\pi} \sin x - \sin x \cos^2 x \, \mathrm{d}x \\
&= 2 - \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

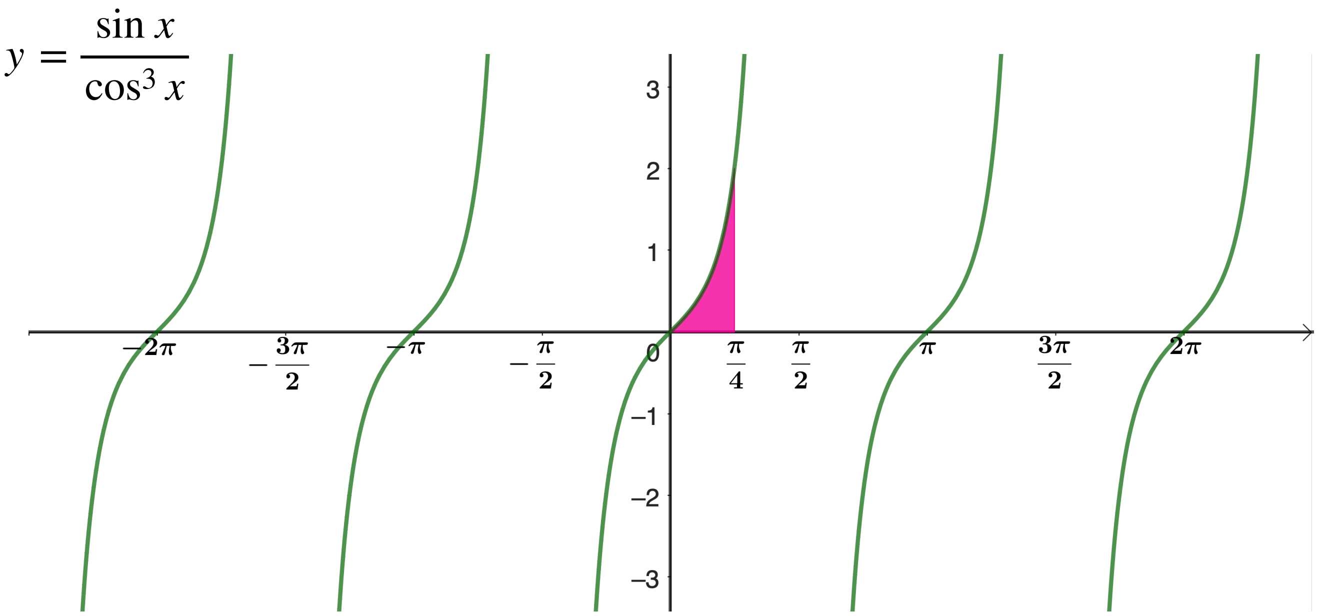
Note also that, when $0 < x < \pi$,

$0 < \sin x < 1 \Rightarrow \sin^3 x < \sin^2 x < \sin x$ which explains why

$$\int_0^{\pi} \sin^3 x \, \mathrm{d}x < \int_0^{\pi} \sin^2 x \, \mathrm{d}x < \int_0^{\pi} \sin x \, \mathrm{d}x$$

$$\text{or } \frac{4}{3} < \frac{\pi}{2} < 2$$

Find this area, and the indefinite integral:



$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-2} x = 2\cos^{-3} x \sin x$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$$

$$= \left[\frac{1}{2\cos^2 x}\right]_0^{\frac{\pi}{4}}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

And for the indefinite integral:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$$

$$= \frac{1}{2\cos^2 x} + c$$

$$= \frac{\sec^2 x}{2} + c$$

Notice, though, that $\sec^2 x = 1 + \tan^2 x$, so

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \frac{\tan^2 x}{2} + c'$$

So we know that $\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \int \tan x \sec^2 x\,\mathrm{d}x = \frac{1}{2}\tan^2 x + c$

but there are also at least three substitutions that will work:

$u = \cos x \qquad u = \sec x \qquad u = \tan x$

Here's one:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \cos x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$= \int_1^{\frac{\sqrt{2}}{2}} \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u^3}\,\mathrm{d}u$$

$$= \left[\frac{1}{2u^2}\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

And for the indefinite integral:

$$\int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int \frac{1}{u^3}\,\mathrm{d}u$$

$$= \frac{1}{2u^2} + c$$

$$= \frac{1}{2\cos^2 x} + c$$

$$= \frac{\sec^2 x}{2} + c$$

Here's another possibility:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \sec x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x$$

$$\begin{aligned}
&= \int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int \frac{\sin x}{\cos^3 x}\frac{1}{\sec x \tan x}\,\mathrm{d}u \\
&= \int \frac{1}{\cos x}\,\mathrm{d}u \\
&= \int u\,\mathrm{d}u \\
&= \frac{u^2}{2} + c \\
&= \frac{\sec^2 x}{2} + c
\end{aligned}$$

And another:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \tan x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$$

$$\begin{aligned}
&= \int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int \frac{\sin x}{\cos^3 x}\frac{1}{\sec^2 x}\,\mathrm{d}u \\
&= \int \tan x\,\mathrm{d}u \\
&= \int u\,\mathrm{d}u \\
&= \frac{u^2}{2} + c' \\
&= \frac{\tan^2 x}{2} + c'
\end{aligned}$$

Here is an integral that is quite a bit more of a challenge:

Find $\displaystyle\int \sec x \, \mathrm{d}x$ using the substitution $u = \sec x + \tan x$.

This is only for very strong students!

There are a number of substitutions that will work.

First, the standard substitution method; this suffers from the fact that it more or less relies on your knowing the answer in advance.

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \sec x \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \\ &= \int \frac{\sec x}{u \sec x} \, \mathrm{d}u \\ &= \int \frac{1}{u} \, \mathrm{d}u \\ &= \ln|u| + c \\ &= \ln|\sec x + \tan x| + c \\ &= \ln\left|\frac{1 + \sin x}{\cos x}\right| + c\end{aligned}$$

$$\begin{aligned} u &= \sec x + \tan x \\ \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= \sec x \tan x + \sec^2 x \\ &= u \sec x\end{aligned}$$

Find $\displaystyle\int \sec x \, \mathrm{d}x$ using the substitution $u = \sin x$.

Here is a version that seems like a more obvious substitution to try. It yields an answer that looks very different indeed, but turns out to be the same.

$$
\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \frac{1}{\cos x} \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \\
&= \int \frac{1}{\cos^2 x} \, \mathrm{d}u \\
&= \int \frac{1}{1 - \sin^2 x} \, \mathrm{d}u \\
&= \int \frac{1}{1 - u^2} \, \mathrm{d}u \\
&= \frac{1}{2} \int \frac{1}{1 - u} + \frac{1}{1 + u} \, \mathrm{d}u \\
&= \frac{1}{2} \Big[-\ln|1 - u| + \ln|1 + u| \Big] \\
&= \frac{1}{2} \ln \left| \frac{1 + u}{1 - u} \right| \\
&= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + c
\end{aligned}
$$

$$u = \sin x$$

$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

You now have two versions of the integral that look very different. Are they equivalent?

$$\begin{aligned}\left(\frac{1+\sin x}{\cos x}\right)^2 &= \frac{(1+\sin x)^2}{1-\sin^2 x} \\ &= \frac{(1+\sin x)^2}{(1+\sin x)(1-\sin x)} \\ &= \frac{1+\sin x}{1-\sin x} \\ \Rightarrow 2\ln|\sec x+\tan x| &= \ln\left|\frac{1+\sin x}{1-\sin x}\right|\end{aligned}$$

Yes, they are!

Strictly for extension only, the next bit!

Here is a standard substitution that is often useful for integration of awkward circular functions, but that your students may well not have come across. You will find two articles on this substitution on my website. Again, the answer looks quite different to the previous two, and again they all turn out to be equivalent.

$$t = \tan\frac{x}{2}$$

$$\Rightarrow \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$$

$$\Rightarrow \cos x = \frac{1-t^2}{1+t^2} \text{ and } \sin x = \frac{2t}{1+t^2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1+t^2}{2}$$

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

$$\begin{aligned}
\int \sec x \,\mathrm{d}x &= \int \sec x \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{1-t^2}\,\mathrm{d}t \\
&= \int \frac{1}{1-t} + \frac{1}{1+t}\,\mathrm{d}t \\
&= -\ln(1-t) + \ln(1+t) + c \\
&= \ln\frac{1+t}{1-t} + c \\
&= \ln\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + c
\end{aligned}$$

This expression is also equivalent to the other versions:

$$
\begin{aligned}
\sec x + \tan x &= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \\
&= \frac{1+2t+t^2}{1-t^2} \\
&= \frac{(1+t)^2}{(1-t)(1+t)} \\
&= \frac{1+t}{1-t} \\
\Rightarrow \ln|\sec x + \tan x| &= \ln\left|\frac{1+t}{1-t}\right|
\end{aligned}
$$

There are a few other variations of the result. For example:

$$
\begin{aligned}
\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) &= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}} \\
&= \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}
\end{aligned}
$$

so we can also write this as

$$
\int \sec x \, \mathrm{d}x = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|
$$

And here are a couple of other versions that I've included just for fun!

$$\ln \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} = \ln \frac{1+\frac{\sin x}{1+\cos x}}{1-\frac{\sin x}{1+\cos x}}$$

$$= \ln \frac{1+\cos x+\sin x}{1+\cos x-\sin x}$$

$$\frac{\sin x}{1+\cos x} = \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$= t$$

$$\ln \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} = \ln \frac{1+\frac{1-\cos x}{\sin x}}{1-\frac{1-\cos x}{\sin x}}$$

$$= \ln \frac{\sin x-\cos x+1}{\sin x+\cos x-1}$$

Early in your students' mathematical careers, they will have happily use the $\sin^{-1}$ key on their calculators to find angles in right-angled triangles when they know side lengths. This is probably one of their earliest encounters with the idea of the inverse of a function, although of course it is not framed this way.

At the start of our work on circular functions, we spent plenty of time separating these functions from right-angled triangles using the unit circle. From there, we generated graphs of the circular functions. It's time to look at a similar process for their inverses. In right-angled triangles, we are only dealing with angles between 0 and $\frac{\pi}{2}$. For each ratio of two sides, there is only one possible angle, so in this range, the inverse functions are well defined. However, when we look at angles outside this range, it is not immediately clear what the inverses of the functions might be. For example:

$\sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = \sin\frac{17\pi}{6} = \ldots = \frac{1}{2}$.

So is $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$ or $\frac{17\pi}{6}$ or … ?

and is $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ or $\frac{11\pi}{6}$ or $-\frac{5\pi}{6}$ or $\frac{7\pi}{6}$ or … ?

Similar questions arise for the inverses of the other circular functions.

The answer is, in general, that we make the simplest possible choice, so, for example:

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6} \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\cos^{-1}\frac{1}{2} = \frac{\pi}{3} \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6} \qquad \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

In other words, for $\sin^{-1}$ and $\tan^{-1}$ we will always choose a value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, whereas for $\cos^{-1}$ we will choose a value between 0 and π.

In the language of functions, we can say:

$$\sin^{-1} : \mathbb{R} \to \left\{ y \middle| -\frac{\pi}{2} \le y \le \frac{\pi}{2} \right\}$$

$$\tan^{-1} : \mathbb{R} \to \left\{ y \middle| -\frac{\pi}{2} \le y \le \frac{\pi}{2} \right\}$$

$$\cos^{-1} : \mathbb{R} \to \left\{ y \middle| 0 \le y \le \pi \right\}$$

Actually, for sin and $\sin^{-1}$ to be inverses of each other, the domain of each has to be the range of the other. This means that the inverse of $\sin^{-1}$ is not sin, but the restriction of sin to the domain $\left\{ x \,\middle|\, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right\}$. This is really a technicality too far at this stage in your students mathematical careers, but it could possibly come up in questions!

One way to approach this with your class would be to tell them all this, and then move on to some questions. But if you want your class to have a deeper insight into the inner workings of these inverse functions, you might consider thinking about them from the point of view of unit circles (the first part of this worksheet) or of graphs (the second part). Or even both!

Whatever you decide, there is still plenty in this worksheet to explore once you have your definitions in place.

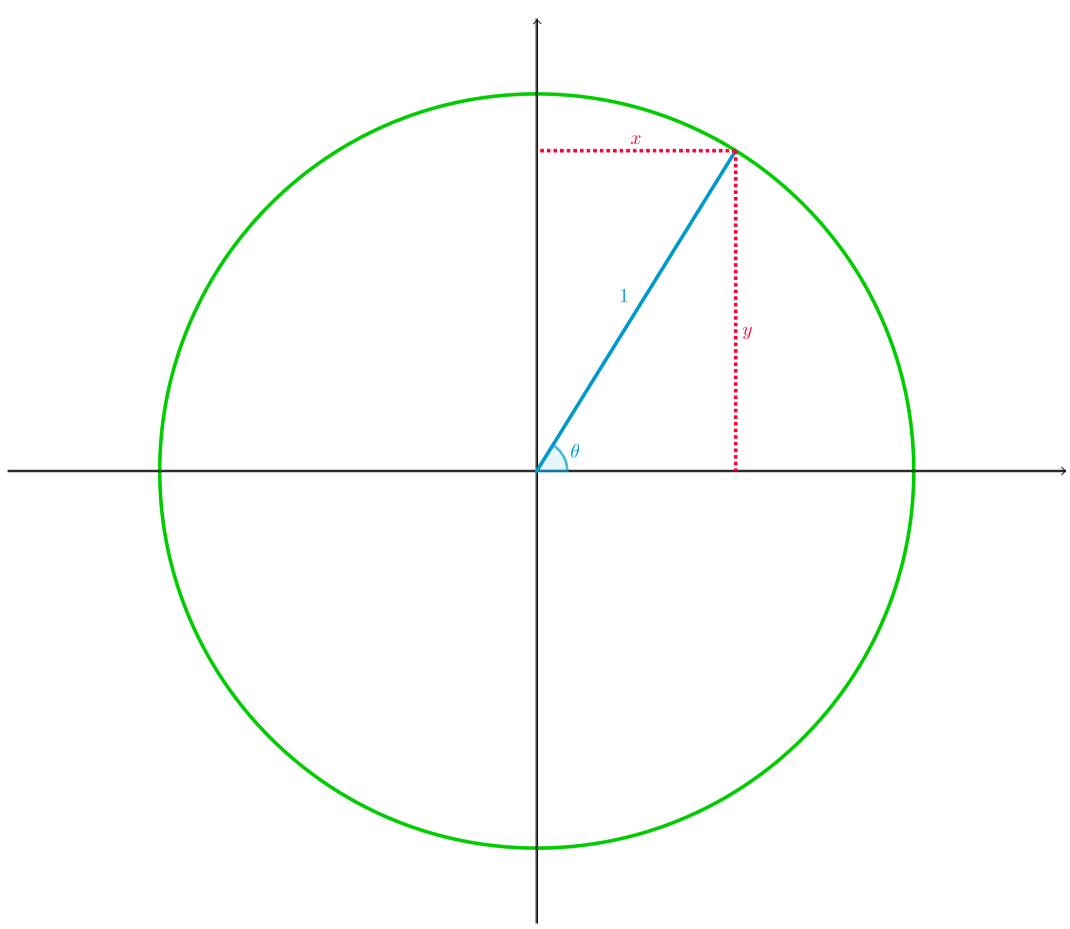
Inverse circular functions

Suppose $0 \leq \theta \leq \frac{\pi}{2}$.

What is θ in terms of x?

What is θ in terms of y?

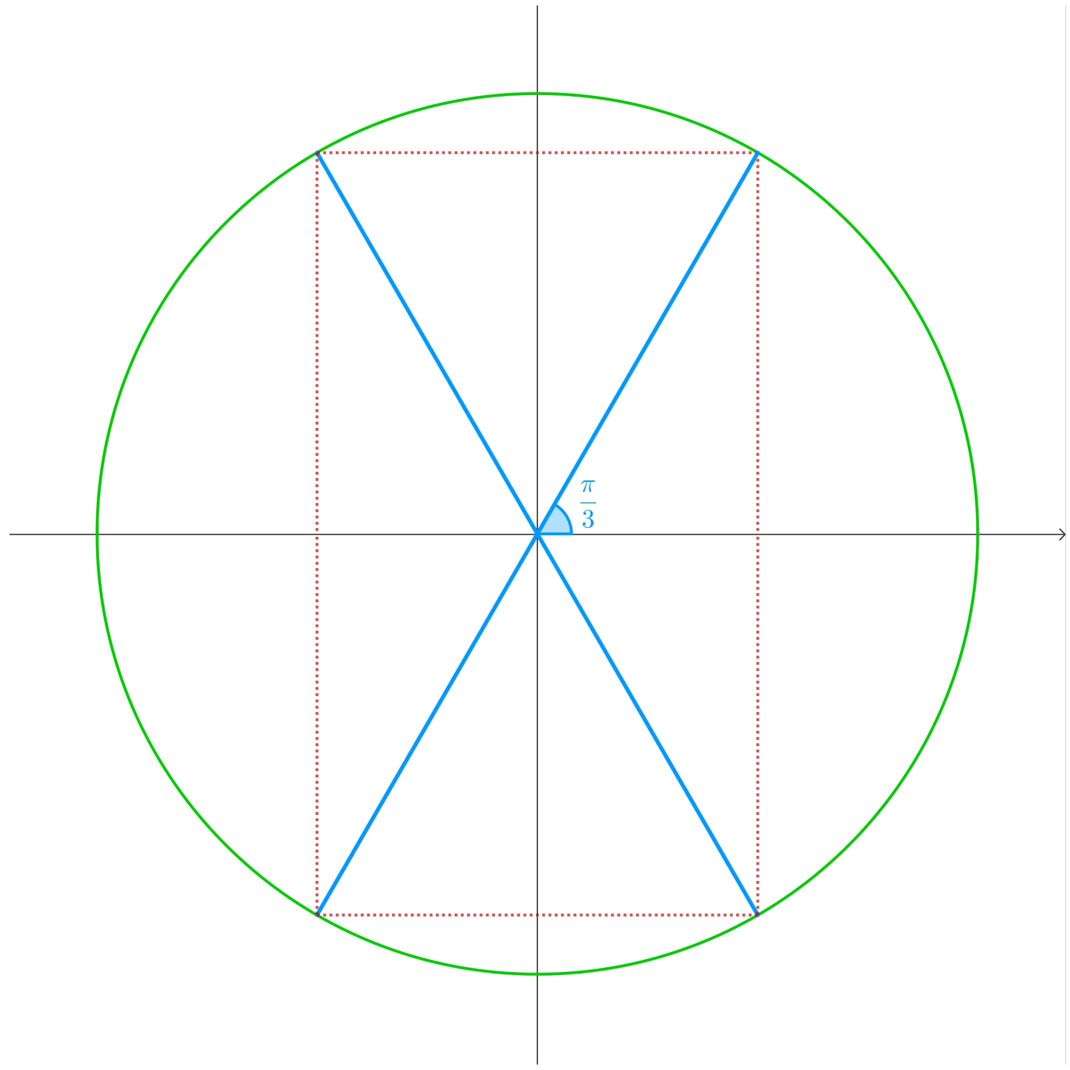
What is θ in terms of $\frac{y}{x}$?



$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x}$$

$$\Rightarrow \theta = \cos^{-1} x = \sin^{-1} y = \tan^{-1} \frac{y}{x}$$

So long as $0 \leq \theta \leq \frac{\pi}{2}$, there is only one value of θ that makes $x = \cos\theta$, $y = \sin\theta$, and $\frac{y}{x} = \tan\theta$. So we know exactly what angle we mean by $\cos^{-1} x$, $\sin^{-1} y$, or $\tan^{-1} \frac{y}{x}$.



What are:

$$\sin\frac{\pi}{3} \qquad \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} \qquad \cos^{-1}\frac{1}{2}$$

$$\tan\frac{\pi}{3} \qquad \tan^{-1}\sqrt{3}$$

$$\sin\left(-\frac{\pi}{3}\right) \qquad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\cos\left(-\frac{\pi}{3}\right) \qquad \cos^{-1}\frac{1}{2}$$

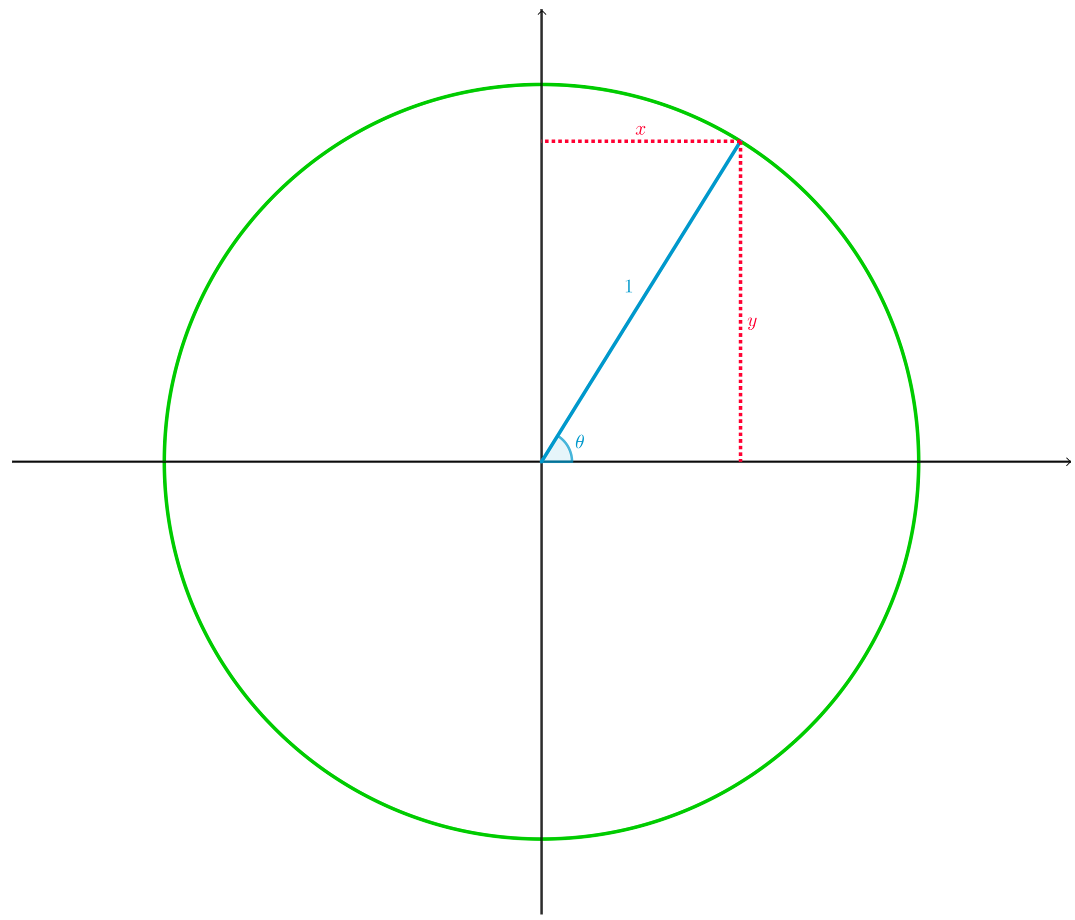
$$\tan\left(-\frac{\pi}{3}\right) \qquad \tan^{-1}\left(-\sqrt{3}\right)$$

$$\sin\frac{2\pi}{3} \qquad \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\cos\frac{2\pi}{3} \qquad \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\tan\frac{2\pi}{3} \qquad \tan^{-1}\left(-\sqrt{3}\right)$$

$\sin \frac{4\pi}{3}$	$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
$\cos \frac{4\pi}{3}$	$\cos^{-1}\left(-\frac{1}{2}\right)$
$\tan \frac{4\pi}{3}$	$\tan^{-1}\sqrt{3}$
$\sin \frac{5\pi}{3}$	$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
$\cos \frac{5\pi}{3}$	$\cos^{-1}\frac{1}{2}$
$\tan \frac{5\pi}{3}$	$\tan^{-1}\left(-\sqrt{3}\right)$
$\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$	$\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$
$\cos^{-1}\left(\cos \frac{2\pi}{3}\right)$	$\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
$\tan^{-1}\left(\tan \frac{2\pi}{3}\right)$	$\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$
$\sin^{-1}\left(\cos \frac{\pi}{3}\right)$	$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$
$\sin^{-1}\left(\cos \frac{4\pi}{3}\right)$	$\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
$\cos^{-1}\left(\sin \frac{\pi}{3}\right)$	$\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$
$\cos^{-1}\left(\sin \frac{4\pi}{3}\right)$	$\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

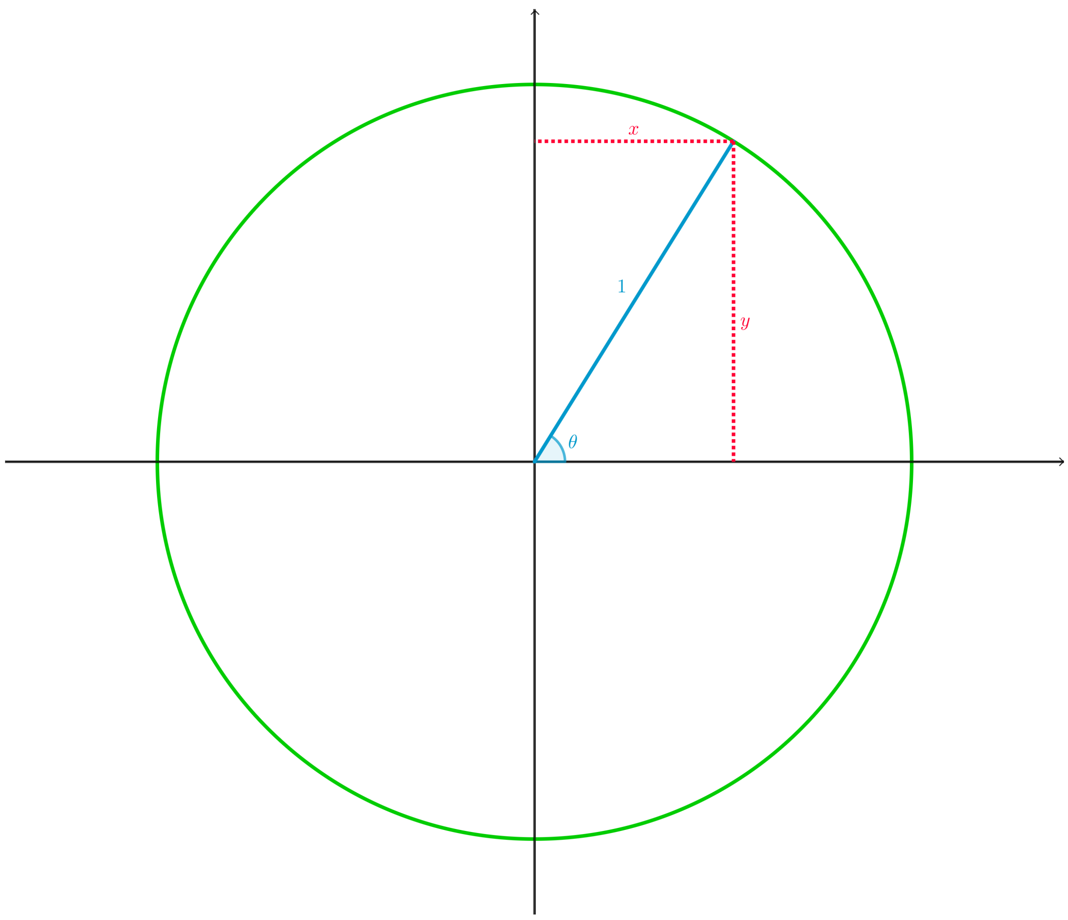


What is $\sin\left(\cos^{-1} x\right)$ in terms of x ?

What is $\cos\left(\sin^{-1} y\right)$ in terms of y ?

What happens when x or y is negative?

$$
\begin{aligned}
& x = \cos\theta \quad y = \sin\theta \\
\Rightarrow\ & \theta = \cos^{-1} x = \sin^{-1} y \\
& \sin\left(\cos^{-1} x\right) = \sin\theta = y = \sqrt{1-x^2} \\
& \cos\left(\sin^{-1} y\right) = \cos\theta = x = \sqrt{1-y^2}
\end{aligned}
$$



What is $\sin^{-1} x$ in terms of θ ?

What is $\sin^{-1}(\cos\theta)$ in terms of θ ?

What is $\cos^{-1} y$ in terms of θ ?

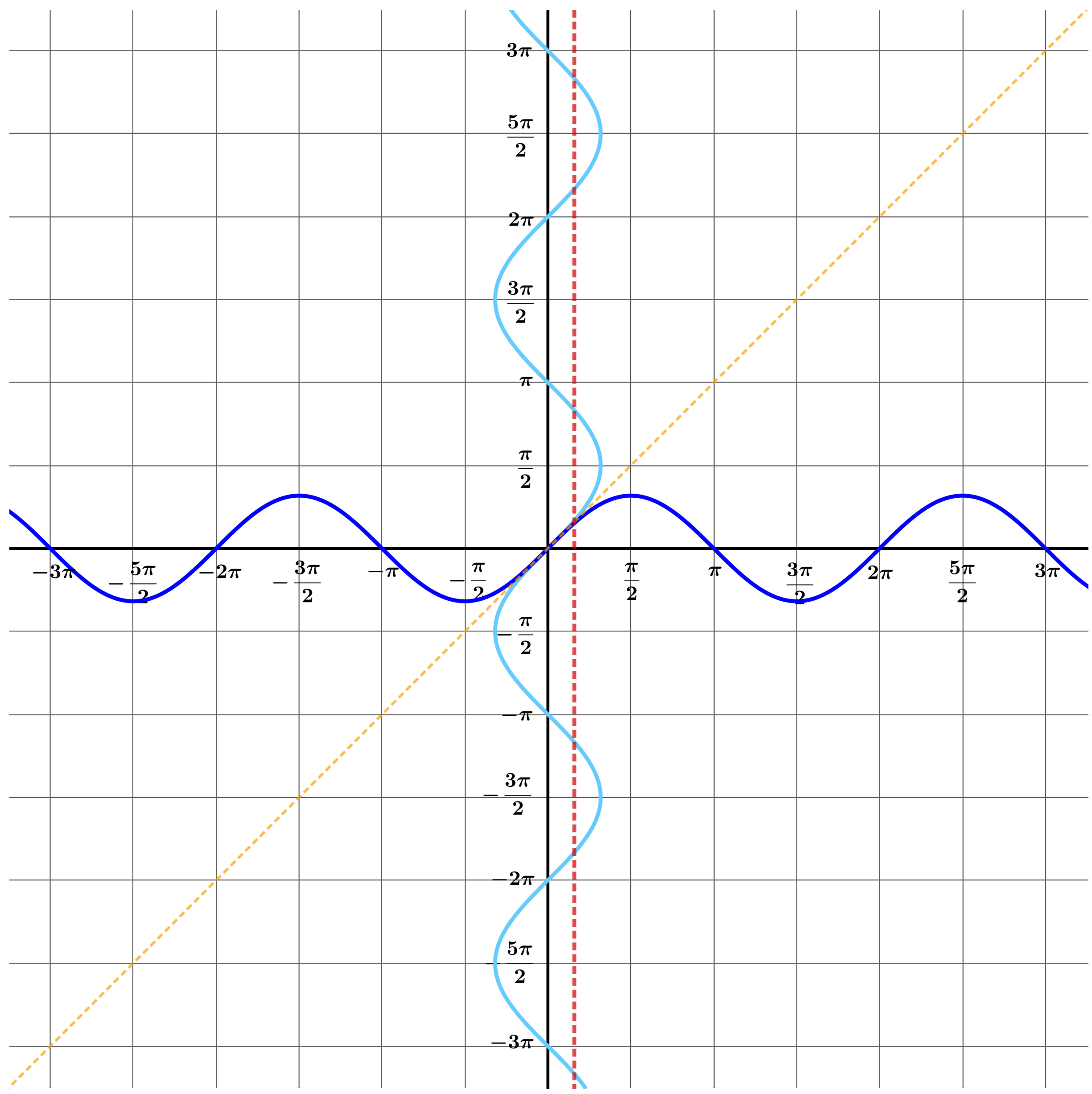
What is $\cos^{-1}(\sin\theta)$ in terms of θ ?

What happens when x or y is negative?

$$\sin^{-1}(\cos\theta) = \sin^{-1} x = \frac{\pi}{2} - \theta \; \cos^{-1}(\sin\theta) = \sin^{-1} y = \frac{\pi}{2} - \theta$$

Next, we'll look at the whole thing again from the point of view of graphs.

Solve the equation $\sin y = \dfrac{1}{2}$.



How does your answer relate to this graph?

How many values do you want for $\sin^{-1} \dfrac{1}{2}$?

How many times do you want the line $x = \dfrac{1}{2}$ to intersect with the graph $y = \sin^{-1} x$?

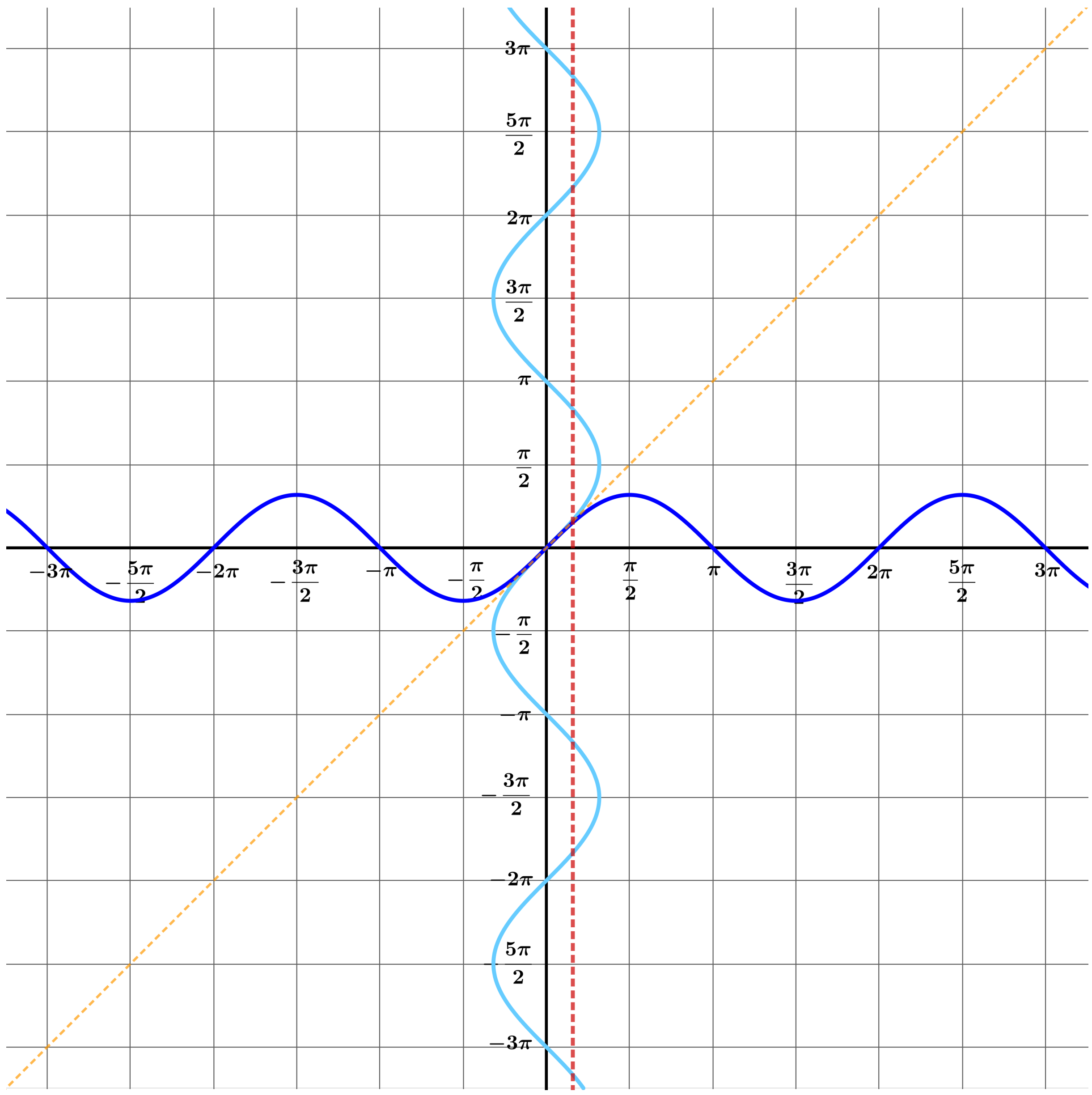
$\sin y = \frac{1}{2}$ has solutions where the graphs $x = \sin y$ and $x = \frac{1}{2}$ intersect. That is,

$$y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ \ldots \ -\frac{7\pi}{6}, \ -\frac{11\pi}{6}, \ -\frac{19\pi}{6}, \ \ldots$$

But only one of these can be the value of $\sin^{-1}\frac{1}{2}$, because, like any function, the function $y = \sin^{-1} x$ must be well-defined (that is, uniquely defined) for all values of x in its domain.

That means that the line $x = \frac{1}{2}$ must intersect with the graph $y = \sin^{-1} x$ exactly once.

How can you adapt the graph $x = \sin y$ to ensure that any vertical line between $x = -1$ and $x = 1$ intersects it exactly once?

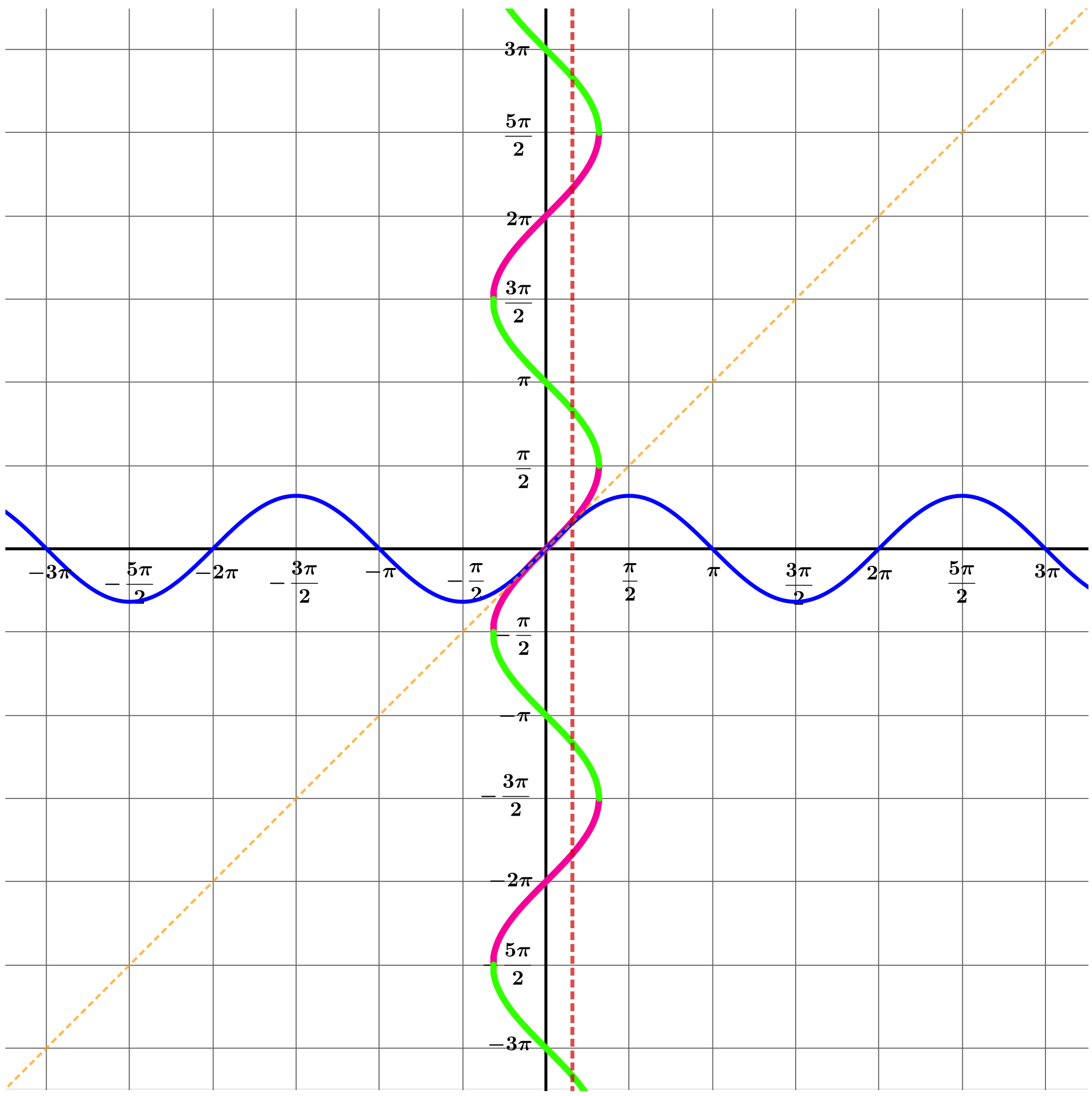


We need to choose just a small part of the curve $x = \sin y$, making sure that it is big enough to intersect every vertical line between $x = -1$ and $x = 1$, but small enough to make sure that no such vertical line intersects this section of the curve more than once.

There are many such segments . . . an infinite number, in fact.

Which of these pink or green segments on the curve would correspond to a possible definition of the function $\sin^{-1} x$?

Which would be the most sensible to choose?



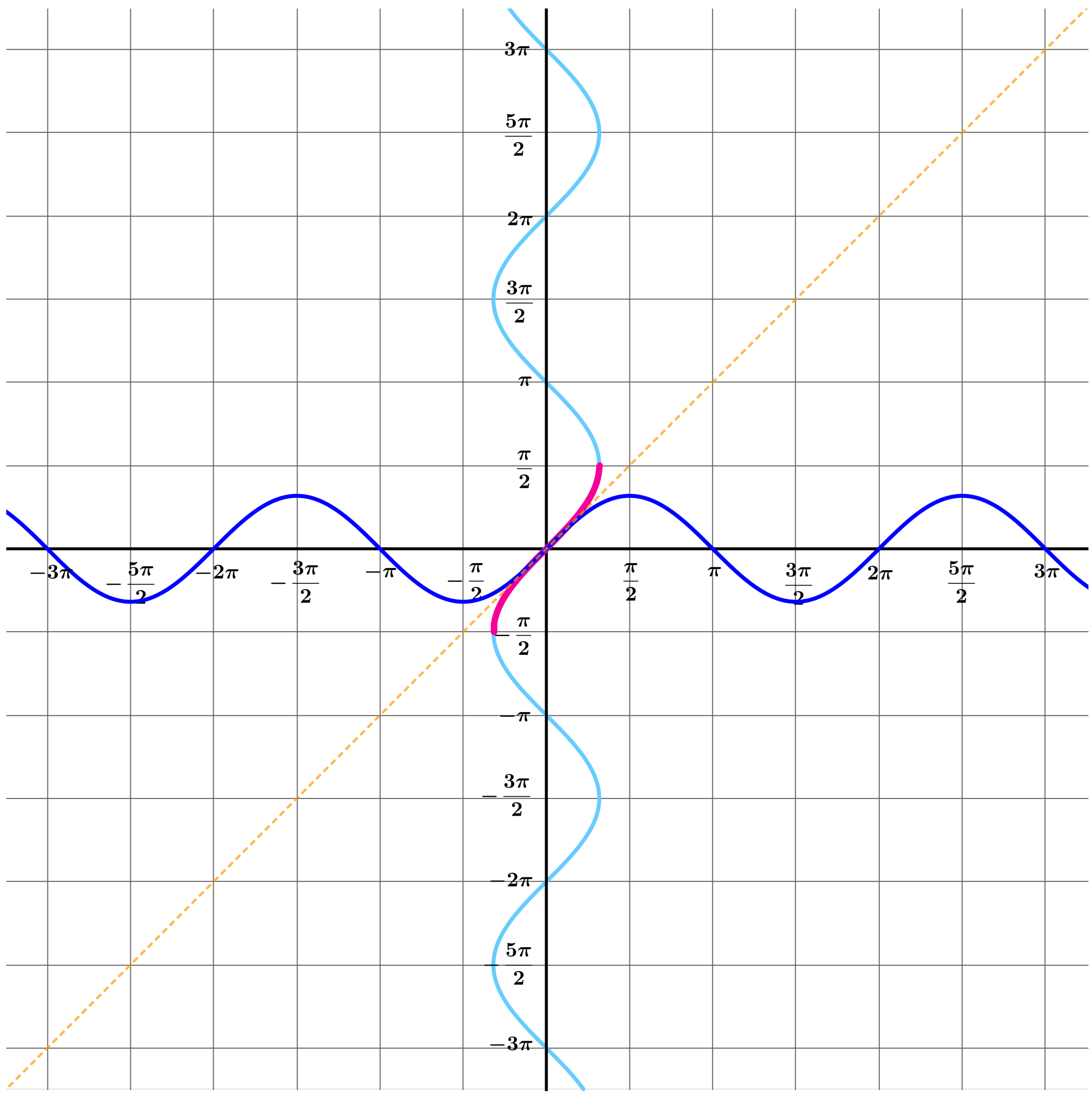
The graph $x = \sin y$ is not the graph of any function, because the y coordinate associated with each value of x is not unique.

$f(x) = \sin^{-1} x$, however, is a function, so it's value for a given value of x must be well-defined (that is, defined without ambiguity).

So to graph the function $f(x) = \sin^{-1} x$, we must make sure that there is only one value of y associated for each value of x between -1 and 1.

We could choose any coloured section of the graph $x = \sin y$, but it would be strange (though not impossible) to choose any segment other than the one that would mean $\sin^{-1} 0 = 0$.

Here it is: the graph $y = \sin^{-1} x$.



What are its domain and range?

domain:

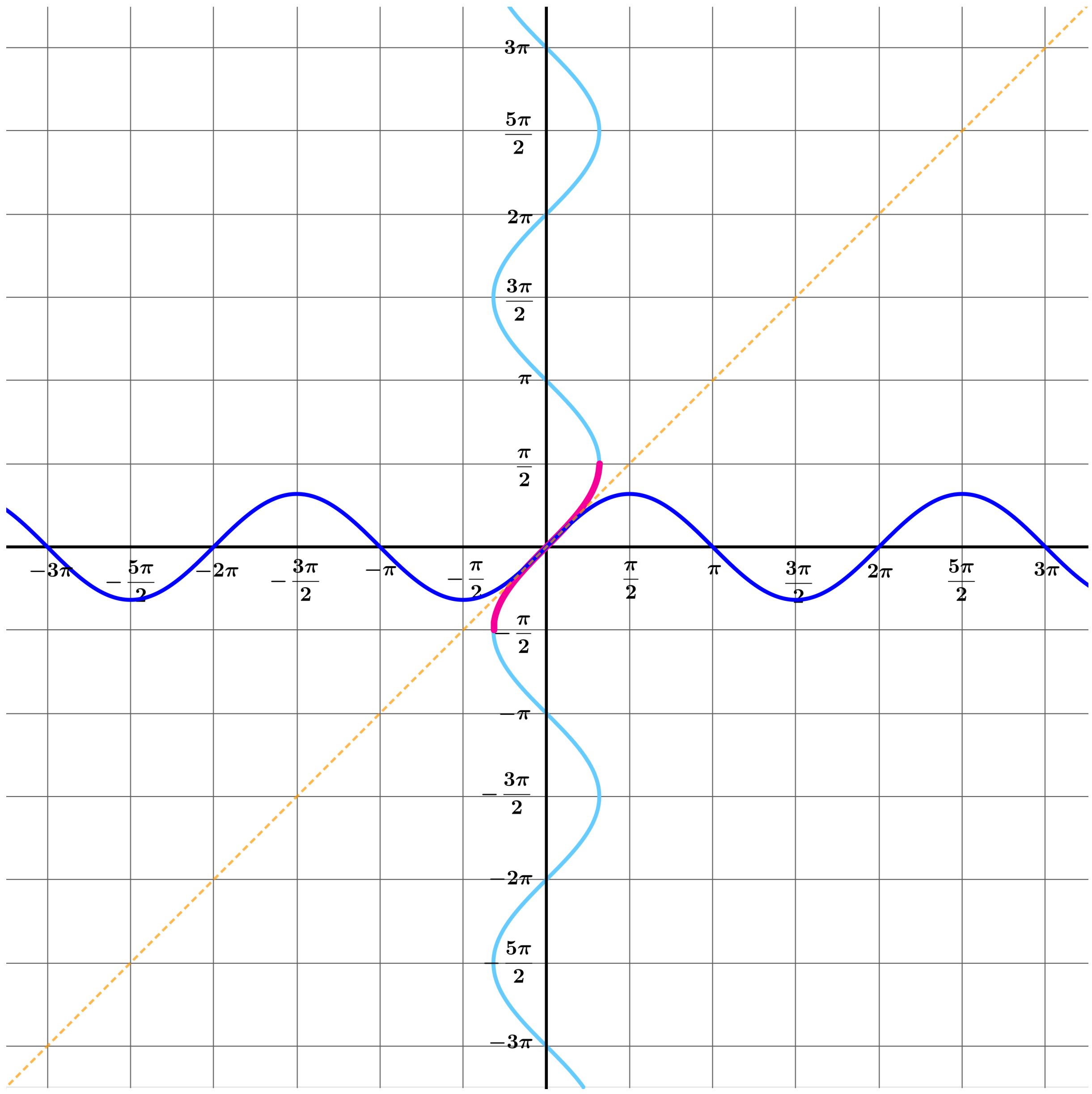
$\{x : -1 \leq x \leq 1\}$

range:

$\left\{y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

Since these are sets, there is no particular reason to choose the letters x and y. Any letters would do. But using x and y help me to keep things straight in my mind, and may help your students, too.

What are $\sin\left(\sin^{-1}\dfrac{1}{2}\right)$ and $\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$?



$\sin\left(\sin^{-1}\dfrac{1}{2}\right) = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$ and $\sin^{-1}\left(\sin\dfrac{\pi}{6}\right) = \sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

Since we want sin and $\sin^{-1}$ are inverses of each other, we would like

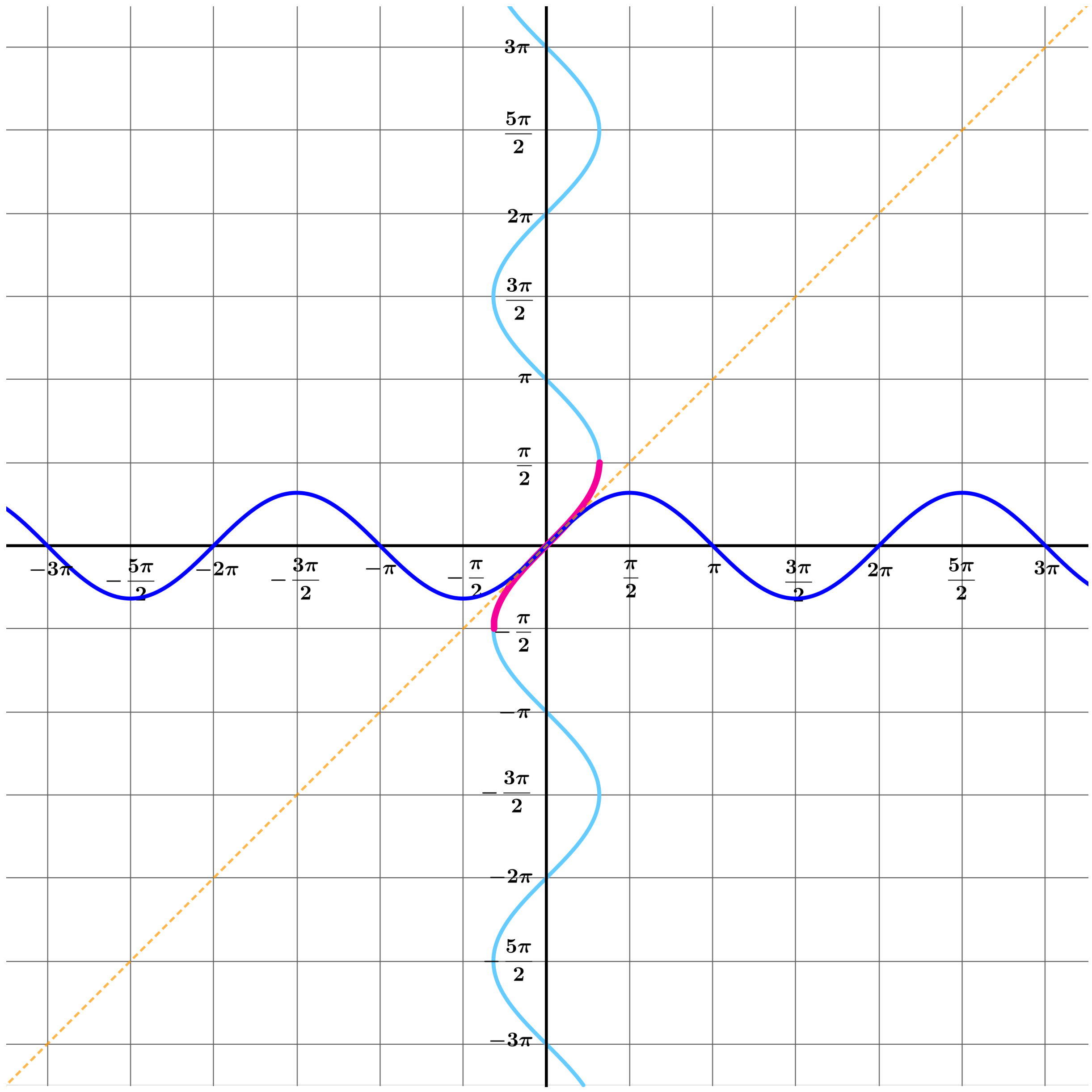
$\sin\left(\sin^{-1}x\right) = x$ and $\sin^{-1}\left(\sin x\right) = x$

for every value of x. And this is true, so long as the domain of one is the range of the other, and vice versa. The range of sin is $\{y : -1 \leq y \leq 1\}$ which is also the domain of $\sin^{-1}$.

However, the sin has domain $\mathbb{R}$ whereas $\sin^{-1}$ has range $\left\{y : -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}\right\}$.

So for the two functions to be inverses of each other, we would technically need to restrict the function sin to the domain $\left\{x : -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}\right\}$.

What are $\sin\left(\sin^{-1}(-1)\right)$ and $\sin^{-1}\left(\sin\frac{3\pi}{2}\right)$?



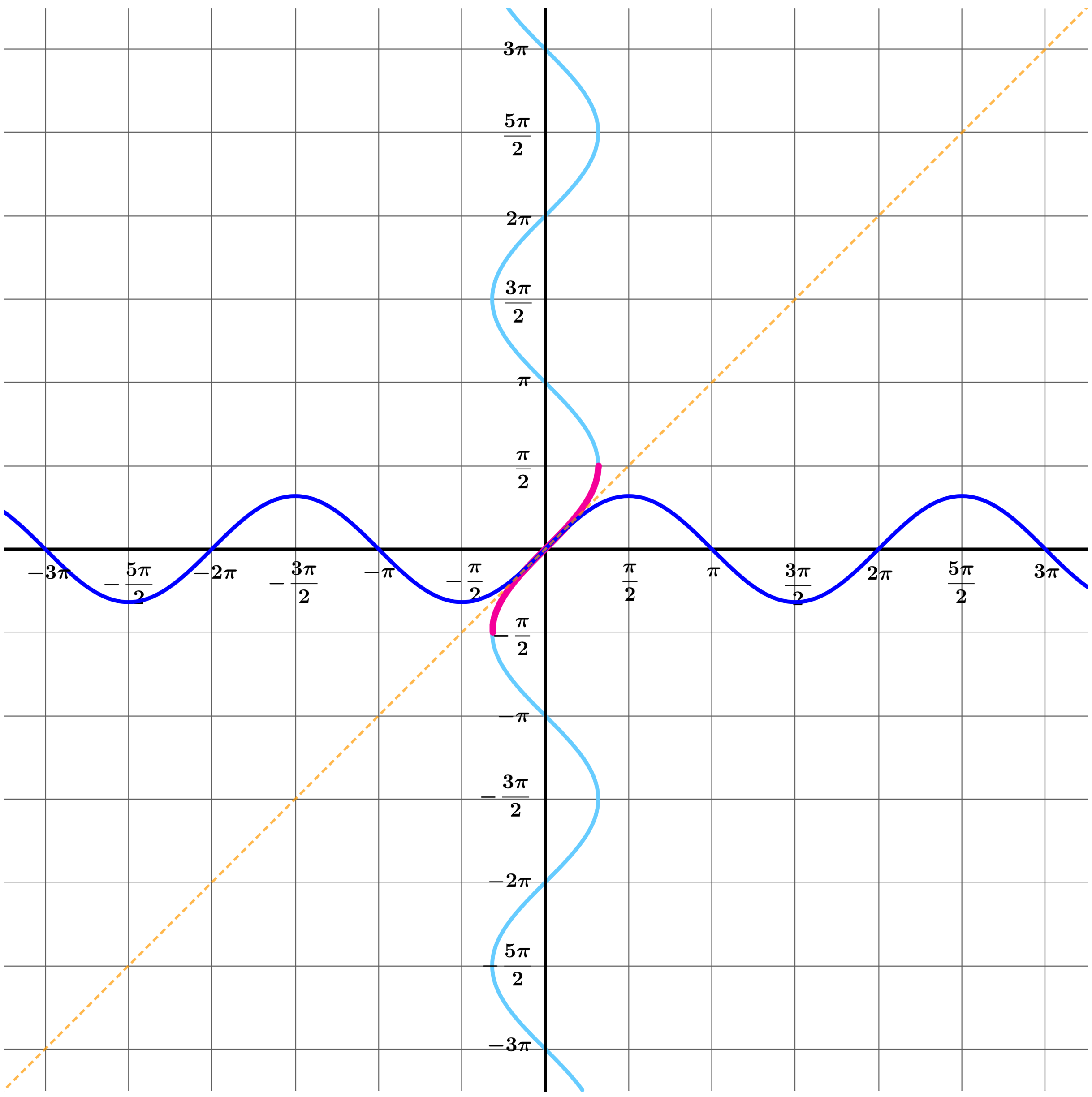
$\sin\left(\sin^{-1}(-1)\right) = \sin\left(-\frac{\pi}{2}\right) = -1$ as expected, but

$\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$

Remember that the equation $\sin\theta = -1$ has multiple solutions, and we need the one that is in

$\left\{\theta : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

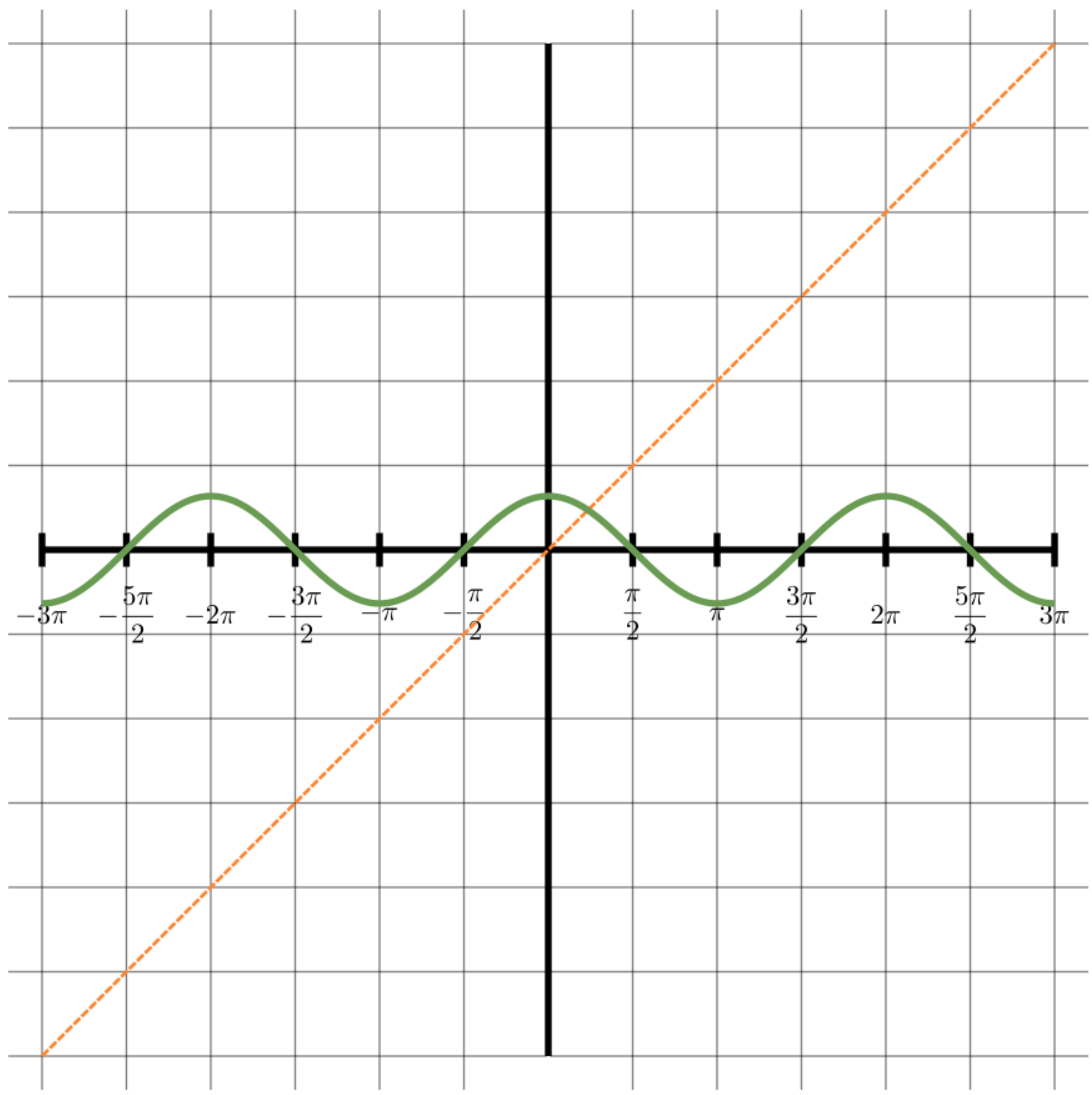
What are $\sin\left(\sin^{-1} x\right)$ and $\sin^{-1}\left(\sin x\right)$?



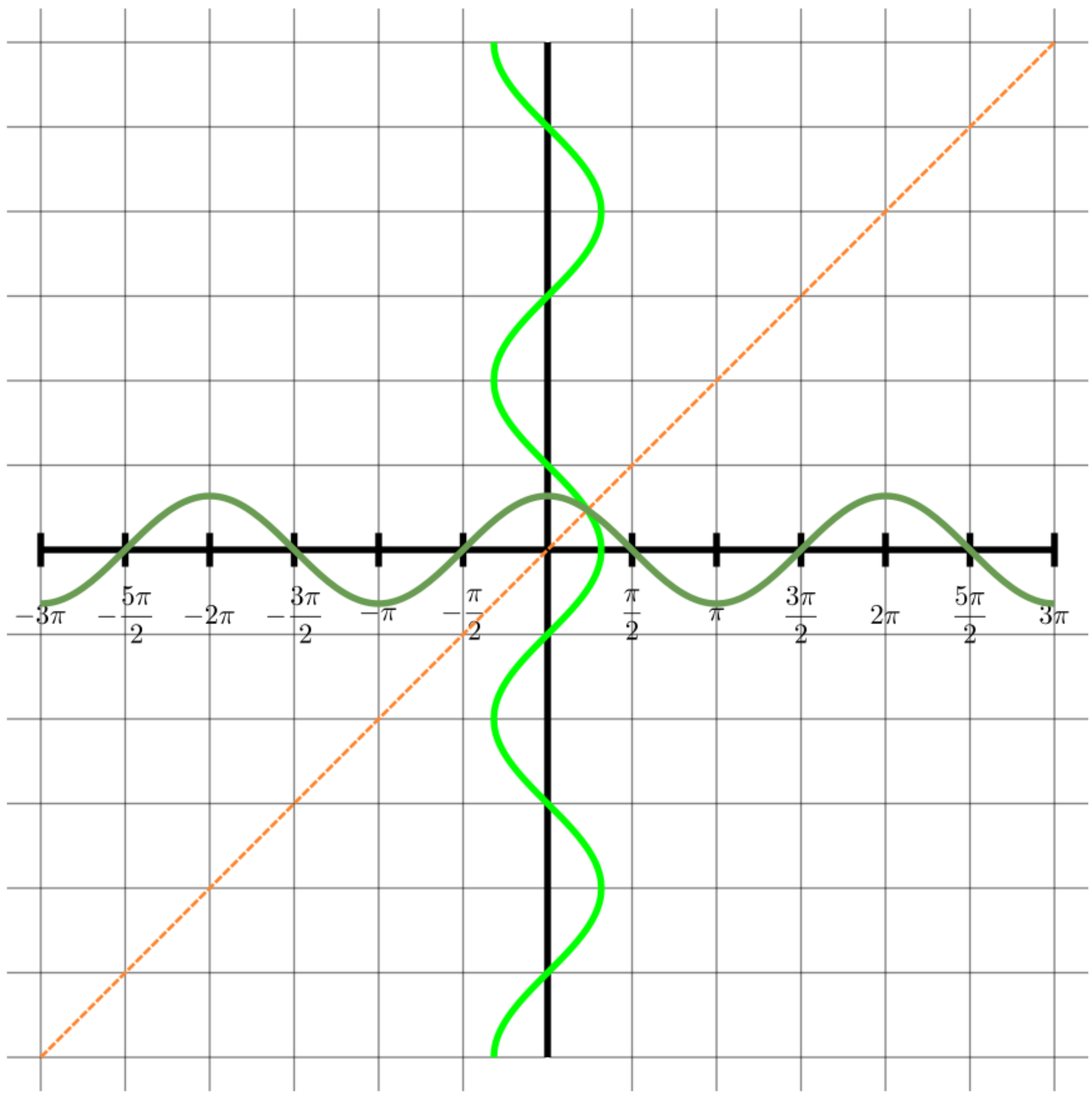
$\sin\left(\sin^{-1} x\right) = x$, but

$\sin^{-1}\left(\sin x\right) = x +$ an integer multiple of 2π where the multiple (possibly negative) is chosen to get a value in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

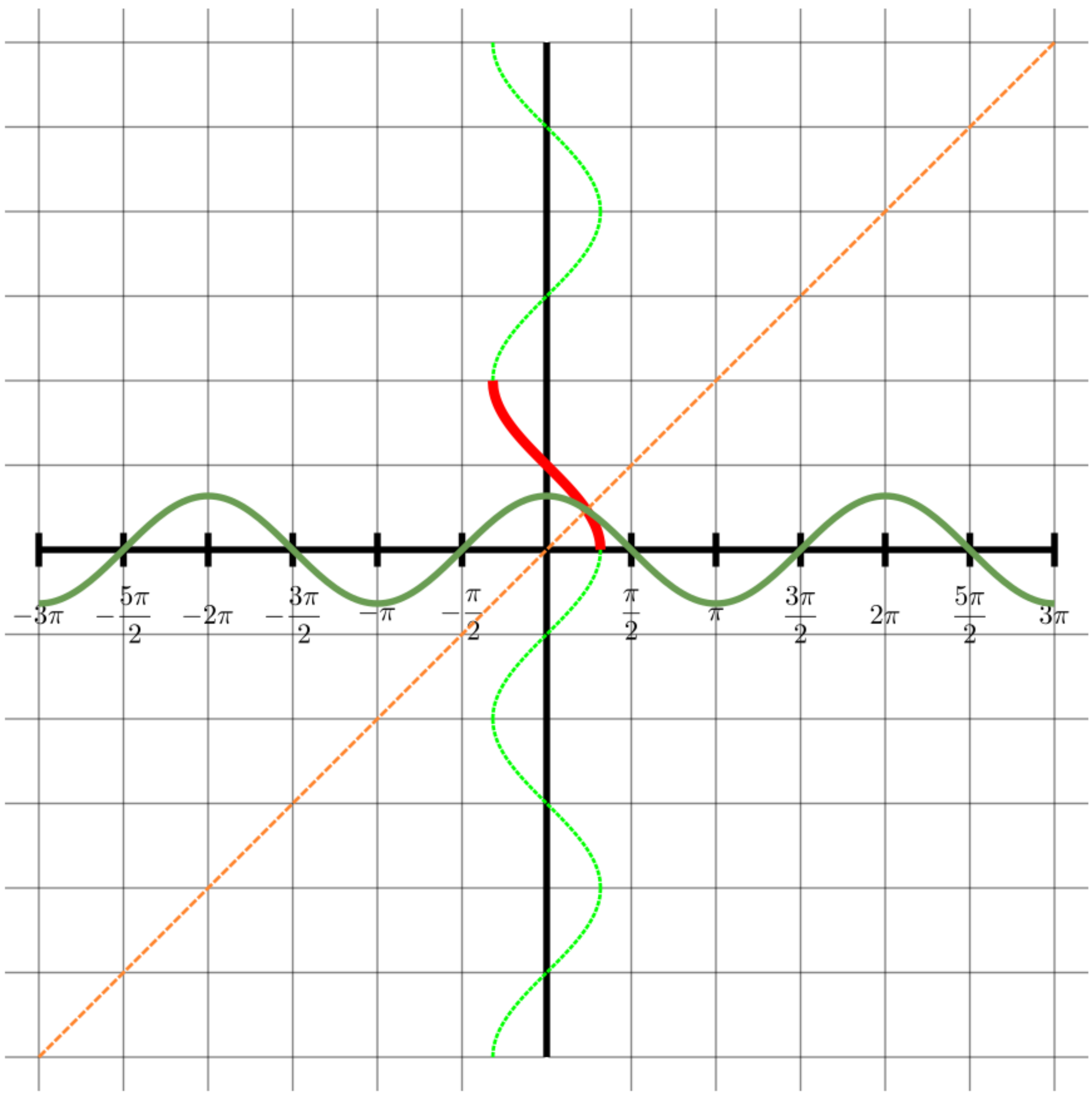
Draw the graph $x = \cos y$.



On the same axes, draw the graph $y = \cos^{-1} x$.



What are the domain and range of the function $f(x) = \cos^{-1} x$



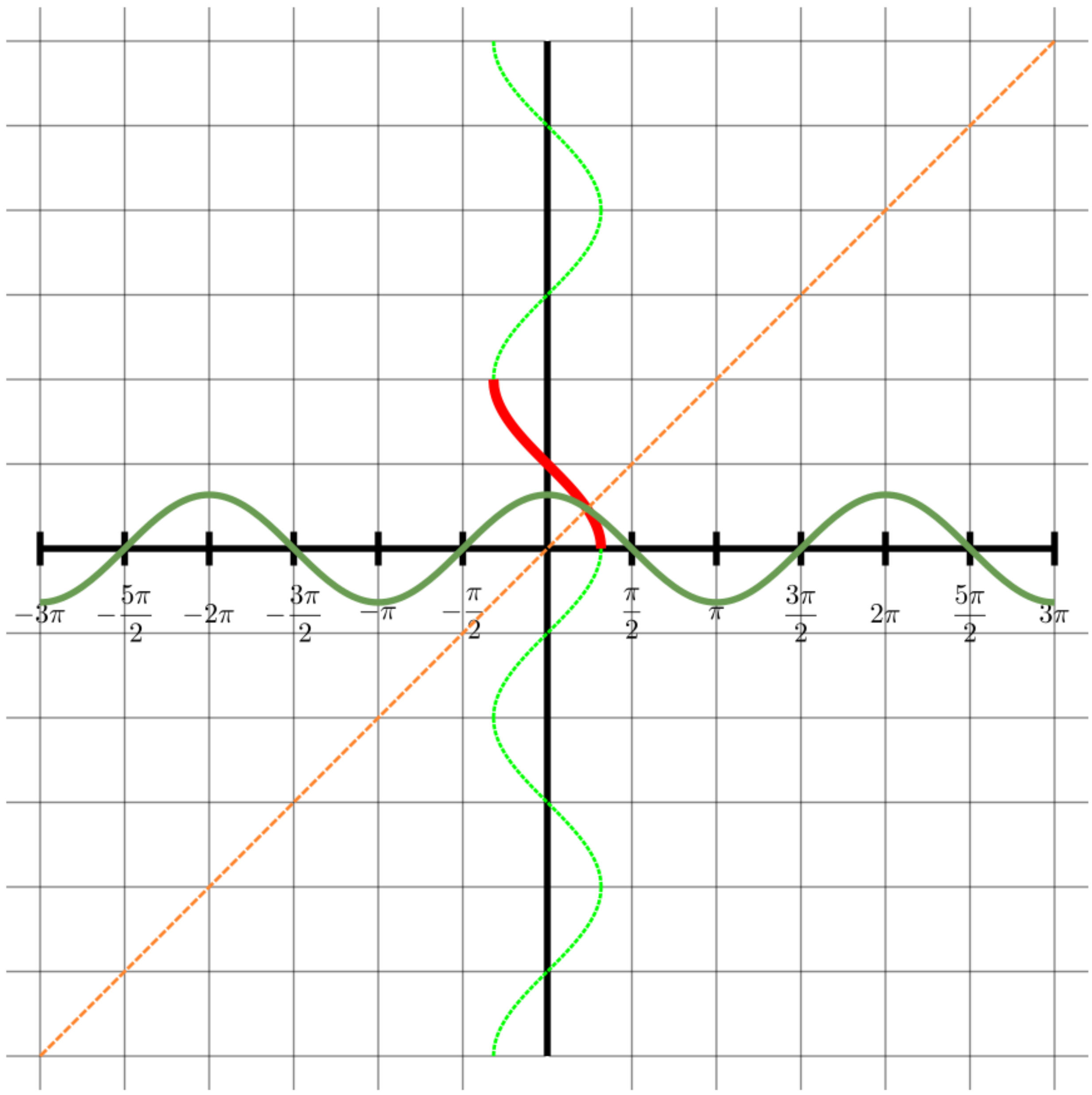
domain:

$\{x : -1 \leq x \leq 1\}$

range:

$\{y : 0 \leq y \leq \pi\}$

What are $\cos\left(\cos^{-1} x\right)$ and $\cos^{-1}(\cos x)$?



Similar considerations apply here to those from $\sin^{-1}$.

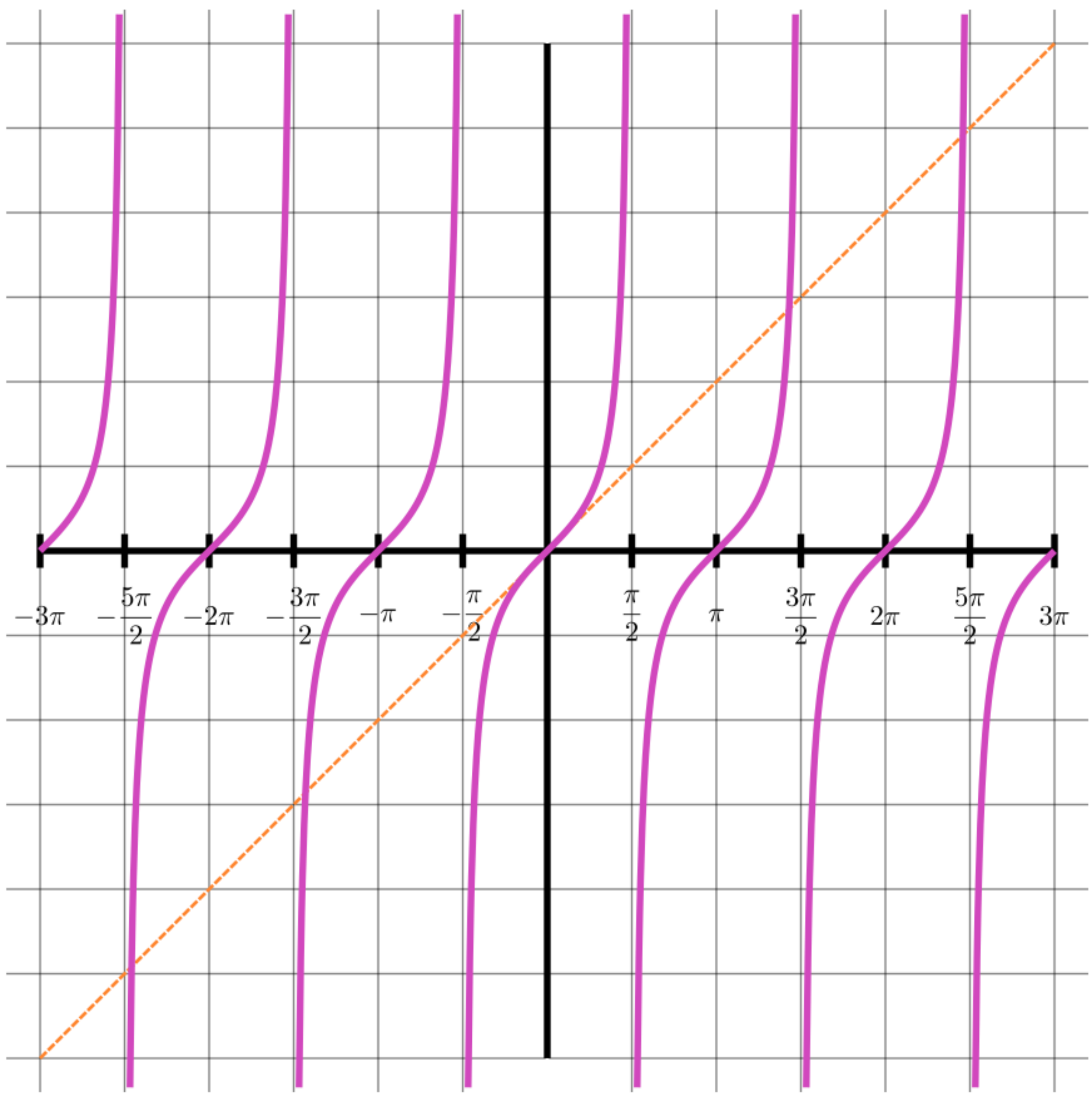
From the graph, it's pretty clear that $\cos\left(\cos^{-1} x\right) = x$.

But, for example, if $x = -3\pi$, then

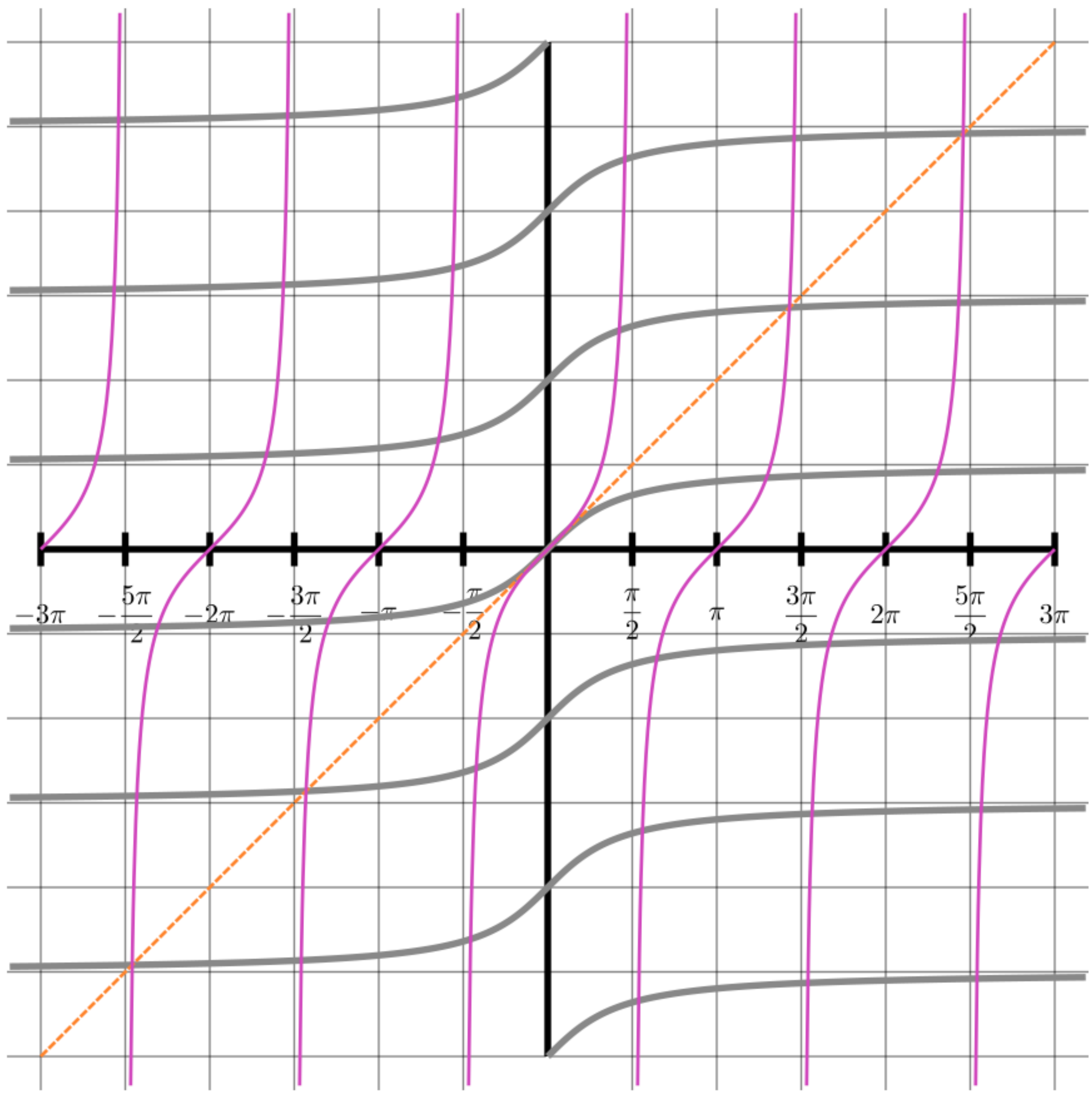
$$\cos^{-1}(\cos x) = \cos^{-1}(-1) = \pi = x + 4\pi.$$

So $\cos\left(\cos^{-1} x\right) = x + 2n\pi$ where $n \in \mathbb{Z}$ is chosen to get a value in the range $\left[0, \pi\right]$.

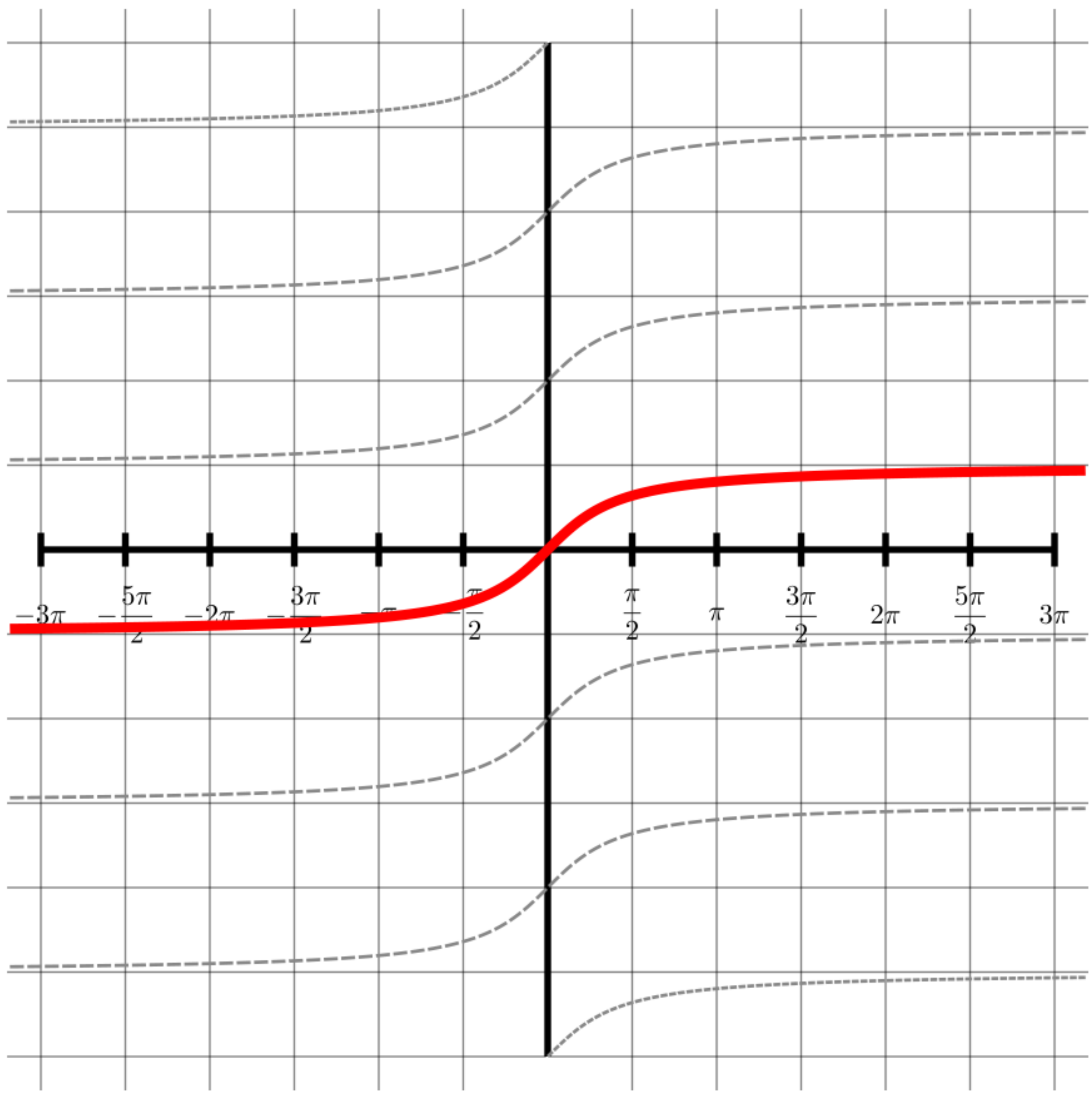
Draw the graph $x = \tan y$.



On the same axes, draw the graph $y = \tan^{-1} x$.



What are the domain and range of the function $f(x) = \tan^{-1} x$



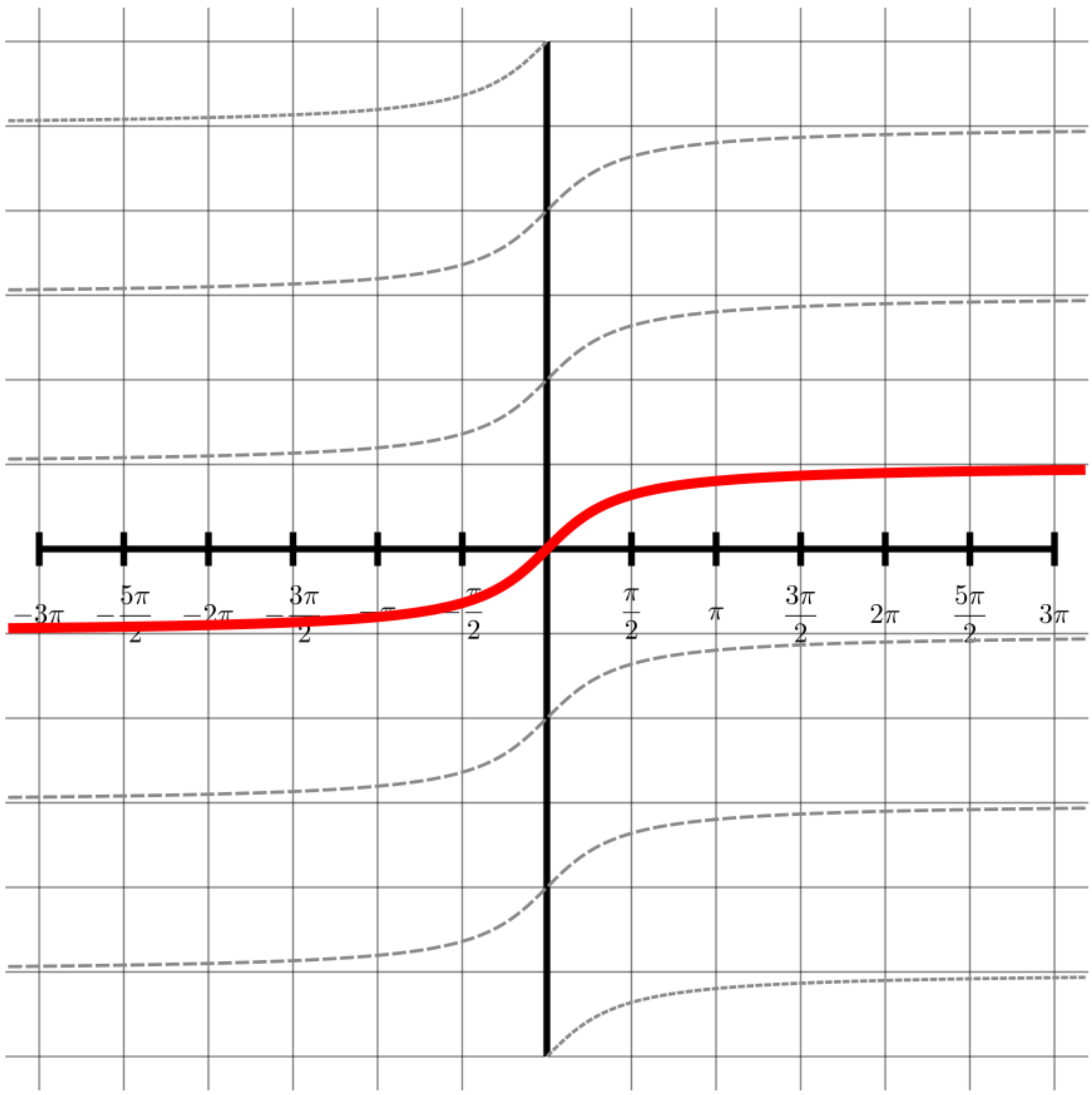
domain:

$\mathbb{R}$

range:

$\left\{ y : -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}$

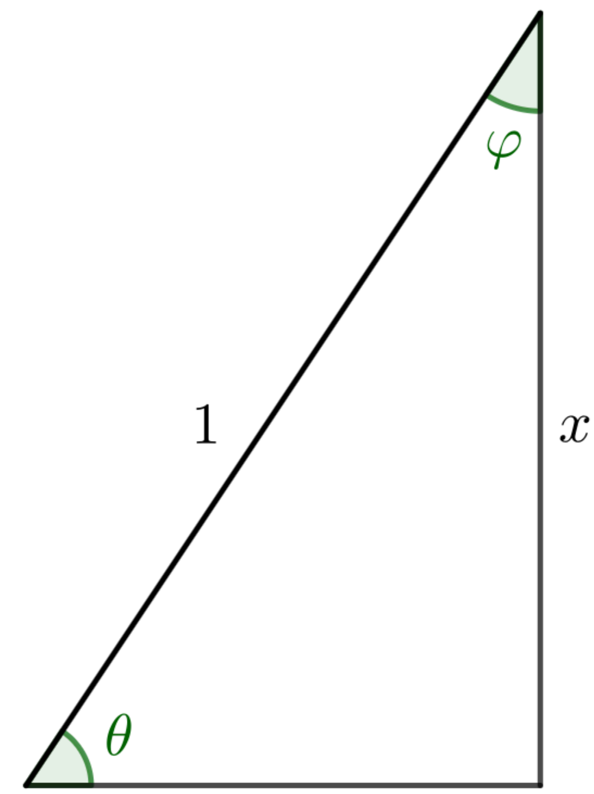
What are $\tan\left(\tan^{-1} x\right)$ and $\tan^{-1}(\tan x)$?



$\tan\left(\tan^{-1} x\right) = x.$

$\tan^{-1}(\tan x) = x + 2n\pi$ where $n \in \mathbb{Z}$ is chosen to get a value in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

What is $\sin^{-1} x + \cos^{-1} x$?

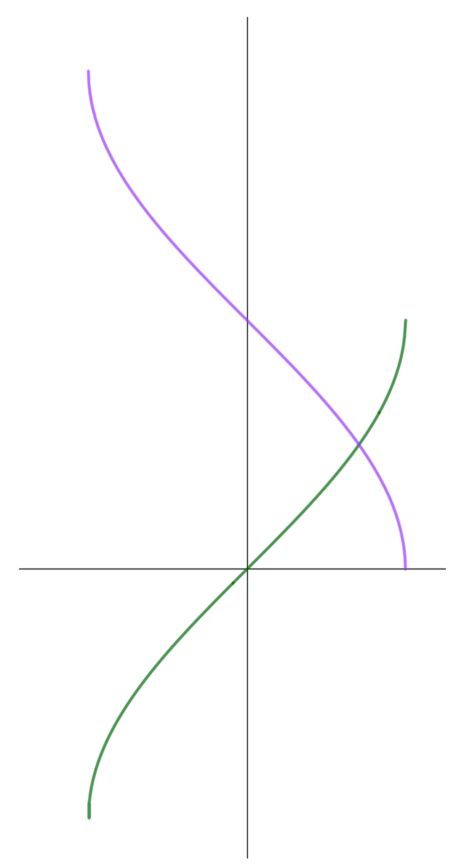


From the diagram:

$\sin^{-1} x + \cos^{-1} x = \theta + \varphi = \dfrac{\pi}{2}$ when x is positive.

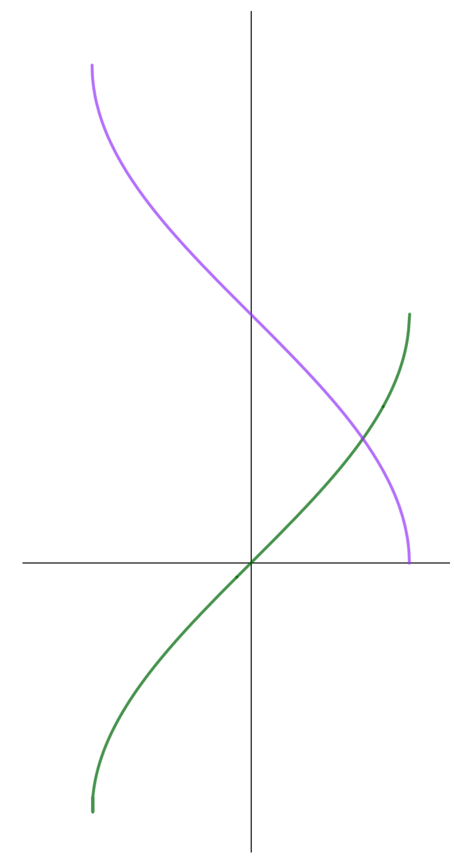
We can also look at graphs for this one.

It's clearly true for $x = -1, 0, 1$, and the symmetry takes care of the rest.



Here are the graphs $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

Where do they cross?



They cross at $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$. The easiest way to see this is by looking at the symmetry of the diagram. Alternatively, note that

$\sin y = \cos y \Rightarrow \tan y = 1 \Rightarrow y = \frac{\pi}{4}$.

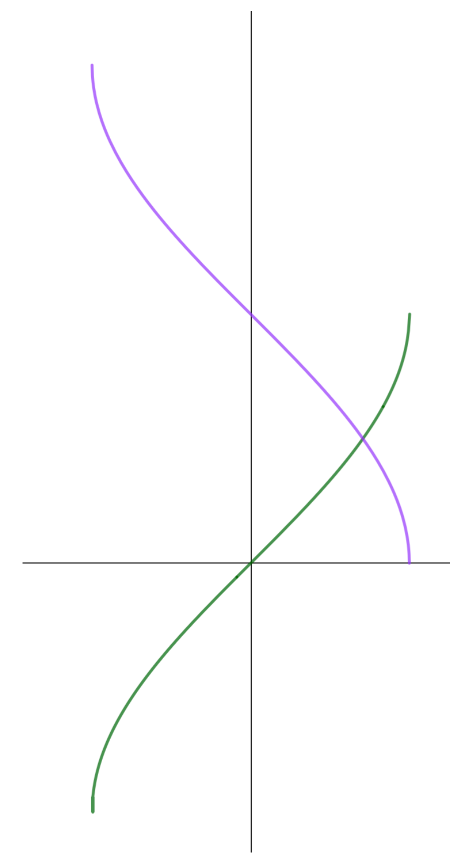
What are

$\sin^{-1} 0 + \cos^{-1} 0$

$\sin^{-1} 1 + \cos^{-1} 1$

$\sin^{-1}(-1) + \cos^{-1}(-1)$

$$\sin^{-1} \frac{\sqrt{2}}{2} + \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



$$\sin^{-1} 0 \ + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\sin^{-1} 1 \ + \cos^{-1} 1 = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

$$\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$$

$$\sin^{-1} \frac{\sqrt{2}}{2} + \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{2} = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\sin^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

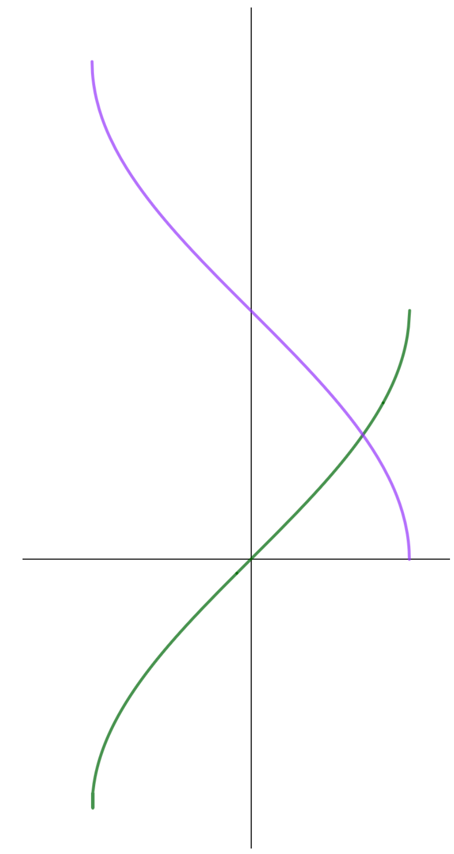
Draw any vertical line that intersects both of the curves.

What is the average of the y coordinates of the intersection points?

What is the sum of the y coordinates of the intersection points?

What is

$\sin^{-1} x + \cos^{-1} x$?



Symmetry shows easily that their sum is $\frac{\pi}{4}$, so their sum is $\frac{\pi}{2}$, which means

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for any x between -1 and 1.

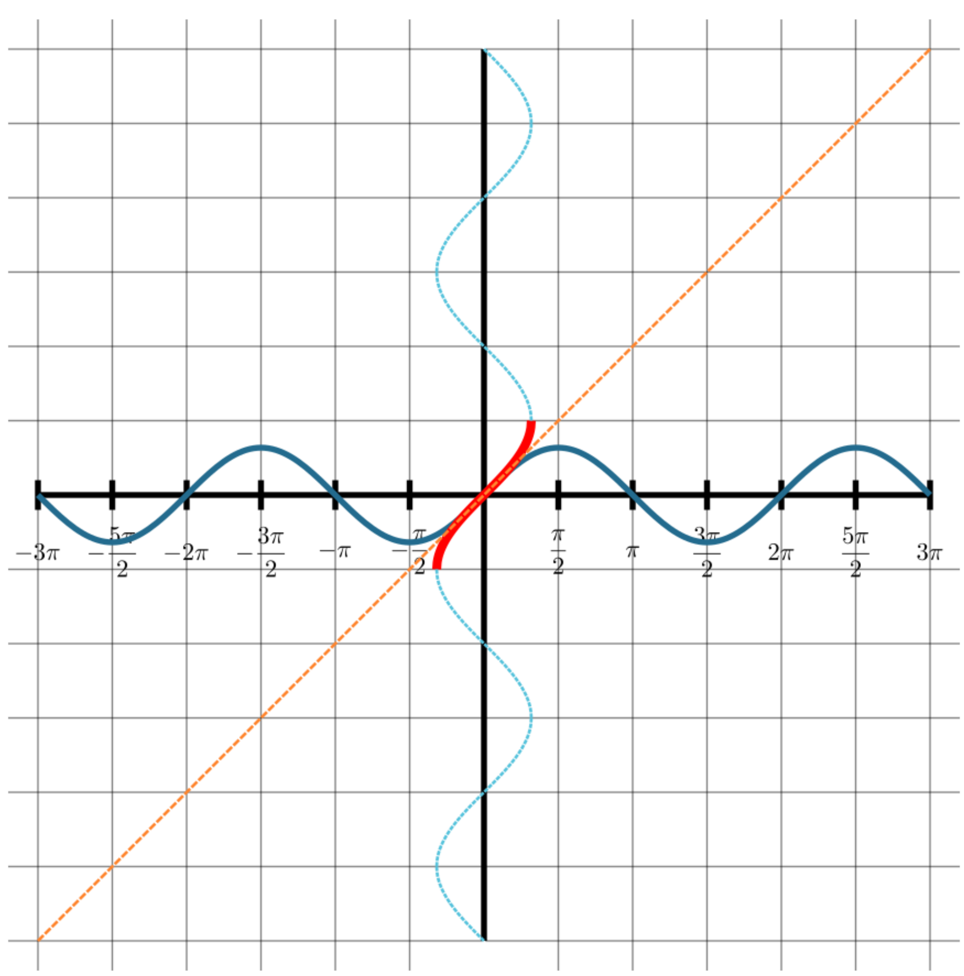
differentials of inverse circular functions

If $x = \sin y$, what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y ?

Use this to what is $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of x ?

Use this to what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x ?

What is $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x$?



Finding $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y is very easy, but finding the differential in terms of x is more delicate.

The usual treatment simply says that

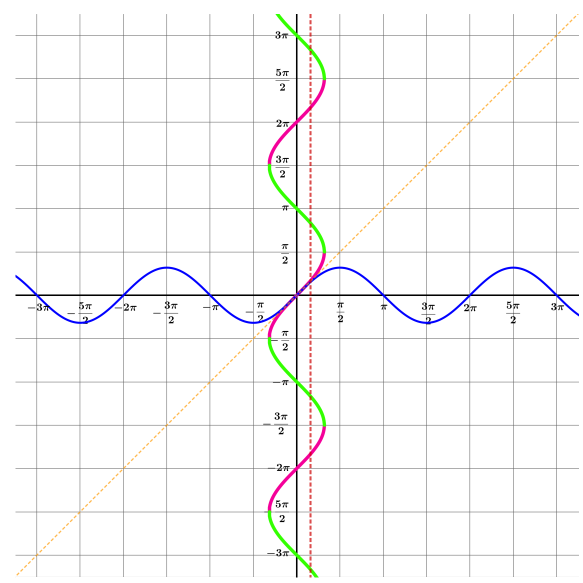
$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

but this is not a strong enough argument. Remember that $\cos^2 y = 1 - \sin^2 y$ but that sin and cos can have either the same sign or opposite signs. This means that

$$\cos y = \pm\sqrt{1-x^2}.$$

So we have to figure out whether to use the square root or its negative. The graph $y = \sin^{-1} x$ is always positive, so we must choose the square root, not its negative.

$$\begin{aligned} x &= \sin y \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \cos y \\ &= \pm\frac{1}{\sqrt{1-x^2}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{1}{\sqrt{1-x^2}} \end{aligned}$$



but gradient of $y = \sin^{-1} x$ is always positive, so

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \sqrt{1-x^2}$$

Notice, though, that this is because of the domain that we chose for the function $\sin^{-1}$. Had we chosen the domain $\left\{x : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}\right\}$, the gradient would always be negative, so that we would have to make the other choice for the differential, namely $-\sqrt{1-x^2}$.

If $x = \cos y$, what is $\dfrac{dx}{dy}$ in terms of y ?

Use this to what is $\dfrac{dx}{dy}$ in terms of x ?

Use this to what is $\dfrac{dy}{dx}$ in terms of x ?

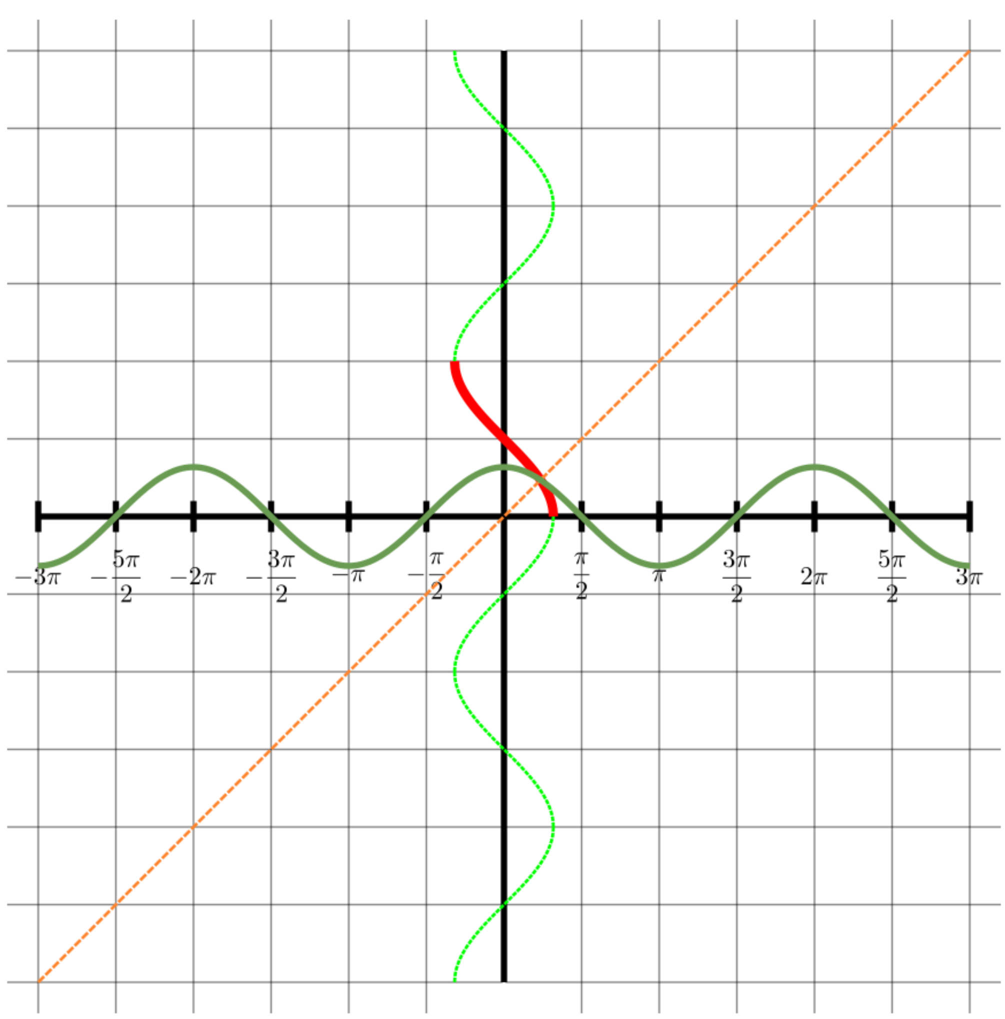
What is $\dfrac{d}{dx}\cos^{-1} x$?

The same considerations regarding signs apply here as they did with $\sin^{-1}$.

$$\begin{aligned} x &= \cos y \\ \Rightarrow \frac{dx}{dy} &= -\sin y \\ &= \pm\sqrt{1-x^2} \\ \Rightarrow \frac{dy}{dx} &= \pm\sqrt{1-x^2} \end{aligned}$$

but gradient of $y = \cos^{-1} x$ is always negative, so

$$\frac{d}{dx}\cos^{-1} x = -\sqrt{1-x^2}$$



If $x = \tan y$, what is $\dfrac{dx}{dy}$ in terms of y ?

$$x = \tan y$$
$$\Rightarrow \frac{dx}{dy} = \sec^2 y$$
$$= 1 + x^2$$

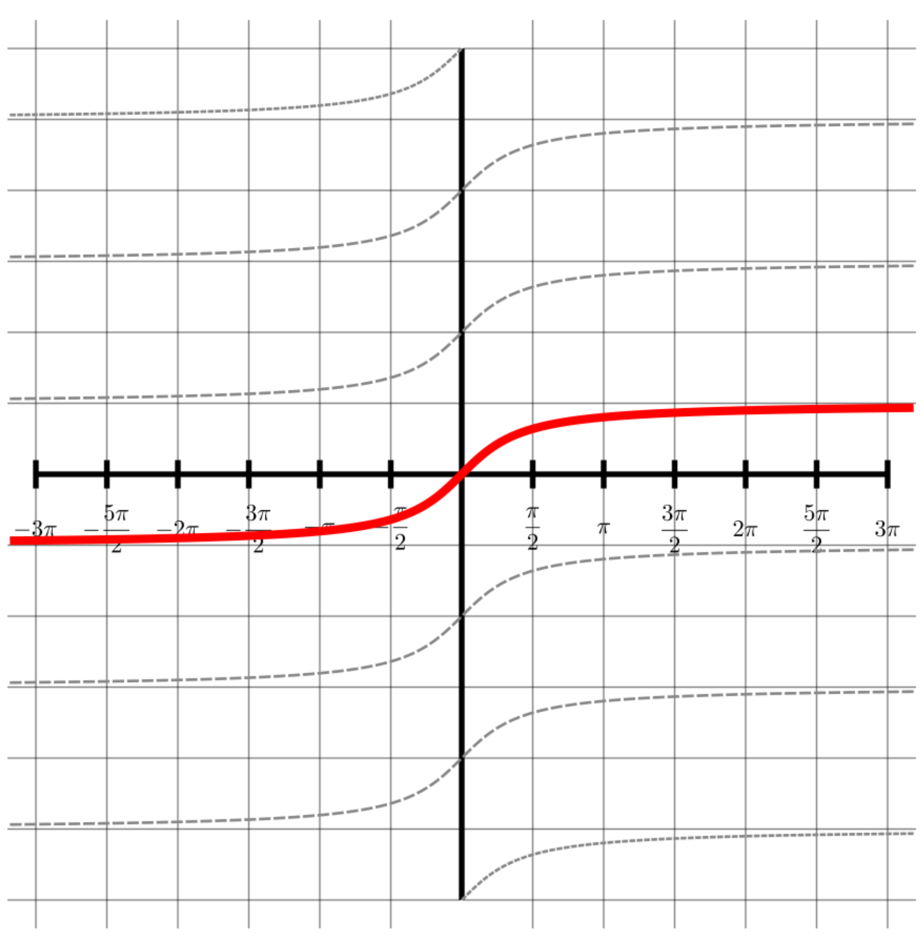
Use this to what is $\dfrac{dx}{dy}$ in terms of x ?

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

Use this to what is $\dfrac{dy}{dx}$ in terms of x ?

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

What is $\dfrac{d}{dx}\tan^{-1} x$?



We can also differentiate the inverse circular functions more directly using the definition of a differential.

Using the fact that

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

and putting $a = \sin A$ and $b = \sin B$, find

$\sin\left(\sin^{-1} a + \sin^{-1} b\right)$ in terms of a and b without using sin or cos in your expression.

Hence find

$\sin^{-1} a + \sin^{-1} b$

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A \sin B \\ \Rightarrow \sin(\sin^{-1} a + \sin^{-1} b) &= \sin(\sin^{-1} a)\cos(\sin^{-1} b) + \cos(\sin^{-1} a)\sin(\sin^{-1} b) \\ &= a\sqrt{1-b^2} + b\sqrt{1-a^2} \\ \Rightarrow \sin^{-1} a + \sin^{-1} b &= \sin^{-1}\left(a\sqrt{1-b^2} + b\sqrt{1-a^2}\right) \end{aligned}$$

Now find

$\sin\left(\sin^{-1}(x+h) - \sin^{-1} x\right)$

and use this to find

$$\lim_{h\to 0} \frac{\sin\left(\sin^{-1}(x+h) - \sin^{-1} x\right)}{h}$$

Put $\theta(h) = \sin\left(\sin^{-1}(x+h) - \sin^{-1} x\right)$ and find

$$\lim_{h\to 0} \frac{\sin^{-1}\theta(h)}{\theta(h)}$$

Hence find

$$\lim_{h\to 0} \frac{\sin^{-1}\theta(h)}{h}$$

Now put $f(x) = \sin^{-1} x$.

Find $f'(x)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x$

$$\begin{aligned}
\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h} &= \sqrt{1-x^2}+\frac{x\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h} \\
&= \sqrt{1-x^2}+x\,\frac{\sqrt{1-x^2}-\sqrt{1-(x+h)^2}}{h} \\
&= \sqrt{1-x^2}+x\,\frac{\left(1-x^2-\left(1-(x+h)^2\right)\right)}{h\left(\sqrt{1-x^2}+\sqrt{1-(x+h)^2}\right)} \\
&= \sqrt{1-x^2}+x\,\frac{2xh+h^2}{h\left(\sqrt{1-x^2}+\sqrt{1-(x+h)^2}\right)} \\
&\to \sqrt{1-x^2}+x\,\frac{2x}{2\sqrt{1-x^2}} \text{ as } h\to 0 \\
&= \sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

$$
\begin{aligned}
f(x) &= \sin^{-1} x \\
f'(x) &= \lim_{h\to 0} \frac{\sin^{-1}(x+h) - \sin^{-1} x}{h} \\
&= \lim_{h\to 0} \frac{\sin^{-1}\left((x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\theta(h)}{h} \times \frac{\sin^{-1}\theta(h)}{\theta(h)} \\
&\qquad \text{where } \theta(h) = (x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2} \\
&= \lim_{h\to 0} \frac{\theta(h)}{h}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h} = \frac{1}{\sqrt{1-x^2}} \\
&\Rightarrow \lim_{h\to 0} \frac{\theta(h)}{h} = \frac{1}{\sqrt{1-x^2}} \\
&\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

integrals using inverse circular functions

Use the substitution $x = \sin u$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-x^2}}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \pm\int \frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}\,\mathrm{d}u \\ &= \pm\int \mathrm{d}u \\ &= \pm u + c \\ &= \pm \sin^{-1} x + c\end{aligned}$$

$$\begin{aligned}x = \sin u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} &= \cos u \\ &= \pm\sqrt{1-\sin^2 u} \\ &= \pm\sqrt{1-x^2}\end{aligned}$$

The problem with the $\pm$ here is more or less always glossed over when teaching integration by substitution, which is why I have included this here. The question arises: should we use the plus or the minus?

If $y = \sin^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

If $y = -\sin^{-1} x$, what is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

Only the first of these differentiates to $\dfrac{1}{\sqrt{1-x^2}}$, so this must be the integral.

Use the substitution $u = \sin^{-1} x$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-x^2}} \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \int \frac{1}{\sqrt{1-x^2}} \sqrt{1-x^2}\,\mathrm{d}u \\ &= \int \mathrm{d}u \\ &= u + c \\ &= \sin^{-1} x + c\end{aligned}$$

$$u = \sin^{1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$$

In this version, there is no question of $\pm$.

In fact, this is the correct substitution. The previous version is just a kind of shorthand.

Use the substitution $x = \cos u$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-x^2}}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \pm\int \frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}\,\mathrm{d}u \\ &= \pm\int \mathrm{d}u \\ &= \pm u + c \\ &= \pm\cos^{-1}x + c\end{aligned}$$

$$\begin{aligned}x = \cos u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} &= -\sin u \\ &= \mp\sqrt{1-\cos^2 u} \\ &= \mp\sqrt{1-x^2}\end{aligned}$$

If $y = \cos^{-1} x$, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

If $y = -\cos^{-1} x$, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Only the second of these differentiates to $\frac{1}{\sqrt{1-x^2}}$, so this must be the integral.

Use the substitution $u = \cos^{-1} x$ for this integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-x^2}}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= -\int \frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}\,\mathrm{d}u \\ &= -\int \mathrm{d}u \\ &= -u + c \\ &= -\cos^{-1} x + c\end{aligned}$$

$$u = \cos^{1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}$$

Show that $\sin^{-1} x + c$ and $-\cos^{-1} x + c$ are equivalent solutions for the integral

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Remember that $\sin^{-1} x = \dfrac{\pi}{2} - \cos^{-1} x$ so the difference is just a difference in the constant c.

Use the substitution $x = \tan u$ for this integral:

$$\int \frac{1}{1+x^2}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{1+x^2}\,\mathrm{d}x &= \int \frac{1}{1+x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\\ &= \int \frac{1}{1+x^2}\left(1+x^2\right)\,\mathrm{d}u\\ &= \int \mathrm{d}u\\ &= u + c\\ &= \tan^{-1} x + c\end{aligned}$$

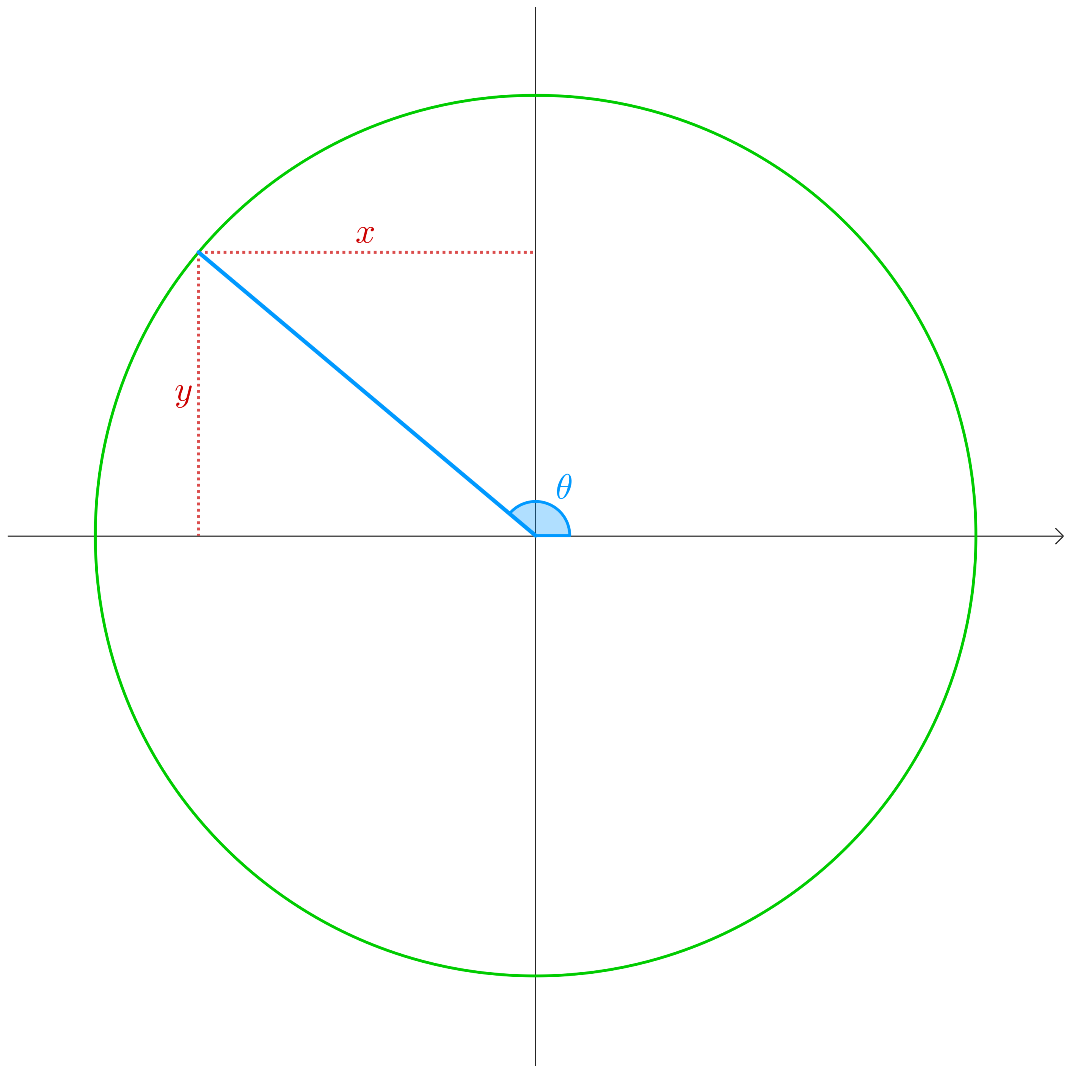
$$\begin{aligned}x = \tan u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} &= \sec^2 u\\ &= 1 + \tan^2 u\\ &= 1 + x^2\end{aligned}$$

Use the substitution $u = \tan^{-1} x$ for this integral:

$$\int \frac{1}{1+x^2}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{1}{1+x^2}\,\mathrm{d}x &= \int \frac{1}{1+x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\\ &= \int \frac{1}{1+x^2}\left(1+x^2\right)\,\mathrm{d}u\\ &= \int \mathrm{d}u\\ &= u + c\\ &= \tan^{-1} x + c\end{aligned}$$

$$u = \tan^{-1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2}$$



What is $\sin\left(\cos^{-1} x\right)$ in terms of x when $x < 0,\ y > 0$?

What is $\cos\left(\sin^{-1} y\right)$ in terms of y when $x < 0,\ y > 0$?

$$
\begin{aligned}
& x = \cos\theta \quad y = \sin\theta \\
\Rightarrow\ & \theta = \cos^{-1} x = \sin^{-1} y \\
& \sin\left(\cos^{-1} x\right) = \sin\theta = y = \sqrt{1 - x^2} \\
& \cos\left(\sin^{-1} y\right) = \cos\theta = x = \sqrt{1 - y^2}
\end{aligned}
$$

What is $\sin\left(\cos^{-1} x\right)$ in terms of x when $x > 0,\ y < 0$?

What is $\cos\left(\sin^{-1} y\right)$ in terms of y when $x > 0,\ y < 0$?

$$x = \cos\theta \quad y = \sin\theta$$

$$\Rightarrow \theta = \cos^{-1} x = \sin^{-1} y$$

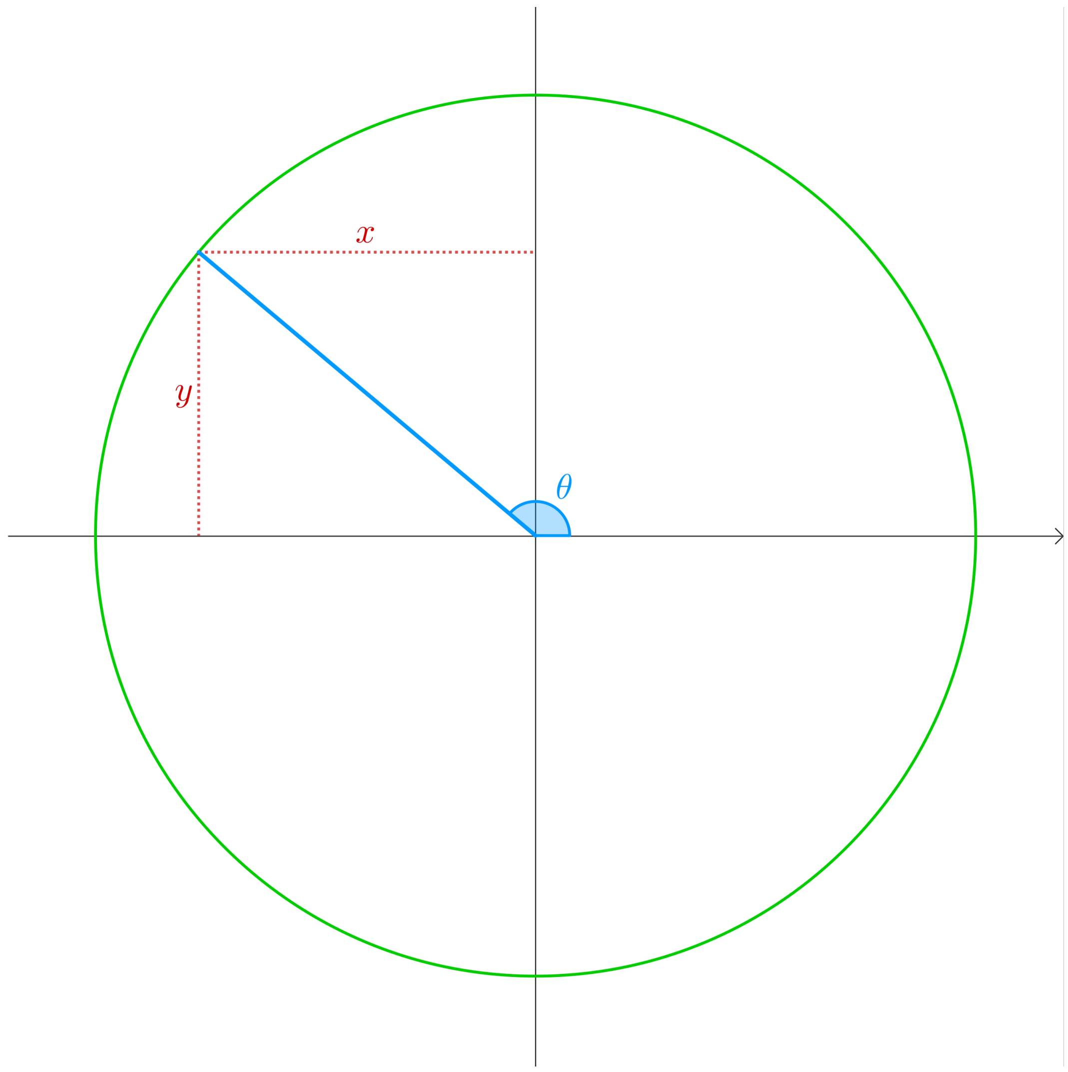
$$\sin\left(\cos^{-1} x\right) = \sin\theta = y = \sqrt{1 - x^2}$$

$$\cos\left(\sin^{-1} y\right) = \cos\theta = x = \sqrt{1 - y^2}$$

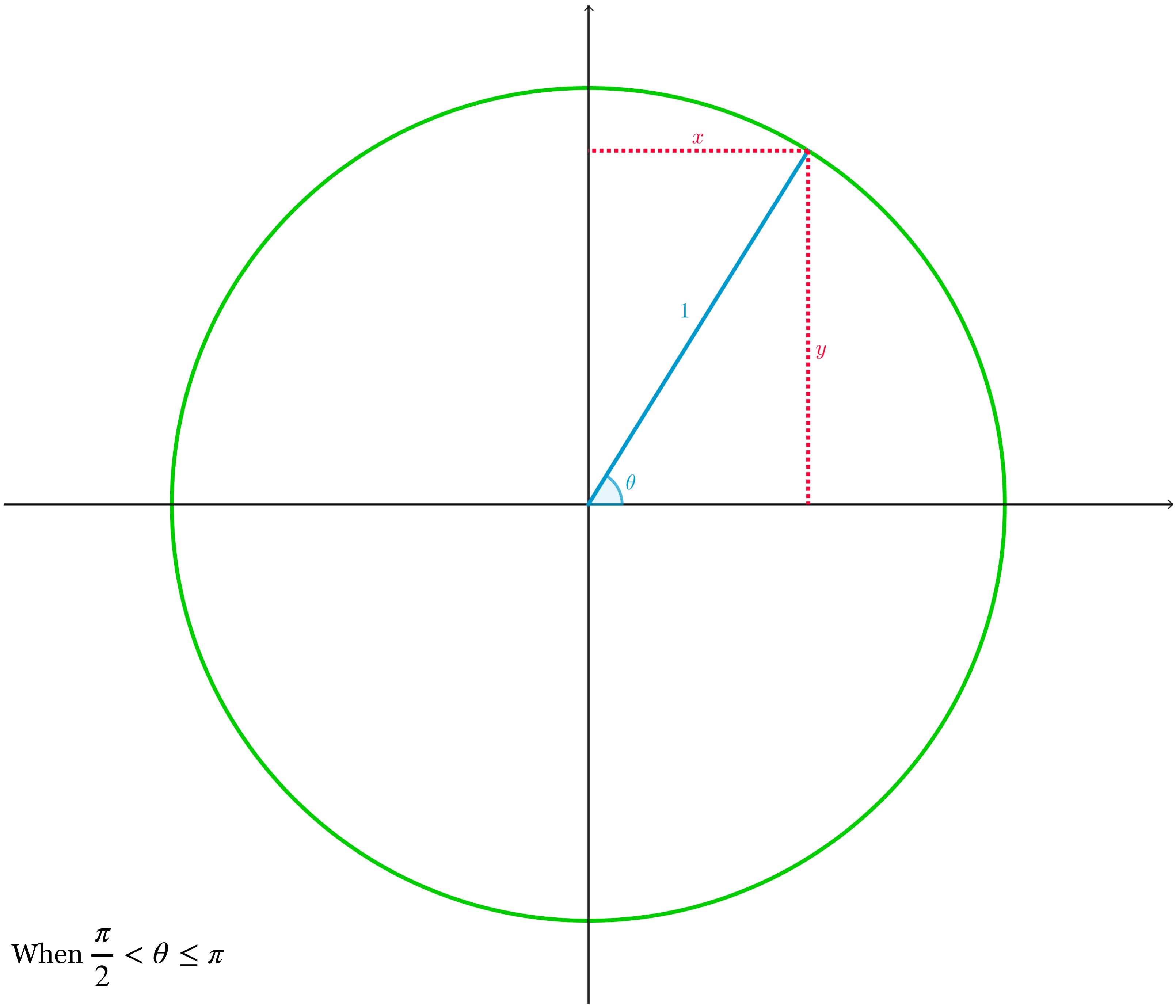
What is $\sin\left(\cos^{-1} x\right)$ in terms of x when $x < 0,\ y < 0$?

What is $\cos\left(\sin^{-1} y\right)$ in terms of y when $x < 0,\ y < 0$?

$$x = \cos\theta \quad y = \sin\theta$$
$$\Rightarrow \theta = \cos^{-1} x = \sin^{-1} y$$
$$\sin\left(\cos^{-1} x\right) = \sin\theta = y = \sqrt{1 - x^2}$$
$$\cos\left(\sin^{-1} y\right) = \cos\theta = x = \sqrt{1 - y^2}$$



x
y
θ



When $\dfrac{\pi}{2} < \theta \le \pi$

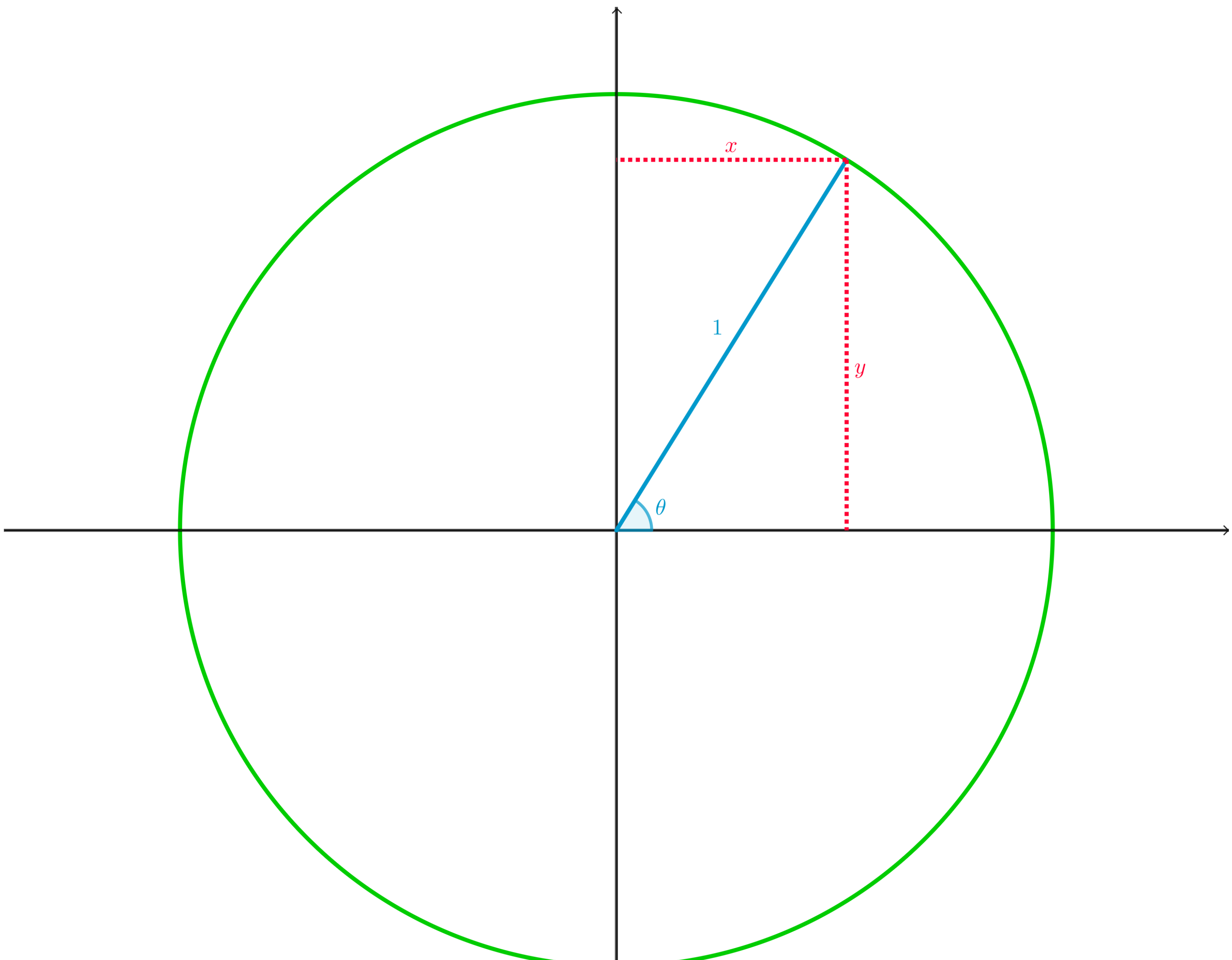
What is $\sin^{-1} x$ in terms of θ ?

What is $\sin^{-1}(\cos\theta)$ in terms of θ ?

What is $\cos^{-1} y$ in terms of θ ?

What is $\cos^{-1}(\sin\theta)$ in terms of θ ?

$\sin^{-1}(\cos\theta) = \sin^{-1} x = \dfrac{\pi}{2} - \theta \ \cos^{-1}(\sin\theta) = \sin^{-1} y = \dfrac{\pi}{2} - \theta$



When $\pi \leq \theta < \dfrac{3\pi}{2}$

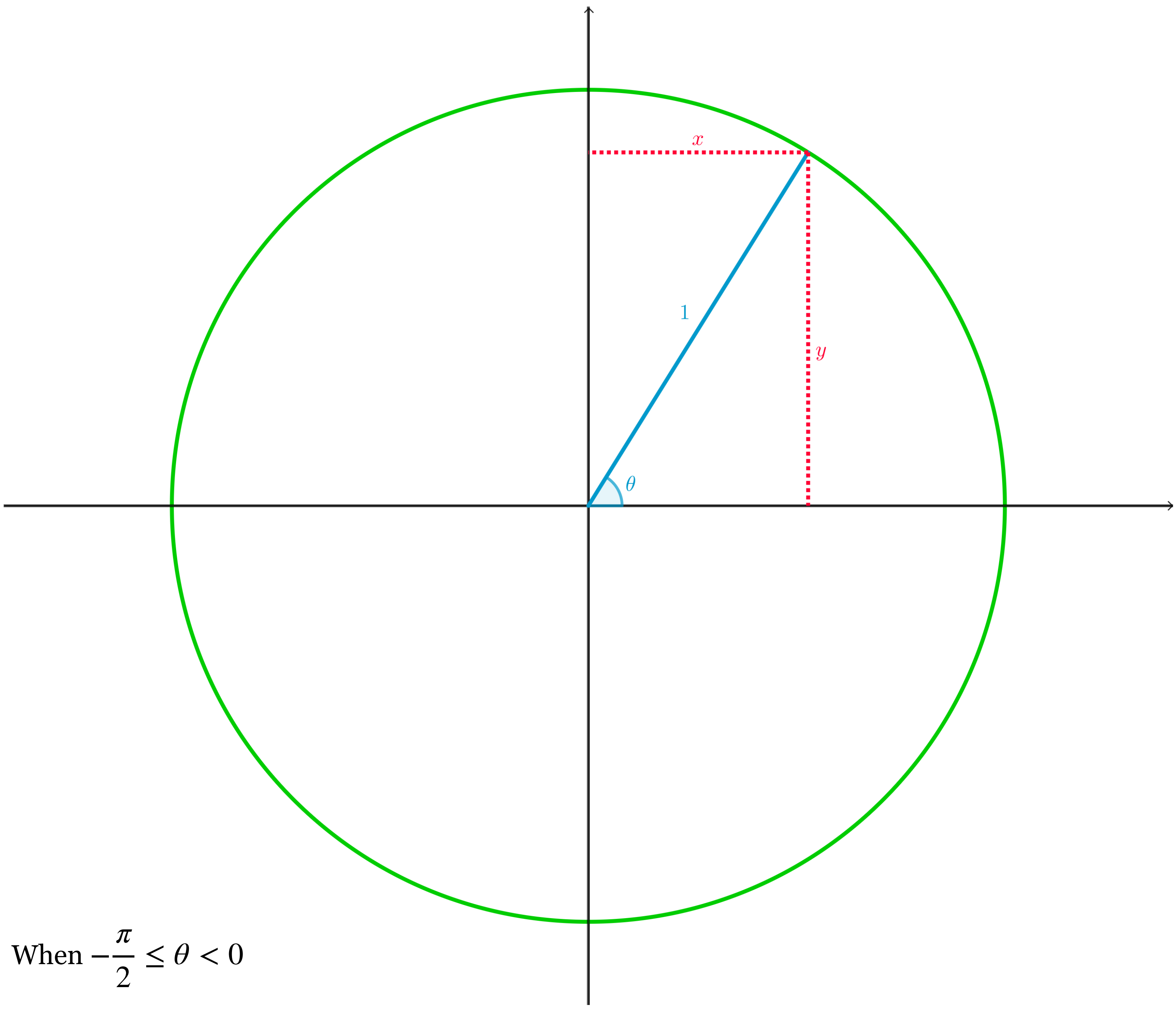
What is $\sin^{-1} x$ in terms of θ ?

What is $\sin^{-1}(\cos\theta)$ in terms of θ ?

What is $\cos^{-1} y$ in terms of θ ?

What is $\cos^{-1}(\sin\theta)$ in terms of θ ?

$$\sin^{-1}(\cos\theta) = \sin^{-1} x = \frac{\pi}{2} - \theta \cos^{-1}(\sin\theta) = \sin^{-1} y = \frac{\pi}{2} - \theta$$



When $-\dfrac{\pi}{2} \le \theta < 0$

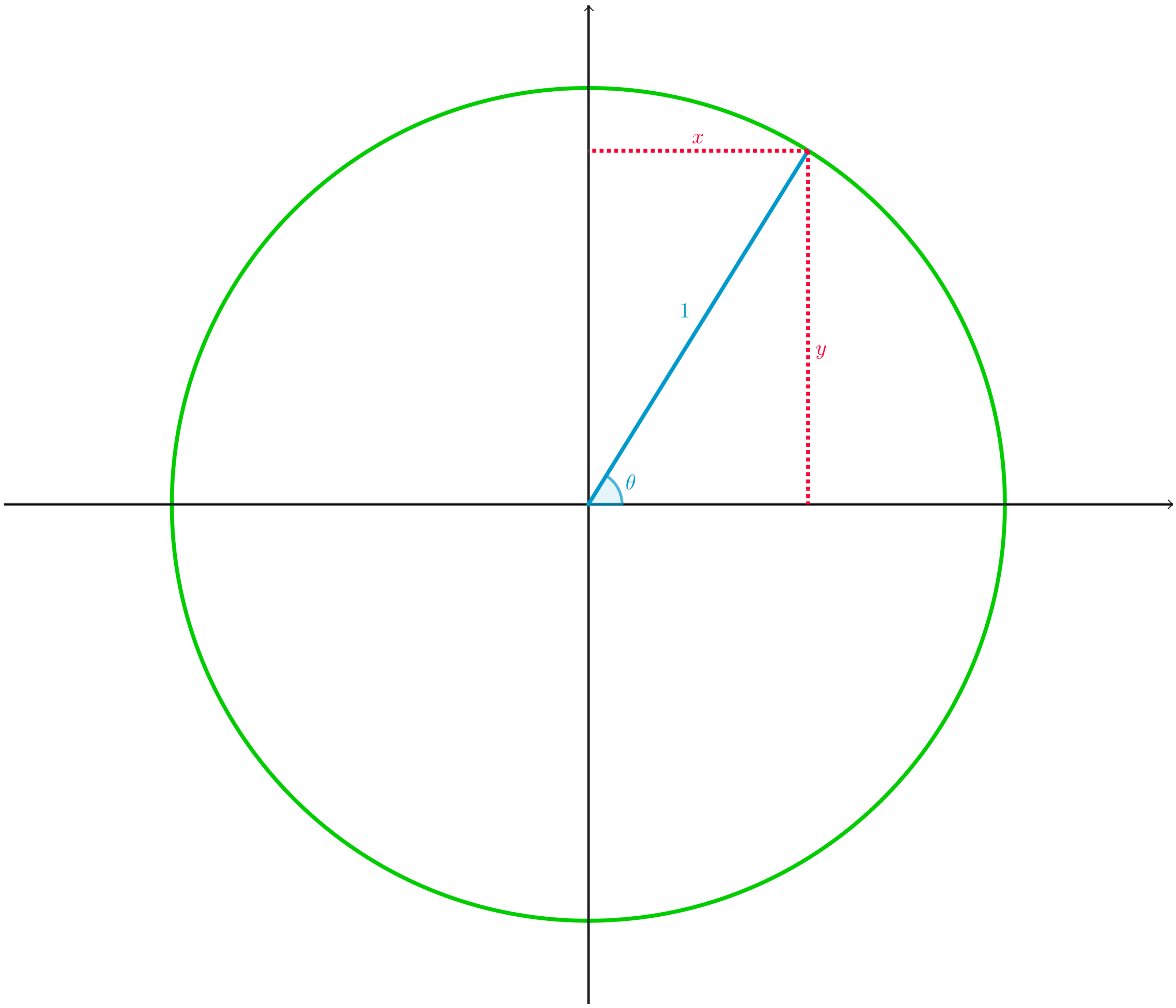
What is $\sin^{-1} x$ in terms of θ ?

What is $\sin^{-1}(\cos\theta)$ in terms of θ ?

What is $\cos^{-1} y$ in terms of θ ?

What is $\cos^{-1}(\sin\theta)$ in terms of θ ?

$\sin^{-1}(\cos\theta) = \sin^{-1} x = \dfrac{\pi}{2} - \theta$ $\cos^{-1}(\sin\theta) = \sin^{-1} y = \dfrac{\pi}{2} - \theta$

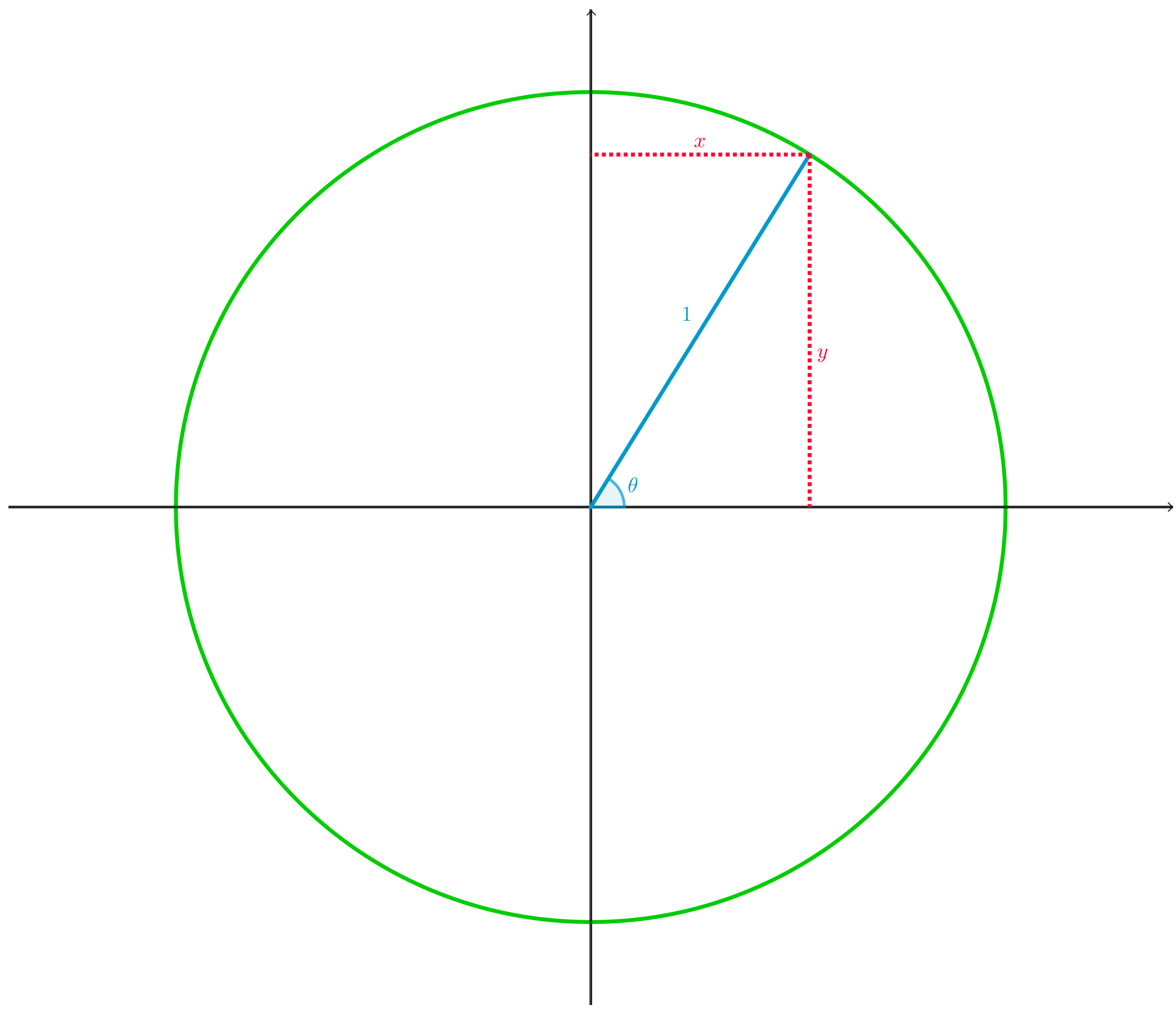


What is $\sin^{-1} x$ in terms of θ ?

What is $\sin^{-1}(\cos\theta)$ in terms of θ ?

What is $\cos^{-1} y$ in terms of θ ?

What is $\cos^{-1}(\sin\theta)$ in terms of θ ?



What is $\tan\theta$ in terms of x ?

What is $\tan\left(\cos^{-1}x\right)$ in terms of x ?

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{1-x^2}}{x} = \frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \tan\left(\cos^{-1}x\right) = \frac{\sqrt{1-x^2}}{x}$$

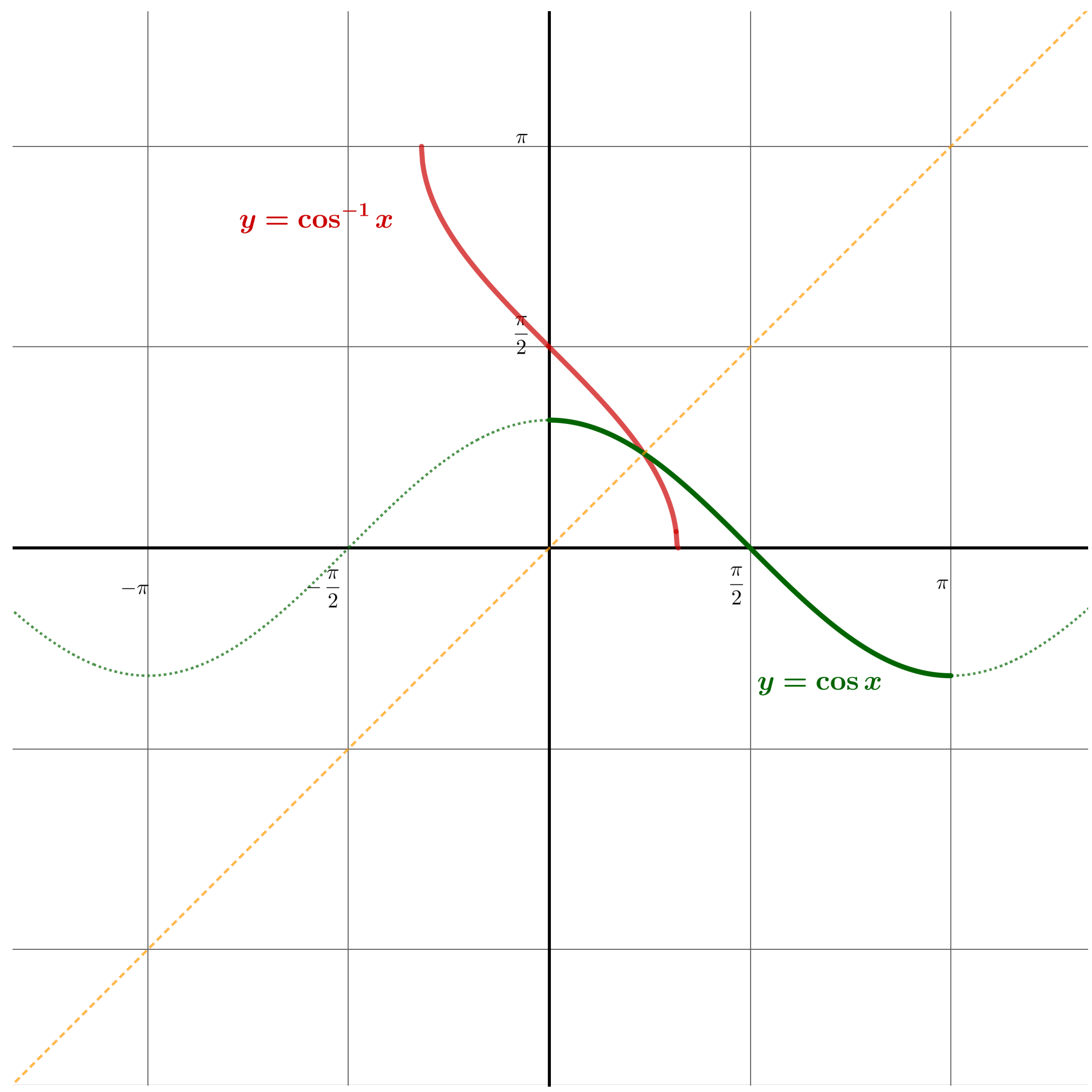
$$\tan\left(\sin^{-1}y\right) = \frac{y}{\sqrt{1-y^2}}$$

What is $\tan\theta$ in terms of y ?

What is $\tan\left(\sin^{-1}y\right)$ in terms of y ?

inverse circular functions: extension

What are the domain and range of $f(x) = \cos^{-1} x$?

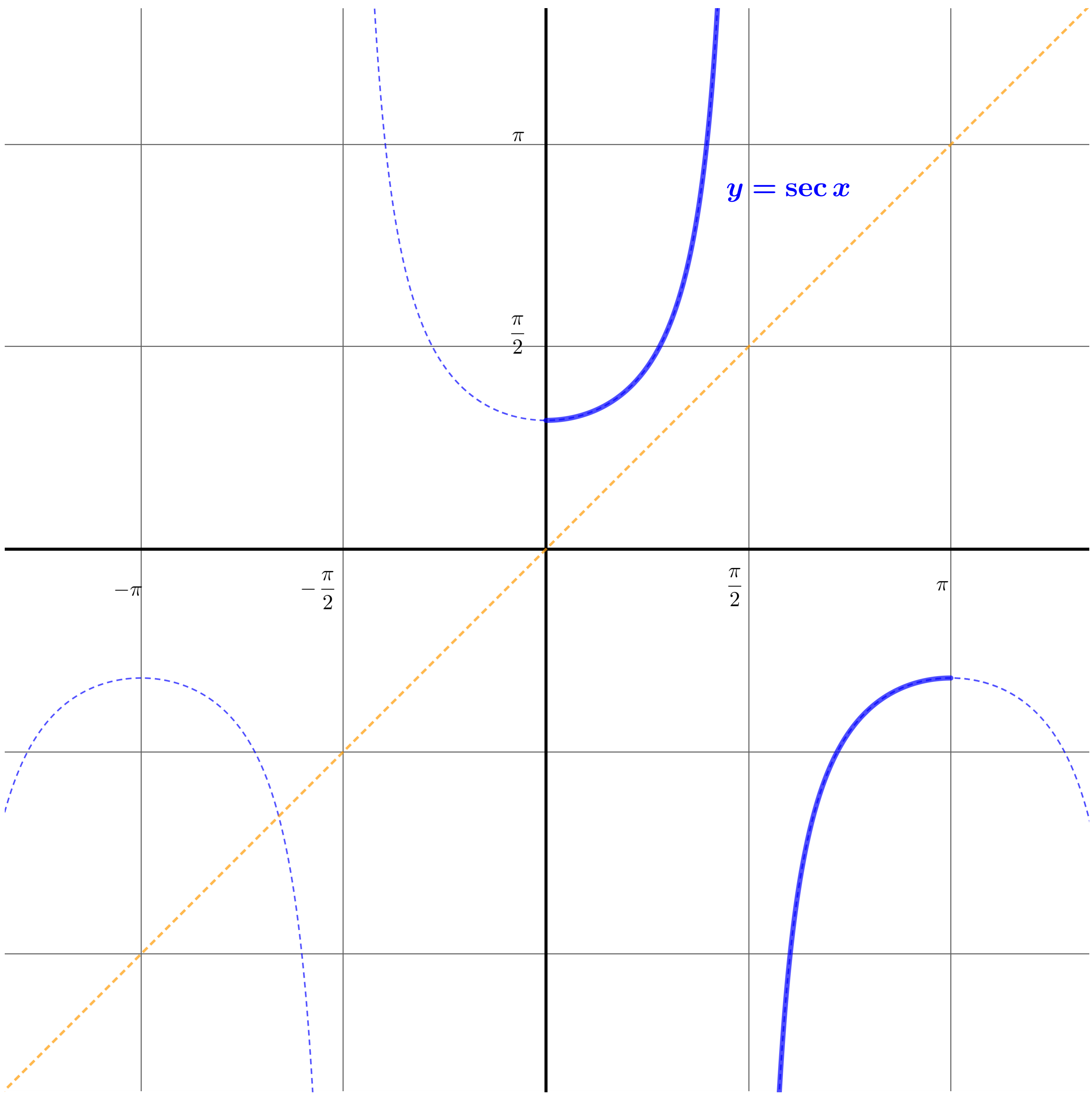


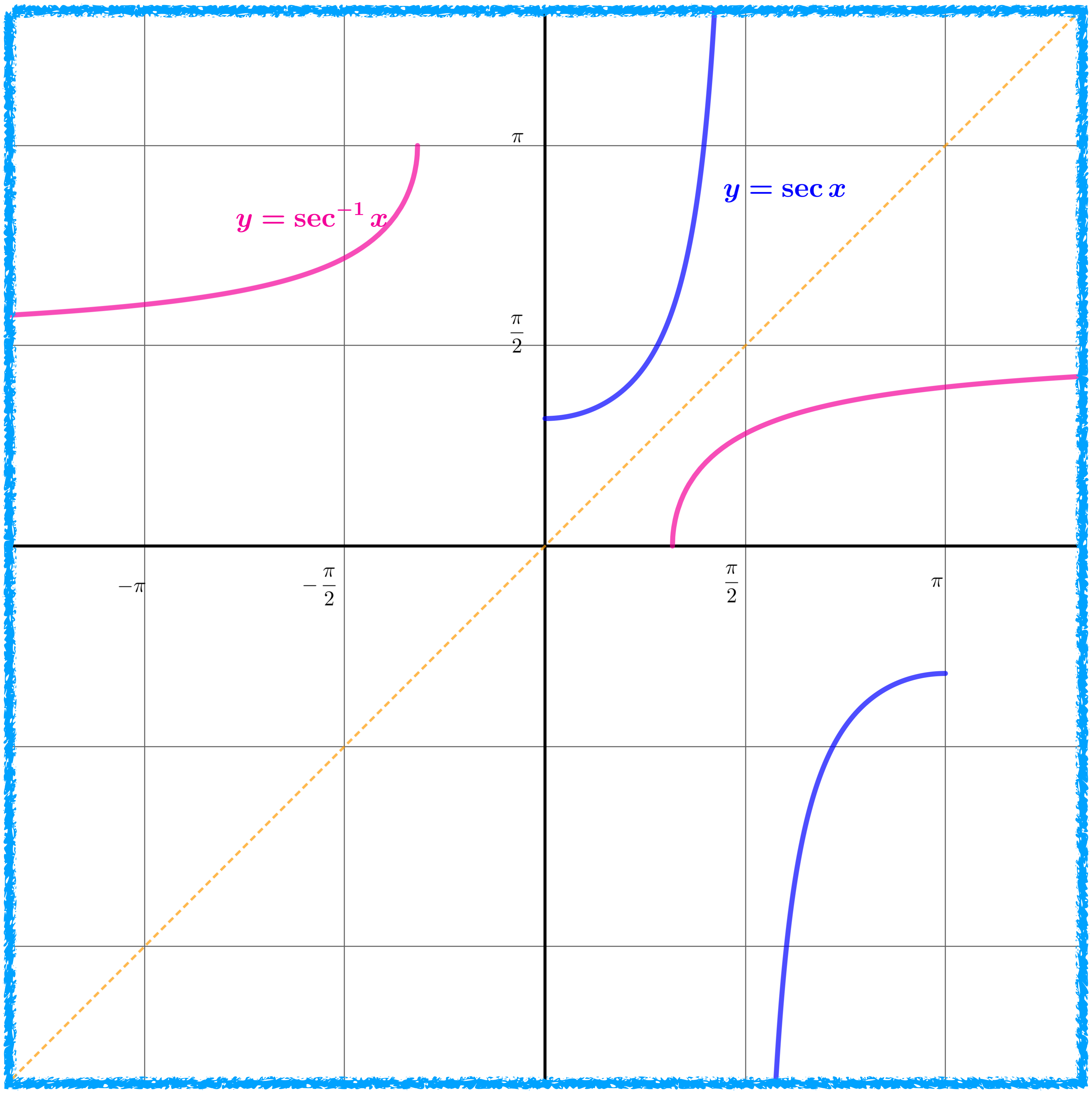
What are the domain and range of $f(x) = \sec^{-1} x$?

Here is the graph $y = \sec x$ over the domain

$$\left\{ x : 0 \le x \le \pi,\ x \ne \frac{\pi}{2} \right\}$$

Draw the graph $y = \sec^{-1} x$.



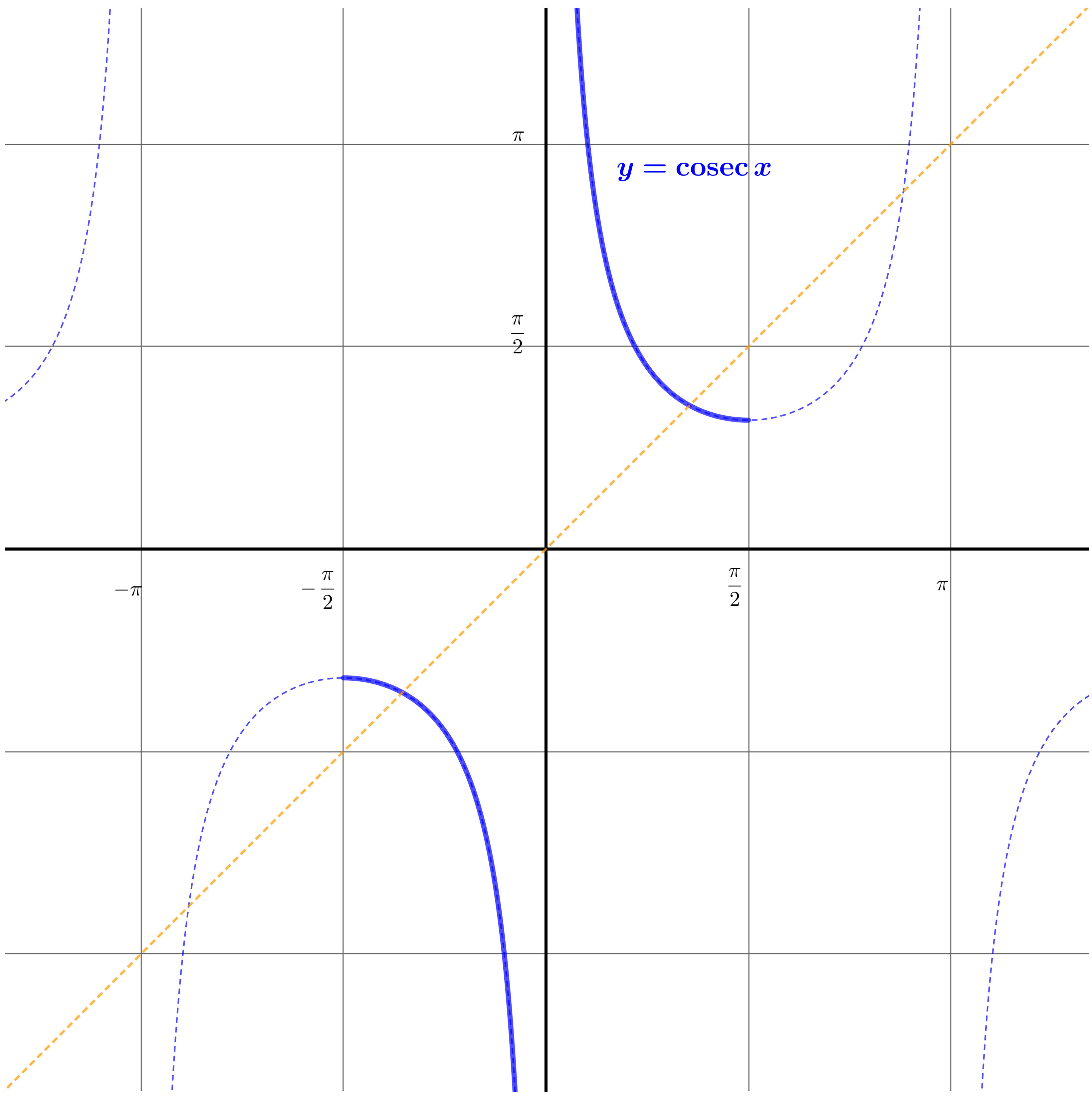


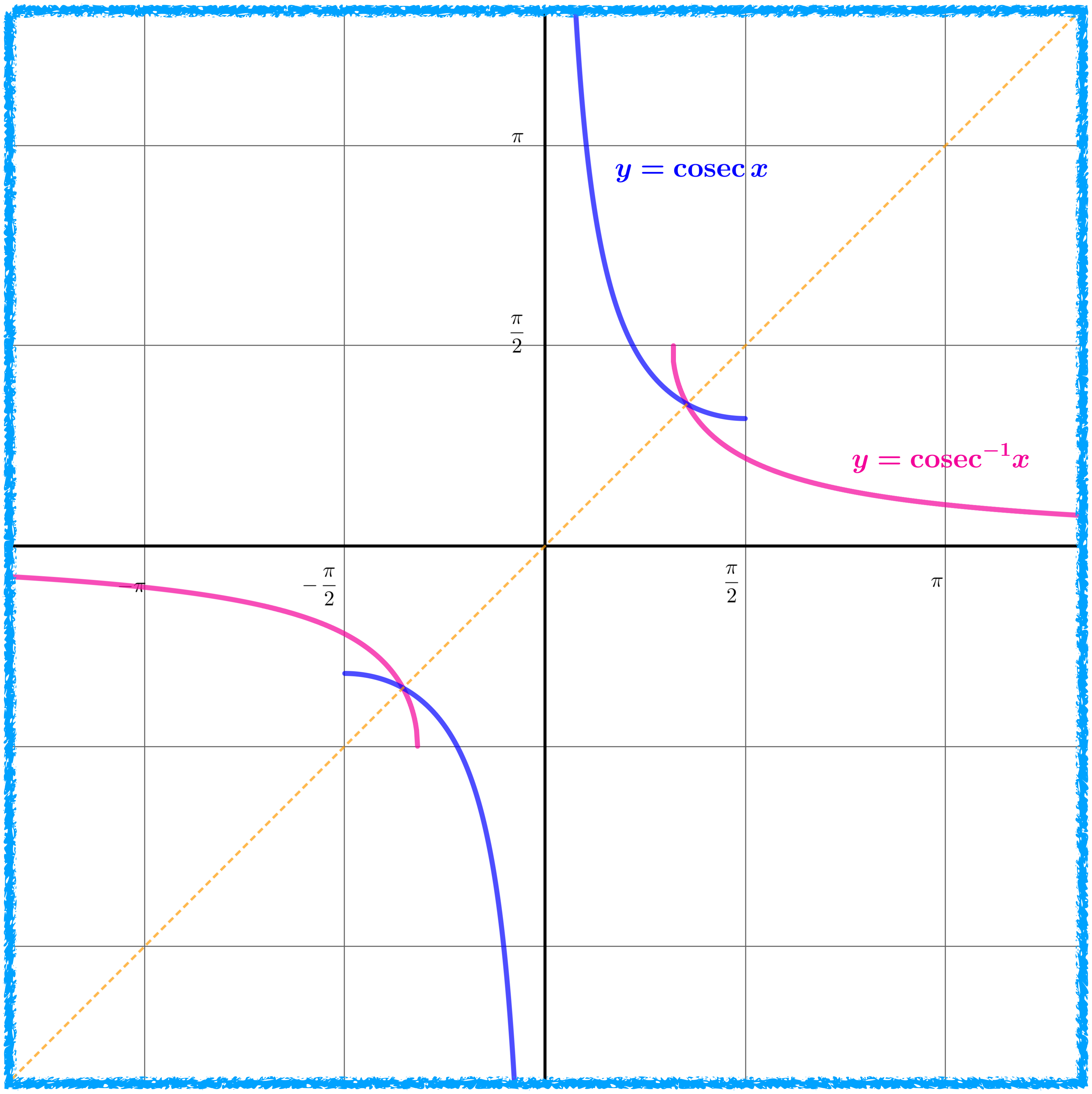
$y = \sec^{-1} x$
$y = \sec x$
π
$\frac{\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π

Here is the graph $y = \operatorname{cosec} x$ over the domain

$$\left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0\right\}$$

Draw the graph $y = \operatorname{cosec}^{-1} x$.





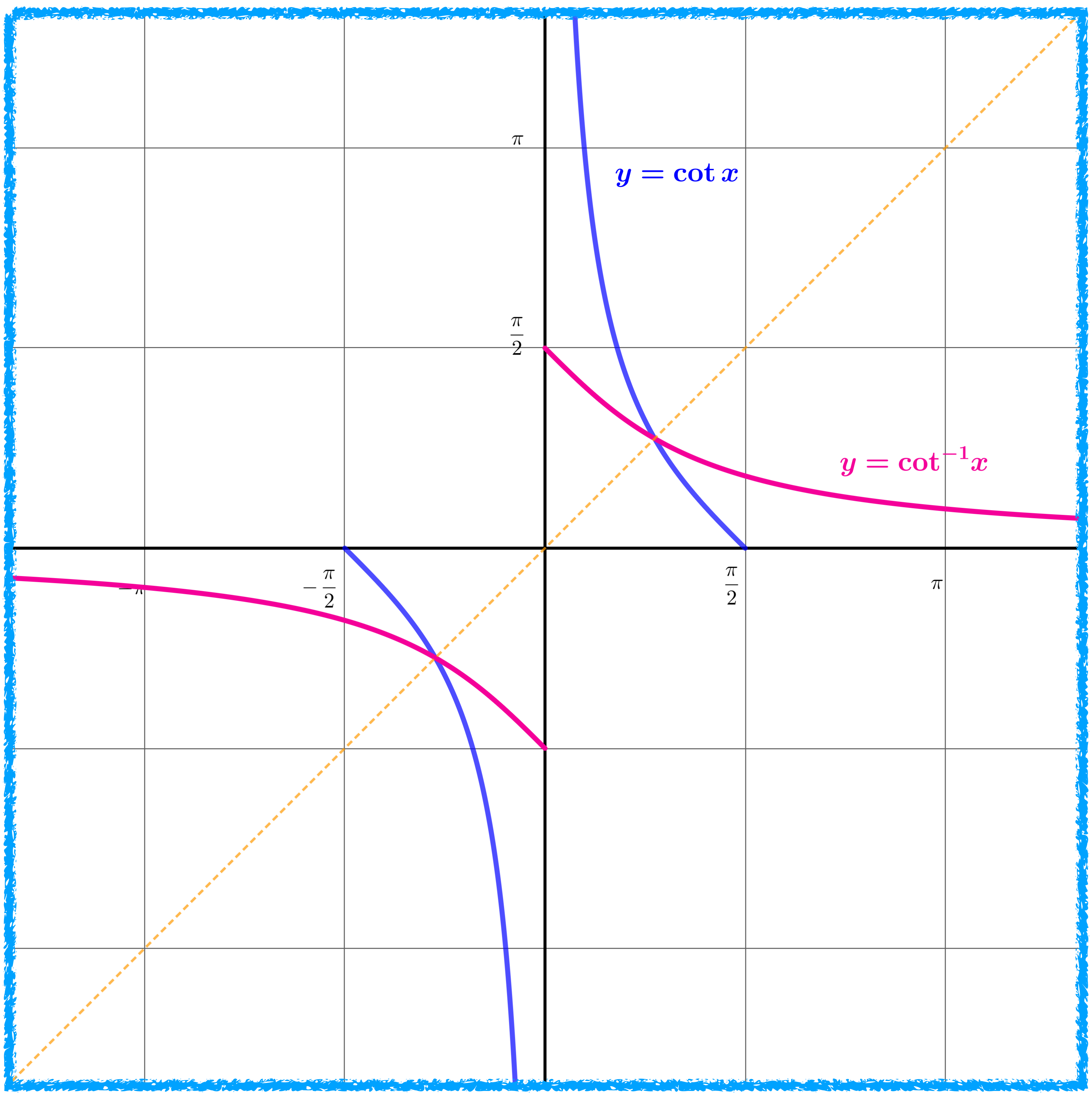
π
$\frac{\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$y = \operatorname{cosec} x$
$y = \operatorname{cosec}^{-1} x$

Here is the graph $y = \cot x$ over the domain

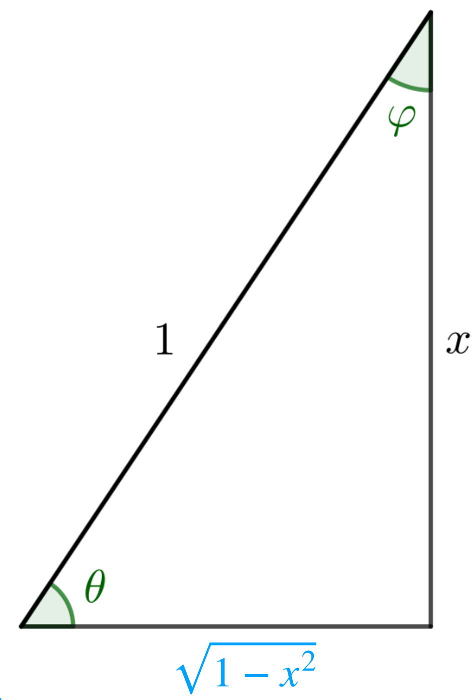
$$\left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0\right\}$$

Draw the graph $y = \cot^{-1} x$.

radical maths


π
$\frac{\pi}{2}$
$y = \cot x$
$y = \cot^{-1} x$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π

What is $\cos\left(\sin^{-1} x\right)$?



From the diagram:

$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1-x^2}$.

You may decide that this is good enough!

However, if $-1 \leq x \leq 0$, we can no longer use the triangle. Whether or not you choose to delve into this with your students will be question of judgement, and probably you will decide against. But just in case:

There are a couple of ways of dealing with the negative values of x. Both are a little subtle.

First

If $x < 0$, look at the graph of $\sin^{-1} x$: it's symmetrical, so the value of $\sin^{-1} x$ must be the negative of the value of $\sin^{-1}|x|$. But $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$.

This means that $\cos\left(\sin^{-1} x\right)$ is always positive, so

$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ for positive and negative values of x.

Second

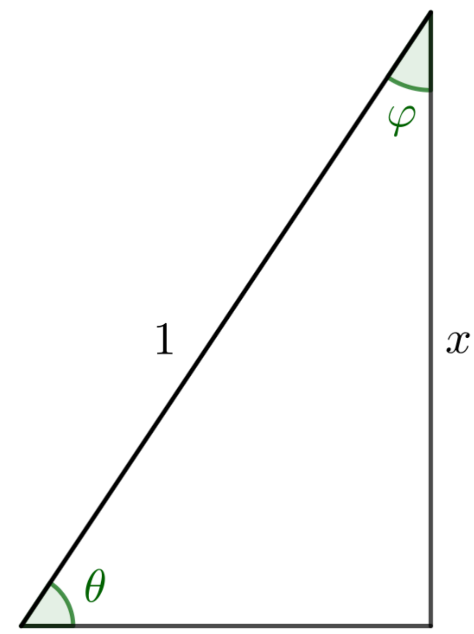
$$\begin{aligned}
&\cos^2 A = 1 - \sin^2 A \\
\Rightarrow\ &\cos^2\left(\sin^{-1} x\right) = 1 - \sin^2\left(\sin^{-1} x\right) \\
&\qquad = \left(1 - \sin\left(\sin^{-1} x\right)\right)^2 \\
&\qquad = 1 - x^2 \\
\Rightarrow\ &\cos\left(\sin^{-1} x\right) = \pm\sqrt{1-x^2}
\end{aligned}$$

But $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$, so $\cos\left(\sin^{-1} x\right) > 0$

So for positive and negative x,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}.$$

What is $\cos^{-1}\left(\sin\theta\right)$?



From the diagram:

$\cos^{-1}\left(\sin\theta\right) = \cos^{-1}x = \varphi = \dfrac{\pi}{2} - \theta.$

You may decide that this is good enough!

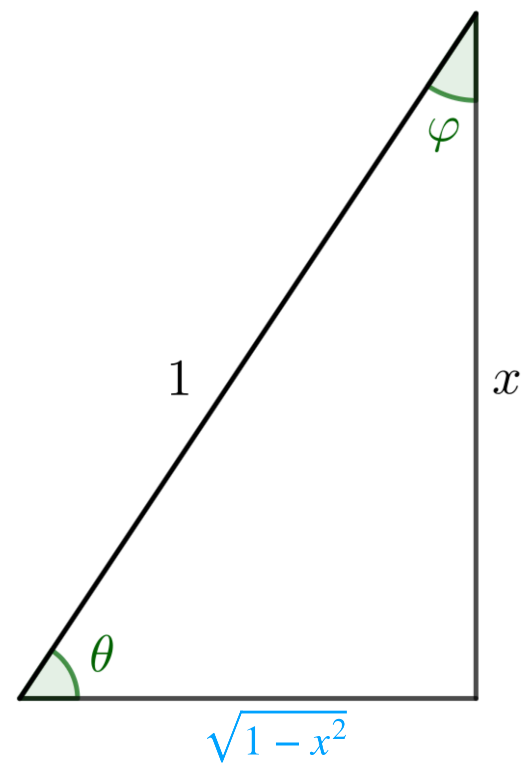
However, if θ is outside the range $\left[0, \dfrac{\pi}{2}\right]$, we can no longer use the triangle.

The range of $\cos^{-1}$ is $\{y : 0 \leq y \leq \pi\}$, so we have to add or subtract multiples of 2π to $\dfrac{\pi}{2} - \theta$ to bring it into this range.

$\cos^{-1}\left(\sin\theta\right) = \cos^{-1}\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\pi}{2} - \theta + 2n\pi$

where $n \in \mathbb{Z}$ is chosen to give a result in the range $\left[0, \pi\right]$.

What are $\sin\left(\cos^{-1} x\right)$ and $\sin^{-1}(\cos\varphi)$?



From the diagram:

$\sin\left(\cos^{-1} x\right) = \sin\varphi = \sqrt{1-x^2}$, so this is true when x is positive.

Again, you will probably decide that this is enough. But just in case (again):

When x is negative, things are a little bit trickier with graphs. As before,

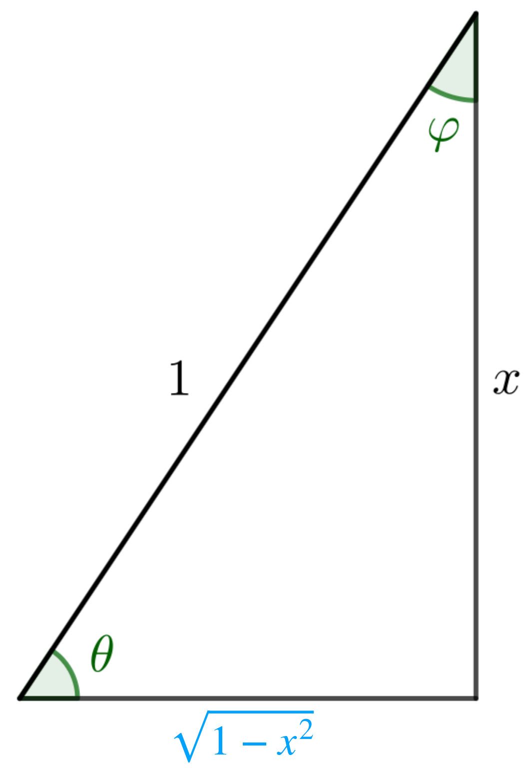
$$\begin{aligned} &\sin^2 A = 1 - \cos^2 A \\ \Rightarrow\ &\sin^2\left(\cos^{-1} x\right) = 1 - \cos^2\left(\cos^{-1} x\right) \\ &\qquad = \left(1 - \cos\left(\cos^{-1} x\right)\right)^2 \\ &\qquad = 1 - x^2 \\ \Rightarrow\ &\sin\left(\cos^{-1} x\right) = \pm\sqrt{1-x^2} \end{aligned}$$

But $0 \le \cos^{-1} x \le \pi$, so $\sin\left(\cos^{-1} x\right) > 0$

So for positive and negative x,

$$\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}.$$

What are $\tan\left(\sin^{-1} x\right)$ and $\tan\left(\cos^{-1} x\right)$?



From the diagram:

$$\tan\left(\sin^{-1} x\right) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\tan\left(\cos^{-1} x\right) = \tan\varphi = \frac{\sqrt{1-x^2}}{x}$$

For negative values of x:

$$\sin^{-1} x < 0 \Rightarrow \tan\left(\sin^{-1} x\right) < 0 \text{ and } \frac{x}{\sqrt{1-x^2}} < 0$$

$$\frac{\pi}{2} < \cos^{-1} x \leq \pi \Rightarrow \tan\left(\cos^{-1} x\right) < 0 \text{ and } \frac{\sqrt{1-x^2}}{x} < 0$$

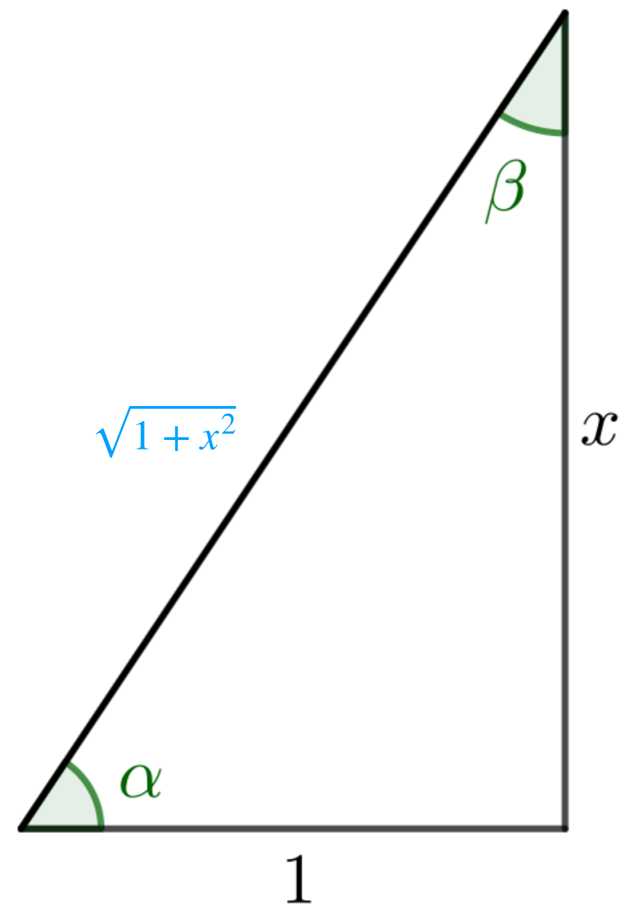
so the results are still true.

Or:

$$\begin{aligned}\tan\left(\sin^{-1} x\right) &= \frac{\sin\left(\sin^{-1} x\right)}{\cos\left(\sin^{-1} x\right)} \\ &= \frac{x}{\sqrt{1-x^2}}\end{aligned}$$

$$\begin{aligned}\tan\left(\cos^{-1} x\right) &= \frac{\sin\left(\cos^{-1} x\right)}{\cos\left(\cos^{-1} x\right)} \\ &= \frac{\sqrt{1-x^2}}{x}\end{aligned}$$

What are $\sin\left(\tan^{-1} x\right)$ and $\cos\left(\tan^{-1} x\right)$?



$$\sin\left(\tan^{-1} x\right) = \sin\alpha = \frac{x}{\sqrt{1+x^2}}$$

When $x > 0$,

$0 < \tan^{-1} x < \frac{\pi}{2} \Rightarrow \sin\left(\tan^{-1} x\right) > 0$

When $x < 0$,

$-\frac{\pi}{2} < \tan^{-1} x < 0 \Rightarrow \sin\left(\tan^{-1} x\right) < 0$

$$\cos\left(\tan^{-1} x\right) = \cos\alpha = \frac{1}{\sqrt{1+x^2}}$$

Since $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$, we know that $\cos\left(\tan^{-1} x\right) > 0$.

$$\begin{aligned} 1 + x^2 = 1 + \tan^2\left(\tan^{-1} x\right) &= \sec^2\left(\tan^{-1} x\right) \\ &= \frac{1}{\cos^2\left(\tan^{-1} x\right)} \\ \Rightarrow \cos\left(\tan^{-1} x\right) &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

$$\begin{aligned} 1 + \frac{1}{x^2} = 1 + \cot^2\left(\tan^{-1} x\right) &= \operatorname{cosec}^2\left(\tan^{-1} x\right) \\ &= \frac{1}{\sin^2\left(\tan^{-1} x\right)} \\ \Rightarrow \sin\left(\tan^{-1} x\right) = \frac{1}{\sqrt{1+\frac{1}{x^2}}} &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

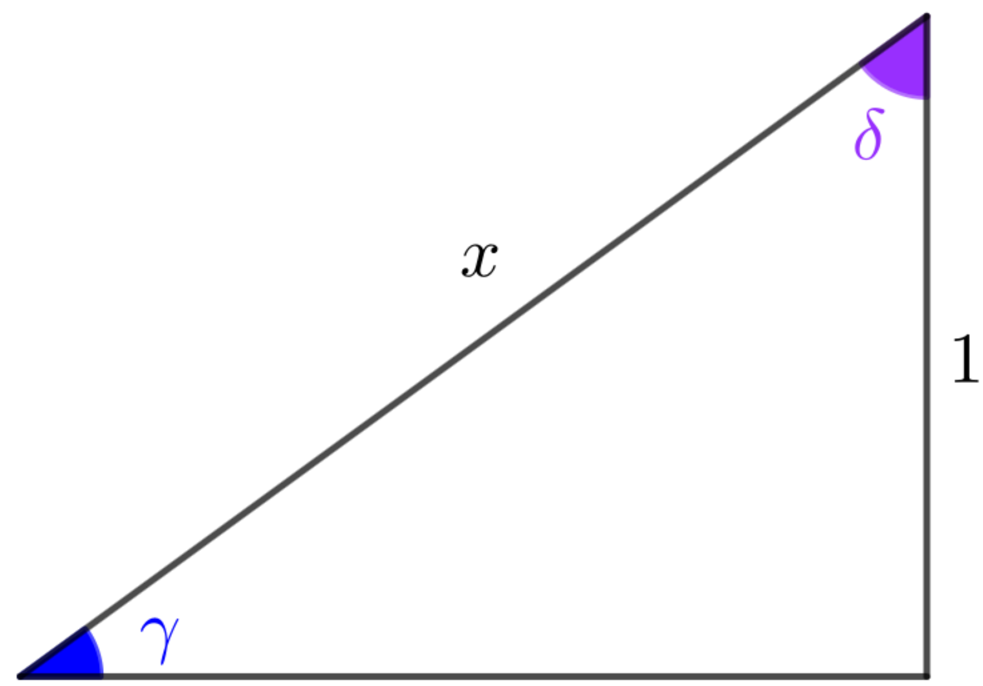
Simplify

$\sin\left(\operatorname{cosec}^{-1} x\right)$

$\operatorname{cosec}\left(\sin^{-1} x\right)$

$\cos\left(\sec^{-1} x\right)$

$\sec\left(\cos^{-1} x\right)$

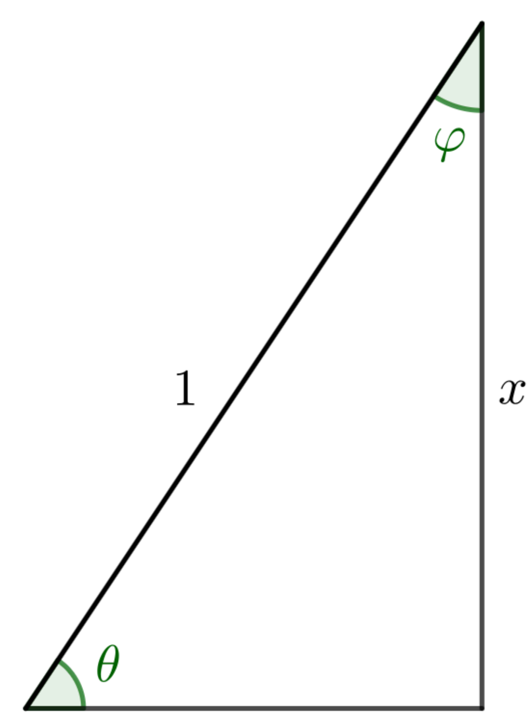


$\sin\left(\operatorname{cosec}^{-1} x\right) = \sin\gamma = \frac{1}{x}$

$\operatorname{cosec}\left(\sin^{-1} x\right) = \operatorname{cosec}\theta = \frac{1}{x}$

$\cos\left(\sec^{-1} x\right) = \cos\delta = \frac{1}{x}$

$\sec\left(\cos^{-1} x\right) = \sec\varphi = \frac{1}{x}$



So

$\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$

$\sec^{-1} x = \cos^{-1}\frac{1}{x}$

These are clearly true when $x > 0$. What about when x is negative?

$$\sec\left(\cos^{-1}\frac{1}{x}\right) = \frac{1}{\cos\left(\cos^{-1}\frac{1}{x}\right)} = x$$

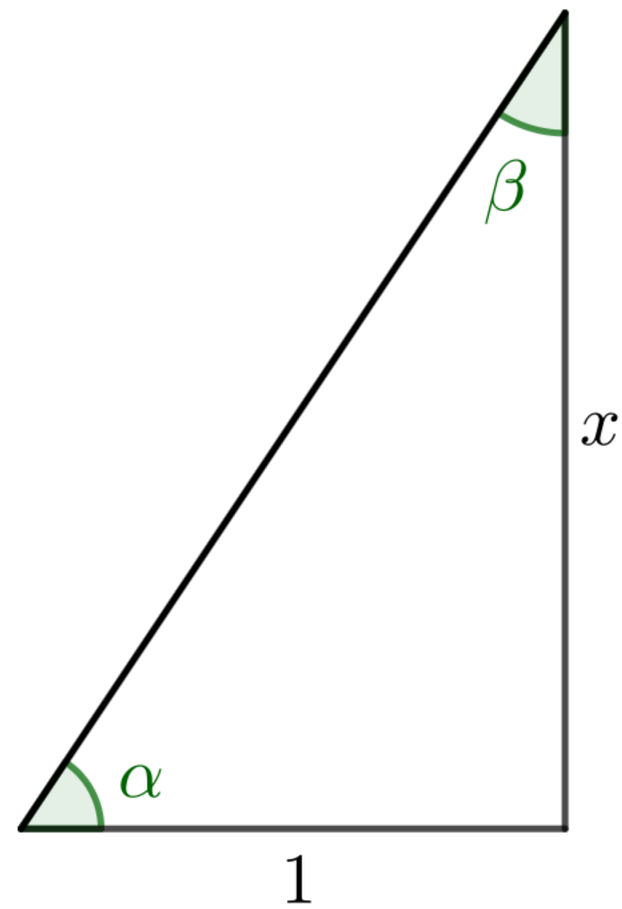
is true for all x, and similarly for sin and cosec.

radical maths

Simplify

$\tan\left(\cot^{-1} x\right)$

$\cot\left(\tan^{-1} x\right)$



$\tan\left(\cot^{-1} x\right) = \tan\beta = \frac{1}{x}$

$\cot\left(\tan^{-1} x\right) = \cot\alpha = \frac{1}{x}$

So

$\cot^{-1} x = \tan^{-1}\frac{1}{x}$

These are clearly true when $x > 0$. What about when x is negative?

$$\cot\left(\tan^{-1}\frac{1}{x}\right) = \frac{1}{\tan\left(\tan^{-1}\frac{1}{x}\right)} = x$$

is true for all x.

What are

$\cot^{-1} x + \tan^{-1} x$

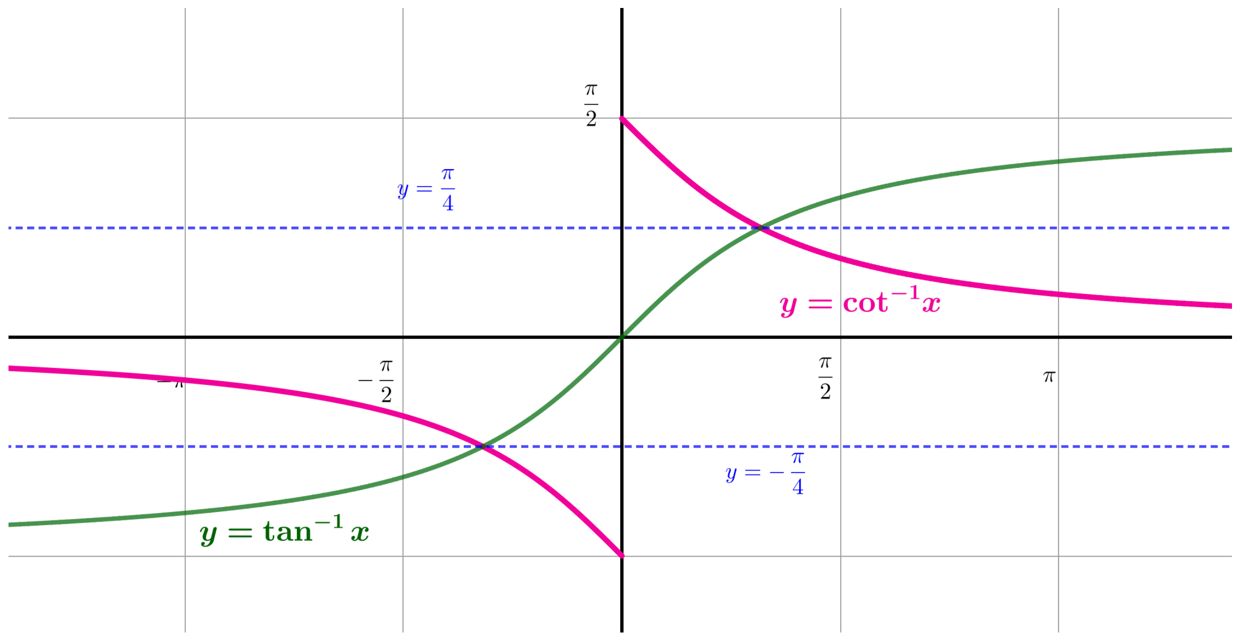
and

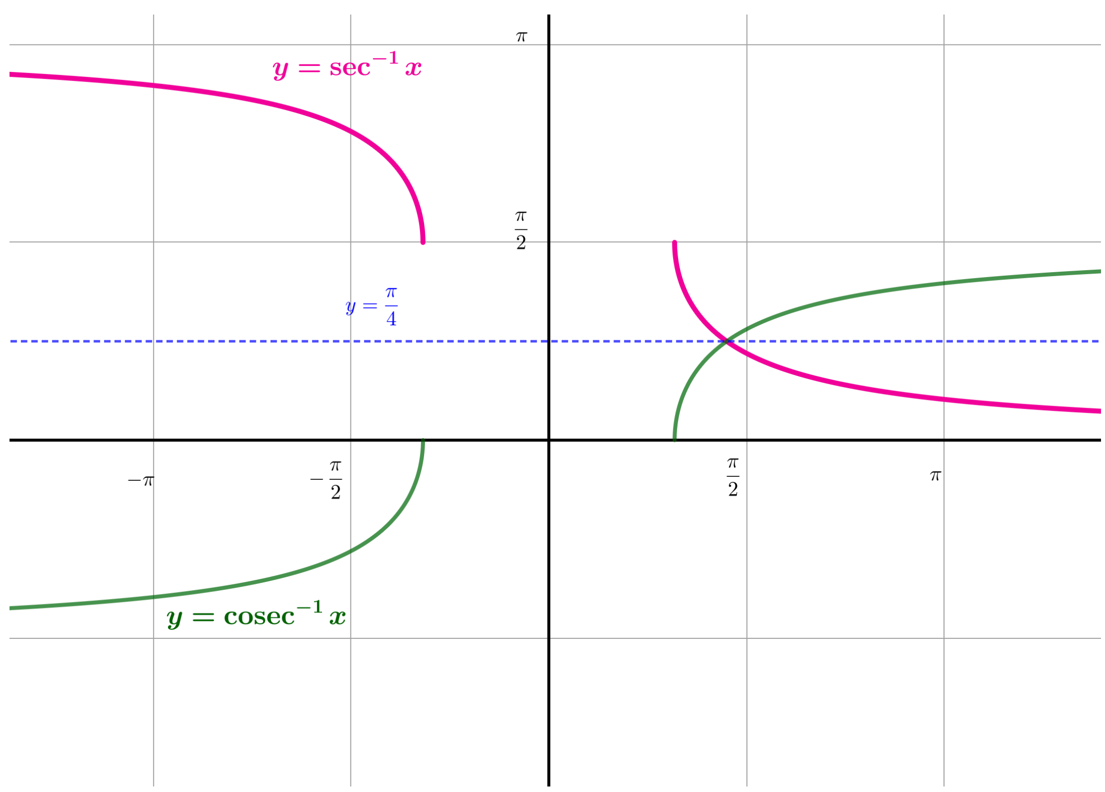
$\sec^{-1} x + \operatorname{cosec}^{-1} x$?

Symmetry:

$\cot^{-1} x + \tan^{-1} x = \pm \dfrac{\pi}{2}$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$





more differentials of inverse circular functions

What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x$?

$$
\begin{aligned}
& y = \sec^{-1} x \\
\Rightarrow\ & x = \sec y \\
\Rightarrow\ & \frac{\mathrm{d}x}{\mathrm{d}y} = \sec y \tan y = \pm\, x\sqrt{x^2-1} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \pm \frac{1}{x\sqrt{x^2-1}}
\end{aligned}
$$

$$
\begin{aligned}
& y = \sec^{-1} x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{x^2-1}}
\end{aligned}
$$

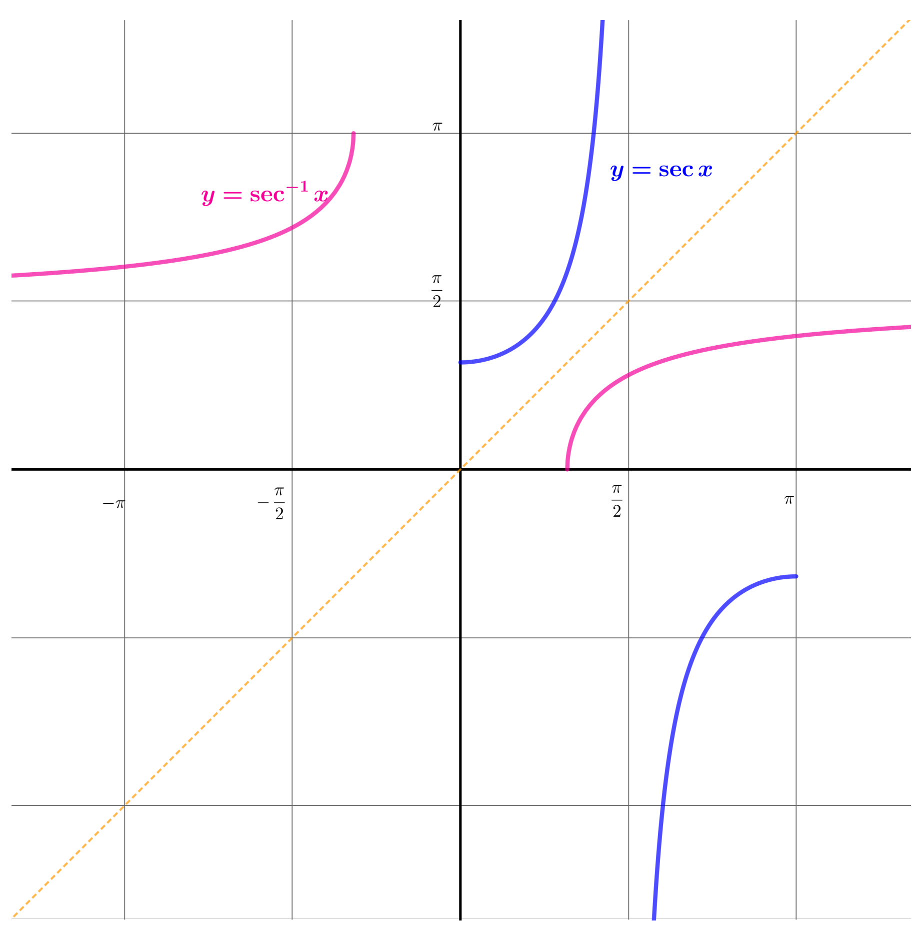
From the graph, gradient is always positive.

But the integral on the right is negative when x is negative. What went wrong?

$\sqrt{a^2b} = |a|b$, not ab

So

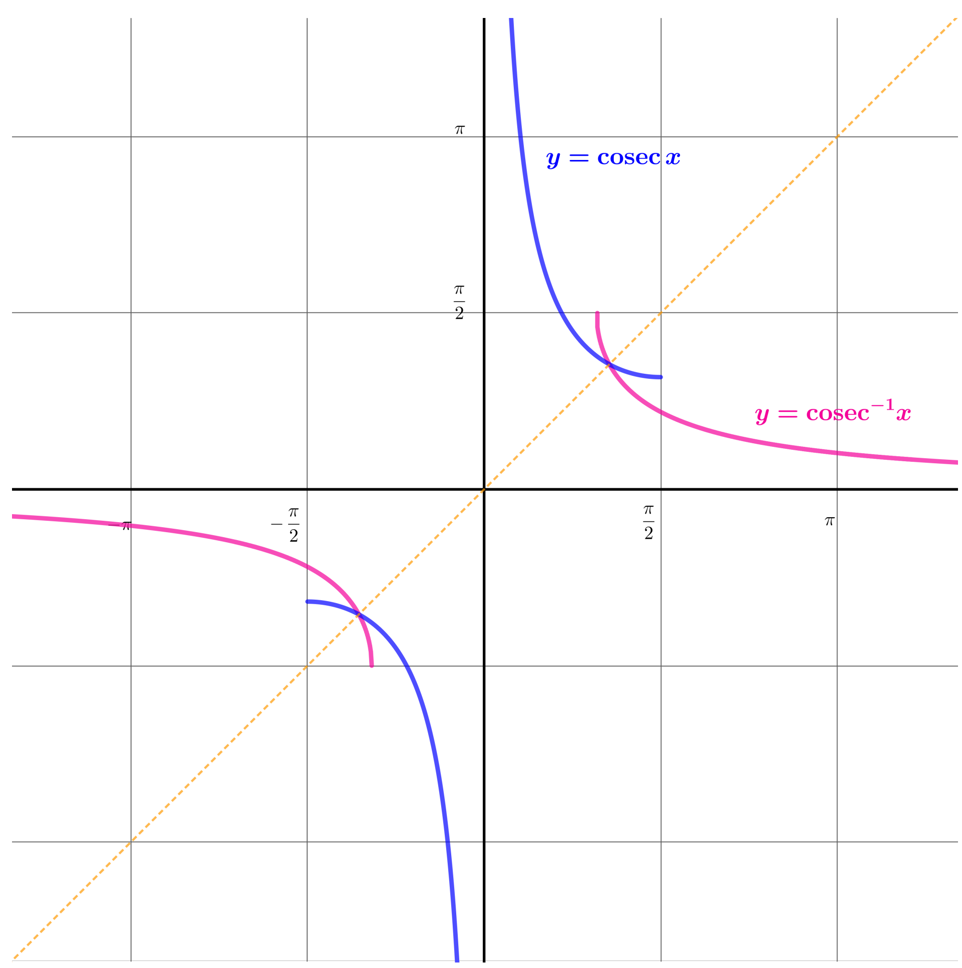
$$
\begin{aligned}
& y = \sec^{-1} x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
& \qquad = \frac{1}{|x|\,|x|}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$



What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec}^{-1} x$?

$$
\begin{aligned}
& y = \operatorname{cosec}^{-1} x \\
\Rightarrow\ & x = \operatorname{cosec} y \\
\Rightarrow\ & \frac{\mathrm{d}x}{\mathrm{d}y} = -\operatorname{cosec} y \cot y = \mp x\sqrt{x^2-1} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \mp \frac{1}{x\sqrt{x^2-1}}
\end{aligned}
$$

$$
\begin{aligned}
& y = \operatorname{cosec}^{-1} x \\
\Rightarrow\ & y = \cos^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}\frac{-1}{\sqrt{1-\frac{1}{x^2}}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$



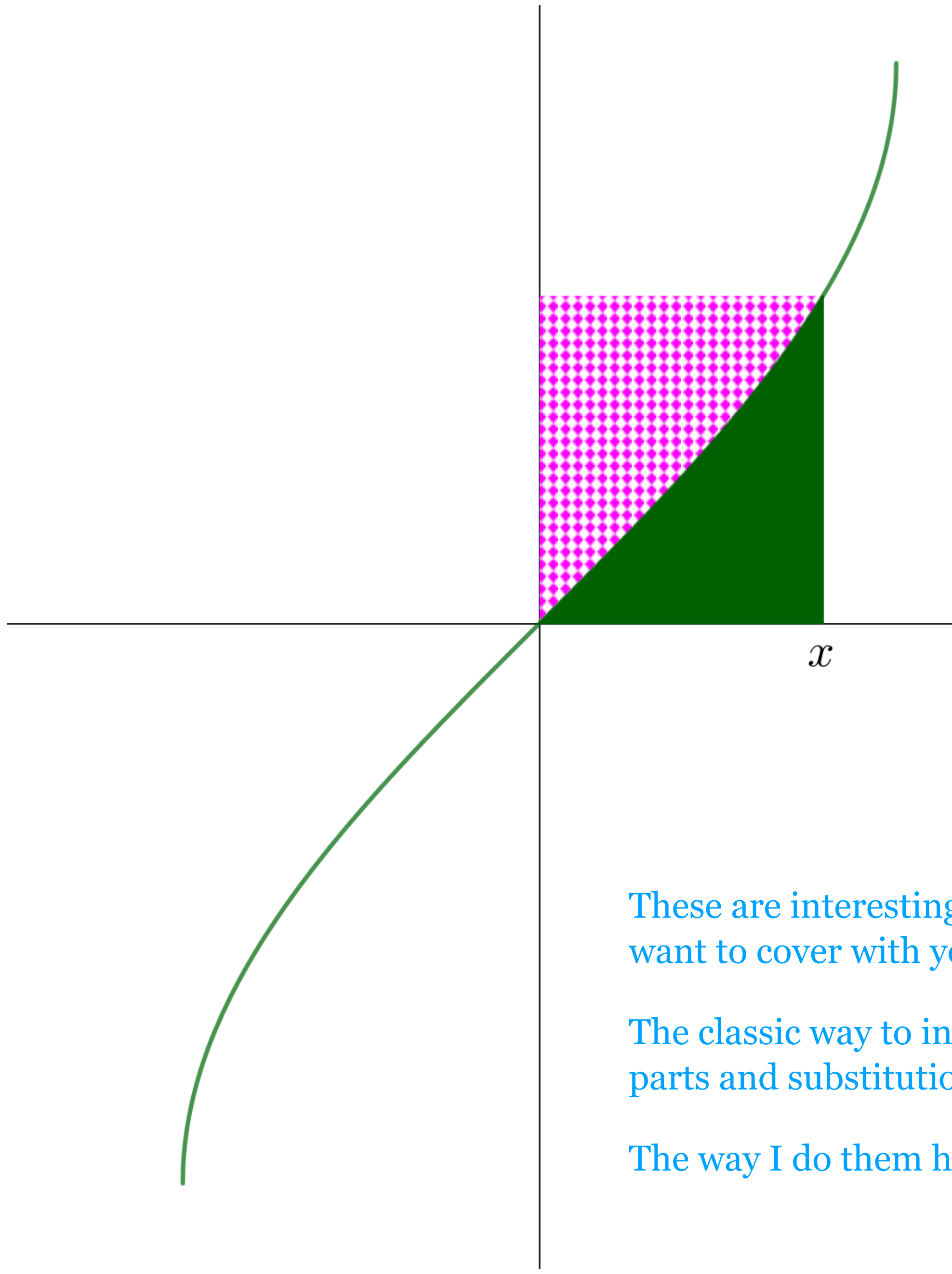
radical maths

What is $\dfrac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x$?

$$
\begin{aligned}
& y = \cot^{-1} x \\
\Rightarrow\ & x = \cot y \\
\Rightarrow\ & \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-\sin^2 y - \cos^2 y}{\sin^2 y} \\
& \qquad = -\operatorname{cosec}^2 y \\
& \qquad = -1 - \cot^2 y \\
& \qquad = -1 - x^2 \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}
\end{aligned}
$$

$$
\begin{aligned}
& y = \cot^{-1} x \\
\Rightarrow\ & y = \tan^{-1}\frac{1}{x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2} \times \frac{1}{1+\frac{1}{x^2}} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}
\end{aligned}
$$

integrals of inverse circular functions



These are interesting, but may well be more than you either need or want to cover with your students.

The classic way to integrate these functions is with a combination of parts and substitution (in either order).

The way I do them here is rather whimsical, but still instructive.

What is the area of the whole shaded rectangle?

The coordinates of the top right-hand vertex of the rectangle are $\left(x, \sin^{-1} x\right)$, so the rectangle area is $x \sin^{-1} x$.

Write the two shaded areas as integrals.

green (solid) area $= \displaystyle\int_0^x \sin^{-1} x \, \mathrm{d}x$

pink (chequered) area $= \displaystyle\int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y$

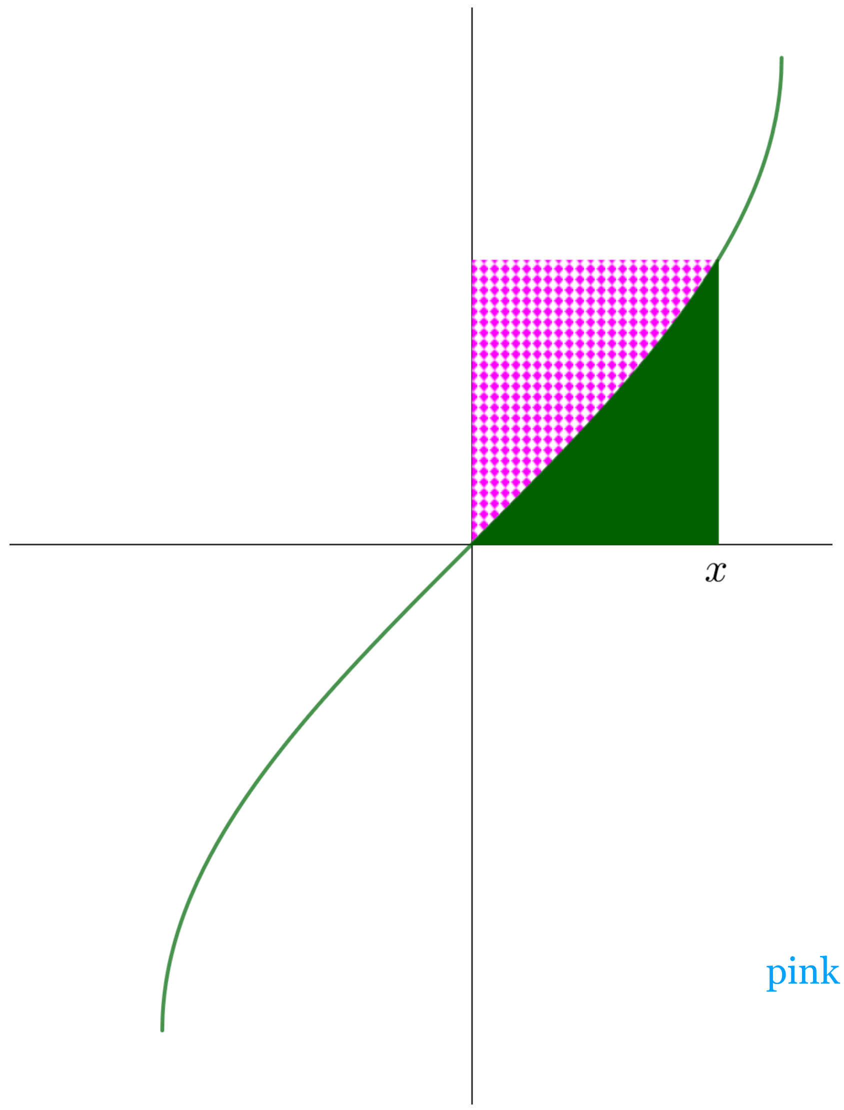
What is the area of the pink (chequered) area?

$$= \int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y$$

pink (chequered) area $= \left[-\cos y\right]_0^{\sin^{-1} x}$

$$= -\cos \sin^{-1} x - -1$$

$$= 1 - \sqrt{1 - x^2}$$



pink (chequered) area $= \int_0^{\sin^{-1} x} \sin y \, \mathrm{d}y$

$= \left[-\cos y\right]_0^{\sin^{-1} x}$

$= -\cos \sin^{-1} x - -1$

$= 1 - \sqrt{1 - x^2}$

Use these results to find

$$\int_0^x \sin^{-1} x \, \mathrm{d}x$$

$=$ green (solid) area $=$ rectangle $-$ pink (chequered) area

$= x \sin^{-1} x - 1 + \sqrt{1 - x^2}$

and

$$\int \sin^{-1} x \, \mathrm{d}x$$

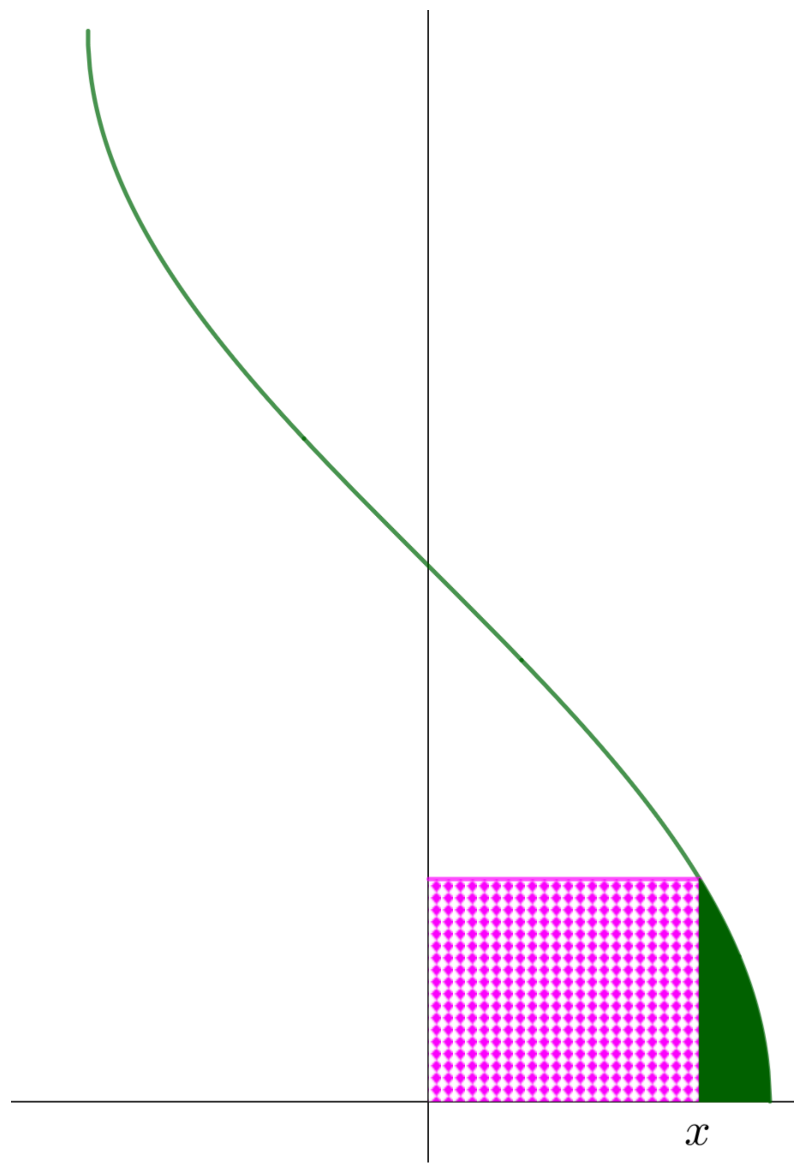
$= x \sin^{-1} x - 1 + \sqrt{1 - x^2} +$ constant

$= x \sin^{-1} x + \sqrt{1 - x^2} + c$

remember: 1 less than a constant is just another constant

radical maths

Here is the graph $y = \cos^{-1} x$.



What is the area of the pink (chequered) rectangle?

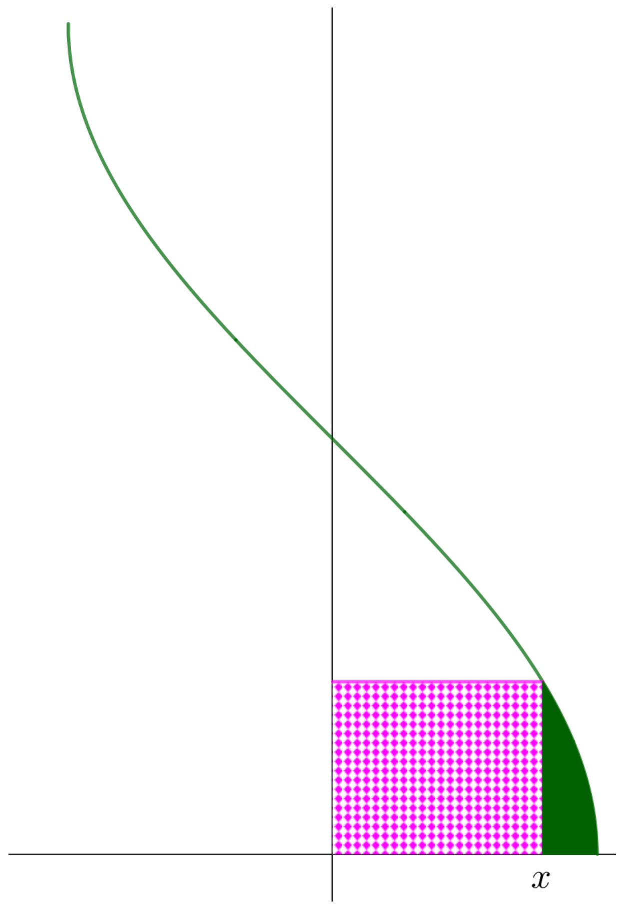
The coordinates of the top right-hand vertex of the rectangle are $\left(x, \cos^{-1} x\right)$, so the rectangle area is $x \cos^{-1} x$.

Write the entire shaded area as an integral.

$$\text{entire shaded area} = \int_0^{\cos^{-1} x} \cos y \, \mathrm{d}y$$

Write the green (solid) shaded area as an integral.

$$\text{green (solid) area} = \int_x^1 \cos^{-1} x \, \mathrm{d}x$$



$$\text{entire shaded area} = \int_{-\cos^{-1}x}^{0} \cos y \, \mathrm{d}y$$
$$= \left[\sin y\right]_{-\cos^{-1}x}^{0}$$
$$= \sin\cos^{-1}x$$
$$= \sqrt{1-x^2}$$

Use these results to find

$$\int_x^1 \cos^{-1}x \, \mathrm{d}x$$

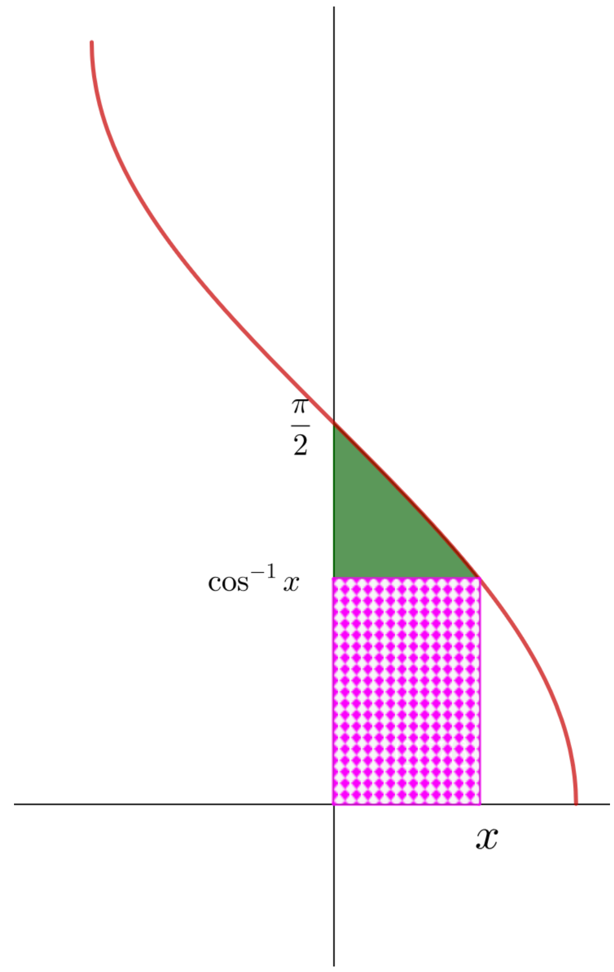
$$= \text{green (solid) area} = \text{entire shaded area} - \text{pink (chequered) area}$$
$$= \sqrt{1-x^2} - x\cos^{-1}x$$

and

$$\int \cos^{-1}x \, \mathrm{d}x$$

$$I(x) = \int \cos^{-1}x \, \mathrm{d}x$$
$$\Rightarrow \int_x^1 \cos^{-1}x \, \mathrm{d}x = \left[I\right]_x^1$$
$$= I(1) - I(x)$$
$$= 1 - I(x)$$
$$\Rightarrow I(x) = 1 + x\cos^{-1}x - \sqrt{1-x^2} + c$$
$$= x\cos^{-1}x - \sqrt{1-x^2} + c'$$

Here is the graph $y = \cos^{-1} x$.



Use these results to find

$$\int_0^x \cos^{-1} x \, \mathrm{d}x$$

$= \text{green (solid) area} + \text{pink (chequered) area}$

$$= \int_0^{\frac{\pi}{2} - \cos^{-1} x} \sin x \, \mathrm{d}x + x \cos^{-1} x$$

and

$$= \int_0^{\sin^{-1} x} \sin x \, \mathrm{d}x + x \cos^{-1} x$$

$$= -\left[\cos x\right]_0^{\sin^{-1} x} + x \cos^{-1} x$$

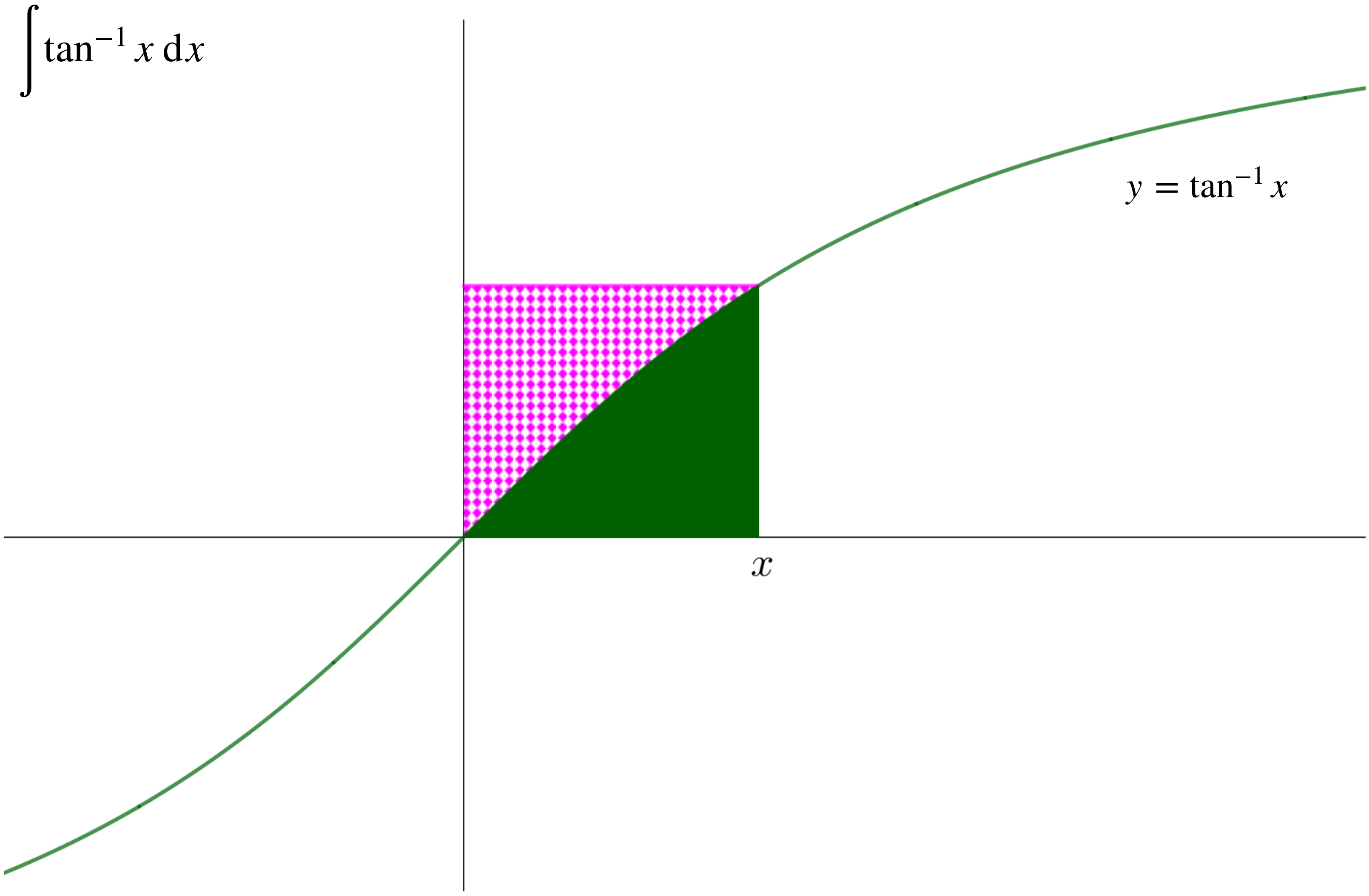
$$= -\cos(\sin^{-1} x) + 1 + x \cos^{-1} x$$

$$= x \cos^{-1} x - \sqrt{1 - x^2} + 1$$

$$\int \cos^{-1} x \, \mathrm{d}x \qquad = x \cos^{-1} x - \sqrt{1 - x^2} + c$$

Use a similar strategy to find

$$\int \tan^{-1} x \, \mathrm{d}x$$



$$\int \tan^{-1} x \, \mathrm{d}x = x \tan^{-1} x - \int_0^{\tan^{-1} x} \tan y \, \mathrm{d}y$$

$$\begin{aligned}\int \tan^{-1} x \, \mathrm{d}x &= x \tan^{-1} x + \left[\ln \cos y\right]_0^{\tan^{-1} x} \\ &= x \tan^{-1} x + \ln\left(\cos\left(\tan^{-1} x\right)\right) \\ &= x \tan^{-1} x + \ln \frac{1}{\sqrt{1+x^2}} \\ &= x \tan^{-1} x - \frac{1}{2} \ln \sqrt{1+x^2}\end{aligned}$$

$$\int \sin^{-1} x \, \mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad v = x$$

$$u = \sin^{-1} x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}$$

$$\int \sin^{-1} x \, \mathrm{d}x = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

$$= x \sin^{-1} x + \sqrt{1 - x^2}$$

where the final integral is done either by substitution or by "intuition".

$$\int \cos^{-1} x \, \mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad v = x$$

$$u = \cos^{-1} x \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{1 - x^2}}$$

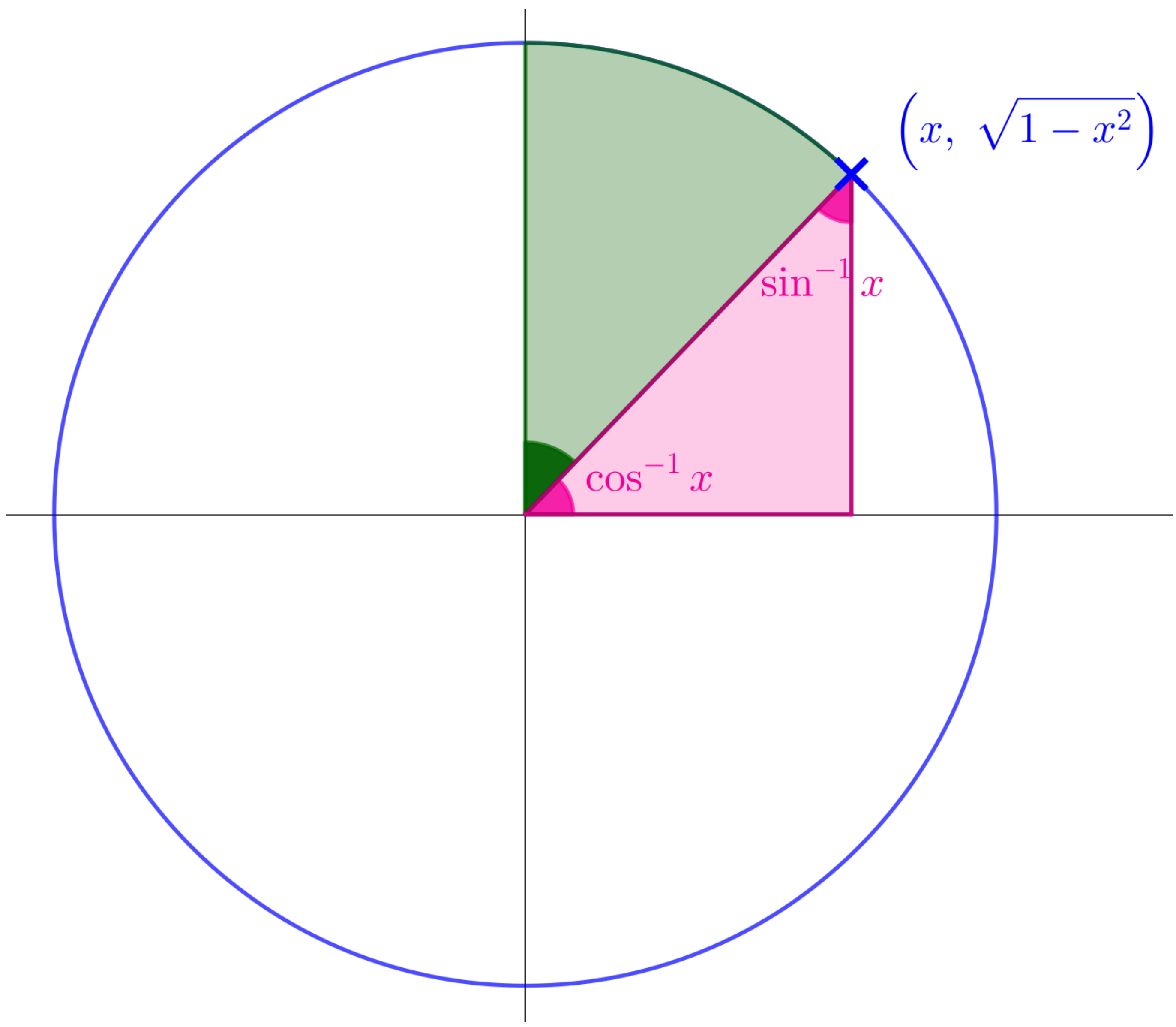
$$\int \cos^{-1} x \, \mathrm{d}x = x \cos^{-1} x + \int \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

$$= x \cos^{-1} x - \sqrt{1 - x^2}$$

Another integral using inverse circular functions

By finding the two shaded areas, find

$\int_0^x \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



$$\int_0^x \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}$$

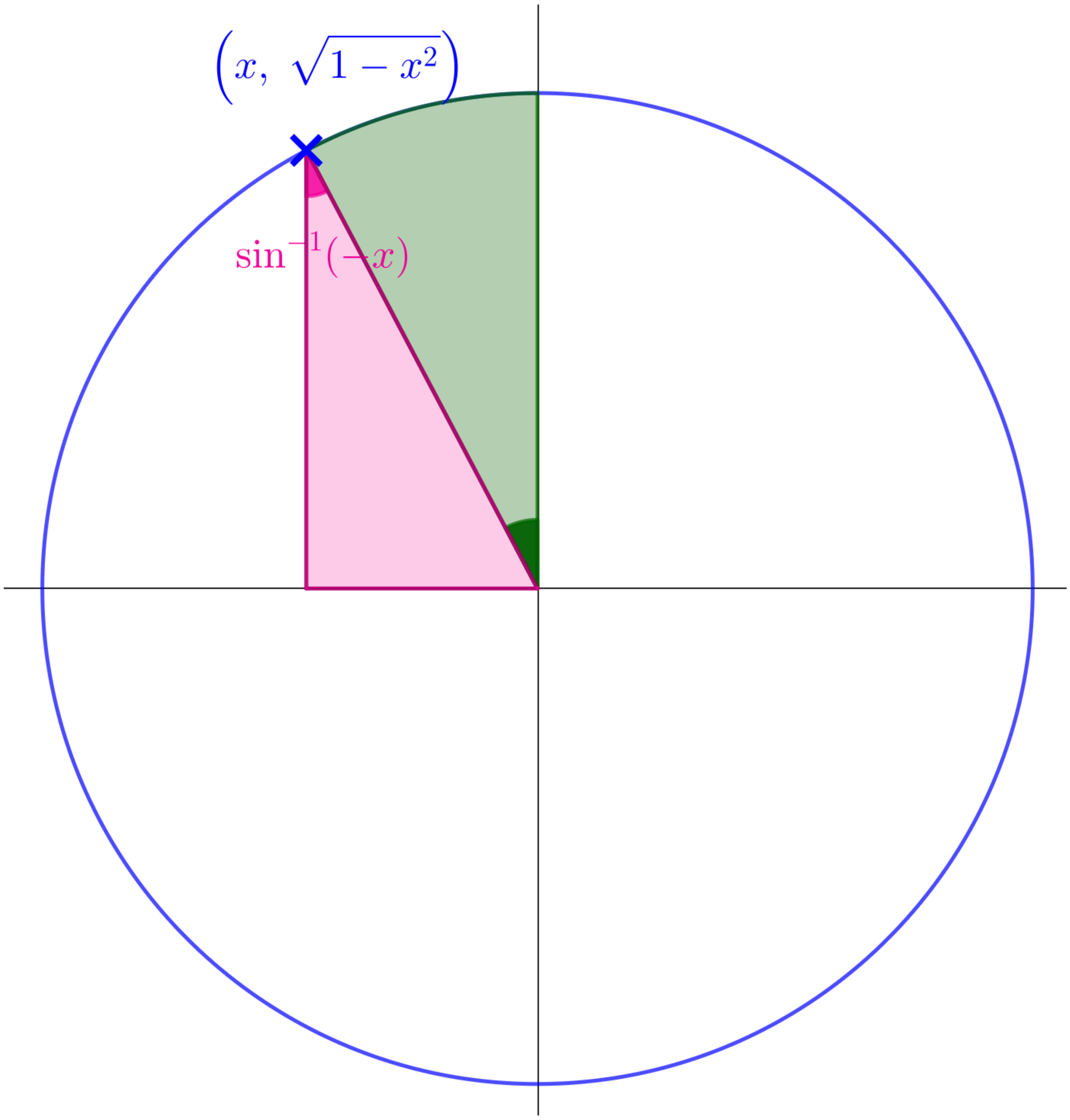
because the green angle is $\sin^{-1} x$ and the area of a segment is $\frac{1}{2}r^2\theta$

and

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + c$$

$x < 0$, by finding the two shaded areas, find

$\int_x^0 \sqrt{1-x^2}\,\mathrm{d}x$ and hence find $\int \sqrt{1-x^2}\,\mathrm{d}x$.



$$\begin{aligned}\int_x^0 \sqrt{1-x^2}\,\mathrm{d}x &= \left[\frac{1}{2}\sin^{-1}(-x) + \frac{1}{2}(-x)\sqrt{1-x^2}\right]_x^0 \\ &= \left[-\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2}\right]_x^0 \\ &= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}\end{aligned}$$

and

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + c$$

Use the substitutions $u = \sin x$ and $u = \cos x$ to find $\int \sqrt{1-x^2}\,\mathrm{d}x$

$$
\begin{aligned}
\int \sqrt{1-x^2}\,\mathrm{d}x &= \int \sqrt{1-x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}x \\
&= \int \cos^2 u\,\mathrm{d}u \\
&= \frac{1}{2}\int \cos 2u + 1\,\mathrm{d}u \\
&= \frac{1}{4}\sin 2u + \frac{1}{2}u + c \\
&= \frac{1}{2}\sin u \cos u + \frac{1}{2}u + c \\
&= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + c
\end{aligned}
$$

$$
\begin{aligned}
\int \sqrt{1-x^2}\,\mathrm{d}x &= \int \sqrt{1-x^2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}x \\
&= -\int \sin^2 u\,\mathrm{d}u \\
&= \frac{1}{2}\int \cos 2u - 1\,\mathrm{d}u \\
&= \frac{1}{4}\sin 2u \frac{1}{2}u + c \\
&= \frac{1}{2}\sin u \cos u + \frac{1}{2}u + c \\
&= \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x + c
\end{aligned}
$$