for independence
for confidence
for creativity
for insight

Circular functions 7

Transforming and adding circular functions

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Transforming and adding circular functions

$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x$, $\cos^{-1} x$, $\cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

What happens when two perfectly regular waveforms interact this each other? Many wonderful things that can be analysed and explained with some rather complicated equations. Here, we will start to think about the simplest examples: when the wavelengths of the two waves are the same, and when they are out of phase by a quarter of a wavelength. Even slightly harder examples will have to wait for university, but they will rely on this first step.

Here, we will investigate what happens when a wave whose equation is something like $y = 2\sin x$ meets a wave whose equation is something like $y = 5\cos x$. We will discover that the combination is another sine wave, and we can easily figure out its amplitude and the difference between its phase and that of the original sine wave.

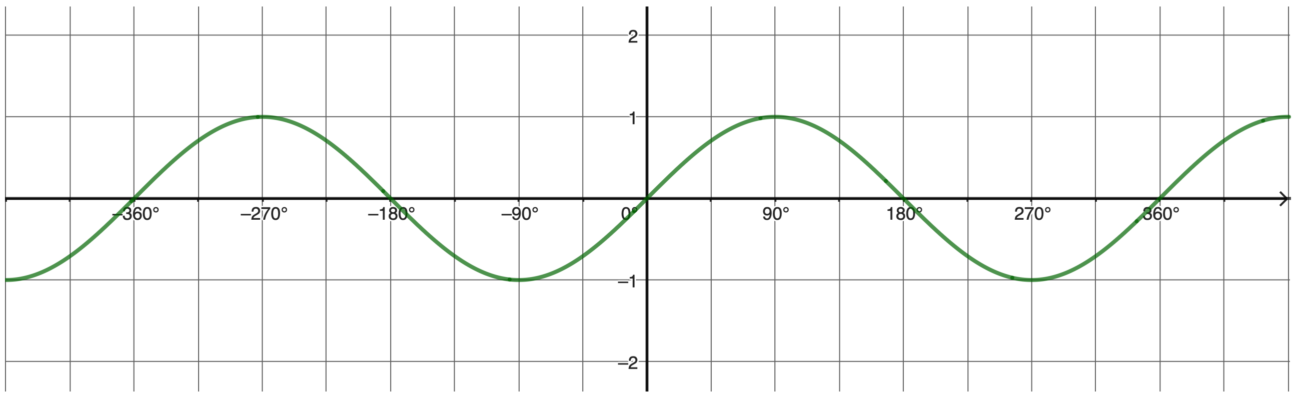
My approach is highly graphical: I explore graph transformations first and then see how these help with the waves superimposition problem.

Investigating functions like $f(x) = 2\sin x + 5\cos x$ through graphs and transformations reinforces the association of circular functions with graphs, and gives an opportunity to explore the relationship between transformations and compound angle formulas.

Here is the graph $y = \sin x$.

Translate the graph left by $45°$.

What is the equation of your new graph?



Now stretch the new graph parallel to the y axis scale factor $\sqrt{2}$.

What is the equation of your latest graph?

This first task will raise the question: why add 45° to x when the graph is translated to the left. The easy way to address this is to look at the point on the translated graph when $x = -45°$:

$$x = -45° \Rightarrow x + 45° = 0 \Rightarrow \sin\left(x + 45°\right) = 0$$

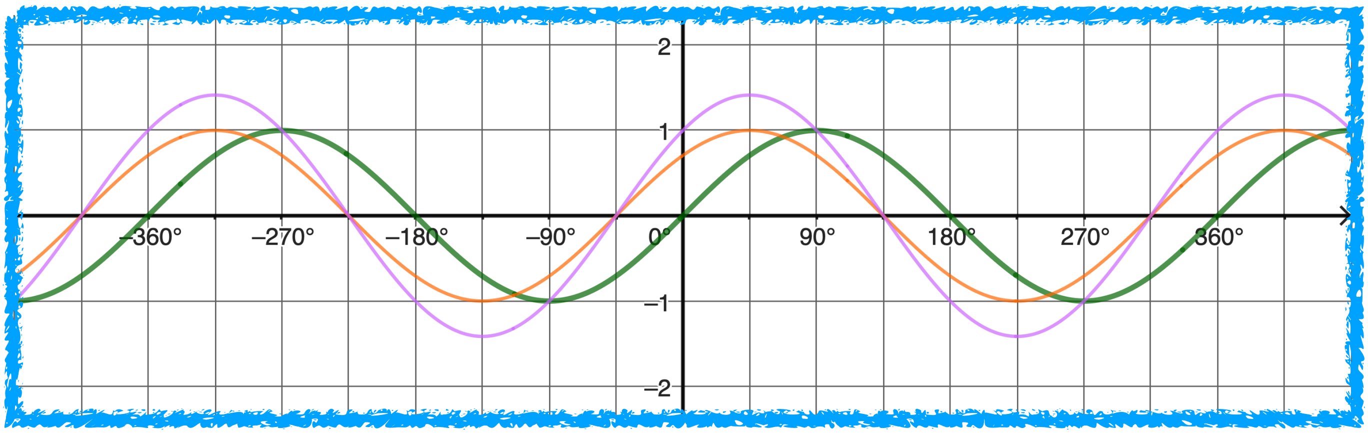
So the first “new”graph is

$$y = \sin\left(x + 45°\right)$$

and the second “new” graph is

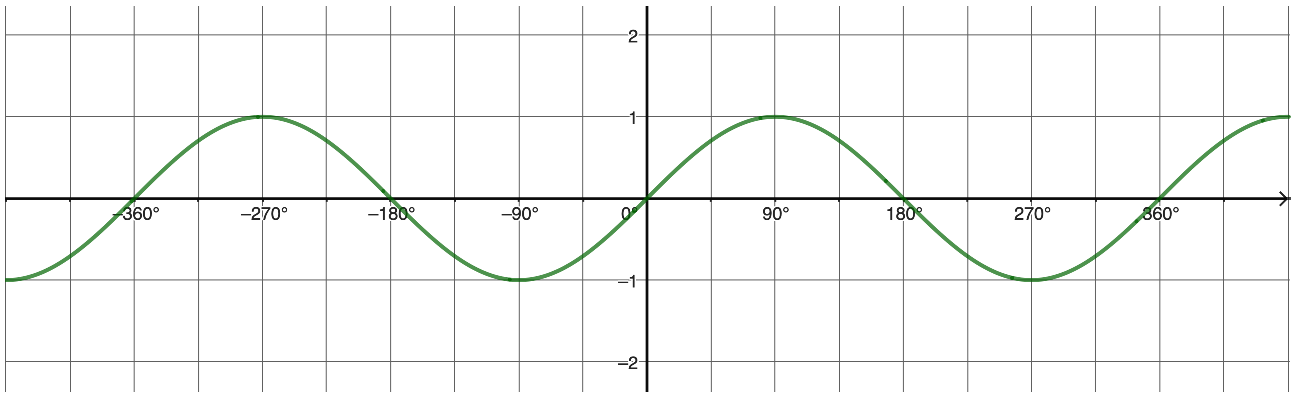
$$y = \sqrt{2}\sin\left(x + 45°\right)$$

Take care when stretching: the points where the graph crosses the x axis do not move! So the pink and orange curves on this diagram intersect on the x axis.



Now use a compound angle formula to expand $\sqrt{2}\sin\left(x + 45^\circ\right)$

Draw the graph $y = \sin x + \cos x$



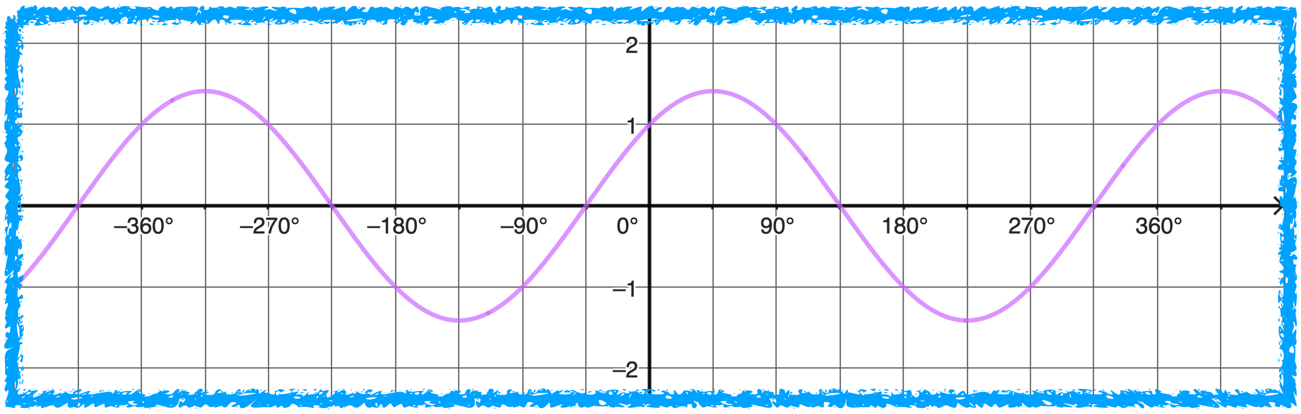
$$\sqrt{2}\sin\left(x+45^\circ\right) = \sqrt{2}\left(\sin x\cos 45^\circ + \cos x\sin 45^\circ\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sin x + \cos x$$

Don't let your students use their calculators for sin and cos of 45°!

Use the fact that $\sin x + \cos x = \sqrt{2}\sin\left(x+45^\circ\right)$

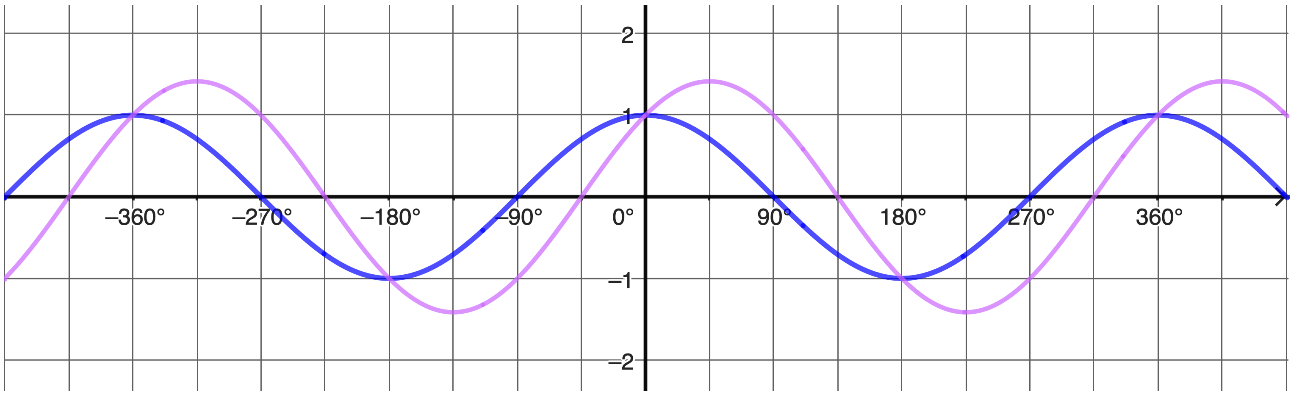


We can achieve the same graph

$y = \sin x + \cos x$

by applying transformations to the graph $y = \cos x$. Here are the two graphs together

What transformations will turn the blue ($y = \cos x$) into the pink ($y = \sin x + \cos x$)?



Here, students explore the transformations idea a bit more deeply by starting with $y = \cos x$ and figuring out how to get from this to the red graph $y = \sin x + \cos x$.

Translate right by 45° and stretch parallel to the y axis scale factor $\sqrt{2}$ to give the equation

$$y = \sqrt{2}\cos(x - 45^\circ)$$

You could also translate right by $45^\circ + 360n^\circ$ and stretch parallel to the y axis scale factor $\sqrt{2}$.

You could even translate right by $45^\circ + (2n + 1)180^\circ$ and stretch parallel to the y axis scale factor $-\sqrt{2}$.

The video looks at these alternatives a bit more closely.

Ultimately, to keep things definite, we will always choose a positive value of R and then the smallest possible value of α.

Expand $\sqrt{2}\cos\left(x - 45^\circ\right)$

$$\begin{aligned}\sqrt{2}\cos\left(x - 45^\circ\right) &= \sqrt{2}\left(\cos x \cos 45^\circ + \sin x \sin 45^\circ\right)\\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)\\ &= \cos x + \sin x\end{aligned}$$

Don't let your students use their calculators for sin and cos of 45°!

What do you notice about the graphs

$y = \sqrt{2}\sin\left(x + 45^\circ\right)$

$y = \sqrt{2}\cos\left(x - 45^\circ\right)$

$y = \sin x + \cos x$

At this point, they have discovered that

$$\sqrt{2}\sin\left(x + 45^{\circ}\right) = \sqrt{2}\cos\left(x - 45^{\circ}\right) = \sin x + \cos x$$

They could have reached this point far more easily simply by using the compound angle formulas, but, by looking closely at graphs, their understanding will have increased so much more.

What transformations of $y = \sin x$ or $y = \cos x$ will result in the graphs

$$y = \sqrt{2}\sin(x + 405^\circ)$$

$$y = \sqrt{2}\cos(x - 405^\circ)$$

$$y = -\sqrt{2}\sin(x + 225^\circ)$$

$$y = -\sqrt{2}\cos(x - 225^\circ)$$

What is the difference between these graphs and the graph $y = \sin x + \cos x$?

Expanding the brackets in each of these shows that they are also the graph $y = \sin x + \cos x$

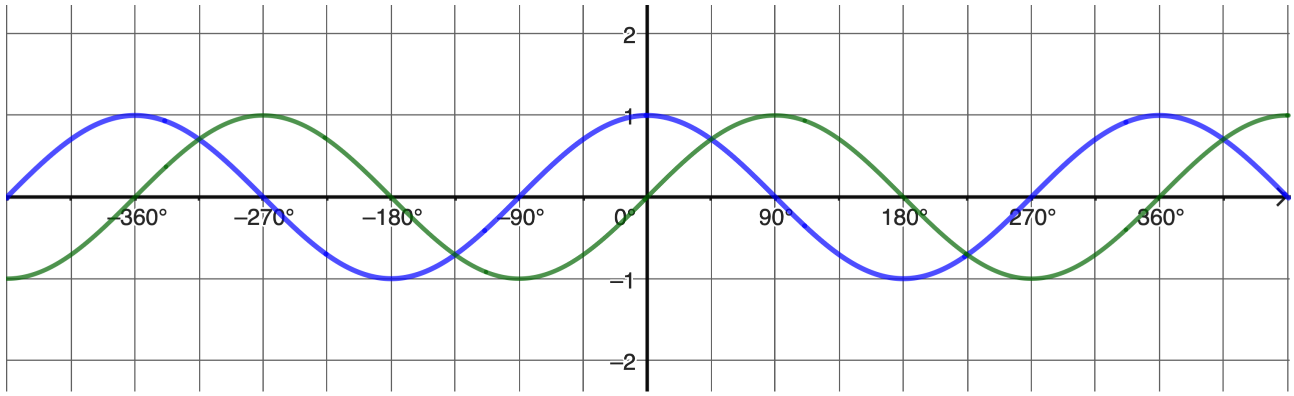
Translating left or right by $45^\circ + 180n^\circ$ and stretching parallel to the y axis scale factor $\sqrt{2}$ when n is even and $-\sqrt{2}$ when n is odd will always yield the same curve, and you can see a demonstration of this on the video.

Ultimately, to keep things definite and straightforward, we will always choose the positive value of R and then the smallest positive value of α.

We can also look at this from the point of view of a molecule being moved around by the two waves $y = \sin x$ and $y = \cos x$.

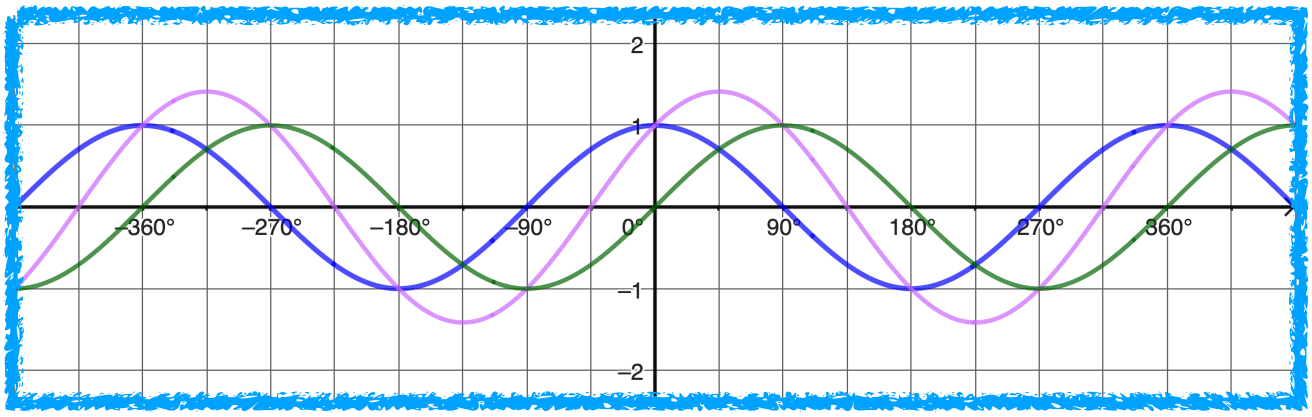
Draw the graph obtained by "adding up" the two graphs.

What is the equation of your new graph?



By "adding", I mean, for each x value, adding the two y values from the green and blue graphs to get a new y value.

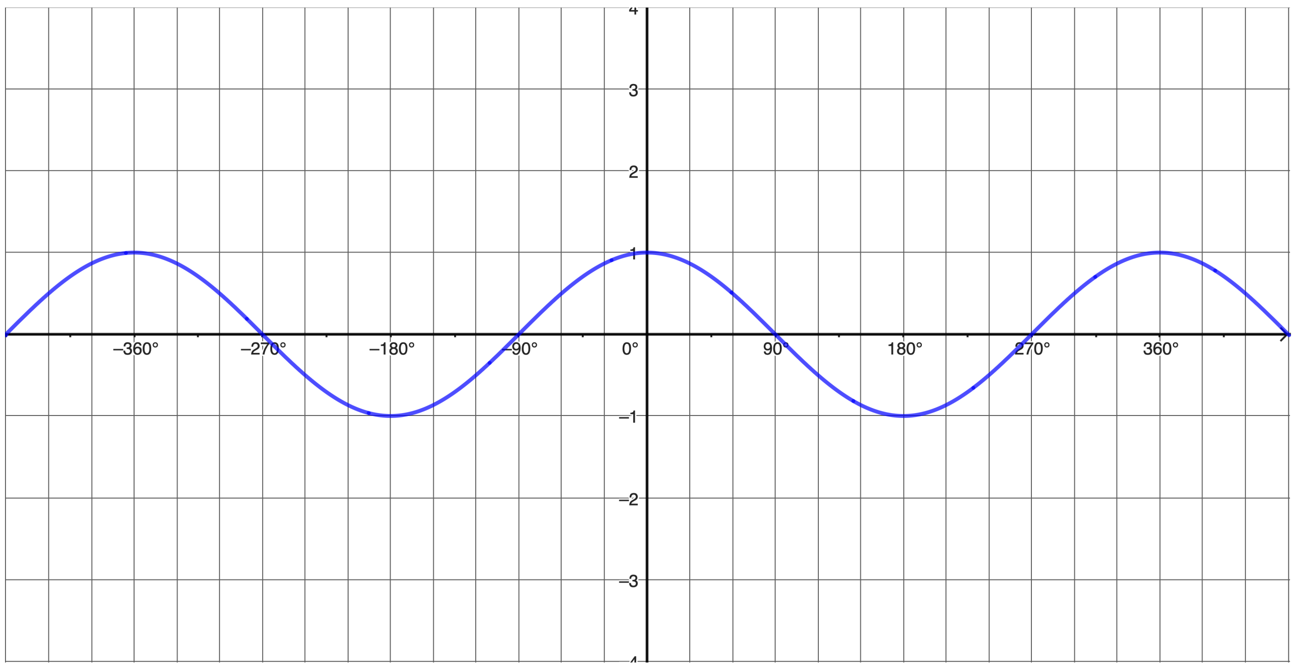
This will generate the graph $y = \sin x + \cos x$ in a more direct way than by transformations—probably not be a curriculum skill, but it does offer another angle on understanding.



Now for a slightly trickier example.

What transformations take the graph $y = \cos x$ to the graph

$y = 2\sqrt{3}\cos(x + 30°)$?



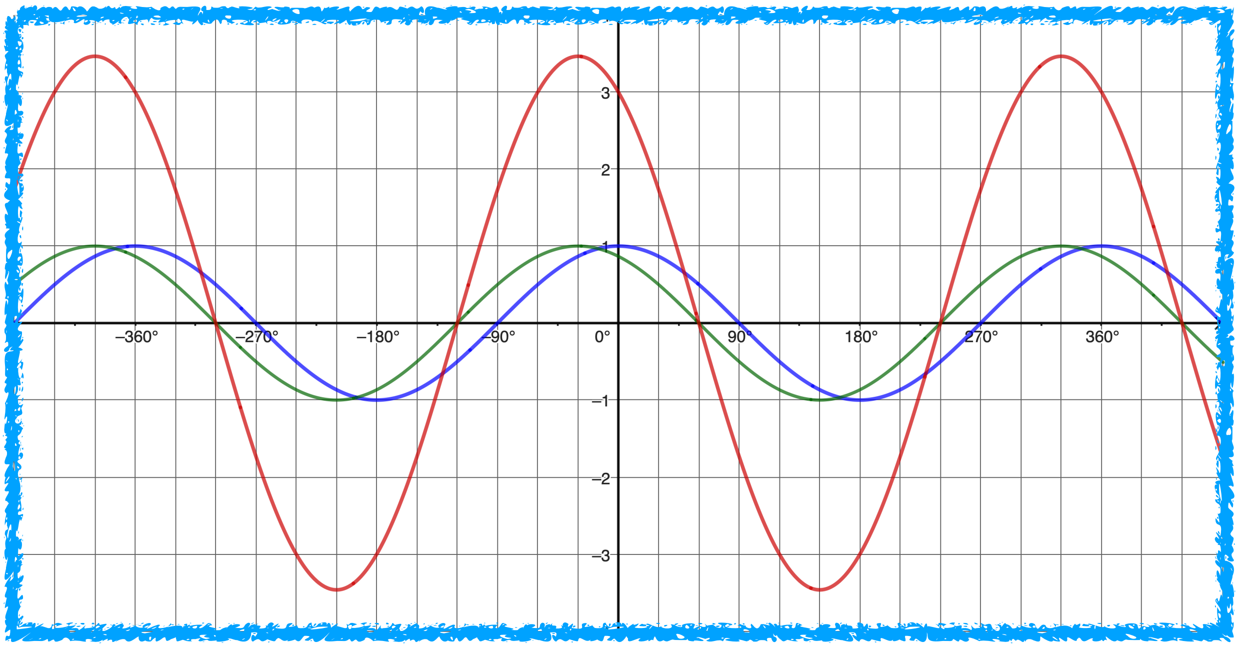
On the same axes, draw the graphs

$y = \cos(x + 30°)$ and $y = 2\sqrt{3}\cos(x + 30°)$.

A slightly harder example, but the principles are the same.

Translate left by 30° and stretch parallel to the y axis scale factor $2\sqrt{3}$.

You could also add or subtract multiples of π, changing the sign of the scale factor if necessary. However, for simplicity's sake, we will always choose the smallest translation and the positive scale factor.



Expand the brackets in $y = 2\sqrt{3}\cos(x + 30^\circ)$.

$$2\sqrt{3}\cos(x+30^\circ) = 2\sqrt{3}\left(\cos x\cos 30^\circ - \sin x\sin 30^\circ\right)$$

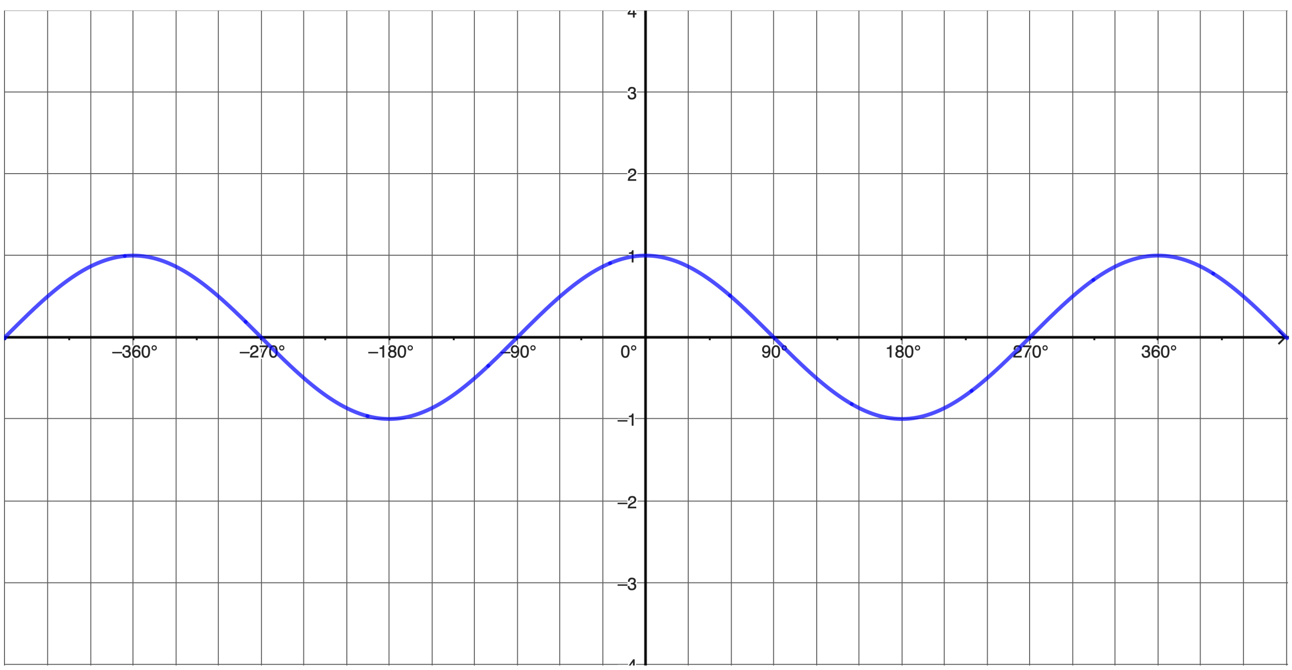
$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$= 3\cos x - \sqrt{3}\sin x$$

Don't let your students use their calculators for sin and cos of 30°!

Draw the graph

$$y = 3\cos x - \sqrt{3}\sin x$$



$$2\sqrt{3}\cos(x+30^\circ) = 2\sqrt{3}\left(\cos x\cos 30^\circ - \sin x \sin 30^\circ\right)$$

$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$

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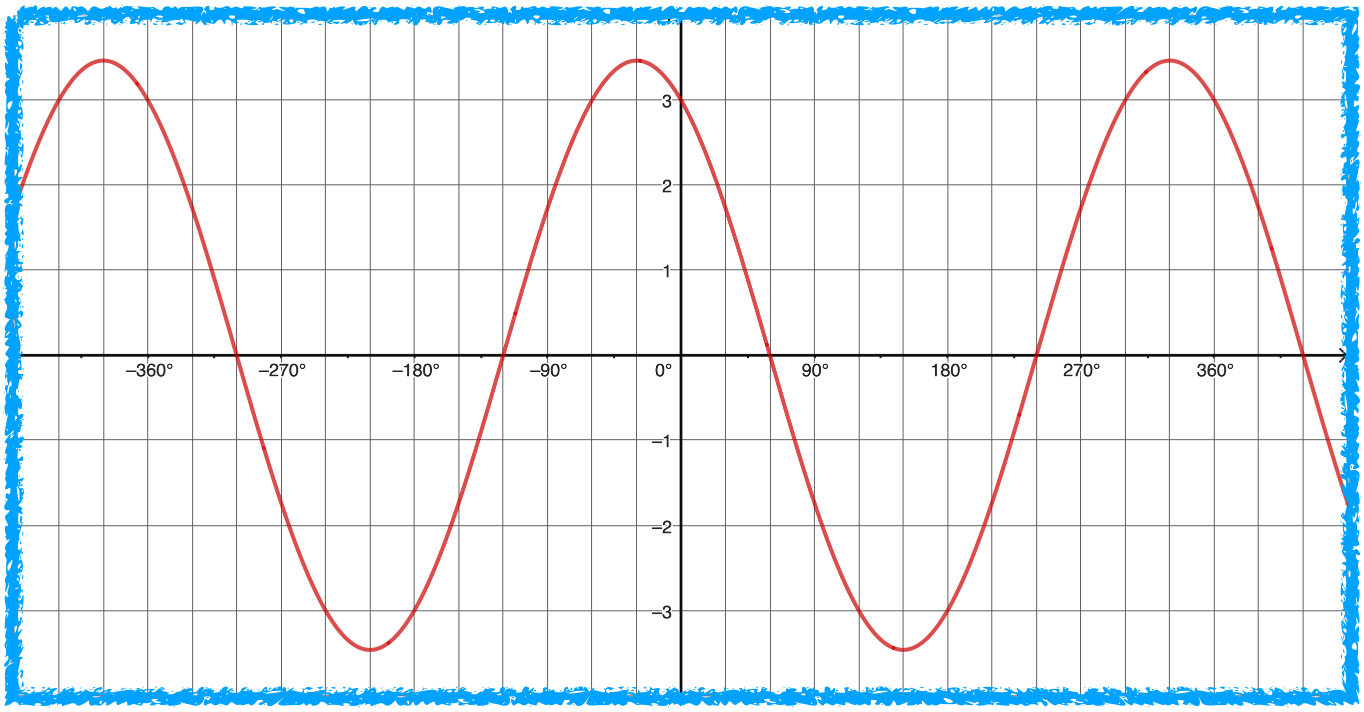


Next, let's take a similar but very slightly different example: the wave $y = 3\sin x$ meets the wave $y = \sqrt{3}\cos x$

Firstly, expand the brackets in $y = R\sin(x + \alpha)$.

$$y = R(\sin x \cos\alpha + \sin\alpha\cos x)$$

$$= (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$$

Obviously brackets are not needed, but they will help in a minute.

If $R\sin(x + \alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

$$R\sin(x + \alpha) = (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$$

$$= 3\sin x + \sqrt{3}\cos x$$

This must be true for every value of x, and the only way this can happen is if

$R\cos\alpha\sin x = 3\sin x \Rightarrow R\cos\alpha = 3$

and

$R\sin\alpha\cos x = \sqrt{3}\cos x \Rightarrow R\sin\alpha = \sqrt{3}$

Use these results to find $\tan\alpha$.

Hence find the smallest positive value of α.

Find $(R\sin\alpha)^2 + (R\cos\alpha)^2$.

Hence find R.

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3} \qquad \Rightarrow \tan\alpha = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \alpha = 30°$$

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = \left(\sqrt{3}\right)^2 + 3^2 = 12$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 12$$

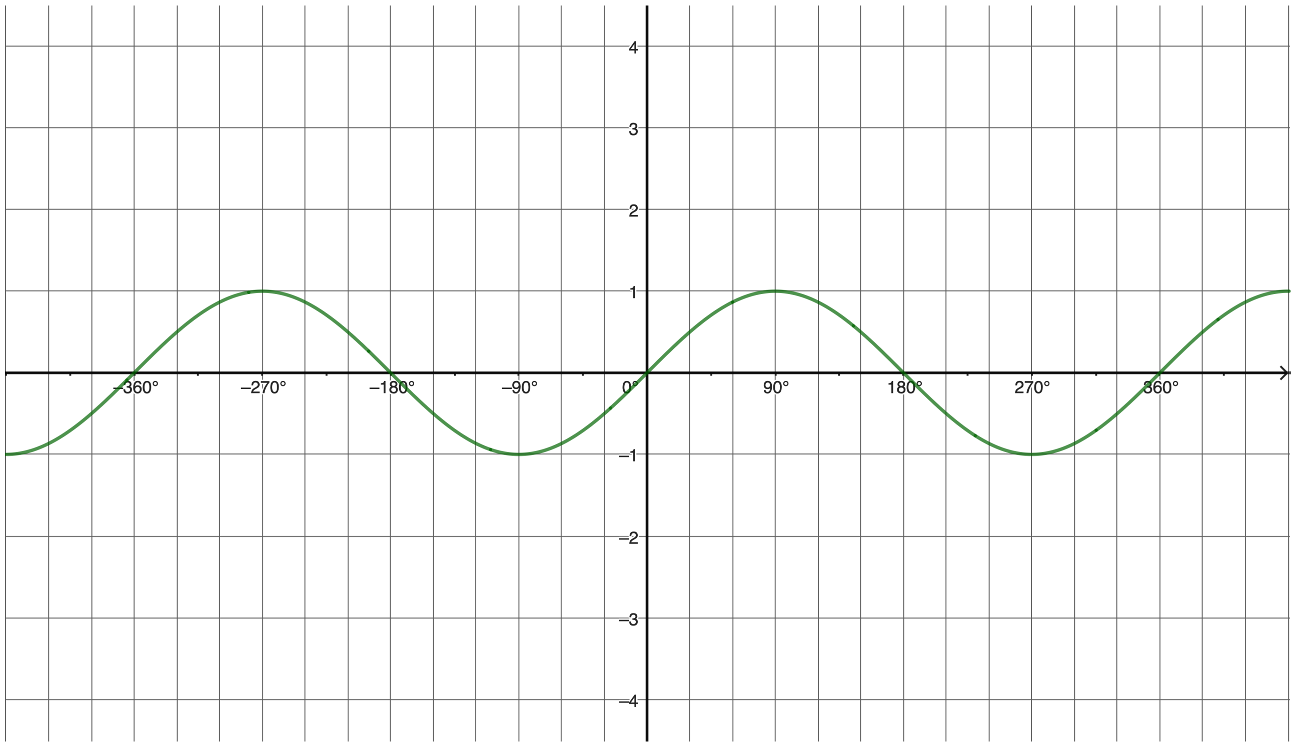
$$\Rightarrow R^2 = 12$$

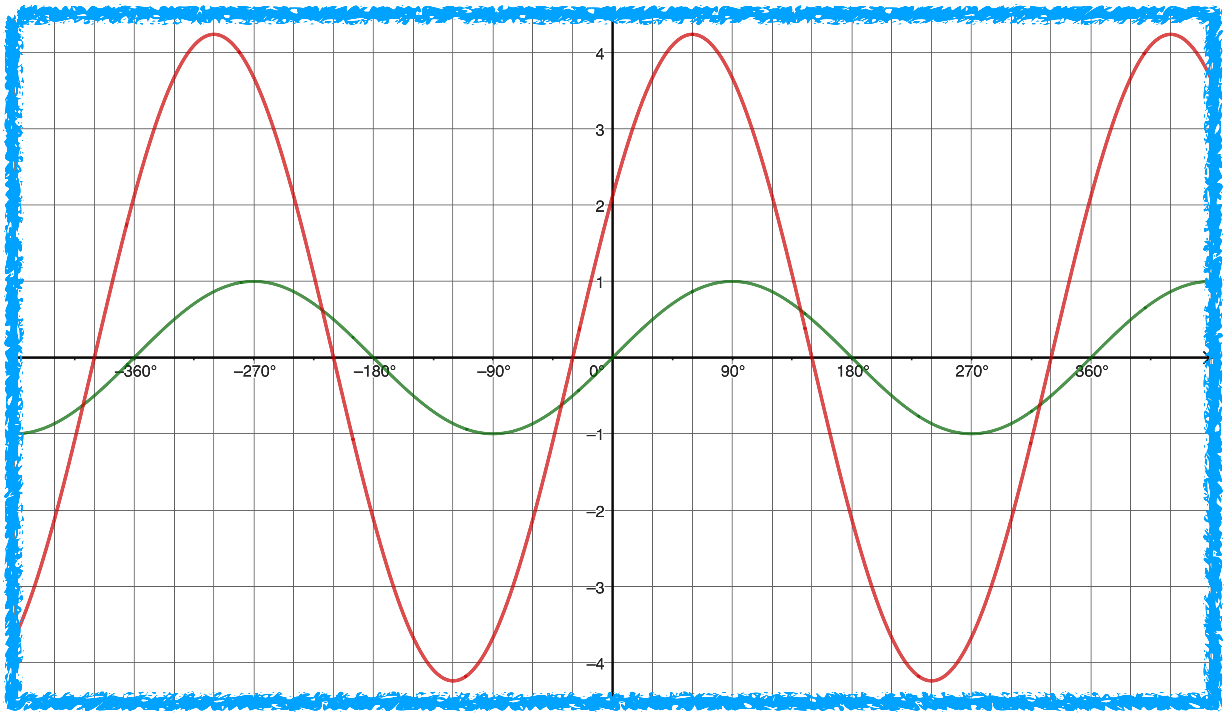
$$\Rightarrow R = 2\sqrt{3}$$

Use these results to find the transformations that take the graph $y = \sin x$ to the graph

$$y = 3\sin x + \sqrt{3}\cos x$$

and draw this graph.





$$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin(x + 30^\circ)$$

Translation by $\begin{pmatrix} -30^\circ \\ 0 \end{pmatrix}$ and stretch parallel to y axis scale factor $3\sqrt{2}$.

We could also do it this way.

If $R\cos(x - \alpha) = 3\sin x + \sqrt{3}\cos x$, find

$R\sin\alpha$

$R\cos\alpha$

Use these results to find $\tan\alpha$.

Hence find the smallest positive value of α.

Find $(R\sin\alpha)^2 + (R\cos\alpha)^2$.

Hence find R.

$$R\cos(x-\alpha) = R(\cos x\cos\alpha + \sin\alpha\sin x)$$

$$= (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

$$R\sin\alpha = 3$$

$$R\cos\alpha = \sqrt{3}$$

$$\Rightarrow \frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{\sqrt{3}}$$

$$\Rightarrow \tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\Rightarrow \alpha = 60^\circ$$

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 3^2 + \left(\sqrt{3}\right)^2 = 12$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 12$$

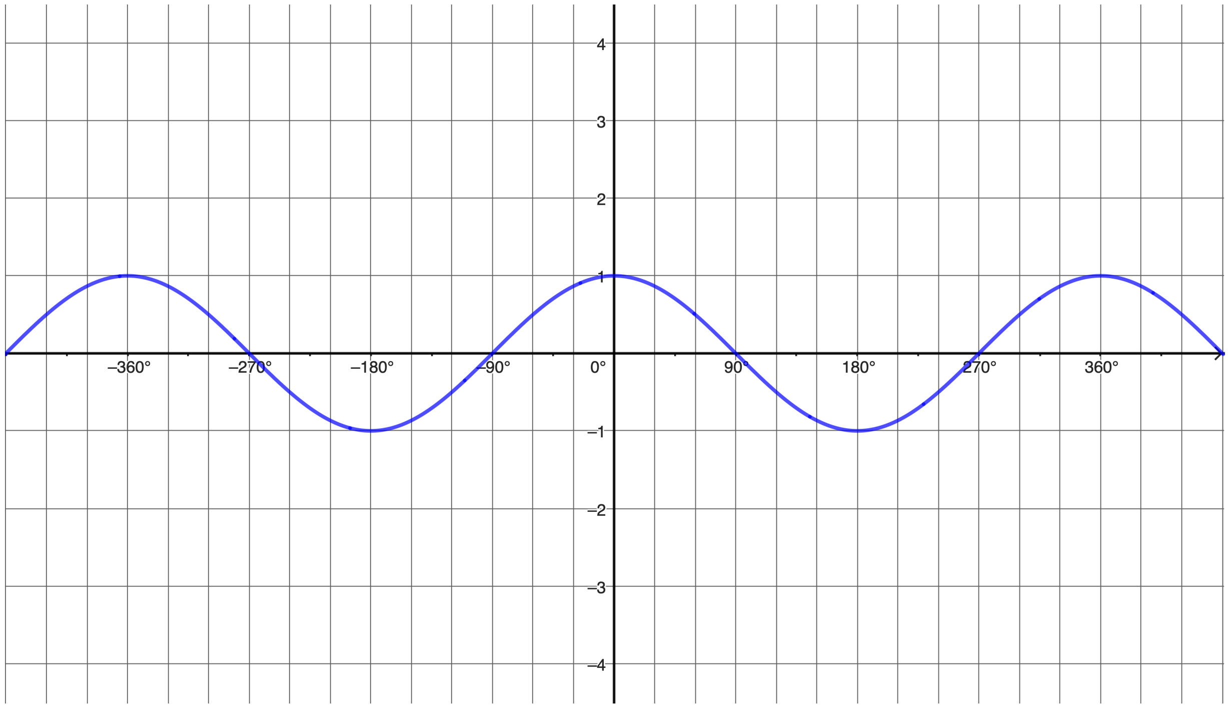
$$\Rightarrow R^2 = 12$$

$$\Rightarrow R = 2\sqrt{3}$$

Use these results to find the transformations that take the graph $y = \cos x$ to the graph

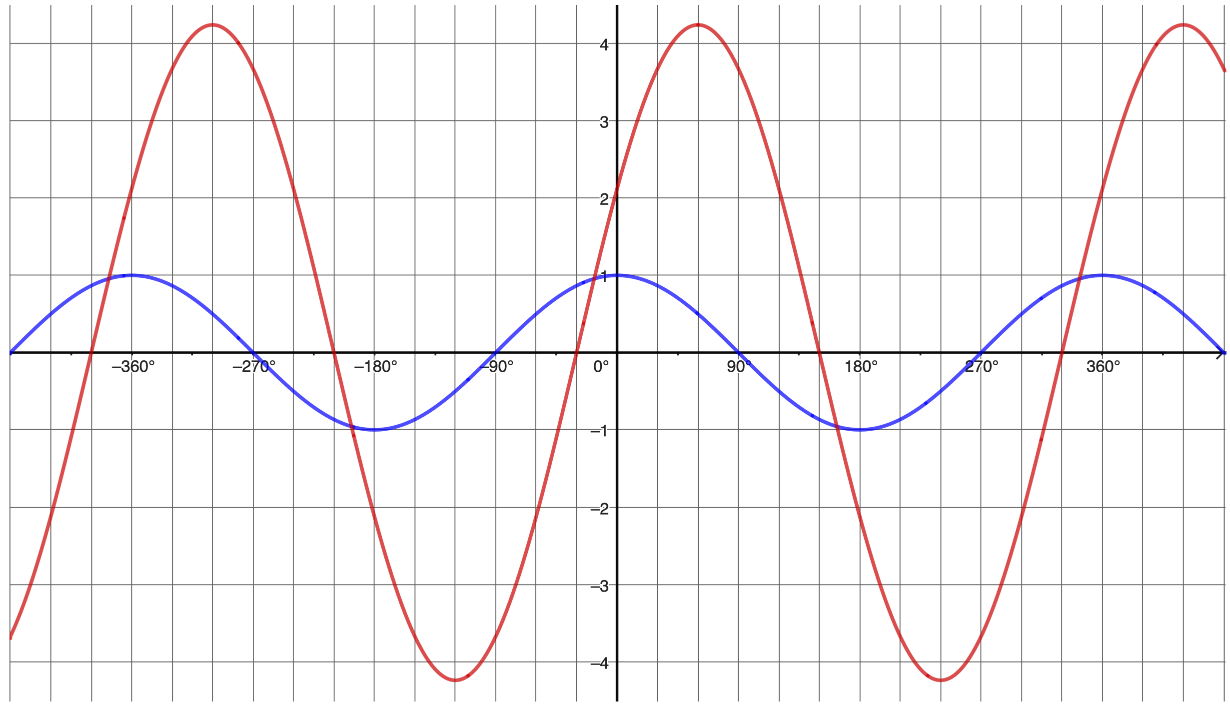
$$y = 3\sin x + \sqrt{3}\cos x$$

and draw this graph.



$$3\sin x + \sqrt{3}\cos x = 3\sqrt{2}\cos\left(x - 60^{\circ}\right)$$

Translation by $\begin{pmatrix} 60^{\circ} \\ 3 \end{pmatrix}$ 0 and stretch parallel to y axis scale factor $3\sqrt{2}$.



Any two waves $y = A\sin x$ and $y = B\cos x$ can be combined (either adding or subtracting) by these methods. Sometimes it's easier to transform the graph $y = \sin x$ to get $y = A\sin x \pm B\cos x$. Other times, starting with $y = \cos x$ is better.

Experiment with the combinations

$y = 5\sin x + 2\cos x$

$y = 5\sin x - 2\cos x$

$y = 2\cos x - 5\sin x$

using each of the following forms:

$R\sin(x + \alpha)$

$R\sin(x - \alpha)$

$R\cos(x + \alpha)$

$R\cos(x - \alpha)$

and taking $\tan 22^\circ$ to be $\dfrac{2}{5}$.

Which R, α forms work best for each of the combinations?

$R\sin(x+\alpha)$ or $R\cos(x-\alpha) = 5\sin x + 2\cos x$

$R\sin(x-\alpha) = 5\sin x - 2\cos x$

$R\cos(x+\alpha) = 2\cos x - 5\sin x$

are the best: they all guarantee the possibility for finding the combination of a positive R with $0 \leq \alpha \leq 90^\circ$.