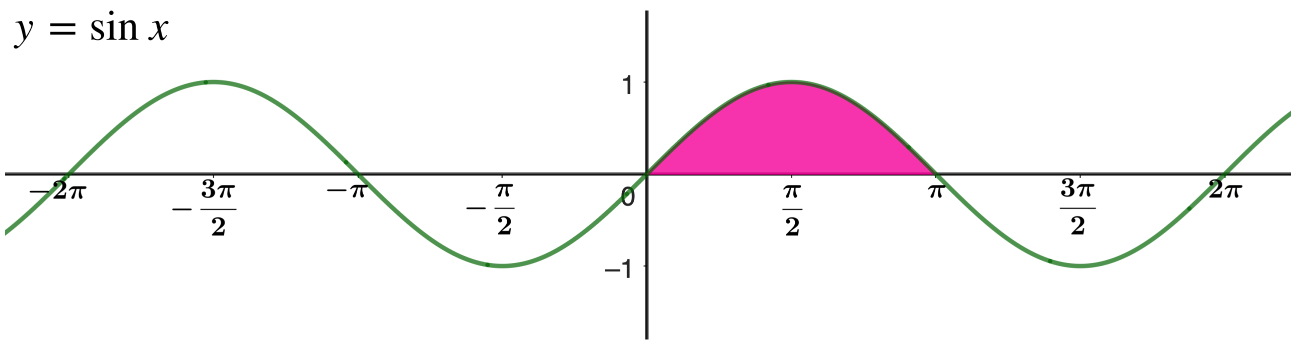
Integrals of circular functions

Find these areas



Since the differential of cos is $-\sin$, it must be the case that the integral of sin is $-\cos$.

$$\text{area} = \int_0^{\pi} \sin x \, \mathrm{d}x$$

$$= \Big[-\cos x\Big]_0^{\pi}$$

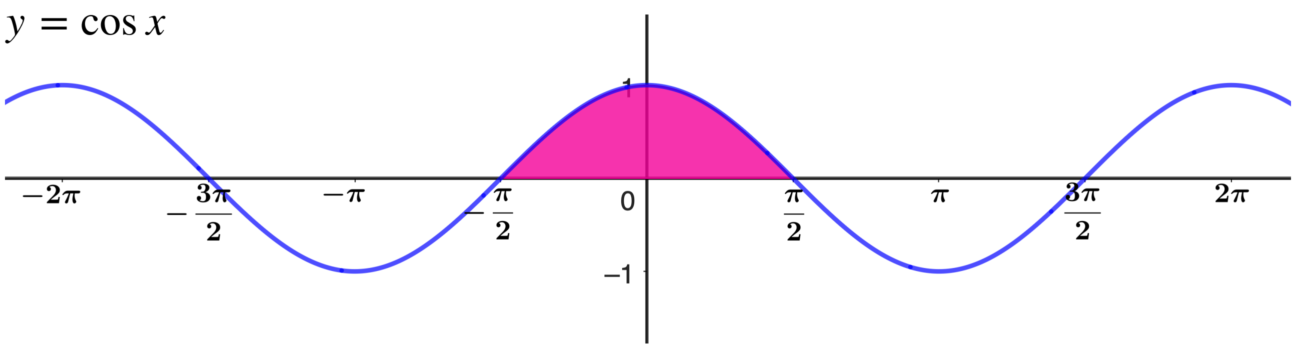
$$= -\cos \pi - (-\cos 0)$$

$$= 1 + 1 = 2$$

Please don't let your students use their calculators for any steps such as

- putting the integral into the calculator
- finding $\cos 0$ or $\cos \pi$

Of course they may be able to do so in an exam, but if they do so now, they will learn nothing.



No calculators!

$$\text{area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, \mathrm{d}x$$

$$= \Big[\sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)$$

$$= 1 + 1 = 2$$

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln|\cos x|$

Integrating $\tan x$ needs integration by substitution, with which your students may not yet be familiar. However, they will surely know the chain rule, so if necessary, this page will provide a fix for this.

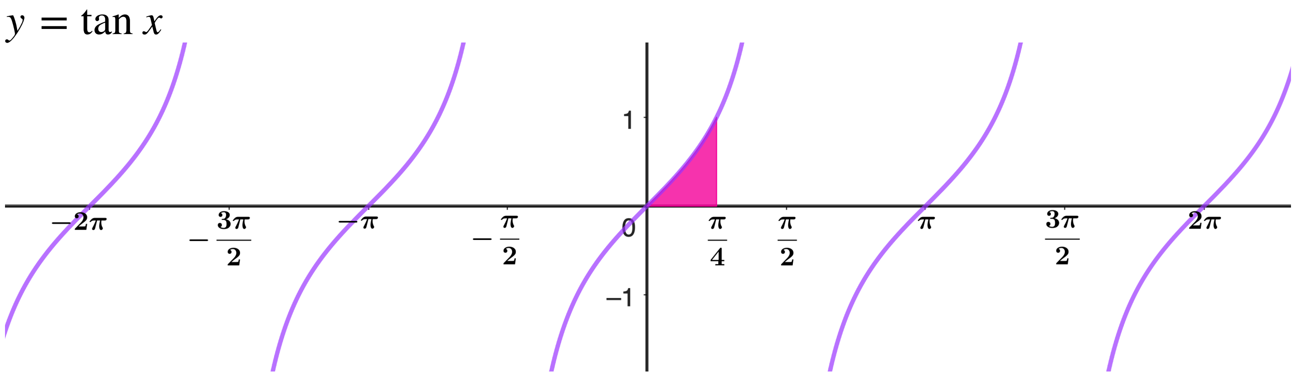
$w = \ln|\cos x| \qquad u = \cos x$

$w = \ln|u|$

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{u} = \frac{1}{\cos x} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\cos x}\sin x = -\tan x$$

Find this area:



For the integral of tan, either use the result on the previous page, or use the substitution $u = \cos x$. Or they might remember the rule for $\int \frac{f'}{f}$, but even so, it's a good idea to do the substitution to remind themselves why this rule works.

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x \qquad u = \cos x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$= -\int \frac{\sin x}{\cos x} \frac{1}{\sin x} \, \mathrm{d}u$$

$$= -\int \frac{1}{u} \, \mathrm{d}u$$

$$= \ln|u|$$

$$= -\ln|\cos x|$$

$$= \ln|\sec x|$$

Often, students will get the final version from the formula book without noticing that $-\ln|\cos x|$ and $\ln|\sec x|$ are the same thing, so it's a good moment to emphasise this.

For the area, we can use either form of the integral of tan, but please, no calculators, even for the manipulation of logs.

$$\text{area} = \int_0^{\frac{\pi}{4}} \tan x \, \mathrm{d}x$$

$$= \Big[-\ln(\cos x) \Big]_0^{\frac{\pi}{4}}$$

$$= -\ln\left(\cos\frac{\pi}{4}\right) - \big(-\ln(\cos 0)\big)$$

$$= -\ln\frac{1}{\sqrt{2}} + \ln 1$$

$$= \ln\sqrt{2}$$

$$= \frac{1}{2}\ln 2$$

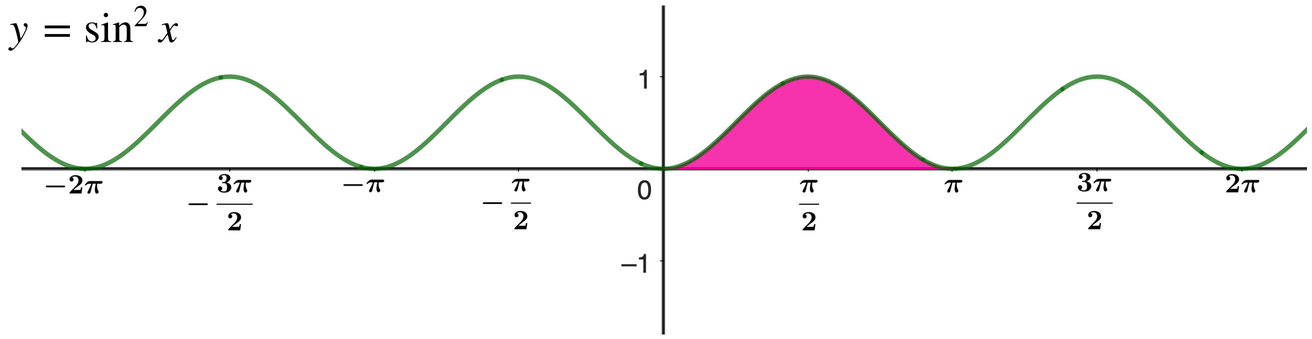
$$\text{area} = \int_0^{\frac{\pi}{4}} \tan x \, \mathrm{d}x$$

$$= \Big[\ln(\sec x) \Big]_0^{\frac{\pi}{4}}$$

$$= \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$$

$$= \ln\sqrt{2} - \ln 1$$

$$= \frac{1}{2}\ln 2$$



Compare your answer with $\int_0^\pi \sin x \,\mathrm{d}x$. Which is bigger? Explain your answer in terms of the graphs.

Here is the most straightforward method, but even once they have seen this idea and used it a few times, students often need reminding to use a double angle formula for $\cos 2x$ to integrate $\sin^2 x$ or $\cos^2 x$.

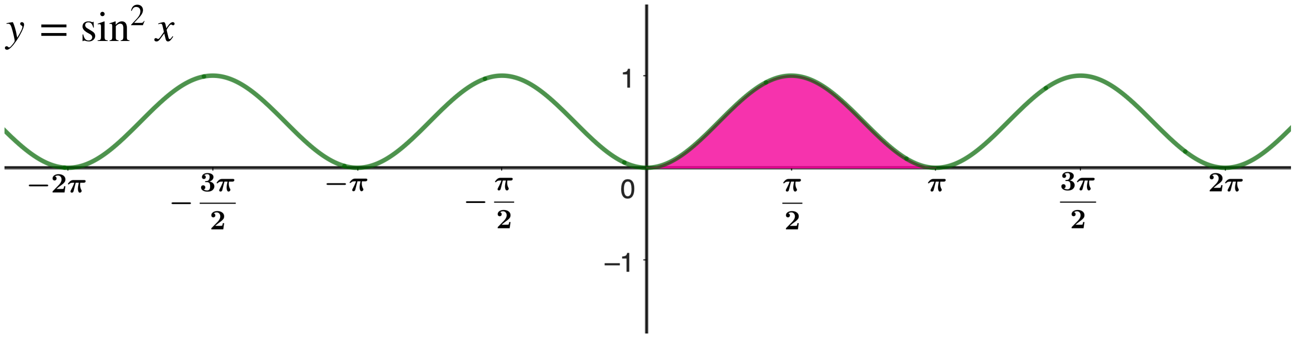
$$\text{area} = \int_0^\pi \sin^2 x \,\mathrm{d}x$$

$$= \int_0^\pi \frac{1 - \cos 2x}{2} \,\mathrm{d}x$$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi$$

$$= \frac{\pi}{2}$$

$\frac{\pi}{2} < 2$. When $0 < x < \pi$, $0 < \sin x < 1 \Rightarrow \sin^2 x < \sin x$, so the area on the $\sin^2$ graph must be less.



Probably the double angle method is the easiest, but, left to their own devices, your students might decide to try integrating by parts. You may or may not want to teach this method, but in either case, it's a good idea to support someone who sets off along these lines:

$$
\begin{aligned}
\int \sin^2 x \, \mathrm{d}x &= -\sin x \cos x + \int \cos x \cos x \, \mathrm{d}x \\
&= -\frac{1}{2}\sin 2x + \int \cos^2 x \, \mathrm{d}x \\
&= -\frac{1}{2}\sin 2x + \int 1 - \sin^2 x \, \mathrm{d}x \\
&= -\frac{1}{2}\sin 2x + x - \int \sin^2 x \, \mathrm{d}x
\end{aligned}
$$

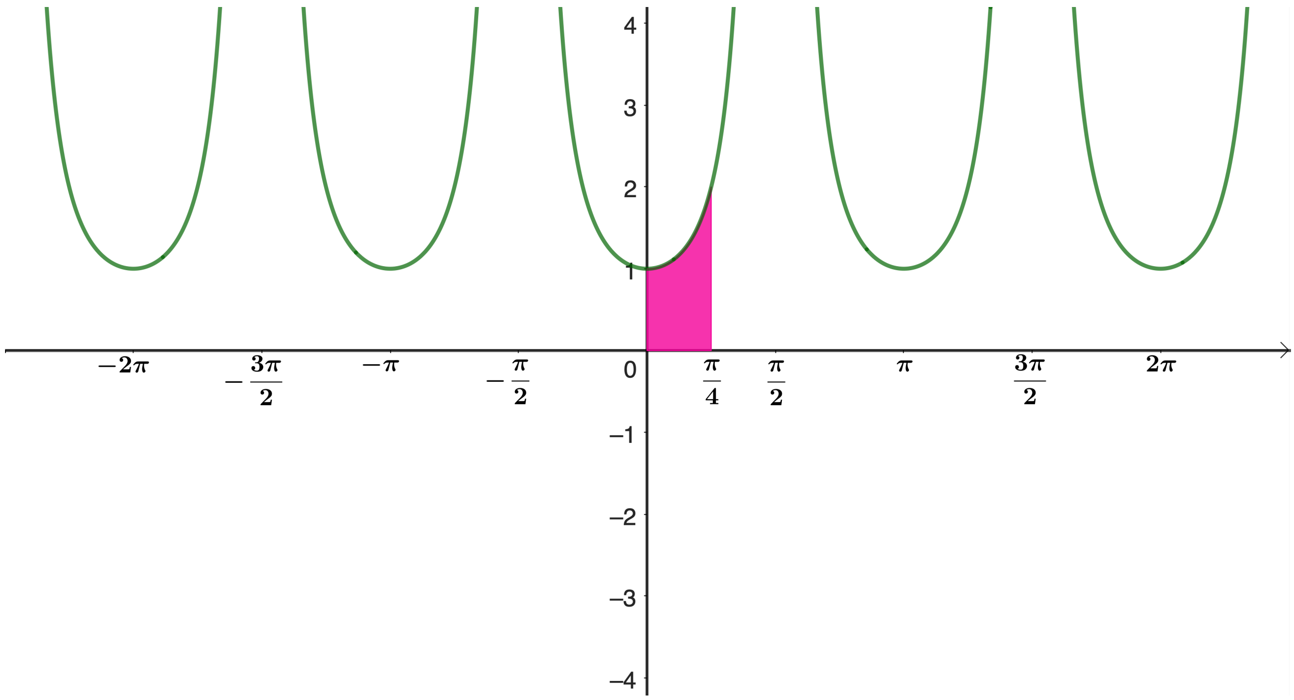
$$\Rightarrow 2\int \sin^2 x \, \mathrm{d}x = x - \frac{1}{2}\sin 2x$$

$$\Rightarrow \int \sin^2 x \, \mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$u = \sin x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \qquad v = -\cos x$$

$y = \sec^2 x$



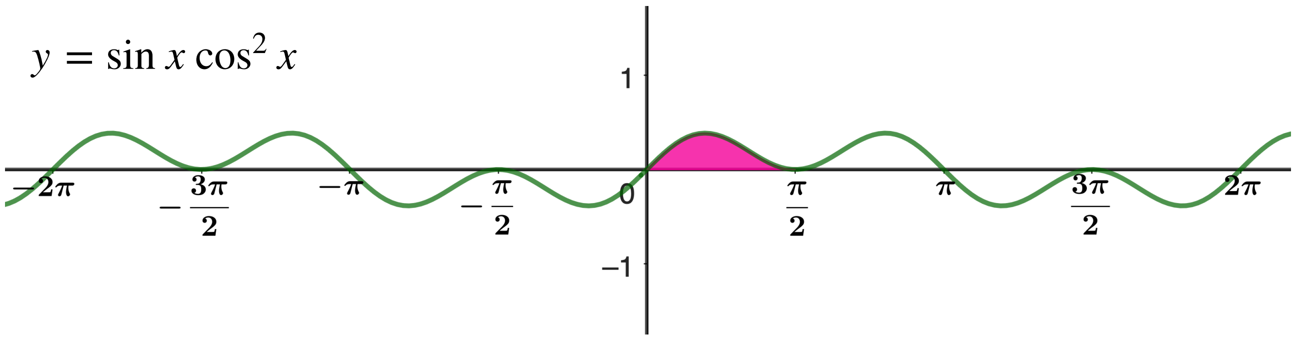
Remember that the differential of tan is $\sec^2$, so

$$\text{area} = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$$

$$= \Big[\tan x\Big]_0^{\frac{\pi}{4}} \quad .$$

$$= \tan \frac{\pi}{4} - \tan 0$$

$$= 1$$



Some integrals look terrifying but actually turn out to be pretty simple. Here, perhaps you can see straight away that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^3 x = -3\sin x\cos^2 x$$

so that the integral is $-\frac{1}{3}\cos^3 x$.

This kind of "integrating by looking" is always something to be on the alert for.

If not, there is always substitution:

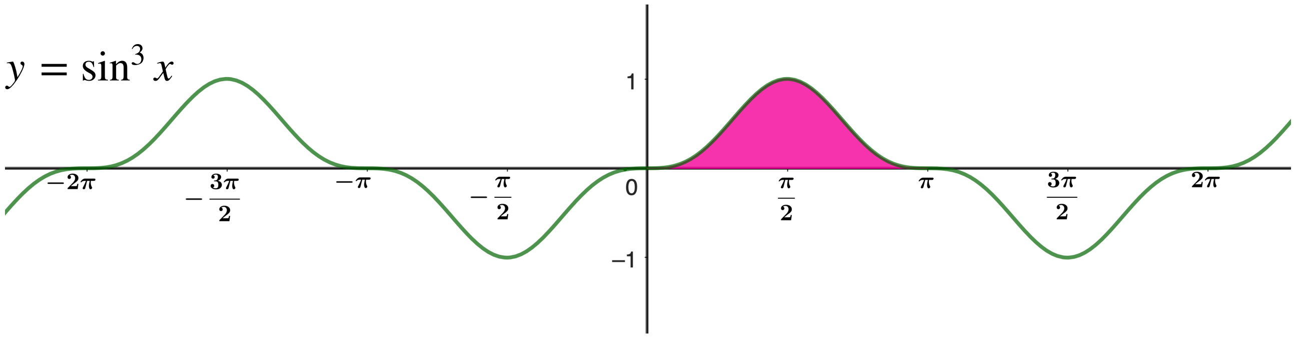
$$u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$\int_0^{\frac{\pi}{2}} \sin x\cos^2 x\,\mathrm{d}x = \int_1^0 \sin x\cos^2 x\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int_1^0 \frac{u^2\sin x}{\sin x}\,\mathrm{d}u$$

$$= \int_0^1 u^2\,\mathrm{d}u = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

Note the change of order in limits in the last line to change the sign.



Here, I've used two results from earlier in the sheet (but doubled one of them)

$$\int_0^\pi \sin^3 x \,\mathrm{d}x = \int_0^\pi \sin x \left(1 - \cos^2 x\right) \,\mathrm{d}x$$
$$= \int_0^\pi \sin x - \sin x \cos^2 x \,\mathrm{d}x$$
$$= 2 - \frac{2}{3}$$
$$= \frac{4}{3}$$

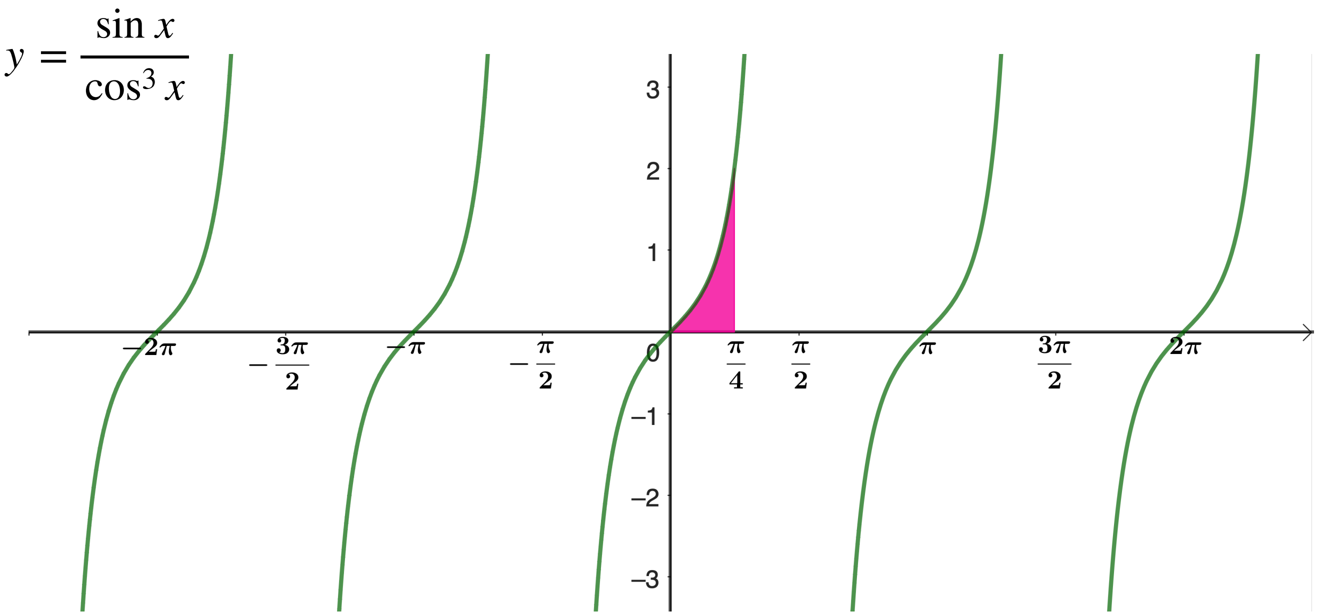
Note also that, when $0 < x < \pi$,

$0 < \sin x < 1 \Rightarrow \sin^3 x < \sin^2 x < \sin x$ which explains why

$$\int_0^\pi \sin^3 x \,\mathrm{d}x < \int_0^\pi \sin^2 x \,\mathrm{d}x < \int_0^\pi \sin x \,\mathrm{d}x$$

$$\text{or } \frac{4}{3} < \frac{\pi}{2} < 2$$

Find this area, and the indefinite integral:



$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-2} x = 2\cos^{-3} x \sin x$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$$

$$= \left[\frac{1}{2\cos^2 x}\right]_0^{\frac{\pi}{4}}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

And for the indefinite integral:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$$

$$= \frac{1}{2\cos^2 x} + c$$

$$= \frac{\sec^2 x}{2} + c$$

Notice, though, that $\sec^2 x = 1 + \tan^2 x$, so

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \frac{\tan^2 x}{2} + c'$$

So we know that $\displaystyle\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \int \tan x \sec^2 x\,\mathrm{d}x = \frac{1}{2}\tan^2 x + c$

but there are also at least three substitutions that will work:

$u = \cos x \qquad u = \sec x \qquad u = \tan x$

Here's one:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \cos x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$

$$= \int_1^{\frac{\sqrt{2}}{2}} \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u^3}\,\mathrm{d}u$$

$$= \left[\frac{1}{2u^2}\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

And for the indefinite integral:

$$\int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= -\int \frac{1}{u^3}\,\mathrm{d}u$$

$$= \frac{1}{2u^2} + c$$

$$= \frac{1}{2\cos^2 x} + c$$

$$= \frac{\sec^2 x}{2} + c$$

Here's another possibility:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \sec x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x$$

$$\begin{aligned}
&= \int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int \frac{\sin x}{\cos^3 x}\frac{1}{\sec x \tan x}\,\mathrm{d}u \\
&= \int \frac{1}{\cos x}\,\mathrm{d}u \\
&= \int u\,\mathrm{d}u \\
&= \frac{u^2}{2} + c \\
&= \frac{\sec^2 x}{2} + c
\end{aligned}$$

And another:

$$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x \qquad u = \tan x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$$

$$\begin{aligned}
&= \int \frac{\sin x}{\cos^3 x}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int \frac{\sin x}{\cos^3 x}\frac{1}{\sec^2 x}\,\mathrm{d}u \\
&= \int \tan x\,\mathrm{d}u \\
&= \int u\,\mathrm{d}u \\
&= \frac{u^2}{2} + c' \\
&= \frac{\tan^2 x}{2} + c'
\end{aligned}$$

Here is an integral that is quite a bit more of a challenge:

Find $\displaystyle\int \sec x \, \mathrm{d}x$ using the substitution $u = \sec x + \tan x$.

This is only for very strong students!

There are a number of substitutions that will work.

First, the standard substitution method; this suffers from the fact that it more or less relies on your knowing the answer in advance.

$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \sec x \, \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \\
&= \int \frac{\sec x}{u \sec x} \, \mathrm{d}u \\
&= \int \frac{1}{u} \, \mathrm{d}u \\
&= \ln|u| + c \\
&= \ln|\sec x + \tan x| + c \\
&= \ln\left|\frac{1 + \sin x}{\cos x}\right| + c
\end{aligned}$$

$$\begin{aligned}
u &= \sec x + \tan x \\
\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= \sec x \tan x + \sec^2 x \\
&= u \sec x
\end{aligned}$$

Find $\displaystyle\int \sec x \, \mathrm{d}x$ using the substitution $u = \sin x$.

Here is a version that seems like a more obvious substitution to try. It yields an answer that looks very different indeed, but turns out to be the same.

$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \frac{1}{\cos x} \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \\
&= \int \frac{1}{\cos^2 x} \, \mathrm{d}u \\
&= \int \frac{1}{1 - \sin^2 x} \, \mathrm{d}u \\
&= \int \frac{1}{1 - u^2} \, \mathrm{d}u \\
&= \frac{1}{2} \int \frac{1}{1-u} + \frac{1}{1+u} \, \mathrm{d}u \\
&= \frac{1}{2} \left[-\ln|1-u| + \ln|1+u| \right] \\
&= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| \\
&= \frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + c
\end{aligned}$$

$$u = \sin x$$

$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

You now have two versions of the integral that look very different. Are they equivalent?

$$\begin{aligned}\left(\frac{1+\sin x}{\cos x}\right)^2 &= \frac{(1+\sin x)^2}{1-\sin^2 x}\\[1em] &= \frac{(1+\sin x)^2}{(1+\sin x)(1-\sin x)}\\[1em] &= \frac{1+\sin x}{1-\sin x}\\[1em] \Rightarrow 2\ln|\sec x+\tan x| &= \ln\left|\frac{1+\sin x}{1-\sin x}\right|\end{aligned}$$

Yes, they are!

Strictly for extension only, the next bit!

Here is a standard substitution that is often useful for integration of awkward circular functions, but that your students may well not have come across. You will find two articles on this substitution on my website. Again, the answer looks quite different to the previous two, and again they all turn out to be equivalent.

$$t = \tan\frac{x}{2}$$

$$\Rightarrow \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$$

$$\Rightarrow \cos x = \frac{1-t^2}{1+t^2} \text{ and } \sin x = \frac{2t}{1+t^2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1+t^2}{2}$$

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

$$\begin{aligned}
\int \sec x \,\mathrm{d}x &= \int \sec x \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{1-t^2}\,\mathrm{d}t \\
&= \int \frac{1}{1-t} + \frac{1}{1+t}\,\mathrm{d}t \\
&= -\ln(1-t) + \ln(1+t) + c \\
&= \ln\frac{1+t}{1-t} + c \\
&= \ln\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + c
\end{aligned}$$

This expression is also equivalent to the other versions:

$$
\begin{aligned}
\sec x + \tan x &= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \\
&= \frac{1+2t+t^2}{1-t^2} \\
&= \frac{(1+t)^2}{(1-t)(1+t)} \\
&= \frac{1+t}{1-t} \\
\Rightarrow \ln|\sec x + \tan x| &= \ln\left|\frac{1+t}{1-t}\right|
\end{aligned}
$$

There are a few other variations of the result. For example:

$$
\begin{aligned}
\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) &= \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}} \\
&= \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}
\end{aligned}
$$

so we can also write this as

$$
\int \sec x \, \mathrm{d}x = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|
$$

And here are a couple of other versions that I've included just for fun!

$$\ln \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} = \ln \frac{1+\frac{\sin x}{1+\cos x}}{1-\frac{\sin x}{1+\cos x}}$$

$$= \ln \frac{1+\cos x+\sin x}{1+\cos x-\sin x}$$

$$\frac{\sin x}{1+\cos x} = \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$= t$$

$$\ln \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} = \ln \frac{1+\frac{1-\cos x}{\sin x}}{1-\frac{1-\cos x}{\sin x}}$$

$$= \ln \frac{\sin x-\cos x+1}{\sin x+\cos x-1}$$