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for insight

Circular functions 8

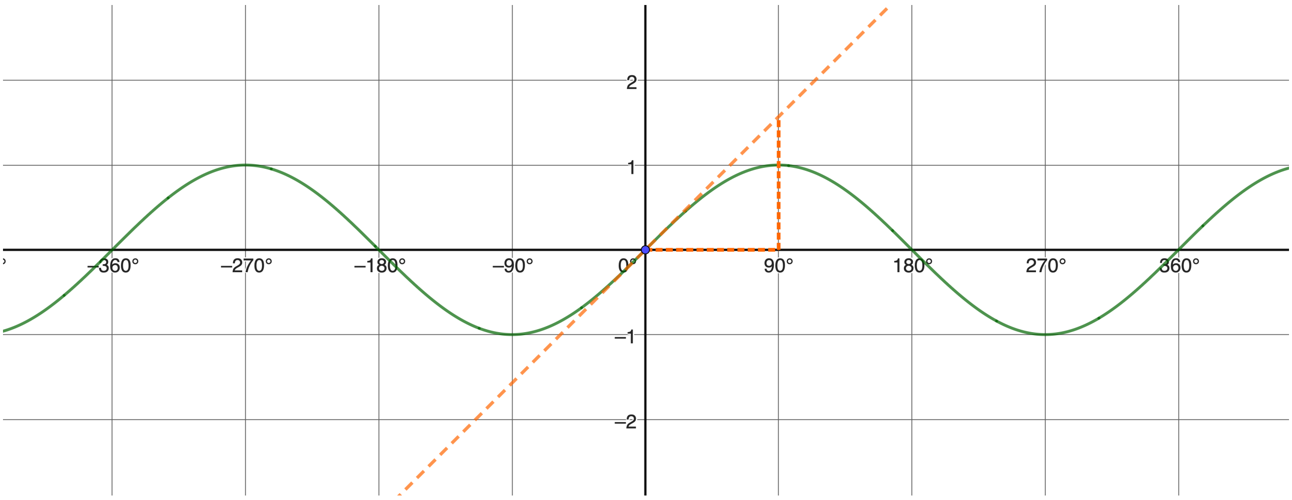
Differentials of circular functions

teacher version

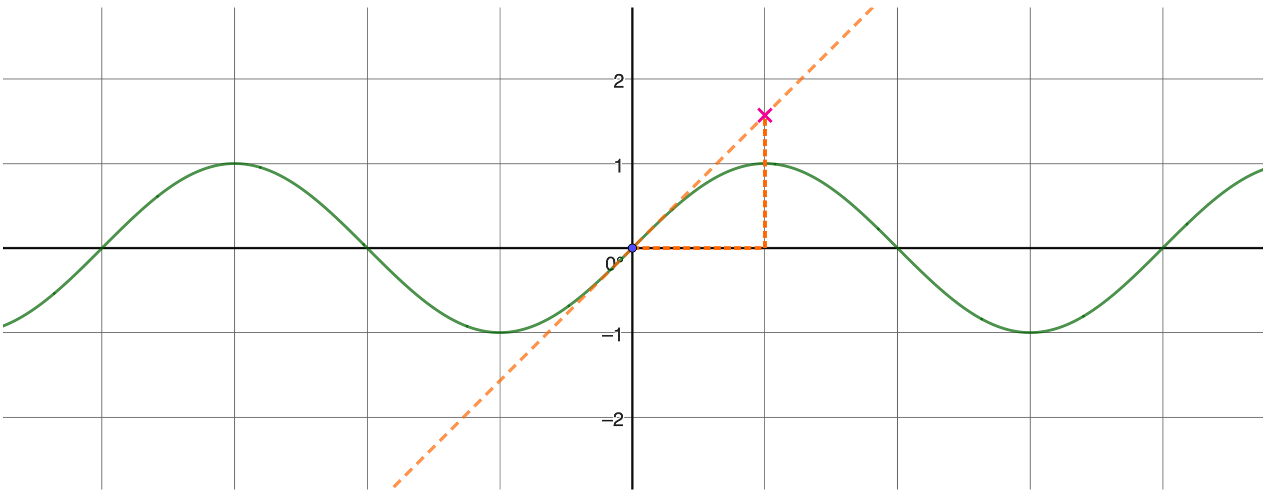
Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90° - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45°\right)$ and so on
Differentiating circular functions	**radians, and tangents to graphs**
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

What (approximately) is the gradient of this tangent?



The gradient may look a bit like 1, but is it? Here is a new version of the graph without a scale on the x axis. If the gradient of this tangent is to be 1, what (approximately) would the x coordinate of the pink cross have to be?



radical maths

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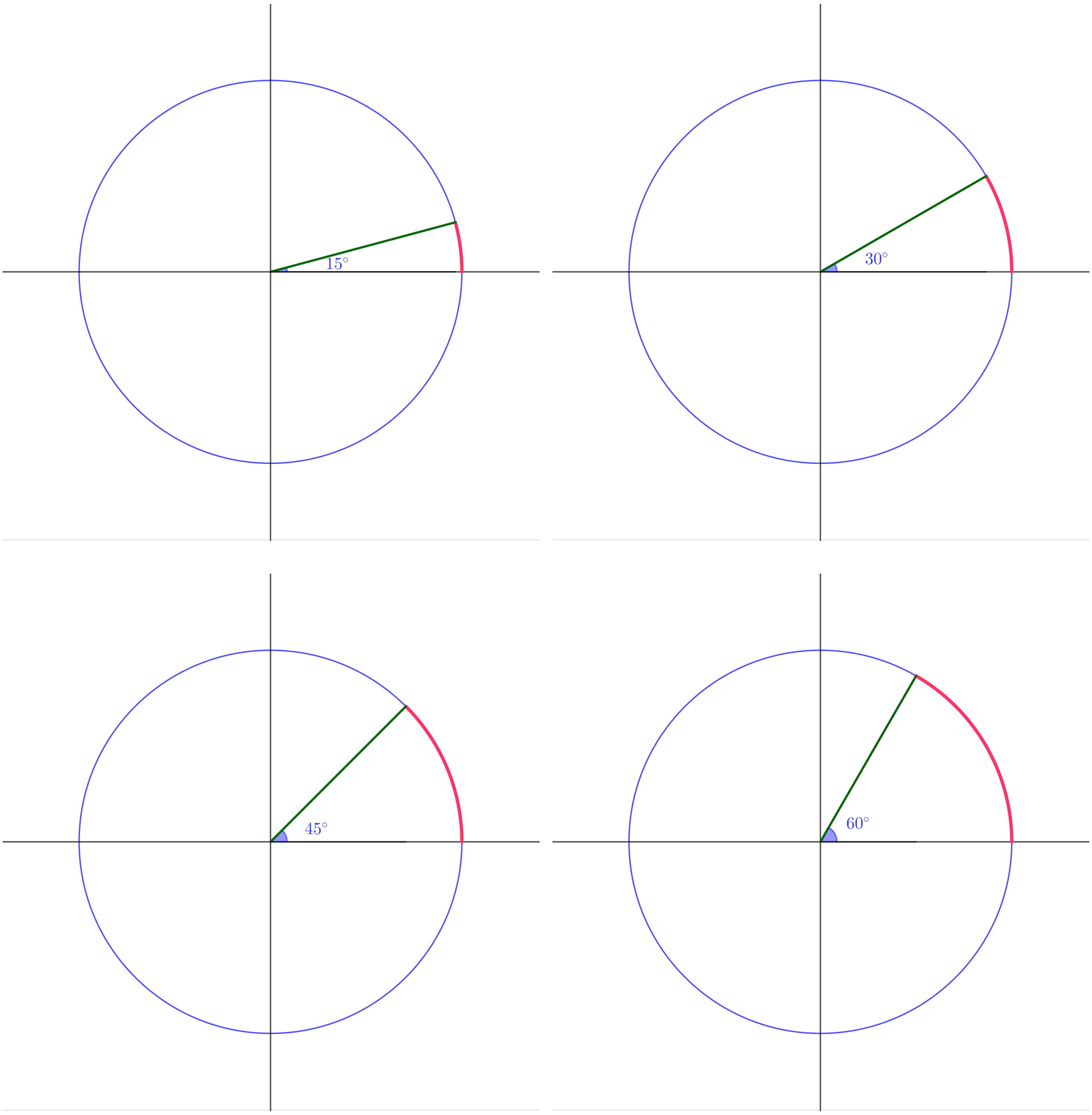
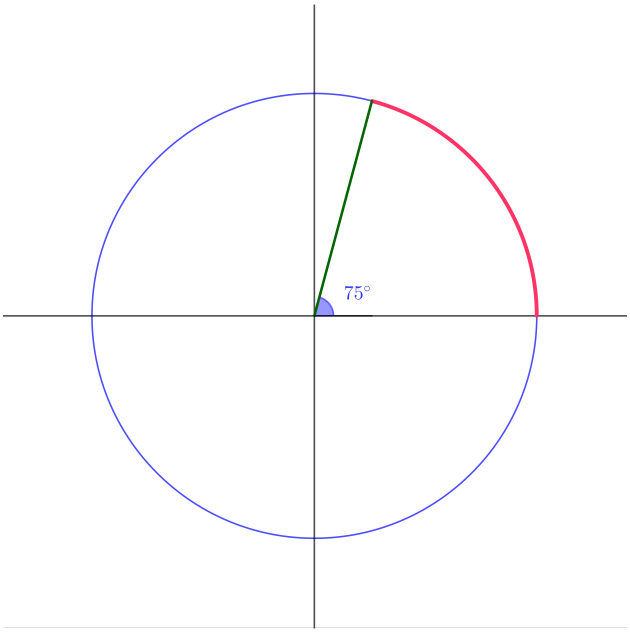
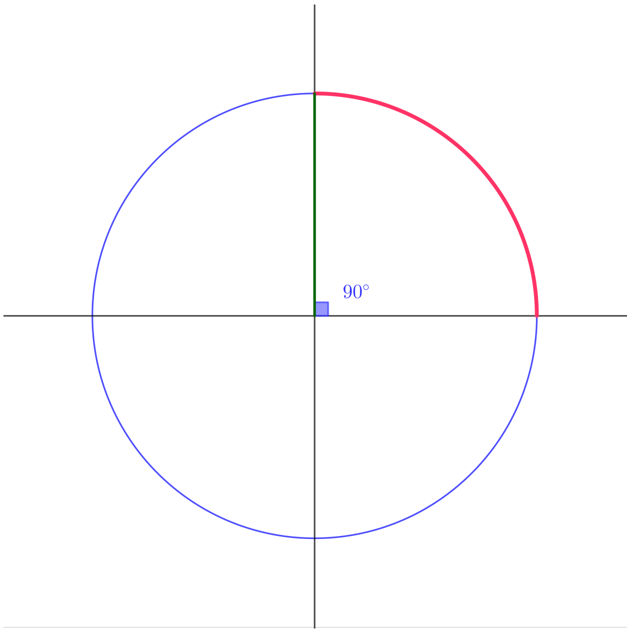
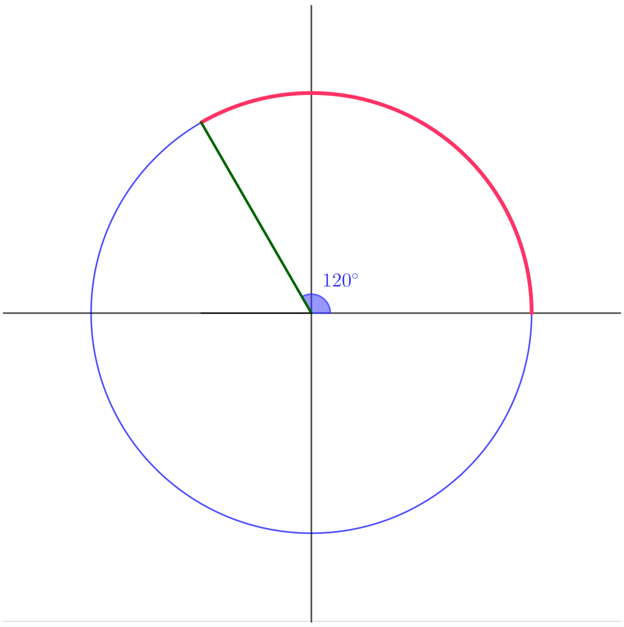
Now, we will try to find this x coordinate exactly. That is to say, we will find units for the x axis that makes this gradient 1. In theory, we can differentiate the circular functions without doing this, but everything works out so much more easily if we do, and that's the way it's done the world over. To do this, we need to go back to the unit circle.

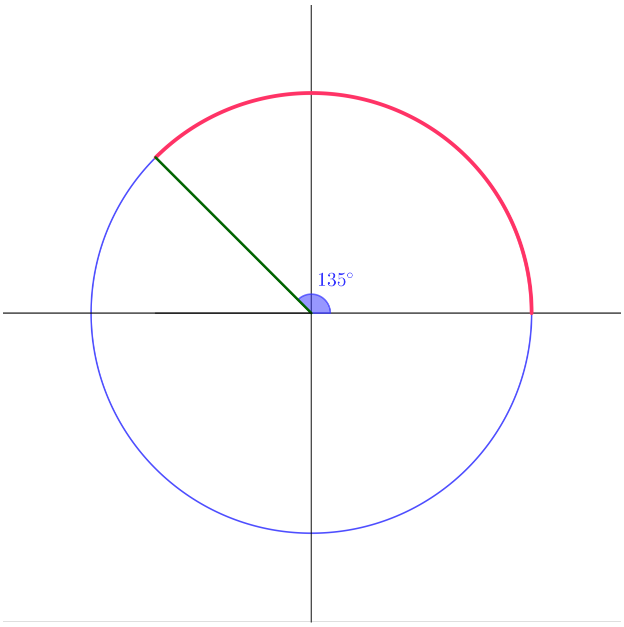
First of all, find the pink arc length on each of these circles with radius 1:

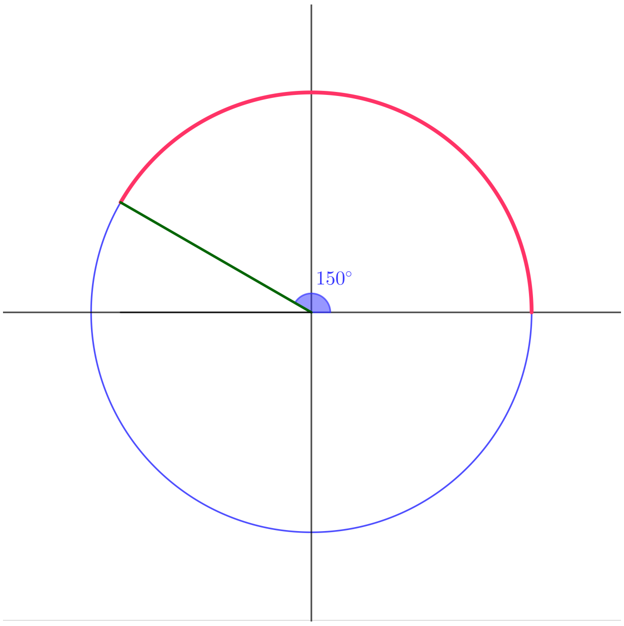


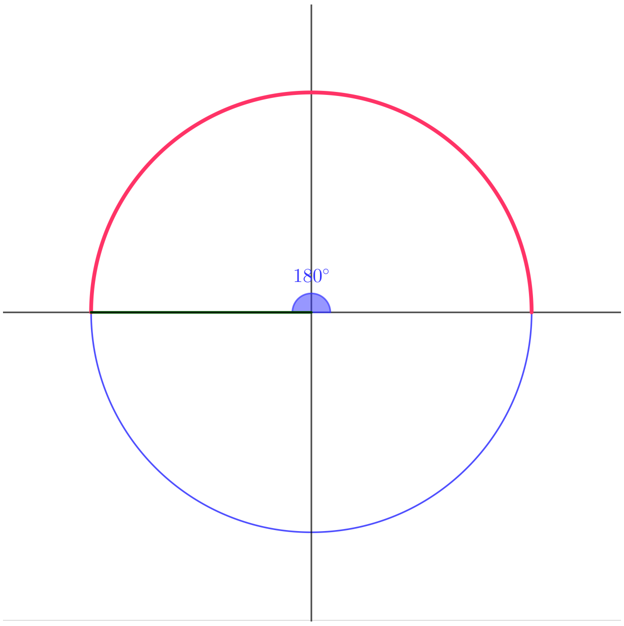

75°

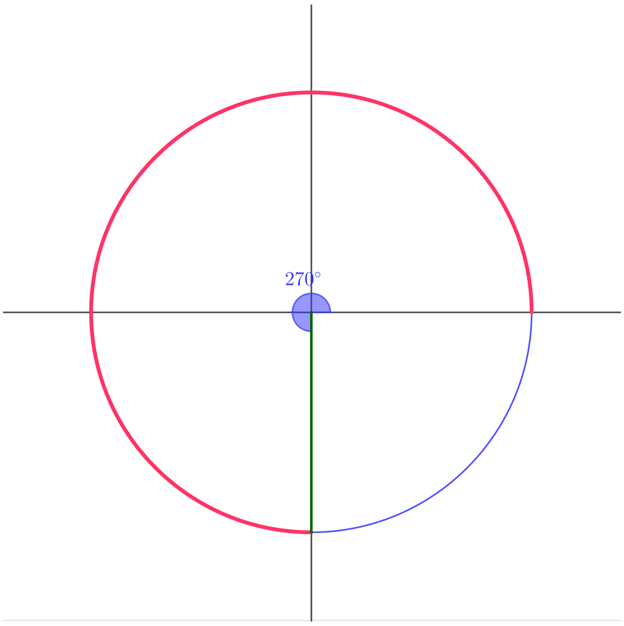

90°

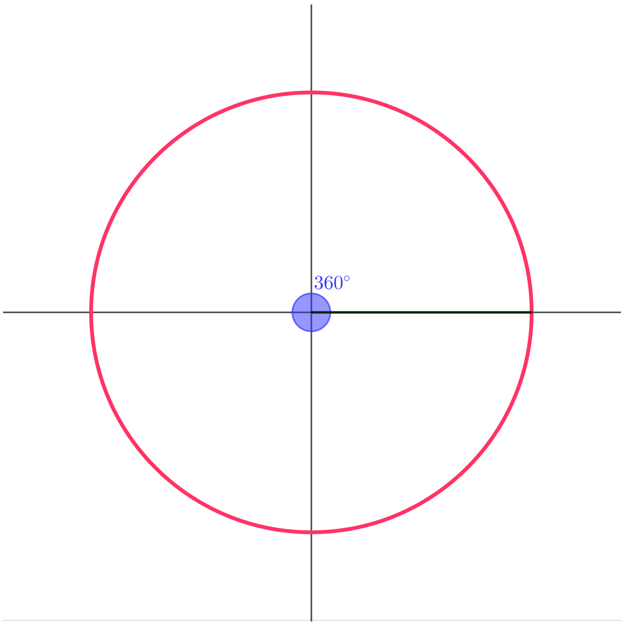

120°


135°


150°

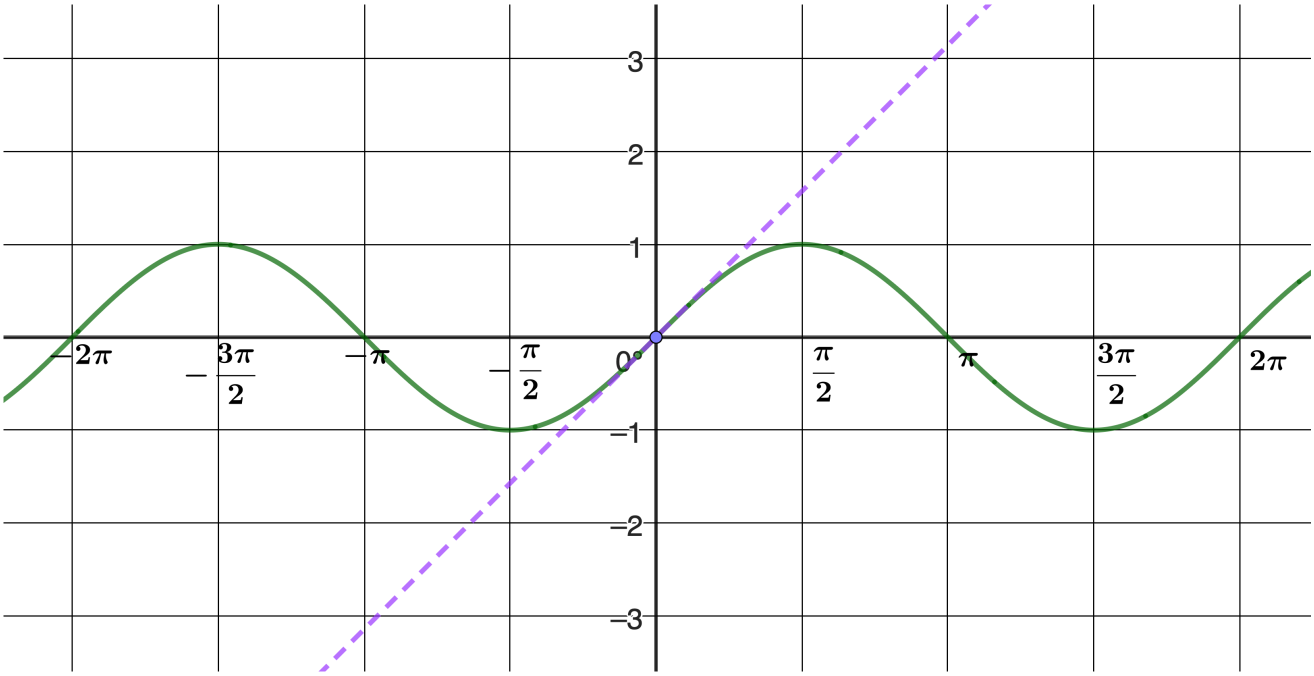

180°


270°


360°

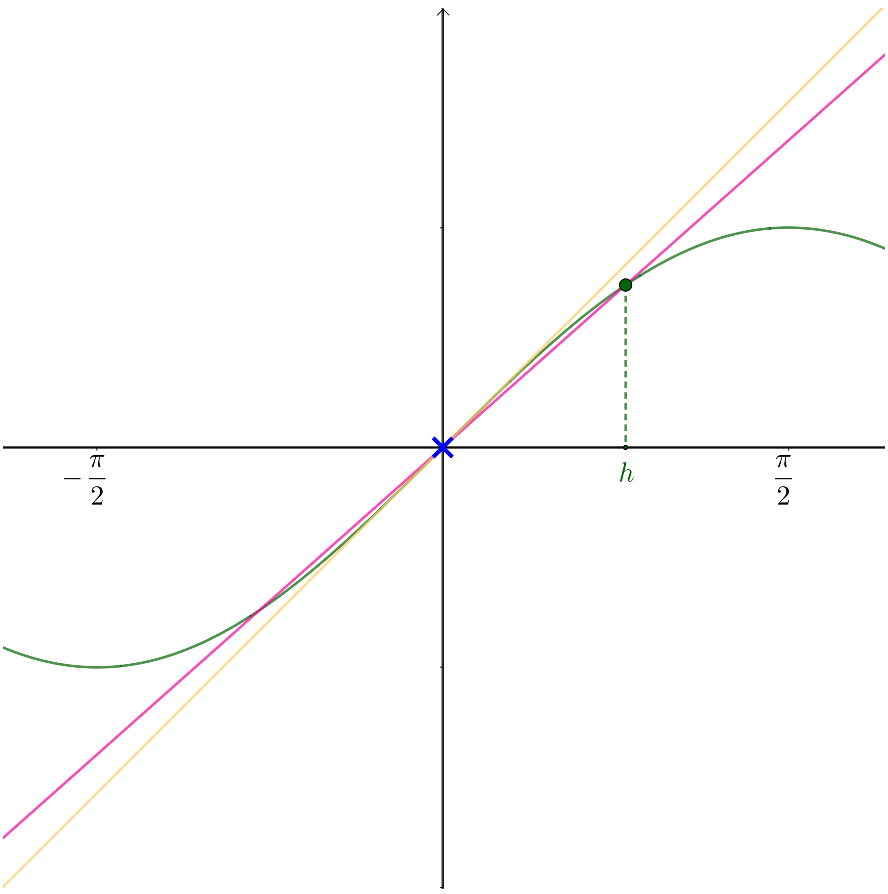
Using $\pi = 3.14159\ldots$, what is $\dfrac{\pi}{2}$ as a decimal?

What does the gradient of the tangent look like now?



To figure out whether this is the right scale to make the gradient of the tangent equal to 1, we will use the idea of the tangent as a limit.

First, what is the gradient of the pink line?

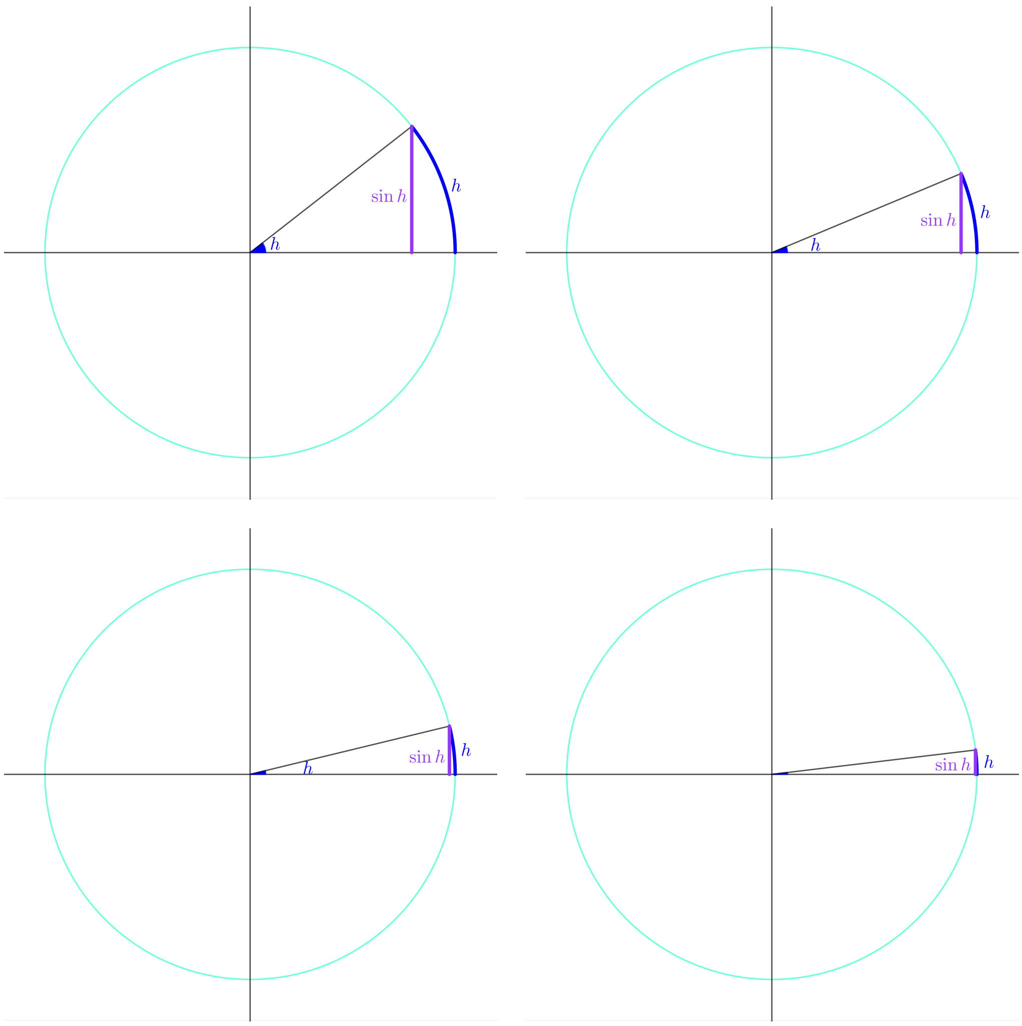


What will happen to the pink line as h gets increasingly small?

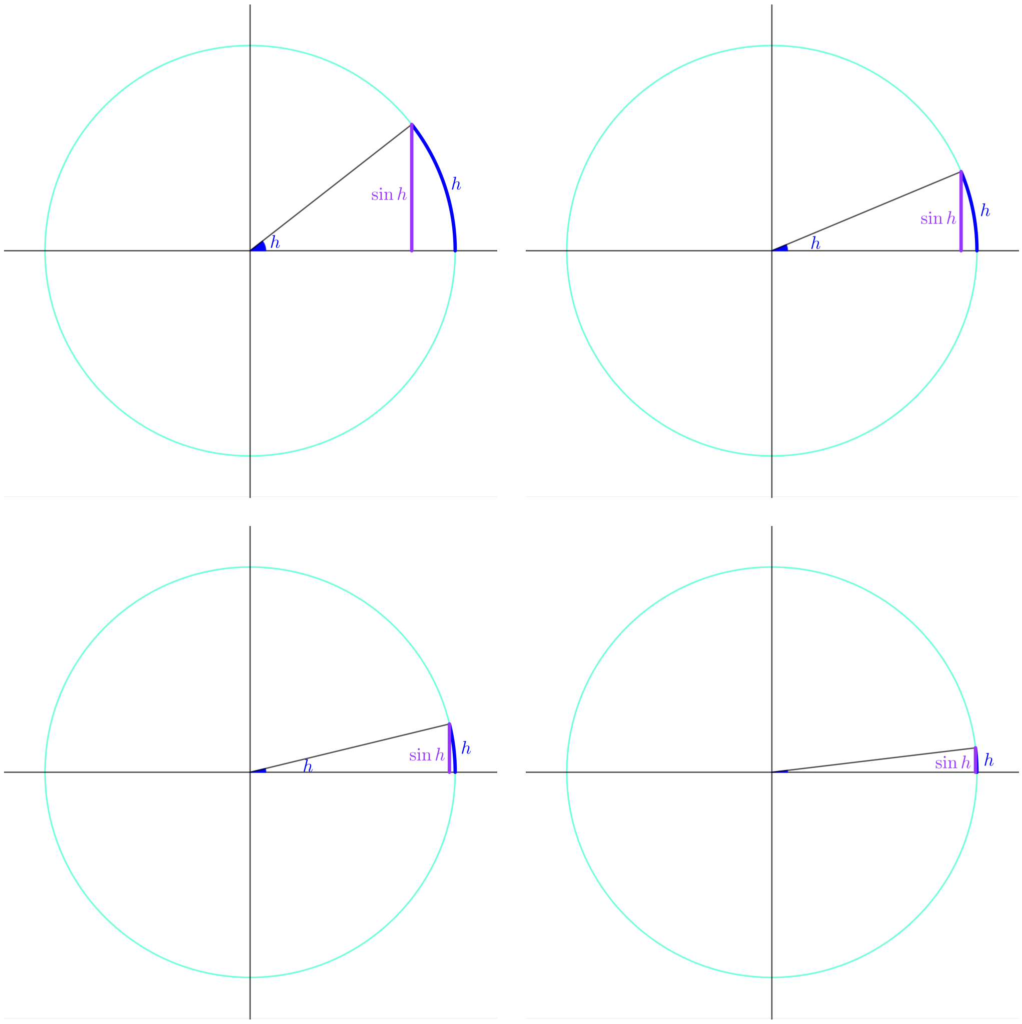
What does this tell us about the gradient of the tangent?

Look at this sequence of diagrams. Why have I used the same letter, h, for both the angles and the arc lengths?

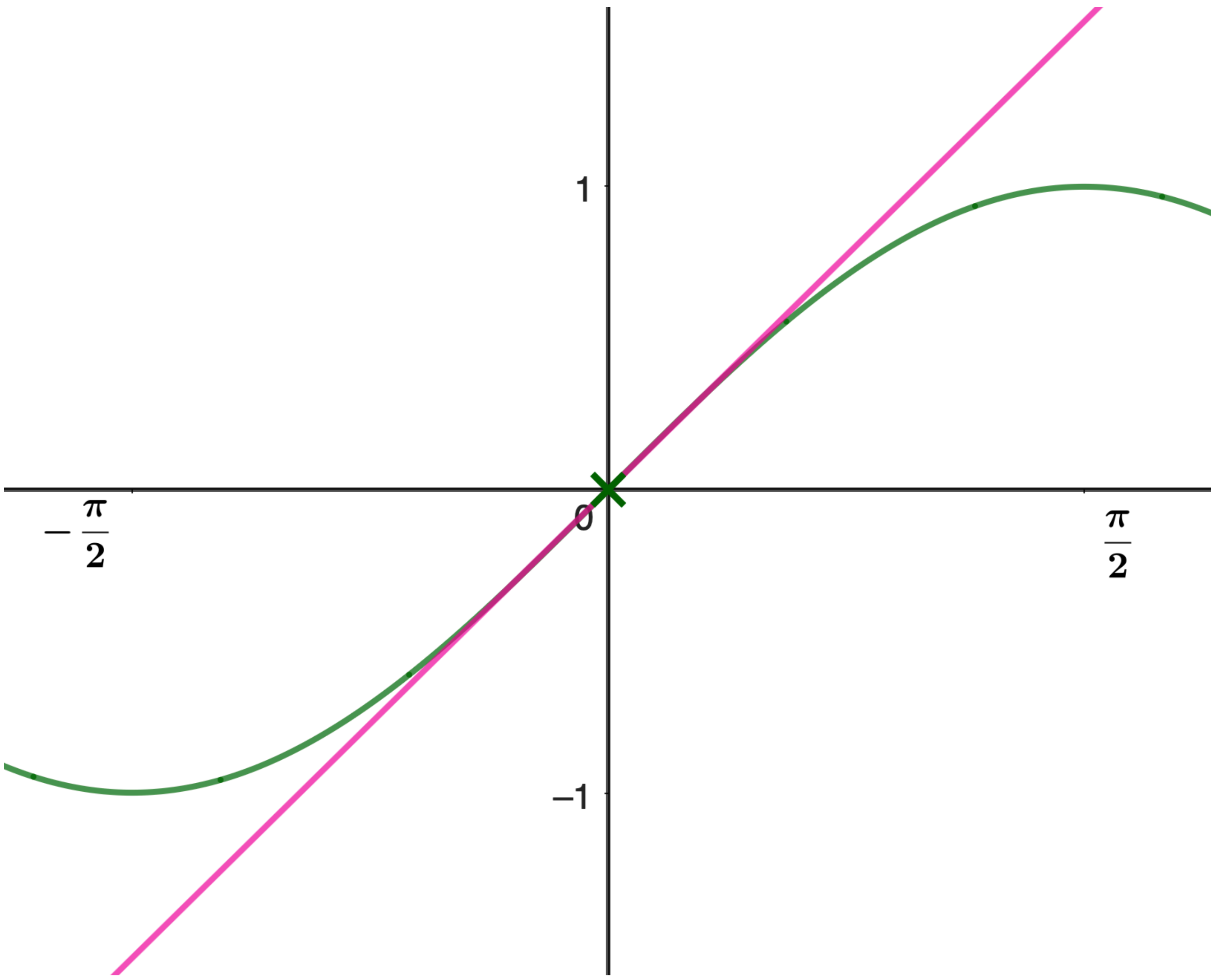
What do you notice about the relationship between the angle size h and the ratio of the lengths of the blue arc and the purple segment?



What does this tell you about the ratio $\dfrac{\sin h}{h}$ as $h \to 0$?

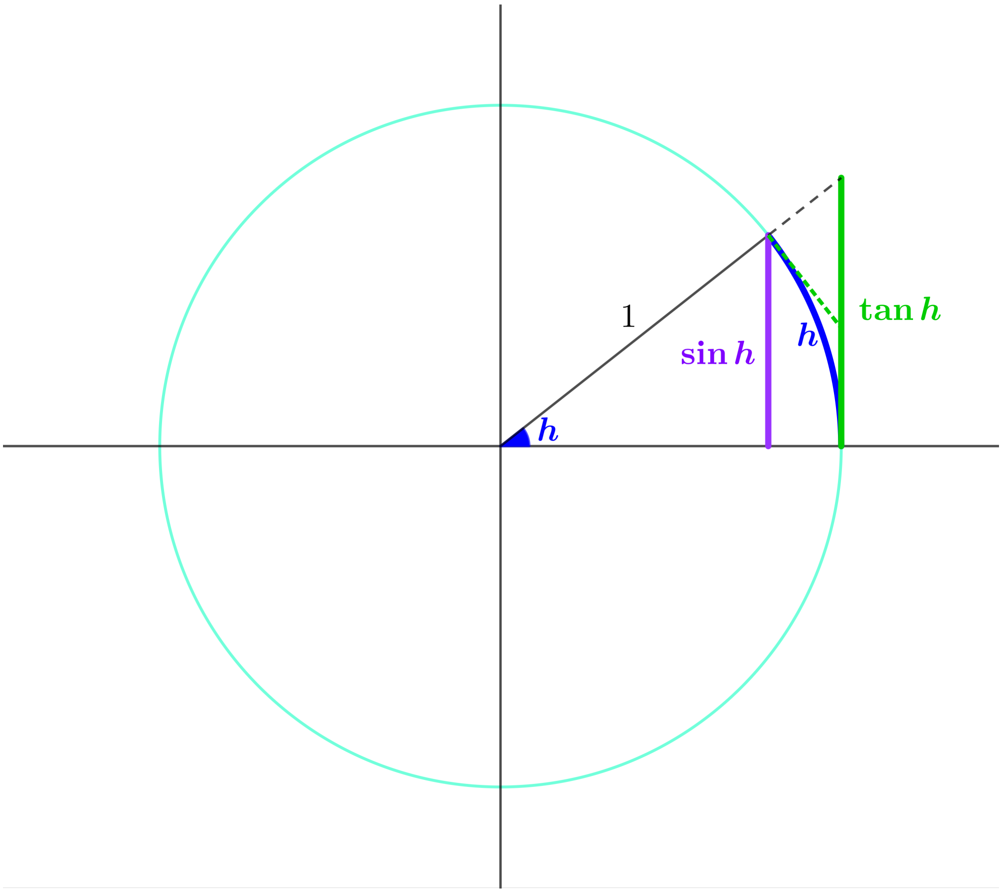

sin h
h
h
sin h
h
h
sin h
h
h
sin h
h

What is the gradient of the tangent to the curve $y = \sin x$ at the origin?



That gives you a sense of why $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$, but it is not quite the whole story.

Here is a more complete “proof”, in case you are interested.



Write down an inequality involving the pink and green line segments and the blue arc.

Use this to find lower and upper bounds for $\dfrac{h}{\sin h}$.

Now find lower and upper bounds for $\dfrac{\sin h}{h}$.

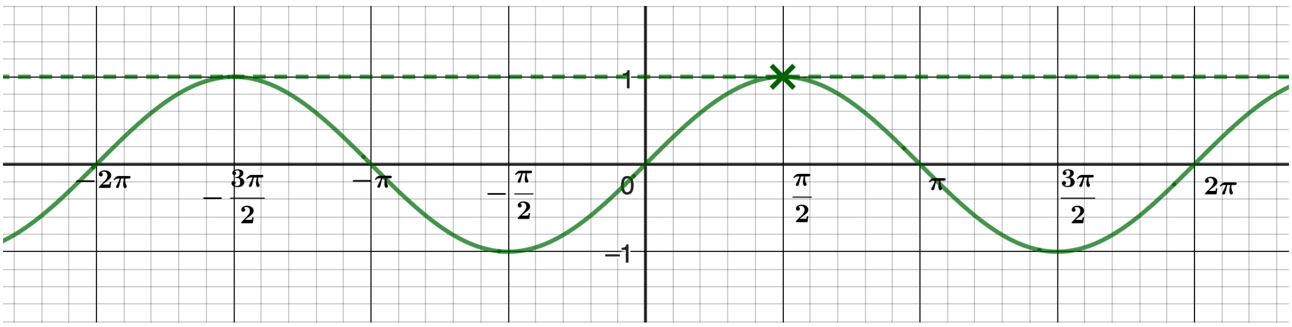
What is $\lim\limits_{h\to 0} \cos h$?

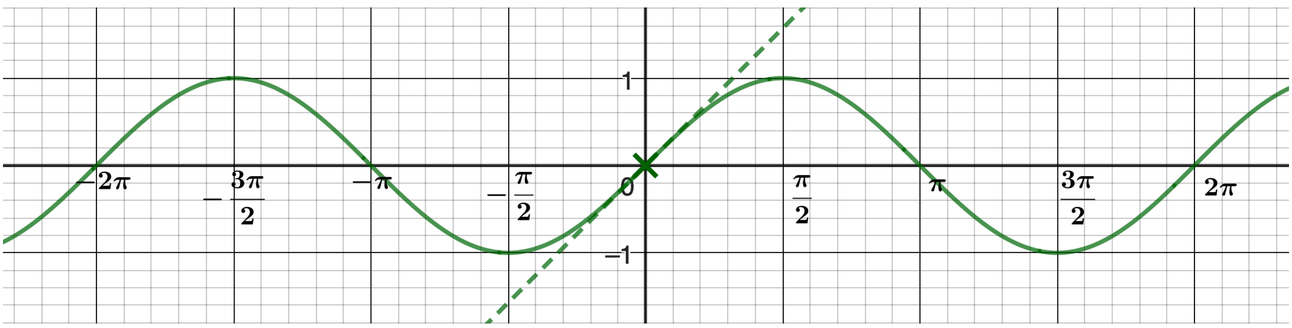
Use this to find $\lim\limits_{h\to 0} \dfrac{\sin h}{h}$.

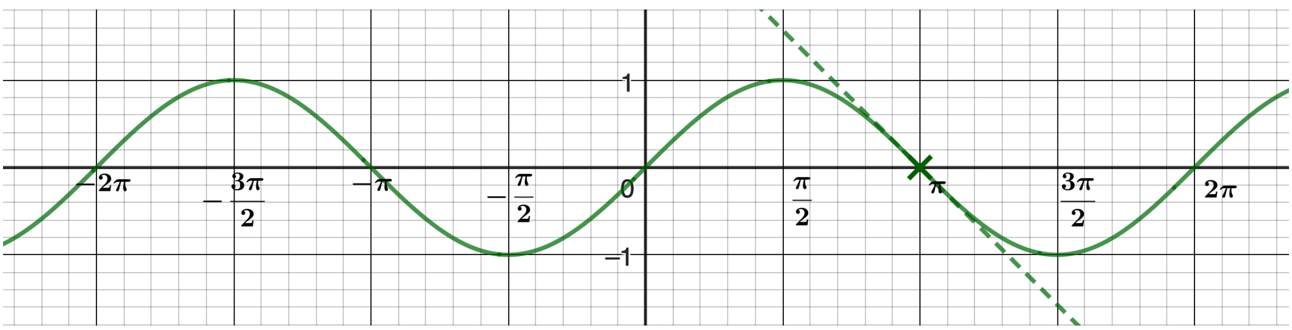
Now we know that the gradient of the tangent to the graph $y = \sin x$ at the origin is 1 (when we use radians as our unit of angles).

Next, we will think about the gradient of the curve at other points.

To start with, what are the gradients of these three tangents?



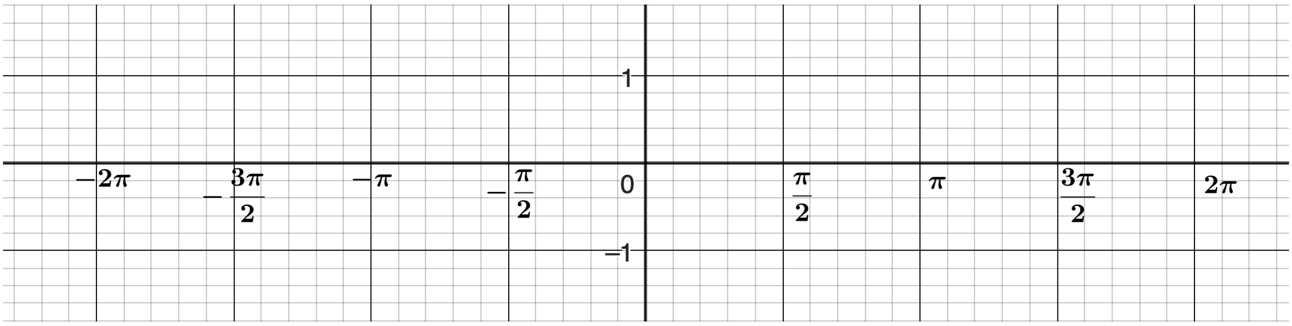




Use the graph to fill in this table:

x	**gradient of tangent**
0	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	
$-\frac{\pi}{2}$	
$-\pi$	
$-\frac{3\pi}{2}$	
-2π	

Now mark the values from the table on these axes with the left-hand column on the x axis and the right-hand column on the y axis.



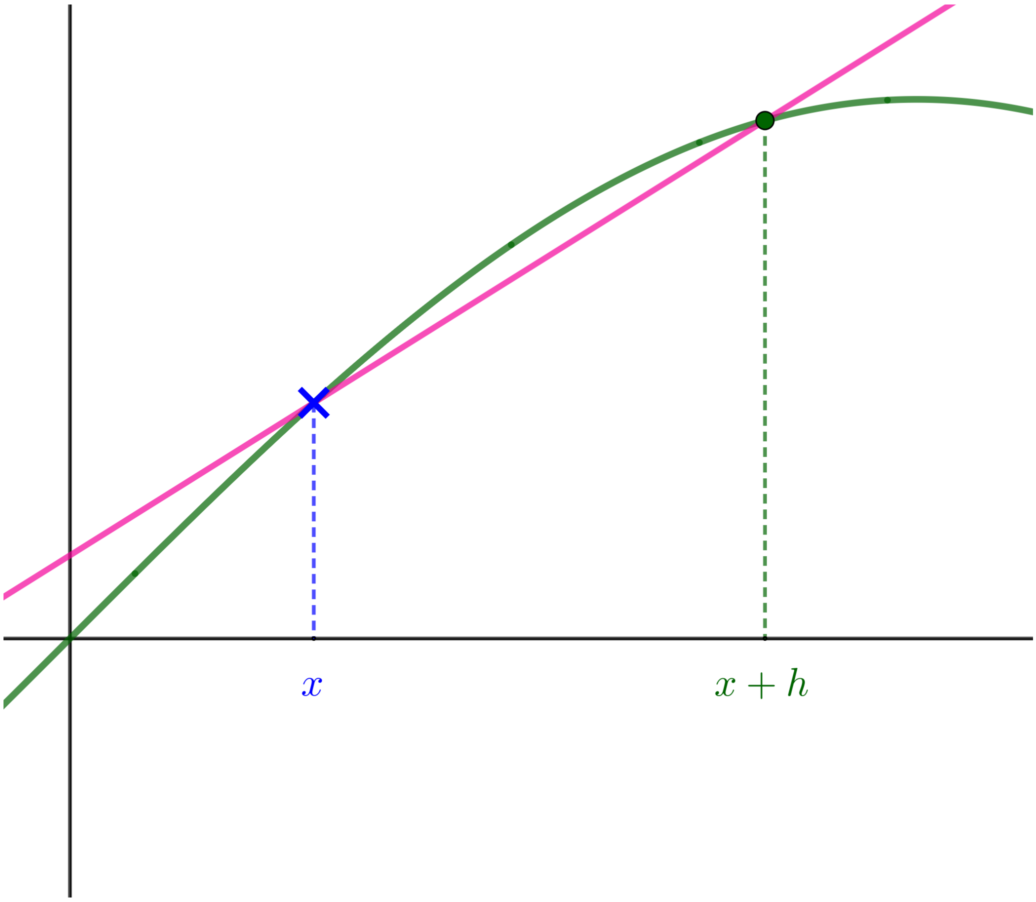
What does this graph look like?

What happens to the gradient between these points? Use this idea to draw the whole curve representing the gradient.

Now we know that the gradient of the sin graph looks rather like cos, and we are in a position to see that the differential of sin really is cos.

What is the gradient of the pink line in terms of x and h ?

What happens to the pink line as h gets increasingly small?



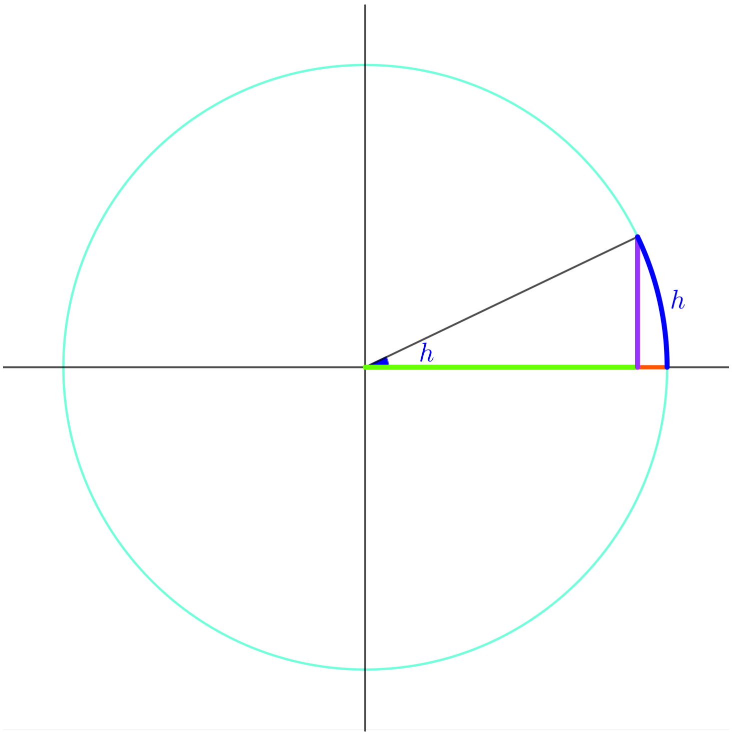
Use this to write the gradient of the tangent as a limit.

By using a compound angle formula and then rearranging, express this gradient as a multiple of $\cos x$ minus a multiple of $\sin x$.

Simplify this using the fact that $\lim_{h \to 0} \dfrac{\sin h}{h} = 1$.

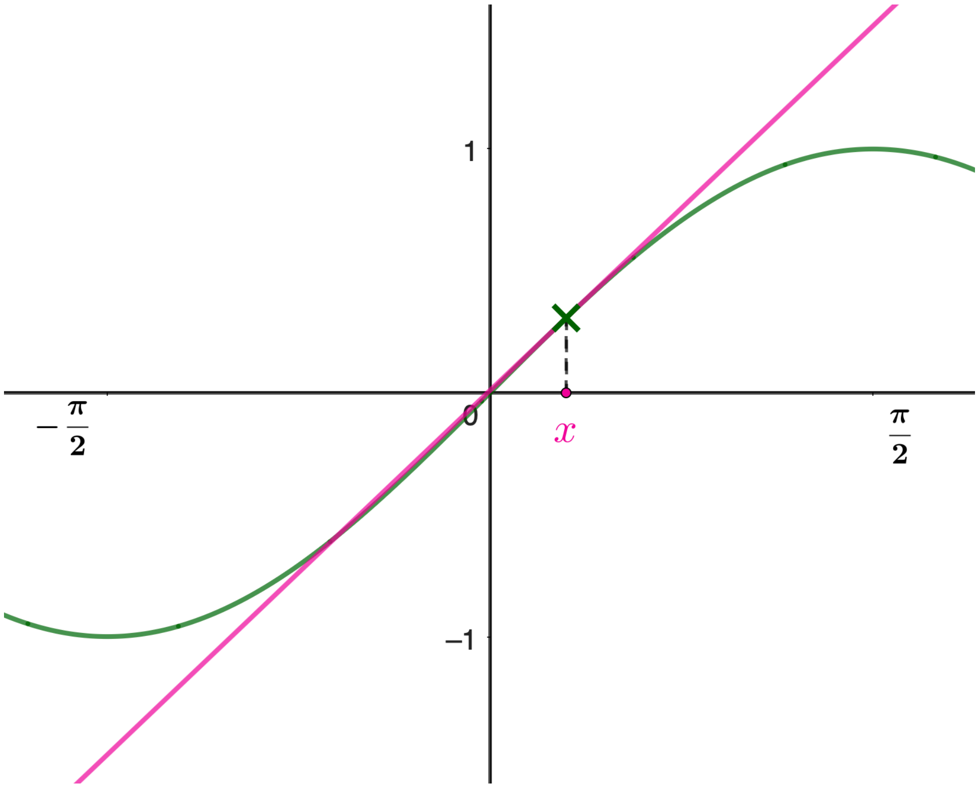
What is the length of the orange line segment?

What happens to the ratio of the orange line segment to the blue arc as h gets increasingly small?



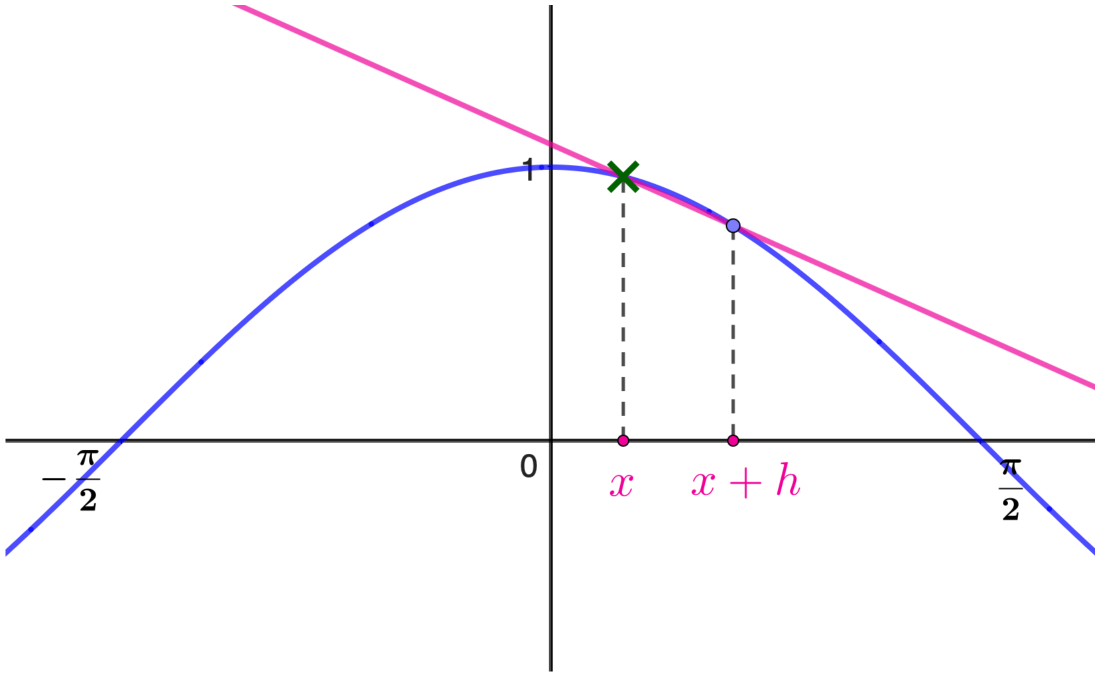
What does this tell you about $\lim_{h \to 0} \dfrac{1 - \cos h}{h}$?

Use this limit to find the gradient of the tangent to the curve $y = \sin x$ at the green cross.

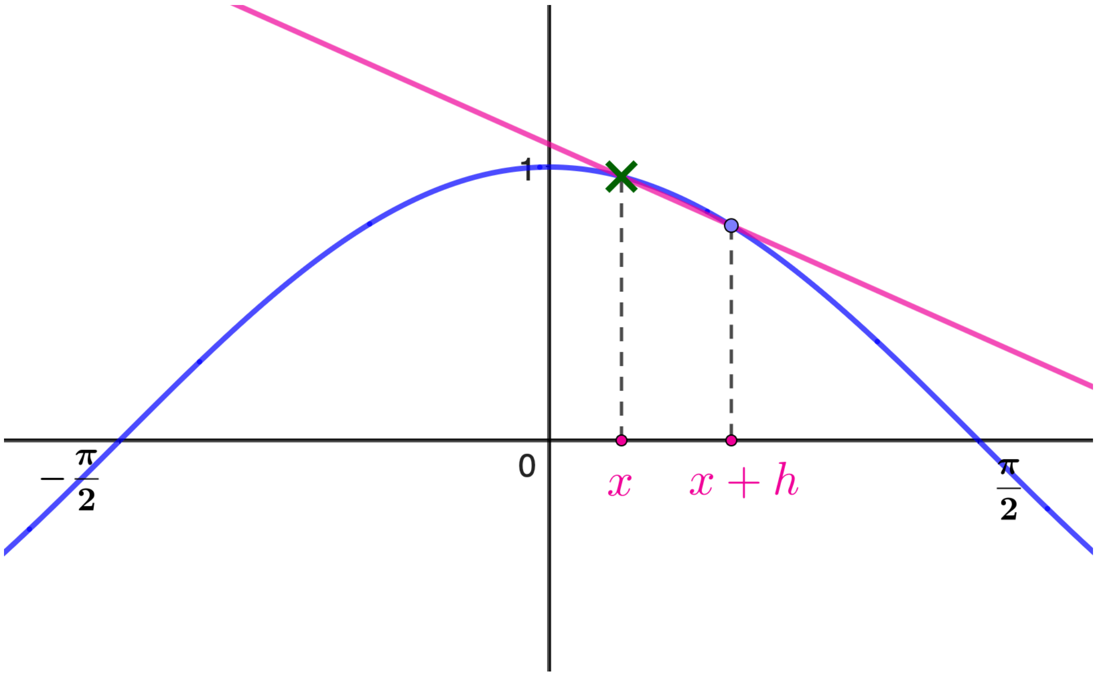


If $f(x) = \sin x$, what is $f'(x)$?

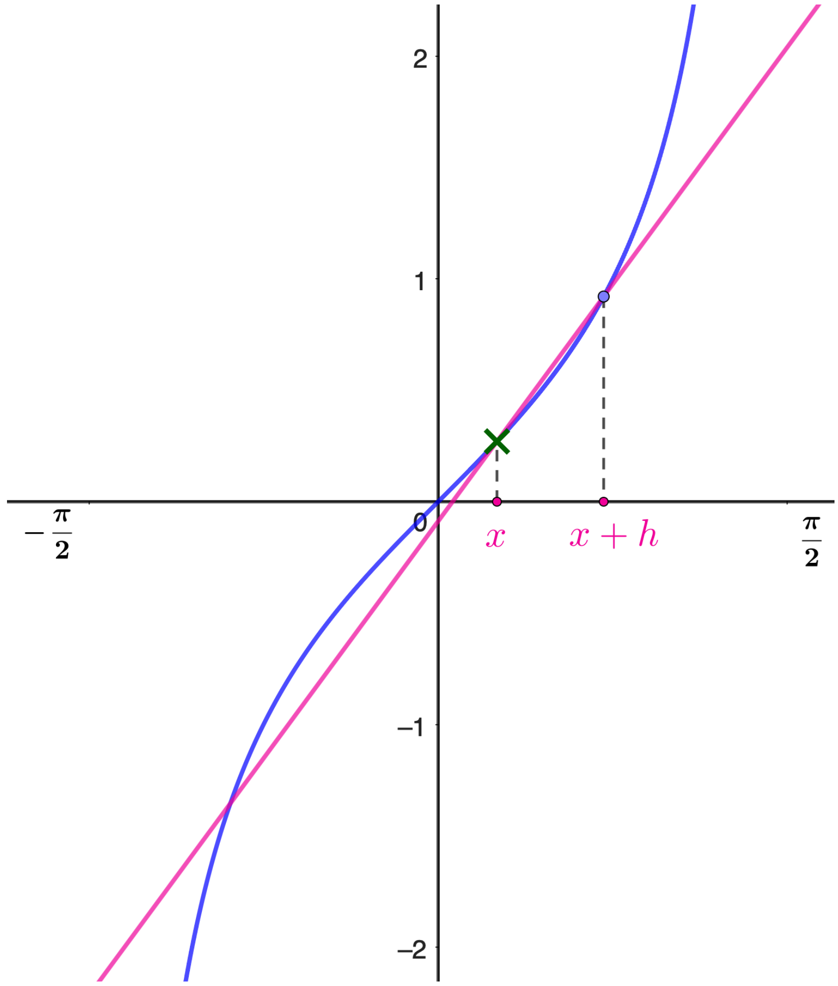
Now we know how to differentiate sine, we can tackle other circular functions.



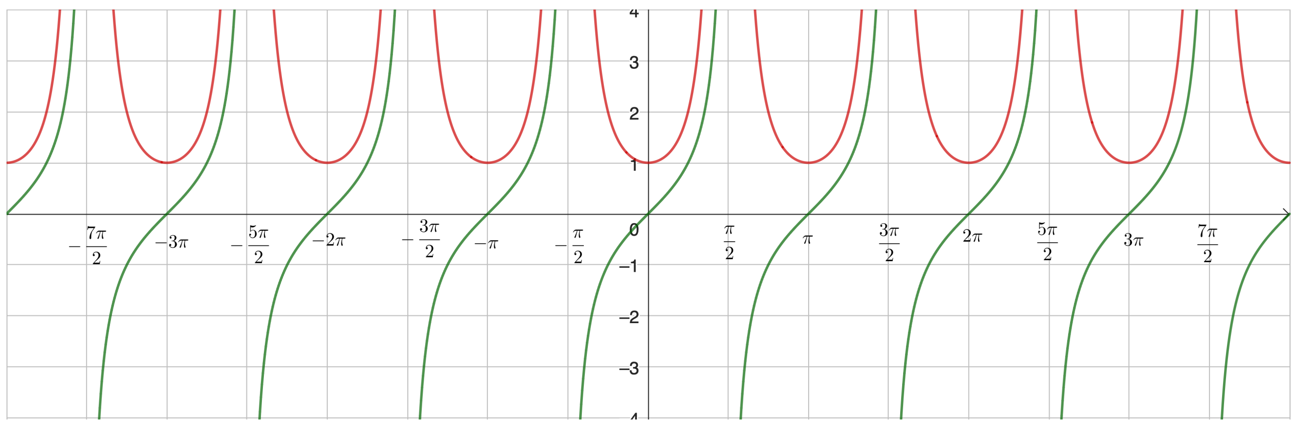
Use the same ideas to find the gradient of the tangent to the curve $y = \cos x$ at the green cross.


1
0
$-\frac{\pi}{2}$
x
$x+h$
$\frac{\pi}{2}$

What is the gradient of the tangent to the curve $y = \tan x$ at the green cross?

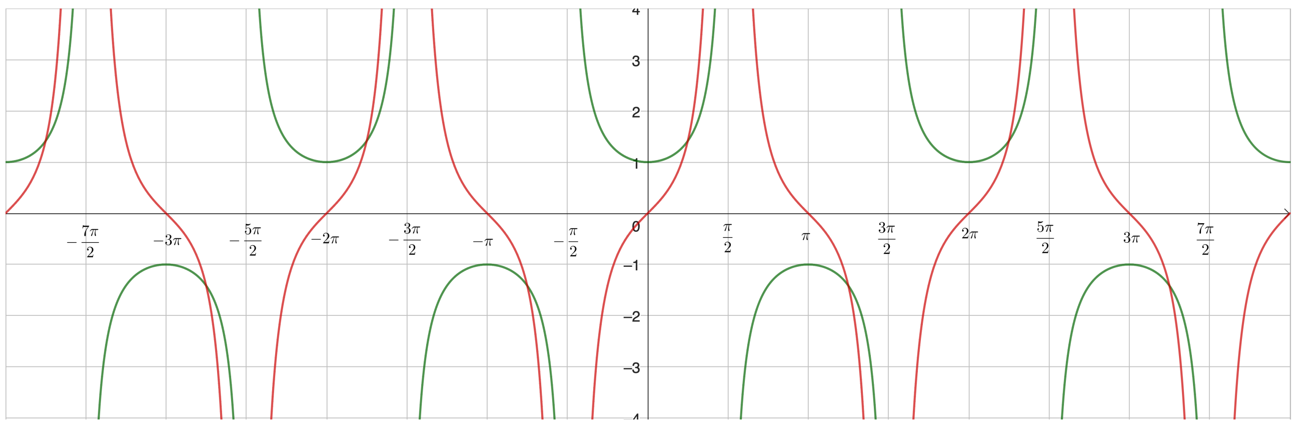


Here is the graph of tan x along with the graph of its differential. How do the two curves relate to each other?



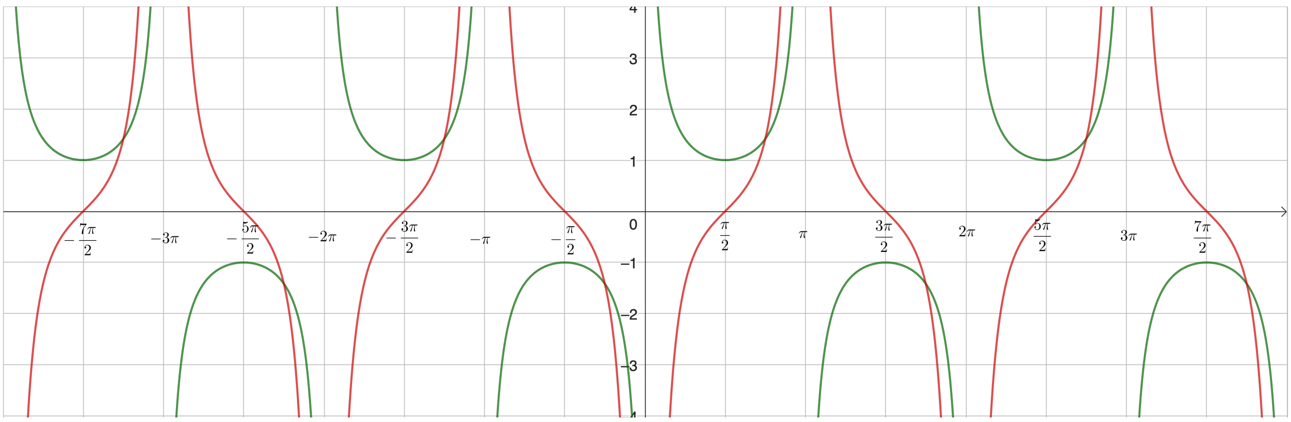
Differentiate $f(x) = \sec x$

Here is the graph of $\sec x$ along with the graph of its differential. How do the two curves relate to each other?



Differentiate $f(x) = \text{cosec}\, x$

Here is the graph of cosec$x$ along with the graph of its differential. How do the two curves relate to each other?



Differentiate $f(x) = \cot x$

Here is the graph of $\cot x$ along with the graph of its differential. How do the two curves relate to each other?