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Circular functions 8

Differentials of circular functions

teacher version

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	like $\sin\theta = 0.4$
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90° - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A+B) = \sin A \cos B + \cos A \sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45°\right)$ and so on
Differentiating circular functions	**radians, and tangents to graphs**
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

My approach here is to begin by getting a sense of how the gradient of a sine graph works, and to see why the fact that the differential is cosine might make sense.

From this, students will see that, for the result to be true, the gradient of the sine graph at the origin must be 1. Then we move on to understanding what this means in terms of the limit as a line approaches the tangent at the origin.

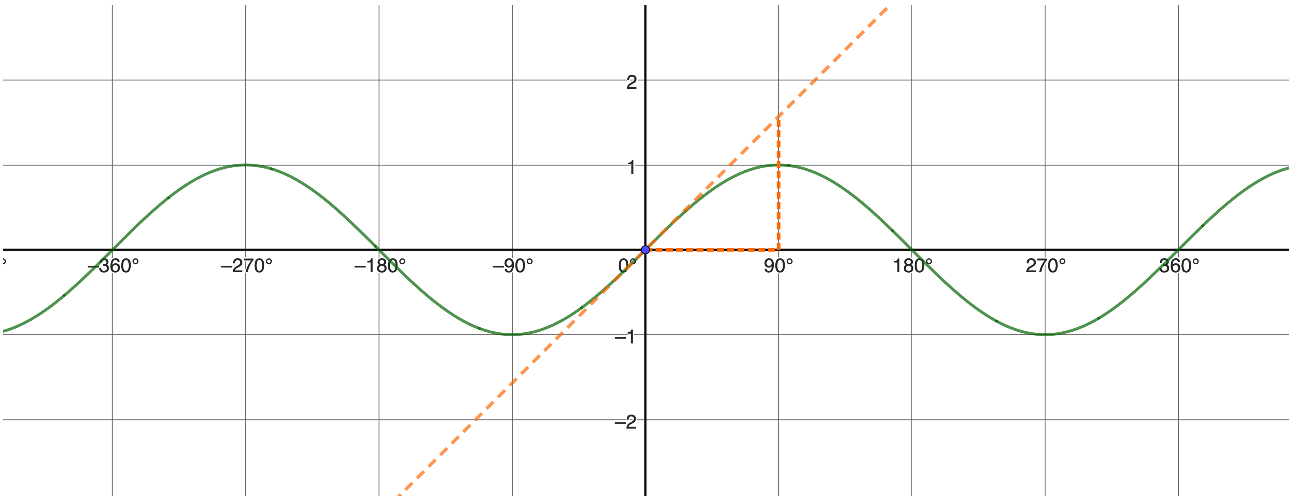
To see why this limit really is 1, we need to switch to a unit circle representation of sine and cosine. There are different degrees of rigour with which we can approach the proof, and how much detail you choose to explore will depend very much on the appetite and aptitude of your students.

The unit circle is the classic way to understand this limit, and a geometrical demonstration is certainly enough for now. However, some of your students may find this less satisfying than they would like, so I've included a more thorough version for your consideration.

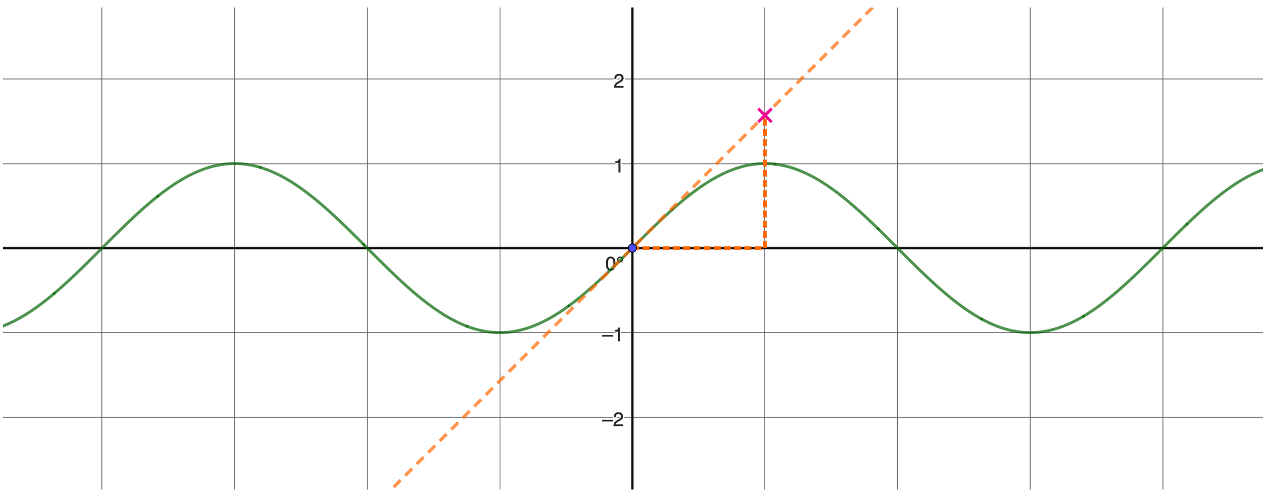
Once we know that the differential of sine is cosine, we can get all the other differentials of circular functions by using the chain, product, and quotient rules. For cosine, we can also use a "first principles" method as we did with sine; we can do this for other functions, too, but beyond sine and cosine, it's probably more trouble than it's worth. I've included a tan version in case you are interested.

The difficulty with capturing all this on a worksheet is that limiting processes really cry out for animations to bring them to life. I recommend using this worksheet in conjunction with my video version of the journey.

What (approximately) is the gradient of this tangent?



The gradient may look a bit like 1, but is it? Here is a new version of the graph without a scale on the x axis. If the gradient of this tangent is to be 1, what (approximately) would the x coordinate of the pink cross have to be?



Look at the orange triangle.

y-step is approximately 1.6.

x-step is 90

so

$$\text{gradient} \approx \frac{1.6}{90}$$

which is very small.

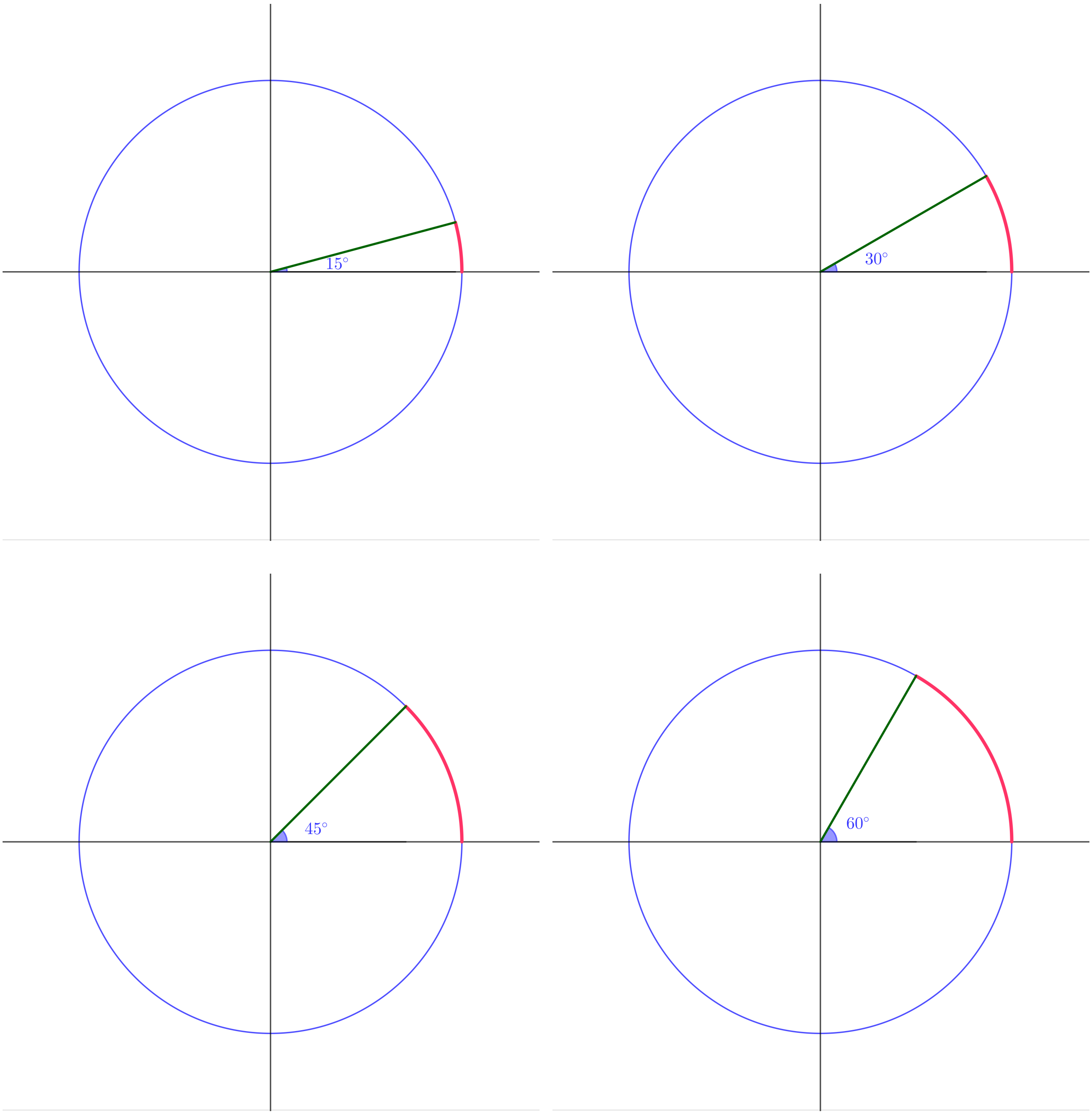
It looks much bigger than this, but this is because the axis scales are so different.

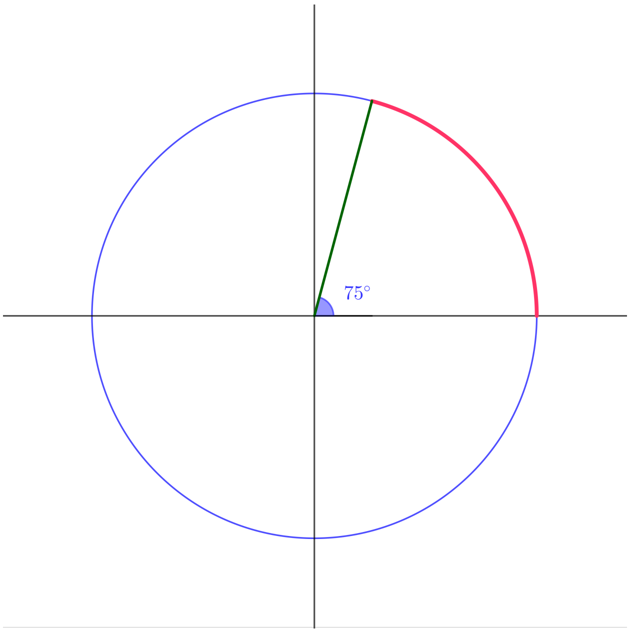
To make the gradient 1, the x coordinate of the pink cross would have to be approximately 1.6.

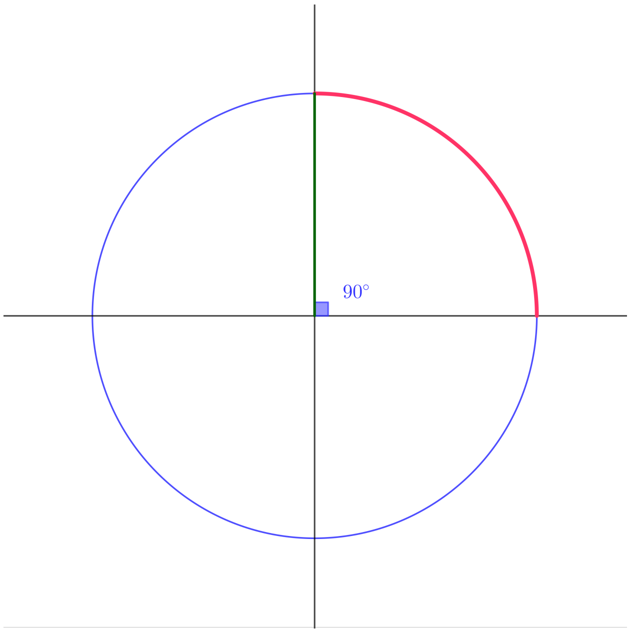
Now we will see how to find this value exactly.

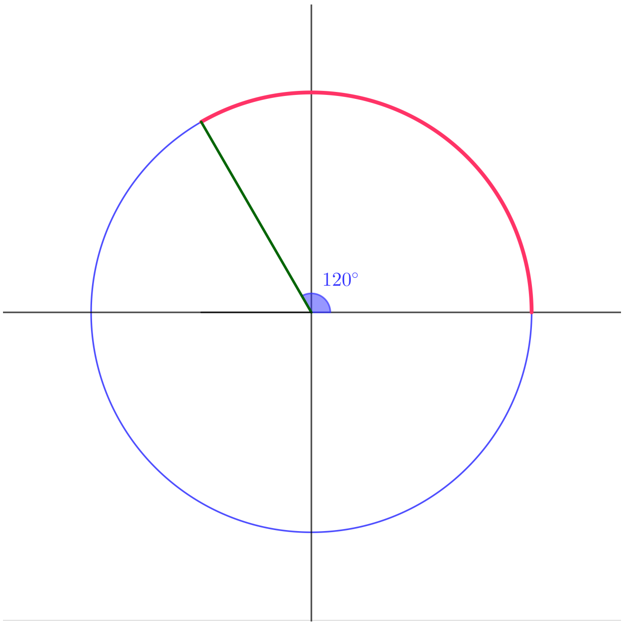
Now, we will try to find this x coordinate exactly. That is to say, we will find units for the x axis that makes this gradient 1. In theory, we can differentiate the circular functions without doing this, but everything works out so much more easily if we do, and that's the way it's done the world over. To do this, we need to go back to the unit circle.

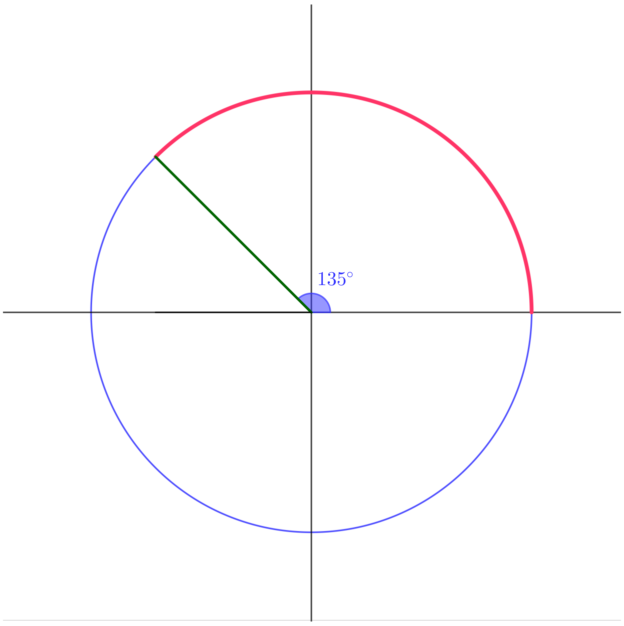
First of all, find the pink arc length on each of these circles with radius 1:

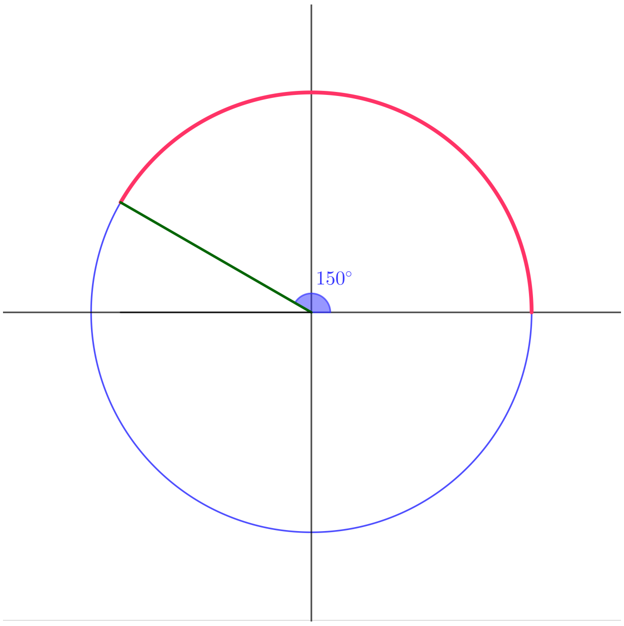


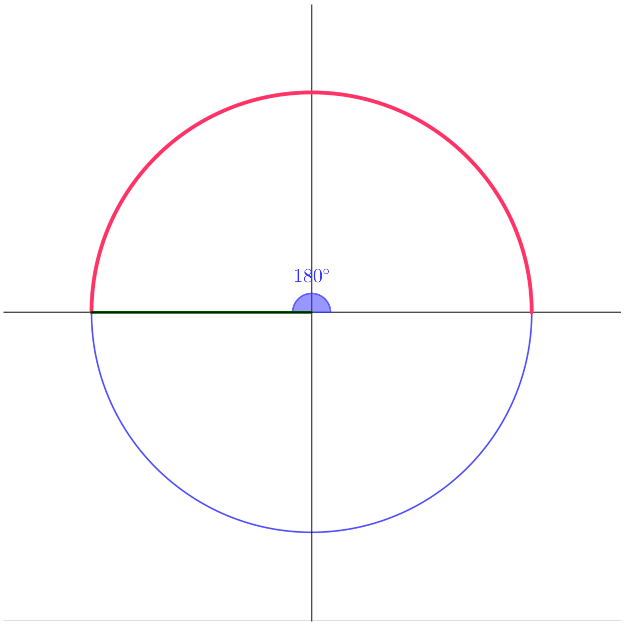

75°

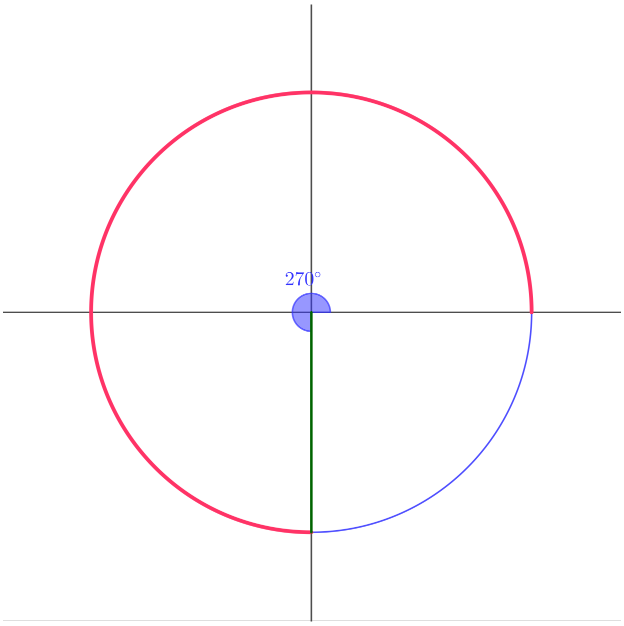

90°

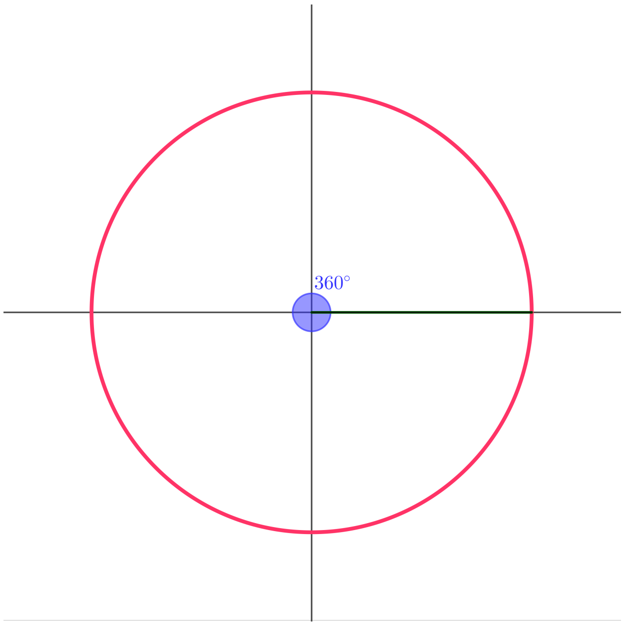

120°


135°


150°

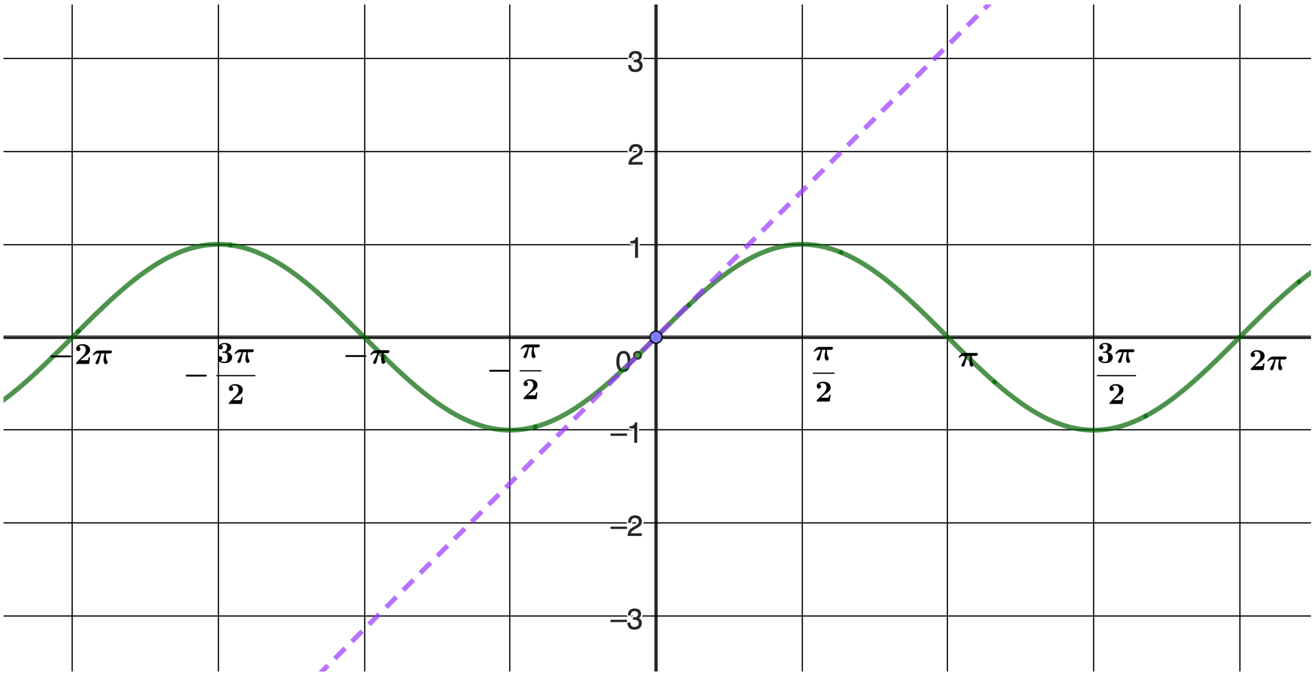

180°


270°


360°

Using $\pi = 3.14159\ldots$, what is $\dfrac{\pi}{2}$ as a decimal?

What does the gradient of the tangent look like now?



This is the place to introduce the idea of radians: instead of measuring angles in degrees, we will use the arc lengths on the unit circle as the measure of the angle itself.

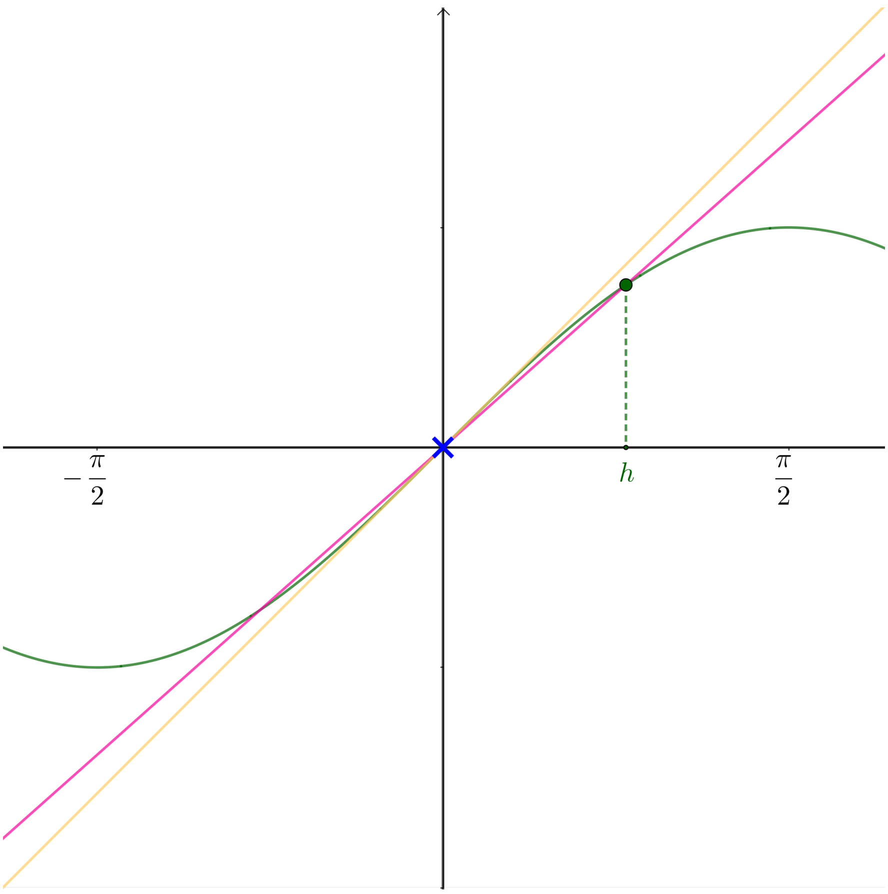
This means that we can still call the equation of the graph

$y = \sin x$

but the units of x are now radians rather than degrees.

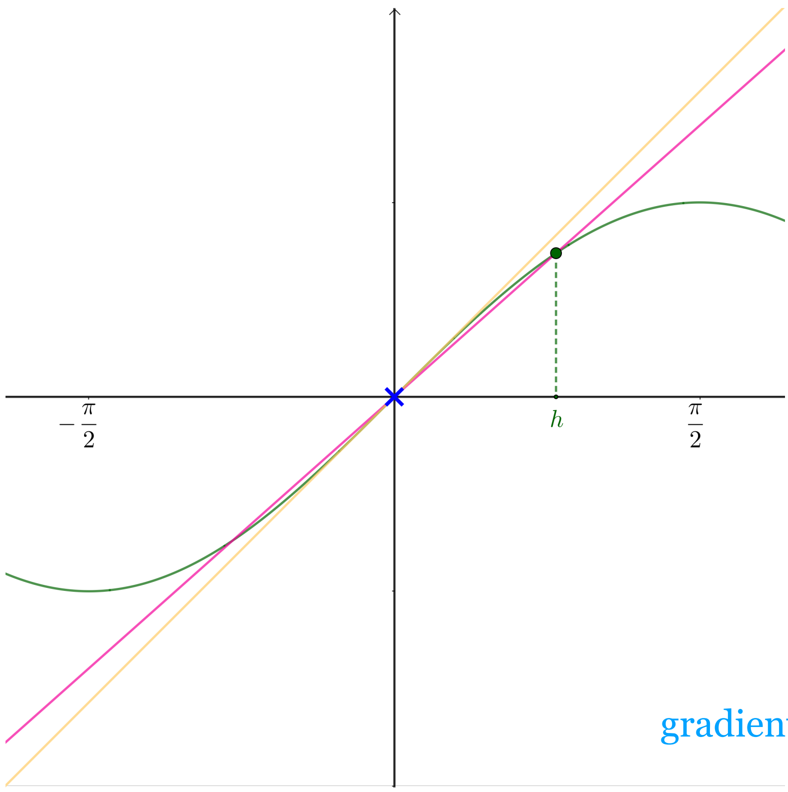
To figure out whether this is the right scale to make the gradient of the tangent equal to 1, we will use the idea of the tangent as a limit.

First, what is the gradient of the pink line?



What will happen to the pink line as h gets increasingly small?

What does this tell us about the gradient of the tangent?



$$\begin{aligned}\text{gradient} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \frac{\sin h - \sin 0}{h}\\ &= \frac{\sin h}{h}\end{aligned}$$

Remember that the units of h is radians.

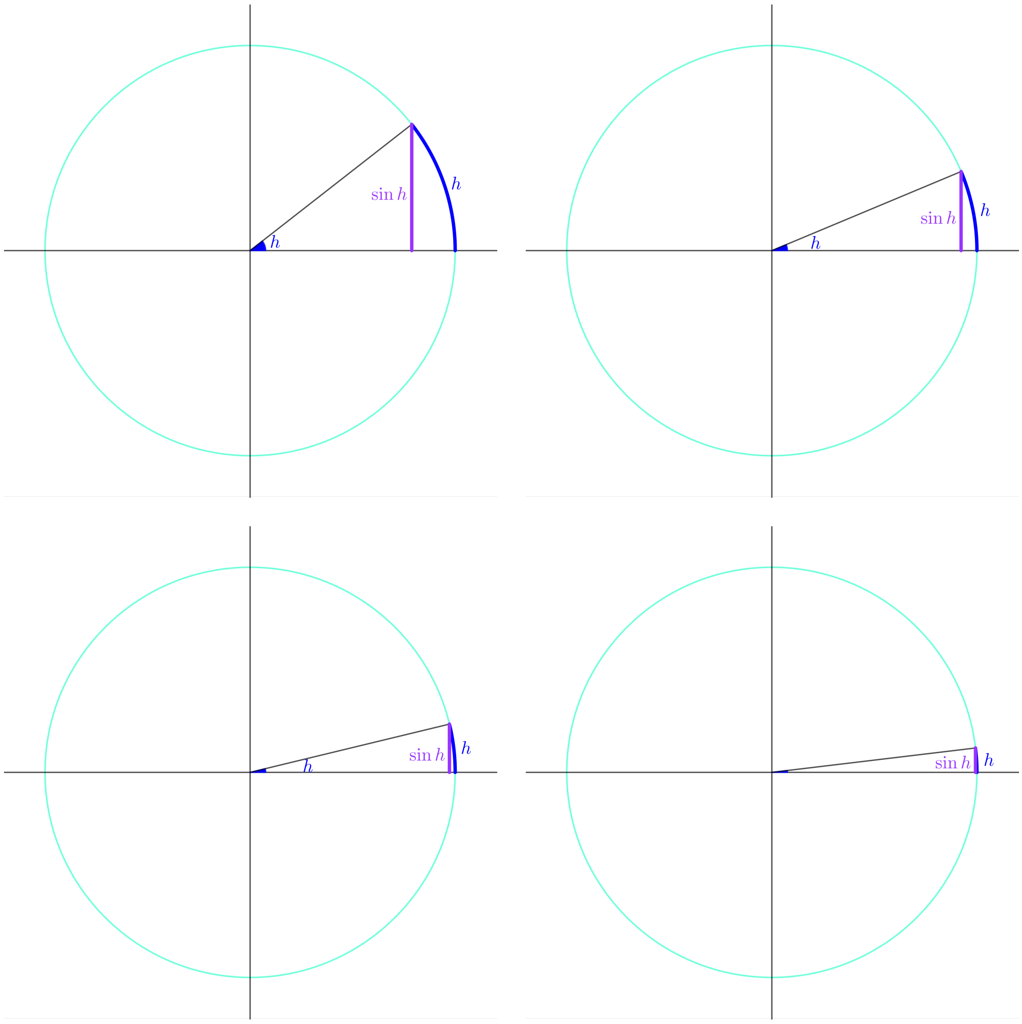
The pink line will get increasingly close to the tangent, and its gradient will get increasingly close to

$$\lim_{h \to 0} \frac{\sin h}{h}$$

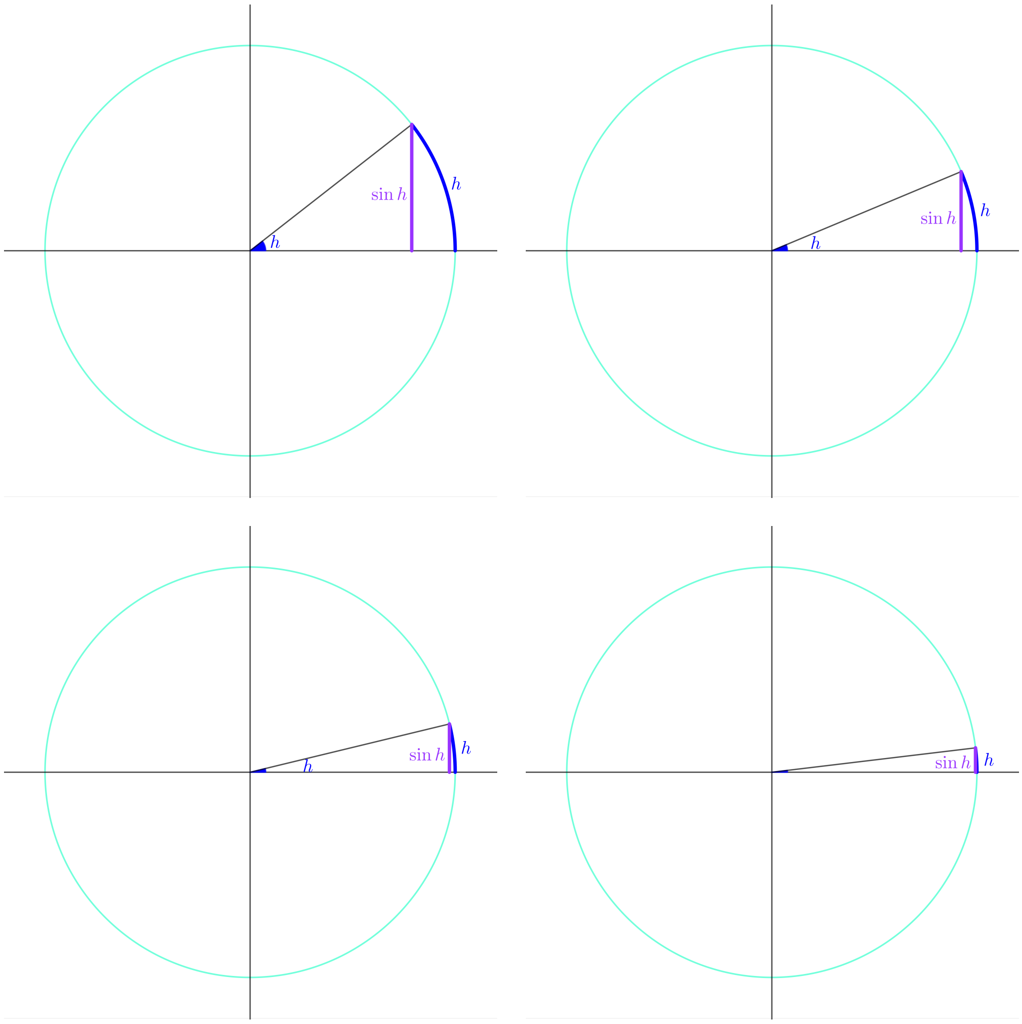
So the gradient of the tangent is this limit. We already know that it looks like 1, and that we would like it to be 1. Next, we will see why it really is 1. To do this, we must turn back yet again to the unit circle.

Look at this sequence of diagrams. Why have I used the same letter, h, for both the angles and the arc lengths?

What do you notice about the relationship between the angle size h and the ratio of the lengths of the blue arc and the purple segment?



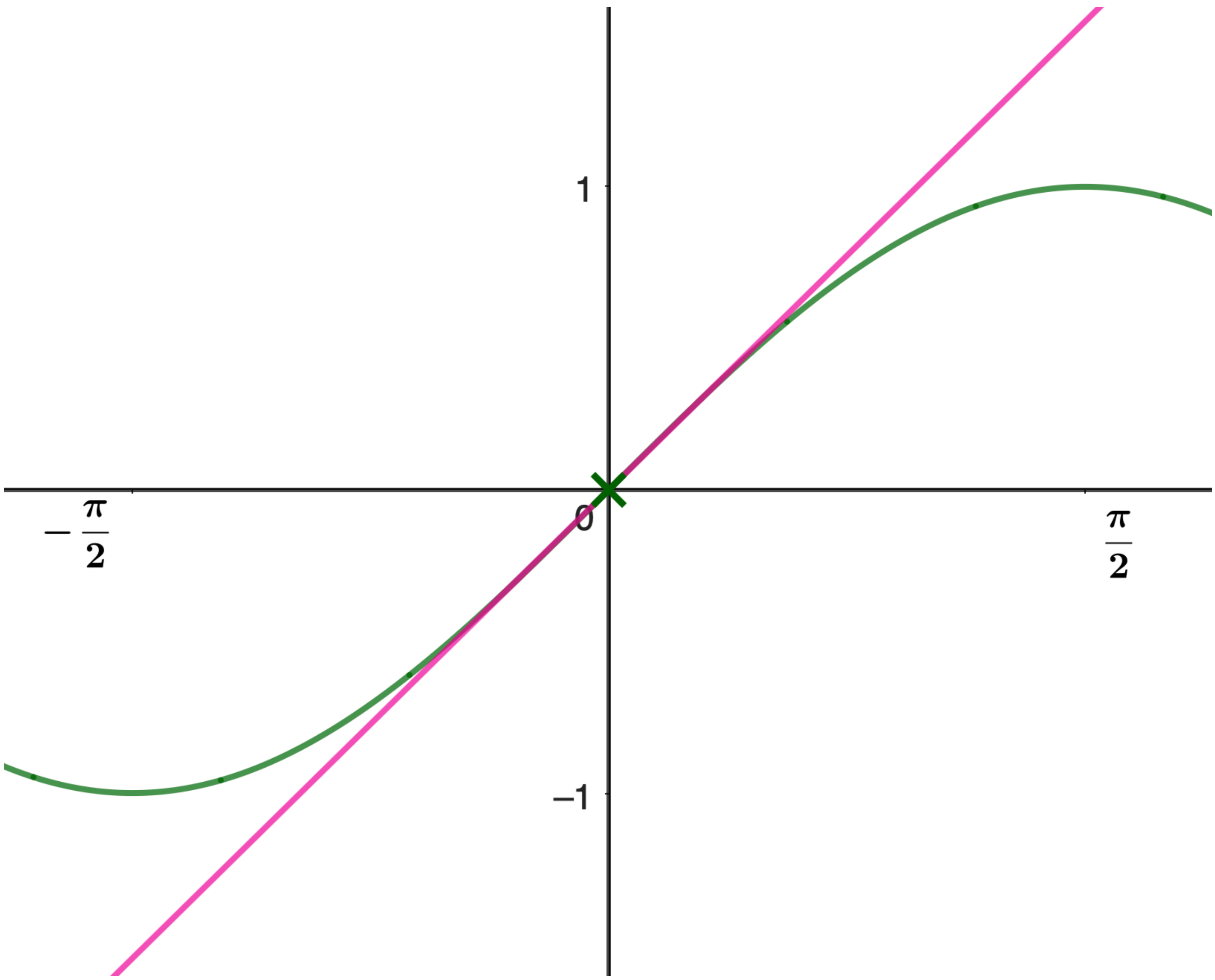
What does this tell you about the ratio $\dfrac{\sin h}{h}$ as $h \to 0$?



The smaller h gets, the closer the lengths are to each other. This is not quite the same as saying that the ratio gets closer to 1. We don't now yet if the arc length is always the same fraction (slightly over 1) of the mauve segment. But this is a subtle technical point.

It turns out that the ratio really does tend to 1, but I would be tempted not to alert your students to this little technicality unless they are going to attempt the more rigorous proof below.

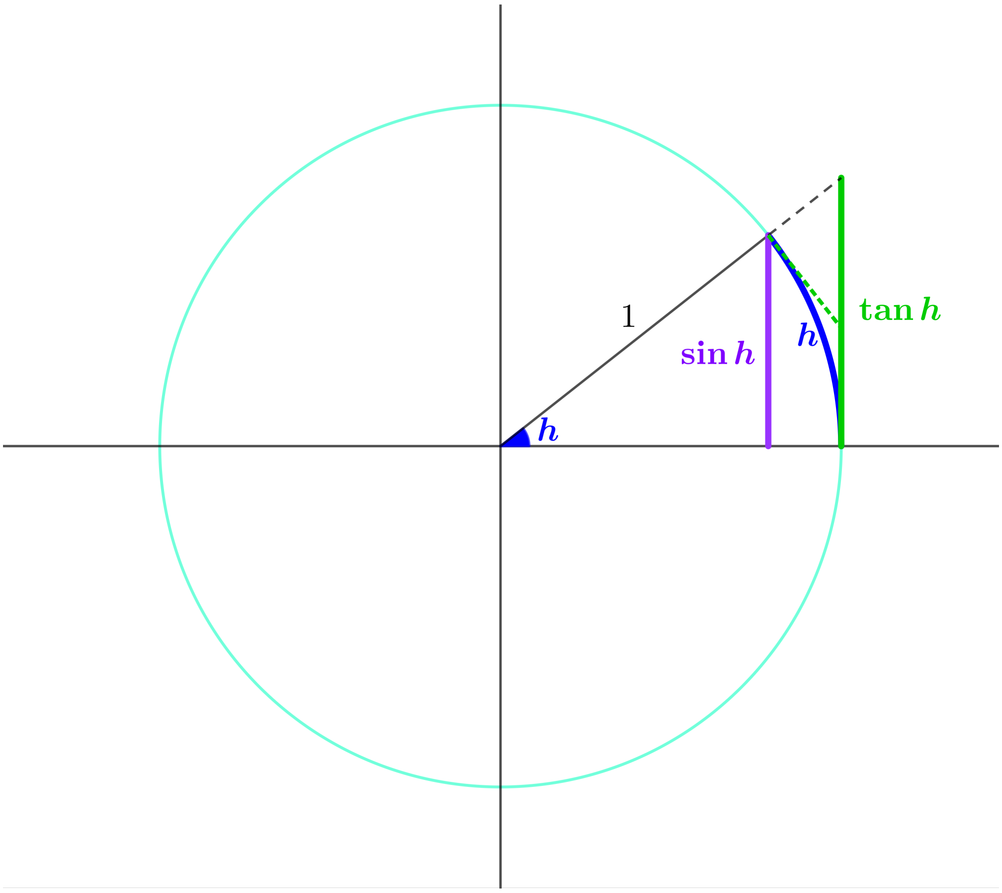
What is the gradient of the tangent to the curve $y = \sin x$ at the origin?



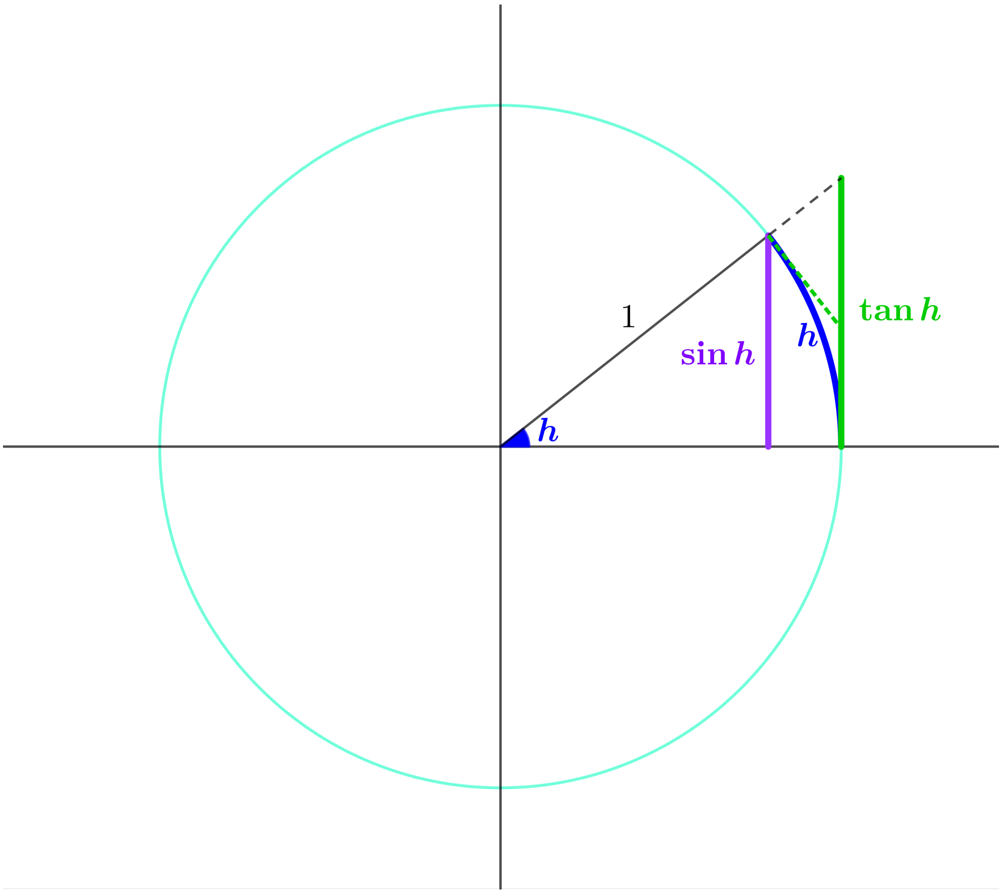
$$\text{gradient of tangent} = \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

That gives you a sense of why $\lim_{h \to 0} \frac{\sin h}{h} = 1$, but it is not quite the whole story.

Here is a more complete “proof”, in case you are interested.



Write down an inequality involving the pink and green line segments and the blue arc.



Write down an inequality involving the pink and green line segments and the blue arc.

These two pages are really for your more able classes. They address the technical problem that I flagged up a couple of pages back.

It looks immediately as though $\sin h < h < \tan h$. Can we be sure that $\tan h$ really is greater than h?

Well, the little triangle at the top right is right angled, and the hypotenuse is half of $\tan h$. So replacing the top half of the tan segment with the green dotted segment makes a path that is definitely longer than the blue arc. This shows pretty convincingly that $\tan h > h$, and hence that

$$\sin h < h < \tan h$$

Use this to find lower and upper bounds for $\dfrac{h}{\sin h}$.

Now find lower and upper bounds for $\dfrac{\sin h}{h}$.

$$\begin{aligned}
&\sin h < h < \tan h \\
\Rightarrow\ &\frac{\sin h}{\sin h} < \frac{h}{\sin h} < \frac{1}{\cos h} \\
\Rightarrow\ &1 < \frac{h}{\sin h} < \frac{1}{\cos h} \\
\Rightarrow\ &\cos h < \frac{\sin h}{h} < 1
\end{aligned}$$

What is $\lim\limits_{h\to 0} \cos h$?

Use this to find $\lim\limits_{h\to 0} \dfrac{\sin h}{h}$.

$$\cos h \to 1 \text{ as } h \to 0 \text{ and } \cos h < \frac{\sin h}{h} < 1$$

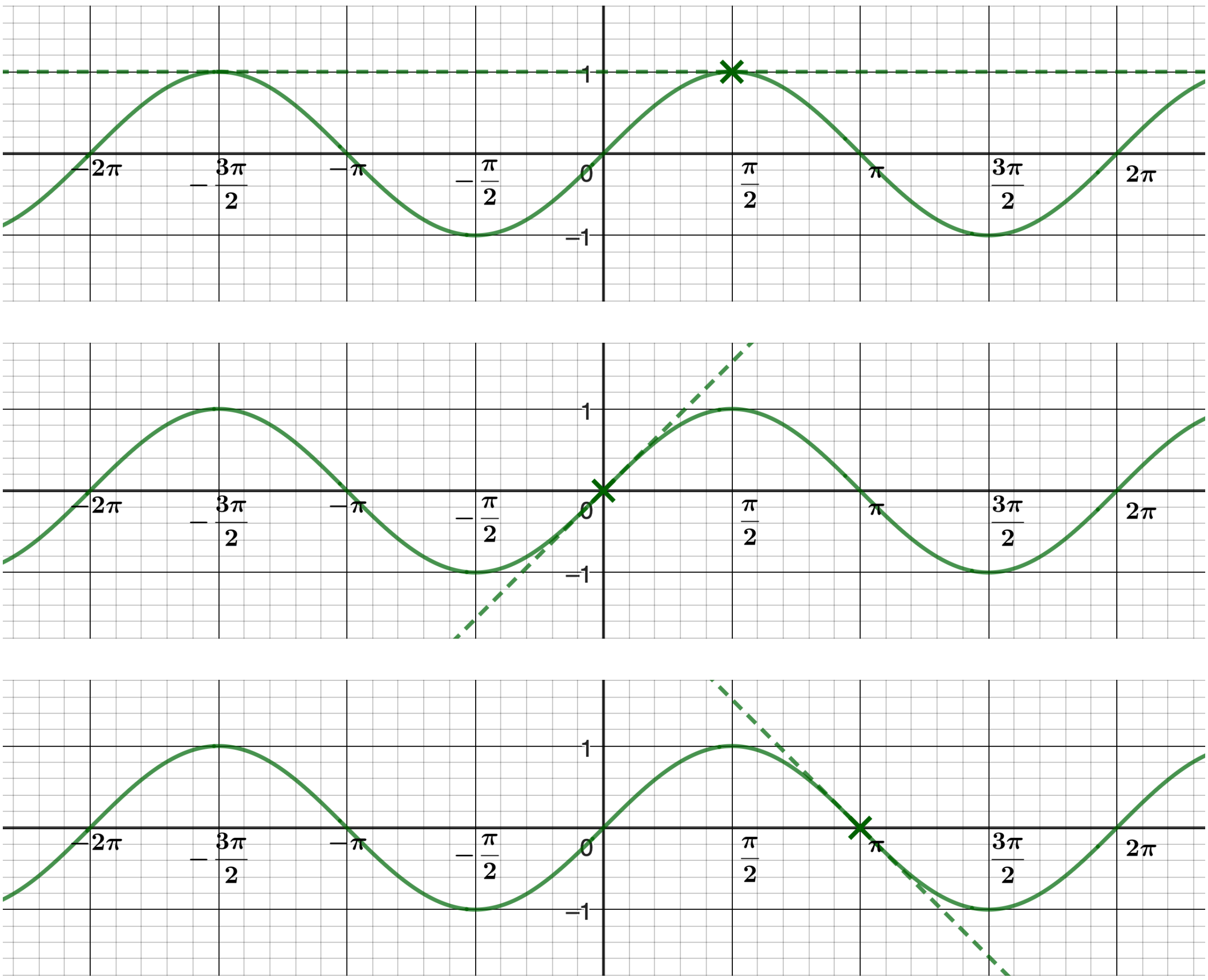
$$\Rightarrow 1 \le \lim_{h\to 0} \frac{\sin h}{h} \le 1$$

$$\Rightarrow \lim_{h\to 0} \frac{\sin h}{h} = 1$$

Now we know that the gradient of the tangent to the graph $y = \sin x$ at the origin is 1 (when we use radians as our unit of angles).

Next, we will think about the gradient of the curve at other points.

To start with, what are the gradients of these three tangents?

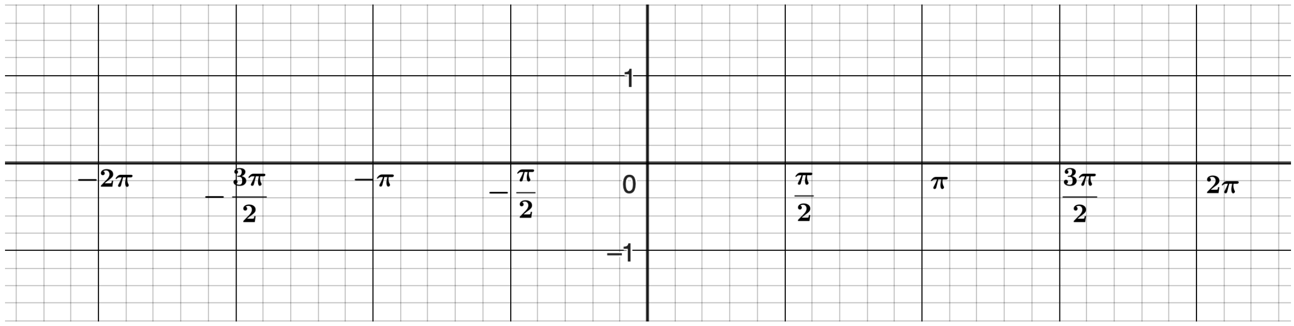


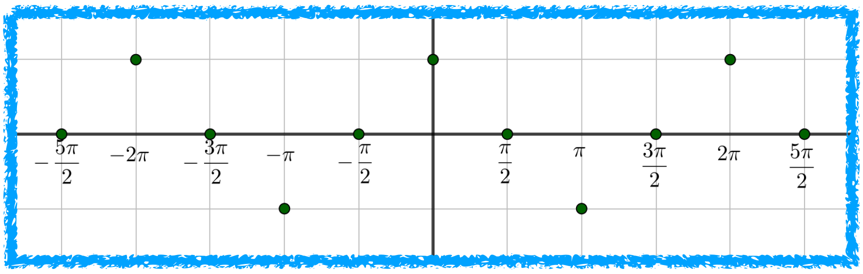
Symmetry tells us that the third of these has gradient -1.

Use the graph to fill in this table:

x	**gradient of tangent**
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1
$-\frac{\pi}{2}$	0
$-\pi$	-1
$-\frac{3\pi}{2}$	0
-2π	1

Now mark the values from the table on these axes with the left-hand column on the x axis and the right-hand column on the y axis.

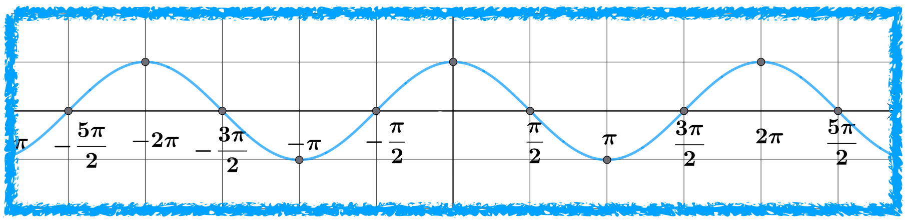




What does this graph look like?

This looks like the outline of a cos graph. The next section of this sheet is devoted to showing that it really is a cos graph.

What happens to the gradient between these points? Use this idea to draw the whole curve representing the gradient.



This now looks like the cos graph. Is it?

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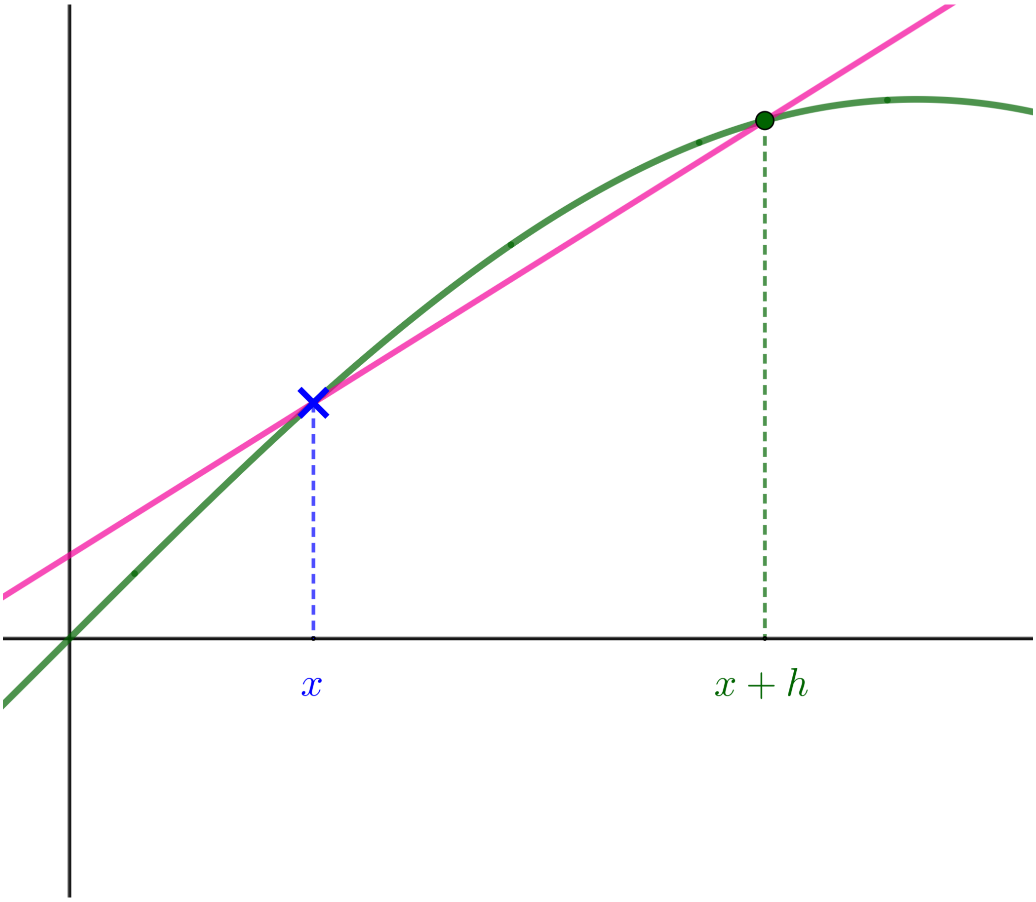
Now we know that the gradient of the sin graph looks rather like cos, and we are in a position to see that the differential of sin really is cos.

What is the gradient of the pink line in terms of x and h ?

What happens to the pink line as h gets increasingly small?



The gradient of the line is

$$\frac{\sin(x+h) - \sin x}{h}$$

and the line gets increasingly close to the tangent at the blue cross as h approaches 0.

Use this to write the gradient of the tangent as a limit.

By using a compound angle formula and then rearranging, express this gradient as a multiple of $\cos x$ minus a multiple of $\sin x$.

$$\frac{\sin(x+h)-\sin x}{h} = \frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

$$= \cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}$$

$$\Rightarrow \text{gradient of tangent} = \lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right]$$

Simplify this using the fact that $\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1$.

$$\lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right] = \lim_{h\to 0}\cos x\frac{\sin h}{h} - \lim_{h\to 0}\sin x\frac{1-\cos h}{h}$$

$$\lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right] = \lim_{h\to 0}\cos x\frac{\sin h}{h} - \lim_{h\to 0}\sin x\frac{1-\cos h}{h}$$

because the limit of a sum (or difference) is the sum (or difference) of two limits. This is a rather technical point, and perhaps it is perfectly obvious, at least for well-behaved functions such as these. It doesn't need much time in class, if any, but it is something lurking in the background that will come up in a first term analysis course of a maths degree.

Now notice that, as far as h is concerned, $\cos x$ is constant, as is $\sin x$. This means that we can "take them out of the limit"--the limit of a multiple of a function is the same multiple of the limit of the function. Again, little or no time needed on this in class. Again, first term analysis.

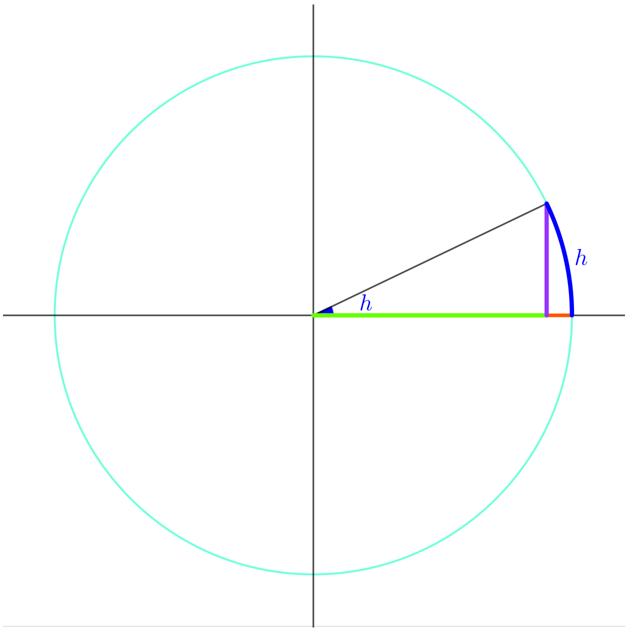
So

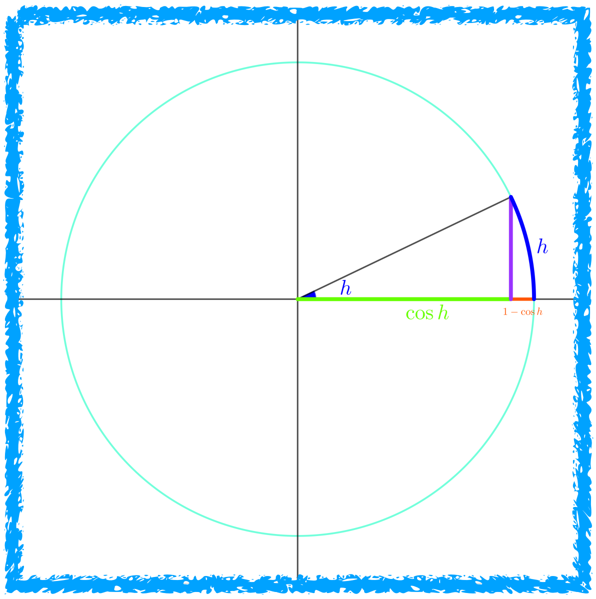
$$\begin{aligned}\text{gradient of tangent} &= \lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right]\\ &= \lim_{h\to 0}\cos x\frac{\sin h}{h} - \lim_{h\to 0}\sin x\frac{1-\cos h}{h}\\ &= \cos x\lim_{h\to 0}\frac{\sin h}{h} - \sin x\lim_{h\to 0}\frac{1-\cos h}{h}\end{aligned}$$

but we already know that $\lim_{h\to 0}\frac{\sin h}{h} = 1.$

What is the length of the orange line segment?

What happens to the ratio of the orange line segment to the blue arc as h gets increasingly small?





The orange segment is $1 - \cos h$.

What does this tell you about $\lim_{h \to 0} \dfrac{1 - \cos h}{h}$?

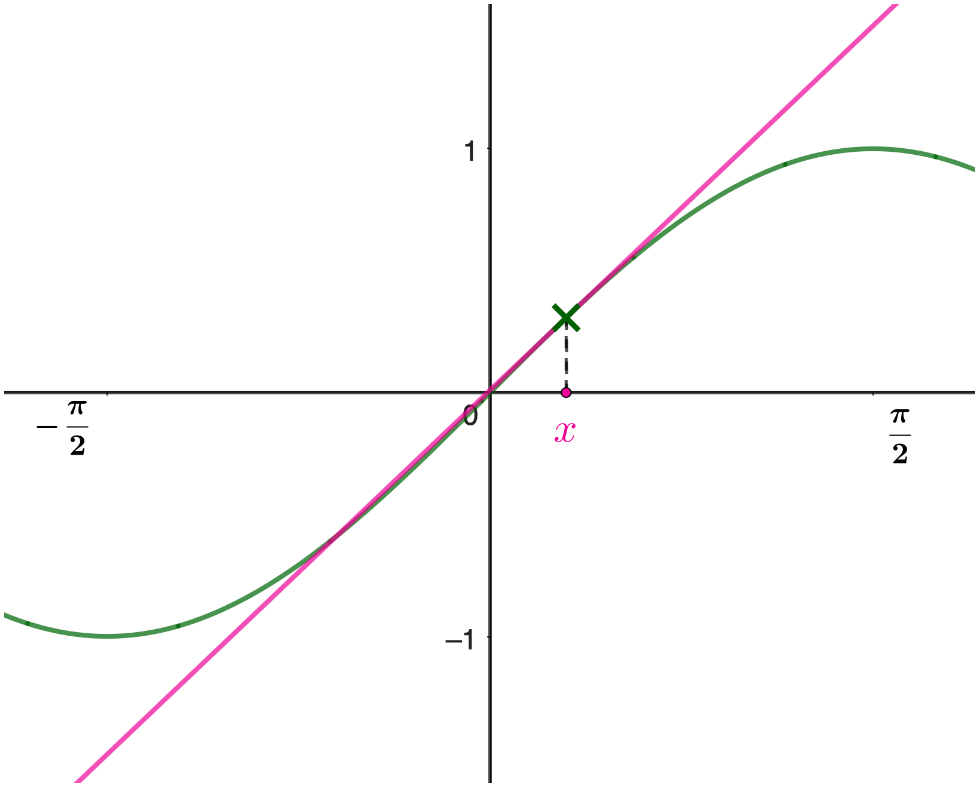
$$\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

It certainly looks as though this ratio tends to 0, and this is really enough for this stage in your students' mathematical career. However, as before, it's not the whole story, and you may want to lead them through a more rigorous proof such as this:

$$
\begin{aligned}
\frac{1-\cos h}{h} &= \frac{1-\cos^2 h}{h(1+\cos h)} \\
&= \frac{\sin^2 h}{h(1+\cos h)} \\
&= \frac{\sin h}{h} \times \frac{\sin h}{1+\cos h}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \lim_{h \to 0} \frac{1-\cos h}{h} &= \lim_{h \to 0} \frac{\sin h}{h} \times \frac{\sin h}{1+\cos h} \\
&= 1 \times 0 \\
&= 1
\end{aligned}
$$

Use this limit to find the gradient of the tangent to the curve $y = \sin x$ at the green cross.



$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\sin h}{h} - \sin x \lim_{h \to 0} \frac{1 - \cos h}{h}$$

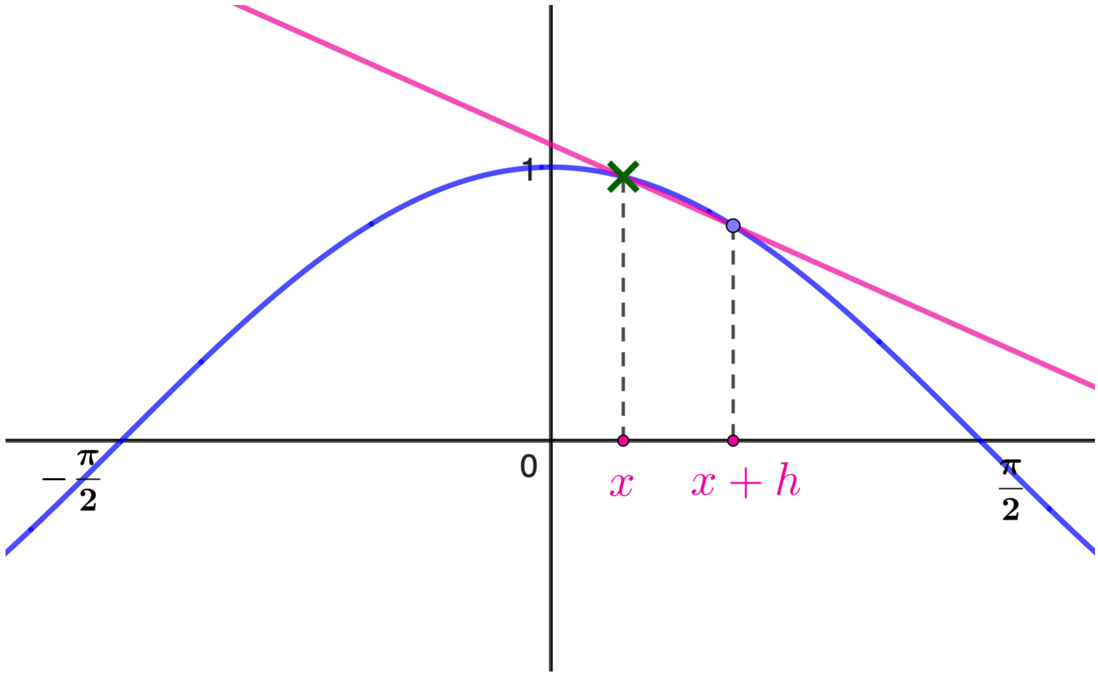
$$= \cos x \times 1 - \sin x \times 0$$

$$= \cos x$$

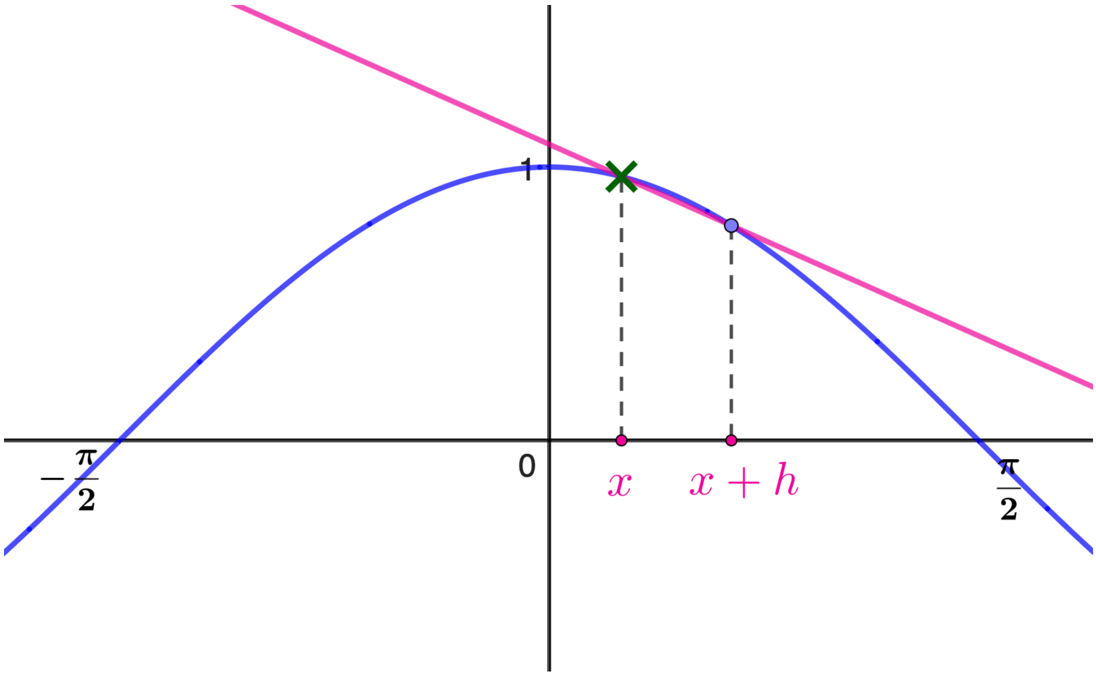
If $f(x) = \sin x$, what is $f'(x)$?

$$f'(x) = \cos x$$

Now we know how to differentiate sine, we can tackle other circular functions.



Use the same ideas to find the gradient of the tangent to the curve $y = \cos x$ at the green cross.



$$\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin h\sin x-\cos x}{h}$$

$$=-\sin x\lim_{h\to 0}\frac{\sin h}{h}-\cos x\lim_{h\to 0}\frac{1-\cos h}{h}$$

$$=-\sin x$$

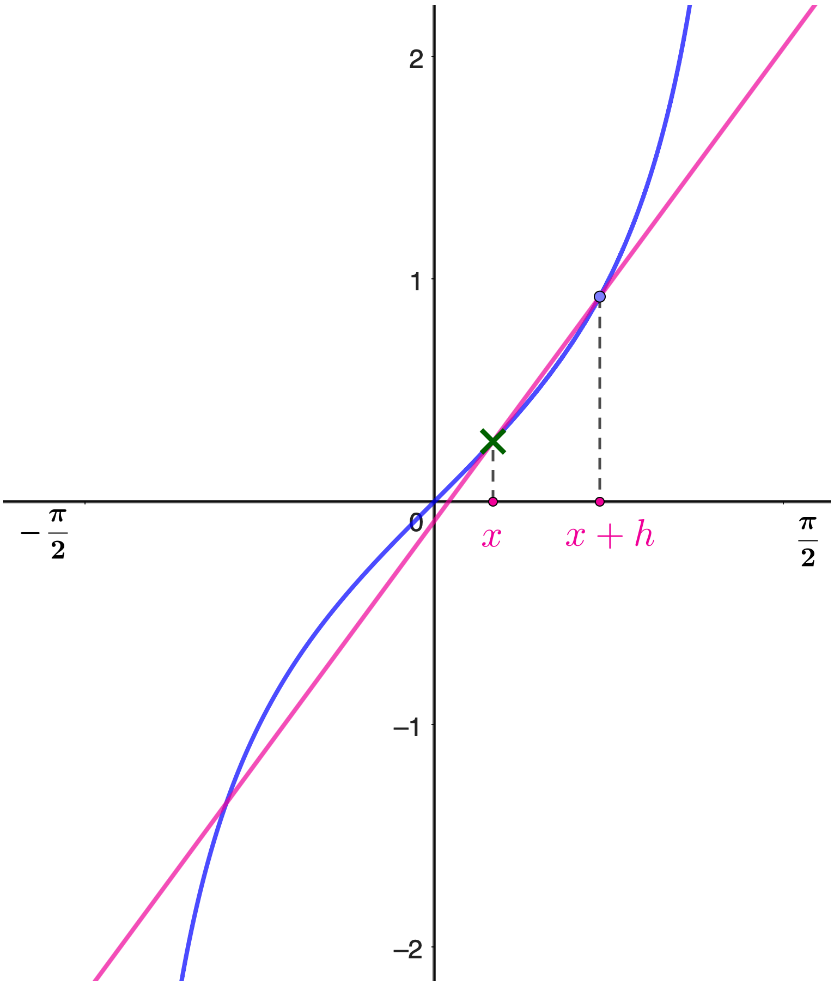
We could also find this differential by translating the cos graph to become the sin curve.

$$\cos x=\sin\left(\frac{\pi}{2}-x\right)$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\cos x=-\cos\left(\frac{\pi}{2}-x\right)\text{ by the chain rule}$$

$$=-\sin x$$

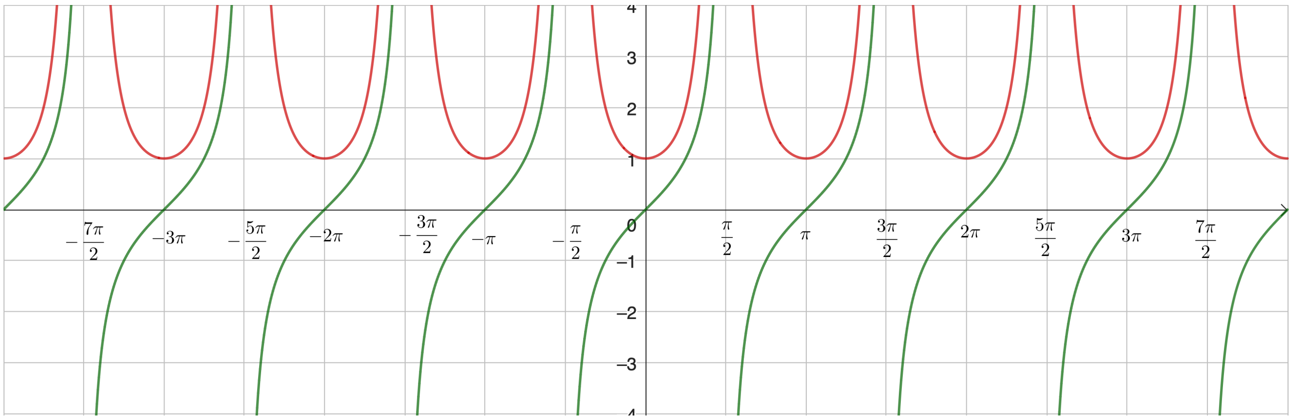
What is the gradient of the tangent to the curve $y = \tan x$ at the green cross?



Using the quotient rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Here is the graph of tan x along with the graph of its differential. How do the two curves relate to each other?



The point here is to see that the expressions for the differential do make some graphical sense. This differential isn't just slightly random expressions, but it does actually relate to something a bit more tangible! The same for the next few functions.

The tan curve (green) has points of inflection at the same time that the differential curve has local minima, and the differential curve tends to $\pm\infty$ when the gradient of the green curve tends to $\pm\infty$.

We can also find this differential by thinking about limits:

$$\begin{aligned}
\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\tan x+\tan h-\tan x+\tan^2 x\tan h}{1-\tan x\tan h}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\tan^2 x\tan h+\tan h}{1-\tan x\tan h}\right)\\
&= \lim_{h\to 0}\left(\frac{\tan h(1+\tan^2 x)}{h(1-\tan x\tan h)}\right)\\
&= \lim_{h\to 0}\frac{\tan h}{h}\left(\frac{\sec^2 x}{1-\tan x\tan h}\right)\\
&= \lim_{h\to 0}\frac{\sin h}{h}\times\frac{1}{\cos h}\times\frac{\sec^2 x}{1-\tan x\tan h}\\
&= \sec^2 x
\end{aligned}$$

because

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \qquad \lim_{h\to 0}\cos h=1 \qquad \lim_{h\to 0}\tan h=0$$

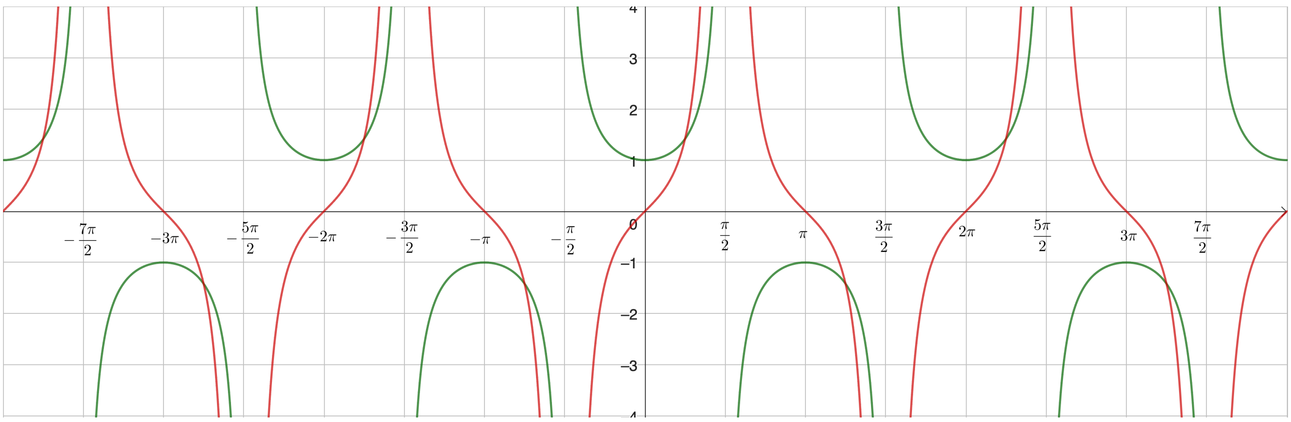
Differentiate $f(x) = \sec x$

$$y = \sec x = \frac{1}{\cos x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x = \sin x \sec^2 x$$

Here is the graph of $\sec x$ along with the graph of its differential. How do the two curves relate to each other?



The sec curve (green) has turning points at the same time that the differential curve crosses the x axis, and the differential curve tends to $\pm\infty$ when the gradient of the red curve tends to $\pm\infty$.

You might also like this implicit differentiation version, although your students will quite possibly not yet be ready for this:

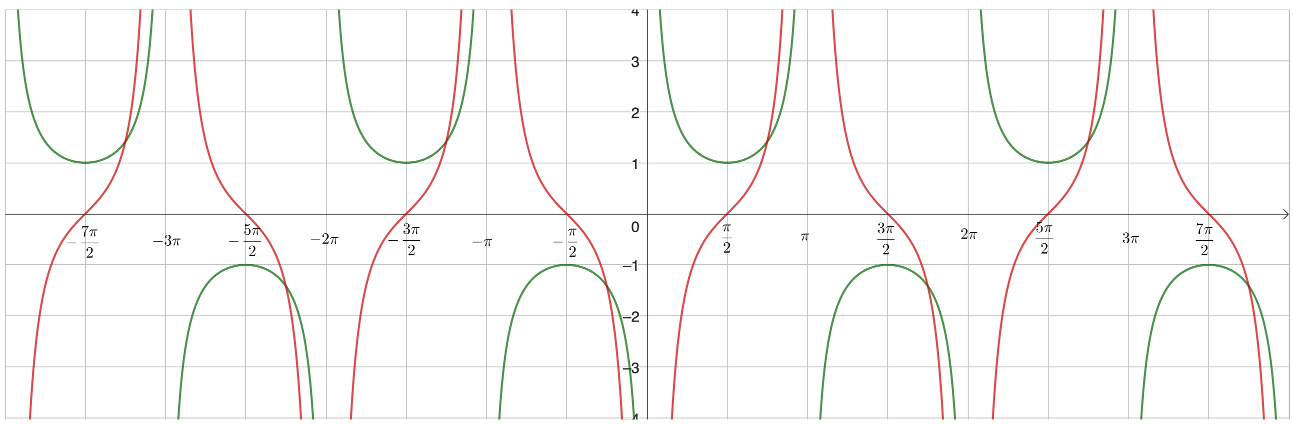
$$
\begin{aligned}
&y\cos x = 1 \\
\Rightarrow\ & -y\sin x + \frac{\mathrm{d}y}{\mathrm{d}x}\cos x = 0 \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\sin x}{\cos x} \\
& \qquad = \sec x \tan x
\end{aligned}
$$

Differentiate $f(x) = \text{cosec}\, x$

$$y = \text{cosec}\, x = \frac{1}{\sin x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos x}{\sin^2 x} = -\cot x \,\text{cosec}\, x = -\cos x \,\text{cosec}^2\, x$$

Here is the graph of cosec$x$ along with the graph of its differential. How do the two curves relate to each other?



The cosec curve (green) has turning points at the same time that the differential curve crosses the x axis, and the differential curve tends to $\pm\infty$ when the gradient of the red curve tends to $\pm\infty$.

Differentiate $f(x) = \cot x$

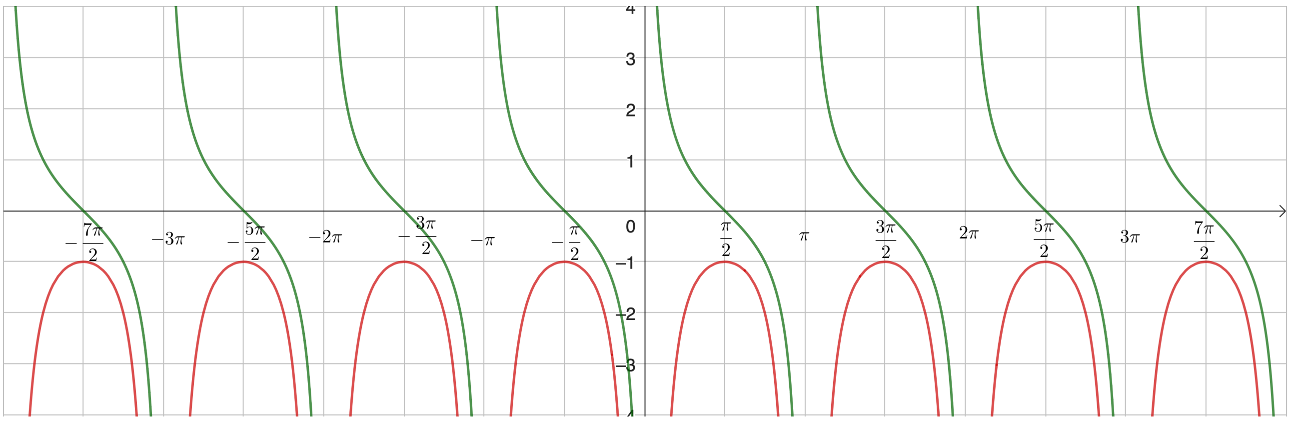
$$y = \cot x$$

$$\Rightarrow y = \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Here is the graph of $\cot x$ along with the graph of its differential. How do the two curves relate to each other?



The cot curve (green) has points of inflection at the same time that the differential curve has local maxima, and the differential curve tends to $\pm\infty$ when the gradient of the green curve tends to $\pm\infty$.