for independence
for confidence
for creativity
for insight

Circular functions 6

$\sin(A + B)$ etc

teacher version

Circular functions

Defining the circular functions — sin, cos, tan and the unit circle

Solving circular function equations — like $\sin\theta = 0.4$

Graphing the circular functions — graphs $y = \cos x$ and the like

Relationships between circular functions — $\sin(90^\circ - x) = \cos x$ and the like

More circular functions — $\sec x = \dfrac{1}{\cos x}$ and so on

Circular functions of sums — **formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$**

Transforming and adding circular functions — $\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on

Differentiating circular functions — radians, and tangents to graphs

Integrating circular functions — areas

Inverses of circular functions — $\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

We all know formulas for $(a+b)^2$ or e^{a+b} and so on, but what about the circular functions? Are there formulas for $\sin(\alpha+\beta)$ and the other circular functions? Yes there are, and they are of fundamental importance in more or less all applications of mathematics to the physical world.

There are so many ways to derive the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. Everyone has their favourite, it seems, and will argue passionately for its supremacy. I find nearly all of them a bit unsatisfactory for one or more of the following reasons:

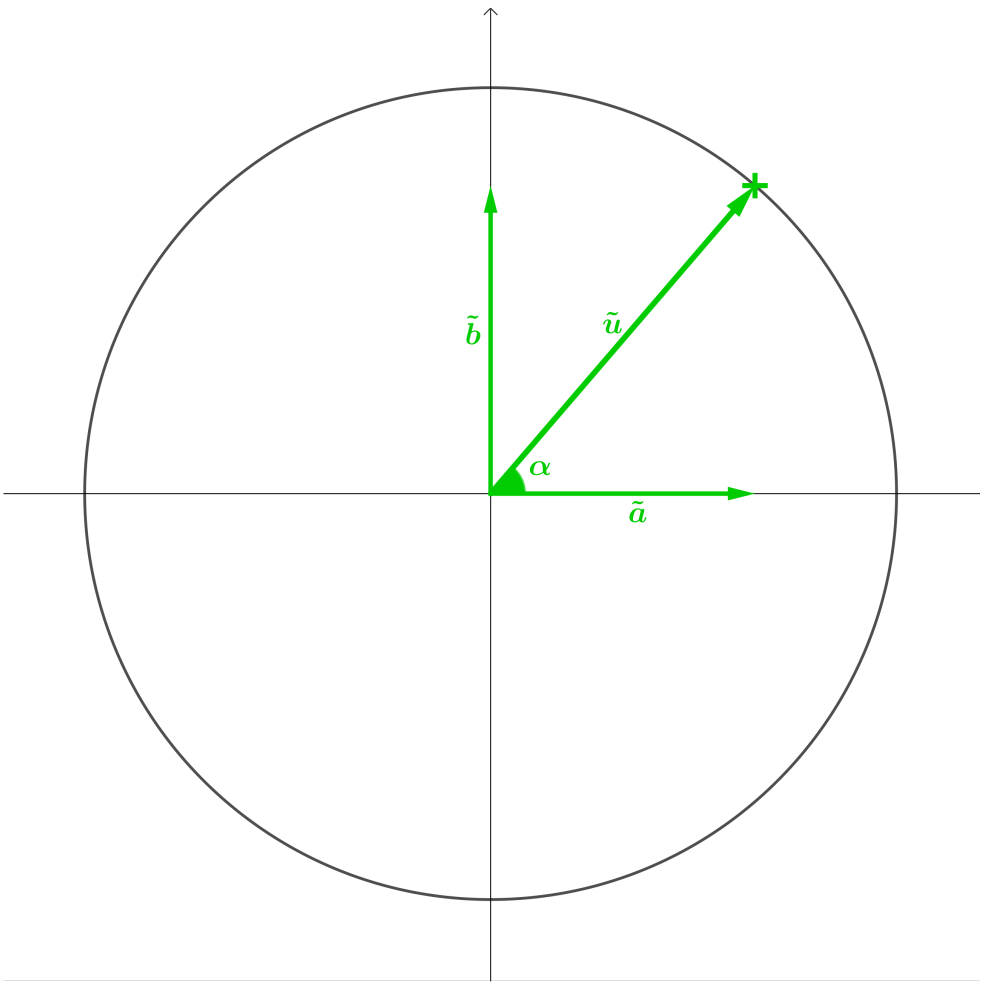
- they only cover cases when $\alpha < 90^\circ, \quad \beta < 90^\circ,$ and perhaps even $\alpha+\beta < 90^\circ$
- they are a bit contrived, often involving either lengths or areas of right-angled triangles in a rather complicated and unilluminating way.

Since I am taking my students on a journey that splits the circular functions away from right-angled triangles and attaches them instead to unit circles, I prefer to rely on the idea of rotations. This method does not rely on the angles being acute—it is quite general (at least, I claim that it is, but perhaps someone will show me why it isn't quite as general as I would like it to be). The downside is that it requires some knowledge of vectors. Even though it doesn't involve very much, it may be prudent, for some classes, to have a quick look at how vectors work before tackling this sheet.

Here is a circle radius 1.

What are the coordinates of the green cross?

Write each of the vectors as column vectors in terms of α.

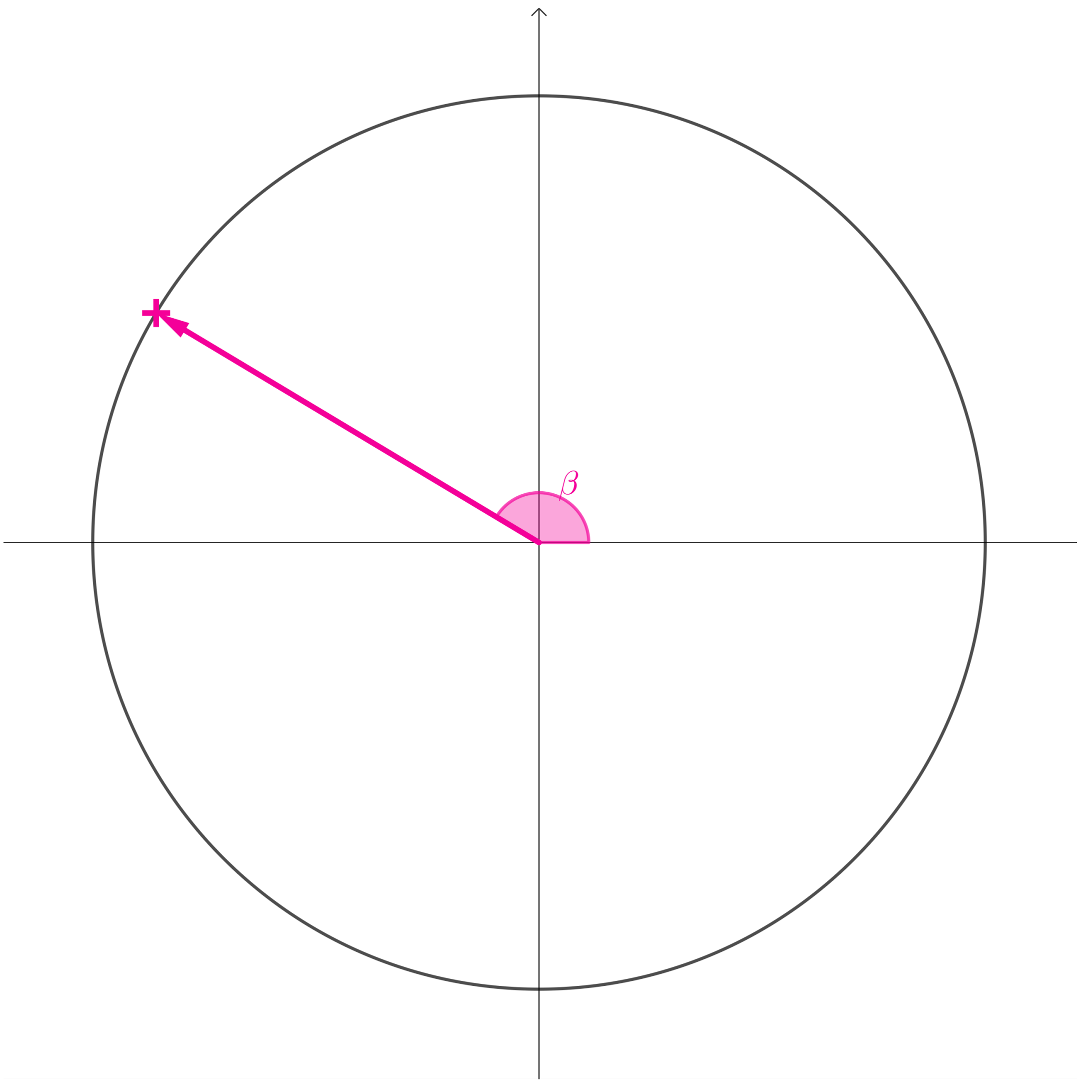


The green cross is at $(\cos\alpha,\ \sin\alpha)$

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}} = \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

What are the coordinates of the red cross?

Write the red vector as column vector in terms of β.

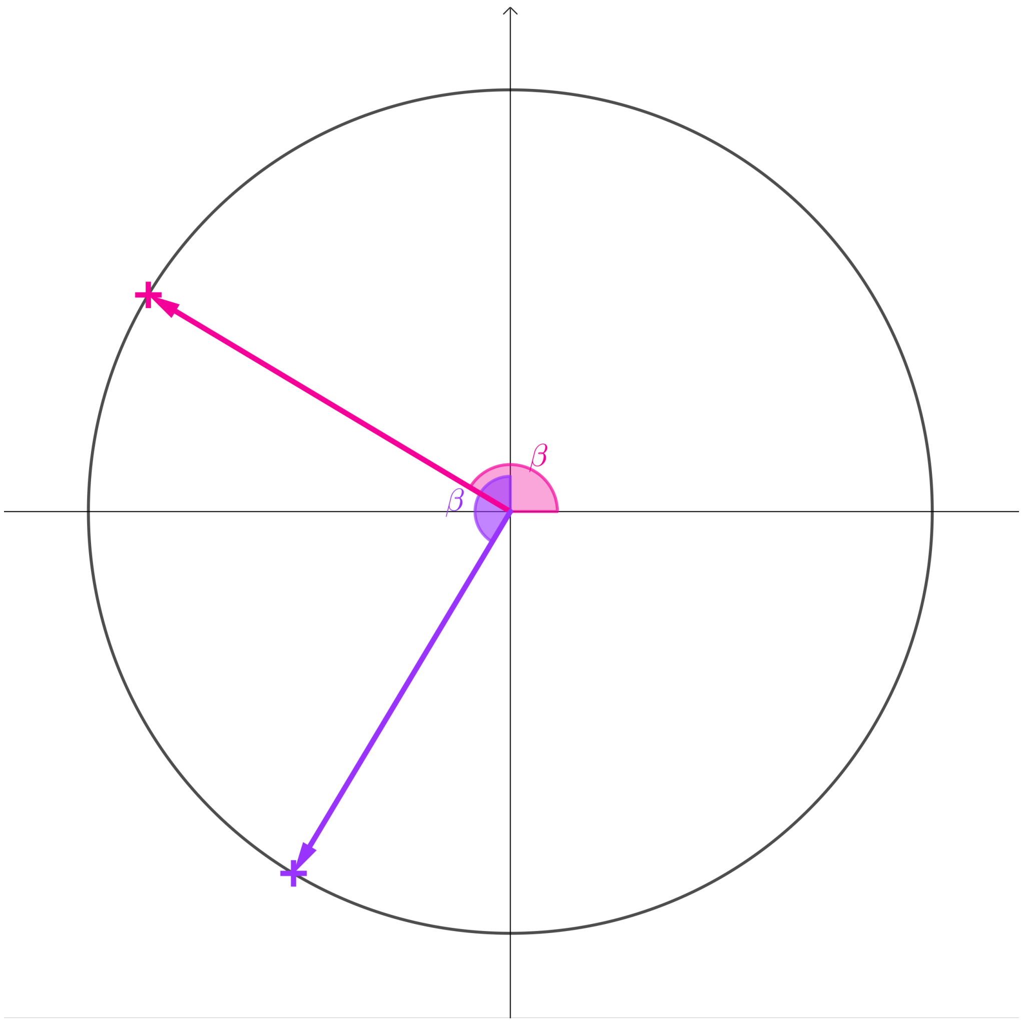


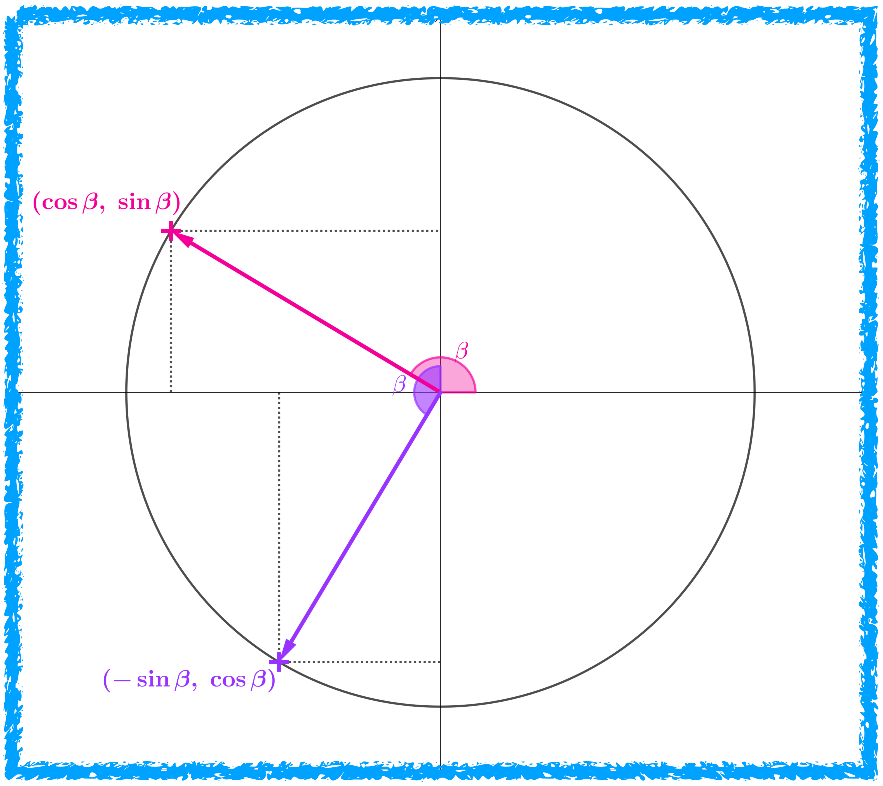
The pink cross is at $(\cos\beta,\ \sin\beta)$. This is just the definition of cos and sine as the x and y coordinates on the unit circle.

The pink vector is $\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$.

What are the coordinates of the darker cross?

Write the purple vector as column vector in terms of β.





The purple cross is at $(-\sin\beta,\ \cos\beta)$.

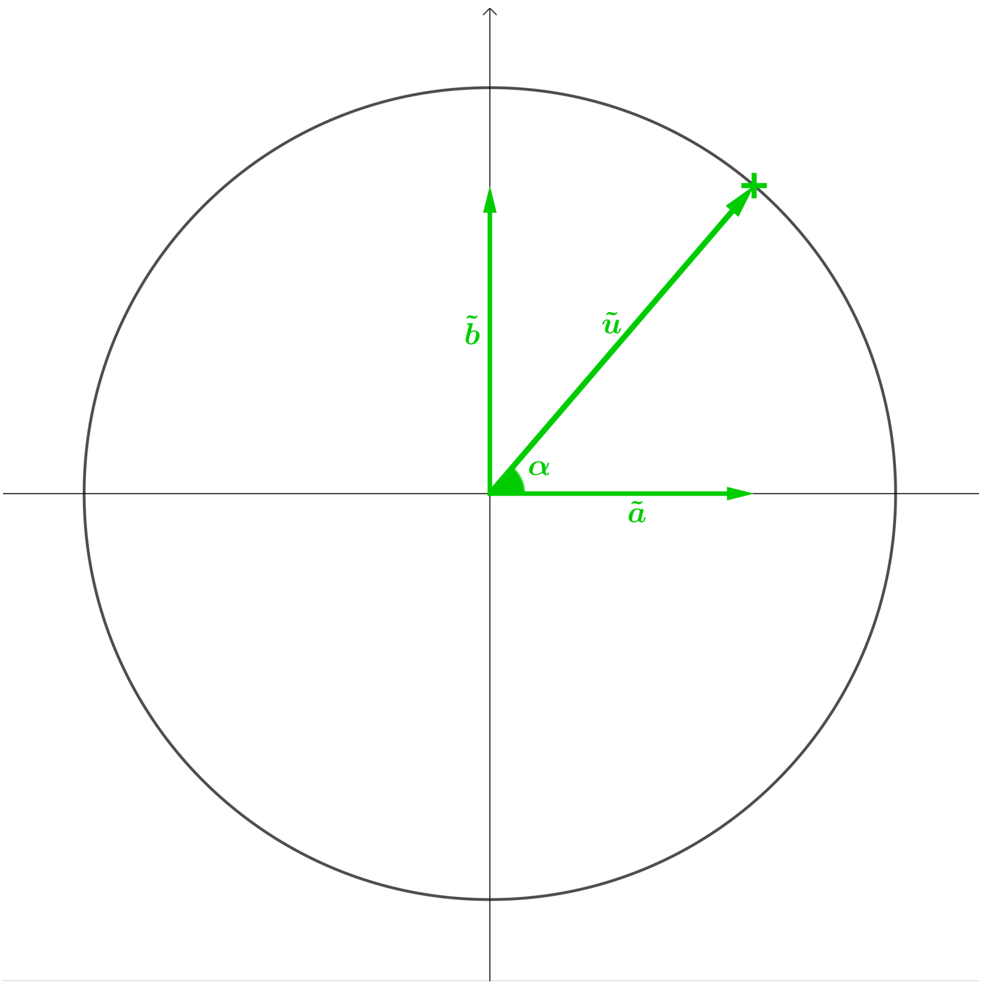
The purple vector is $\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$

The sizes of the x and y components of the purple vector are the sizes of the y and x components respectively of the pink vector.

Notice that, in this case, $\sin\beta > 0$ and $\cos\beta < 0$, and both components of the rotated purple vector are negative. It might be worth spending a bit of time showing that these signs work out correctly no matter what the size of β.

Quick reminder:

Write each of the vectors as column vectors in terms of α.

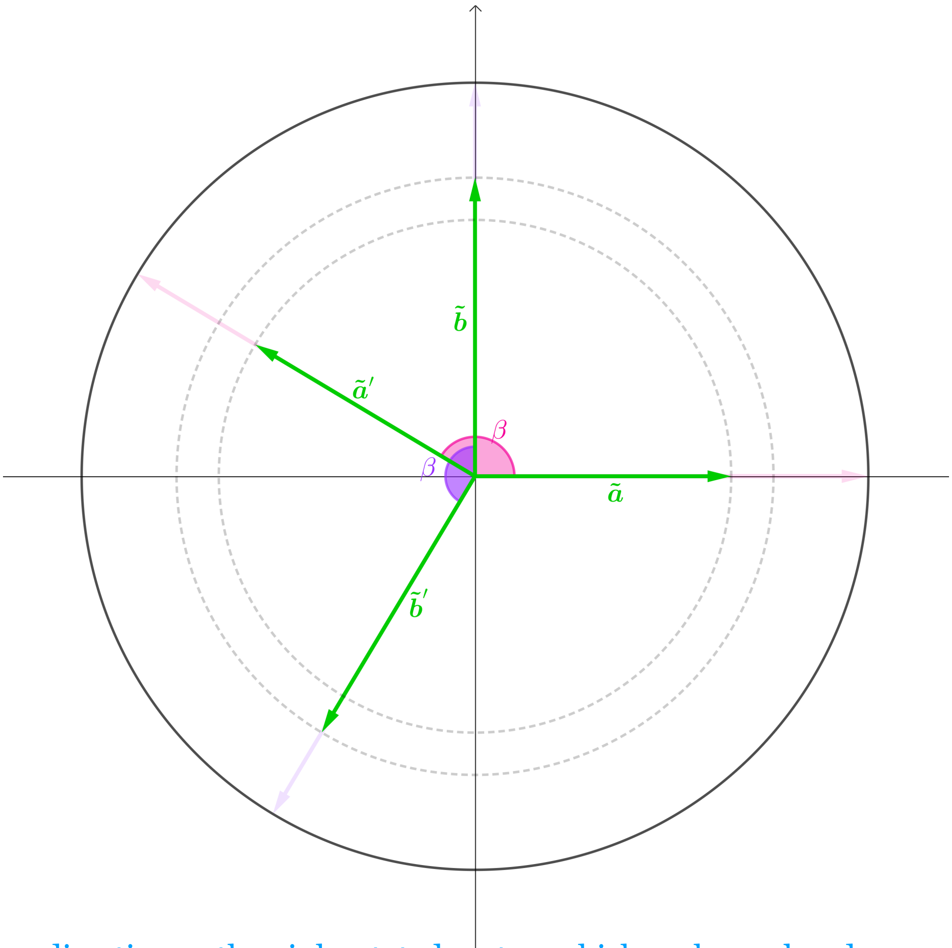


The green cross is at $(\cos\alpha,\ \sin\alpha)$

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}} = \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

What are the magnitudes and directions of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?

Write the vectors $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$ as column vectors in terms of α and β.



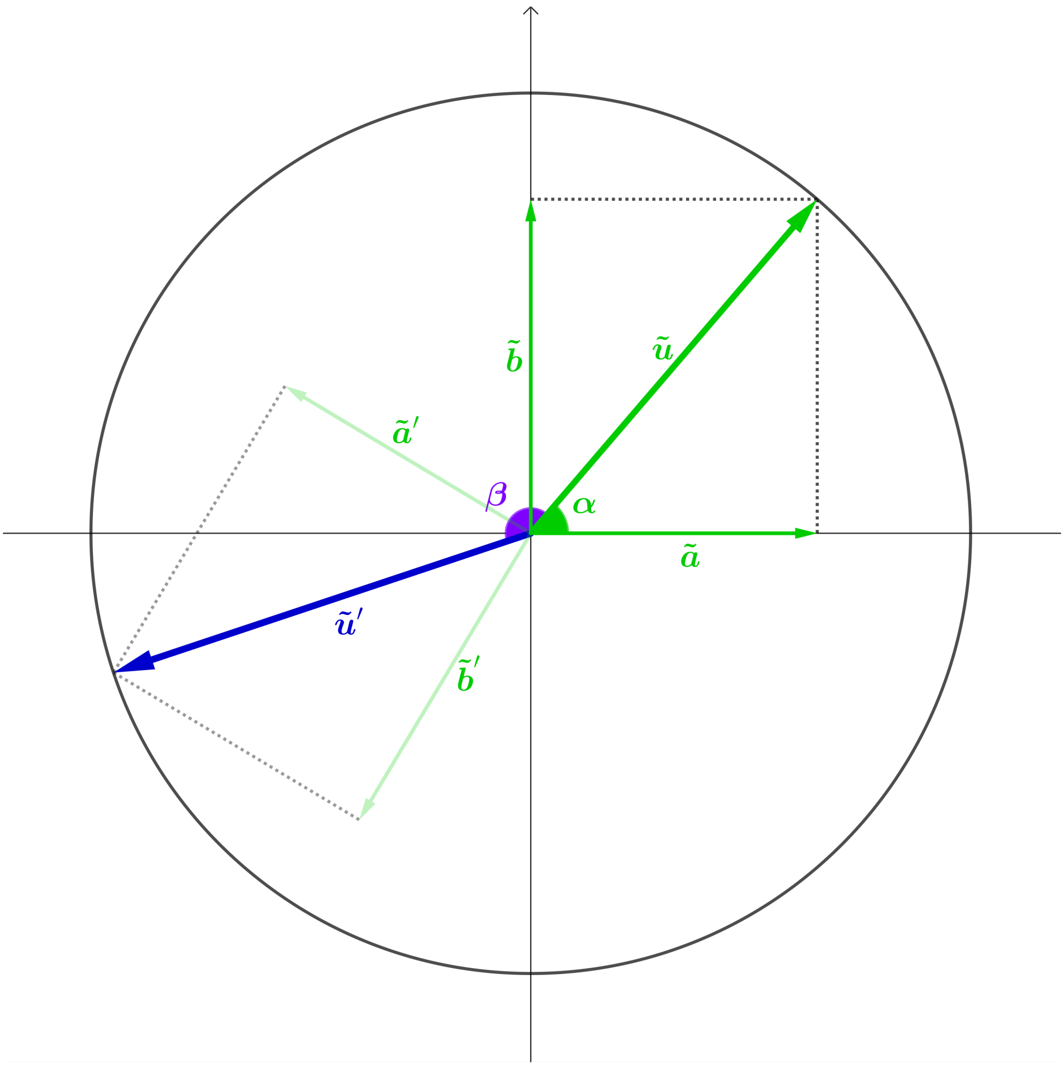
$\tilde{\boldsymbol{a}}'$ has the same direction as the pink rotated vector, which we have already seen is $\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$, and it has the same magnitude as $\tilde{\boldsymbol{a}}$, which is $\cos\alpha$. So

$$|\tilde{\boldsymbol{a}}'| = |\tilde{\boldsymbol{a}}| \Rightarrow \tilde{\boldsymbol{a}}' = \cos\alpha\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$$

and similarly,

$$|\tilde{\boldsymbol{b}}'| = |\tilde{\boldsymbol{b}}| \Rightarrow \tilde{\boldsymbol{b}}' = \sin\alpha\begin{pmatrix}-\sin\beta\\ \cos\beta\end{pmatrix}$$

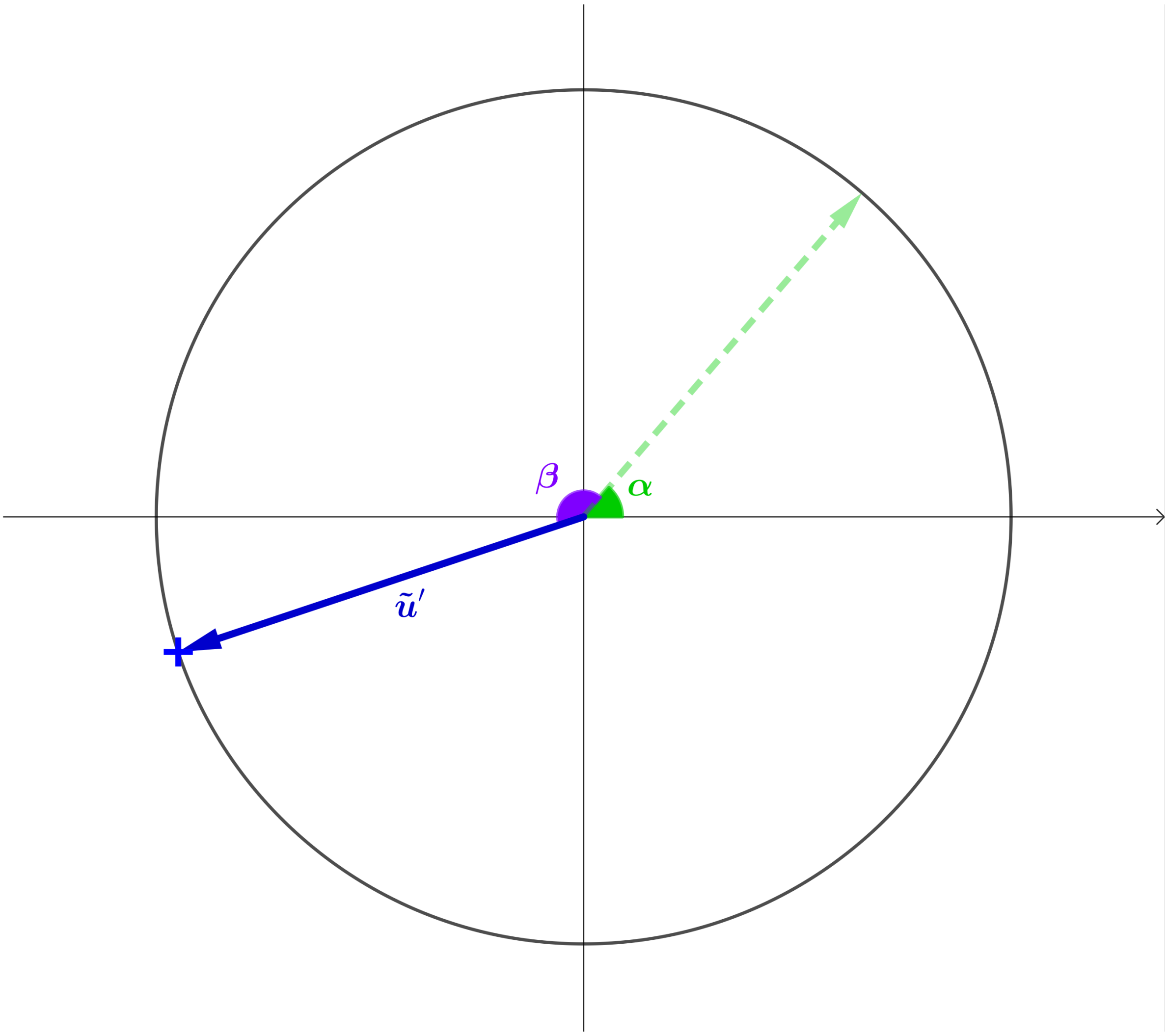
What is $\tilde{\boldsymbol{u}}'$ in terms of $\tilde{\boldsymbol{a}}'$ and $\tilde{\boldsymbol{b}}'$?



Use this to write $\tilde{\boldsymbol{u}}'$ in terms of α and β.

$$\begin{aligned}\tilde{\boldsymbol{u}}' &= \tilde{\boldsymbol{a}}' + \tilde{\boldsymbol{b}}' \\ &= \cos\alpha \begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix} + \sin\alpha \begin{pmatrix}-\sin\beta \\ \cos\beta\end{pmatrix} \\ &= \begin{pmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta\end{pmatrix}\end{aligned}$$

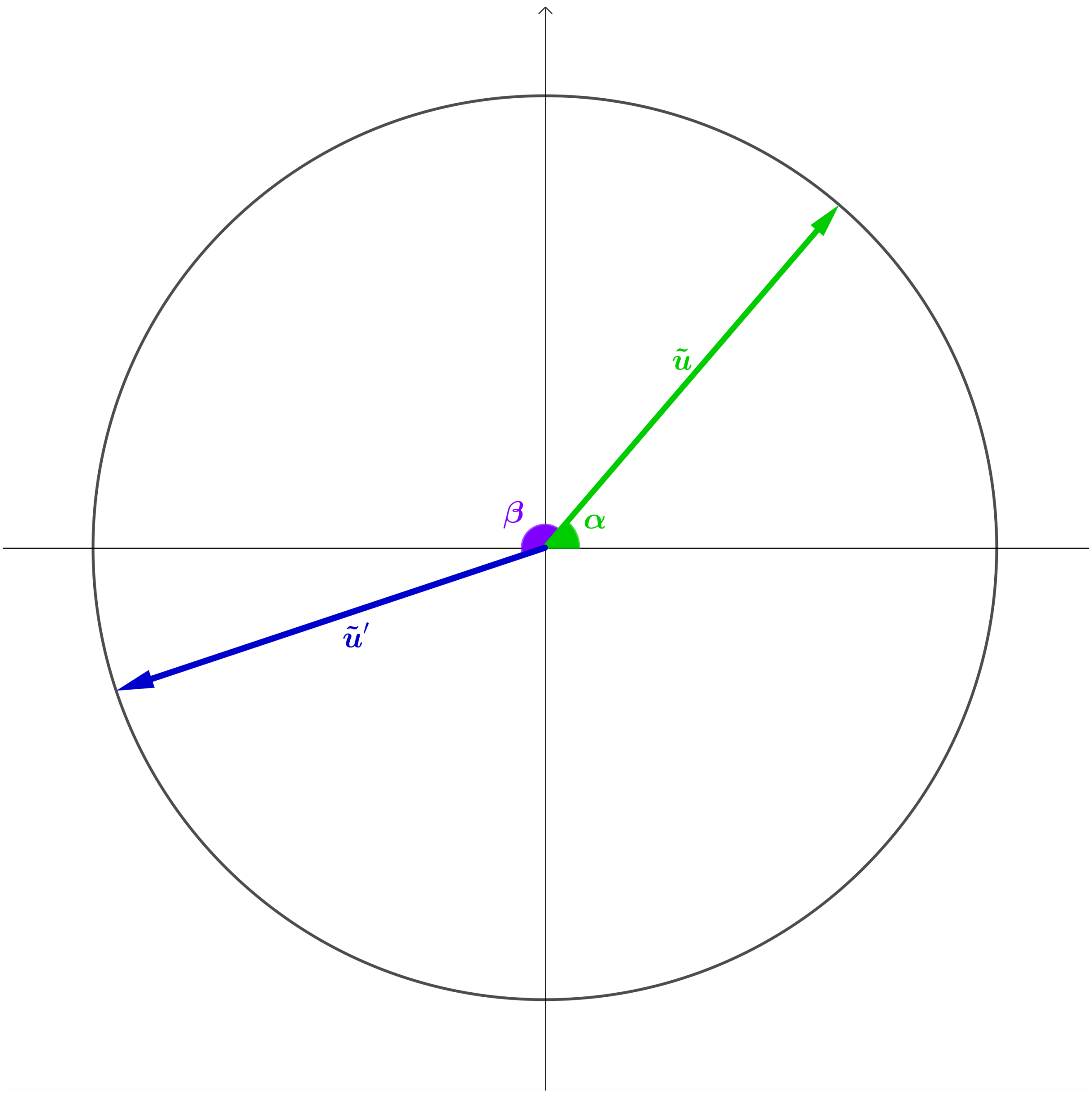
What are the coordinates of the blue cross?



The coordinates of the blue point are $\big(\cos(\alpha+\beta),\ \sin(\alpha+\beta)\big)$. Again, this is just the definition of cos and sine. So:

$$\tilde{\boldsymbol{u}}' = \begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix}$$

Use the two expressions for $\tilde{\boldsymbol{u}}'$ together to find new expressions for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.



$$\tilde{\boldsymbol{u}}' = \begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta \end{pmatrix}$$

which gives the familiar formulas right away.

Use these results to find $\tan(\alpha+\beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

Use these results to find sin, cos, and tan of $\alpha - \beta$.

Remember

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

so

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + -\beta) \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

In the same way,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Use these results to find $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$.

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha + \alpha)\\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha\\ &= 2\sin\alpha\cos\alpha\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos(\alpha + \alpha)\\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\ &= \cos^2\alpha - \sin^2\alpha\end{aligned}$$

$$\begin{aligned}\tan 2\alpha &= \tan(\alpha + \alpha)\\ &= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}\\ &= \frac{2\tan\alpha}{1 - \tan^2\alpha}\end{aligned}$$

Use $\cos^2\alpha + \sin^2\alpha = 1$ to find two different formulas for $\cos 2\alpha$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1 - \cos^2\alpha) \\ &= 2\cos^2\alpha - 1\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= (1 - \sin^2\alpha) - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Find $\sin 75^\circ$, $\cos 75^\circ$, and $\tan 75^\circ$.

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\cos 75^\circ = \cos(45^\circ + 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\tan 75^\circ = \tan(45^\circ + 30^\circ)$$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Find $\sin 15^\circ$, $\cos 15^\circ$, and $\tan 15^\circ$.

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\end{aligned}$$

or

$$\begin{aligned}\sin 15^\circ &= \cos(90^\circ - 15^\circ)\\ &= \cos 75^\circ\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 15^\circ &= \sin(90^\circ - 15^\circ)\\ &= \sin 75^\circ\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\tan 15^\circ &= \frac{1}{\tan(90^\circ - 15^\circ)}\\ &= \frac{1}{\tan 75^\circ}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\end{aligned}$$

Use the formula $\cos 2\theta = 2\cos^2\theta - 1$ to find $\cos 15°$.

$$\cos 30° = 2\cos^2 15° - 1$$

$$\Rightarrow \cos^2 15° = \frac{\cos 30° + 1}{2}$$

$$= \frac{\frac{\sqrt{3}}{2} + 1}{2}$$

$$= \frac{\sqrt{3} + 2}{4}$$

$$\Rightarrow \cos 15° = \frac{\sqrt{\sqrt{3} + 2}}{2}$$

Compare the two expressions you now have for $\cos 15°$.

$$\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2 = \frac{8 + 2\sqrt{12}}{16}$$

$$= \frac{8 + 4\sqrt{3}}{16}$$

$$= \frac{2 + \sqrt{3}}{4}$$

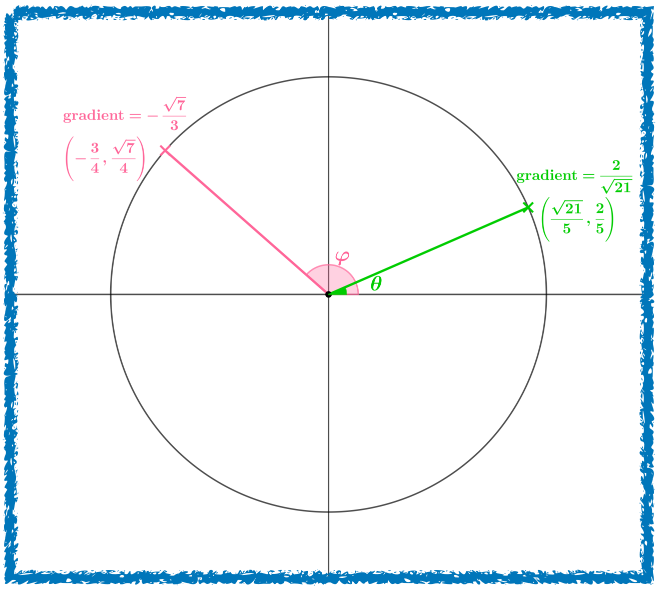
so they are the same.

Find $\displaystyle\int \sin^2 x \, \mathrm{d}x$

$$\begin{aligned}\int \sin^2 x \, \mathrm{d}x &= \int \frac{1 - \cos 2x}{2} \, \mathrm{d}x \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + c\end{aligned}$$

If $\sin\theta = \dfrac{2}{5}$ (θ is acute) and $\cos\varphi = -\dfrac{3}{4}$ (φ is obtuse)

find, **without using your calculator**:



If $\sin\theta = \dfrac{2}{5}$ (θ is acute) and $\cos\varphi = -\dfrac{3}{4}$ (φ is obtuse)

find, **without using your calculator**:

$\cos\theta$

$\cos^2\theta = 1 - \sin^2\theta$ and

$\cos\theta > 0 \Rightarrow \cos\theta = \dfrac{\sqrt{21}}{5}$

$\tan 2\theta$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{4\sqrt{21}}{17}$$

$\tan\theta$

$$\tan\theta = \text{gradient} = \frac{2}{\sqrt{21}}$$

$\cos\dfrac{\theta}{2}$

$$\cos\frac{\theta}{2} > 0$$

$$\Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{5+\sqrt{21}}{10}}$$

$\cos 2\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \frac{21}{25} - \frac{4}{25} = \frac{17}{25}$$

$\sin\dfrac{\theta}{2}$

$$\sin\frac{\theta}{2} > 0$$

$$\Rightarrow \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{5-\sqrt{21}}{10}}$$

$\sin 2\theta$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{2}{5}\frac{\sqrt{21}}{5} = \frac{4\sqrt{21}}{25}$$

$\tan\varphi$

$$\tan\varphi = \text{gradient} = -\frac{\sqrt{7}}{3}$$

$\tan\frac{\theta}{2}$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \sqrt{\frac{5-\sqrt{21}}{5+\sqrt{21}}}$$

$\cos 2\varphi$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$
$$= \frac{9}{16} - \frac{7}{16} = \frac{1}{8}$$

$\sin\varphi$

$\sin^2\varphi = 1 - \cos^2\varphi$ and

$$\sin\varphi > 0 \Rightarrow \sin\varphi = \frac{\sqrt{7}}{4}$$

$\sin 2\varphi$

$$\sin 2\varphi = 2\sin\varphi\cos\varphi$$
$$= 2 \times \frac{\sqrt{7}}{4}\frac{-3}{4} = -\frac{3\sqrt{7}}{8}$$

$\tan 2\varphi$

$\tan\dfrac{\varphi}{2}$

$$\tan 2\varphi = \frac{\sin 2\varphi}{\cos 2\varphi} = -3\sqrt{7}$$

$$\tan\frac{\varphi}{2} = \frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} = \sqrt{7}$$

$\cos\dfrac{\varphi}{2}$

$$\cos\frac{\varphi}{2} > 0 \Rightarrow \cos\frac{\varphi}{2} = \sqrt{\frac{1+\cos\varphi}{2}} = \frac{\sqrt{2}}{4}$$

$\sin\dfrac{\varphi}{2}$

$$\sin\frac{\varphi}{2} > 0 \Rightarrow \sin\frac{\varphi}{2} = \sqrt{\frac{1-\cos\varphi}{2}} = \frac{\sqrt{14}}{4}$$

$$\cos(\theta + \varphi)$$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

$$= -\frac{\sqrt{21}}{5}\frac{3}{4} - \frac{2}{5}\frac{\sqrt{7}}{4}$$

$$= -\frac{\sqrt{7}\left(3\sqrt{3}+2\right)}{20}$$

$$\cos(\theta - \varphi)$$

$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$

$$= -\frac{\sqrt{21}}{5}\frac{3}{4} + \frac{2}{5}\frac{\sqrt{7}}{4}$$

$$= \frac{\sqrt{7}\left(2-3\sqrt{3}\right)}{20}$$

$$\sin(\theta + \varphi)$$

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$

$$= -\frac{2}{5}\frac{3}{4} + \frac{\sqrt{21}}{5}\frac{\sqrt{7}}{4}$$

$$= \frac{7\sqrt{3}-6}{20}$$

$$\sin(\theta - \varphi)$$

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$$

$$= -\frac{2}{5}\frac{3}{4} - \frac{\sqrt{21}}{5}\frac{\sqrt{7}}{4}$$

$$= -\frac{7\sqrt{3}+6}{20}$$

$$\tan(\theta + \varphi)$$

$$\tan(\theta + \varphi) = \frac{\sin(\theta + \varphi)}{\cos(\theta + \varphi)}$$

$$= \frac{6-7\sqrt{3}}{\sqrt{7}\left(3\sqrt{3}+2\right)}$$

$$\tan(\theta - \varphi)$$

$$\tan(\theta - \varphi) = \frac{\sin(\theta - \varphi)}{\cos(\theta - \varphi)}$$

$$= \frac{6+7\sqrt{3}}{\sqrt{7}\left(3\sqrt{3}-2\right)}$$

In the last two, you could rationalise the denominator, but I don't think you gain much, if anything.