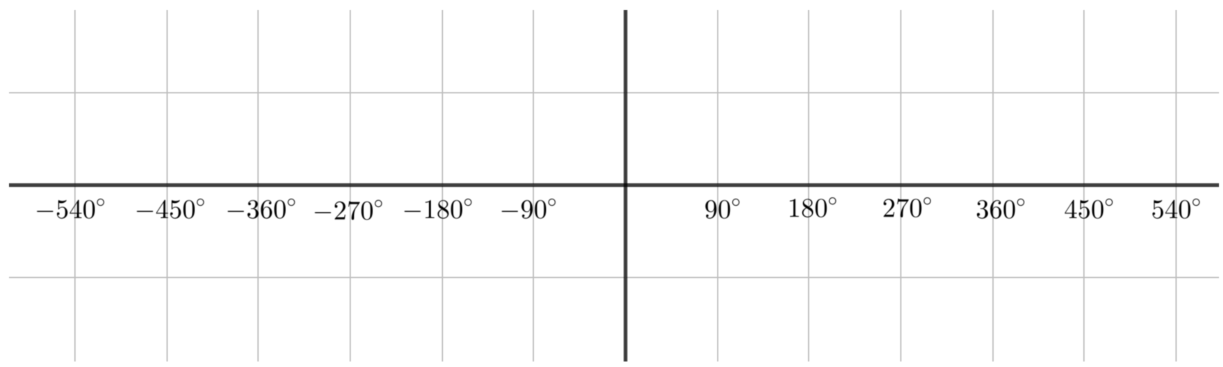
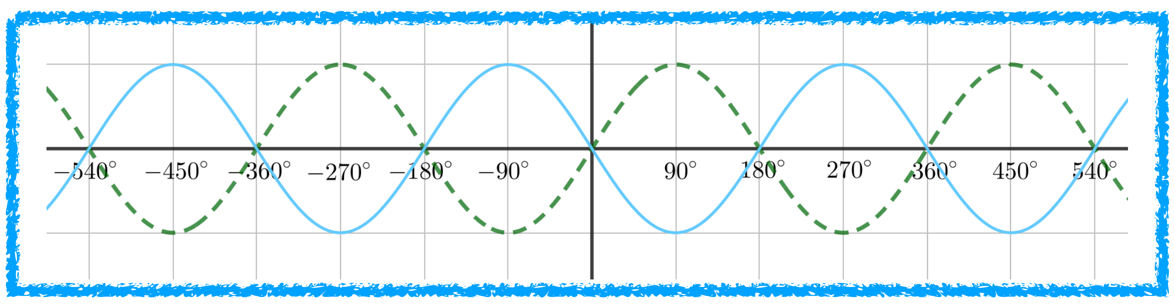
Relationships between circular functions

Draw the following graphs:

$y = -\sin x$

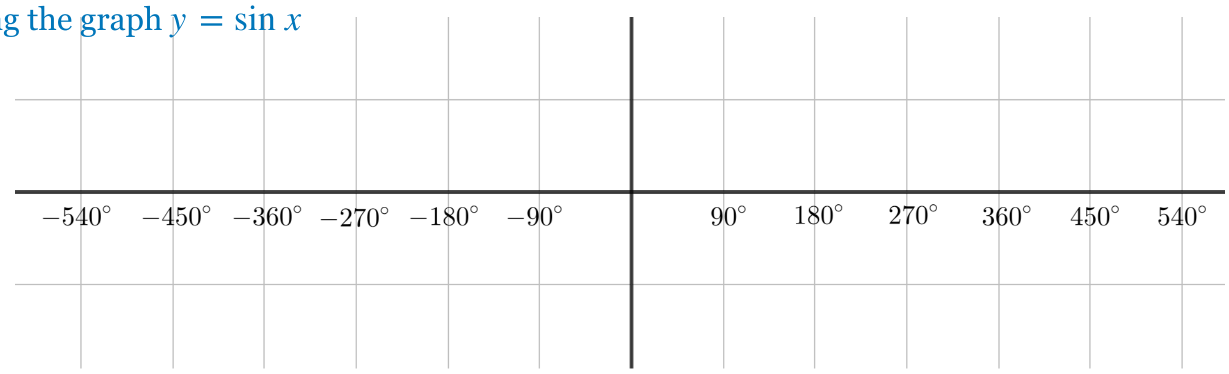




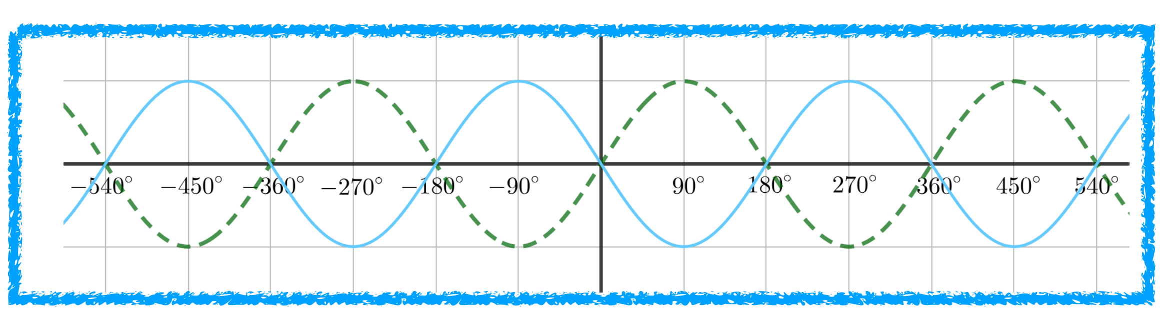
Reflect sin graph in the x axis.

It's easiest if they start by drawing the graph $y = \sin x$

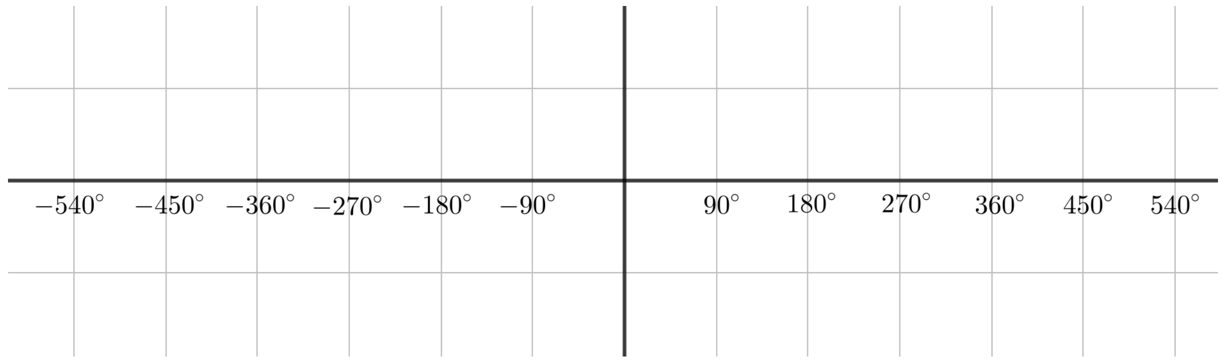
$y = \sin(-x)$

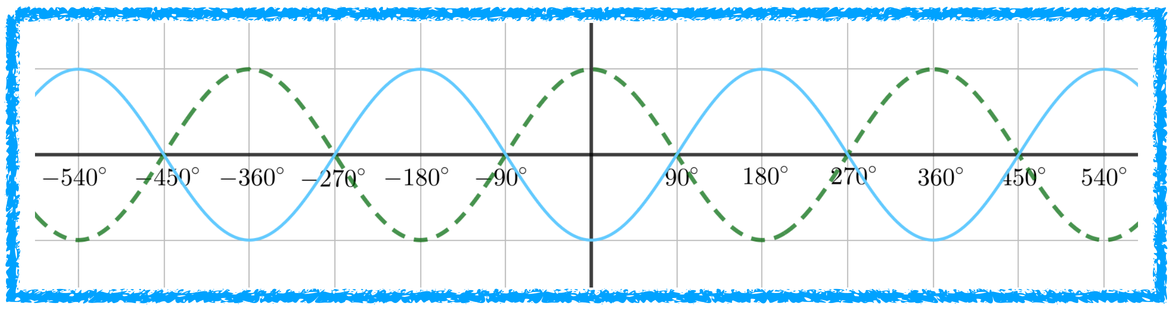


Reflect sin graph in the y axis.



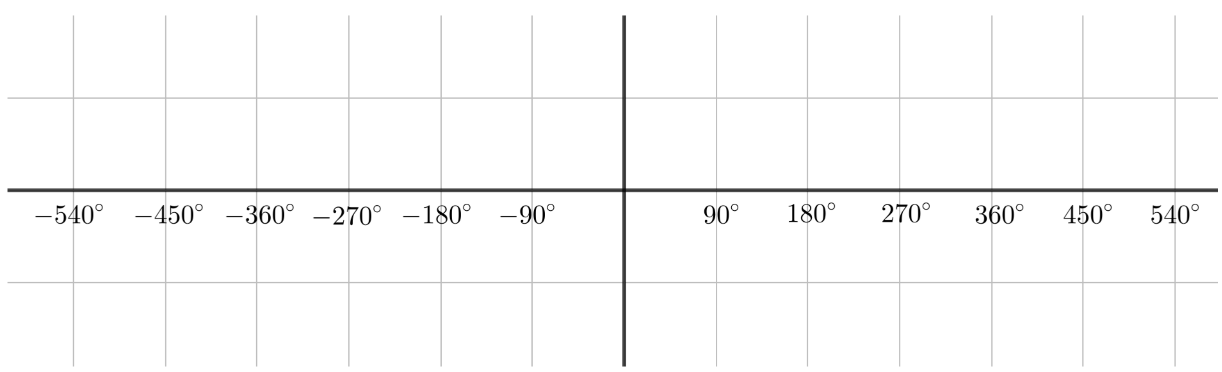
$y = -\cos x$

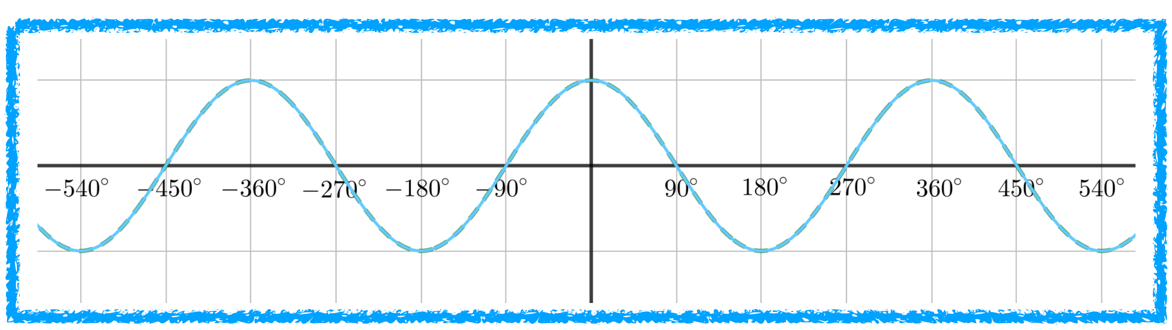




Reflect cos graph in the x axis.

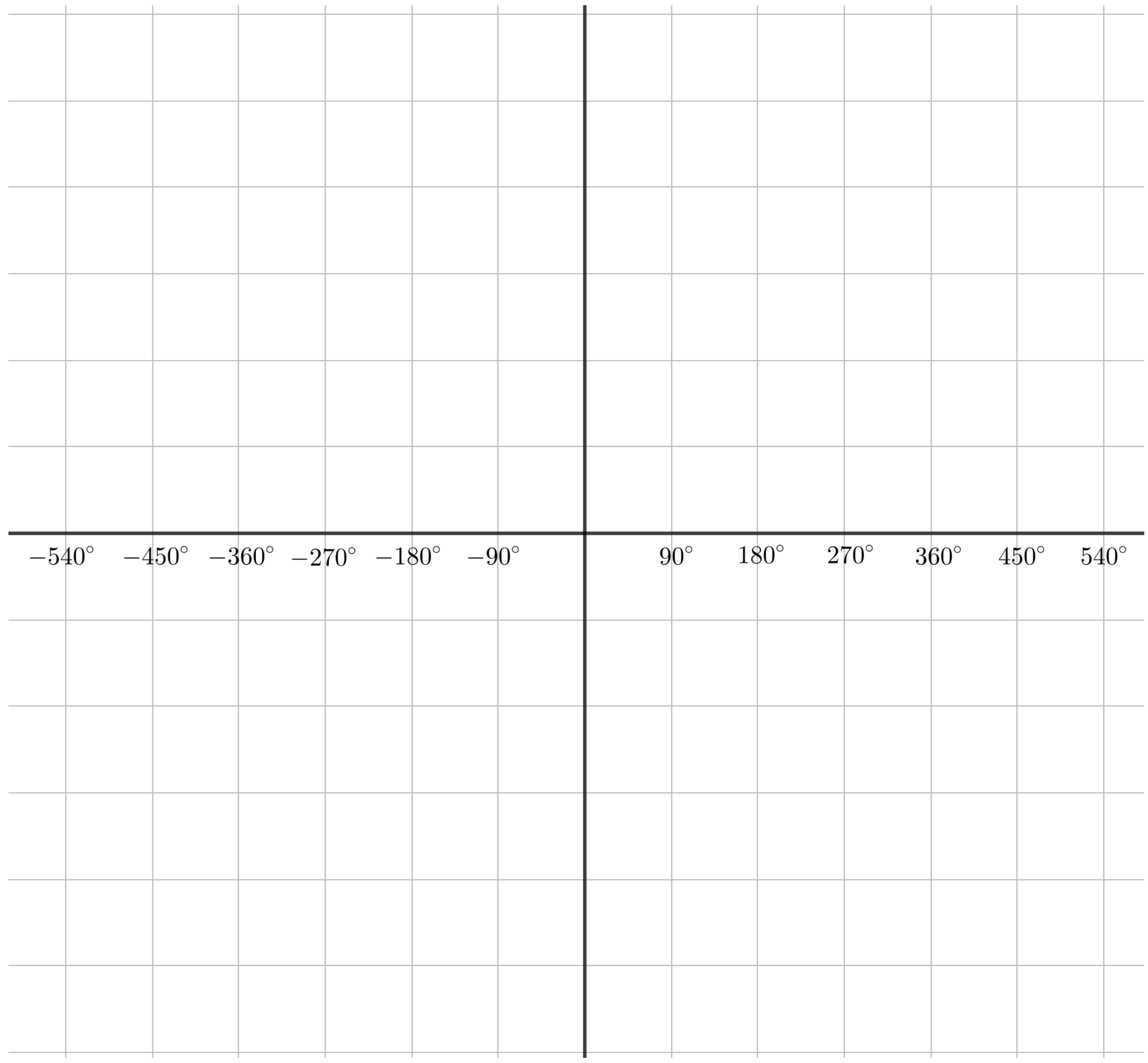
$y = \cos(-x)$





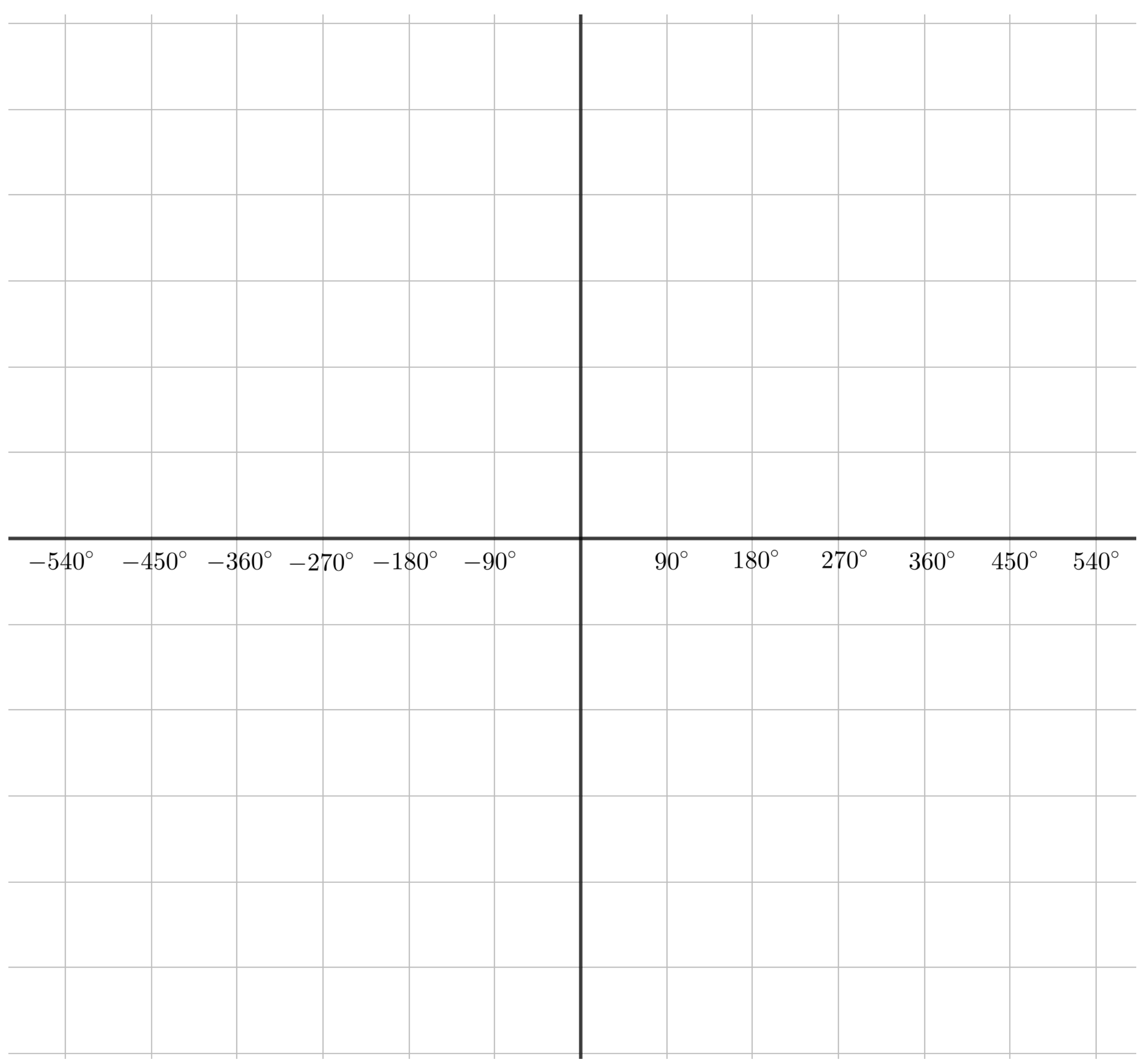
Reflect cos graph in the y axis.

$y = -\tan x$

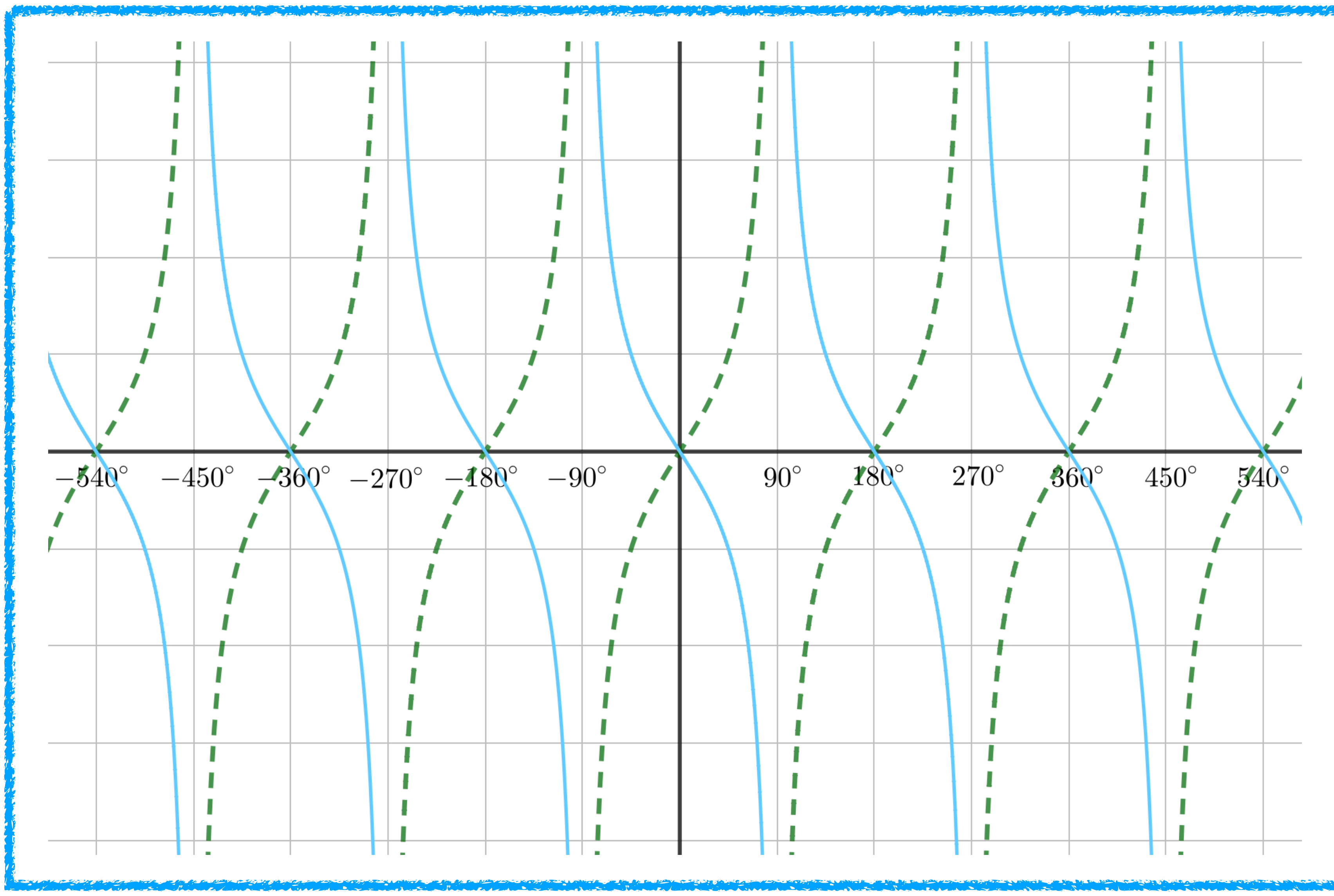


−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°

$y = \tan(-x)$



−540°
−450°
−360°
−270°
−180°
−90°
90°
180°
270°
360°
450°
540°



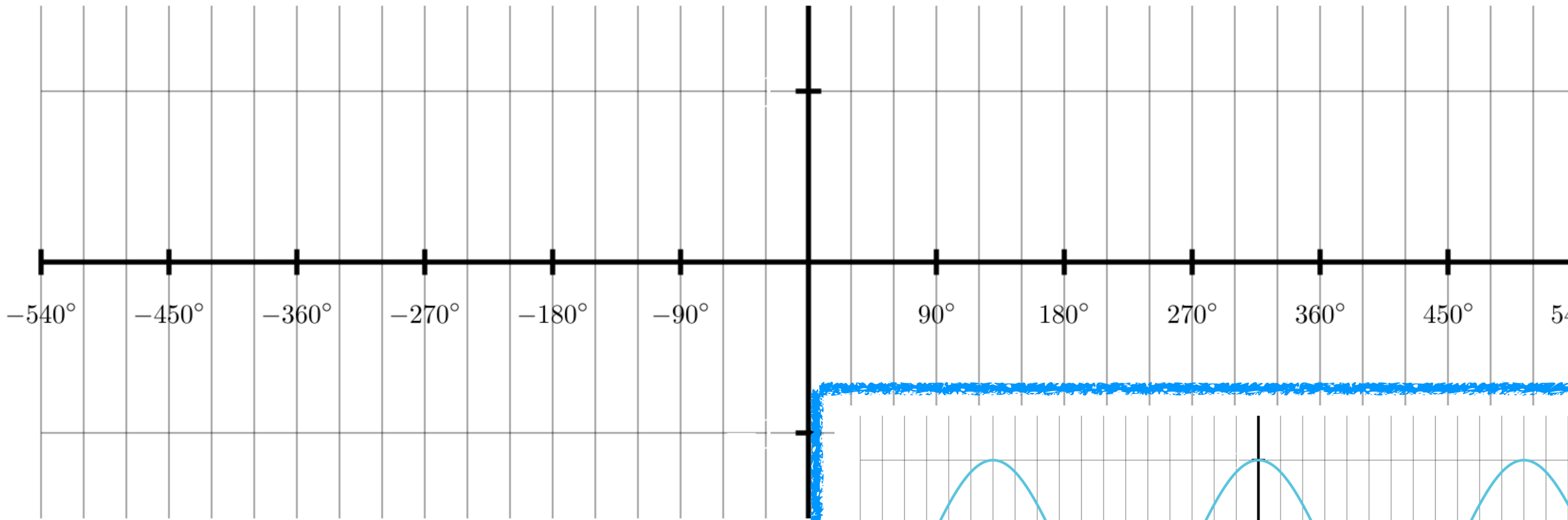
Reflecting the tan graph in the x axis and reflecting the tan graph in the y axis result in the same graph.

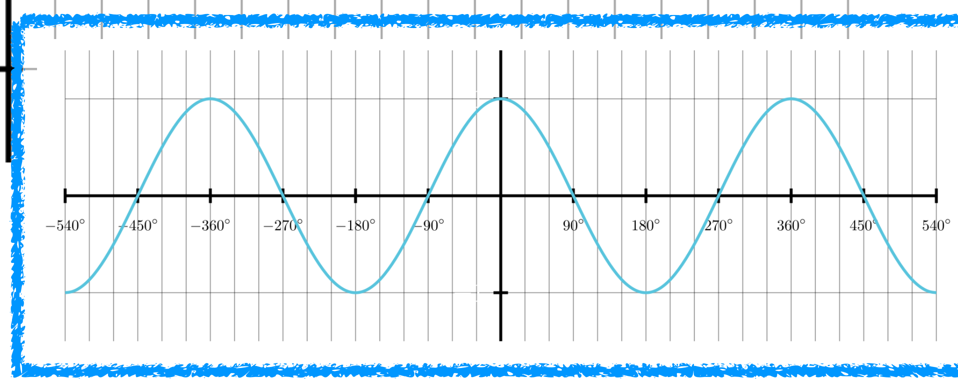
The identities

$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$

are fundamental bites of subject knowledge that your students should have at their fingertips. This last sequence of graphs reinforces these facts, and it's worth discussing them again and how they relate to the unit circle.

$y = \sin(90^\circ - x)$



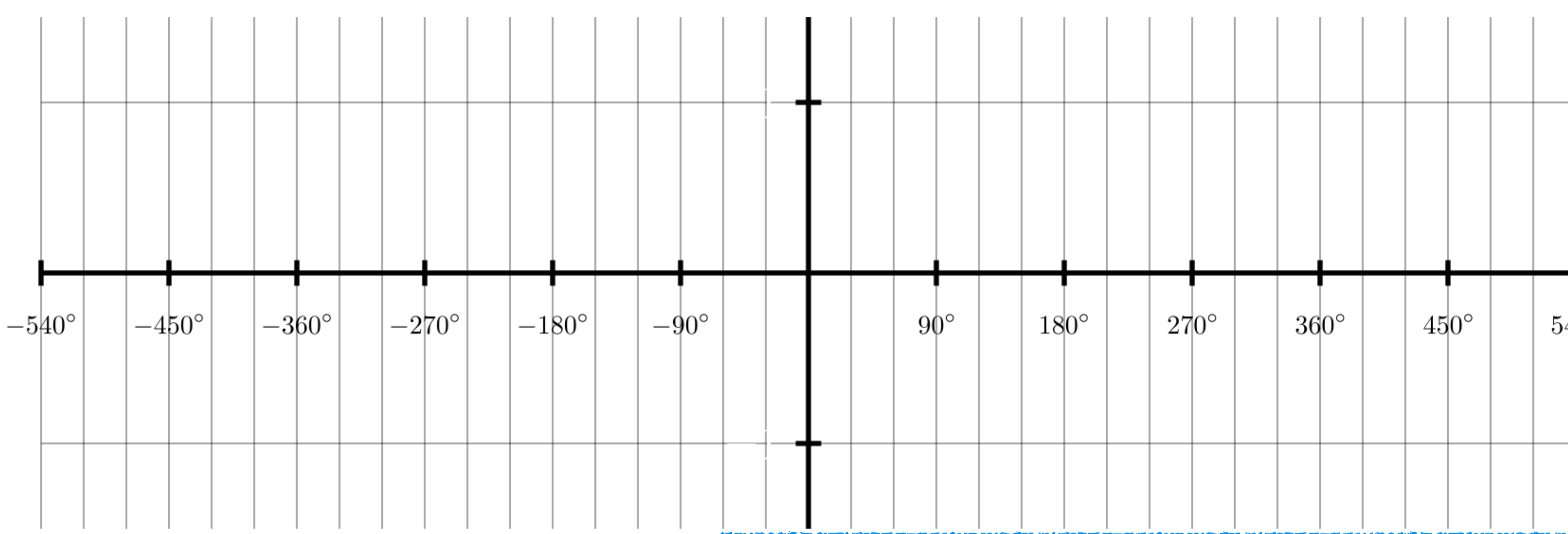


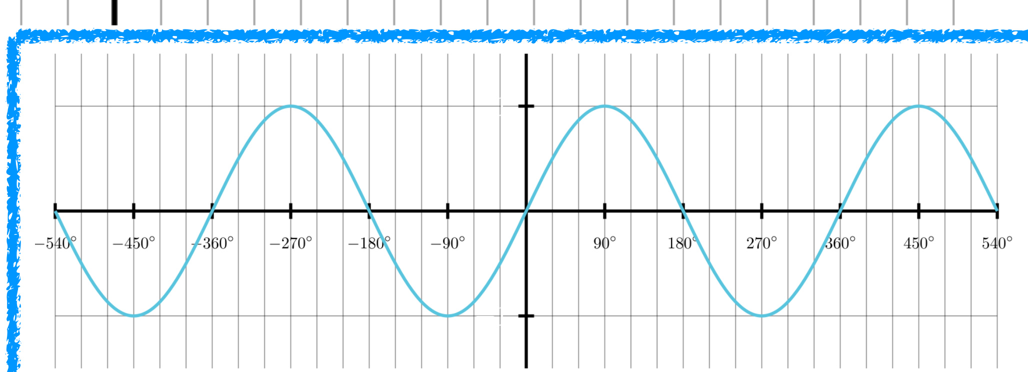
The easiest way to see this is to put some values into the function such as $0, 90, -90\ldots$

You could, in theory, use compound angle formulas, but that is not really in the spirit of this worksheet.

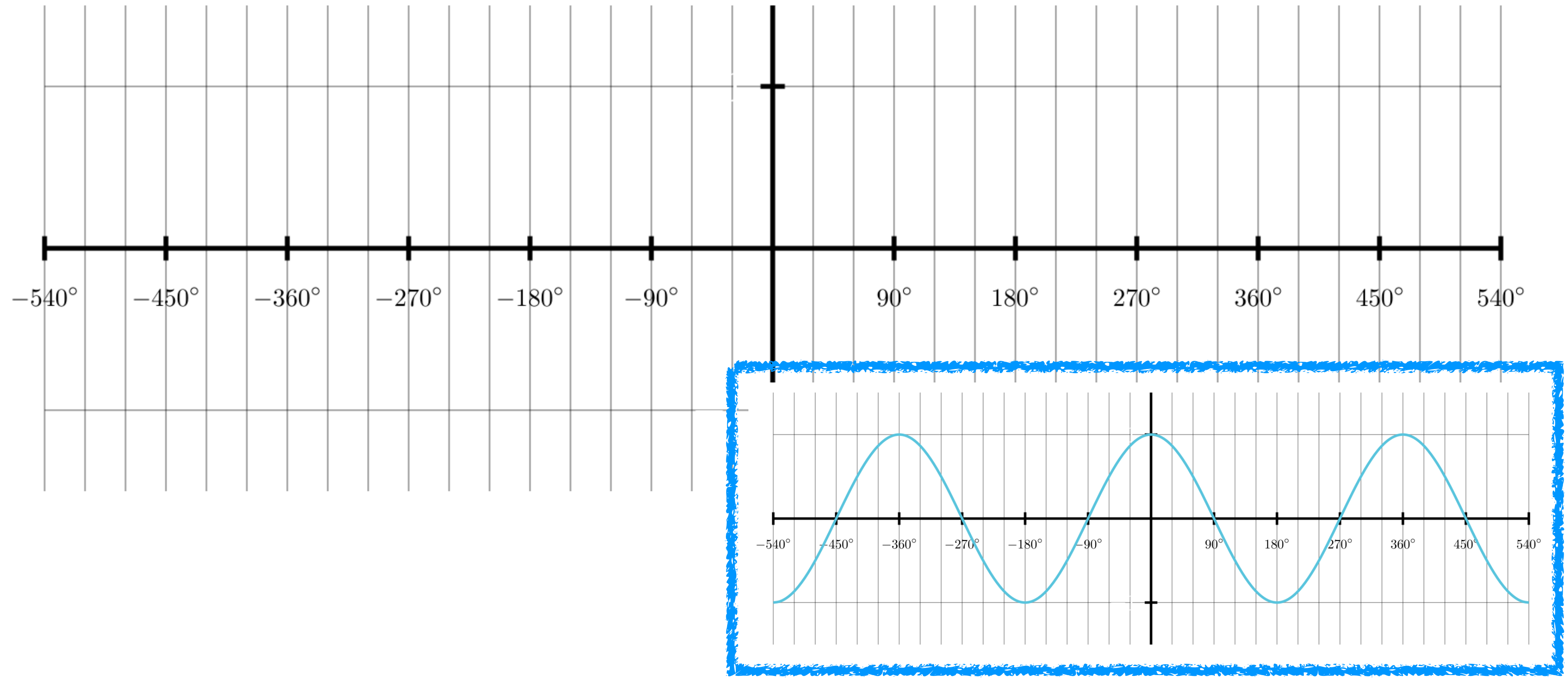
You could, in theory, use transformations, but this example is a bit tricky and would take you off course.

$y = \cos(90^\circ - x)$





$y = \sin(90^\circ + x)$

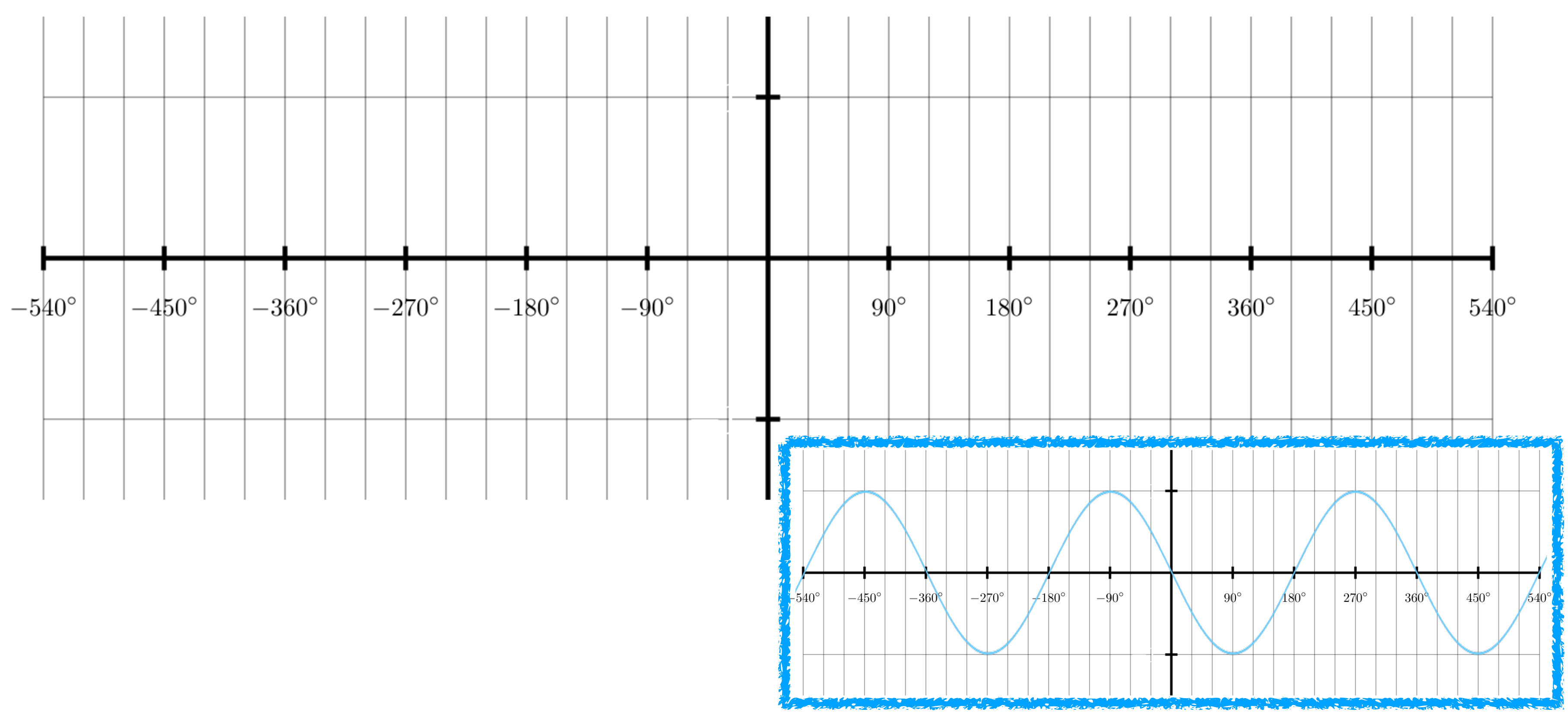


The easiest way to see this is to put some values into the function such as 0, 90, $-90\ldots$

You could, in theory, use compound angle formulas, but that is not really in the spirit of this worksheet.

You could, in theory, use transformations, but this example is a bit tricky and would take you off course.

$y = \cos(90^\circ + x)$



Use these graphs to find some relationships between sin and cos.

$\sin(90° - x) = \sin(90° + x) = \cos x$ $\cos(90° - x) = \cos(x - 90°) = \sin x$ $\cos(90° + x) = \sin(x - 90°) = -\sin x$

$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$

Show that

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

whenever θ is not a multiple of 90°.

If you were to ask your students how to prove this identity before you have discussed it, you might see something like this:

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

which is true, so the original identity is true.

We all know what they mean, and in fact, if only they had written

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\iff \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\iff \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

$$\iff \sin^2\theta + \cos^2\theta = 1$$

then the argument would have been logically sound.

This is a moment, however, when our responsibilities as trainers of successful takers of exams kicks in. An argument set out like this is logically as clear as clear can be:

$$\begin{aligned}\tan\theta + \frac{1}{\tan\theta} &= \frac{\sin\theta}{\cos\theta} + 1 \div \frac{\sin\theta}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \\ &= \frac{1}{\cos\theta\sin\theta}\end{aligned}$$

No examiner could possibly deny a candidate their marks for this. The key difference here is that the left-hand side of the identity appears on the left of the top line, from then on there are only expressions on the right-hand side of the equals signs, and the last of these expressions is the right-hand side of the required identity.

I don't particularly like having to set out solutions like this, but it's not really about what I like!

Very often, almost always in fact, questions on identities avoid references to the exceptions where one or other side of the identity is not well defined.

Here, whenever θ is a multiple of 90°, neither $\tan\theta$ nor $\dfrac{1}{\cos\theta\sin\theta}$ is defined since $\cos\theta = 0$.

I wouldn't go so far as to ask my students to justify the fact that these are exceptions in their solutions, but I would discuss them in class.

Show that

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$$

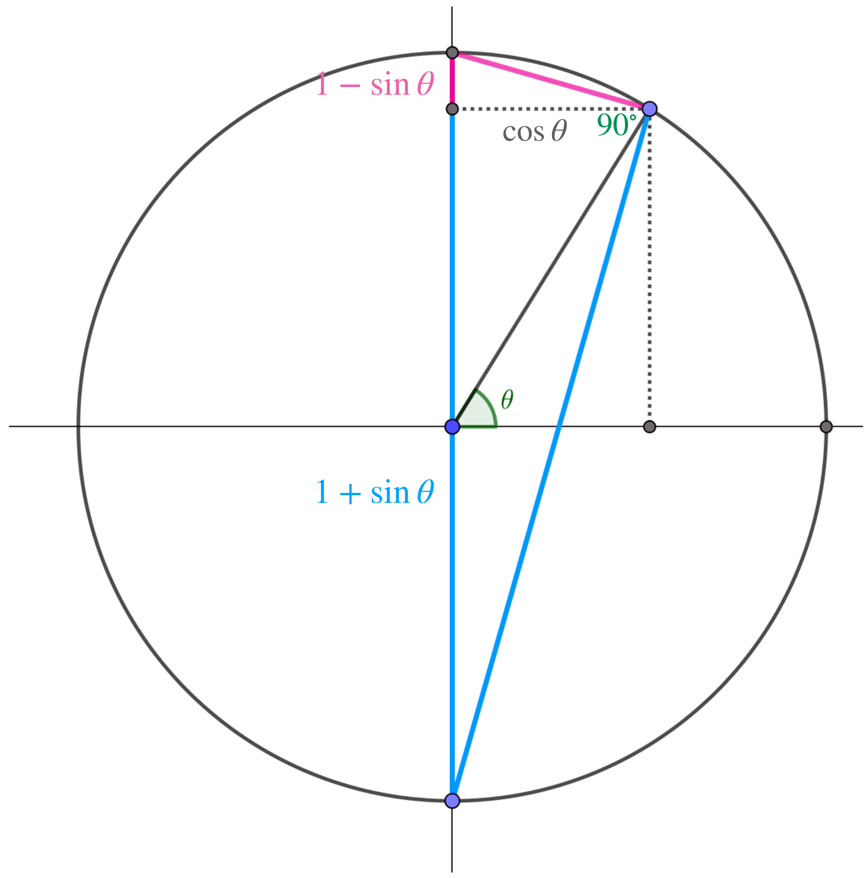
whenever θ is not a multiple of 90°.

There are many routes to proving an identity like this. Here are two possibilities.

$$\begin{aligned}
\frac{\cos\theta}{1+\sin\theta} &= \frac{\cos\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta} \\
&= \frac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \\
&= \frac{\cos\theta(1-\sin\theta)}{(1-\sin^{\theta})} \\
&= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} \\
&= \frac{1-\sin\theta}{\cos\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{\cos\theta}{1+\sin\theta} &= \frac{\cos\theta}{1+\sin\theta} \times \frac{\cos\theta}{\cos\theta} \\
&= \frac{\cos^2\theta}{(1+\sin\theta)\cos\theta} \\
&= \frac{1-\sin^2\theta}{(1+\sin\theta)\cos\theta} \\
&= \frac{(1+\sin\theta)(1-\sin\theta)}{(1+\sin\theta)\cos\theta} \\
&= \frac{1-\sin\theta}{\cos\theta}
\end{aligned}$$

Here's a lovely geometrical proof, at least for angles between 0 and 90°.



Angle in a semicircle is a right angle, so the pink and blue triangles are similar. Hence:

$$\frac{1 - \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 + \sin\theta}$$

This is pretty easily adapted for obtuse angles, and slightly less easily adapted for reflex.

Show that

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} \equiv \frac{1 + \sin\theta}{\cos\theta}$$

Here's a hard example that shows up the limitations of the obsessive style of setting out of proofs. Before attempting this question, though, your students will need quite a bit of practice with simpler examples! They may never want to think about something as tricky as this, but some will, so here it is, just in case.

What does that $\equiv$ sign mean?

We can read it as "is equivalent to", meaning "is equal to for every value of θ.

In fact, it is not quite true, because when the bottom of either fraction is 0, the fraction is not defined.

If $\theta = 90^\circ + n \times 360^\circ$, then both fractions have denominator 0.

If $\theta = -90^\circ + n \times 360^\circ$, then the denominator of the right-hand side is 0.

θ is an odd multiple of 90° covers both of these, and when this is true, the "identity" is not true.

Show that

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} \equiv \frac{1 + \sin\theta}{\cos\theta}$$

Make life a bit easier by writing $s = \sin\theta \quad c = \cos\theta$.

Now we need to show that

$$\frac{s - c + 1}{s + c - 1} = \frac{1 + s}{c}$$

The easiest way, I reckon, is to say

$$\begin{aligned}
\frac{s - c + 1}{s + c - 1} \times \frac{c}{1 + s} &= \frac{sc - c^2 + c}{s + c - 1 + s^2 + sc - s} \\
&= \frac{sc - 1 + s^2 + c}{c - 1 + sc + s^2} \\
&= 1
\end{aligned}$$

which seems perfectly adequate to me, because $x \times \dfrac{1}{y} = 1 \Rightarrow x = y$.

Alternatively:

$$\begin{aligned}
\frac{1 + s - c}{c} &= \frac{1 + s}{c} - 1 \\
&= \frac{c}{1 - s} - 1 \\
&= \frac{c - 1 + s}{1 - s} \\
\Rightarrow \frac{1 + s - c}{c - 1 + s} &= \frac{c}{1 - s} = \frac{1 + s}{c}
\end{aligned}$$

or

$$\begin{aligned}
\frac{1 + s - c}{1 + s} &= 1 - \frac{c}{1 + s} \\
&= 1 - \frac{1 - s}{c} \\
&= \frac{c - 1 + s}{c} \\
\Rightarrow \frac{1 + s - c}{c - 1 + s} &= \frac{1 + s}{c}
\end{aligned}$$

I could turn this proof into something that the obsessive examiner would approve of, but I wouldn't bother. No one will object to this proof outside the context of a school examination.

By the way, whether you use the equivalence sign $\equiv$ is really a matter of personal style. It distinguishes between an equation to solve ($=$) and an identity to prove ($\equiv$), but I tend to rely on the context to make the difference and stick to $=$, otherwise I am certain to be inconsistent sooner or later.