radical maths

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Circular functions 2

Solving circular functions equations

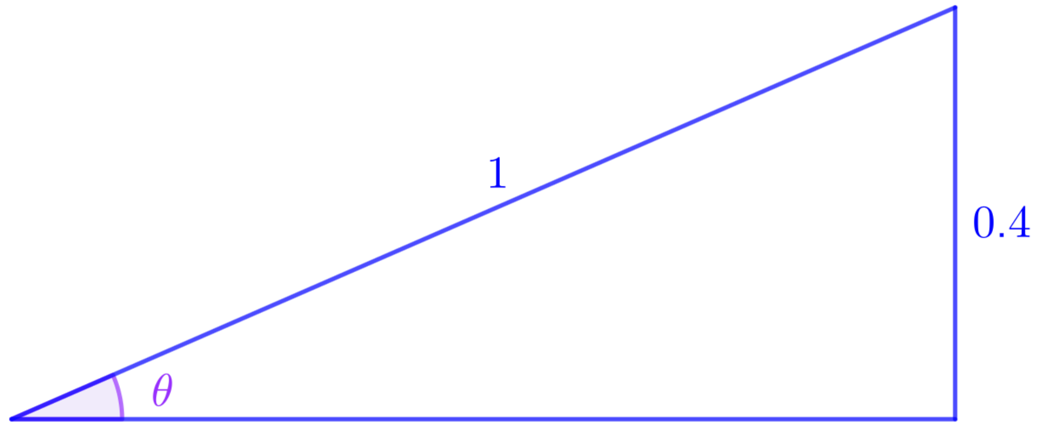
teacher version

Circular functions

Defining the circular functions	sin, cos, tan and the unit circle
Solving circular function equations	**like $\sin\theta = 0.4$**
Graphing the circular functions	graphs $y = \cos x$ and the like
Relationships between circular functions	$\sin(90^\circ - x) = \cos x$ and the like
More circular functions	$\sec x = \dfrac{1}{\cos x}$ and so on
Circular functions of sums	formulas like $\sin(A + B) = \sin A \cos B + \cos A \sin B$
Transforming and adding circular functions	$\sin x + \cos x = \sqrt{2}\sin\left(x + 45^\circ\right)$ and so on
Differentiating circular functions	radians, and tangents to graphs
Integrating circular functions	areas
Inverses of circular functions	$\arcsin x,\ \cos^{-1} x,\ \cot^{-1} x$ and the like, including graphs, differentials, integrals, and integration by substitution

Solving equations with circular functions

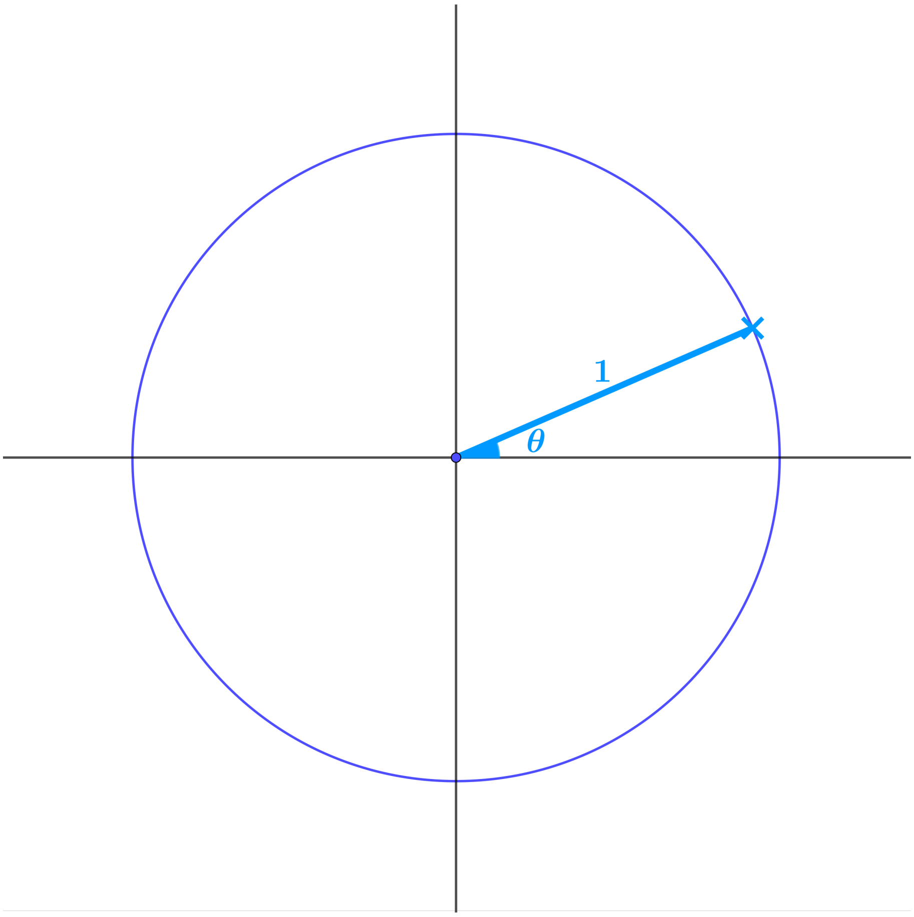
Use your calculator to find the angle θ.



There are an infinite number of solutions to an equation like $\sin\theta = 0.4$, and this is only the smallest positive solution. It's worth spending the time on this sheet, as this is the next step in the process of divorcing the circular functions from triangles. Trigonometry means "measurement of three-sided shapes", but sin, cos, tan and so on are really functions rather than ratios of sides of "trigons", and their power is far greater than that needed for the measurement of three-sided shapes.

Finding other solutions is a question of symmetry, and you can either use this unit circle to do your visualisation or you can use graphs. I am a great believer in the advantages of the unit circle over graphs, and that is the approach I take here.

If the y coordinate of the blue point is 0.4, find the angle θ.

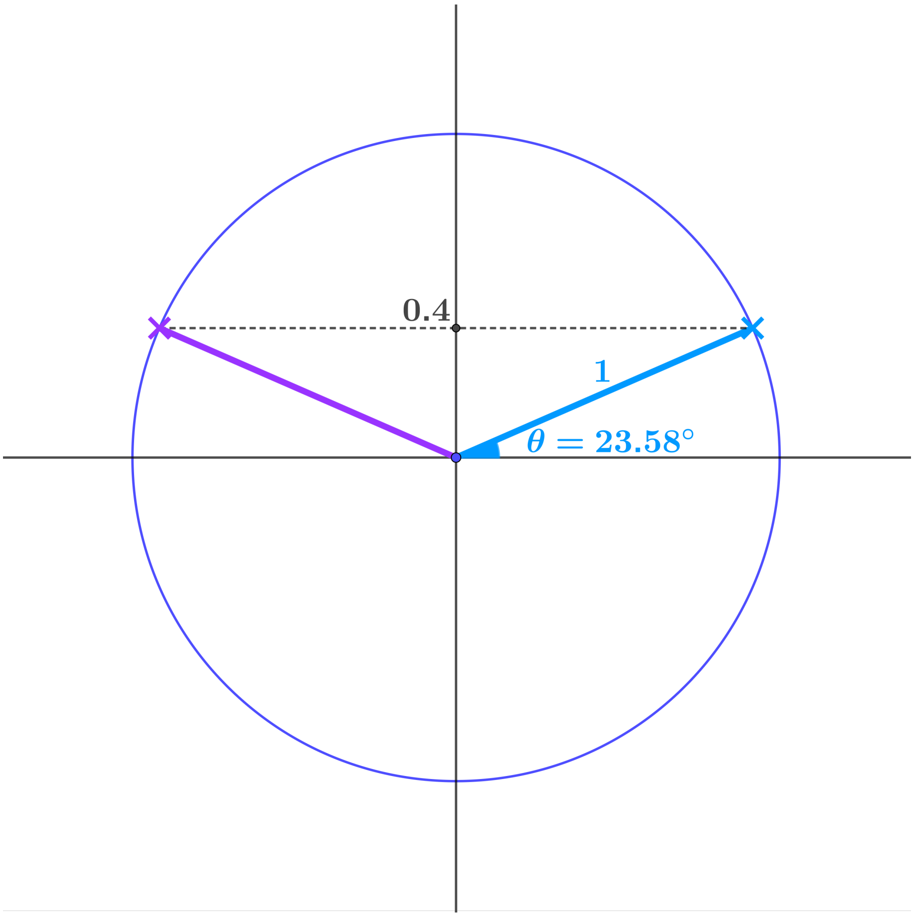


What other point on the circle has the same y coordinate as the blue point?

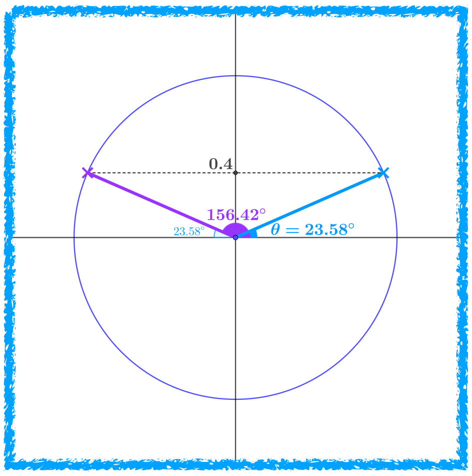
If you have always taught your students to use graphs to find solutions to equations like this, I strongly recommend that you give this way a go. It has many advantages: for example, the unit circle is easy to sketch when solving a problem, and easy to visualise without sketching. It's easy to remember that the coordinates of a point are $(\cos\theta, \sin\theta)$ and that the gradient of the radius is $\tan\theta$. On the other hand, for graphs, you have to remember quite a bit of detail about three different graphs, and then it's much harder to use symmetry to read off the solutions.

You might find it takes some getting used to, but I don't think you will ever look back!

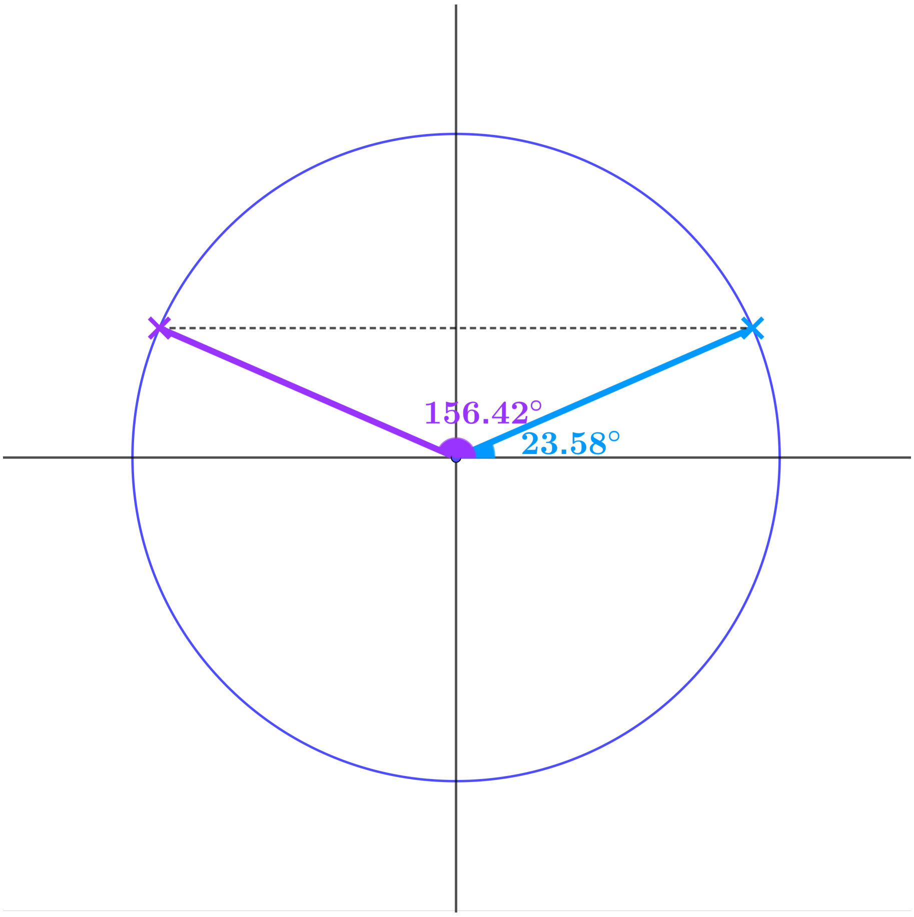
What other positive angle between 0° and 360° is a solution of
$\sin\theta = 0.4$?



Symmetry is the key (as it is with graphs), so
$180 - 23.58 = 156.42°$ is another solution of $\sin\theta = 0.4$

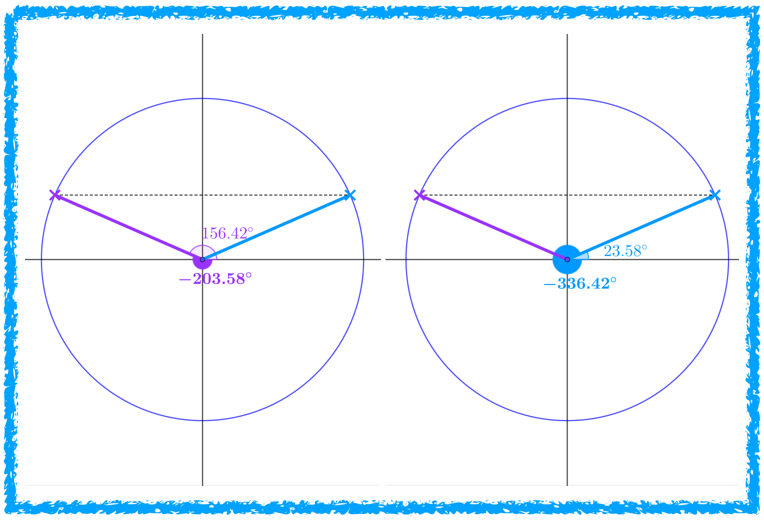


What negative angles between $-360°$ and $0°$ are solutions of $\sin\theta = 0.4$?

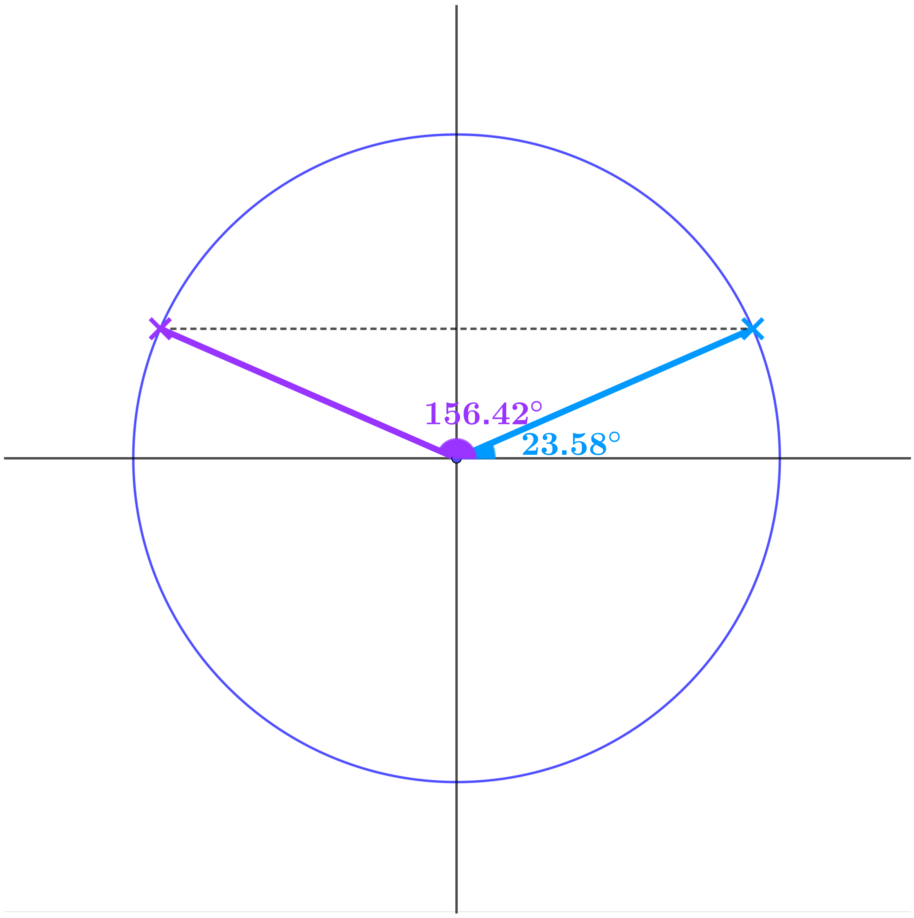


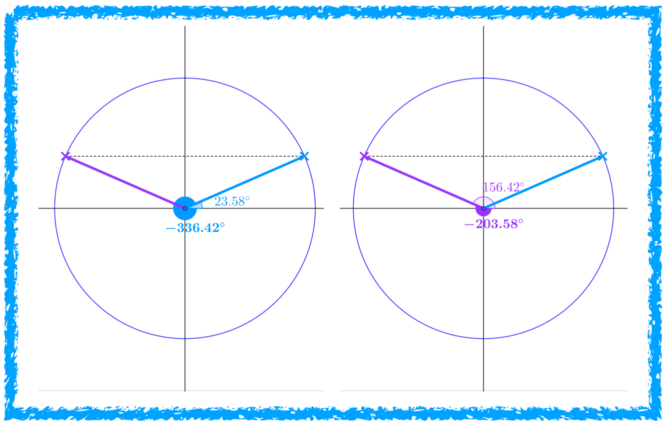
To find the size of the angles, subtract the positive angles from 360°.

But they are measured clockwise from the positive x axis, so they are negative.



Solve the equation $\sin\theta = 0.4$





$\theta = 23.58°, 156.42°, 383.58°, 516.42° \ \ldots$

$\quad -203.58°, \ -336.42°, \ -563.58°, \ -696.42° \ \ldots$

Keep adding or subtracting 360° as often as you like.

If α is any solution of the equation $\sin\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

$180 - \alpha$

$\alpha + 360$

$\alpha - 360$

The point of asking this question is to get to some easy rules for solving sin, cos, and tan equations that don't even need the unit circle, let alone graphs.

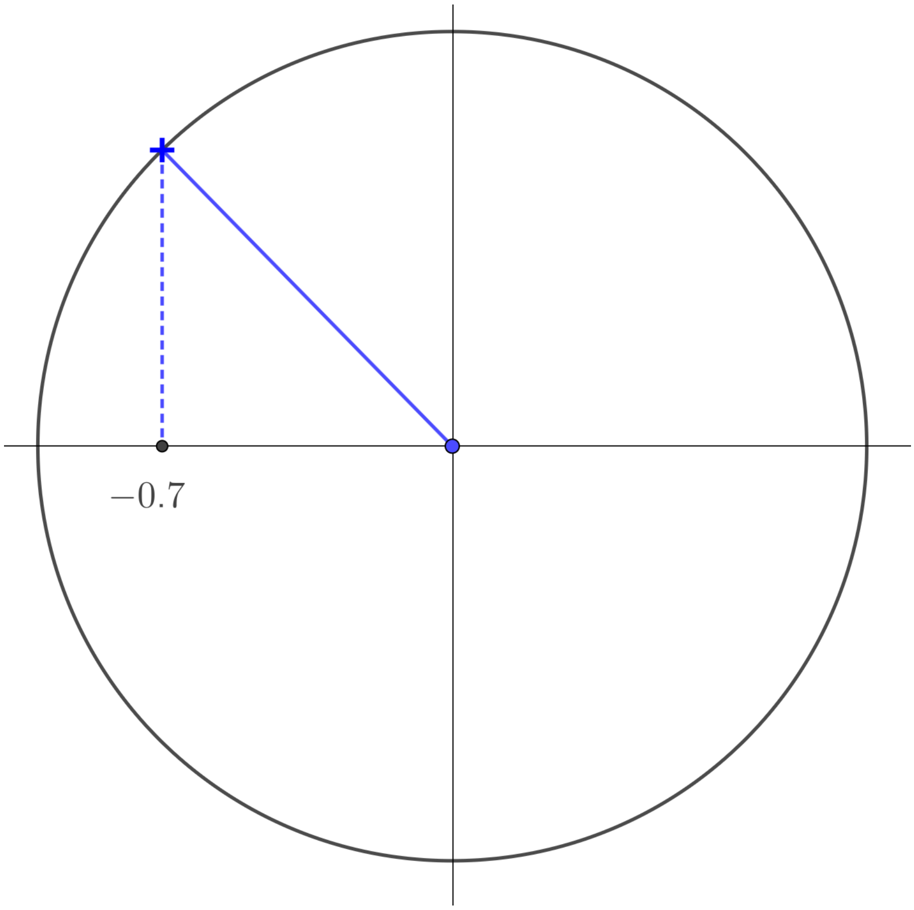
Here is the first easy rule to remember:

- for sin, take your answer from 180.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

Later, we will see that $180 + \alpha$ works for tan and $-\alpha$ works for cos.

The last two options, $\alpha \pm 360$, work for sin, cos, and tan.

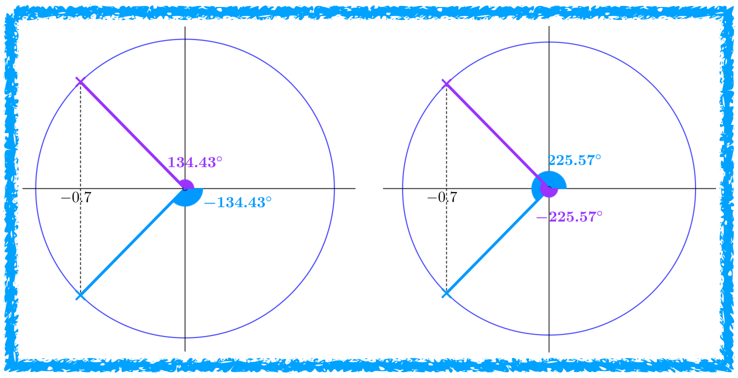
Adapt the previous method to solve the equation $\cos\theta = -0.7$.



This may seem like a bit leap, but it's just the same method, using x instead of y coordinates. Give your students plenty of time to figure this out by working together, rather than pointing the way. If they have had that time, then your help later will make far more sense.

$\theta = 134.43^\circ, 225.57^\circ, 494.43^\circ, 585.57^\circ\ \dots$

$-134.43^\circ,\ -225.57^\circ,\ -494.43^\circ,\ -585.57^\circ\ \dots$



If α is any solution of the equation $\cos\theta = k$, which of the following are also solutions of the equation:

$180-\alpha$

$180+\alpha$

$-\alpha$

$\alpha+360$

$\alpha-360$

$-\alpha$

$\alpha+360$

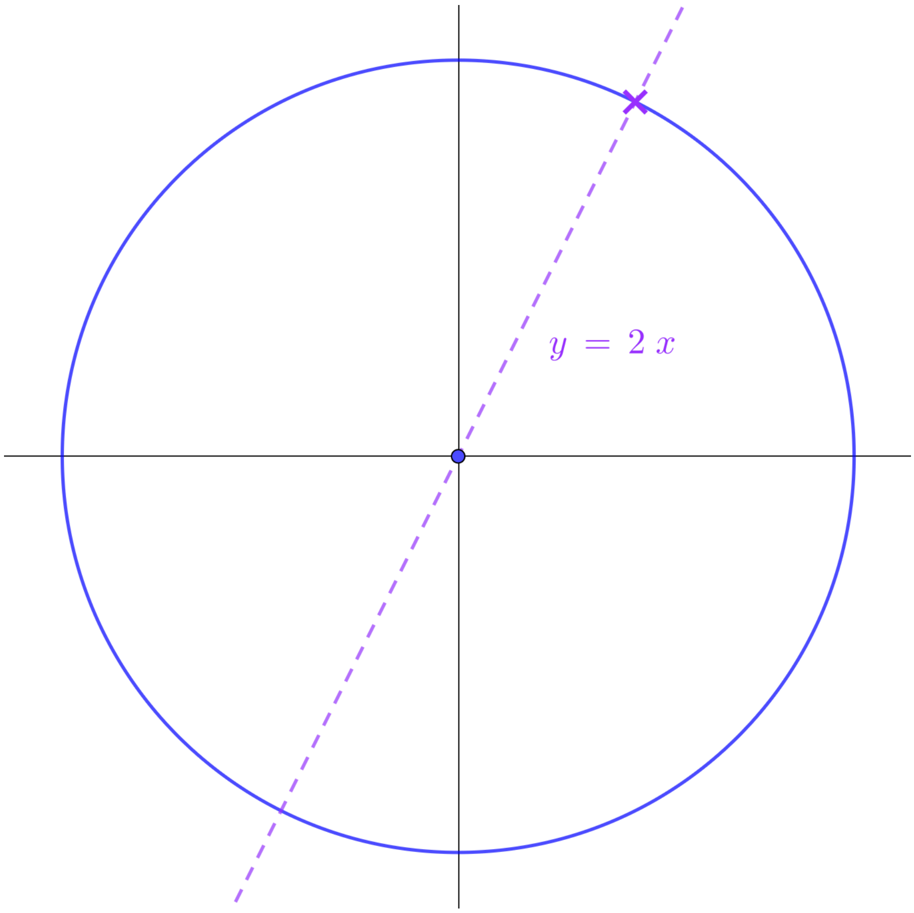
$\alpha-360$

This is the second easy rule to remember:

- for cos, take the negative of your answer.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

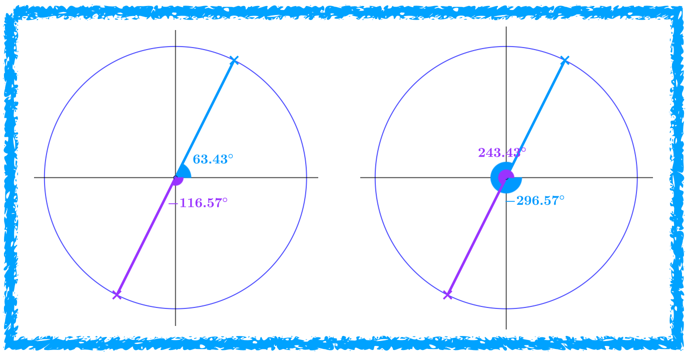
Use this diagram and a calculator to solve the equation

$$\tan\theta = 2$$



$\theta = 63.43°,\ 243.43°,\ 423.43°,\ 603.43°\ \ldots$

$-116.57°,\ -296.57°,\ -476.57°,\ -656.57°\ \ldots$



If α is any solution of the equation $\tan\theta = k$, which of the following are also solutions of the equation:

$180 - \alpha$

$180 + \alpha$

$-\alpha$

$\alpha + 360$

$\alpha - 360$

$180 + \alpha$

$\alpha + 360$

$\alpha - 360$

Here is the third easy rule to remember:

- for tan, add 180 to your answer.
- then keep adding or subtracting 360° as often as you like to each of your solutions.

Now recap the three rules:

sin: find one solution, take it away from 180, and then repeatedly ± 360.

cos: find one solution, take it's negative, and then repeatedly ± 360.

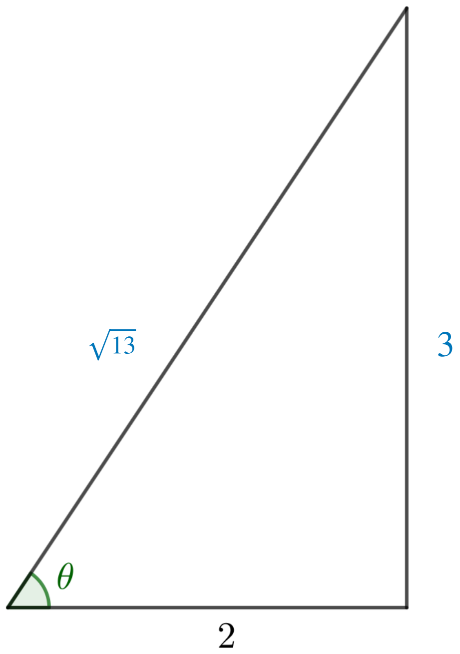
tan: find one solution, add it to 180, and then repeatedly ± 360.

At this point, your students will need plenty of basic practice. When they first encounter an equation like $\sin(2\theta) = 0.4$, they will probably find $\sin^{-1} 0.4$, then divide this by two, and then use the $\pm 180 \pm 360$ rule to find what they think are the other solutions.

I like to let them do this once, use their calculator to find sine of one of their non-solutions, see that it is not 0.4 and ask them to figure out what went wrong.

They will still need a bit of prompting to get the right method, which is to find all the possible values of 2θ and then divide these all by 2.

If $\tan\theta = \dfrac{3}{2}$, find $\sin\theta$ and $\cos\theta$.



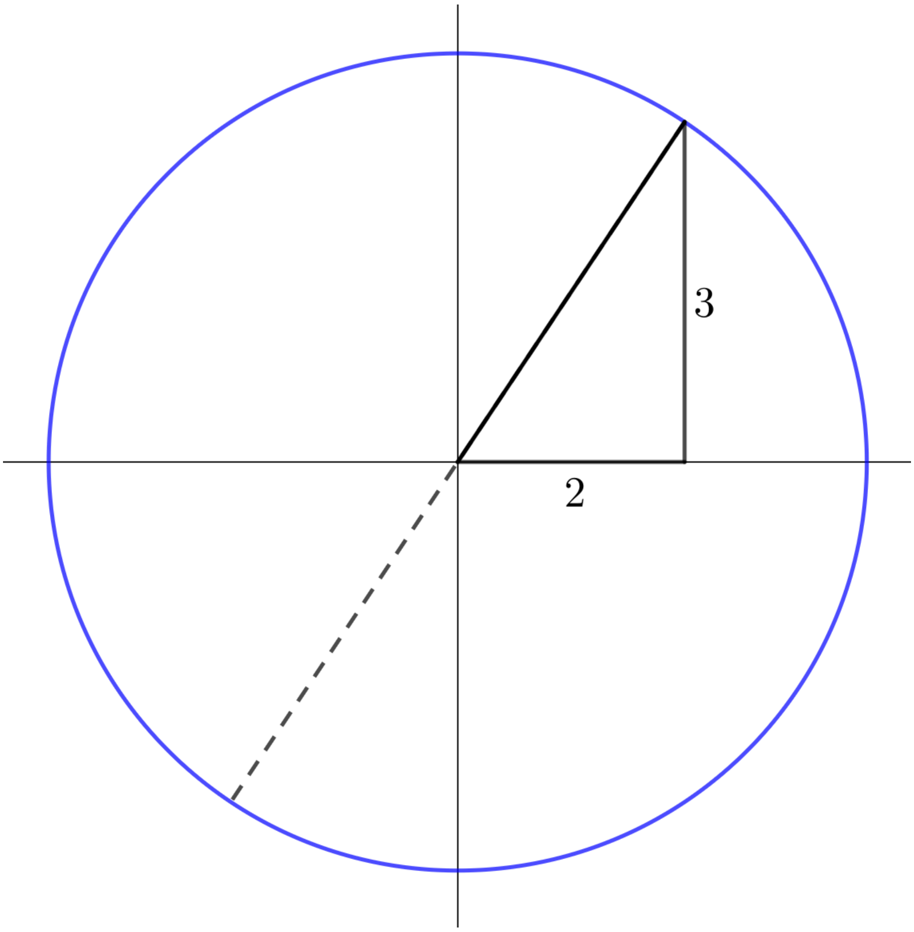
This is a right-angled triangle question.

$$\sin\theta = \frac{3}{\sqrt{13}} \qquad \cos\theta = \frac{2}{\sqrt{13}}$$

Here is a very useful non-calculator technique that tends to get overlooked. Of course, you can use the calculator to find θ and then find sin and cos. This, however, teaches you nothing about the circular functions, how they work, and how they relate to each other.

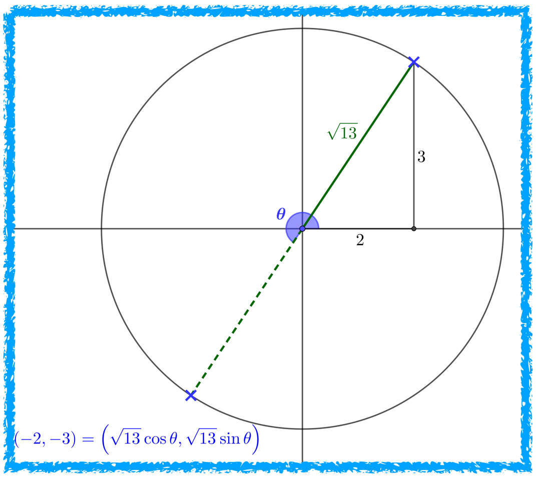
So don't let you students use their calculators for these!

If $\tan\theta = \dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.



Here, I am using a circle that no longer has radius 1. However, all the same considerations from earlier apply. In this case, the whole unit-circle diagram has been enlarged by scale factor $\sqrt{13}$.

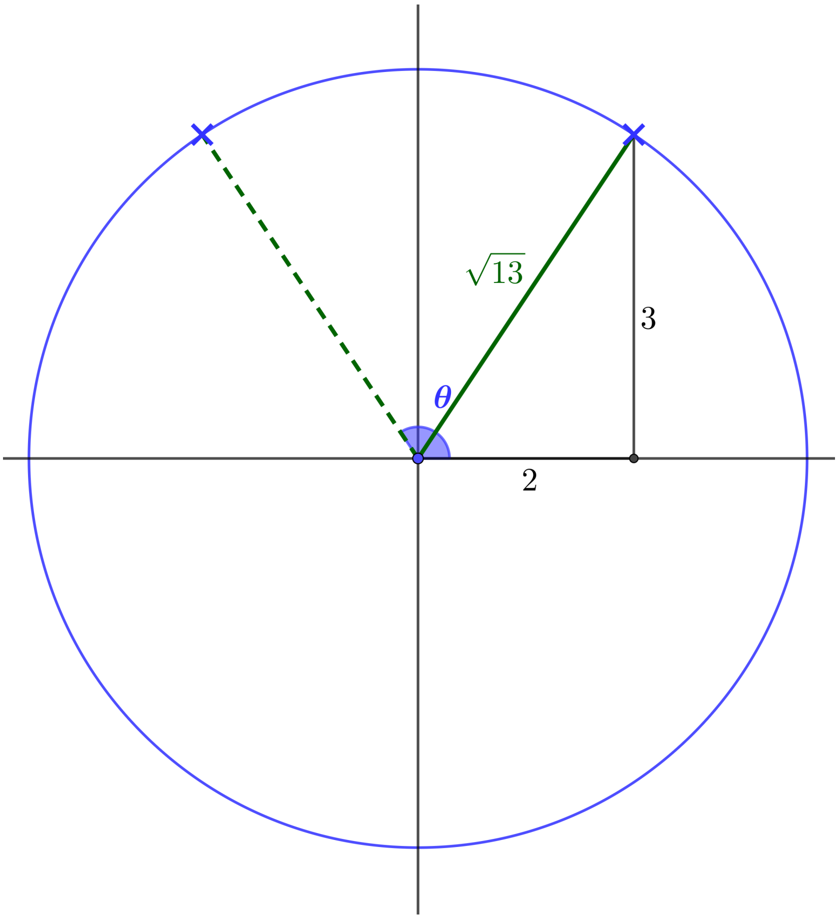
We could redraw the diagram with radius 1, but it's really more trouble than it's worth.



Here, $\tan\theta$ is still $\dfrac{3}{2}$, but

$$\sin\theta = -\frac{3}{\sqrt{13}} \qquad \cos\theta = -\frac{2}{\sqrt{13}}$$

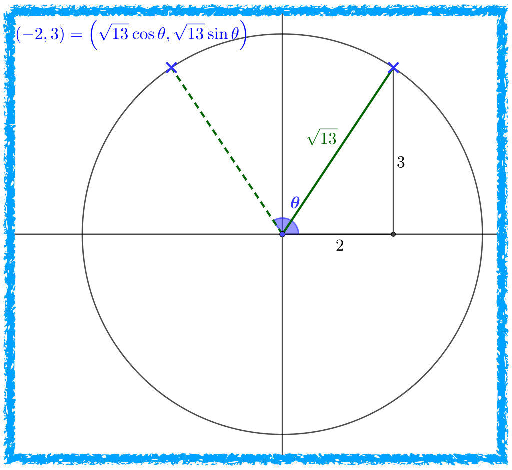
If $\tan\theta = -\dfrac{3}{2}$, and θ is obtuse, find $\sin\theta$ and $\cos\theta$.



Now we need the point where the tan of the angle is the same size as before, but negative.

Here, $\tan\theta = -\dfrac{3}{2}$ and

$\sin\theta = \dfrac{3}{\sqrt{13}}$ $\qquad$ $\cos\theta = -\dfrac{2}{\sqrt{13}}$



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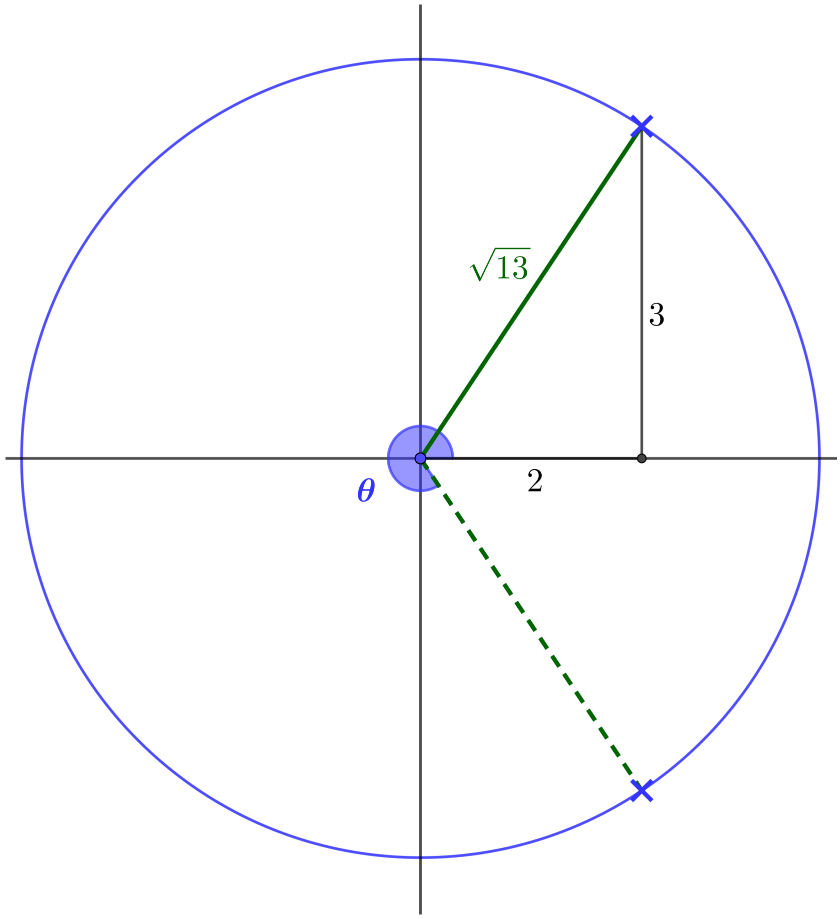
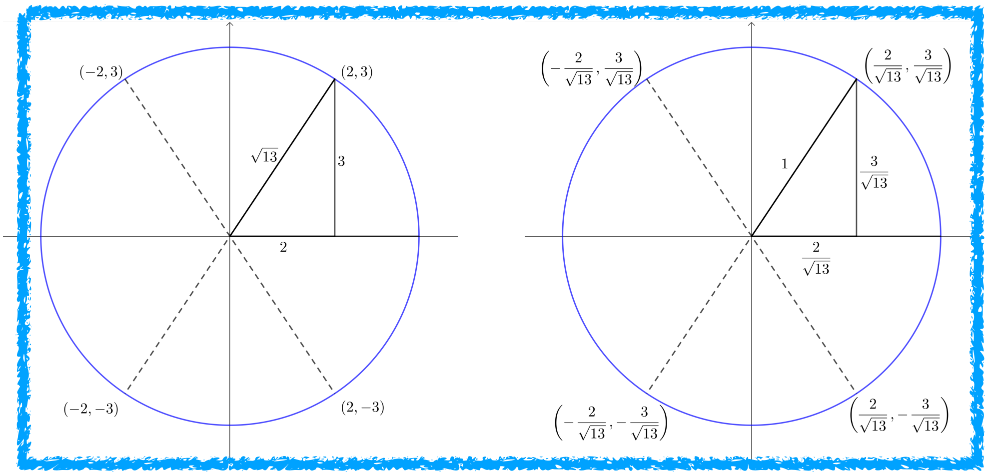
If $\tan\theta = -\dfrac{3}{2}$, and θ is reflex, find $\sin\theta$ and $\cos\theta$.

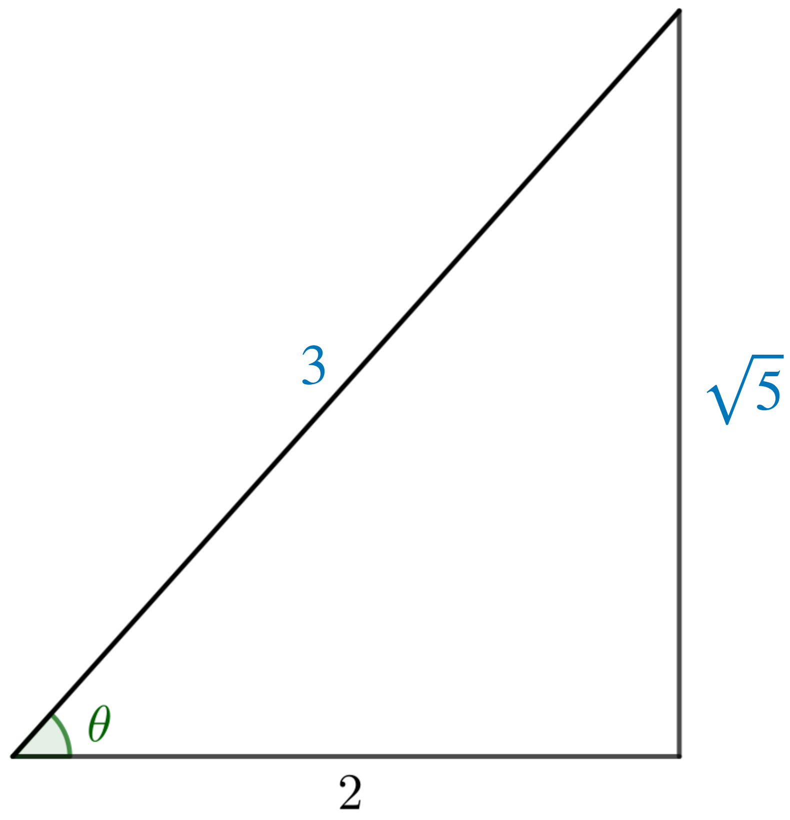


Here,

$\tan\theta = -\dfrac{3}{2}$ and $\sin\theta = -\dfrac{3}{\sqrt{13}}$ $\qquad \cos\theta = \dfrac{2}{\sqrt{13}}$



If $\cos\theta = \dfrac{2}{3}$, find $\tan\theta$ and $\sin\theta$.



$\tan\theta = \dfrac{\sqrt{5}}{2}$ $\qquad$ $\sin\theta = \dfrac{\sqrt{5}}{3}$

Find $\tan\theta$ and $\sin\theta$ when:

$$\cos\theta = \frac{2}{3}, \text{ and } \theta \text{ is between } 270^\circ \text{ and } 360^\circ$$

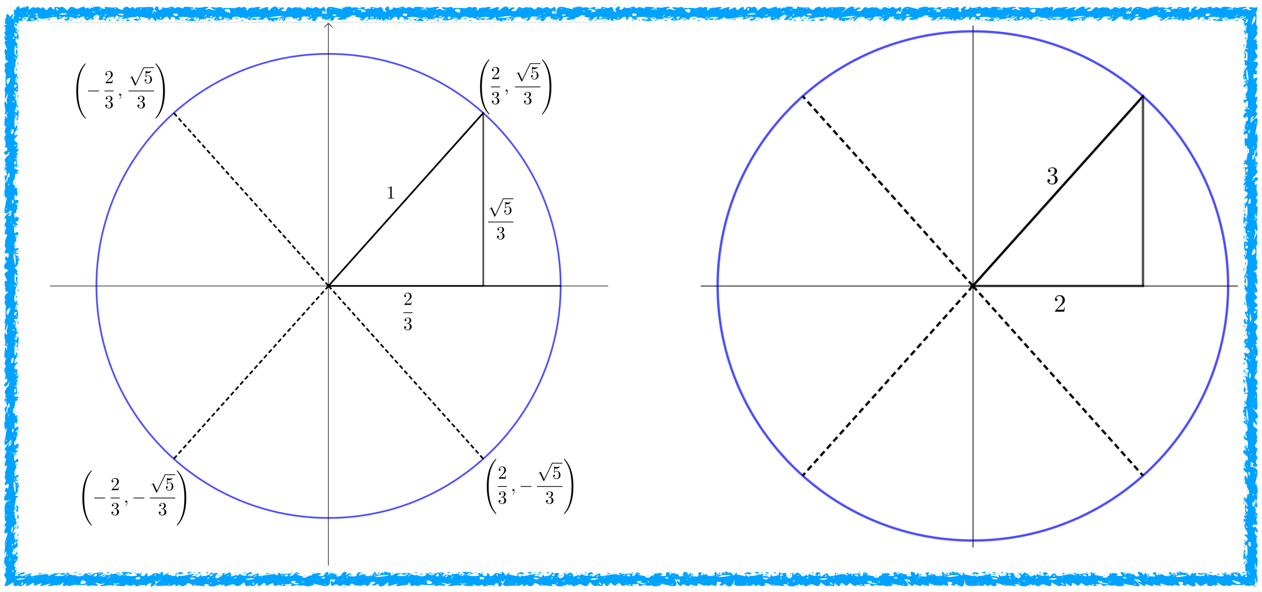
$$\tan\theta = -\frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$

$$\cos\theta = -\frac{2}{3}, \text{ and } \theta \text{ is between } 180^\circ \text{ and } 270^\circ$$

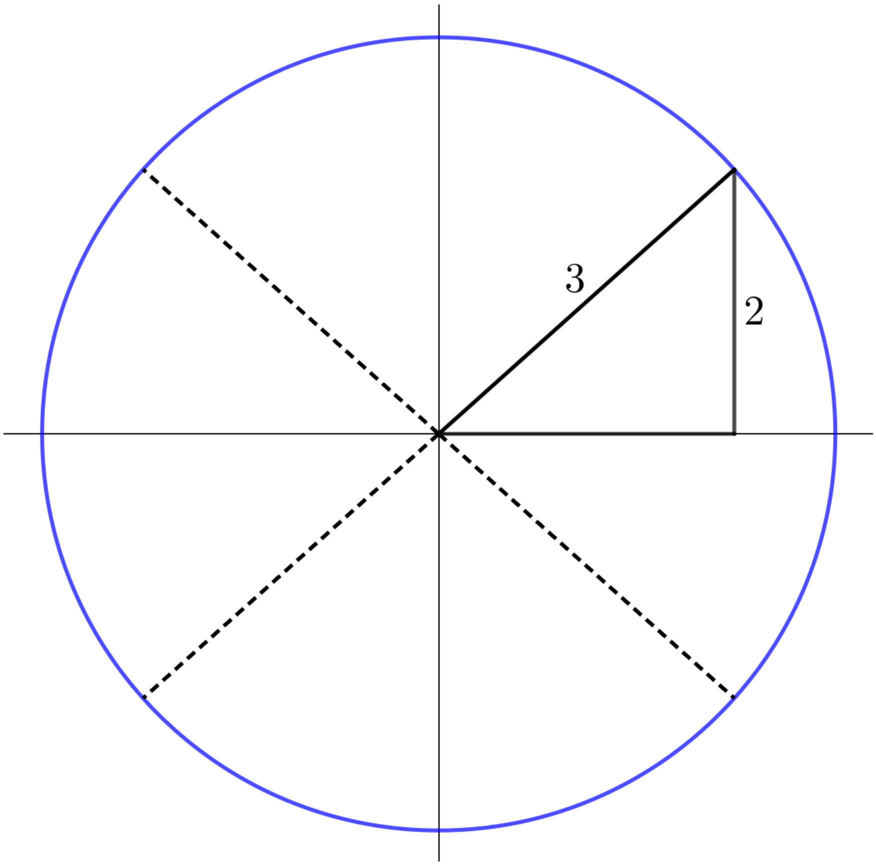
$$\tan\theta = \frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$

$$\cos\theta = -\frac{2}{3}, \text{ and } \theta \text{ is obtuse.}$$

$$\tan\theta = \frac{\sqrt{5}}{2} \qquad \sin\theta = -\frac{\sqrt{5}}{3}$$



Find $\tan\theta$ and $\cos\theta$ when $\sin\theta = \pm\frac{2}{3}$ for the various possible values of θ.



$0 < \theta < 90^\circ$ $\quad \sin\theta = \frac{2}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$ $\quad \cos\theta = \frac{\sqrt{5}}{3}$

$90^\circ < \theta < 180^\circ$ $\quad \sin\theta = \frac{2}{3}$ $\quad \tan\theta = -\frac{2}{\sqrt{5}}$ $\quad \cos\theta = -\frac{\sqrt{5}}{3}$

$180^\circ < \theta < 270^\circ$ $\quad \sin\theta = -\frac{2}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$ $\quad \cos\theta = -\frac{\sqrt{5}}{3}$

$270^\circ < \theta < 360^\circ$ $\quad \sin\theta = -\frac{2}{3}$ $\quad \tan\theta = -\frac{2}{\sqrt{5}}$ $\quad \cos\theta = \frac{\sqrt{5}}{3}$