

Hilbert modules over semicrossed products of the disk algebra

by

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ABSTRACT

Given the disk algebra $\mathcal{A}(\mathbb{D})$ and an automorphism α , there is associated a non-self-adjoint norm closed subalgebra $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ of the crossed product $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$ called the semicrossed product of $\mathcal{A}(\mathbb{D})$ with α . It is well known that the automorphisms of $\mathcal{A}(\mathbb{D})$ arise via composition with conformal bijections φ of \mathbb{D} . These automorphisms are labeled according to the corresponding conformal maps as parabolic, hyperbolic, or elliptic and each case is studied. The contractive and completely contractive representations of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ on a Hilbert space \mathcal{H} (i.e. contractive Hilbert modules) are found to be in a one-to-one correspondence with pairs of contractions S and T on \mathcal{H} satisfying $TS = S\varphi(T)$. To this end, a noncommutative dilation result is obtained. It states that given a pair of contractions S and T on \mathcal{H} satisfying $TS = S\varphi(T)$ there exist a pair of unitaries U and V on $\mathcal{K} \supseteq \mathcal{H}$ satisfying $VU = U\varphi(V)$ and dilating S and T respectively. Some concrete representations of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ are then found in order to compute the characters, the maximal ideal space, and the strong radical. The Shilov and orthoprojective Hilbert modules over $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ are shown to correspond to pairs of isometries S and T satisfying $TS = S\varphi(T)$.

CHAPTER 1

INTRODUCTION

Just as nineteenth century physics was the motivating factor behind operator theory, twentieth century physics in the form of quantum theory motivated the study of algebras of operators. Within this framework, the main interest for many years had been in C^* -algebras - that is, the self-adjoint case. However, in the fifties and sixties researchers began considering non-self-adjoint algebras. Much of this early work culminated in the model theory of Sz. Nagy and Foias [Sz-NF] for a contraction. In many ways, the study of contractions is equivalent to the study of the (non-self-adjoint) disk algebra $\mathcal{A}(\mathbb{D})$ and its contractive representations.

Many efforts have been made at extending this theory to commuting n -tuples of contractions without much success. The main problem is the vast difference one encounters in moving from functions of a single complex variable to several complex variables. One might in fact classify what we are about to study under the heading “non-commutative complex variables”.

One point of view, the one which we take, is a module approach to model theory. Just as an algebraist might study a ring by “the modules it keeps”, so too might an analyst study an operator algebra. To this end, a two step program from homological algebra has been adopted. In the first step, one studies a “nice” or “elementary” class of modules. In the second step, a general module is studied in terms of its resolutions via these “nice” modules. However, the analogous role of a projective module (a “nice” module) from homological algebra is not completely understood in the context of operator algebras. The “elementary” modules which we consider in this work are the Shilov modules of Douglas and Paulsen [DP] and the orthogonally projective modules of Muhly and Solel [MS2]. For the disk algebra, these modules coincide. It is not known what the correct notion of an “elementary” module over an operator

algebra is, but it is our hope that the study of several examples will aid in this development.

As Douglas and Paulsen [DP] point out, much of the dilation theory begun by Sz. Nagy can be reformulated in terms of Shilov module resolutions. For example, in the successful Sz. Nagy model theory for a contraction, all the information needed to classify the contractions on a Hilbert space \mathcal{H} up to unitary equivalence is contained in a module map between two Shilov modules. This module map corresponds to the so-called “characteristic operator function” for a contraction. The goal of this viewpoint is to extract information about a Hilbert module from a module map between the Shilov modules occurring in the Shilov resolution of this module.

The context in which we study Hilbert modules is over a semicrossed product, a (non-self-adjoint) subalgebra of a C^* -crossed product. C^* -crossed products have dynamical systems as their origin.

In quantum physics, the observables can sometimes be modeled by a C^* -algebra. Time evolution or spatial translation of the observables is then described by a triple $(\mathfrak{A}, G, \alpha)$, called a C^* -dynamical system, where \mathfrak{A} is a C^* -algebra, G is a locally compact group, and $\alpha : G \rightarrow \text{Aut}(\mathfrak{A})$ is a homomorphism ($\text{Aut}(\mathfrak{A})$ is the group of automorphisms of \mathfrak{A} with topology stemming from pointwise convergence). The C^* -crossed product $G \times_{\alpha} \mathfrak{A}$ is then obtained from the C^* -dynamical system $(\mathfrak{A}, G, \alpha)$ and becomes of interest since the study of this C^* -algebra aids in the study of the C^* -dynamical system.

A semicrossed product is a non-self-adjoint version of a C^* -crossed product. One can think of it as emanating from triples $(\mathfrak{A}, S, \alpha)$ where \mathfrak{A} is a norm closed subalgebra of a C^* -algebra (i.e. an operator algebra) and S is a semigroup. Our interest in this work is the case where \mathfrak{A} is the disk algebra $\mathcal{A}(\mathbb{D})$, S is the semigroup \mathbb{Z}^+ , and α acts in the obvious fashion. From a Hilbert module point of view, these algebras are of interest because the associated contractive Hilbert modules correspond to pairs of contractions S and T satisfying the relationship $TS = S\alpha(T)$. We characterize the Shilov and orthogonally projective modules over the semicrossed product $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ as those corresponding to isometries S and T .

CHAPTER 2

BACKGROUND

2.1 Operator Algebras

A Banach space \mathfrak{X} is a complete normed linear space. If \mathfrak{X} is also an algebra over \mathbb{R} or \mathbb{C} satisfying $\|xy\| \leq \|x\|\|y\|$ for all x, y in \mathfrak{X} , then we call \mathfrak{X} a Banach algebra. An automorphism of a Banach algebra \mathfrak{X} is a continuous isomorphism of \mathfrak{X} onto \mathfrak{X} . In a Banach algebra \mathfrak{X} , an involution is a map $x \mapsto x^*$ of \mathfrak{X} into \mathfrak{X} such that $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $(\alpha x + y)^* = \overline{\alpha}x^* + y^*$ for all $\alpha \in \mathbb{C}$ and $x, y \in \mathfrak{X}$. A Banach *-algebra is a Banach algebra with an involution. A C^* -algebra is a Banach algebra \mathfrak{X} with involution $*$ such that $\|x^*x\| = \|x\|^2 \forall x \in \mathfrak{X}$.

Example 2.1.1. Let \mathbb{T} denote the complex unit circle and $C(\mathbb{T})$ the continuous complex valued functions on \mathbb{T} . If $f(z)^* = \overline{f(z)}$, then $C(\mathbb{T})$ is a C^* -algebra.

Example 2.1.2. For a Hilbert space \mathcal{H} , a linear transformation $T : \mathcal{H} \rightarrow \mathcal{H}$ is bounded if $\sup\{\|Th\| : \|h\| = 1\} < \infty$. Denote by $B(\mathcal{H})$ the collection of bounded linear operators on \mathcal{H} . The adjoint T^* of $T \in B(\mathcal{H})$ is the unique operator $S \in B(\mathcal{H})$ satisfying $\langle Sh, k \rangle = \langle h, Tk \rangle \forall h, k \in \mathcal{H}$ [Con, p. 31]. The operator norm for $T \in B(\mathcal{H})$ is given by $\|T\| = \sup\{\|Th\| : \|h\| = 1\}$. Equipped with the operator norm and the adjoint as involution, $B(\mathcal{H})$ is a C^* -algebra.

A *-homomorphism ϕ of a Banach *-algebra \mathfrak{A} into another Banach *-algebra \mathfrak{B} is an algebraic homomorphism which preserves the involution. If ϕ is a bijection we say it is a *-isomorphism. A representation π of a Banach algebra \mathfrak{A} on a Hilbert space \mathcal{H} is a homomorphism of \mathfrak{A} into $B(\mathcal{H})$. If \mathfrak{A} is involutive, then π is required to be a *-homomorphism and is called a *-representation. The algebra $B(\mathcal{H})$ of Example 2.1.2 is very important in the study

of C^* -algebras. In fact, in a certain sense, all C^* -algebras are subalgebras of $B(\mathcal{H})$. This is the Gelfand-Neumark theorem which is stated as Theorem 2.1.4. For a proof, see Kadison and Ringrose [KR, Theorem 4.5.6] or Murphy [Mur, Theorem 3.4.1].

Definition 2.1.3. A representation π of a Banach algebra \mathfrak{A} on \mathcal{H} is faithful if π is one-to-one.

Theorem 2.1.4 (Gelfand-Neumark). Each C^* -algebra \mathfrak{A} has a faithful representation on a Hilbert space \mathcal{H} .

The Gelfand-Neumark theorem can be restated as follows: If \mathfrak{A} is a C^* -algebra, there is a Hilbert space \mathcal{H} such that \mathfrak{A} is $*$ -isomorphic to a C^* -subalgebra of $B(\mathcal{H})$.

Definition 2.1.5. A contractive representation π of a Banach algebra \mathfrak{A} on \mathcal{H} is one for which $\|\pi(a)\| \leq \|a\| \ \forall a \in \mathfrak{A}$.

It is a well known fact that any representation π of a C^* -algebra on a Hilbert space \mathcal{H} is a contractive representation [KR, Theorem 4.1.8]. It is also the case that the nonzero representations of a Banach algebra on \mathbb{C} (the multiplicative linear functionals or characters) are contractive [Zhu, Proposition 4.1].

Definition 2.1.6. An operator algebra \mathfrak{A} is a norm closed subalgebra of a C^* -algebra or, equivalently, of $B(\mathcal{H})$.

Example 2.1.7. The disk algebra $\mathcal{A}(\mathbb{D})$ is the algebra of continuous functions on the closed unit disk $\overline{\mathbb{D}}$ in \mathbb{C} which are analytic on the open disk \mathbb{D} . Alternatively, $\mathcal{A}(\mathbb{D})$ is the uniform closure of the polynomials $p(z)$. Identifying $f \in \mathcal{A}(\mathbb{D})$ with its boundary values, $\mathcal{A}(\mathbb{D})$ consists of functions $f \in C(\mathbb{T})$ whose Fourier coefficients vanish on the negative integers. $\mathcal{A}(\mathbb{D})$ is an operator algebra as it is a norm closed subalgebra of the C^* -algebra $C(\mathbb{T})$.

An Banach $*$ -algebra \mathfrak{A} is self-adjoint if $T \in \mathfrak{A} \Rightarrow T^* \in \mathfrak{A}$. An operator algebra is a C^* -algebra if and only if it is self-adjoint. The interest in contractive representations of operator algebras is that they coincide with $*$ -representations when the operator algebra is a C^* -algebra.

Definition 2.1.8. A subspace \mathcal{M} of a Hilbert space \mathcal{H} is invariant under $T \in B(\mathcal{H})$ if $T(\mathcal{M}) \subseteq \mathcal{M}$. The subspace \mathcal{M} is reducing for T if it is invariant for T and T^* . A subspace \mathcal{M} is invariant for an operator algebra $\mathfrak{A} \subseteq B(\mathcal{H})$ if $T(\mathcal{M}) \subseteq \mathcal{M} \ \forall T \in \mathfrak{A}$.

Example 2.1.9. The disk algebra $\mathcal{A}(\mathbb{D})$ can be considered as an operator algebra on $L^2(\mathbb{T})$ by multiplication. Let $H^2(\mathbb{D})$ denote the Hardy space of analytic, square integrable functions on \mathbb{D} . For symbol $f \in H^\infty(\mathbb{T})$ (*resp.* $L^\infty(\mathbb{T})$), define the multiplication operator M_f on $H^2(\mathbb{D})$ (*resp.* $L^2(\mathbb{T})$) by $M_f(g) = fg$. Then $H^2(\mathbb{D})$ is a closed invariant subspace for $\mathcal{A}(\mathbb{D})$. This is equivalent to saying that $H^2(\mathbb{D})$ is a closed invariant subspace for the multiplication operator M_z on $L^2(\mathbb{T})$. In fact, all the closed invariant subspaces of $L^2(\mathbb{T})$ under the multiplication operator M_z have been characterized by Beurling [Hof].

Theorem 2.1.10 (Beurling). Let \mathcal{M} be a closed subspace of $L^2(\mathbb{T})$ which is invariant under the multiplication operator M_z . If $M_z(\mathcal{M}) = \mathcal{M}$, then \mathcal{M} is the set of functions on $L^2(\mathbb{T})$ vanishing on some Borel subset of \mathbb{T} . If $M_z(\mathcal{M}) \neq \mathcal{M}$, then $\mathcal{M} = F(z) \cdot H^2(\mathbb{D})$ where $|F(z)| = 1$ is a measurable function.

2.2 Ideals and Radicals

Definition 2.2.1. A subalgebra \mathcal{I} of an algebra \mathfrak{A} is called a left (*resp.* right) ideal if $a\mathcal{I} \subseteq \mathcal{I}$ (*resp.* $\mathcal{I}a \subseteq \mathcal{I}$) $\forall a \in \mathfrak{A}$. A (two-sided) ideal is a subalgebra \mathcal{I} which is both a left and right ideal.

Definition 2.2.2. An ideal \mathcal{M} in an algebra \mathfrak{A} is maximal if $\mathcal{M} \neq \mathfrak{A}$ and if whenever \mathcal{I} is an ideal of \mathfrak{A} such that $\mathcal{M} \subseteq \mathcal{I} \subseteq \mathfrak{A}$, then either $\mathcal{I} = \mathfrak{A}$ or $\mathcal{I} = \mathcal{M}$.

For a commutative, unital Banach algebra \mathfrak{A} , there is a one-to-one correspondence between the characters of \mathfrak{A} , $\hat{\mathfrak{A}}$, and the set $\mathcal{M}(\mathfrak{A})$ of maximal ideals of \mathfrak{A} .

Proposition 2.2.3. [Ped2, Proposition 4.2.2] Let \mathfrak{A} be a commutative, unital Banach algebra. Then the map $\hat{\mathfrak{A}} \rightarrow \mathcal{M}(\mathfrak{A})$ given by $\gamma \mapsto \ker \gamma$ is a bijection.

Proof. Let \mathcal{J} be a maximal ideal in \mathfrak{A} . It is easy to see that \mathcal{J} is closed. Let $0 \neq B \in \mathfrak{A}/\mathcal{J}$, the quotient algebra. If B is not invertible in \mathfrak{A}/\mathcal{J} , then B is contained in a proper ideal $0 \neq \mathcal{I} \neq \mathfrak{A}/\mathcal{J}$ by the commutativity of \mathfrak{A} . Let $\pi : \mathfrak{A} \rightarrow \mathfrak{A}/\mathcal{J}$ be the quotient map so that $\pi^{-1}(\mathcal{I})$ is an ideal in \mathfrak{A} . However, $\mathcal{J} \subsetneq \pi^{-1}(\mathcal{I}) \subsetneq \mathfrak{A}$ then contradicts the maximality of \mathcal{J} . Thus every nonzero element of \mathfrak{A}/\mathcal{J} is invertible. By the Gelfand-Mazur theorem [KR, Cor.

3.2.4], $\mathfrak{A}/\mathcal{J} \cong \mathbb{C}$. Hence from the quotient map $\pi : \mathfrak{A} \rightarrow \mathfrak{A}/\mathcal{J}$, we can form the character $\hat{\pi} : \mathfrak{A} \rightarrow \mathbb{C}$ where it is clear that $\mathcal{J} = \ker \hat{\pi}$. On the other hand, if $\gamma \in \hat{\mathfrak{A}}$, it is easy to verify that $\ker \gamma$ is an ideal of codimension 1 and hence maximal. \square

The assumption of commutativity is crucial in the above argument; this will be exhibited in Theorem 4.4.15 with an example of a noncommutative algebra in which the maximal ideals and kernels of characters do not coincide.

Example 2.2.4. Every maximal ideal of $\mathcal{A}(\mathbb{D})$ is of the form

$$M_\lambda = \{f \in \mathcal{A}(\mathbb{D}) : f(\lambda) = 0\}$$

for some point $\lambda \in \overline{\mathbb{D}}$. Note also that by Proposition 2.2.3 all the characters on $\mathcal{A}(\mathbb{D})$, γ_λ , are given by evaluation at λ for some $\lambda \in \overline{\mathbb{D}}$ [Hof].

Definition 2.2.5. An ideal \mathcal{P} in an algebra \mathfrak{A} is prime if $\mathcal{P} \neq \mathfrak{A}$ and for any ideals \mathcal{I}, \mathcal{J} in \mathfrak{A}

$$\mathcal{I}\mathcal{J} \subseteq \mathcal{P} \Rightarrow \mathcal{I} \subseteq \mathcal{P} \text{ or } \mathcal{J} \subseteq \mathcal{P}.$$

The following theorem can be of use in characterizing the maximal ideals of an algebra \mathfrak{A} .

Theorem 2.2.6. [Hun, Theorem 2.19] If \mathfrak{A} is a commutative, unital algebra then every maximal ideal is prime.

Definition 2.2.7. The strong radical \mathfrak{A}_S of a unital algebra \mathfrak{A} is the intersection of all maximal ideals of \mathfrak{A} unless there are no such ideals, in which case $\mathfrak{A}_S = \mathfrak{A}$. If $\mathfrak{A}_S = (0)$, then \mathfrak{A} is said to be strongly semi-simple.

Definition 2.2.8. An operator algebra $\mathfrak{A} \subseteq B(\mathcal{H})$ is algebraically irreducible provided (0) and \mathcal{H} are the only invariant (not necessarily closed) subspaces. The Jacobson radical, \mathfrak{A}_J , of an operator algebra \mathfrak{A} is defined as the intersection of the kernels of all algebraically irreducible representations of \mathfrak{A} . If $\mathfrak{A}_J = (0)$, then \mathfrak{A} is called semi-simple.

Remark 2.2.9. [Dav, Corollary I.9.13] All C^* -algebras are semi-simple.

Definition 2.2.10. $T \in \mathfrak{A}$, a Banach algebra, is quasinilpotent if $\lim_{n \rightarrow \infty} \|T^n\|^{1/n} = 0$.

The Jacobson radical of a Banach algebra \mathfrak{A} can be characterized in terms of the quasinilpotent elements of \mathfrak{A} .

Theorem 2.2.11. For a Banach algebra \mathfrak{A} , the Jacobson radical

$$\begin{aligned}\mathfrak{A}_J &= \{T \in \mathfrak{A} : TS \text{ is quasinilpotent } \forall S \in \mathfrak{A}\} \\ &= \{T \in \mathfrak{A} : ST \text{ is quasinilpotent } \forall S \in \mathfrak{A}\}.\end{aligned}$$

Proof. First, note that the Jacobson radical \mathfrak{A}_J is a (2-sided) ideal in \mathfrak{A} . Since every element in \mathfrak{A}_J is quasinilpotent [Ric, Theorem 2.3.4], one direction is clear. On the other hand, \mathfrak{A}_J is the sum of all quasinilpotent ideals in \mathfrak{A} [Ric, Theorem 2.3.5(ii)]. Also, if \mathcal{I} is a quasinilpotent ideal in \mathfrak{A} , then $\mathcal{I} \subseteq \{T \in \mathfrak{A} : TS \text{ is quasinilpotent } \forall S \in \mathfrak{A}\}$ which completes the proof. \square

The set \mathcal{Q} of quasinilpotents of a Banach algebra \mathfrak{A} is directly related to the Jacobson radical in some cases. For example, if the Banach algebra is commutative or if \mathcal{Q} is an ideal then $\mathfrak{A}_J = \mathcal{Q}$ [Ric, Corollary 2.3.6].

Theorem 2.2.12. [Ric, Theorem 2.3.11] The strong radical always contains the (Jacobson) radical.

Example 2.2.13. Example 2.2.4 shows that the strong radical of the disk algebra $\mathcal{A}(\mathbb{D})$ is (0) . Hence by Theorem 2.2.12 the Jacobson radical of $\mathcal{A}(\mathbb{D})$ is also (0) .

2.3 C^* -envelopes

For a single normal operator T in $B(\mathcal{H})$, the C^* -algebra generated by T and the identity I , denoted $C^*(T)$, is unique and understood so far as $C^*(T) \cong C(sp(T))$ [Ped2, Proposition 4.3.15]. However, the C^* -algebra generated by an operator algebra \mathfrak{A} need not be unique. In this section we discuss the C^* -envelope of an operator algebra \mathfrak{A} which is an essentially unique C^* -algebra containing \mathfrak{A} .

Definition 2.3.1. For C^* -algebras \mathfrak{B}_1 and \mathfrak{B}_2 , let $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}_2$ be a linear map from an operator algebra $\mathfrak{A} \subseteq \mathfrak{B}_1$. Consider $\mathfrak{A} \otimes M_n(\mathbb{C})$ as a subalgebra of the C^* -algebra $\mathfrak{B}_1 \otimes M_n(\mathbb{C})$.

$\mathfrak{A} \otimes M_n(\mathbb{C})$ can be thought of as matrices with entries in \mathfrak{A} with norm inherited from the C^* -algebra $\mathfrak{B}_1 \otimes M_n(\mathbb{C})$. Let $\varphi_n : \mathfrak{A} \otimes M_n(\mathbb{C}) \rightarrow \mathfrak{B}_2 \otimes M_n(\mathbb{C})$ be determined by applying φ to each element of each matrix over \mathfrak{A} . We say that φ is completely isometric if each φ_n is isometric.

Let now \mathfrak{A} be an operator algebra in a unital C^* -algebra \mathfrak{B} which contains the identity and generates \mathfrak{B} as a C^* -algebra. A closed (two-sided) ideal \mathcal{J} of \mathfrak{B} is a boundary ideal for \mathfrak{A} if the quotient map $q : \mathfrak{B} \rightarrow \mathfrak{B}/\mathcal{J}$ is completely isometric on \mathfrak{A} . A boundary ideal which contains every other boundary ideal is the Shilov boundary for \mathfrak{A} . The existence of a Shilov boundary for \mathfrak{A} is due to Hamana [Ham].

Theorem 2.3.2. [Ham, Theorem 4.4] If \mathfrak{A} is an operator algebra in a unital C^* -algebra \mathfrak{B} which contains the identity and generates \mathfrak{B} as a C^* -algebra, then \mathfrak{A} has a Shilov boundary \mathcal{J} .

Definition 2.3.3. For $\mathfrak{A}, \mathfrak{B}$, and \mathcal{J} as in Theorem 2.3.2, we call \mathfrak{B}/\mathcal{J} the C^* -envelope of \mathfrak{A} denoted $C^*(\mathfrak{A})$.

In general, the C^* -envelope of an operator algebra \mathfrak{A} differs from the C^* -algebra generated by \mathfrak{A} . In Theorem 2.3.5, conditions are given for which the two coincide.

Example 2.3.4. Consider the disk algebra $\mathcal{A}(\mathbb{D})$ as a subalgebra of $C(\overline{\mathbb{D}})$. Note that $C^*(\mathcal{A}(\mathbb{D})) = C(\overline{\mathbb{D}})$. The Shilov boundary for $\mathcal{A}(\mathbb{D})$ is then $\mathcal{J} = \{f \in C(\overline{\mathbb{D}}) : f(\mathbb{T}) = (0)\}$. On the other hand, if $\mathcal{A}(\mathbb{D})$ is considered to be a subalgebra of $C(\mathbb{T})$, then the Shilov boundary \mathcal{J} is (0) . In fact, for certain operator algebras \mathfrak{A} (such as $\mathcal{A}(\mathbb{D})$), the C^* -algebra generated by \mathfrak{A} is completely determined once one has factored by the Shilov boundary [Arv].

Theorem 2.3.5. [Arv, Corollary 2.2.8] Let $\mathfrak{A}, \mathfrak{B}$, and \mathcal{J} be as in Theorem 2.3.2. If $\mathfrak{A} + \mathfrak{A}^*$ is dense in \mathfrak{B} , then $\mathcal{J} = 0$.

2.4 Isometries

A bounded linear operator $T \in B(\mathcal{H})$ is called a contraction if $\|T\| \leq 1$ and an isometry if $\|Th\| = \|h\| \ \forall h \in \mathcal{H}$. Equivalently, $T \in B(\mathcal{H})$ is an isometry if and only if $T^*T = I_{\mathcal{H}}$. An

operator $U \in B(\mathcal{H})$ is called a unitary if it is a surjective isometry. Equivalently, $U \in B(\mathcal{H})$ is a unitary if and only if $U^*U = UU^* = I_{\mathcal{H}}$.

Example 2.4.1. Let $\mathcal{H} = \ell_+^2(\mathbb{C})$, the Hilbert space of square summable sequences of complex numbers $\{\xi_n\}_{n=0}^\infty$. Define the simple unilateral shift U_+ on \mathcal{H} by $U_+(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \xi_2, \dots)$. Note that $U_+^*(\xi_0, \xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \xi_3, \dots)$ so that $U_+^*U_+ = I_{\mathcal{H}}$ and U_+ is an isometry. However, U_+ is not a unitary as $U_+U_+^* \neq I_{\mathcal{H}}$.

Example 2.4.2. Let $\mathcal{H} = \ell^2(\mathbb{C})$ denote the Hilbert space of square summable sequences $\{\xi_n\}_{n=-\infty}^\infty$. Define the simple bilateral shift U on \mathcal{H} by $U(\dots, \xi_{-1}, \xi_0, \xi_1, \dots) = (\dots, \xi_{-2}, \xi_{-1}, \xi_0, \dots)$. Then $U^*U = UU^* = I_{\mathcal{H}}$ and U is unitary.

There is a model for isometries on a Hilbert space \mathcal{H} called the Wold decomposition. Consider an isometry T on \mathcal{H} . A subspace \mathcal{L} of \mathcal{H} is called wandering for T if $T^p\mathcal{L} \perp T^q\mathcal{L}$ (i.e. $T^p\mathcal{L}$ is orthogonal to $T^q\mathcal{L}$) for all nonnegative integers $p \neq q$. Since $T^*T = I_{\mathcal{H}}$, \mathcal{L} is wandering if $T^n\mathcal{L} \perp \mathcal{L}$ for $n \geq 1$. The orthogonal sum $M_+(\mathcal{L}) = \bigoplus_0^\infty T^n(\mathcal{L})$ is then well-defined. An isometry $T \in B(\mathcal{H})$ is called a unilateral shift if there exists in \mathcal{H} a subspace \mathcal{L} wandering for T such that $\mathcal{H} = M_+(\mathcal{L})$. The dimension of \mathcal{L} is called the multiplicity of the unilateral shift T . A proof of Theorem 2.4.3 can be found in Sz. Nagy and Foias [Sz-NF] and Fillmore [Fil1].

Theorem 2.4.3 (Wold Decomposition). Let T be an isometry on \mathcal{H} . Then $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_t$ where \mathcal{H}_u and \mathcal{H}_t reduce T , $T|_{\mathcal{H}_u}$ is a unitary, and $T|_{\mathcal{H}_t}$ is a unilateral shift of multiplicity $\dim(\ker T^*)$. Furthermore, this decomposition is unique up to unitary equivalence,

$$\mathcal{H}_u = \bigcap_{n \geq 0} T^n \mathcal{H}, \quad \text{and} \quad \mathcal{H}_t = M_+(\mathcal{H} \ominus T\mathcal{H}).$$

Remark 2.4.4. The operator $T|_{\mathcal{H}_t}$ in the Wold decomposition is also called a pure isometry.

Example 2.4.5. If U_+ is the simple unilateral shift on $\mathcal{H} = \ell_+^2(\mathbb{C})$, then $\mathcal{L} = \ker U_+$ is a wandering subspace and $\mathcal{H} = \mathcal{H}_t$.

Example 2.4.6. If U is the simple bilateral shift on $\mathcal{H} = \ell^2(\mathbb{C})$, then $\mathcal{H} = \mathcal{H}_u$.

Example 2.4.7. Let $|\mu| = 1$ and $|a| < 1$ be complex numbers and consider the conformal bijection of \mathbb{D} given by $\varphi(z) = \mu \frac{z-a}{1-\bar{a}z}$. Define the composition operator C_φ on $H^2(\mathbb{D})$ by

$C_\varphi(f) = f \circ \varphi$. Note that C_φ is an isometry since $\|f \circ \varphi\| = \|f\| \ \forall f \in H^2(\mathbb{D})$. If $\varphi(z) = \mu z, |\mu| = 1$, then C_φ is clearly unitary. In fact, C_φ is unitary for any such conformal bijection. For a complete account on composition operators, see Cowen and MacCluer [CM].

Example 2.4.8. The multiplication operator M_f satisfies $\|M_f\| = \|f\|_\infty$ so that M_f is a contraction if $\|f\|_\infty \leq 1$. Furthermore, $M_f^* = M_{\bar{f}}$ so that M_f is an isometry if and only if M_f is a unitary if and only if $f\bar{f} = |f|^2 = 1$. Note that the multiplication operator M_z on $H^2(\mathbb{D})$ is a unilateral shift of multiplicity 1.

2.5 Crossed Products $\mathbb{Z} \times_\alpha C(\mathbb{T})$

Let α be an automorphism of $C(\mathbb{T})$. Consider the Banach space $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ consisting of all formal sums $\sum_{-\infty}^\infty \delta_n \otimes f_n$ with $f_n \in C(\mathbb{T})$, δ_n the Kronecker delta on \mathbb{Z} , $\sum_{-\infty}^\infty \|f_n\| < \infty$, and with norm $\|\sum_{-\infty}^\infty \delta_n \otimes f_n\| = \sum_{-\infty}^\infty \|f_n\|$. On simple tensors, define a multiplication by $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes \alpha^m(f)g$ and an involution by $(\delta_n \otimes f)^* = \delta_{-n} \otimes \alpha^{-n}(f)$. These definitions make $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ into a Banach $*$ -algebra.

Definition 2.5.1. Let $\pi : C(\mathbb{T}) \rightarrow B(\mathcal{H})$ be a C^* -representation, α an automorphism of $C(\mathbb{T})$, and $V \in B(\mathcal{H})$ a unitary on \mathcal{H} . We say that (V, π) is a covariant representation of $(C(\mathbb{T}), \alpha)$ if $V\pi(\alpha(f)) = \pi(f)V \ \forall f \in C(\mathbb{T})$.

Covariant representations of $C(\mathbb{T})$ exist as Example 2.5.2 shows.

Example 2.5.2. For an automorphism α of $C(\mathbb{T})$ and a C^* -representation π of $C(\mathbb{T})$ on \mathcal{H} , define a C^* -representation ρ of $C(\mathbb{T})$ on $\mathcal{K} = \oplus_{-\infty}^\infty \mathcal{H}$ by

$$\rho(f)(\dots, \xi_{-1}, \xi_0, \xi_1, \xi_2, \dots) = (\dots, \pi(\alpha^{-1}(f))\xi_{-1}, \pi(f)\xi_0, \pi(\alpha(f))\xi_1, \pi(\alpha^2(f))\xi_2, \dots).$$

If U is the bilateral shift on \mathcal{K} , then (U, ρ) is a covariant representation of $(C(\mathbb{T}), \alpha)$. Denote the C^* -representation ρ constructed in this fashion by $\tilde{\pi}$.

It is well-known that there is a one-to-one correspondence between $*$ -representations of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ and covariant representations of $(C(\mathbb{T}), \alpha)$ on \mathcal{H} given by $\rho \leftrightarrow (V, \pi)$ where

$\rho(\sum_{-\infty}^{\infty} \delta_n \otimes f_n) = \sum_{-\infty}^{\infty} V^n \pi(f_n)$ [Ped1, Proposition 7.6.4]. Denote this representation ρ by $V \times \pi$.

Proposition 2.5.3. Let α be an automorphism of $C(\mathbb{T})$. The Banach $*$ -algebra $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ admits a faithful $*$ -representation.

Proof. Let π be a faithful representation of $C(\mathbb{T})$ on \mathcal{H} ; such a representation exists by the Gelfand-Neumark theorem. Let U be the bilateral shift on $\mathcal{K} = \oplus_{-\infty}^{\infty} \mathcal{H}$ and let $\tilde{\pi}$ be constructed as in Example 2.5.2. Then $U \times \tilde{\pi}$ is a $*$ -representation of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ on \mathcal{K} . Suppose $(U \times \tilde{\pi})(\sum_{-\infty}^{\infty} \delta_n \otimes f_n) = 0$. Then $(U \times \tilde{\pi})(\sum_{-\infty}^{\infty} \delta_n \otimes f_n)(\dots, 0, \xi_0, 0, \dots) = (\dots, 0, 0, 0, \dots) \forall \xi_0 \in \mathcal{H}$. Hence $(\dots, \pi(f_{-1})\xi_0, \pi(f_0)\xi_0, \pi(f_1)\xi_0, \pi(f_2)\xi_0, \dots) = (\dots, 0, 0, 0, \dots)$. However, π is faithful so that $f_n = 0 \forall n \in \mathbb{Z}$ and thus $U \times \tilde{\pi}$ is faithful. \square

The crossed product $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$ could then be defined as the C^* -enveloping algebra of the Banach $*$ -algebra $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ [Ped1]. However, since $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ admits a faithful $*$ -representation, there is an equivalent definition [MM] better suited for our purposes.

Definition 2.5.4. The crossed product of $C(\mathbb{T})$ by α , denoted $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$, is the completion of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ under the norm

$$\|F\| = \sup \{ \|\pi(F)\| : \pi \text{ is a (contractive) } * \text{-representation of } \ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha) \}$$

for $F \in \ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$.

Example 2.5.5 (Irrational Rotation Algebra). Let α be an automorphism on $C(\mathbb{T})$ given by composition with the map $\varphi(z) = \mu z$ where $\mu = e^{2\pi i \alpha}$ and $\alpha \in (0, 1)$ is irrational. In this case $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$ is called the irrational rotation algebra. This algebra is characterized as the unique C^* -algebra generated by any two unitaries, U and V , satisfying the twisted commutation relation $VU = \mu UV$ [Bre][Dav]. Concrete representations of this C^* -algebra appear in Example 2.5.6 and Example 2.5.7.

Example 2.5.6. For the map $\varphi(z) = \mu z$ with $\mu = e^{2\pi i \alpha}$ and $\alpha \in (0, 1)$ irrational, consider the C^* -algebra generated by the composition operator $C_{\varphi^{-1}}$ and the multiplication operator

M_z on $L^2(\mathbb{T})$. Since $M_z C_{\varphi^{-1}} = \mu C_{\varphi^{-1}} M_z$ and $C_{\varphi^{-1}}$ and M_z are unitaries on $L^2(\mathbb{T})$ it follows that this is the irrational rotation algebra $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$.

Example 2.5.7. Let α be the automorphism of composition with the map φ given in Example 2.5.6. Let $\mathcal{H} = \bigoplus_{-\infty}^{\infty} L^2(\mathbb{T})$. For $f \in C(\mathbb{T})$, define

$$D_f(\dots, \xi_{-1}, \xi_0, \xi_1, \xi_2, \dots) = (\dots, \alpha^{-1}(f)\xi_{-1}, f\xi_0, \alpha(f)\xi_1, \alpha^2(f)\xi_2, \dots).$$

If U is the bilateral shift on \mathcal{H} then it is easy to check that U and D_z are unitaries on \mathcal{H} satisfying $D_z U = \mu U D_z$. Hence the C^* -algebra generated by U and D_z is the irrational rotation algebra $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$.

The irrational rotation algebra has been well-studied and many interesting facts are known about them [Rie1] [Rie2]. A good overview can be found in Fillmore [Fil2] or Davidson [Dav].

2.6 Ergodic Theory

Let X be a set, \mathcal{B} a σ -algebra of subsets of X , and $m : \mathcal{B} \rightarrow \mathbb{R}^+$ a measure with $m(X) = 1$ so that (X, \mathcal{B}, m) is a probability space. A map $\varphi : X \rightarrow X$ is measurable if $\varphi^{-1}(B) \in \mathcal{B} \forall B \in \mathcal{B}$. φ is measure preserving if φ is measurable and $m(\varphi^{-1}(B)) = m(B) \forall B \in \mathcal{B}$. If $\varphi^{-1}(B) = B$ for $B \in \mathcal{B}$, then $\varphi^{-1}(X \setminus B) = X \setminus B$ and φ can be studied in two parts; namely $\varphi|_B$ and $\varphi|_{X \setminus B}$. Ergodicity is a concept of irreducibility for measure preserving transformations φ such that if φ has this property then the study of φ cannot be split into two parts.

Definition 2.6.1. $\varphi : (X, \mathcal{B}, m) \rightarrow (X, \mathcal{B}, m)$ is ergodic if for $B \in \mathcal{B}$, $\varphi^{-1}B = B \Rightarrow m(B) = 0$ or $m(B) = 1$.

For $\varphi : (X, \mathcal{B}, m) \rightarrow (X, \mathcal{B}, m)$ measure preserving, it can be shown that φ is ergodic if and only if whenever $f \in L^2(m)$ and $(f \circ \varphi)(x) = f(x) \forall x \in X$ then f is constant on X .

Example 2.6.2. Consider the map $\varphi : \mathbb{T} \rightarrow \mathbb{T}$ given by $\varphi(z) = \mu z$ where $\mu = e^{2\pi i \alpha}$ and $\alpha \in (0, 1)$ is irrational. With m as normalized Lebesgue measure on \mathbb{T} , φ is clearly measure preserving as it is just a rotation of \mathbb{T} . Moreover, φ can be shown to be ergodic by using the fact that $f \in L^2(m)$ and $(f \circ \varphi)(z) = f(z)$ on \mathbb{T} implies f is constant on \mathbb{T} .

Theorem 2.6.3 (Birkhoff Ergodic Theorem). Let $\varphi : (X, \mathcal{B}, m) \rightarrow (X, \mathcal{B}, m)$ be measure preserving. Let $f \in L^1(m)$. Define the time mean of f to be

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))$$

and the space mean of f to be

$$\frac{1}{m(X)} \int_X f \, dm.$$

Then, if φ is ergodic, the time mean and the space mean are equal.

Example 2.6.4. For φ as in Example 2.6.2, and $f \in \mathcal{A}(\mathbb{D})$, Theorem 2.6.3 applies since $f \in L^1(m)$. Hence, $\forall z \in \overline{\mathbb{D}}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\mu^i z) \longrightarrow \int_{\mathbb{T}} f \, dm(z) = f(0).$$

In fact, this convergence is uniform since φ is uniquely ergodic [Wal, Theorem 5.17].

For a more detailed account of ergodic theory and topological dynamics, see Walters [Wal].

CHAPTER 3

DILATION THEORY

3.1 Hilbert Modules

Definition 3.1.1. Let \mathfrak{A} be a unital algebra over \mathbb{C} . A unitary (left) module \mathfrak{X} over \mathfrak{A} is an additive abelian group together with a function $\mathfrak{A} \times \mathfrak{X} \rightarrow \mathfrak{X}$ (the image of (a, x) being denoted $a \cdot x$) such that $\forall \alpha \in \mathbb{C}, a, b \in \mathfrak{A}$, and $x, y \in \mathfrak{X}$

- (i) $a \cdot (x + y) = a \cdot x + a \cdot y$
- (ii) $(a + b) \cdot x = a \cdot x + b \cdot x$
- (iii) $a \cdot (b \cdot x) = (ab) \cdot x$
- (iv) $1_{\mathfrak{A}} \cdot x = x$
- (v) $\alpha(a \cdot x) = (\alpha a) \cdot x = a \cdot (\alpha x)$

Let \mathfrak{X} and \mathfrak{Y} be modules over a unital algebra \mathfrak{A} . A module homomorphism is a function $\phi : \mathfrak{X} \rightarrow \mathfrak{Y}$ such that $\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$ and $\phi(a \cdot x_1) = a \cdot \phi(x_1) \forall x_1, x_2 \in \mathfrak{X}, a \in \mathfrak{A}$. If ϕ is a bijection, we call ϕ a module isomorphism. A pair of module homomorphisms, $\mathfrak{X} \xrightarrow{\phi} \mathfrak{Y} \xrightarrow{\psi} \mathfrak{Z}$, is exact at \mathfrak{Y} provided $\text{Im } \phi = \text{Ker } \psi$. A finite sequence of module homomorphisms, $\mathfrak{X}_0 \xrightarrow{\phi_1} \mathfrak{X}_1 \xrightarrow{\phi_2} \mathfrak{X}_2 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{n-1}} \mathfrak{X}_{n-1} \xrightarrow{\phi_n}$ is exact provided $\text{Im } \phi_i = \text{Ker } \phi_{i+1}$ for $i = 1, 2, \dots, n-1$. An exact sequence of the form $0 \rightarrow \mathfrak{X} \xrightarrow{\phi} \mathfrak{Y} \xrightarrow{\psi} \mathfrak{Z} \rightarrow 0$ is a short exact sequence. If there exists a module homomorphism $\tilde{\psi} : \mathfrak{Z} \rightarrow \mathfrak{Y}$ with $\psi\tilde{\psi} = 1_{\mathfrak{Z}}$, then we say that this short exact sequence splits or is split exact. If every short exact sequence $0 \rightarrow \mathfrak{X} \xrightarrow{\phi} \mathfrak{Y} \xrightarrow{\psi} \mathcal{P} \rightarrow 0$ ending at \mathcal{P} is split exact we say that \mathcal{P} is projective. Equivalently, a module \mathcal{P} is projective if given any diagram of module homomorphisms

$$\begin{array}{ccccc}
& & \mathcal{P} & & \\
& & \downarrow \phi & & \\
\mathfrak{X} & \xrightarrow{\psi} & \mathfrak{Y} & \longrightarrow & 0
\end{array}$$

with bottom row exact, there exists a module homomorphism $\tilde{\psi} : \mathcal{P} \rightarrow \mathfrak{X}$ such that the diagram commutes.

Definition 3.1.2. A Hilbert module over a (unital) operator algebra \mathfrak{A} is a Hilbert space \mathcal{H} which is an algebraic (left) module over \mathfrak{A} such that there exists $K > 0$ with $\|f \cdot \xi\| \leq K \|f\| \|\xi\| \forall f \in \mathfrak{A}$ and $\xi \in \mathcal{H}$.

Note that the norm condition ensures that the module multiplication is continuous. If the above inequality holds for $K = 1$ we call \mathcal{H} a contractive Hilbert module.

Recall that a representation of a (unital) operator algebra \mathfrak{A} on a Hilbert space \mathcal{H} is a continuous homomorphism $\rho : \mathfrak{A} \rightarrow B(\mathcal{H})$ such that $\rho(1_{\mathfrak{A}}) = I_{\mathcal{H}}$. The continuity, equivalently boundedness, of ρ implies there exists $K \geq 1$ such that $\|\rho(f)\xi\| \leq K \|f\| \|\xi\| \forall f \in \mathfrak{A}$ and $\xi \in \mathcal{H}$. It then becomes clear that there is a natural correspondence between (contractive) Hilbert modules \mathcal{H} over \mathfrak{A} and (contractive) representations of \mathfrak{A} on \mathcal{H} . A (contractive) representation ρ of \mathfrak{A} on \mathcal{H} makes \mathcal{H} into a (contractive) Hilbert module if the module action is defined by $f \cdot \xi = \rho(f)\xi$ for $f \in \mathfrak{A}$ and $\xi \in \mathcal{H}$. Conversely, given a (contractive) Hilbert module \mathcal{H} over \mathfrak{A} , a (contractive) representation ρ of \mathfrak{A} on \mathcal{H} can be constructed by $\rho(f)\xi = f \cdot \xi$.

Some contractive representations satisfy the following stronger condition:

Definition 3.1.3. Let \mathfrak{A} be an operator algebra and ρ a contractive representation of \mathfrak{A} on \mathcal{H} . For $n \geq 1$, form the tensor product $\mathfrak{A} \otimes M_n(\mathbb{C})$ (as a subalgebra of $C^*(\mathfrak{A}) \otimes M_n(\mathbb{C})$). Define $\rho_n : \mathfrak{A} \otimes M_n(\mathbb{C}) \rightarrow B(\mathcal{H}) \otimes M_n(\mathbb{C})$ by $(a_{ij}) \mapsto (\rho(a_{ij}))$. Then ρ is said to be completely contractive if $\|\rho_n\| = 1 \forall n \geq 1$.

Thinking of $\mathfrak{A} \otimes M_n(\mathbb{C})$ as matrices with entries from \mathfrak{A} we view $\rho_n(a_{ij})$ as an operator on $\oplus_1^n \mathcal{H}$ by $\rho_n(a_{ij})(\xi_1, \xi_2, \dots, \xi_n) = [\rho(a_{ij})]_{n \times n}(\xi_1, \xi_2, \dots, \xi_n)$. The norm on $\mathfrak{A} \otimes M_n(\mathbb{C})$ is that unique norm it inherits from $C^*(\mathfrak{A}) \otimes M_n(\mathbb{C})$ [Mur, Theorem 3.4.2].

Example 3.1.4. Let $T \in B(\mathcal{H})$ and let \mathcal{P} denote the algebra of polynomials in $\mathcal{A}(\mathbb{D})$. Then \mathcal{H} becomes a module over \mathcal{P} with module action given by $p \cdot \xi = p(T)\xi$ for $p \in \mathcal{P}$ and $\xi \in \mathcal{H}$. This extends to make \mathcal{H} into a contractive Hilbert module over $\mathcal{A}(\mathbb{D})$ if and only if $\|p(T)\| \leq \|p(z)\|_{\mathcal{A}(\mathbb{D})} \forall p \in \mathcal{P}$. Theorem 3.2.5 will show that this condition holds if and only if T is a contraction on \mathcal{H} . Hence it will follow that the contractive Hilbert modules \mathcal{H} over $\mathcal{A}(\mathbb{D})$ are in one-to-one correspondence with contractions T on \mathcal{H} . In fact, by Theorem 3.5.9 all the contractive Hilbert modules over $\mathcal{A}(\mathbb{D})$ are completely contractive. To prove Theorem 3.2.5, the notion of a dilation is introduced.

3.2 Dilating a Contraction

Definition 3.2.1. Let $T \in B(\mathcal{H})$ and $U \in B(\mathcal{K})$ where $\mathcal{H} \subseteq \mathcal{K}$, $P_{\mathcal{H}} \in B(\mathcal{K})$ is the projection operator onto \mathcal{H} , and $T^n h = P_{\mathcal{H}} U^n h \forall h \in \mathcal{H}$ and $n \geq 1$. U is called a power dilation of T . Furthermore, U is an isometric (resp. unitary) dilation if U is an isometry (resp. unitary).

From this point forward the term dilation will be used for power dilation without confusion.

Definition 3.2.2. $U \in B(\mathcal{K})$ is a minimal isometric dilation of $T \in B(\mathcal{H})$ if U is an isometric dilation of T and $\mathcal{K} = \bigvee_0^\infty U^n \mathcal{H}$. $U \in B(\mathcal{K})$ is a minimal unitary dilation of $T \in B(\mathcal{H})$ if U is a unitary dilation of T and $\mathcal{K} = \bigvee_{-\infty}^\infty U^n \mathcal{H}$.

The existence of minimal and isometric dilations is the consequence of Theorem 3.2.3. For a proof, as well as a short history of the problem, the interested reader should see Chapter I of Sz. Nagy and Foias [Sz-NF].

Theorem 3.2.3. Let T be a contraction on a Hilbert space \mathcal{H} . Then there exists a minimal isometric (resp. unitary) dilation U on $\mathcal{K} \supseteq \mathcal{H}$ which is unique up to unitary equivalence.

Remark 3.2.4. Let φ be a conformal bijection of \mathbb{D} of the form $\varphi(z) = \frac{z-a}{1-\bar{a}z}$ for $|a| < 1$. The operator $\varphi(T)$ defined by $(T-aI)(I-\bar{a}T)^{-1}$ is a contraction. If T is an isometry (resp. unitary) then $\varphi(T)$ is an isometry (resp. unitary). Moreover, if U is an isometric (resp. unitary) dilation of T then $\varphi(U)$ is an isometric (resp. unitary) dilation of $\varphi(T)$. In fact, if U is the minimal such dilation, then so is $\varphi(U)$ [Sz-NF, Proposition 4.3].

Theorem 3.2.5 (von Neumann's Inequality). Let $T \in B(\mathcal{H})$. Then $\|p(T)\| \leq \|p(z)\|_{\mathcal{A}(\mathbb{D})}$ for all polynomials $p(z)$ if and only if T is a contraction.

Proof. One direction of the proof is clear. Let then $T \in B(\mathcal{H})$ be a contraction with unitary dilation $U \in B(\mathcal{K})$, $\mathcal{K} \supseteq \mathcal{H}$. Since $T^n h = P_{\mathcal{H}} U^n h \ \forall \ n \geq 1$ and $h \in \mathcal{H}$, it follows that $p(T)h = P_{\mathcal{H}} p(U)h$ for any polynomial $p(z)$ and $h \in \mathcal{H}$. If $sp(U)$ denotes the spectrum of U , then

$$\|p(T)\| \leq \|p(U)\| = \max_{\lambda \in sp(U)} |p(\lambda)| \leq \max_{|\lambda|=1} |p(\lambda)| = \|p(z)\|_{\mathcal{A}(\mathbb{D})}$$

by the spectral representation of a unitary operator and the maximum modulus theorem. \square

A partial answer to the problem of characterizing the Hilbert modules \mathcal{H} over $\mathcal{A}(\mathbb{D})$ described in Example 3.1.4 can now be given. These modules are in one-to-one correspondence with contractions on \mathcal{H} . The problem of classifying the contractive Hilbert modules over $\mathcal{A}(\mathbb{D})$ is then equivalent to classifying contractions up to unitary equivalence.

3.3 Dilating Commuting Pairs of Contractions

Example 3.3.1. The bidisk algebra $\mathcal{A}(\mathbb{D}^2)$ is the closure in $C(\overline{\mathbb{D}^2})$ of the polynomials in the coordinate functions z_1 and z_2 . If T_1 and T_2 are operators on a Hilbert space \mathcal{H} which commute, then \mathcal{H} becomes a module over the algebra \mathcal{P} of polynomials $p(z_1, z_2)$ by defining the module action $p(z_1, z_2) \cdot \xi = p(T_1, T_2)\xi$ for $\xi \in \mathcal{H}$. In order to make this module into a contractive Hilbert module over $\mathcal{A}(\mathbb{D}^2)$, it is clearly necessary that T_1 and T_2 are contractions. That the condition is sufficient as well results from Theorem 3.3.3 due to Andô [And].

Definition 3.3.2. Let $\{T_1, T_2\}$ be commuting operators in $B(\mathcal{H})$. The pair of operators $\{U_1, U_2\}$ in $B(\mathcal{K})$ is called a (power) dilation of $\{T_1, T_2\}$ if $\mathcal{H} \subseteq \mathcal{K}$, $U_2 U_1 = U_1 U_2$, and $T_2^n T_1^m = P_{\mathcal{H}} U_2^n U_1^m \ \forall \ m, n \geq 0$. The pair $\{U_1, U_2\}$ is said to be an isometric (resp. unitary) dilation if U_1 and U_2 are isometries (resp. unitaries).

The existence of an isometric dilation $\{U_1, U_2\}$ of a pair of commuting contractions $\{T_1, T_2\}$ follows from Theorem 3.3.3. Andô's theorem will follow as a special case of Theorem 3.4.4, the proof of which is modified from Andô's work.

Theorem 3.3.3 (Andô's Theorem). Every commuting pair of contractions has an isometric (*resp.* unitary) dilation.

Remark 3.3.4. For commuting contractions $\{T_1, T_2\}$ on \mathcal{H} , the isometric dilation $\{U_1, U_2\}$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ guaranteed from Theorem 3.3.3 can be taken to be minimal in the sense that $\mathcal{K} = \bigvee_{m,n \geq 0} U_1^m U_2^n \mathcal{H}$. However, unlike the case of a single contraction, not all minimal isometric dilations of $\{T_1, T_2\}$ are unitarily equivalent [Sz-NF].

Remark 3.3.5. A suspected generalization of Andô's Theorem does not hold. If $\{T_1, T_2, T_3\}$ is a system of commuting contractions on \mathcal{H} , there is not necessarily a system $\{U_1, U_2, U_3\}$ of commuting unitaries on $\mathcal{K} \supseteq \mathcal{H}$ satisfying $T_i h = P_{\mathcal{H}} U_i h \ \forall h \in \mathcal{H}$ and $i = 1, 2, 3$ [Par].

Remark 3.3.6. A generalization of von Neumann's inequality to two commuting contractions $\{T_1, T_2\}$ holds; that is, if $p(z_1, z_2)$ is a polynomial in two variables, then

$$\|p(T_1, T_2)\| \leq \sup\{|p(z_1, z_2)| : |z_1| \leq 1, |z_2| \leq 1\}.$$

Note that this yields the fact that the contractive Hilbert modules \mathcal{H} over $\mathcal{A}(\mathbb{D}^2)$ are in one-to-one correspondence with pairs of commuting contractions on \mathcal{H} . In general, von Neumann's inequality does not hold for a system $\{T_1, T_2, T_3\}$ of commuting contractions. That is to say that there exists three such commuting contractions and a polynomial $p(z_1, z_2, z_3)$ satisfying

$$\|p(T_1, T_2, T_3)\| > \sup\{|p(z_1, z_2, z_3)| : |z_i| \leq 1, i = 1, 2, 3\}.$$

For the construction of these contractions, see Crabb and Davie [CD].

One of the best known results of dilation theory is the commutant lifting theorem. It is an amazing fact that Andô's theorem and the commutant lifting theorem can each be used to prove each other [FF]. Before the commutant lifting theorem can be stated, a few definitions are needed.

Definition 3.3.7. An operator $B : \mathcal{K} \rightarrow \mathcal{K}'$ is a lifting of $A : \mathcal{H} \rightarrow \mathcal{H}'$ if $\mathcal{H} \subseteq \mathcal{K}$, $\mathcal{H}' \subseteq \mathcal{K}'$, and $P_{\mathcal{H}'} B = A P_{\mathcal{H}}$.

The operator B lifts A if and only if $B : \mathcal{H} \oplus \mathcal{H}^\perp \rightarrow \mathcal{H}' \oplus \mathcal{H}'^\perp$ admits a matrix representation of the form

$$\begin{pmatrix} A & 0 \\ D & C \end{pmatrix}.$$

We say that B extends A if $B|_{\mathcal{H}} = A$; this occurs when B has matrix representation

$$\begin{pmatrix} A & D \\ 0 & C \end{pmatrix}.$$

Hence, B extends A if and only if B^* lifts A^* .

Definition 3.3.8. Let $T \in B(\mathcal{H})$ and $T' \in B(\mathcal{H}')$ be contractions. Define the intertwining set of T and T' by

$$I(T, T') = \{A : \mathcal{H} \rightarrow \mathcal{H}' : \|A\| \leq 1 \text{ and } T'A = AT\}.$$

For $U \in B(\mathcal{K})$ and $U' \in B(\mathcal{K}')$ the minimal isometric dilations of T and T' respectively, we say that B is a contractive intertwining lifting of A if

- (i) $B \in I(U, U')$
- (ii) B is contractive with $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$ (i.e. B is a lifting of A).

The existence of a contractive intertwining lifting of a pair of contractions T and T' is the substance of the commutant lifting theorem. Four different proofs of this result, one of which relies upon Andô's theorem, can be found in Chapter VII of Foias and Frazho [FF].

Theorem 3.3.9 (Commutant Lifting). Let $T \in B(\mathcal{H})$ and $T' \in B(\mathcal{H}')$ be contractions. If $A \in I(T, T')$, then there exists a contractive intertwining lifting B of A .

3.4 Dilating Noncommuting Pairs of Contractions

In this section we show that if φ is a conformal bijection of the disk and S and T are contractions satisfying $TS = S\varphi(T)$ then $\{S, T\}$ has a power dilation $\{U, V\}$. First, two different proofs are given for the case where φ is a rotation.

Definition 3.4.1. Let $\{S, T\}$ be a pair of contractions satisfying $TS = S\varphi(T)$ where φ is a conformal bijection of \mathbb{D} . A pair of operators $\{U, V\}$ in $B(\mathcal{K})$ will be called a power φ -dilation of $\{S, T\}$ provided $\mathcal{H} \subseteq \mathcal{K}$, $VU = U\varphi(V)$, and $T^n S^m h = P_{\mathcal{H}} V^n U^m h \forall m, n \geq 0$ and $h \in \mathcal{H}$. A power φ -dilation will be called isometric (resp. unitary) if U and V are isometries (resp. unitaries).

Definition 3.4.2. Let $T \in B(\mathcal{H})$ be a contraction. Denote by D_T the defect operator given by $D_T = (I_{\mathcal{H}} - T^*T)^{1/2}$.

Note that D_T is well-defined as $T^*T \leq I_{\mathcal{H}}$. Also, $D_T^* = D_T$ and $0 \leq D_T \leq I_{\mathcal{H}}$. In some sense, D_T measures how far T is from being isometric.

Lemma 3.4.3. $\|D_T h\|^2 = \|h\|^2 - \|Th\|^2$.

Proof. Since $D_T^* = D_T$, $\|D_T h\|^2 = \langle D_T h, D_T h \rangle = \langle D_T^2 h, h \rangle = \langle h - T^*Th, h \rangle = \|h\|^2 - \|Th\|^2$. □

The proof of Theorem 3.4.4 is a simple modification of Andô's proof of Theorem 3.3.3.

Theorem 3.4.4. Let $\varphi(z) = \mu z$ where $|\mu| = 1$. Then any set of contractions $\{S, T\}$ in $B(\mathcal{H})$ satisfying $TS = S\varphi(T)$ has an *isometric φ -dilation* $\{U, V\}$ on \mathcal{K} .

Proof. Define $\mathcal{H}_+ = \bigoplus_{i=0}^{\infty} \mathcal{H}$. Define on \mathcal{H}_+ the operators \tilde{S} and \tilde{T} by

$$\begin{aligned} \tilde{S}\{h_0, h_1, h_2, \dots\} &= \{Sh_0, D_S h_0, 0, h_1, h_2, \dots\} \\ \tilde{T}\{h_0, h_1, h_2, \dots\} &= \{Th_0, D_T h_0, 0, h_1, h_2, \dots\} \end{aligned}$$

It follows by Lemma 3.4.3 that \tilde{S} and \tilde{T} are isometric, but \tilde{S} and \tilde{T} do not satisfy $\tilde{T}\tilde{S} = \mu\tilde{S}\tilde{T}$.

Form the space $\mathcal{G} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. By the natural identification

$$\{h_0, h_1, h_2, \dots\} = \{h_0, \{h_1, h_2, h_3, h_4\}, \{h_5, h_6, h_7, h_8\}, \dots\}$$

we have

$$\mathcal{H}_+ = \mathcal{H} \oplus \mathcal{G} \oplus \mathcal{G} \oplus \dots$$

For $n \geq 1$, let G_n be a unitary operator on \mathcal{G} (to be specified later in the proof). Let $G : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ be given by

$$G\{h_0, h_1, h_2, \dots\} = \{h_0, G_1\{h_1, h_2, h_3, h_4\}, G_2\{h_5, h_6, h_7, h_8\}, G_3\{h_9, \dots, h_{12}\}, \dots\}.$$

Then, G is also unitary and its inverse is given by

$$G^{-1}\{h_0, h_1, h_2, \dots\} = \{h_0, G_1^{-1}\{h_1, h_2, h_3, h_4\}, G_2^{-1}\{h_5, h_6, h_7, h_8\}, G_3^{-1}\{h_9, \dots, h_{12}\}, \dots\}.$$

Define operators U and V on \mathcal{H}_+ by

$$U = G\tilde{S} \text{ and } V = \tilde{T}G^{-1}$$

Note that these are both isometries on \mathcal{H}_+ . A simple computation shows that

$$\begin{aligned} UV\{h_0, h_1, h_2, \dots\} &= G\tilde{S}\tilde{T}G^{-1}\{h_0, h_1, h_2, \dots\} \\ &= G\tilde{S}\tilde{T}\{h_0, G_1^{-1}\{h_1, h_2, h_3, h_4\}, G_2^{-1}\{h_5, h_6, h_7, h_8\}, \dots\} \\ &= G\tilde{S}\{Th_0, D_Th_0, 0, G_1^{-1}\{h_1, h_2, h_3, h_4\}, G_2^{-1}\{h_5, h_6, h_7, h_8\}, \dots\} \\ &= G\{STh_0, D_STh_0, 0, D_Th_0, 0, G_1^{-1}\{h_1, h_2, h_3, h_4\}, G_2^{-1}\{h_5, h_6, h_7, h_8\}, \dots\} \\ &= \{STh_0, G_1\{D_STh_0, 0, D_Th_0, 0\}, G_2G_1^{-1}\{h_1, h_2, h_3, h_4\}, \\ &\quad G_3G_2^{-1}\{h_5, h_6, h_7, h_8\}, \dots\} \end{aligned}$$

and

$$\begin{aligned} VU\{h_0, h_1, h_2, \dots\} &= \tilde{T}GG^{-1}\tilde{S}\{h_0, h_1, h_2, \dots\} \\ &= \tilde{T}\{Sh_0, D_Sh_0, 0, h_1, h_2, \dots\} \\ &= \{TSh_0, D_TSh_0, 0, D_Sh_0, 0, h_1, h_2, \dots\} \end{aligned}$$

Since $TS = \mu ST$ by assumption, the relation $VU = \mu UV$ will hold if and only if we can choose G_n ($n \geq 1$) to satisfy

$$\mu G_1\{D_STh_0, 0, D_Th_0, 0\} = \{D_TSh_0, 0, D_Sh_0, 0\} \quad (3.1)$$

and $\forall n \geq 2$,

$$\mu G_n G_{n-1}^{-1}\{h_{4n-7}, h_{4n-6}, h_{4n-5}, h_{4n-4}\} = \{h_{4n-7}, h_{4n-6}, h_{4n-5}, h_{4n-4}\}. \quad (3.2)$$

For $n \geq 2$, the choice of unitary G_n satisfying (3.2) is clear; it is $G_n = \bar{\mu}G_{n-1}$. So, we need only show we can choose a unitary G_1 to satisfy (3.1). But, by Lemma 3.4.3,

$$\begin{aligned}
\|D_STh_0\|^2 + \|D_Th_0\|^2 &= \|Th_0\|^2 - \|STh_0\|^2 + \|h_0\|^2 - \|Th_0\|^2 \\
&= \|h_0\|^2 - \|STh_0\|^2 \\
&= \|h_0\|^2 - \|\bar{\mu}TSh_0\|^2 \\
&= \|\bar{\mu}Sh_0\|^2 - \|\bar{\mu}TSh_0\|^2 + \|\bar{\mu}h_0\|^2 - \|\bar{\mu}Sh_0\|^2 \\
&= \|\bar{\mu}D_TSh_0\|^2 + \|\bar{\mu}D_SSh_0\|^2
\end{aligned}$$

So, $\|\{D_STh, 0, D_Th, 0\}\| = \|\bar{\mu}\{D_TSh, 0, D_SSh, 0\}\|$ for all $h \in \mathcal{H}$. Let \mathcal{L}_1 be the subspace of vectors of the form $\{D_STh, 0, D_Th, 0\}$. Let \mathcal{L}_2 be the subspace of vectors of the form $\bar{\mu}\{D_TSh, 0, D_SSh, 0\}$. Then, by defining $G_1 : \mathcal{L}_1 \rightarrow \mathcal{G}$ by

$$G_1\{D_STh, 0, D_Th, 0\} = \bar{\mu}\{D_TSh, 0, D_SSh, 0\}$$

we see that G_1 is isometric onto \mathcal{L}_2 . By continuity, G_1 can be extended to an isometry from $\mathcal{M}_1 = \overline{\mathcal{L}_1}$ onto $\mathcal{M}_2 = \overline{\mathcal{L}_2}$, the closures of \mathcal{L}_1 and \mathcal{L}_2 respectively.

We now show that G_1 can be extended to an isometry of \mathcal{G} onto itself. For this, we need only show that $\mathcal{M}_1^\perp = \mathcal{G} \ominus \mathcal{M}_1$ and $\mathcal{M}_2^\perp = \mathcal{G} \ominus \mathcal{M}_2$ have the same dimension. If \mathcal{H} has finite dimension, so does \mathcal{G} . But then $\dim \mathcal{M}_1 = \dim \mathcal{M}_2 \Rightarrow \dim \mathcal{M}_1^\perp = \dim \mathcal{M}_2^\perp$. If $\dim \mathcal{H}$ is infinite, then $\dim \mathcal{H} = \dim \mathcal{G} \geq \dim \mathcal{M}_i^\perp \geq \dim \mathcal{H}$ ($i = 1, 2$) since both \mathcal{M}_1^\perp and \mathcal{M}_2^\perp contain the subspace of all vectors of the form $\{\{0, h, 0, 0\} : h \in \mathcal{H}\}$. Hence, $\dim \mathcal{H} = \dim \mathcal{M}_1^\perp = \dim \mathcal{M}_2^\perp$. So, if the unitary G_1 is determined (and hence so to the unitaries G_n for $n \geq 2$), then U and V will be isometries on $\mathcal{H}_+ \supseteq \mathcal{H}$ satisfying $VU = \mu UV$. Further,

$$\begin{aligned}
U\{h_0, h_1, h_2, \dots\} &= \{Sh_0, \dots\} \\
V\{h_0, h_1, h_2, \dots\} &= \{Th_0, \dots\}
\end{aligned}$$

and so

$$V^n U^m \{h_0, h_1, h_2, \dots\} = \{T^n S^m h_0, \dots\}$$

for $m, n \geq 0$. Hence,

$$\begin{aligned} P_{\mathcal{H}} V^n U^m \{h, 0, 0, \dots\} &= P_{\mathcal{H}} \{T^n S^m h, \dots\} \\ &= T^n S^m h \end{aligned}$$

$\forall h \in \mathcal{H}$ and $m, n \geq 0$. The pair $\{U, V\}$ on \mathcal{H}_+ is then a power isometric φ -dilation of $\{S, T\}$. \square

It is not clear how to generalize the proof of Theorem 3.4.4 to dilate contractions satisfying $TS = S\varphi(T)$ for *any* conformal bijection φ of the disk. One might surmise that the commutant lifting theorem might be of some use. In fact, Theorem 3.4.4 can also be proven by two applications of the commutant lifting theorem. However, since it is not clear how T intertwines any suitable functions of S for contractions S and T satisfying $TS = S\varphi(T)$, this proof too seems to fail to yield a more general result.

Second proof of Theorem 3.4.4. Let \tilde{T} be the minimal isometric dilation of T on $\mathcal{K} = \bigvee_0^\infty \tilde{T}(\mathcal{H})$. By Remark 3.2.4, $\mu\tilde{T}$ is the minimal isometric dilation of μT . Since $S \in I(\mu T, T)$, the commutant lifting theorem yields the existence of a contractive intertwining lifting \tilde{S} of S on \mathcal{K} . In particular, $\tilde{T}\tilde{S} = \mu\tilde{S}\tilde{T}$.

Let U be the minimal isometric dilation of \tilde{S} on $\mathcal{L} = \bigvee_0^\infty U^n(\mathcal{K})$. Since $\tilde{T} \in I(\tilde{S}, \mu\tilde{S})$, a second application of the commutant lifting theorem guarantees a contractive lifting $V \in I(U, \mu U)$ of \tilde{T} on \mathcal{L} . In particular, $VU = \mu UV$ and V is a lifting of \tilde{T} (i.e. $P_{\mathcal{K}}V = \tilde{T}P_{\mathcal{K}}$). Note that $\|k\| = \|\tilde{T}P_{\mathcal{K}}k\| = \|P_{\mathcal{K}}V k\| \leq \|V k\| \leq \|k\| \ \forall k \in \mathcal{K} \Rightarrow \|V k\| = \|k\| \ \forall k \in \mathcal{K}$. Thus,

$$\|V U^n k\| = \|\mu^n U^n V k\| = \|V k\| = \|k\| = \|U^n k\|$$

for all $k \in \mathcal{K}$ and $n \geq 0$. By Lemma 3.4.3, $\|D_V U^n k\|^2 = \|U^n k\|^2 - \|V U^n k\|^2 = 0$ so that $D_V = 0$ on \mathcal{L} . Hence, V is an isometry on \mathcal{L} . Also note that V lifts T since V lifts \tilde{T} and \tilde{T} lifts T . Similarly, U lifts S . It follows that

$$\begin{aligned} T^n S^m h &= T^n P_{\mathcal{H}} U^m h \\ &= P_{\mathcal{H}} V^n U^m h \end{aligned}$$

$\forall h \in \mathcal{H}$ so that $\{U, V\}$ is a power φ -dilation of $\{S, T\}$. \square

Theorem 3.4.5. Let $\varphi(z) = \mu z$ where $|\mu| = 1$. Then contractions $\{S, T\}$ satisfying $TS = S\varphi(T)$ have a power unitary φ -dilation $\{U, V\}$.

Proof. Let $\{\tilde{S}, \tilde{T}\}$ in $B(\mathcal{K})$ be a strong isometric φ -dilation of $\{S, T\}$ (guaranteed by Theorem 3.4.4). First, extend \tilde{S} to a unitary operator \tilde{U} on some $\mathcal{L} \supseteq \mathcal{K}$. Choose this extension to be minimal (see Theorem 3.2.3) in the sense that

$$\mathcal{L} = \bigvee_{-\infty}^{\infty} \tilde{U}^n(\mathcal{K}). \quad (3.3)$$

Then, for every finite sum $\sum_n (\mu \tilde{U})^n \tilde{T} k_n$ where $k_n \in \mathcal{K}$ we have

$$\begin{aligned} \left\| \sum_n (\mu \tilde{U})^n \tilde{T} k_n \right\|^2 &= \sum_n \sum_m \left\langle \mu^n \tilde{U}^n \tilde{T} k_n, \mu^m \tilde{U}^m \tilde{T} k_m \right\rangle \\ &= \sum_{n \geq m} \left\langle \mu^n \tilde{U}^n \tilde{T} k_n, \mu^m \tilde{U}^m \tilde{T} k_m \right\rangle + \sum_{n < m} \left\langle \mu^n \tilde{U}^n \tilde{T} k_n, \mu^m \tilde{U}^m \tilde{T} k_m \right\rangle \\ &= \sum_{n \geq m} \left\langle \mu^{n-m} \tilde{U}^{n-m} \tilde{T} k_n, \tilde{T} k_m \right\rangle + \sum_{n < m} \left\langle \tilde{T} k_n, \mu^{m-n} \tilde{U}^{m-n} \tilde{T} k_m \right\rangle \\ &= \sum_{n \geq m} \left\langle \mu^{n-m} \tilde{S}^{n-m} \tilde{T} k_n, \tilde{T} k_m \right\rangle + \sum_{n < m} \left\langle \tilde{T} k_n, \mu^{m-n} \tilde{S}^{m-n} \tilde{T} k_m \right\rangle \end{aligned}$$

since \tilde{U} extends \tilde{S} and $\tilde{S} = \tilde{U}$ on \mathcal{K} . But, for $p \geq 0$, $\tilde{T} \tilde{S}^p = \mu^p \tilde{S}^p \tilde{T}$ since $\{\tilde{S}, \tilde{T}\}$ is a power isometric φ -dilation of $\{S, T\}$. Hence, since \tilde{T} is an isometry,

$$\begin{aligned} \left\| \sum_n (\mu \tilde{U})^n \tilde{T} k_n \right\|^2 &= \sum_{n \geq m} \left\langle \tilde{T} \tilde{S}^{n-m} k_n, \tilde{T} k_m \right\rangle + \sum_{n < m} \left\langle \tilde{T} k_n, \tilde{T} \tilde{S}^{m-n} k_m \right\rangle \\ &= \sum_{n \geq m} \left\langle \tilde{S}^{n-m} k_n, k_m \right\rangle + \sum_{n < m} \left\langle k_n, \tilde{S}^{m-n} k_m \right\rangle \\ &= \sum_{n \geq m} \left\langle \tilde{U}^{n-m} k_n, k_m \right\rangle + \sum_{n < m} \left\langle k_n, \tilde{U}^{m-n} k_m \right\rangle \\ &= \left\| \sum_n \tilde{U}^n k_n \right\|^2 \end{aligned}$$

Define a linear subspace

$$\mathcal{M} = \left\{ \sum_n \tilde{U}^n k_n : k_n \in \mathcal{K}, n \text{ belonging to a finite subset of } \mathbb{Z} \right\}.$$

Define $\tilde{V} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\tilde{V} \left(\sum_n \tilde{U}^n k_n \right) = \sum_n \tilde{U}^n (\tilde{T} \mu^n k_n) \quad (3.4)$$

The preceeding calculations show that \tilde{V} is isometric. Further, $\overline{\mathcal{M}} = \mathcal{L}$ by (3.3). Thus, \tilde{V} extends by continuity to an isometry on \mathcal{L} . For $k \in \mathcal{K}$ and $n \in \mathbb{Z}$, we have by (3.4),

$$\begin{aligned} \mu \tilde{U} \tilde{V}(\tilde{U}^n k) &= \mu \tilde{U}(\tilde{U}^n \tilde{T} \mu^n k) \\ &= \tilde{U}^{n+1}(\tilde{T} \mu^{n+1} k) \\ &= \tilde{V}(\tilde{U}^{n+1} k) \\ &= \tilde{V} \tilde{U}(\tilde{U}^n k) \end{aligned}$$

So, for finite sums of the form $\sum_n \tilde{U}^n k_n$ ($n \in \mathbb{Z}, k_n \in \mathcal{K}$), we have

$$\mu \tilde{U} \tilde{V} \left(\sum_n \tilde{U}^n k_n \right) = \tilde{V} \tilde{U} \left(\sum_n \tilde{U}^n k_n \right).$$

Thus, by (3.3) we have $\tilde{V} \tilde{U} = \mu \tilde{U} \tilde{V}$ on \mathcal{L} showing that $\{\tilde{U}, \tilde{V}\}$ is another power φ -dilation of $\{S, T\}$.

Next, extend \tilde{V} to a unitary operator V on $\mathcal{M} \supseteq \mathcal{L}$. Again, choose this extension to be minimal in the sense that

$$\mathcal{M} = \bigvee_{-\infty}^{\infty} V^n(\mathcal{L}). \quad (3.5)$$

As before with

$$\mathcal{N} = \left\{ \sum_n V^n l_n : l_n \in \mathcal{L}, n \text{ belonging to a finite subset of } \mathbb{Z} \right\}$$

define $U : \mathcal{N} \rightarrow \mathcal{N}$ by

$$U \left(\sum_n V^n l_n \right) = \sum_n V^n (\tilde{U} \mu^n l_n) \quad (3.6)$$

and verify that U is isometric. Further, $\overline{\mathcal{N}} = \mathcal{M}$ by (3.5), so that U extends by continuity to an isometry on \mathcal{M} . Note that \tilde{U} is unitary on \mathcal{L} so that $\tilde{U}(\mathcal{L}) = \mathcal{L}$. Thus $U(\mathcal{N}) = \mathcal{N}$ by (3.6), and hence $U(\mathcal{M}) = \mathcal{M}$ (i.e. U is unitary on \mathcal{M}) by (3.5). Since $VU = \mu UV$ on finite sums of the form $\sum_n V^n l_n$, (3.5) shows that $VU = \mu UV$ on \mathcal{M} where both U and V are unitaries. Hence, $\{U, V\}$ is a power unitary φ -dilation of $\{S, T\}$ on \mathcal{M} . \square

The proof of Theorem 3.4.5 can be extended to give a unitary φ -dilation of contractions satisfying $TS = S\varphi(T)$ for *any* conformal bijection of \mathbb{D} provided these contractions have an

isometric φ -dilation. However, it is not clear how such a dilation can be found. We now give a different lifting argument to show that a pair of contractions S and T satisfying $TS = S\varphi(T)$ where φ is *any* conformal bijection of \mathbb{D} can be φ -dilated to unitaries U and V satisfying $VU = U\varphi(V)$. The idea here is to dilate one of the operators to a *unitary* first and then do the same with the other operator. In the proofs of Theorem 3.4.4 both operators were dilated to *isometries* and then the resulting pair was dilated to unitaries. Proofs in this section closely resemble those in Sebestyén [Seb1]. Lemma 3.4.7 is directly lifted from Sebestyén [Seb1].

Lemma 3.4.6. Let S and T be contractions satisfying $TS = S\varphi(T)$ for some conformal bijection φ of \mathbb{D} . Then $\varphi^{-1}(T)S = ST$.

Proof. Let $p_n(z)$ be a sequence of polynomials such that $p_n(z) \rightarrow \varphi^{-1}(z)$ in $\|\cdot\|_\infty$. Then, it is easy to verify that $p_n(T)S = S(p_n \circ \varphi)(T)$. But $(p_n \circ \varphi)(z) \rightarrow z$ in $\|\cdot\|_\infty$ then shows via the functional calculus for contractions [Sz-NF] that $p_n(T)S \rightarrow \varphi^{-1}(T)S$ since $\|p_n(T)S - \varphi^{-1}(T)S\| \leq \|p_n(T) - \varphi^{-1}(T)\|\|S\| \rightarrow 0$. Similarly, $S(p_n \circ \varphi)(T) \rightarrow ST$ so that $\varphi^{-1}(T)S = ST$. \square

Lemma 3.4.7. [Seb1, Lemma 1] Let \mathcal{K} and \mathcal{K}' be Hilbert spaces, $\mathcal{H} \subseteq \mathcal{K}$ and $\mathcal{H}' \subseteq \mathcal{K}'$ be subspaces and $X : \mathcal{H} \rightarrow \mathcal{K}'$ and $X' : \mathcal{H}' \rightarrow \mathcal{K}$ be given bounded linear transformations. Then, there exists an operator $Y : \mathcal{K} \rightarrow \mathcal{K}'$ extending X so that Y^* extends X' if and only if $\langle Xh, h' \rangle = \langle h, X'h' \rangle \ \forall h \in \mathcal{H}, h' \in \mathcal{H}'$. Moreover, $\|Y\| \leq \max\{\|X\|, \|X'\|\}$.

Suppose now that S and T are contractions on \mathcal{H} which satisfy $TS = S\varphi(T)$ for some conformal bijection φ of \mathbb{D} . Note that $\varphi(T)$ is a well defined contraction by the functional calculus found in Sz. Nagy and Foias [Sz-NF]. The statement of Lemma 3.4.8 should be compared with the commutant lifting theorem (Theorem 3.3.9).

Lemma 3.4.8. Let S and T be contractions on \mathcal{H} such that $TS = S\varphi(T)$. If U is the minimal *isometric* dilation of S acting on \mathcal{K} , then there exists an operator T_φ on \mathcal{K} such that T_φ^* extends T^* (i.e. T_φ lifts T), $\|T_\varphi\| \leq 1$, and $T_\varphi U = U\varphi(T_\varphi)$.

Proof. Let U be the minimal isometric dilation of S acting on a Hilbert space \mathcal{K} containing \mathcal{H} . Then, U^* extends S^* , where $\mathcal{K} \equiv \bigvee_{n=0}^{\infty} \mathcal{K}_n$ and $\mathcal{K}_n \equiv \bigvee_{k=0}^n U^k(\mathcal{H})$. Note also that the orthogonal

projections $P_n : \mathcal{K} \rightarrow \mathcal{K}_n$ satisfy the equation $P_{n+1}U = UP_n$ for any $n \geq 0$. We use induction on n to construct a sequence T_n of contractions with the properties sought in the Lemma. Consider the maps

$$\begin{aligned} U\varphi(T)U^*|_{\mathcal{H}_0} & : \mathcal{H}_0 \equiv U(\mathcal{H}) \rightarrow \mathcal{K}_1 \\ T^* & : \mathcal{K}_0 \equiv \mathcal{H} \rightarrow \mathcal{K}_1 \end{aligned}$$

By an application of Lemma 3.4.8, there exists $T_1 : \mathcal{K}_1 \rightarrow \mathcal{K}_1$ such that

$$\begin{aligned} T_1|_{\mathcal{H}_0} & = U\varphi(T)U^*|_{\mathcal{H}_0} \\ T_1^*|_{\mathcal{K}_0} & = T^* \end{aligned}$$

if and only if $\langle U\varphi(T)U^*(Uh), k \rangle = \langle Uh, T^*k \rangle \ \forall h, k \in \mathcal{H}$. But, if $h, k \in \mathcal{H}$,

$$\begin{aligned} \langle U\varphi(T)U^*(Uh), k \rangle & = \langle U\varphi(T)h, k \rangle \\ & = \langle \varphi(T)h, U^*k \rangle \\ & = \langle \varphi(T)h, S^*k \rangle \quad (\text{since } U^* \text{ extends } S^*) \\ & = \langle S\varphi(T)h, k \rangle \\ & = \langle TSh, k \rangle \\ & = \langle P_0Sh, T^*k \rangle \quad (\text{since } \mathcal{K}_0 = \mathcal{H}) \\ & = \langle Uh, T^*k \rangle \quad (\text{since } U|_{\mathcal{H}} = S) \end{aligned}$$

Since T is a contraction, Lemma 3.4.8 also guarantees

$$\begin{aligned} \|T_1\| & \leq \max \{ \|U\varphi(T)U^*|_{\mathcal{H}_0}\|, \|T^*\| \} \\ & \leq \max \{ \|\varphi(T)\|, \|T^*\| \} \\ & \leq 1 \end{aligned}$$

Since $P_1U = UP_0$, it also follows that

$$U\varphi(T)P_0 = U\varphi(T)U^*(UP_0) = T_1(UP_0) = T_1P_1U. \quad (3.7)$$

For the second step in the induction, consider the maps

$$\begin{aligned} U\varphi(T_1)U^*|_{\mathcal{H}_1} &: \mathcal{H}_1 \equiv U(\mathcal{K}_1) \rightarrow \mathcal{K}_2 \\ T_1^* &: \mathcal{K}_1 \rightarrow \mathcal{K}_2 \end{aligned}$$

Once again an application of Lemma 3.4.8 yields the existence of $T_2 : \mathcal{K}_2 \rightarrow \mathcal{K}_2$ such that

$$\begin{aligned} T_2|_{\mathcal{H}_1} &= U\varphi(T_1)U^*|_{\mathcal{H}_1} \\ T_2^*|_{\mathcal{K}_1} &= T_1^* \end{aligned}$$

if and only if $\langle U\varphi(T_1)U^*(Uh), k \rangle = \langle Uh, T_1^*, k \rangle \ \forall \ h, k \in \mathcal{K}_1$. But, if $h, k \in \mathcal{K}_1$,

$$\begin{aligned} \langle U\varphi(T_1)U^*(Uh), k \rangle &= \langle U\varphi(T_1)h, P_1k \rangle \\ &= \langle P_1U\varphi(T_1)h, k \rangle \\ &= \langle UP_0\varphi(T_1)h, k \rangle \quad (\text{since } P_1U = UP_0) \\ &= \langle h, \varphi(T_1)^*P_0^*U^*k \rangle \end{aligned}$$

Furthermore, for $x \in \mathcal{K}_1, y \in \mathcal{K}_0$, and $\varphi(z) = \mu\varphi_a(z) = \mu\frac{z-a}{1-\bar{a}z}$,

$$\begin{aligned} \langle x, [\mu\varphi_a(T_1)]^*y \rangle &= \langle x, \bar{\mu}\varphi_{\bar{a}}(T_1^*)y \rangle \quad (\text{see [Sz-NF, p.14]}) \\ &= \langle x, \bar{\mu}\varphi_{\bar{a}}(T^*)y \rangle \quad (\text{since } T_1^*|_{\mathcal{K}_0} = T^* \text{ and } \varphi_{\bar{a}} \text{ is analytic}) \\ &= \langle x, [\mu\varphi_a(T)]^*y \rangle \end{aligned}$$

Hence,

$$\begin{aligned} \langle U\varphi(T_1)U^*(Uh), k \rangle &= \langle h, \varphi(T)^*P_0U^*k \rangle \\ &= \langle h, (P_0\varphi(T)^*P_0)U^*k \rangle \\ &= \langle UP_0\varphi(T)P_0h, k \rangle \\ &= \langle U\varphi(T)P_0h, P_1k \rangle \\ &= \langle T_1P_1Uh, k \rangle \quad \text{by (3.7)} \\ &= \langle Uh, P_1T_1^*k \rangle \\ &= \langle Uh, T_1^*k \rangle \end{aligned}$$

Since $\|T_1\| \leq 1$, it also follows from Lemma 3.4.8 that

$$\|T_2\| \leq \max\{\|U\varphi(T_1)U^*|_{\mathcal{H}_1}\|, \|T_1^*\|\} \leq 1$$

and

$$U\varphi(T_1)P_1 = U\varphi(T_1)U^*(UP_1) = T_2(UP_1) = T_2P_2U$$

For step n in the induction, consider the maps

$$\begin{aligned} U\varphi(T_{n-1})U^*|_{\mathcal{H}_{n-1}} &: \mathcal{H}_{n-1} \equiv U(\mathcal{K}_{n-1}) \rightarrow \mathcal{K}_n \\ T_{n-1}^* &: \mathcal{K}_{n-1} \rightarrow \mathcal{K}_n \end{aligned}$$

By Lemma 3.4.8, there exists $T_n : \mathcal{K}_n \rightarrow \mathcal{K}_n$ with

$$\begin{aligned} T_n|_{\mathcal{H}_{n-1}} &= U\varphi(T_{n-1})U^*|_{\mathcal{H}_{n-1}} \\ T_n^*|_{\mathcal{K}_{n-1}} &= T_{n-1}^* \end{aligned}$$

if and only if $\langle U\varphi(T_{n-1})U^*(Uh), k \rangle = \langle Uh, T_{n-1}^*k \rangle \quad \forall h, k \in \mathcal{K}_{n-1}$. But, if $h, k \in \mathcal{K}_{n-1}$,

$$\begin{aligned} \langle U\varphi(T_{n-1})U^*(Uh), k \rangle &= \langle U\varphi(T_{n-1})h, P_{n-1}k \rangle \\ &= \langle P_{n-1}U\varphi(T_{n-1})h, k \rangle \\ &= \langle UP_{n-2}\varphi(T_{n-1})h, k \rangle \quad (P_{n-1}U = UP_{n-2} \text{ by induction hypothesis}) \\ &= \langle h, \varphi(T_{n-1})^*P_{n-2}^*U^*k \rangle \\ &= \langle h, \varphi(T_{n-2})^*P_{n-2}^*U^*k \rangle \quad (\text{since } T_{n-1}^*|_{\mathcal{K}_{n-2}} = T_{n-2}^*) \\ &= \langle h, (P_{n-2}\varphi(T_{n-2})^*P_{n-2})U^*k \rangle \quad (\text{where } T_0 \equiv T) \\ &= \langle U\varphi(T_{n-2})P_{n-2}h, k \rangle \\ &= \langle T_{n-1}P_{n-1}Uh, k \rangle \quad (\text{by induction hypothesis}) \\ &= \langle Uh, P_{n-1}T_{n-1}^*k \rangle \\ &= \langle Uh, T_{n-1}^*k \rangle \end{aligned}$$

Further,

$$\|T_n\| \leq \max\{\|U\varphi(T_{n-1})U^*|_{\mathcal{H}_{n-1}}\|, \|T_{n-1}^*\|\} \leq 1 \quad \text{by induction.}$$

Also,

$$U\varphi(T_{n-1})P_{n-1} = U\varphi(T_{n-1})U^*(UP_{n-1}) = T_n(UP_{n-1}) = T_nP_nU$$

As a result of this process, by the minimality of the dilation, we arrive at an operator $T_\varphi^* : \mathcal{K} \rightarrow \mathcal{K}$ as the extension of each $\{T_n^*\}_{n=0}^\infty$ such that $\|T_\varphi\| \leq 1$. Since $T_nP_n \rightarrow T_\varphi$ in the strong operator topology, the identity $U\varphi(T_{n-1})P_{n-1} = T_nP_nU$ yields $T_\varphi U = U\varphi(T_\varphi)$. \square

Lemma 3.4.9. Let S and T be contractions on \mathcal{H} such that $TS = S\varphi(T)$. If U is the minimal *unitary* dilation of S acting on a Hilbert space \mathcal{K} , then there exists T_φ on \mathcal{K} which is a dilation of T with $\|T_\varphi\| \leq 1$ and $T_\varphi U = U\varphi(T_\varphi)$.

Proof. First, let U_+ be the minimal isometric dilation of S acting on \mathcal{K}_+ ; $\mathcal{H} \subseteq \mathcal{K}_+ \subseteq \mathcal{K}$. By Lemma 3.4.8 there exists an extension $T_{\varphi+}^*$ of T^* on \mathcal{K}_+ with $\|T_{\varphi+}\| \leq 1$ and

$$T_{\varphi+}U_+ = U_+\varphi(T_{\varphi+}).$$

By Lemma 3.4.6 it follows that

$$\varphi^{-1}(T_{\varphi+})U_+ = U_+T_{\varphi+} \quad (3.8)$$

Taking the adjoint of (3.8) gives $T_{\varphi+}^*U_+^* = U_+^*\varphi^{-1}(T_{\varphi+})^* = U_+^*\bar{\mu}\psi_{\bar{a}}(T_{\varphi+}^*)$ where $\varphi^{-1} \equiv \mu\psi_a = \mu\frac{z-a}{1-\bar{a}z}$. Recall that when U is the minimal *unitary* dilation of S , U^* is the unique minimal isometric dilation of U_+^* [DMP, Lemma 6.1]. It follows by Lemma 3.4.8 that there exists an operator T_φ^* on \mathcal{K} such that T_φ extends $T_{\varphi+}$, $\|T_\varphi^*\| \leq 1$, and $T_\varphi^*U^* = U^*\bar{\mu}\psi_{\bar{a}}(T_\varphi^*) = U^*\varphi^{-1}(T_\varphi)^*$. That is to say, by taking adjoints and applying Lemma 3.4.6, that $T_\varphi U = U\varphi(T_\varphi)$. \square

Theorem 3.4.10. Let S and T be contractions on \mathcal{H} such that $TS = S\varphi(T)$. Then there exists a pair of unitaries U and V on some $\mathcal{K} \supseteq \mathcal{H}$ such that $VU = U\varphi(V)$ and $S^mT^n = P_{\mathcal{H}}U^mV^n|_{\mathcal{H}}$ for every $m, n \in \mathbb{N}$. That is, $\{U, V\}$ is a power φ -dilation of $\{S, T\}$.

Proof. By Lemma 3.4.9, there exists a dilation of $\{S, T\}$, say $\{U_0, T_0\}$ acting on a Hilbert space \mathcal{K}_0 so that U_0 is the minimal *unitary* dilation of S and T_0 is a contractive dilation of T on \mathcal{K}_0 , with

$$T_0U_0 = U_0\varphi(T_0). \quad (3.9)$$

Let then V be the minimal *unitary* dilation of T_0 that acts on a Hilbert space \mathcal{K} . We extend U_0 to a unitary U on \mathcal{K} so that $VU = U\varphi(V)$. By minimality, and the fact that $\varphi^{-1}(V)$ is the minimal unitary dilation of $\varphi^{-1}(T_0)$ by Remark 3.2.4,

$$\mathcal{K} = \bigvee_{-\infty}^{\infty} V^n(\mathcal{K}_0) = \bigvee_{-\infty}^{\infty} \varphi^{-1}(V)^n(\mathcal{K}_0). \quad (3.10)$$

Define then U on \mathcal{K} by the assignment

$$U(V^n k_0) = \varphi^{-1}(V)^n U_0 k_0$$

for $n \in \mathbb{Z}$ and $k_0 \in \mathcal{K}_0$. Note that U is onto by (3.10). Hence, we need only to show that U is isometric (and thus unitary). But, for every finite sum $\sum_n V^n U_0 k_0$ where $k_0 \in \mathcal{K}_0$, we can use (3.9) to show

$$\begin{aligned} \left\| \sum_n V^n k_0 \right\|^2 &= \sum_n \sum_m \langle V^n U_0 k_0, V^m U_0 k_0 \rangle \\ &= \sum_{n \geq m} \langle V^{n-m} k_0, k_0 \rangle + \sum_{n < m} \langle k_0, V^{m-n} k_0 \rangle \\ &= \sum_{n \geq m} \langle T_0^{n-m} k_0, k_0 \rangle + \sum_{n < m} \langle k_0, T_0^{m-n} k_0 \rangle \\ &= \sum_{n \geq m} \langle U_0 T_0^{n-m} k_0, U_0 k_0 \rangle + \sum_{n < m} \langle U_0 k_0, U_0 T_0^{m-n} k_0 \rangle \\ &= \sum_{n \geq m} \langle \varphi^{-1}(T_0)^{n-m} U_0 k_0, U_0 k_0 \rangle + \sum_{n < m} \langle U_0 k_0, \varphi^{-1}(T_0)^{m-n} U_0 k_0 \rangle \\ &= \sum_{n \geq m} \langle \varphi^{-1}(V)^{n-m} U_0 k_0, U_0 k_0 \rangle + \sum_{n < m} \langle U_0 k_0, \varphi^{-1}(V)^{m-n} U_0 k_0 \rangle \\ &= \sum_{n \geq m} \langle \varphi^{-1}(V)^n U_0 k_0, \varphi^{-1}(V)^m U_0 k_0 \rangle + \sum_{n < m} \langle \varphi^{-1}(V)^n U_0 k_0, \varphi^{-1}(V)^m U_0 k_0 \rangle \\ &= \left\| \sum_n \varphi^{-1}(V)^n U_0 k_0 \right\|^2 \end{aligned}$$

and hence U is unitary.

Finally, we show that

$$S^m T^n = P_{\mathcal{H}} U^m V^n|_{\mathcal{H}}$$

for all $m, n \geq 0$. An identity from [DMP, Theorem 6] shows that

$$\begin{aligned} S^m T^n &= (P_{\mathcal{H}} U_0^m|_{\mathcal{H}})(P_{\mathcal{H}} T_0^n|_{\mathcal{H}}) \\ &= P_{\mathcal{H}} U_0^m T_0^n|_{\mathcal{H}} \end{aligned}$$

But, for $h_0, k_0 \in \mathcal{H}$, we have

$$\begin{aligned} \langle U_0^m T_0^n h_0, k_0 \rangle &= \langle T_0^n h_0, U_0^{m*} k_0 \rangle \\ &= \langle V^n h_0, U^{m*} k_0 \rangle \\ &= \langle U^m V^n h_0, k_0 \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} S^m T^n &= P_{\mathcal{H}} U_0^m T_0^n|_{\mathcal{H}} \\ &= P_{\mathcal{H}} U^m V^n|_{\mathcal{H}} \end{aligned}$$

□

3.5 Shilov Modules

Definition 3.5.1. Let \mathfrak{A} be an operator algebra with C^* -envelope $C^*(\mathfrak{A})$. A Hilbert module \mathcal{H} for \mathfrak{A} is a Shilov module if there exists a $C^*(\mathfrak{A})$ -module \mathcal{K} such that \mathcal{H} is isomorphic to a submodule of \mathcal{K} viewed as an \mathfrak{A} -module.

Example 3.5.2. For the disk algebra $\mathcal{A}(\mathbb{D})$, note that $C^*(\mathcal{A}(\mathbb{D})) = C(\mathbb{T})$ by Example 2.3.4 and $L^2(\mathbb{T})$ is a $C(\mathbb{T})$ -module with action given by pointwise multiplication. Of course, $L^2(\mathbb{T})$ can also be viewed as an $\mathcal{A}(\mathbb{D})$ -module under the same module action. Then $H^2(\mathbb{D})$ is a Shilov module for $\mathcal{A}(\mathbb{D})$ since it is a submodule for $L^2(\mathbb{T})$ invariant under multiplication by functions in $\mathcal{A}(\mathbb{D})$.

Example 3.5.3. Let $\mathfrak{A} = T_2$, the upper triangular 2×2 matrices. Then $C^*(\mathfrak{A}) = M_2(\mathbb{C})$ and $\mathbb{C} \oplus \mathbb{C}$ is a $M_2(\mathbb{C})$ -module with action given by matrix multiplication. But, $\mathbb{C} \oplus \mathbb{C}$ can also be considered as a T_2 module under the same action. It is easy to verify that $\mathbb{C} \oplus 0$ is a Shilov module for \mathfrak{A} since it is invariant for \mathfrak{A} .

Definition 3.5.4. A sequence of Hilbert modules over \mathfrak{A}

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{M} \rightarrow \mathcal{H} \rightarrow 0 \tag{3.11}$$

is a short exact isometric sequence if \mathcal{K} is isomorphic to a submodule of \mathcal{M} having quotient isomorphic to \mathcal{H} .

Remark 3.5.5. A submodule of a Hilbert module \mathcal{M} is a closed subspace of \mathcal{M} invariant for \mathfrak{A} . If \mathcal{K} is a submodule of \mathcal{M} , it does not follow in general that \mathcal{K}^\perp is a submodule of \mathcal{M} . However, if \mathcal{K}^\perp is a submodule of \mathcal{M} , then (3.11) with $\mathcal{H} = \mathcal{K}^\perp$ is a short exact isometric sequence.

Definition 3.5.6. A Hilbert module \mathcal{H} over \mathfrak{A} admits a Shilov resolution if there is a short exact isometric sequence

$$0 \rightarrow \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi} \mathcal{H} \rightarrow 0 \quad (3.12)$$

where \mathcal{K} and \mathcal{M} are Shilov modules. If such a resolution exists, then (\mathcal{M}, Φ) is called a Shilov dominant for \mathcal{H} .

Definition 3.5.7. Let $\rho_{\mathcal{H}}$ be a contractive representation of an operator algebra \mathfrak{A} on \mathcal{H} . Then $\rho_{\mathcal{H}}$ is said to have a Stinespring dilation if there is a triple (\mathcal{K}, π, V) where \mathcal{K} is a Hilbert space, π is a C^* -representation of $C^*(\mathfrak{A})$ on \mathcal{K} and $V : \mathcal{H} \rightarrow \mathcal{K}$ is an isometry such that

$$\rho_{\mathcal{H}}(a) = V^* \pi(a) V \quad (3.13)$$

for all $a \in \mathfrak{A}$.

Example 3.5.8. Let $\rho_{\mathcal{H}}$ be a contractive representation of $\mathcal{A}(\mathbb{D})$ on \mathcal{H} determined by a contraction T . Let U be the minimal isometric dilation of T on $\mathcal{K} = \bigvee_{-\infty}^{\infty} U^n(\mathcal{H})$. Then $\rho_{\mathcal{H}}$ has a Stinespring dilation (\mathcal{K}, π, V) where $\pi : C(\mathbb{T}) \rightarrow B(\mathcal{K})$ is the C^* -representation determined by U and $V : \mathcal{H} \rightarrow \mathcal{K}$ is the inclusion map.

The following theorem is a key result in determining the existence of Shilov resolutions for contractive Hilbert modules \mathcal{H} over operator algebras.

Theorem 3.5.9 (Stinespring-Arveson). *[MS2, Theorem 2.13]* Let \mathcal{H} be a contractive Hilbert module over an operator algebra \mathfrak{A} . The following are equivalent:

- (i) \mathcal{H} admits a Shilov resolution.
- (ii) \mathcal{H} is a completely contractive Hilbert module over \mathfrak{A} .
- (iii) The representation $\rho_{\mathcal{H}}$ associated with \mathcal{H} has a Stinespring dilation.

Definition 3.5.10. Let

$$0 \rightarrow \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi} \mathcal{H} \rightarrow 0$$

be a Shilov resolution. A Shilov dominant (\mathcal{M}, Φ) for \mathcal{H} is strongly minimal if there is no submodule \mathcal{N} of \mathcal{M} such that Φ maps \mathcal{N} onto \mathcal{H} .

Definition 3.5.11. Let \mathcal{H} be a contractive Hilbert module over \mathfrak{A} . Let $0 \rightarrow \mathcal{K}_i \rightarrow \mathcal{M}_i \rightarrow \mathcal{H} \rightarrow 0$ ($i=1,2$) be two Shilov resolutions of \mathcal{H} . These resolutions are said to be isomorphic if there exist module isomorphisms $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ and $\psi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \phi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \end{array}$$

Theorem 3.5.12 (Sz.-Nagy). *[DP, Theorem 3.15]* Every contractive Hilbert module \mathcal{H} over $\mathcal{A}(\mathbb{D})$ has a Shilov resolution. The Shilov modules correspond to isometries on \mathcal{H} and, moreover, any two strongly minimal Shilov resolutions of \mathcal{H} are isomorphic.

Proof. That Shilov resolutions exist follows from Theorem 3.2.3. Since Shilov modules for $\mathcal{A}(\mathbb{D})$ are determined by restrictions of unitaries, there is a one-to-one correspondence between Shilov modules and isometries on \mathcal{H} . Since the contraction T determining a contractive Hilbert module \mathcal{H} has a minimal unitary dilation U which is unique up to unitary equivalence, it follows that any two strongly minimal Shilov resolutions of \mathcal{H} are isomorphic. \square

Recall that the Shilov modules \mathcal{H} for $\mathcal{A}(\mathbb{D})$ (i.e. isometries on \mathcal{H}) can be classified according to the Wold decomposition. So, in some sense, all the information for classifying the contractive Hilbert modules \mathcal{H} over $\mathcal{A}(\mathbb{D})$ is contained in the module maps Φ and Ψ where $0 \rightarrow \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi}$

$\mathcal{H} \rightarrow 0$ is a Shilov resolution. However, assuming Φ is the quotient map, we see that \mathcal{H} is really completely determined from the module map Ψ . Part of the goal of thinking in terms of contractive Hilbert modules \mathcal{H} over \mathfrak{A} rather than in terms of contractive representations of \mathfrak{A} on \mathcal{H} is to extract information about \mathcal{H} from this map Ψ [DP].

Example 3.5.13. By Andô's Theorem, it is clear that every contractive Hilbert module \mathcal{H} over $\mathcal{A}(\mathbb{D}^2)$ has a Shilov resolution. However, strongly minimal resolutions are no longer unique (since the minimal unitary dilation of two commuting contractions is not unique up to unitary equivalence) [DP].

Shilov modules are the building blocks for (completely) contractive Hilbert modules in the sense that completely contractive Hilbert modules can be resolved in terms of Shilov modules. This is similar to the role that projective modules play in ring theory. Muhly and Solel [MS2] have introduced the concept of “orthogonally projective” which they believe is the proper operator algebraic analogue of “projective” found in ring theory.

Definition 3.5.14. Let

$$0 \rightarrow \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi} \mathcal{P} \rightarrow 0 \quad (3.14)$$

be a short exact isometric sequence of contractive Hilbert modules over an operator algebra \mathfrak{A} . Then \mathcal{P} is orthogonally projective if (3.14) is split by a contraction. That is, there exists a module map $\tilde{\Phi} : \mathcal{P} \rightarrow \mathcal{M}$ with $\|\tilde{\Phi}\| \leq 1$.

Example 3.5.15. The orthogonally projective modules \mathcal{P} over $\mathcal{A}(\mathbb{D})$ correspond to isometries T on \mathcal{P} [MS2]. Hence the orthogonally projective and Shilov modules over $\mathcal{A}(\mathbb{D})$ coincide.

Proposition 3.5.16. [MS2, Theorem 3.1] For C^* -algebras, every contractive Hilbert module is orthogonally projective.

Proposition 3.5.17. [MS2, Proposition 3.2] Completely contractive orthogonally projective Hilbert modules are Shilov.

The converse of Proposition 3.5.17 is also true when all contractive Hilbert modules over the operator algebra are completely contractive and the notion of “commutant lifting” (which we do not discuss) holds for the algebra [MS2].

CHAPTER 4

SEMICROSSED PRODUCTS OF THE DISK ALGEBRA

4.1 The Semicrossed Product of the Disk Algebra

Definition 4.1.1. Let α be an automorphism of an operator algebra \mathfrak{A} , ρ a contractive representation of \mathfrak{A} on \mathcal{H} , and V a contraction (*resp.* isometry) on \mathcal{H} . We say that (ρ, V) is a contractive (*resp.* isometric) covariant representation of (\mathfrak{A}, α) if $V\rho(\alpha(f)) = \rho(f)V \ \forall f \in \mathfrak{A}$.

Example 4.1.2. Let \mathfrak{A} be an operator algebra and α an automorphism of \mathfrak{A} . Let ρ be a contractive representation of \mathfrak{A} on \mathcal{K} and let $\mathcal{H} = H^2(\mathcal{K})$, the space of square summable elements in $\bigoplus_0^\infty \mathcal{K}$. Define a contractive representation π of \mathfrak{A} on \mathcal{H} by $\pi(f)(\xi_0, \xi_1, \xi_2, \dots) = (\rho(f)\xi_0, \rho(\alpha(f))\xi_1, \rho(\alpha^2(f))\xi_2, \dots)$. If U_+ is the unilateral shift on \mathcal{H} , then $U_+\pi(\alpha(f)) = \pi(f)U_+ \ \forall f \in \mathfrak{A}$. Write $\pi = \tilde{\rho}$ for the contractive representation constructed above. Since $(\tilde{\rho}, U_+)$ is a contractive covariant representation of (\mathfrak{A}, α) , it follows that contractive covariant representations always exist.

Theorem 4.1.3. [Hof] Every automorphism α of $\mathcal{A}(\mathbb{D})$ is of the form $\alpha(f) = f \circ \varphi$ for some conformal bijection φ of \mathbb{D} .

Proof. Let α be an automorphism of \mathbb{D} . For $f \in \mathcal{A}(\mathbb{D})$ and $\lambda \in \mathbb{C}$, note that $\lambda \in \text{Ran } f$ if and only if $f(z) - \lambda$ is not invertible. Thus, since $\alpha(1) = 1$, it follows that $\lambda \in \text{Ran } f$ if and only if $\lambda \in \text{Ran } \alpha(f)$. If $\varphi(z)$ is defined by $\alpha(z) = \varphi(z)$ it follows that $\text{Ran } \varphi = \mathbb{D}$. For $|\lambda| < 1$ and $f \in \mathcal{A}(\mathbb{D})$ we have $|\varphi(\lambda)| < 1$ and

$$f(z) - f(\varphi(\lambda)) = (z - \varphi(\lambda))g(z) \tag{4.1}$$

for some g analytic on \mathbb{D} . Applying α to (4.1) gives

$$\alpha(f(z)) - f(\varphi(\lambda)) = (\varphi(z) - \varphi(\lambda))\alpha(g(z))$$

so that $\alpha(f(z)) - f(\varphi(\lambda))$ vanishes at $z = \lambda$. Since $|\lambda| < 1$ is arbitrary, $\alpha(f(\lambda)) = f(\varphi(\lambda))$ on \mathbb{D} . Since α maps $\mathcal{A}(\mathbb{D})$ onto $\mathcal{A}(\mathbb{D})$, φ has an inverse (that function that α maps to z) and is conformal. \square

Example 4.1.4. The disk algebra $\mathcal{A}(\mathbb{D})$ can be considered as an operator algebra acting (via multiplication) on the Hilbert space $L^2(\mathbb{T})$. Let $\alpha : \mathcal{A}(\mathbb{D}) \rightarrow \mathcal{A}(\mathbb{D})$ be an (isometric) automorphism of $\mathcal{A}(\mathbb{D})$ so that $\alpha(f) = f \circ \varphi$ for some conformal mapping φ of \mathbb{D} by Theorem 4.1.3. Let U be the forward unilateral shift on $\oplus_{i=0}^{\infty} L^2(\mathbb{T})$,

$$U(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \dots).$$

For $f \in \mathcal{A}(\mathbb{D})$, let D_f be the diagonal operator on $\oplus_{i=0}^{\infty} L^2(\mathbb{T})$ given by

$$D_f(\xi_0, \xi_1, \xi_2, \dots) = (f\xi_0, \alpha(f)\xi_1, \alpha^2(f)\xi_2, \alpha^3(f)\xi_3, \dots).$$

Then $UD_{\alpha(f)} = D_fU \ \forall f \in \mathcal{A}(\mathbb{D})$. If $\rho : \mathcal{A}(\mathbb{D}) \rightarrow B(\oplus_{i=0}^{\infty} L^2(\mathbb{T}))$ is the contractive representation of $\mathcal{A}(\mathbb{D})$ determined by $\rho(f) = D_f$, then (ρ, U) is an isometric covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$. We let \mathfrak{A}_α denote the norm closed subalgebra of $B(\oplus_{i=0}^{\infty} L^2(\mathbb{T}))$ generated by U and $D_f, f \in \mathcal{A}(\mathbb{D})$. Note that $UD_{\alpha(f)} = D_fU$ so that \mathfrak{A}_α is commutative if and only if α (and hence φ) is the identity. Further, we remark that every conformal bijection of \mathbb{D} has the form $\varphi(z) = \mu\varphi_a(z)$ where $\mu \in \mathbb{C}, |\mu| = 1$, and $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ ($a \in \mathbb{D}$). We classify these according to their fixed points as hyperbolic, parabolic, or elliptic [Bur]. In the hyperbolic case, $\varphi_a : \mathbb{D} \rightarrow \mathbb{D}$ has two distinct fixed points which lie on $\partial\mathbb{D}$. If φ_a is parabolic, it has a unique fixed point lying in $\partial\mathbb{D}$. In the elliptic case, φ_a has one fixed point in \mathbb{D} (and one outside \mathbb{D}). By Corollary 4.3.4, the study of \mathfrak{A}_α can be reduced to three specific cases.

Example 4.1.5. For $f \in \mathcal{A}(\mathbb{D})$ and α as in Example 4.1.4, the composition operator $C_{\varphi^{-1}}$ and the Toeplitz operator T_f on $H^2(\mathbb{D})$ satisfy $C_{\varphi^{-1}}T_{\alpha(f)} = T_fC_{\varphi^{-1}}$. We define \mathcal{B}_α to be the norm closed subalgebra of $B(H^2(\mathbb{D}))$ generated by $C_{\varphi^{-1}}$ and T_f ($f \in \mathcal{A}(\mathbb{D})$) where $\alpha(f)(z) = f(\varphi(z))$.

Note that if ρ is the contractive representation of $\mathcal{A}(\mathbb{D})$ on $H^2(\mathbb{D})$ given by $\rho(f) = T_f$, then $(\rho, C_{\varphi^{-1}})$ is an isometric covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$.

Remark 4.1.6. Unlike covariant representations of C^* -algebras, there are isometric covariant representations (ρ, V) satisfying $V\rho(\alpha(f)) = \rho(f)V$ but not $\rho(\alpha(f))V^* = V^*\rho(f)$. (Examples 4.1.2, 4.1.4, and 4.1.5 satisfy both conditions). Examples 4.1.7 and 4.1.8 are such isometric covariant representations.

Example 4.1.7. Let α be an elliptic automorphism of $\mathcal{A}(\mathbb{D})$. That is, $\alpha(f)(z) = f(\mu z)$ for some $|\mu| = 1$. Let ρ be the contractive (unital) representation of $\mathcal{A}(\mathbb{D})$ on $\ell^2(\mathbb{C})$ given by

$$\rho(z)(\xi_0, \xi_1, \xi_2, \dots) = (0, \mu\xi_0, \mu^2\xi_1, \mu^3\xi_2, \dots)$$

where z denotes the function $f(z) = z$ and V the unilateral shift. Then

$$\begin{aligned} \rho(z)V(\xi_0, \xi_1, \xi_2, \dots) &= \rho(z)(0, \xi_0, \xi_1, \dots) \\ &= (0, 0, \mu^2\xi_0, \mu^3\xi_1, \dots) \\ &= V(0, \mu^2\xi_0, \mu^3\xi_1, \dots) \\ &= V\rho(\alpha(z))(\xi_0, \xi_1, \xi_2, \dots). \end{aligned}$$

It follows that $V\rho(\alpha(q(z))) = \rho(q(z))V$ for all polynomials $q(z)$. However,

$$\begin{aligned} V^*\rho(z)(\xi_0, \xi_1, \xi_2, \dots) &= V^*(0, \mu\xi_0, \mu^2\xi_1, \mu^3\xi_2, \dots) \\ &= (\mu\xi_0, \mu^2\xi_1, \mu^3\xi_2, \dots) \end{aligned}$$

while

$$\begin{aligned} \rho(\alpha(z))V^*(\xi_0, \xi_1, \xi_2, \dots) &= \rho(\alpha(z))(\xi_1, \xi_2, \xi_3, \dots) \\ &= (0, \mu^2\xi_1, \mu^3\xi_2, \dots) \end{aligned}$$

so that $V^*\rho(z) \neq \rho(\alpha(z))V^*$.

Example 4.1.8. Let α be the automorphism of Example 4.1.7. Let $E_{i,j}$ denote the 4×4 matrix with (i, j) entry 1 and all other entries 0. Let ρ be the contractive representation of $\mathcal{A}(\mathbb{D})$ on \mathbb{C}^4

determined by the contraction $E_{4,3}$. Define a contraction $V = E_{4,1}$. If $q(z)$ is any polynomial, $q(z) = a_0 + a_1z + \dots + a_nz^n$, then $\rho(q(z)) = \rho(a_0 + a_1z) = a_0I + a_1E_{4,3}$ since $\rho(z)^2 = 0$. Note also that $\rho(\alpha(q(z))) = \rho(a_0 + a_1\mu z)$. Hence, $V\rho(\alpha(q(z))) = E_{4,1} \cdot (a_0I + a_1\mu E_{4,3}) = a_0E_{4,1} = (a_0I + a_1E_{4,3}) \cdot E_{4,1}$. However, $V^*\rho(z) = E_{1,4} \cdot E_{4,3} = E_{1,3}$ and $\rho(\alpha(z))V^* = \mu E_{4,3} \cdot E_{1,4} = 0$.

Consider an automorphism α of $C(\mathbb{T})$ given by composition with the restricted Möbius transformation $\varphi|_{\mathbb{T}}$. If δ_n denotes the Kronecker delta on \mathbb{Z} , the algebra $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ consists of all formal sums $\sum_{-\infty}^{\infty} \delta_n \otimes f_n$ with $f_n \in C(\mathbb{T})$, $\sum_{-\infty}^{\infty} \|f_n\| < \infty$. An adjoint and multiplication can be defined (on simple tensors) by $(\delta_n \otimes f)^* = \delta_{-n} \otimes \alpha^{-n}(\overline{f})$ and $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes f\alpha^n(g)$.

A multiplication could also be defined by letting \mathbb{Z} act on the left side by $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes \alpha^m(f)g$. If the Banach space $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ is provided with this alternate multiplication, and the adjoint is left unchanged, we obtain a new Banach algebra denoted $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$. The Banach algebras $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ and $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ are isomorphic [Pet1].

Define the Banach algebra $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ to be the subalgebra of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ consisting of elements of the form $F = \sum_{n \geq 0} \delta_n \otimes f_n$ with $f_n \in \mathcal{A}(\mathbb{D})$ and $\|F\|_1 = \sum_{n \geq 0} \|f_n\| < \infty$. Endow $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ with a multiplication $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes \alpha^m(f)g$ so that it is a Banach algebra without adjoint.

If (ρ, V) is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$ on \mathcal{H} , then $\pi : \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha) \rightarrow B(\mathcal{H})$ defined by

$$\pi \left(\sum_{n \geq 0} \delta_n \otimes f_n \right) = \sum_{n \geq 0} V^n \rho(f_n)$$

is a contractive representation. Denote this representation by $\pi = V \times \rho$.

Proposition 4.1.9. The correspondence $(\rho, V) \leftrightarrow V \times \rho$ is a bijection between the collection of contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$ and contractive representations of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$.

Proof. By the preceding remarks, we need only show that π , a contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on \mathcal{H} , gives rise to a contractive covariant pair (ρ, V) of $(\mathcal{A}(\mathbb{D}), \alpha)$ and that

$\pi = V \times \rho$. Define a contraction V on \mathcal{H} by $V = \pi(\delta_1 \otimes 1)$. Define a (contractive) representation ρ of $\mathcal{A}(\mathbb{D})$ by $\rho(f) = \pi(\delta_0 \otimes f)$. Then (ρ, V) is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$ since $\rho(f)V = \pi(\delta_0 \otimes f \cdot \delta_1 \otimes 1) = \pi(\delta_1 \otimes \alpha(f)) = \pi(\delta_1 \otimes 1 \cdot \delta_0 \otimes \alpha(f)) = V\rho(\alpha(f))$. To complete the proof, note that $\pi = V \times \rho$ on a dense subset of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ and hence everywhere. \square

Proposition 4.1.10. The Banach algebra $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ admits a faithful contractive representation.

Proof. Let ρ be the (faithful) contractive representation of $\mathcal{A}(\mathbb{D})$ on $L^2(\mathbb{T})$ given by multiplication. As in Example 4.1.2, $(\tilde{\rho}, U_+)$ is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$. Thus, $U_+ \times \tilde{\rho}$ is a contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on $\bigoplus_0^\infty L^2(\mathbb{T})$. Suppose $(U_+ \times \tilde{\rho})(\sum_{n \geq 0} \delta_n \otimes f_n) = 0$. Then $\sum_{n \geq 0} U_+^n \rho(f_n) = 0$ and hence

$$\left(\sum_{n \geq 0} U_+^n \rho(f_n) \right) (\xi_0, 0, 0, \dots) = (0, 0, 0, \dots)$$

$\forall \xi_0 \in L^2(\mathbb{T})$. It follows that $\forall k \geq 0, f_k \cdot \xi_0 = 0 \forall \xi_0 \in L^2(\mathbb{T})$. By the faithfulness of $\rho, f_k \equiv 0 \forall k \geq 0$. Thus, $U_+ \times \tilde{\rho}$ is faithful. \square

With Definition 2.5.4 and Propositions 4.1.9 and 4.1.10 in mind, we define an operator enveloping norm on $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$.

Definition 4.1.11. For $F \in \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$, set

$$\|F\| = \sup \{ \|(V \times \rho)(F)\| : (\rho, V) \text{ is a contractive covariant representation of } (\mathcal{A}(\mathbb{D}), \alpha) \}.$$

Define the semicrossed product of $\mathcal{A}(\mathbb{D})$ with α , denoted $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, to be the completion of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ with respect to this norm.

4.2 Contractive Representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$

Theorem 4.2.1. The contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ are in a one-to-one correspondence with pairs of contractions S and T satisfying $TS = S\varphi(T)$.

Proof. Since contractive representations ρ of $\mathcal{A}(\mathbb{D})$ on \mathcal{H} correspond bijectively with contractions on \mathcal{H} [Sz-NF], it follows from Proposition 4.1.9 that there is a bijection between the contractive representations of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ and the contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ is dense in $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, the contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$ give rise to all contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$. Conversely, all pairs of contractions S and T satisfying $TS = S\varphi(T)$ give rise to a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$. \square

Corollary 4.2.2. The character space of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is

$$\mathcal{M} = \{(z_0, \xi_0) \in \mathbb{C}^2 : |z_0| \leq 1, |\xi_0| \leq 1 \text{ and either } \xi_0 = 0 \text{ or } \varphi(z_0) = z_0\}.$$

Proof. Any character γ (a contractive representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ on \mathbb{C}) is determined by a pair $(z_0, \xi_0) \in \mathbb{C}^2$ satisfying $|z_0| \leq 1, |\xi_0| \leq 1$, and $z_0\xi_0 = \xi_0\varphi(z_0)$ by Theorem 4.2.1. However, $z_0\xi_0 = \xi_0\varphi(z_0)$ if and only if $\xi_0 = 0$ or z_0 is a fixed point of φ . \square

Proposition 4.2.3. The semicrossed product $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is isomorphic to a non-self-adjoint subalgebra of $\mathbb{Z} \times_\alpha C(\mathbb{T})$.

Proof. Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ can be considered to be a subalgebra of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$, there exists an embedding ι of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ into $\mathbb{Z} \times_\alpha C(\mathbb{T})$. If, for $F \in \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$, $\|F\| = \|\iota(F)\|$, then ι can be extended to an isometric isomorphism $\hat{\iota} : \mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) \rightarrow \mathbb{Z} \times_\alpha C(\mathbb{T})$ so that

$$\begin{array}{ccc} \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha) & \xrightarrow{\iota} & \mathbb{Z} \times_\alpha C(\mathbb{T}) \\ \subseteq \downarrow & & \downarrow id \\ \mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) & \xrightarrow[\hat{\iota}]{} & \mathbb{Z} \times_\alpha C(\mathbb{T}) \end{array}$$

commutes. Since every covariant representation (π, V) of $(C(\mathbb{T}), \alpha)$ restricts to a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$, it follows that $\|F\| \geq \|\iota(F)\|$. We show $\|\iota(F)\| \geq \|F\|$ to complete the proof. If π is any contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on \mathcal{H} , it is determined by two contractions $S = \pi(\delta_1 \otimes 1)$ and $T = \pi(\delta_0 \otimes z)$ which satisfy $TS = S\varphi(T)$. By Theorem 3.4.10, there exist unitaries U and V on $\mathcal{K} \supseteq \mathcal{H}$ such that $VU = U\varphi(V)$ and

$T^n S^m = P_{\mathcal{H}} V^n U^m|_{\mathcal{H}} \forall m, n \in \mathbb{N}$. Hence π can be extended to a contractive Banach $*$ -representation $\tilde{\pi}$ of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ on \mathcal{K} by defining

$$\tilde{\pi} \left(\sum_{n=-\infty}^{\infty} \delta_n \otimes f_n \right) = \sum_{n=-\infty}^{\infty} U^n f_n(V).$$

Hence $\|F\| \leq \|\iota(F)\|$. □

Remark 4.2.4. Recall that the semicrossed product norm (Definition 4.1.11) was defined by taking a supremum over the collection of contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. By Theorem 3.4.10, since each contractive covariant pair can be dilated to a covariant representation of $(C(\mathbb{T}), \alpha)$ which restricts to an isometric covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$, we could equally well have defined this norm by taking a supremum over the (smaller) collection of isometric covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. In fact, by Proposition 4.3.1, we could also have defined this norm by taking a supremum over the pure isometric covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. Moreover, as Theorem 4.2.6 shows, this norm makes every representation of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ completely contractive.

Lemma 4.2.5. The C^* -envelope of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$, $C^*(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$, is isometrically isomorphic to $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$.

Proof. Let $\hat{\iota}$ be as in Proposition 4.2.3. For $\delta_0 \otimes g \in \hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$, $\delta_0 \otimes \bar{g}$ is in the C^* -algebra generated by $\hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$. Since the trigonometric polynomials are dense in $C(\mathbb{T})$, it follows that the C^* -algebra generated by $\hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$ contains all elements of the form $\delta_0 \otimes f$, $f \in C(\mathbb{T})$. Thus, $(\delta_0 \otimes f)(\delta_n \otimes 1) = \delta_n \otimes f$ is in the C^* -algebra generated by $\hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$ for all $f \in C(\mathbb{T}), n \in \mathbb{Z}$ so this algebra contains the closure of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ in the operator norm. Note that the closure of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ in the operator norm is $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$ by Definition 2.5.4. Since $\hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})) + \hat{\iota}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))^*$ is dense in $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$, an application of Theorem 2.3.5 completes the proof. □

Theorem 4.2.6. Every contractive representation of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ is completely contractive.

Proof. By Theorem 3.5.9, a contractive representation ρ of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ on \mathcal{H} is completely contractive if and only if there exists a triple (\mathcal{K}, π, X) where π is a $*$ -representation of $C^*(\mathbb{Z}^+ \times_{\alpha}$

$\mathcal{A}(\mathbb{D}) = \mathbb{Z} \times_{\alpha} C(\mathbb{T})$ on \mathcal{K} and an isometry $X : \mathcal{H} \rightarrow \mathcal{K}$ such that $\rho(F) = X^* \pi(F) X \ \forall F \in \mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$. However, each contractive representation ρ of $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ on \mathcal{H} is completely determined by a pair of contractions S and T satisfying $TS = S\varphi(T)$. Let U and V be the pair of unitaries on \mathcal{K} generated by Theorem 3.4.10 and take $X : \mathcal{H} \rightarrow \mathcal{K}$ to be the inclusion map. Define then π on a dense subset of $\mathbb{Z} \times_{\alpha} C(\mathbb{T})$ by

$$\pi \left(\sum_{-\infty}^{\infty} \delta_n \otimes f_n \right) = \sum_{-\infty}^{\infty} U^n f_n(V).$$

□

Corollary 4.2.7. Every contractive Hilbert module over $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ has a Shilov resolution.

Proof. Apply Theorem 3.5.9. □

4.3 Concrete Representations

In this section, we investigate the relationship between $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ and the two concrete representations given in Examples 4.1.4 and 4.1.5. It will be shown that $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ is completely isometrically isomorphic to \mathfrak{A}_{α} . For α elliptic and similar to an irrational rotation via conjugation, it also follows that $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ is completely isometrically isomorphic to \mathfrak{B}_{α} .

Proposition 4.3.1. $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$ is completely isometrically isomorphic to \mathfrak{A}_{α} .

Proof. Let

$$\pi : \mathbb{Z} \times_{\alpha} C(\mathbb{T}) \rightarrow B \left(\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T}) \right)$$

be defined on the dense subset $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ by

$$\sum_{-\infty}^{\infty} \delta_n \otimes f_n \mapsto \sum_{-\infty}^{\infty} U^n M_{f_n}$$

where U is the bilateral shift and

$$M_f(\dots, \xi_{-1}, \xi_0, \xi_1, \dots) = (\dots, \alpha^{-1}(f)\xi_{-1}, f\xi_0, \alpha(f)\xi_1, \dots).$$

By [Ped1, 7.7.5], π is an isometry since $\rho(f) \cdot \xi = f \cdot \xi$ ($\xi \in L^2(\mathbb{T})$) is a faithful representation of $\mathcal{A}(\mathbb{D})$ on $L^2(\mathbb{T})$. In fact, π is completely isometric as it is a $*$ -homomorphism [Pau, p. 28].

Hence $\tilde{\pi} \equiv \pi|_{\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})}$ is completely isometric. Note that the subspace $\bigoplus_0^{\infty} L^2(\mathbb{T})$ is invariant under $\tilde{\pi}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$ so that

$$\hat{\pi} : \mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}) \rightarrow B \left(\bigoplus_0^{\infty} L^2(\mathbb{T}) \right)$$

defined on a dense subset by

$$\hat{\pi} \left(\sum_{n \geq 0} \delta_n \otimes f_n \right) = \tilde{\pi} \left(\sum_{n \geq 0} \delta_n \otimes f_n \right) \Big|_{\bigoplus_0^{\infty} L^2(\mathbb{T})}$$

is clearly completely contractive onto its range \mathfrak{A}_{α} . It is completely isometric if the induced map $\hat{\pi}_k : (\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})) \otimes M_k(\mathbb{C}) \rightarrow \mathfrak{A}_{\alpha} \otimes M_k(\mathbb{C})$ is isometric for all $k \geq 1$. Let $F = (F_{ij}) \in (\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})) \otimes M_k(\mathbb{C})$. We show that

$$\|\hat{\pi}(F)\| = \|(\tilde{\pi}(F_{ij}))\| = \left\| (\tilde{\pi}(F_{ij})) \Big|_{\bigoplus_1^k \left(\bigoplus_0^{\infty} L^2(\mathbb{T}) \right)} \right\|.$$

Define

$$\ell_N^2(L^2(\mathbb{T})) = \left\{ \xi = (\xi_k)_{k=-\infty}^{\infty} \in \bigoplus_{-\infty}^{\infty} L^2(\mathbb{T}) : \xi_k = 0 \text{ for } k < -N \right\}.$$

Then each $\ell_N^2(L^2(\mathbb{T}))$ is invariant under $\tilde{\pi}(\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D}))$ and $\bigcup_{N \geq 0} \ell_N^2(L^2(\mathbb{T}))$ is dense in $\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T})$.

Furthermore, $\bigcup_{N \geq 0} \bigoplus_1^k \ell_N^2(L^2(\mathbb{T}))$ is dense in $\bigoplus_1^k \left(\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T}) \right)$. It follows that

$$\begin{aligned} \|(\tilde{\pi}(F_{ij}))\| &= \sup_{N \geq 0} \sup_{\substack{\xi \in \bigoplus_1^k \ell_N^2(L^2(\mathbb{T})) \\ \|\xi\|=1}} \|(\tilde{\pi}(F_{ij}))\xi\| \\ &= \sup_{N \geq 0} \sup_{\substack{\xi \in \bigoplus_1^k \ell_N^2(L^2(\mathbb{T})) \\ \|\xi\|=1}} \|(\tilde{\pi}(F_{ij}))\xi_U\| \end{aligned}$$

where $\xi_U = (U^N \xi_1, U^N \xi_2, \dots, U^N \xi_k)$, U is the bilateral shift, and $\xi_i \in \ell_N^2(L^2(\mathbb{T}))$. Thus,

$$\begin{aligned} \|(\tilde{\pi}(F_{ij}))\| &= \sup_{\substack{\eta \in \bigoplus_1^k \ell_0^2(L^2(\mathbb{T})) \\ \|\eta\|=1}} \|(\tilde{\pi}(F_{ij}))\eta\| \\ &= \left\| (\tilde{\pi}(F_{ij})) \Big|_{\bigoplus_1^k \left(\bigoplus_0^{\infty} L^2(\mathbb{T}) \right)} \right\|. \end{aligned}$$

□

Lemma 4.3.2. [HPW, Lemma 10] Let φ and ψ be conformal bijections of \mathbb{D} . Let α and β be automorphisms of $\mathcal{A}(\mathbb{D})$ given by $\alpha(f) = f \circ \varphi$ and $\beta(f) = f \circ \psi^{-1} \circ \varphi \circ \psi$. Let $\Phi : \mathfrak{A}_\alpha \rightarrow \mathfrak{A}_\beta$ be defined on a dense subalgebra by

$$\Phi \left(\sum_{k=0}^n U^k D_{f_k} \right) = \sum_{k=0}^n U^k D_{f_k \circ \psi}.$$

Then Φ is an isometric isomorphism on the subalgebra, and hence extends to an isometric isomorphism of \mathfrak{A}_α onto \mathfrak{A}_β .

By an appropriate choice of a conformal map ψ , Lemma 4.3.2 allows us to reduce the study of \mathfrak{A}_α , in some sense, to three specific cases according to the conjugacy class of α . When α is hyperbolic, we can assume φ fixes 1 and -1 . When α is parabolic, we can assume φ fixes 1 and when α is elliptic we can assume φ fixes 0.

Proposition 4.3.3. Let $\overline{\mathbb{D}}_1 = \{(0, x_2, x_3) : x_2^2 + x_3^2 \leq 1\}$, $\overline{\mathbb{D}}_2 = \{(x_1, 1, x_3) : x_1^2 + x_3^2 \leq 1\}$, $\overline{\mathbb{D}}_3 = \{(x_1, -1, x_3) : x_1^2 + x_3^2 \leq 1\}$, and $\overline{P} = \{(x_1, x_2, x_3) : x_1 = x_2^2 + x_3^2, x_2^2 + x_3^2 \leq 1\}$. Let \mathcal{M} denote the character space of \mathfrak{A}_α with the wk-* topology. Then, with the relative topology from \mathbb{R}^3 ,

- (i) $\mathcal{M} \cong \overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2$ if α is parabolic.
- (ii) $\mathcal{M} \cong \overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2 \cup \overline{\mathbb{D}}_3$ if α is hyperbolic.
- (iii) $\mathcal{M} \cong \overline{\mathbb{D}}_1 \cup \overline{P}$ if α is elliptic.

Proof. We prove only the parabolic case (i) where 1 is the unique fixed point under φ . Arguments for cases (ii) and (iii) are similar. Recall that

$$\mathcal{M} = \left\{ \gamma_z^{(\xi)} : |z| \leq 1, |\xi| \leq 1, \text{ and either } \xi = 0 \text{ or } \varphi(z) = z \right\}$$

where $\gamma_z^{(\xi)}$ is defined on a dense subset of \mathfrak{A}_α by

$$\gamma_z^{(\xi)} \left(\sum_{j=0}^n U^j D_{f_j} \right) = \sum_{j=0}^n f_j(z) \xi^j.$$

Let $\Phi : \overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2 \rightarrow \mathcal{M}$ be defined by

$$\left\{ \begin{array}{ll} (0, x_2, x_3) \mapsto \gamma_{x_2+ix_3}^{(0)} & \text{for } (0, x_2, x_3) \in \overline{\mathbb{D}}_1 \\ (x_1, 1, x_3) \mapsto \gamma_1^{(x_1+ix_3)} & \text{for } (x_1, 1, x_3) \in \overline{\mathbb{D}}_2 \end{array} \right\}.$$

It is easy to check that Φ is a well-defined bijection. Since \mathcal{M} with the wk-* topology is compact Hausdorff and $\overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2$ with the relative topology is also compact Hausdorff, we need only show that Φ is continuous. Suppose then that $\mathbf{x}_\alpha \rightarrow \mathbf{y}$ in $\overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2$. W.l.o.g., assume that all the \mathbf{x}_α 's are in one of $\overline{\mathbb{D}}_1$ or $\overline{\mathbb{D}}_2$. If the \mathbf{x}_α 's are all in $\overline{\mathbb{D}}_1$, then $\mathbf{y} \in \overline{\mathbb{D}}_1$ and by the continuity of f_0 ,

$$\begin{aligned} \Phi(\mathbf{x}_\alpha) \left(\sum_{j=0}^n U^j D_{f_j} \right) &= \gamma_{(\mathbf{x}_\alpha)_2+i(\mathbf{x}_\alpha)_3}^{(0)} \left(\sum_{j=0}^n U^j D_{f_j} \right) \\ &= f_0((\mathbf{x}_\alpha)_2 + i(\mathbf{x}_\alpha)_3) \\ &\rightarrow f_0(\mathbf{y}_2 + i\mathbf{y}_3) \\ &= \gamma_{\mathbf{y}_2+i\mathbf{y}_3}^{(0)} \left(\sum_{j=0}^n U^j D_{f_j} \right) \\ &= \Phi(\mathbf{y}) \left(\sum_{j=0}^n U^j D_{f_j} \right). \end{aligned}$$

On the other hand, if the \mathbf{x}_α 's are all in $\overline{\mathbb{D}}_2$, then $\mathbf{y} \in \overline{\mathbb{D}}_2$ and

$$\begin{aligned} \Phi(\mathbf{x}_\alpha) \left(\sum_{j=0}^n U^j D_{f_j} \right) &= \gamma_1^{(\mathbf{x}_\alpha)_1+i(\mathbf{x}_\alpha)_3} \left(\sum_{j=0}^n U^j D_{f_j} \right) \\ &= \sum_{j=0}^n f_j(1) ((\mathbf{x}_\alpha)_1 + i(\mathbf{x}_\alpha)_3)^j \\ &\rightarrow \sum_{j=0}^n f_j(1) (\mathbf{y}_1 + i\mathbf{y}_3)^j \\ &= \Phi(\mathbf{y}) \left(\sum_{j=0}^n U^j D_{f_j} \right) \end{aligned}$$

showing that Φ is continuous. □

Corollary 4.3.4. If α and β belong to different conjugacy classes (i.e., elliptic, parabolic, and hyperbolic), then the algebras \mathfrak{A}_α and \mathfrak{B}_α , and hence $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ and $\mathbb{Z}^+ \times_\beta \mathcal{A}(\mathbb{D})$, are not isomorphic.

Proof. When α is elliptic and β is either parabolic or hyperbolic, it is known that \mathfrak{A}_α is not isomorphic to \mathfrak{A}_β . This follows since the (Jacobson) radical in the elliptic case is $\{0\}$ whereas in the other cases the radical is the nontrivial set of quasinilpotents (Theorems 11 and 12 of [HPW]). For α parabolic and β hyperbolic, consider the character spaces \mathcal{M}_α and \mathcal{M}_β (under the wk-* topology) of \mathfrak{A}_α and \mathfrak{A}_β respectively. By Proposition 4.3.3, \mathcal{M}_α and \mathcal{M}_β are not homeomorphic. Hence \mathfrak{A}_α and \mathfrak{A}_β are not isomorphic; for if such an isomorphism Γ existed, it would induce a homeomorphism τ of the character spaces defined by $\tau(\gamma)(F) = \gamma(\Gamma(F))$. \square

Corollary 4.2.2 is but one example of the usefulness of the characterization of the contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ given in Theorem 4.2.1. We give now a direct proof, using Proposition 4.3.1, to this corollary. The technique is not new; it is similar to that found in Hoover [Hoo]. First a lemma is stated that will also be used in the characterization of the maximal ideals of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$.

To each $F \in \mathfrak{A}_\alpha$ we may associate a unique Fourier series, $F \sim \sum_{n=0}^{\infty} U^n D_{f_n}$. We denote by $\pi_n(F)$ the n th Fourier coefficient of F . Some useful properties of these Fourier coefficients are listed in Lemma 4.3.5.

Lemma 4.3.5. [HPW] For $n = 0, 1, 2, \dots$, there is a linear mapping $\pi_n : \mathfrak{A}_\alpha \rightarrow \mathcal{A}(\mathbb{D})$ satisfying

- (i) $\|\pi_n(F)\| \leq \|F\|$, for $F \in \mathfrak{A}_\alpha$.
- (ii) $\pi_0(FG) = \pi_0(F)\pi_0(G)$ for $F, G \in \mathfrak{A}_\alpha$.
- (iii) $\pi_n \left(\sum_{k=0}^N U^k D_{f_k} \right) = \begin{cases} f_n & 0 \leq n \leq N \\ 0 & n > N \end{cases}$.
- (iv) $\pi_n(F) = 0 \ \forall n \geq 0 \Rightarrow F \equiv 0$.
- (v) $\pi_n(FG) = \sum_{k=0}^n \alpha^k(\pi_{n-k}(F))\pi_k(G) \ \forall F, G \in \mathfrak{A}_\alpha$.

Let \mathcal{M} denote the character space of \mathfrak{A}_α . For $z \in \overline{\mathbb{D}}$, the maximal ideal space of $\mathcal{A}(\mathbb{D})$, we define the fibres of \mathcal{M} over z by

$$\mathcal{M}_z \equiv \{ \gamma \in \mathcal{M} : \gamma(D_f) = \pi_0(D_f)(z) \ \forall f \in \mathcal{A}(\mathbb{D}) \}.$$

Note that if $\gamma_z^{(0)} : \mathfrak{A}_\alpha \rightarrow \mathbb{C}$ is defined by $\gamma_z^{(0)}(F) = \pi_0(F)(z)$ for $z \in \overline{\mathbb{D}}$ then $\gamma_z^{(0)}$ is a character by Lemma 4.3.5. Furthermore, $\gamma_z^{(0)} \in \mathcal{M}_z$ so that $\mathcal{M}_z \neq \emptyset$.

Proposition 4.3.6. If $z \in \overline{\mathbb{D}}$ is not a fixed point of φ then $\mathcal{M}_z = \{\gamma_z\}$.

Proof. If z is not a fixed point of φ , then $\exists f \in \mathcal{A}(\mathbb{D})$ such that $f(\varphi(z)) \neq f(z)$. But, for $\gamma \in \mathcal{M}_z$,

$$\begin{aligned} 0 &= \gamma(D_f U - U D_{f \circ \varphi}) \\ &= \gamma(D_f) \gamma(U) - \gamma(U) \gamma(D_{f \circ \varphi}) \\ &= \gamma(U) [f(z) - f(\varphi(z))]. \end{aligned}$$

Hence, $\gamma(U) = 0$ for $\gamma \in \mathcal{M}_z$ and so $\gamma = \gamma_z^{(0)}$ on elements of the form $\sum_{i=0}^n U^i D_{f_i}$ which are dense in \mathfrak{A}_α . \square

We now investigate the structure of the fibre(s) \mathcal{M}_{z_0} where z_0 is a fixed point of φ . For any $0 \leq i \leq n$, $f_i \in \mathcal{A}(\mathbb{D})$, note that

$$\begin{aligned} |f_i(z_0)| &\leq \|f_i\| \\ &= \left\| \pi_i \left(\sum_{j=0}^n U^j D_{f_j} \right) \right\| \\ &\leq \left\| \sum_{j=0}^n U^j D_{f_j} \right\| \end{aligned}$$

So, if $|\xi_0| < 1$,

$$\begin{aligned} \left| \sum_{i=0}^n f_i(z_0) \xi_0^i \right| &\leq \sum_{i=0}^n \left\| \sum_{j=0}^n U^j D_{f_j} \right\| |\xi_0|^i \\ &\leq \left\| \sum_{j=0}^n U^j D_{f_j} \right\| \frac{1}{1 - |\xi_0|}. \end{aligned}$$

Hence, the map $\gamma_{z_0}^{(\xi_0)} : \mathfrak{A}_\alpha \rightarrow \mathbb{C}$ defined by

$$\gamma_{z_0}^{(\xi_0)} \left(\sum_{i=0}^n U^i D_{f_i} \right) = \sum_{i=0}^n f_i(z_0) \xi_0^i$$

is a bounded linear functional which by density extends to all of \mathfrak{A}_α . This extension, $\gamma_{z_0}^{(\xi_0)}$, is multiplicative since z_0 is a fixed point of φ and

$$\begin{aligned}\gamma_{z_0}^{(\xi_0)}(U^k D_{f_k} U^l D_{f_l}) &= \gamma_{z_0}^{(\xi_0)}(U^{k+l} D_{f_l \circ f_k \circ \varphi^l}) \\ &= f_l(z_0) f_k(\varphi^l(z_0)) \xi_0^{k+l} \\ &= f_k(z_0) \xi_0^k \cdot f_l(z_0) \xi_0^l \\ &= \gamma_{z_0}^{(\xi_0)}(U^k D_{f_k}) \gamma_{z_0}^{(\xi_0)}(U^l D_{f_l}).\end{aligned}$$

Let \mathcal{A} denote the dense subalgebra of \mathfrak{A}_α consisting of sums of the form $\sum_{i=0}^n U^i D_{f_i}$ ($f_i \in \mathcal{A}(\mathbb{D})$). Define $\rho_{z_0} : \mathcal{A} \rightarrow \mathcal{A}(\mathbb{D})$ by

$$\rho_{z_0} \left(\sum_{i=0}^n U^i D_{f_i} \right) (z) \equiv \sum_{i=0}^n f_i(z_0) z^i$$

so that ρ_{z_0} is clearly an algebra homomorphism. We seek to extend ρ_{z_0} to a continuous homomorphism on \mathfrak{A}_α .

Suppose $\{\gamma_n\}_{n=0}^\infty \subseteq \mathcal{M}_{z_0}$ and $\gamma_n \rightarrow \gamma \in \mathcal{M}$ in the wk-* topology. Then $\gamma(D_f) = \lim_{n \rightarrow \infty} \gamma_n(D_f) = \lim_{n \rightarrow \infty} f(z_0) = f(z_0)$ for all $f \in \mathcal{A}(\mathbb{D})$. Thus, \mathcal{M}_{z_0} is closed in \mathcal{M} and hence, since \mathcal{M} is wk-* compact, \mathcal{M}_{z_0} is wk-* compact. The map $\Phi : \mathcal{M} \rightarrow \mathbb{C}$ defined by $\gamma \mapsto \gamma(U)$ is clearly continuous. Hence, $\Phi(\mathcal{M}_{z_0}) = \{\gamma(U) : \gamma \in \mathcal{M}_{z_0}\}$ is compact in \mathbb{C} since \mathcal{M}_{z_0} is wk-* compact. Since $\mathbb{D} \subseteq \{\gamma(U) : \gamma \in \mathcal{M}_{z_0}\}$, we have that $\forall \xi \in \mathbb{T}, \exists \gamma_{z_0}^{(\xi)} \in \mathcal{M}_{z_0}$ such that $\gamma_{z_0}^{(\xi)}(U) = \xi$. Hence, $\forall |\xi| \leq 1$

$$\begin{aligned}\rho_{z_0} \left(\sum_{i=0}^n U^i D_{f_i} \right) (\xi) &= \sum_{i=0}^n f_i(z_0) \xi^i \\ &= \sum_{i=0}^n \gamma_{z_0}^{(\xi)}(U)^i \gamma_{z_0}^{(\xi)}(D_{f_i}) \\ &= \gamma_{z_0}^{(\xi)} \left(\sum_{i=0}^n U^i D_{f_i} \right)\end{aligned}$$

and

$$\begin{aligned}
\left\| \rho_{z_0} \left(\sum_{i=0}^n U^i D_{f_i} \right) \right\| &= \sup_{|\xi| \leq 1} \left| \gamma_{z_0}^{(\xi)} \left(\sum_{i=0}^n U^i D_{f_i} \right) \right| \\
&\leq \sup_{|\xi| \leq 1} \left\| \gamma_{z_0}^{(\xi)} \right\| \cdot \left\| \sum_{i=0}^n U^i D_{f_i} \right\| \\
&= \left\| \sum_{i=0}^n U^i D_{f_i} \right\|
\end{aligned}$$

since $\gamma_{z_0}^{(\xi)}$ is a character forcing $\|\gamma_{z_0}^{(\xi)}\| = 1$ for $|\xi| < 1$. By density, ρ_{z_0} can be extended to a continuous algebra homomorphism on \mathfrak{A}_α . That ρ_{z_0} is onto follows since if $g \in \mathcal{A}(\mathbb{D})$, then $g(U) \in \mathfrak{A}_\alpha$ and $\rho_{z_0}(g(U)) = g$. We summarize the above arguments in Proposition 4.3.7.

Proposition 4.3.7. There exists a bounded algebra homomorphism ρ of \mathfrak{A}_α onto $\mathcal{A}(\mathbb{D})$ such that $\gamma \in \mathcal{M}_{z_0}$ if and only if $\exists \xi \in \overline{\mathbb{D}}$ such that $\gamma(F) = \rho(F)(\xi) \forall F \in \mathfrak{A}_\alpha$.

Propositions 4.3.6 and 4.3.7 combine to give an alternate proof of Corollary 4.2.2. This alternate method of computation will be used later in the characterization of the maximal ideal space of \mathfrak{A}_α .

Let us now reconsider the algebra \mathfrak{B}_α defined in Example 4.1.5. In what follows we discuss the isomorphism question regarding $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ and \mathfrak{B}_α in the case where α is elliptic. Recall that we can assume that $\alpha(f)(z) = f(\mu z)$ where $|\mu| = 1$.

Proposition 4.3.8. If α is elliptic and conjugate to a nonperiodic rotation, then \mathfrak{A}_α is completely isometrically isomorphic to \mathfrak{B}_α .

Proof. Recall from Example 2.5.5 that the irrational rotation algebra can be realized as $\mathbb{Z} \times_\mu C(\mathbb{T})$ or as the C^* -algebra of operators on $B(L^2(\mathbb{T}))$ generated by the composition operator $C_{\varphi^{-1}}$ (where $\varphi(z) = \mu z$) and multiplication operators M_f ($f \in C(\mathbb{T})$). Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ can be isometrically embedded in $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$, it follows that $\rho : \mathfrak{A}_\alpha \rightarrow \mathbb{Z} \times_\mu C(\mathbb{T})$ defined on a dense subset by

$$\rho \left(\sum_{n=0}^N U^n D_{f_n} \right) = \sum_{n=0}^N C_{\varphi^{-1}}^n M_{f_n}$$

is an isometric representation on $L^2(\mathbb{T})$. Let $\rho_{H^2} : \mathfrak{A}_\alpha \rightarrow B(H^2(\mathbb{D}))$ be given by $\rho_{H^2}(F) = \rho(F)|_{H^2(\mathbb{D})}$. Then ρ_{H^2} is a contractive representation of \mathfrak{A}_α onto \mathfrak{B}_α . To show that ρ_{H^2} is isometric, we show $\|\rho(F)\| = \|\rho(F)|_{H^2(\mathbb{D})}\| \ \forall F \in \mathfrak{A}_\alpha$. This follows as in the proof of Proposition 4.3.1.

The fact that ρ_{H^2} is an isometry implies that $\rho_{H^2}^{-1}$ is a contractive representation. We show that both ρ_{H^2} and $\rho_{H^2}^{-1}$ are completely contractive.

First, note that by Proposition 4.3.1 and Lemma 4.2.5, $C^*(\mathfrak{A}_\alpha) \cong \mathbb{Z} \times_\alpha C(\mathbb{T})$. Let π be a C^* -representation of $C^*(\mathfrak{A}_\alpha)$ on $L^2(\mathbb{T})$ defined by $\pi(U) = C_{\varphi^{-1}}$ and $\pi(D_f) = M_f$ where U is the bilateral shift on $\oplus_{-\infty}^\infty L^2(\mathbb{T})$ and $D_f(\dots, \xi_{-1}, \xi_0, \xi_1, \dots) = (\dots, \alpha^{-1}(f)\xi_{-1}, f\xi_0, \alpha(f)\xi_1, \dots)$. Let $X : H^2(\mathbb{D}) \rightarrow L^2(\mathbb{T})$ be inclusion. Then $\rho_{H^2}(F) = X^*\pi(F)X \ \forall F \in \mathfrak{A}_\alpha$, and ρ_{H^2} is completely contractive by Theorem 3.5.9.

By the above comments and Lemma 4.2.5, $C^*(\mathfrak{B}_\alpha) \cong \mathbb{Z} \times_\alpha C(\mathbb{T})$. Let π' be a C^* -representation of $C^*(\mathfrak{B}_\alpha)$ on $\oplus_{-\infty}^\infty L^2(\mathbb{T})$ defined by $\pi'(C_{\varphi^{-1}}) = U$ and $\pi'(M_f) = D_f$ where U and D_f are as above. Let $X : \oplus_0^\infty L^2(\mathbb{T}) \rightarrow \oplus_{-\infty}^\infty L^2(\mathbb{T})$ be inclusion. Then $\rho_{H^2}^{-1}(F) = X^*\pi'(F)X \ \forall F \in \mathfrak{B}_\alpha$ and $\rho_{H^2}^{-1}$ is also completely contractive. \square

When α is elliptic and periodic we can construct a contractive, but not faithful, representation of \mathfrak{A}_α onto \mathfrak{B}_α .

Proposition 4.3.9. $\pi : \mathfrak{A}_\alpha \rightarrow \mathfrak{B}_\alpha$ determined by $U \mapsto C_{\varphi^{-1}}$ and $D_f \mapsto T_f$ is a contractive, surjective homomorphism.

Proof. The result follows by Proposition 4.3.1 and the fact that a contractive representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is completely determined by two contractions S and T satisfying $TS = S\varphi(T)$. \square

Remark 4.3.10. Proposition 4.3.9 shows that algebraically $\mathfrak{A}_\alpha / \ker \pi \cong \mathfrak{B}_\alpha$. However, $\ker \pi \neq (0)$. For example, if $f \in \mathcal{A}(\mathbb{D})$ then $0 = C_{\varphi^{-1}}T_f + C_{\varphi^{-1}}^{K+1}T_{-f} = \pi(UD_f + U^{K+1}D_{-f})$. This algebraic isomorphism explains the disparity in the character spaces of \mathfrak{A}_α and \mathfrak{B}_α when α is periodic [Hoo].

Example 4.3.11. There is a class of representations of \mathfrak{A}_α that exhibit the “lower triangularity” of the algebra. By Lemma 4.3.5, it is easy to show that $\forall n \geq 0$ and $z \in \overline{\mathbb{D}}$, the

map

$$F \longmapsto \begin{pmatrix} \pi_0(F)(z) \\ \pi_1(F)(z) & \alpha(\pi_0(F))(z) \\ \pi_2(F)(z) & \alpha(\pi_1(F))(z) & \alpha^2(\pi_0(F))(z) \\ \vdots & \vdots & \vdots & \ddots \\ \pi_{n-1}(F)(z) & \alpha(\pi_{n-2}(F))(z) & \alpha^2(\pi_{n-3}(F))(z) & \dots & \alpha^{n-1}(\pi_0(F))(z) \end{pmatrix}$$

is a contractive representation of \mathfrak{A}_α on \mathbb{C}^n . Similarly, for all $n \geq 0$, the map

$$F \longmapsto \begin{pmatrix} \pi_0(F) \\ \pi_1(F) & \alpha(\pi_0(F)) \\ \pi_2(F) & \alpha(\pi_1(F)) & \alpha^2(\pi_0(F)) \\ \vdots & \vdots & \vdots & \ddots \\ \pi_{n-1}(F) & \alpha(\pi_{n-2}(F)) & \alpha^2(\pi_{n-3}(F)) & \dots & \alpha^{n-1}(\pi_0(F)) \end{pmatrix}$$

is a contractive representation on $\oplus_0^{n-1} L^2(\mathbb{T})$.

4.4 The Maximal Ideal Space of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$

In this section we show that there is a one-to-one correspondence between the maximal ideals of \mathfrak{A}_α and the kernels of the characters on \mathfrak{A}_α except in the case where α is elliptic and periodic. We use an ergodic argument for the nonperiodic elliptic case and a spectral argument for the hyperbolic and parabolic cases. We then characterize the maximal ideal space in the case where α is periodic using the fact that $M_K(\mathbb{C})$ is simple.

Recall from Corollary 4.2.2 that the maps $\gamma_z^{(\xi)} : \mathfrak{A}_\alpha \rightarrow \mathbb{C}$ (where $|z| \leq 1$, $|\xi| \leq 1$, and $\varphi(z) = z$ if $\xi \neq 0$) defined on a dense subset by

$$\gamma_z^{(\xi)} \left(\sum_{i=0}^n U^i D_{f_i} \right) = \sum_{i=0}^n f_i(z) \xi^i$$

are the characters of \mathfrak{A}_α .

Consider the case where α is either parabolic or hyperbolic. We show by way of contradiction that every maximal ideal \mathcal{M} in \mathfrak{A}_α contains the commutator ideal, denoted \mathcal{C} , and hence is of codimension one.

Lemma 4.4.1. If \mathcal{B} is a (unital) Banach algebra and \mathcal{M} is a maximal ideal in \mathcal{B} not containing the commutator ideal \mathcal{C} , then $\mathcal{B} = \mathcal{M} + \mathcal{C}$.

Proof. By the maximality of \mathcal{M} , we can find $b_0 = m_0 + c_0 \in (\mathcal{M} + \mathcal{C}) \cap \{b \in \mathcal{B} : \|b - 1\| < \frac{1}{2}\}$ where $m_0 \in \mathcal{M}$ and $c_0 \in \mathcal{C}$. Since b_0 is invertible, $1 = b_0^{-1}m_0 + b_0^{-1}c_0 \in \mathcal{M} + \mathcal{C}$ and hence $\mathcal{B} = \mathcal{M} + \mathcal{C}$. \square

Theorem 4.4.2. Let α be parabolic or hyperbolic. The maximal ideals of \mathfrak{A}_α are precisely the kernels of its characters.

Proof. We show that any maximal ideal \mathcal{M} contains the commutator ideal. Suppose it does not. By Lemma 4.4.1, there exists $F \in \mathcal{M}$ and $C \in \mathcal{C}$ such that $D_1 = F + C$. Since $\gamma_z^{(0)}(C) = 0$ it follows that $\gamma_z^{(0)}(F) = 1 \forall z \in \overline{\mathbb{D}}$. Write $F = D_1 + G$ so that $\pi_0(G) \equiv 0$ and $\pi_n(F) \equiv \pi_n(G)$ for $n \geq 1$. Let z_0 be a fixed point of φ . Since $\gamma_{z_0}^{(\xi)}(C) = 0$ it follows that $\gamma_{z_0}^{(\xi)}(F) = \gamma_{z_0}^{(\xi)}(D_1) = 1 \forall \xi \in \overline{\mathbb{D}}$. Hence, $\gamma_{z_0}^{(\xi)}(F - D_1) = \sum_{n=1}^{\infty} \pi_n(F)(z_0)\xi^n = 0 \forall \xi \in \overline{\mathbb{D}}$. Thus, $\pi_n(F)(z_0) = 0$ for $n \geq 1$, and so $\pi_0(G) \equiv 0$ and $\pi_n(G)(z_0) = 0$ for $n \geq 1$. From Theorem 12 of [HPW], the Jacobson radical is $\text{Rad}(\mathfrak{A}_\alpha) = \{F \in \mathfrak{A}_\alpha : \pi_0(F) = 0 \text{ and } \pi_n(F)(z_0) = 0 \text{ for } \varphi(z_0) = z_0\}$; that is, the set of quasinilpotents. Hence, $G \in \text{Rad}(\mathfrak{A}_\alpha)$. We conclude that $sp(G) = \{0\}$ so that $sp(F) = sp(D_1 + G) = \{1\}$ by the spectral mapping theorem. But then $F \in \mathcal{M}$ is invertible, contradicting the maximality of \mathcal{M} . \square

We now consider the case where α is elliptic. Recall that we are assuming w.l.o.g. that $\alpha(f) = f \circ \varphi$ where $\varphi(z) = \mu z$ for some $|\mu| = 1$. The structure of \mathfrak{A}_α is closely tied to whether μ is a root of unity or not. Lemma 4.4.3 holds for any automorphism α of $\mathcal{A}(\mathbb{D})$.

Lemma 4.4.3. The map $\tilde{\alpha} : \mathfrak{A}_\alpha \rightarrow \mathfrak{A}_\alpha$ defined on a dense subset by

$$\tilde{\alpha} \left(\sum_{i=0}^n U^i D_{f_i} \right) = \sum_{i=0}^n U^i D_{\alpha(f_i)}$$

is an isometric automorphism.

Proof. Since α is a homomorphism, it is easy to see that $\tilde{\alpha}$ is as well since

$$\begin{aligned} \tilde{\alpha} \left(\left(\sum_{i=0}^n U^i D_{f_i} \right) \left(\sum_{j=0}^m U^j D_{g_j} \right) \right) &= \tilde{\alpha} \left(\sum_{i=0}^{m+n} \sum_{j=0}^i U^i D_{\alpha^{i-j}(f_i)g_{i-j}} \right) \\ &= \sum_{i=0}^{m+n} \sum_{j=0}^i U^i D_{\alpha^{i-j+1}(f_j)\alpha(g_{i-j})} \\ &= \left(\sum_{i=0}^n U^i D_{\alpha(f_i)} \right) \left(\sum_{j=0}^m U^j D_{\alpha(g_j)} \right). \end{aligned}$$

By using the representation Π_z of elements in $\mathbb{Z}^+ \times_{\alpha} C(\mathbb{T})$ given in Peters [Pet2],

$$\begin{aligned} \left\| \tilde{\alpha} \left(\sum_{i=0}^n U^i D_{g_i} \right) \right\|_{\mathfrak{A}_{\alpha}} &= \left\| \sum_{i=0}^n U^i D_{\alpha(g_i)} \right\|_{\mathbb{Z}^+ \times_{\alpha} C(\mathbb{T})} \\ &= \sup_{z \in \mathbb{T}} \left\| \Pi_z \left(\sum_{i=0}^n U^i D_{\alpha(g_i)} \right) \right\|_{B(\bigoplus_0^{\infty} L^2(\mathbb{T}))} \\ &= \sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| (\alpha(g_0)(z)\xi_0, \alpha(g_1)(z)\xi_0 + \alpha^2(g_0)(z)\xi_1, \dots) \right\|_{\bigoplus_0^{\infty} L^2(\mathbb{T})} \\ &= \sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| (g_0(z)\xi_0, g_1(z)\xi_0 + \alpha(g_0)(z)\xi_1, \dots) \right\|_{\bigoplus_0^{\infty} L^2(\mathbb{T})} \end{aligned}$$

The last equality holds since any conformal bijection φ of $\overline{\mathbb{D}}$ maps \mathbb{T} onto itself in a one-to-one fashion and so

$$\begin{aligned} &\sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| (\alpha(g_0)(z)\xi_0, \alpha(g_1)(z)\xi_0 + \alpha^2(g_0)(z)\xi_1, \dots) \right\|_{\bigoplus_0^{\infty} L^2(\mathbb{T})}^2 \\ &= \sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| \alpha(g_0)(z)\xi_0 \right\|_{L^2(\mathbb{T})}^2 + \left\| \alpha(g_1)(z)\xi_0 + \alpha^2(g_0)(z)\xi_1 \right\|_{L^2(\mathbb{T})}^2 + \dots \\ &= \sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| g_0(z)\xi_0 \right\|_{L^2(\mathbb{T})}^2 + \left\| g_1(z)\xi_0 + \alpha(g_0)(z)\xi_1 \right\|_{L^2(\mathbb{T})}^2 + \dots \\ &= \sup_{z \in \mathbb{T}} \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_0^{\infty} L^2(\mathbb{T})}} \left\| (g_0(z)\xi_0, g_1(z)\xi_0 + \alpha(g_0)(z)\xi_1, \dots) \right\|_{\bigoplus_0^{\infty} L^2(\mathbb{T})}^2 \end{aligned}$$

Hence,

$$\begin{aligned}
\left\| \tilde{\alpha} \left(\sum_{i=0}^n U^i D_{g_i} \right) \right\| &= \sup_{z \in \mathbb{T}} \left\| \Pi_z \left(\sum_{i=0}^n U^i D_{g_i} \right) \right\|_{B(\bigoplus_0^\infty L^2(\mathbb{T}))} \\
&= \left\| \sum_{i=0}^n U^i D_{g_i} \right\|_{\mathbb{Z}^+ \times_\alpha C(\mathbb{T})} \\
&= \left\| \sum_{i=0}^n U^i D_{g_i} \right\|_{\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})}
\end{aligned}$$

Thus, $\tilde{\alpha}$ is isometric on a dense subset of \mathfrak{A}_α . That $\tilde{\alpha}$ is one-to-one and onto is also clear since $\tilde{\alpha}$ is invertible. \square

If μ is a K th root of unity, define $\#: \mathfrak{A}_\alpha \rightarrow \mathfrak{A}_\alpha$ by $F \mapsto \frac{1}{K} \sum_{k=0}^{K-1} \tilde{\alpha}^k(F)$. Note that $\#$ is $\tilde{\alpha}$ -invariant. As in [Pet2, V.8] we can define a map $\#$ with similar properties when α is nonperiodic. For $f \in \mathcal{A}(\mathbb{D})$, note that $1/(n+1) \sum_{k=0}^n \alpha^k(f)$ converges uniformly to $\int_{\mathbb{T}} f dm(z)$. For $n \geq 0$, define Q_n on \mathfrak{A}_α by $Q_n(F) = \frac{1}{n+1} \sum_{k=0}^n \tilde{\alpha}^k(F)$. Since $\tilde{\alpha}$ is isometric, each Q_n is a contraction. Moreover, if $F = \sum_{i=0}^N U^i D_{f_i}$, then $\{Q_n(F)\}_{n=0}^\infty$ converges in \mathfrak{A}_α . This follows since

$$Q_n(F) = Q_n \left(\sum_{i=0}^N U^i D_{f_i} \right) = \sum_{i=0}^N U^i D_{\frac{1}{n+1} \sum_{k=0}^n \alpha^k(f_i)}$$

so that

$$\|Q_n(F) - \sum_{i=0}^N U^i D_{\int_{\mathbb{T}} f_i dm(z)}\| \leq \sum_{i=0}^N \left\| \frac{1}{n+1} \sum_{k=0}^n \alpha^k(f_i) - \int_{\mathbb{T}} f_i dm(z) \right\| \rightarrow 0.$$

We define

$$\#(F) = \sum_{i=0}^n U^i D_{\int_{\mathbb{T}} f_i dm(z)} = \sum_{i=0}^n U^i D_{f_i(0)}$$

since each $f_i \in \mathcal{A}(\mathbb{D})$. We then extend the definition of $\#$ to all of \mathfrak{A}_α . Let $F \in \mathfrak{A}_\alpha$ and $\epsilon > 0$ be given. Find $G = \sum_{i=0}^N U^i D_{g_i}$ such that $\|F - G\| < \frac{\epsilon}{4}$. Then, by the above argument, find $M > 0$ so that $n \geq M$ implies $\|Q_n(G) - \#(G)\| < \frac{\epsilon}{4}$. It follows that

$$\|Q_n(F) - \#(G)\| \leq \|Q_n(F - G)\| + \|Q_n(G) - \#(G)\| < \frac{\epsilon}{2}. \quad (4.2)$$

So, if $n, m \geq M$ then $\|Q_n(F) - Q_m(F)\| < \epsilon$ and $\{Q_n(F)\}_{n=0}^\infty$ is Cauchy in \mathfrak{A}_α . Also, by (4.2), since $\tilde{\alpha}(\#(G)) = \#(G)$, $\|Q_n(F) - \tilde{\alpha}^k(Q_n(F))\| < \epsilon \forall k \geq 0$ and $n \geq M$. So, $Q_n(F)$ converges

to an $\tilde{\alpha}$ -invariant element in \mathfrak{A}_α . We define $\#(F)$ to be this element. In either case, define \mathfrak{A}_0 to be the closed subalgebra of \mathfrak{A}_α generated by $\{U, D_f : f \text{ is } \alpha\text{-invariant}\}$.

Proposition 4.4.4. If α is nonperiodic, then \mathfrak{A}_0 is the subalgebra of \mathfrak{A}_α generated by $\{U, D_1\}$. Furthermore, $\#$ is a linear projection onto the maximal abelian subalgebra \mathfrak{A}_0 of \mathfrak{A}_α .

Proof. The map $\#$ is clearly linear as $\tilde{\alpha}$ is. Also, $\#$ is continuous as $\tilde{\alpha}$ is and $\#^2(F) = \#(F)$ since $\#(F)$ is invariant under $\tilde{\alpha}$. That $\mathfrak{A}_0 \subseteq \#(\mathfrak{A}_\alpha)$ is clear. If $F \in \#(\mathfrak{A}_\alpha)$, $F = \#(G)$ for some $G \in \mathfrak{A}_\alpha$ so that F is a limit of elements of the form $\sum_{i=0}^N U^i D_{g_i}$, where each g_i is α -invariant. Hence $\#(\mathfrak{A}_\alpha) = \mathfrak{A}_0$. Let $F = \sum_{i=0}^n U^i D_{f_i} \in \mathfrak{A}_0$. Then by α -invariance $f_i(\mu^k \cdot \frac{1}{2}) = f_i(\frac{1}{2}) \quad \forall k \in \mathbb{N}, 0 \leq i \leq n$, so that each $f_i \in \mathcal{A}(\mathbb{D})$, the nonperiodicity of α , and the ergodic theorem (Theorem 2.6.3) gives $f_i \equiv f_i(0)$ on \mathbb{D} and hence $\overline{\mathbb{D}}$. We need only show \mathfrak{A}_0 is a maximal abelian subalgebra of \mathfrak{A}_α . By definition, \mathfrak{A}_0 is commutative. Suppose that $F \in \mathfrak{A}_\alpha$, $F \sim \sum_{n=0}^{\infty} U^n D_{f_n}$, commutes with \mathfrak{A}_0 . Then $FU = UF$ and $\alpha(f_n) = f_n \quad \forall n \geq 0$. Each f_n is then constant by the nonperiodicity of α . \square

Proposition 4.4.5. If \mathcal{R} is a closed, $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α , then $\#(\mathcal{R})$ is a closed ideal in \mathfrak{A}_0 and moreover $\#(\mathcal{R}) = \mathcal{R} \cap \mathfrak{A}_0$.

Proof. It is a routine exercise to show that $\#(\mathcal{R})$ is an ideal in \mathfrak{A}_0 . If $F \in \mathcal{R}$, then $\#(F) \in \mathcal{R}$ since \mathcal{R} is closed and $\tilde{\alpha}$ -invariant. So, $\#(\mathcal{R}) \subseteq \mathcal{R} \cap \mathfrak{A}_0$. But, $\#$ is a projection so that $\mathcal{R} \cap \mathfrak{A}_0 \subseteq \#(\mathcal{R})$. \mathcal{R} and \mathfrak{A}_0 closed then yield $\#(\mathcal{R})$ closed. \square

The characters of \mathfrak{A}_0 , which are easy to compute, will be used to characterize the maximal ideals in \mathfrak{A}_α . Since \mathfrak{A}_0 is a commutative Banach algebra, its maximal ideal space corresponds in a one-to-one fashion with the kernels of its characters (Theorem 2.2.3). If α is nonperiodic, then $\mathfrak{A}_0 \cong \mathcal{A}(\mathbb{D})$ (given by $U \mapsto z$) and its characters are determined by $U \mapsto \xi \in \overline{\mathbb{D}}$. Denote these by $\gamma_{\mathfrak{A}_0}^{(\xi)}$. If α is periodic with period K , there are more characters. In fact, if we denote by $\gamma_{\mathfrak{A}_0, z_0}^{(\xi)}$ the map determined by $U \mapsto \xi$ and $D_z \mapsto z_0$, the maximal ideal space of \mathfrak{A}_0 can be computed as the characters of \mathfrak{A}_α were using the technique found in Propositions 4.3.6 and 4.3.7. If

$$\psi = \min \left\{ \theta : e^{i\theta} = \mu^k, 0 \leq k \leq K-1, 0 < \theta < 2\pi \right\},$$

then

$$\mathcal{M}_{\mathfrak{A}_0} = \left\{ \gamma_{\mathfrak{A}_0, r e^{i\theta_0}}^{(\xi)} : 0 \leq r \leq 1, |\xi| \leq 1, 0 \leq \theta_0 < \psi \right\}$$

is the set of characters on \mathfrak{A}_0 .

As in [Pet2, V.9], for an ideal $\mathcal{I} \subseteq \mathfrak{A}_0$ define

$$\tilde{\mathcal{I}} = \{F \in \mathfrak{A}_\alpha : \#(H\tilde{\alpha}^n(F)G) \in \mathcal{I} \ \forall \ H, G \in \mathfrak{A}_\alpha, n \geq 0\}.$$

Using $\#$ we can then construct a one-to-one correspondence between the maximal ideals in \mathfrak{A}_0 and the maximal $\tilde{\alpha}$ -invariant ideals in \mathfrak{A}_α .

Proposition 4.4.6. If \mathcal{I} is a closed ideal in \mathfrak{A}_0 , $\tilde{\mathcal{I}}$ is a closed $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α . Moreover, $\tilde{\mathcal{I}}$ is the unique maximal element of the set $\{\mathcal{J} : \mathcal{J} \text{ is an } \tilde{\alpha}\text{-invariant ideal in } \mathfrak{A}_\alpha, \#(\mathcal{J}) \subseteq \mathcal{I}\}$.

Proof. If $F \in \tilde{\mathcal{I}}$ and $J \in \mathfrak{A}_\alpha$, then $FJ \in \tilde{\mathcal{I}}$ since $\#(H\tilde{\alpha}^n(FJ)G) = \#(H\tilde{\alpha}^n(F)(\tilde{\alpha}^n(J)G)) \in \mathcal{I} \ \forall \ n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$. Similarly, $JF \in \tilde{\mathcal{I}}$. The continuity of $\#$ shows that $\tilde{\mathcal{I}}$ is closed if \mathcal{I} is. Further, $\tilde{\mathcal{I}}$ is $\tilde{\alpha}$ -invariant since if $F \in \tilde{\mathcal{I}}$, then $\#(H\tilde{\alpha}(\tilde{\alpha}^n(F))G) = \#(H\tilde{\alpha}^{n+1}(F)G) \in \mathcal{I} \ \forall \ n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$. Let \mathcal{R} be any $\tilde{\alpha}$ -invariant ideal such that $\#(\mathcal{R}) \subseteq \mathcal{I}$. If $F \in \mathcal{R}$, then $H\tilde{\alpha}^n(F)G \in \mathcal{R} \ \forall \ n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$ so that $\#(H\tilde{\alpha}^n(F)G) \in \#(\mathcal{R}) \subseteq \mathcal{I}$ and hence $\mathcal{R} \subseteq \tilde{\mathcal{I}}$. \square

Proposition 4.4.7. (i) If $\mathcal{M}_0 \subseteq \mathfrak{A}_0$ is a maximal ideal, then $\widetilde{\mathcal{M}_0} \subseteq \mathfrak{A}_\alpha$ is a maximal $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α .

(ii) If \mathcal{R} is a maximal $\tilde{\alpha}$ -invariant ideal, then $\#(\mathcal{R}) \subseteq \mathfrak{A}_0$ is a maximal ideal. Furthermore, $\mathcal{R} = \widetilde{\#(\mathcal{R})}$.

Proof. By Proposition 4.4.6, \mathcal{M}_0 closed implies $\widetilde{\mathcal{M}_0}$ is a maximal ideal such that $\#(\widetilde{\mathcal{M}_0}) \subseteq \mathcal{M}_0$. But, $\mathcal{M}_0 \subseteq \#(\widetilde{\mathcal{M}_0})$ by $\tilde{\alpha}$ -invariance of \mathfrak{A}_0 so that $\#(\widetilde{\mathcal{M}_0}) = \mathcal{M}_0$ and $\widetilde{\mathcal{M}_0}$ is proper in \mathfrak{A}_α . Let $\mathcal{R} \supseteq \widetilde{\mathcal{M}_0}$ be a closed $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α . If $\#(\mathcal{R}) = \mathfrak{A}_0$, then $\mathcal{R} \cap \mathfrak{A}_0 = \mathfrak{A}_0$ by Proposition 4.4.5 and thus $D_1 \in \mathcal{R}$, a clear contradiction. Hence, $\#(\mathcal{R})$ is a proper ideal in \mathfrak{A}_0 . Now, $\#(\mathcal{R}) \supseteq \#(\widetilde{\mathcal{M}_0}) = \mathcal{M}_0$ so maximality of \mathcal{M}_0 shows that $\#(\mathcal{R}) = \mathcal{M}_0$ and by uniqueness of Proposition 4.4.6, $\mathcal{R} = \widetilde{\mathcal{M}_0}$. Hence $\widetilde{\mathcal{M}_0}$ is a maximal $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α and (i) follows.

If $F \in \mathcal{R}$, then $\#(H\tilde{\alpha}^n(F)G) = \#(\#(H\tilde{\alpha}^n(F)G)) \subseteq \#(\mathcal{R}) \forall n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$ since \mathcal{R} is an $\tilde{\alpha}$ -invariant ideal. Hence, $F \in \widetilde{\#(\mathcal{R})}$. \mathcal{R} a maximal $\tilde{\alpha}$ -invariant ideal gives $\mathcal{R} = \widetilde{\#(\mathcal{R})}$ by Proposition 4.4.6.

Let $\mathcal{M}_0 \supseteq \#(\mathcal{R})$ be a maximal ideal in \mathfrak{A}_0 . By (i), $\widetilde{\mathcal{M}_0}$ is a maximal $\tilde{\alpha}$ -invariant ideal in \mathfrak{A}_α . Since $\widetilde{\mathcal{M}_0} \supseteq \widetilde{\#(\mathcal{R})} = \mathcal{R}$ by the above paragraph, we have that $\mathcal{R} = \widetilde{\mathcal{M}_0}$ by maximality of \mathcal{R} . Thus, $\mathcal{M}_0 = \#(\widetilde{\mathcal{M}_0}) = \#(\mathcal{R})$ is maximal in \mathfrak{A}_0 . \square

Having constructed a one-to-one correspondence between the maximal $\tilde{\alpha}$ -invariant ideals in \mathfrak{A}_α and the maximal ideals in \mathfrak{A}_0 . We now characterize the maximal ideals in \mathfrak{A}_α that are $\tilde{\alpha}$ -invariant and those that are not.

Let \mathcal{M} be a maximal ideal in \mathfrak{A}_α . Let $\langle U \rangle$ denote the closed ideal in \mathfrak{A}_α generated by U . Then, $\mathcal{M} \cdot \langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$. Furthermore, $\tilde{\alpha}(\mathcal{M})$ maximal implies $\tilde{\alpha}(\mathcal{M})$ is prime (Theorem 2.2.6) so that either $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$ or $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$.

Theorem 4.4.8. If α is nonperiodic, the maximal ideal space of \mathfrak{A}_α is precisely the space of characters.

Proof. Let \mathcal{M} be maximal in \mathfrak{A}_α . If $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$ then $\mathcal{M} = \tilde{\alpha}(\mathcal{M})$. Thus, \mathcal{M} is $\tilde{\alpha}$ -invariant and by Proposition 4.4.7,

$$\mathcal{M} = \widetilde{\#(\mathcal{M})} = \{F \in \mathfrak{A}_\alpha : \#(H\tilde{\alpha}^n(F)G) \in \ker \gamma \forall n \geq 0 \text{ and } H, G \in \mathfrak{A}_\alpha\}$$

for some γ a character on \mathfrak{A}_0 . To show that $\mathcal{M} = \ker \gamma_0^{(\xi)}$ for some $\xi \in \overline{\mathbb{D}}$, we need only show $\mathcal{M} \subseteq \ker \gamma_0^{(\xi)}$. But $F \sim \sum_{n=0}^{\infty} U^n D_{f_n}, F \in \mathcal{M}$ implies

$$0 = \gamma(\#(F)) = \gamma\left(\sum_{n=0}^{\infty} U^n D_{f_n(0)}\right) = \sum_{n=0}^{\infty} f_n(0)\xi^n = \gamma_0^{(\xi)}(F)$$

for some $\xi \in \overline{\mathbb{D}}$.

If $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$, then by applying $\tilde{\alpha}^{-1}$ it follows that $\langle U \rangle \subseteq \mathcal{M}$. $\mathcal{M}/\langle U \rangle$ is then a maximal ideal in $\mathfrak{A}_\alpha/\langle U \rangle$. It is not hard to show that $\langle U^n \rangle = \cap_{i=0}^{n-1} \ker \pi_i$ for all $n \geq 1$ where π_i is the projection onto the i th Fourier coefficient. Hence $\mathcal{A}(\mathbb{D}) \cong \mathfrak{A}_\alpha/\langle U \rangle$ via the map $f \mapsto D_f + \langle U \rangle$. So, $\mathcal{M}/\langle U \rangle$ corresponds to a maximal ideal in $\mathcal{A}(\mathbb{D})$; namely a kernel of point evaluation (Example 2.2.4). So, $\mathcal{M}/\langle U \rangle = \ker \gamma_z^{(0)}$ for some $z \in \overline{\mathbb{D}}$. \square

We now show that if α has period K , there are maximal ideals in \mathfrak{A}_α of codimension 1 and K^2 . Define S_K to be the $K \times K$ shift matrix given by

$$S_{ij} = \begin{cases} 1 & \text{if } i - j = 1 \pmod{K} \\ 0 & \text{otherwise} \end{cases}$$

and $T(f, \mu)$ to be the $K \times K$ diagonal matrix given by $T(f, \mu)_{j,j} = f(\mu^{j-1}z_0)$. For $|w_0| \leq 1$ and $|z_0| \leq 1$, define $\rho_{z_0, w_0} : \mathfrak{A}_\alpha \rightarrow M_K(\mathbb{C})$ on a dense subset by

$$\sum_{\ell=0}^{KL-1} U^\ell D_{f_\ell} \mapsto \sum_{\ell=0}^{KL-1} w_0^\ell S_K^\ell T(f_\ell, \mu)$$

Lemma 4.4.9. If $|z_0| \leq 1$, $|w_0| \leq 1$, then ρ_{z_0, w_0} is a contractive representation.

Proof. This follows by Theorem 4.2.1 since ρ_{z_0, w_0} is determined by two contractions S_K and $T(z, \mu)$ satisfying $T(z, \mu)S_K = \mu S_K T(z, \mu)$. \square

Once again the characterization of the contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is used. In fact, a direct proof of the Lemma 4.4.9 is possible but quite a bit more complicated. We present such a direct proof in the following four lemmas.

Lemma 4.4.10. For $|w_0| = 1$, define τ_{w_0} on a dense subset of \mathfrak{A}_α by

$$\tau_{w_0} \left(\sum_{n=0}^N U^n D_{f_n} \right) = \sum_{n=0}^N U^n w_0^n D_{f_n}.$$

Then τ_{w_0} is isometric and hence extends to an isometric automorphism of \mathfrak{A}_α .

Proof. Define a Hilbert space isomorphism Λ_{w_0} on $\bigoplus_{n=0}^\infty L^2(\mathbb{T})$ by $\{\xi_n\}_{n=0}^\infty \mapsto \{\bar{w}_0^n \xi_n\}_{n=0}^\infty$. Let

$F = \sum_{n=0}^N U^n D_{f_n}$. Then

$$\begin{aligned}
\|\tau_{w_0}(F)\|^2 &= \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{T})}} \left\| \left(\sum_{n=0}^N U^n w_0^n D_{f_n} \right) (\xi_0, \xi_1, \xi_2, \dots) \right\|^2 \\
&= \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{T})}} \|f_0 \xi_0\|^2 + \|\alpha(f_0) \xi_1 + f_1 w_0 \xi_0\|^2 + \|\alpha^2(f_0) \xi_2 + \alpha(f_1) w_0 \xi_1 + f_2 w_0^2 \xi_0\|^2 + \dots \\
&= \sup_{\substack{\|\xi\|=1 \\ \xi \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{T})}} \|f_0 \xi_0\|^2 + \|\alpha(f_0)(\bar{w}_0 \xi_1) + f_1 \xi_0\|^2 + \|\alpha^2(f_0)(\bar{w}_0^2 \xi_2) + \alpha(f_1)(\bar{w}_0 \xi_1) + f_2 \xi_0\|^2 + \dots \\
&= \sup_{\substack{\|\Lambda_{w_0}(\xi)\|=1 \\ \xi \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{T})}} \|f_0 \Lambda_{w_0}(\xi)_0\|^2 + \|\alpha(f_0) \Lambda_{w_0}(\xi)_2 + \alpha(f_1) \Lambda_{w_0}(\xi)_1 + f_0 \Lambda_{w_0}(\xi)_0\|^2 + \dots
\end{aligned}$$

and since Λ_{w_0} is a Hilbert space isomorphism, the last supremum is equal to $\|F\|^2$. Since τ_{w_0} is isometric in a dense subset, it extends to an isometry on \mathfrak{A}_α . It is easy to see that τ_{w_0} is a homomorphism onto and hence an isometric automorphism of \mathfrak{A}_α . \square

Lemma 4.4.11. If $|w_0| = 1$ and $|z_0| \leq 1$, then ρ_{z_0, w_0} is contractive if $\rho_{z_0, 1}$ is contractive.

Proof. By Lemma 4.4.10,

$$\begin{aligned}
\left\| \rho_{z_0, w_0} \left(\sum_{\ell=0}^{KL-1} U^\ell D_{f_\ell} \right) \right\| &= \left\| \rho_{z_0, 1} \left(\tau_{w_0} \left(\sum_{\ell=0}^{KL-1} U^\ell D_\ell \right) \right) \right\| \\
&\leq \left\| \tau_{w_0} \left(\sum_{\ell=0}^{KL-1} U^\ell D_{f_\ell} \right) \right\| \\
&= \left\| \sum_{\ell=0}^{KL-1} U^\ell D_{f_\ell} \right\|
\end{aligned}$$

\square

Lemma 4.4.12. If $|z_0| \leq 1$, then $\rho_{z_0, 1}$ is a contractive representation of \mathfrak{A}_α .

Proof. That $\rho_{z_0, 1}$ is a homomorphism is easy to verify. Consider an element of the form

$\sum_{l=0}^{KL-1} U^l D_{f_l}$. Then,

$$\begin{aligned} & \left\| \rho_{z_0,1} \left(\sum_{l=0}^{KL-1} U^l D_{f_l} \right) \right\|^2 \\ &= \sup_{\|c\|_2=1} \left\| \begin{pmatrix} \sum_{l=0}^{L-1} f_{Kl}(z_0) & \sum_{l=0}^{L-1} f_{Kl+K-1}(\mu z_0) & \cdots & \sum_{l=0}^{L-1} f_{Kl+1}(\mu^{K-1} z_0) \\ \sum_{l=0}^{L-1} f_{Kl+1}(z_0) & \sum_{l=0}^{L-1} f_{Kl}(\mu z_0) & \cdots & \sum_{l=0}^{L-1} f_{Kl+2}(\mu^{K-1} z_0) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{l=0}^{L-1} f_{Kl+K-1}(z_0) & \sum_{l=0}^{L-1} f_{Kl+K-2}(\mu z_0) & \cdots & \sum_{l=0}^{L-1} f_{Kl}(\mu^{K-1} z_0) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} \right\|^2 \\ &= \sup_{\sum_{k=0}^{K-1} |c_k|^2=1} \sum_{j=0}^{K-1} \left| \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} f_{Kl+k}(\mu^{K-k+j} z_0) c_{K-k+j \pmod K} \right|^2 \end{aligned}$$

Suppose the supremum is achieved at $c = (c_0, c_1, \dots, c_{K-1})$. The K functions

$$F_j(z) \triangleq \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} \alpha^{K-k+j} (f_{Kl+k})(z) c_{K-k+j \pmod K}$$

($j = 0, 1, \dots, K-1$) are all analytic on \mathbb{D} and continuous on $\overline{\mathbb{D}}$. Hence, by the maximum modulus theorem, $\forall j = 0, 1, \dots, K-1$, $\exists z_0^{(j)} \in \mathbb{T}$ such that

$$|F_j(z_0)| \leq |F_j(z_0^{(j)})|. \quad (4.3)$$

Let $\tilde{\epsilon} > 0$ be given. By continuity of the square of the absolute value of these K functions at $z_0^{(j)}$ ($j = 0, 1, \dots, K-1$), $\exists \delta > 0$ such that $|z - z_0^{(j)}| < \delta/2$ implies

$$\left| |F_j(z)|^2 - |F_j(z_0^{(j)})|^2 \right| < \frac{\tilde{\epsilon}}{K} \quad (4.4)$$

for $j = 0, 1, \dots, K-1$. Let $\epsilon > 0$ also be given. Choose N such that $\frac{KL}{N} < \epsilon$. Define ξ_{Ki+j} on \mathbb{T} by

$$\xi_{Ki+j}(z) = \begin{cases} \frac{c_j}{\sqrt{N\delta}} & |z - z_0^{(j)}| < \frac{\delta}{2} \\ 0 & |z - z_0^{(j)}| \geq \frac{\delta}{2} \end{cases}$$

for $i = 0, 1, \dots$ and $j = 0, 1, \dots, K-1$ except to take $\xi_n = 0$ for $n = 0$ or $n \geq KL + KN$.

Then, $\|\xi_{Ki+j}\|^2 = \int_{\mathbb{T}} |\xi_{Ki+j}(z)|^2 d\mu(z) = \delta \cdot \frac{|c_j|^2}{N\delta} = \frac{|c_j|^2}{N}$ for $i = 0, 1, \dots$ and $j = 0, 1, \dots, K-1$

(except, of course $\|\xi_n\| = 0$ if $n = 0$ or $n \geq KL + KN$). It follows that

$$\begin{aligned}
\frac{1}{N}|F_j(z_0)|^2 &\leq \frac{1}{N}|F_j(z_0^{(j)})|^2 \quad \text{by (4.3)} \\
&= \frac{1}{\delta} \int_{|z-z_0^{(j)}| < \frac{\delta}{2}} \frac{1}{N}|F_j(z_0^{(j)})|^2 d\mu(z) \\
&< \frac{1}{\delta} \int_{|z-z_0^{(j)}| < \frac{\delta}{2}} \frac{1}{N}|F_j(z)|^2 + \frac{\tilde{\epsilon}}{KN} d\mu(z) \quad \text{by (4.4)} \\
&= \left\| \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} \alpha^{K-k+j} (f_{Kl+k}) \xi_{KL-Kl-k+Ki+j} \right\|^2 + \frac{\tilde{\epsilon}}{KN}
\end{aligned}$$

for $i = 0, 1, \dots$ and $j = 0, 1, \dots, K-1$ and $0 < KL - Kl - k + Ki + j < KL + KN$. Hence,

$$\sum_{j=0}^{K-1} |F_j(z_0)|^2 \leq \sum_{j=0}^{K-1} \sum_{i=0}^{N-1} \left\| \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} \alpha^{K-k+j} (f_{Kl+k}) \xi_{KL-Kl-k+Ki+j} \right\|^2 + \tilde{\epsilon} \quad (4.5)$$

Now,

$$\sum_{j=0}^{K-1} \sum_{i=0}^{N-1} \|\xi_{KL-Kl-k+Ki+j}\|^2 + \sum_{l=1}^{KL-1} \|\xi_{KL-l}\|^2 \leq \sum_{j=0}^{K-1} |c_j|^2 + \frac{KL-1}{N} < 1 + \epsilon \quad (4.6)$$

since $\frac{KL}{N} < \epsilon$ and $\sum_{j=0}^{K-1} |c_j|^2 = 1$. It follows by (4.5) and (4.6) that

$$\begin{aligned}
\left\| \rho_{z_0,1} \left(\sum_{l=0}^{KL-1} U^l D_{f_l} \right) \right\|^2 &= \sum_{j=0}^{K-1} |F_j(z_0)|^2 \\
&\leq \sup_{\|\xi\| \leq 1+\epsilon} \left\| \left(\sum_{l=0}^{KL-1} U^l D_{f_l} \right) (\xi_0, \xi_1, \xi_2, \dots) \right\|^2 + \tilde{\epsilon}
\end{aligned}$$

Letting $\tilde{\epsilon} \downarrow 0^+$ and $\epsilon \downarrow 0^+$ then gives the result. \square

Lemma 4.4.13. If $|w_0| < 1$ and $|z_0| \leq 1$, then ρ_{z_0, w_0} is a contractive representation.

Proof. Let S_K and $T(f, \mu)$ be the $K \times K$ matrices found in the definition of ρ_{z_0, w_0} . Note that

$$\begin{aligned}
\left\| \rho_{z_0, w_0} \left(\sum_{i=0}^N U^i D_{f_i} \right) \right\| &= \sup_{\substack{\|h\|=1 \\ \|k\|=1}} \left| \left\langle \sum_{i=0}^N w_0^i S_K^i T(f_i, \mu) h, k \right\rangle \right| \\
&= \left| \left\langle \sum_{i=0}^N w_0^i S_K^i T(f_i, \mu) h_0, k_0 \right\rangle \right|
\end{aligned}$$

for some $\|h_0\| = 1 = \|k_0\|$. We define a function $g(w_0)$ on \mathbb{D} to be this quantity. Then $g(w_0)$ is analytic on $\overline{\mathbb{D}}$ and by the maximum modulus theorem, $|g(w_0)| \leq |g(w'_0)|$ for some $w'_0 \in \mathbb{T}$.

Hence,

$$\begin{aligned}
\left\| \rho_{z_0, w_0} \left(\sum_{i=0}^n U^i D_{f_i} \right) \right\| &\leq |g(w'_0)| \\
&\leq \sup_{\substack{\|h\|=1 \\ \|k\|=1}} \left| \left\langle \sum_{i=0}^N (w'_0)^i S_K^i T(f_i, \mu) h, k \right\rangle \right| \\
&= \left\| \rho_{z_0, w'_0} \left(\sum_{i=0}^N U^i D_{f_i} \right) \right\| \\
&\leq \left\| \sum_{i=0}^N U^i D_{f_i} \right\|
\end{aligned}$$

by Lemma 4.4.12. □

We see that Lemma 4.4.9 now follows from Lemmas 4.4.10, 4.4.11, 4.4.12, and 4.4.13 by a direct argument. The effectiveness of Theorem 3.4.10 again becomes clear.

By the simplicity of $M_K(\mathbb{C})$ and Lemma 4.4.14, $\ker \rho_{z_0, w_0}$ is a maximal ideal in \mathfrak{A}_α if $z_0 \neq 0$ and $w_0 \neq 0$.

Lemma 4.4.14. If $z_0 \neq 0$ and $w_0 \neq 0$, then ρ_{z_0, w_0} is a contractive representation of \mathfrak{A}_α onto $M_K(\mathbb{C})$.

Proof. We need only show that if $z_0 \neq 0$ and $w_0 \neq 0$, then ρ_{z_0, w_0} is onto. For $0 \leq i, j \leq K-1$, define

$$f_{i,j}(z) = \frac{1}{w_0^{K+i-j(\bmod K)}} \cdot \frac{\prod_{\substack{l=0 \\ l \neq K-i}}^{K-1} (\mu^l z - z_0)}{\prod_{l=1}^{K-1} (\mu^l z_0 - z_0)}$$

Then, for $0 \leq k \leq K-1$,

$$\begin{aligned}
f_{i,j}(\mu^k z_0) &= \frac{1}{w_0^{K+i-j(\bmod K)}} \cdot \frac{\prod_{\substack{l=0 \\ l \neq K-i(\bmod K)}}^{K-1} (\mu^{k+l} z_0 - z_0)}{\prod_{l=1}^{K-1} (\mu^l z_0 - z_0)} \\
&= \frac{1}{w_0^{K+i-j(\bmod K)}} \delta_{k,i}
\end{aligned}$$

where $\delta_{k,i}$ is the Kronecker function (i.e. $\delta_{k,i} = 1$ if $k = i$ and 0 otherwise). Hence, ρ_{z_0, w_0} is onto $M_K(\mathbb{C})$ as $E_{ij} = \rho_{z_0, w_0}(U^{K+i-j(\bmod K)} D_{f_{i,j}})$. □

Theorem 4.4.15. If α has period K and \mathcal{M} is a maximal ideal in \mathfrak{A}_α , then $\mathcal{M} = \ker \rho_{z_0, w_0}$ for some $z_0, w_0 \in \overline{\mathbb{D}}$.

Proof. As in the nonperiodic case, $\tilde{\alpha}(\mathcal{M})$ is maximal and hence prime with either $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$ or $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$. If $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$, then since $\langle U \rangle$ is $\tilde{\alpha}$ -invariant and α is periodic, $\langle U \rangle \subseteq \tilde{\alpha}^K(\mathcal{M}) = \mathcal{M}$. Thus, $\mathcal{M} = \ker \rho_{z_0, 0}$ for some $z_0 \in \overline{\mathbb{D}}$. Suppose then $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$ so that $\mathcal{M} = \tilde{\alpha}(\mathcal{M})$. By Proposition 4.4.7, $\#(\mathcal{M}) = \ker \gamma$ for some character γ on \mathfrak{A}_0 and hence $\mathcal{M} = \widetilde{\#(\mathcal{M})} = \widetilde{\ker \gamma_{\mathfrak{A}_0, z'_0}^{(\xi')}} = \ker \gamma_{\mathfrak{A}_0, z'_0}^{(\xi')}$ for some $\xi' \in \overline{\mathbb{D}}$ and $z'_0 = re^{i\theta}$ where $0 \leq r \leq 1$, $0 \leq \theta < \psi$, and $\psi = \min \{ \theta : e^{i\theta} = \mu^k, 0 \leq k \leq K-1, 0 < \theta < 2\pi \}$. Since $\ker \rho_{z_0, w_0}$ is maximal in \mathfrak{A}_α , we need only show that $\ker \rho_{z_0, w_0} \subseteq \mathcal{M}$ for some z_0, w_0 . But, $\ker \rho_{z'_0, \xi'}$ is $\tilde{\alpha}$ -invariant so that $\rho_{z'_0, \xi'}(\tilde{\alpha}^n(F)) = 0 \ \forall \ n \geq 0$ and $F \in \ker \rho_{z'_0, \xi'}$. Hence $\rho_{z'_0, \xi'}(\#(H\tilde{\alpha}^n(F)G)) = 0 \ \forall \ n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$ yielding $\gamma_{z'_0}^{(\xi')}(\#(H\tilde{\alpha}^n(F)G)) = 0$ and $F \in \mathcal{M}$. \square

4.5 The Strong Radical

Having computed the maximal ideal space of \mathfrak{A}_α , we can now compute its strong radical and compare it to its Jacobson radical. In this section, α will remain fixed.

Theorem 4.5.1. Let \mathfrak{A}_J and \mathfrak{A}_S denote the Jacobson and strong radicals of \mathfrak{A}_α respectively.

- (i) if α is parabolic or hyperbolic, $\mathfrak{A}_J = \mathfrak{A}_S$.
- (ii) if α is elliptic and nonperiodic, $\mathfrak{A}_J \subsetneq \mathfrak{A}_S$.
- (iii) if α is elliptic and periodic, $\mathfrak{A}_J = \mathfrak{A}_S = (0)$.

Proof. From [HPW], the Jacobson radical is precisely the set of quasinilpotents. If α is parabolic or hyperbolic, then

$$\mathfrak{A}_J = \{ F \in \mathfrak{A} : \pi_0(F) \equiv 0 \text{ and } \pi_n(F)(z_0) = 0 \ \forall \ n \geq 1 \text{ for } z_0 \text{ fixed by } \alpha \}.$$

Since the maximal ideals are precisely the kernels of the characters in these cases, (i) follows as

$$\begin{aligned}
\mathfrak{A}_S &= \left\{ F \in \mathfrak{A}_\alpha : F \in \ker \gamma_z^{(0)} \ \forall z \in \overline{\mathbb{D}} \text{ and } F \in \ker \gamma_{z_0}^{(\xi)} \ \forall \xi \in \overline{\mathbb{D}} \text{ for } z_0 \text{ fixed by } \alpha \right\} \\
&= \left\{ F : \pi_0(F) \equiv 0 \text{ and } \sum_{n \geq 1} \pi_n(F)(z_0) \xi^n = 0 \ \forall \xi \in \overline{\mathbb{D}} \right\} \\
&= \{ F : \pi_0(F) \equiv 0 \text{ and } \pi_n(F)(z_0) = 0 \ \forall n \geq 1 \text{ for } z_0 \text{ fixed by } \alpha \} \\
&= \mathfrak{A}_J.
\end{aligned}$$

If α is elliptic, $\mathfrak{A}_J = (0)$. When α is nonperiodic, $\mathfrak{A}_S \supsetneq (0)$ as $UD_z \in \mathfrak{A}_S$ for example. In fact,

$$\mathfrak{A}_S = \{ F : \pi_0(F) \equiv 0 \text{ and } \pi_n(F)(0) = 0 \ \forall n \geq 1 \}.$$

If α is periodic of period K , we show $\mathfrak{A}_S = (0)$ to complete the proof. First, note that the Fourier series of $F \in \mathfrak{A}_\alpha$ is Cesàro summable [Pet2]. Hence,

$$\lim_{N \rightarrow \infty} \left\| \sum_{l=0}^{KN-1} \frac{1}{KN} \left(\sum_{m=0}^l U^m D_{\pi_m(F)} \right) - F \right\| = 0.$$

Let $F \in \mathfrak{A}_S$ and $\epsilon > 0$ be given. We show that $\pi_l(F) \equiv 0 \ \forall l \geq 0$ so that $F = 0$. Choose M such that if $N \geq M$ we have

$$\left\| \sum_{l=0}^{KN-1} \frac{1}{KN} \left(\sum_{m=0}^l U^m D_{\pi_m(F)} \right) - F \right\| = \left\| \sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} - F \right\| < \epsilon.$$

Then,

$$\left\| \rho_{z_0, w} \left(\sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} - F \right) \right\| < \epsilon$$

$\forall z_0, w \in \overline{\mathbb{D}}$ by Lemma 4.4.9. Since $F \in \mathfrak{A}_S$,

$$\left\| \rho_{z_0, w} \left(\sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} \right) \right\| < \epsilon.$$

In particular, for $0 \leq k \leq K-1$ we have

$$\left| \sum_{l=0}^{N-1} \left(1 - \frac{Kl+k}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| < \epsilon$$

$\forall z_0 \in \overline{\mathbb{D}}, w \in \mathbb{T}$. Fix $l_0 \geq 0$. Note that

$$\int_{\mathbb{T}} \left| \sum_{l=0}^{N-1} \left(1 - \frac{Kl+k}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| dm(w) < \epsilon$$

$\forall z_0 \in \overline{\mathbb{D}}$. It follows, since $\int_{\mathbb{T}} z^l dm(z) = 0$ unless $l = -1$, that

$$\begin{aligned} & \left| 1 - \frac{Kl_0+k}{KN} \right| |\pi_{Kl_0+k}(F)(z_0)| \left| \int_{\mathbb{T}} w^{-1} dm(w) \right| \\ &= \left| \int_{\mathbb{T}} \left(1 - \frac{Kl_0+k}{KN} \right) \pi_{Kl_0+k}(F)(z_0) w^{-1} dm(w) \right| \\ &= \left| \int_{\mathbb{T}} \sum_{l=0}^{N-1} \left(1 - \frac{(Kl+k)}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{K(l-l_0)-1} dm(w) \right| \\ &\leq \int_{\mathbb{T}} \left| \sum_{l=0}^{N-1} \left(1 - \frac{(Kl+k)}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| |w^{-Kl_0-k-1}| dm(w) \\ &= \int_{\mathbb{T}} \left| \sum_{l=0}^{N-1} \left(1 - \frac{(Kl+k)}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| dm(w) < \epsilon \end{aligned}$$

$\forall z_0 \in \overline{\mathbb{D}}$. Choosing $N \geq M$ large enough so that $\frac{Kl_0+k}{KN} < \frac{1}{2}$ it follows that $|\pi_{Kl_0+k}(F)(z_0)|$ is arbitrarily small $\forall z_0 \in \overline{\mathbb{D}}$ so that $\pi_{Kl_0+k}(F) \equiv 0$ for $0 \leq k \leq K-1$ and hence $F = 0$. \square

4.6 Shilov Modules for $\mathbb{Z}^+ \times_{\alpha} \mathcal{A}(\mathbb{D})$

Definition 4.6.1. Let \mathcal{P} and \mathcal{K} be contractive Hilbert modules over an operator algebra \mathfrak{A} . A $(\mathcal{P}, \mathcal{K})$ -derivation is a continuous complex linear map $D : \mathfrak{A} \rightarrow B(\mathcal{P}, \mathcal{K})$ such that $D(FG) = D(F)\rho_{\mathcal{P}}(G) + \rho_{\mathcal{K}}(F)D(G)$ for all $F, G \in \mathfrak{A}$.

Lemma 4.6.2. [MS2, Proposition 2.6] Let

$$0 \rightarrow \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi} \mathcal{P} \rightarrow 0$$

be a short exact isometric sequence of contractive Hilbert modules. The representation $\rho_{\mathcal{M}}$ is unitarily equivalent to

$$F \longmapsto \begin{pmatrix} \rho_{\mathcal{K}}(F) & D(F) \\ 0 & \rho_{\mathcal{P}}(F) \end{pmatrix}$$

for some derivation D . If the only derivation for which this occurs is $D = 0$ then \mathcal{P} is orthogonally projective.

Example 4.6.3. Let $\mathcal{P} = \mathcal{K} = \mathbb{C}$ be contractive Hilbert modules over \mathfrak{A}_α determined by the character $\gamma_z^{(0)}$. Then $D : \mathfrak{A}_\alpha \rightarrow B(\mathbb{C})$ defined by $D(FG) = \pi_1(FG)(z)$ is a $(\mathcal{P}, \mathcal{K})$ -derivation by Lemma 4.3.5.

Recall that if \mathcal{P} is a contractive Hilbert module over \mathfrak{A}_α , then $S_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(U)$ and $T_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(D_z)$ are contractions satisfying the relation $T_{\mathcal{P}}S_{\mathcal{P}} = S_{\mathcal{P}}\varphi(T_{\mathcal{P}})$ (Theorem 4.2.1).

Theorem 4.6.4. Let \mathcal{P} be a contractive Hilbert module over \mathfrak{A}_α . If $S_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(U)$ and $T_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(D_z)$ are isometries, then \mathcal{P} is orthogonally projective.

Proof. Let $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow 0$ be a short exact isometric sequence of contractive Hilbert modules ending at \mathcal{P} . Let D be a $(\mathcal{P}, \mathcal{K})$ -derivation such that $\rho_{\mathcal{M}}$ is unitarily equivalent to the representation ρ on $\mathcal{K} \oplus \mathcal{P}$ given by

$$F \mapsto \begin{pmatrix} \rho_{\mathcal{K}}(F) & D(F) \\ 0 & \rho_{\mathcal{P}}(F) \end{pmatrix}.$$

By Lemma 4.6.2, we need only show that $D \equiv 0$ for it to follow that \mathcal{P} is orthogonally projective. Since \mathcal{P}, \mathcal{M} , and \mathcal{K} are contractive Hilbert modules and $\rho_{\mathcal{M}}$ is unitarily equivalent to ρ , it follows that

$$\begin{pmatrix} \rho_{\mathcal{K}}(D_z) & D(D_z) \\ 0 & \rho_{\mathcal{P}}(D_z) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_{\mathcal{K}}(U) & D(U) \\ 0 & \rho_{\mathcal{P}}(U) \end{pmatrix}$$

are contractions. Since $T_{\mathcal{P}}$ is an isometry,

$$\begin{aligned} \left\| \begin{pmatrix} \rho_{\mathcal{K}}(D_z) & D(D_z) \\ 0 & T_{\mathcal{P}} \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} * & * \\ * & D(D_z)^*D(D_z) + T_{\mathcal{P}}^*T_{\mathcal{P}} \end{pmatrix} \right\| \\ &\geq \|D(D_z)^*D(D_z) + I_{\mathcal{P}}\| > 1 \end{aligned}$$

if $D(D_z) \neq 0$. Similarly,

$$\left\| \begin{pmatrix} \rho_{\mathcal{K}}(U) & D(U) \\ 0 & S_{\mathcal{P}} \end{pmatrix} \right\|^2 > 1$$

if $D(U) \neq 0$. Hence, $D(D_z) = 0 = D(U)$. Since D is a derivation, $D(\sum_{i=0}^n U^i D_{p_i}) = 0$ for all polynomials p_i and hence $D \equiv 0$ by continuity. \square

Corollary 4.6.5. Let \mathcal{P} be a contractive Hilbert module over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$. If $S_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(U)$ and $T_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(D_z)$ are isometries, then \mathcal{P} is Shilov.

Proof. Since \mathcal{P} is contractive, it is completely contractive by Theorem 4.2.6. Hence by Proposition 3.5.17, \mathcal{P} is Shilov. \square

Theorem 4.6.6. Let \mathcal{P} be a contractive Hilbert module over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ determined by $S_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(U)$ and $T_{\mathcal{P}} \equiv \rho_{\mathcal{P}}(D_z)$. Then, the following are equivalent:

- (i) $S_{\mathcal{P}}$ and $T_{\mathcal{P}}$ are isometries satisfying $T_{\mathcal{P}}S_{\mathcal{P}} = S_{\mathcal{P}}\varphi(T_{\mathcal{P}})$.
- (ii) \mathcal{P} is an orthogonally projective Hilbert module for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$.
- (iii) \mathcal{P} is a Shilov module for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$.

Proof. That (i) \Rightarrow (ii) follows from Theorem 4.6.4 and (ii) \Rightarrow (iii) by Corollary 4.6.5. We show (iii) \Rightarrow (i). By Lemma 4.2.5 and the fact the \mathcal{P} is Shilov, there exists a module \mathcal{M} over $\mathbb{Z} \times_\alpha C(\mathbb{T})$ such that \mathcal{P} is isomorphic to a submodule of \mathcal{M} where \mathcal{M} is viewed as a module over $\widehat{\imath}(\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}))$ and $\widehat{\imath}$ is the embedding given in Proposition 4.2.3. We investigate the modules \mathcal{M} over $\mathbb{Z} \times_\alpha C(\mathbb{T})$; that is, the C^* -representations $\rho_{\mathcal{M}}$ of $\mathbb{Z} \times_\alpha C(\mathbb{T})$ on \mathcal{M} . But, the C^* -representations $\rho_{\mathcal{M}}$ of $\mathbb{Z} \times_\alpha C(\mathbb{T})$ on \mathcal{M} correspond bijectively to the covariant representations of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ on \mathcal{M} . These covariant representations (π, U) of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ on \mathcal{M} in turn correspond to pairs of unitaries $V \equiv \pi(z)$ and U on \mathcal{M} satisfying $VU = U\varphi(V)$. Hence, $\rho_{\mathcal{M}}$ is determined by two unitaries $V \equiv \rho_{\mathcal{M}}(\delta_0 \otimes z)$ and $U \equiv \rho_{\mathcal{M}}(\delta_1 \otimes 1)$ satisfying $VU = U\varphi(V)$. By the structure of the embedding $\widehat{\imath}$, $\rho_{\mathcal{P}}$ is determined by two isometries $\rho_{\mathcal{P}}(\delta_0 \otimes z) \equiv V|_{\mathcal{P}}$ and $\rho_{\mathcal{P}}(\delta_1 \otimes 1) \equiv U|_{\mathcal{P}}$. \square

4.7 An Invariant Subspace Problem

The characterization of the Shilov modules over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ in the previous section is far from ideal. Ideally one would like a characterization much like that given by the Wold decomposition for the Shilov modules over $\mathcal{A}(\mathbb{D})$. A more limited problem is to classify the

subspaces of $H^2(\mathbb{D})$ and $L^2(\mathbb{T})$ invariant under $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$. The purpose of this section is to do this for the case where the automorphism α is irrational.

Recall by Proposition 4.3.8 that when α is irrational we have $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) \cong \mathfrak{B}_\alpha$.

Proposition 4.7.1. The nontrivial invariant subspaces \mathcal{M} of $H^2(\mathbb{D})$ under \mathfrak{B}_α are of the form $\mathcal{M} = z^{m_0} H^2(\mathbb{D})$ for $m_0 \in \mathbb{N}$.

Proof. Note that \mathfrak{B}_α is the norm closed subalgebra of $B(H^2(\mathbb{D}))$ generated by the Toeplitz operator T_z and composition operator $C_{\bar{\mu}}$. Denote by \mathfrak{B}_{sc} the strongly closed algebra generated by the Toeplitz operator T_z and composition operator $C_{\bar{\mu}}$. Let $(0) \neq \mathcal{M} \subseteq H^2(\mathbb{D})$ be a closed invariant subspace under \mathfrak{B}_α . Then \mathcal{M} is also invariant under \mathfrak{B}_{sc} . To see this, note that if $f \in \mathcal{M}$ and $A \in \mathfrak{B}_{sc}$, then there exists a sequence $\{A_n\}_{n=1}^\infty \subseteq \mathfrak{B}_\alpha$ such that $\|A_n f - A f\| \rightarrow 0$. But, $A_n f \in \mathcal{M}$ for $n \geq 0$ and \mathcal{M} closed show that $A f \in \mathcal{M}$.

As in [NRW], we see that \mathfrak{B}_{sc} contains the projections $P_m : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ defined by $\sum_0^\infty a_n z^n \mapsto a_m z^m$ for each $m \in \mathbb{N}$. It follows that $P_m \mathcal{M} \neq (0)$ for some $m \in \mathbb{N}$ and hence that \mathcal{M} contains z^m for some $m \in \mathbb{N}$. Define $m_0 = \inf_{m \in \mathbb{N}} \{m : z^m \in \mathcal{M}\}$. Note that $z^{m_0} \in \mathcal{M}$ and $z^{m_0+k} \in \mathcal{M} \forall k \in \mathbb{N}$ by the invariance of \mathcal{M} under T_z . Since \mathcal{M} is a closed subspace of $H^2(\mathbb{D})$ invariant under T_z , it is clear that $z^{m_0} H^2(\mathbb{D}) \subseteq \mathcal{M}$. Let $f(z) = \sum_0^\infty a_n z^n \in \mathcal{M}$. Then $P_k(f(z)) = a_k z^k \in \mathcal{M} \forall k \in \mathbb{N}$. Hence $a_k = 0$ for $0 \leq k \leq m_0 - 1$ by the definition of m_0 and $f(z) = \sum_{n=m_0}^\infty a_n z^n = z^{m_0} \sum_{n=0}^\infty a_{n+m_0} z^n \in z^{m_0} H^2(\mathbb{D})$. Thus $\mathcal{M} \subseteq z^{m_0} H^2(\mathbb{D})$ and so $\mathcal{M} = z^{m_0} H^2(\mathbb{D})$. \square

Example 4.7.2. For α irrational, consider the Shilov resolution of \mathfrak{B}_α given by

$$0 \longrightarrow H_0^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}) \xrightarrow{\Phi} \mathbb{C}_{(1,0)} \longrightarrow 0$$

where $H_0^2(\mathbb{D}) = \{\xi \in H^2(\mathbb{D}) : \xi(0) = 0\}$, $\Phi(\xi) = \xi(0)$, and $\mathbb{C}_{(1,0)}$ is the one dimensional Hilbert module \mathbb{C} determined by the character $\gamma_0^{(1)}$. That this is a Shilov resolution follows since $H^2(\mathbb{D})$ is seen to be a Shilov module for \mathfrak{B}_α by considering Example 2.5.6. Note that Φ is a module

map since

$$\begin{aligned}
\Phi \left(\sum_n C_{\bar{\mu}}^n T_{f_n} \cdot \xi(z) \right) &= \Phi \left(\sum_n f_n(\bar{\mu}^n z) \xi(\bar{\mu}^n z) \right) \\
&= \sum_n f_n(0) \xi(0) \\
&= \left(\sum_n C_{\bar{\mu}}^n T_{f_n} \right) \cdot \xi(0) \\
&= \left(\sum_n C_{\bar{\mu}}^n T_{f_n} \right) \cdot \Phi(\xi(z)).
\end{aligned}$$

Since the Hilbert submodules of $H^2(\mathbb{D})$ are of the form $z^n H^2(\mathbb{D})$ and $\Phi(z^n H^2(\mathbb{D})) = 0$ for $n \geq 1$, it follows that $H^2(\mathbb{D})$ is strongly minimal for $\mathbb{C}_{(1,0)}$ by Definition 3.5.10.

Proposition 4.7.3. Let $\mu \in \mathbb{T}$ be irrational, M_z the multiplication operator on $L^2(\mathbb{T})$, and $C_{\bar{\mu}}$ the composition operator given by $C_{\bar{\mu}}(f(z)) = f(\bar{\mu}z)$. Then, there are no nontrivial invariant subspaces \mathcal{M} under the norm closed subalgebra of $B(L^2(\mathbb{T}))$ generated by $M_z, M_{\bar{z}}$, and $C_{\bar{\mu}}$.

Proof. Suppose $\mathcal{M} \subseteq L^2(\mathbb{T})$ is invariant under the norm closed algebra generated by these operators. Then, by Beurling's theory (Theorem 2.1.10), $\mathcal{M} = \chi_B(z)L^2(\mathbb{T})$ for some Borel set B in \mathbb{T} . If \mathcal{M} is invariant under $C_{\bar{\mu}}$, then in particular, $C_{\bar{\mu}}(\chi_B(z) \cdot 1) = \chi_B(\bar{\mu}) \in \chi_B(z)L^2(\mathbb{T})$. Hence $\chi_{\mu B}(z) \in \chi_B(z)L^2(\mathbb{T})$ and $B \supseteq \mu B$. By applying $C_{\bar{\mu}}^n$ for $n \geq 1$, we see that $B = \bigcup_{n \geq 0} \mu^n B$. The map $\varphi : \mathbb{T} \rightarrow \mathbb{T}$ given by $\varphi(z) = \mu z$ is ergodic by Example 2.6.2. Further, $\varphi^{-1}(B) = \varphi^{-1} \left(\bigcup_{n \geq 0} \mu^n B \right) \supseteq B$. But φ^{-1} is just a rotation so that $\varphi^{-1}(B) = (B)$. Hence, by Definition 2.6.1, $m(B) = 0$ or $m(B) = 1$ and $\mathcal{M} = (0)$ or $\mathcal{M} = L^2(\mathbb{T})$. \square

Remark 4.7.4. If α is irrational and $\mathbb{Z}^+ \times_{\alpha} C(\mathbb{T})$ [Pet1] is viewed as the norm closed subalgebra of $B(L^2(\mathbb{T}))$ generated by $M_z, M_{\bar{z}}$, and $C_{\bar{\mu}}$, then it has no proper Shilov modules.

Lemma 4.7.5. The algebra \mathfrak{B}_{sc} of Proposition 4.7.1 contains the projection $P_m \left(\sum_{n=-\infty}^{\infty} a_n z^n \right) = a_m z^m \ \forall m \in \mathbb{Z}$.

Proof. We first show $P_{-1} \in \mathfrak{B}_{sc}$. Define $C_{-1} = \bar{\mu} C_{\bar{\mu}}$ and $A_k = 1/k \sum_{j=1}^k C_{-1}^j$ for $k \geq 1$. Let

$$S = \left\{ f \in L^2(\mathbb{T}) : \lim_{k \rightarrow \infty} (A_k - P_{-1})f = 0 \right\}.$$

Note that

$$\lim_{k \rightarrow \infty} (A_k - P_{-1})z^{-1} = \lim_{k \rightarrow \infty} \left(\frac{1}{k} \sum_{j=1}^k \bar{\mu}^j \bar{\mu}^{-j} z^{-1} - z^{-1} \right) = 0.$$

Also, if $i \neq -1$, then

$$\begin{aligned} \lim_{k \rightarrow \infty} (A_k - P_{-1})z^i &= \lim_{k \rightarrow \infty} \left(\frac{1}{k} \sum_{j=1}^k \bar{\mu}^j \bar{\mu}^{ij} z^i \right) \\ &= \lim_{k \rightarrow \infty} \left(\frac{1}{k} \sum_{j=1}^k \bar{\mu}^{(i+1)j} z^i \right) \\ &= \int_{\mathbb{T}} z^i dm(z) \\ &= 0 \end{aligned}$$

by the ergodic theorem. Since S is a closed subspace of $L^2(\mathbb{T})$ containing $z^i \forall i \in \mathbb{Z}$, it follows that $S = L^2(\mathbb{T})$ and $P_{-1} \in \mathfrak{B}_{sc}$.

We now show that $P_m \in \mathfrak{B}_{sc}$ for $m \neq -1$. Fix $m \neq -1$. Define $C_m = \bar{\mu}^{-m} C_{\bar{\mu}}$ and $A_k = 1/k \sum_{j=1}^k C_m^j$ for $k \geq 1$. Let

$$S = \left\{ f \in L^2(\mathbb{T}) : \lim_{k \rightarrow \infty} (A_k(I - P_{-1}) - P_m)f = 0 \right\}.$$

Since $m \neq -1$, $(A_k(I - P_{-1}) - P_m)z^m = 1/k \sum_{j=1}^k \bar{\mu}^{-mj} \bar{\mu}^{mj} z^m - z^m = 0$. If $i = -1$ (so that $i \neq m$), then $(A_k(I - P_{-1}) - P_m)z^i = 0$. If $i \neq -1$ and $i \neq m$, then

$$\begin{aligned} (A_k(I - P_{-1}) - P_m)z^i &= \frac{1}{k} \sum_{j=1}^k \bar{\mu}^{-mj} \bar{\mu}^{ij} z^i \\ &= \frac{1}{k} \sum_{j=1}^k \bar{\mu}^{(i-m)j} z^i \\ &\rightarrow \int_{\mathbb{T}} z^i dm(z) \\ &= 0 \end{aligned}$$

by the ergodic theorem. Hence, since S is a closed subspace of $L^2(\mathbb{T})$ containing $z^i \forall i \in \mathbb{Z}$, $S = L^2(\mathbb{T})$ and $P_m \in \mathfrak{B}_{sc}$ as $A_k(I - P_{-1}) \in \mathfrak{B}_{sc} \forall k \geq 1$. \square

Proposition 4.7.6. Consider \mathfrak{B}_{α} as an operator algebra on $L^2(\mathbb{T})$. The nontrivial invariant subspaces \mathcal{M} of $L^2(\mathbb{T})$ under \mathfrak{B}_{α} are of the form $\mathcal{M} = z^{m_0} H^2(\mathbb{D})$ for $m_0 \in \mathbb{Z}$.

Proof. Let $(0) \neq \mathcal{M} \subseteq L^2(\mathbb{T})$ be a closed invariant subspace under \mathfrak{B}_α and hence under \mathfrak{B}_{sc} . Then $P_m \mathcal{M} \neq (0)$ for some $m \in \mathbb{Z}$ and \mathcal{M} contains z^m for some $m \in \mathbb{Z}$. Define $m_0 = \inf \{m \in \mathbb{Z} : z^m \in \mathcal{M}\}$. Note that if $m_0 = -\infty$, then the invariance of \mathcal{M} under M_z yields $\mathcal{M} = L^2(\mathbb{T})$. If $m_0 \neq -\infty$, we claim that $\mathcal{M} = z^{m_0} H^2(\mathbb{D})$. The invariance of \mathcal{M} under M_z yields $z^{m_0} H^2(\mathbb{D}) \subseteq \mathcal{M}$. Let then $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{M}$. Then $P_k(f(z)) = a_k z^k \in \mathcal{M} \forall k \in \mathbb{Z}$ by Lemma 4.7.5. But then $a_k = 0$ for $k \leq m_0 - 1$ by the definition of m_0 . Hence, $f(z) = \sum_{n=m_0}^{\infty} a_n z^n = z^{m_0} \sum_{n=0}^{\infty} a_{n+m_0} z^n \in z^{m_0} H^2(\mathbb{D})$. \square

4.8 Wold-type Decompositions

One major goal of this study is to develop a model for the Shilov modules over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$. In developing a model for the Shilov modules over $\mathcal{A}(\mathbb{D})$ the Wold decomposition was a crucial step. For $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, one should try to construct a Wold-type decomposition for pairs of isometries S and T satisfying $TS = \mu ST$. In this section, we show the most canonical extension of the Wold decomposition fails to hold for all these pairs. The idea of this type of Wold decomposition is due to Slocinski [Slo]. More on the development of a decomposition will be discussed in Chapter 5.

Definition 4.8.1. Let S and T be isometries on \mathcal{H} satisfying $TS = \mu ST$ where $|\mu| = 1$ is irrational. A decomposition

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss} \quad (4.7)$$

will be called the Wold decomposition of S and T if the following all hold:

- (i) $\mathcal{H}_{uu}, \mathcal{H}_{us}, \mathcal{H}_{su}$, and \mathcal{H}_{ss} reduce S and T .
- (ii) $S|_{\mathcal{H}_{uu}}$ and $T|_{\mathcal{H}_{uu}}$ are unitary operators.
- (iii) $S|_{\mathcal{H}_{us}}$ is unitary and $T|_{\mathcal{H}_{us}}$ is a shift.
- (iv) $S|_{\mathcal{H}_{su}}$ is a shift and $T|_{\mathcal{H}_{su}}$ is unitary.
- (v) $S|_{\mathcal{H}_{ss}}$ and $T|_{\mathcal{H}_{ss}}$ are shifts.

Note that if $\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss}$ is the Wold decomposition of the pair $\{S, T\}$ and $\mathcal{H}_1 = \mathcal{H}_{uu} \oplus \mathcal{H}_{us}$ and $\mathcal{H}_2 = \mathcal{H}_{su} \oplus \mathcal{H}_{ss}$, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is the Wold decomposition for S .

The following two propositions carry over from the commutative case; that is, the proofs are essentially the same and can be found in Slocinski [Slo].

Proposition 4.8.2. If a Wold decomposition for S and T satisfying $TS = \mu ST$ exists it is unique.

Proposition 4.8.3. Suppose S and T are isometries satisfying $TS = \mu ST$ and having the Wold decomposition. Then \mathcal{H}_1 reduces S and \mathcal{H}_2 reduces T where the decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{K}_1$ and $\mathcal{K}_1 = \mathcal{H}_2 \oplus \mathcal{K}_2$ are the Wold decompositions of the single isometries T and $S|_{\mathcal{K}_1}$ respectively.

Example 4.8.4 shows that there are pairs of isometries $\{S, T\}$ satisfying $TS = \mu ST$ without the Wold decomposition.

Example 4.8.4. Let $(e_{i,j})_{(i,j) \in I}$ be an orthonormal set in a Hilbert space \mathcal{K} where $I = \{(i, j) : i \geq 0 \text{ or } j \geq 0\}$. Let \mathcal{H} be the closed linear span $\bigvee_{(i,j) \in I} e_{i,j}$. Define isometries S and T on \mathcal{H} by $Se_{i,j} = e_{i+1,j}$ and $Te_{i,j} = \mu^i e_{i,j+1}$. Then, it is easy to verify that $TS = \mu ST$. Let $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$ be the Wold decomposition of T . Then $\mathcal{H}_u = \bigcap_{n=0}^{\infty} T^n \mathcal{H} = \bigvee_{i \geq 0} e_{i,j}$. Hence, $e_{0,1} \in \mathcal{H}_u$ and $e_{-1,1} \notin \mathcal{H}_u$. Note that $S^* e_{0,1} = e_{-1,1}$ so that \mathcal{H}_u does not reduce S . Hence, by Proposition 4.8.3, S and T do not have the Wold decomposition.

CHAPTER 5

CONCLUDING REMARKS

One major goal in studying the contractive Hilbert modules over any operator algebra is to develop a model similar to that developed for $\mathcal{A}(\mathbb{D})$. This model consists of essentially two facts - that the Shilov modules over $\mathcal{A}(\mathbb{D})$ correspond to isometries and that by the Wold decomposition these isometries can be decomposed as direct sums of unitaries and pure isometries.

Example 4.8.4 shows that the Shilov modules over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ do not have a Wold-type decomposition. However, we are still interested in describing a model for these Shilov modules. Together with the notion of a Suciuc-type decomposition [Suc], there is some promise along these lines which is now sketched.

Let $\mathcal{I} = \{T^n S^m : m, n \geq 0\}$ denote a collection of isometries on \mathcal{H} where $TS = \mu ST$. A closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is \mathcal{I} -invariant if $A\mathcal{M} \subseteq \mathcal{M} \forall A \in \mathcal{I}$. It is \mathcal{I} -reducing if $A\mathcal{M} \subseteq \mathcal{M}$ and $A^*\mathcal{M} \subseteq \mathcal{M} \forall A \in \mathcal{I}$. The collection \mathcal{I} is unitary if A is unitary on $\mathcal{H} \forall A \in \mathcal{I}$. The collection \mathcal{I} is completely non-unitary if for any \mathcal{I} -reducing subspace \mathcal{M} for which $\mathcal{I}|_{\mathcal{M}} \equiv \{A|_{\mathcal{M}} : A \in \mathcal{I}\}$ is unitary, we have $\mathcal{M} = (0)$. We say that \mathcal{I} is quasi-unitary if

$$\bigvee_{\substack{m_1 - m_2 > 0 \\ \text{or } n_1 - n_2 > 0}} T^{*n_2} S^{*m_2} S^{m_1} T^{n_1} \mathcal{H} = \mathcal{H}.$$

The collection \mathcal{I} is totally non-unitary if for any \mathcal{I} -reducing subspace \mathcal{M} for which $\mathcal{I}|_{\mathcal{M}}$ is quasi-unitary we have $\mathcal{M} = (0)$. The collection \mathcal{I} is strange if it is completely non-unitary and quasi-unitary.

For pairs of contractions $TS = \mu ST$ on \mathcal{H} and \mathcal{I} defined as above, a Suciuc-type decomposition exists. This is to say that $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{H}_t$ where $\mathcal{H}_u, \mathcal{H}_s$, and \mathcal{H}_t are \mathcal{I} -reducing

and moreover $\mathcal{I}|_{\mathcal{H}_u}$ is unitary, $\mathcal{I}|_{\mathcal{H}_s}$ is strange, and $\mathcal{I}|_{\mathcal{H}_t}$ is totally non-unitary. Furthermore, a model can be found for the unitary part and the totally non-unitary part [Suc]. As Examples 5.1.5, 5.1.6, and 5.1.7 show, each term in this decomposition can exist.

Example 5.1.5. Let $\{e_{i,j} : i \geq 0 \text{ and } j \geq 0\}$ be an orthonormal basis for \mathcal{H} . Define operators $Se_{i,j} = e_{i+1,j}$ and $Te_{i,j} = \mu^i e_{i,j+1}$. Then it can be shown that $\mathcal{H} = \mathcal{H}_t$.

Example 5.1.6. Let $\{e_{i,j} : i \geq 0 \text{ or } j \geq 0\}$ be an orthonormal basis for \mathcal{H} . Define S and T as in Example 5.1.5. Then $\mathcal{H} = \mathcal{H}_s$.

Example 5.1.7. Let $\{e_{i,j} : i, j \in \mathbb{Z}\}$ be an orthonormal basis for \mathcal{H} . With S and T defined as in Example 5.1.5, $\mathcal{H} = \mathcal{H}_u$.

Developing a model for the contractive Hilbert modules over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ would be useful by many accounts including the classification of the strongly orthogonally projective modules first introduced by Douglas and Paulsen [DP].

Definition 5.1.8. [MS1] A contractive Hilbert module \mathcal{P} is strongly orthogonally projective if given a diagram

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & & \downarrow \Phi & & \\ \mathcal{M} & \xrightarrow{\Psi} & \mathcal{N} & \longrightarrow & 0 \end{array}$$

of contractive Hilbert modules with a coisometric module map Φ , there is a module map $\tilde{\Psi} : \mathcal{M} \rightarrow \mathcal{P}$ such that $\Psi = \Phi\tilde{\Psi}$ and $\|\tilde{\Psi}\| = \|\Psi\|$.

Definition 5.1.9. [DP] Let \mathcal{H} be a contractive Hilbert module over \mathfrak{A} with Shilov dominant (\mathcal{M}, Φ) . We say that commutant lifting holds for (\mathcal{M}, Φ) if for every contractive Hilbert module \mathcal{K} , every Shilov dominant (\mathcal{N}, Ψ) for \mathcal{K} , and every $\phi \in \text{Hom}(\mathcal{H}, \mathcal{K}) \equiv$ the module maps from \mathcal{H} to \mathcal{K} , there is a $\tilde{\phi} \in \text{Hom}(\mathcal{M}, \mathcal{N})$ with $\|\tilde{\phi}\| = \|\phi\|$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{\phi}} & \mathcal{N} \\ \Phi \downarrow & & \downarrow \Psi \\ \mathcal{H} & \xrightarrow{\phi} & \mathcal{K} \end{array}$$

We say that commutant lifting holds for \mathfrak{A} if commutant lifting holds for each strongly minimal Shilov dominant for each completely contractive Hilbert module over \mathfrak{A} .

Since strongly orthogonally projective Hilbert modules over an operator algebra \mathfrak{A} are also orthogonally projective [MS2, Proposition 2.11], one could try to determine from the set of orthogonally projective Hilbert modules over $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ those that are strongly orthogonally projective. To answer this question, one might ask if commutant lifting holds for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ since if it did then it would follow that all the Shilov modules (which are precisely the orthogonally projective modules) are strongly orthogonally projective [MS2, Theorem 3.3]. However, it would seem unlikely that commutant lifting holds for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ as it fails to hold for $\mathcal{A}(\mathbb{D}^2)$. Indeed, as a counterexample for the elliptic case, one should look to the example of Parrott [Par]. If, however, one could show by a direct argument that all the Shilov modules for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ are strongly orthogonally projective, then it would follow that commutant lifting does hold for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ [MS2, Theorem 3.3]. Thus, it seems unlikely that all Shilov modules for $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ are strongly orthogonally projective. It would be of interest to characterize those which are.

Along other lines, the relationship between $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ and \mathfrak{B}_α has been characterized when α is elliptic. It should not be too difficult to find a relationship between these algebras when α is parabolic or hyperbolic. In connection with this, it is of interest to calculate the invariant subspaces of $H^2(\mathbb{D})$ or $L^2(\mathbb{T})$ under the norm closed algebra generated by $C_{\varphi^{-1}}$ and M_z when φ is parabolic or hyperbolic in order to gain a better understanding of the Shilov modules in these cases. The existence of these invariant subspaces is related to Bishop's operators and has been well-studied.

There are also many algebraic questions about $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ that have yet to be answered. For example, along the same lines as a characterization of the maximal ideals, one could ask for a characterization of the prime ideals. In general one can ask which properties of $\mathcal{A}(\mathbb{D})$ does the semicrossed product $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ retain?

In much the same fashion, it appears that one could define and do a similar analysis on semicrossed products of the disk algebra by endomorphisms rather than automorphisms. In

many ways, this seems to be a more natural and useful construction.

Finally, there is the ambitious goal of determining the proper definition for the semicrossed product of a general operator algebra \mathfrak{A} . Once this is done, the obvious questions of what properties of \mathfrak{A} the semicrossed product of \mathfrak{A} retain can be considered. The new definition might prove fruitful in the construction of new and interesting (non-self-adjoint) operator algebras from old.

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