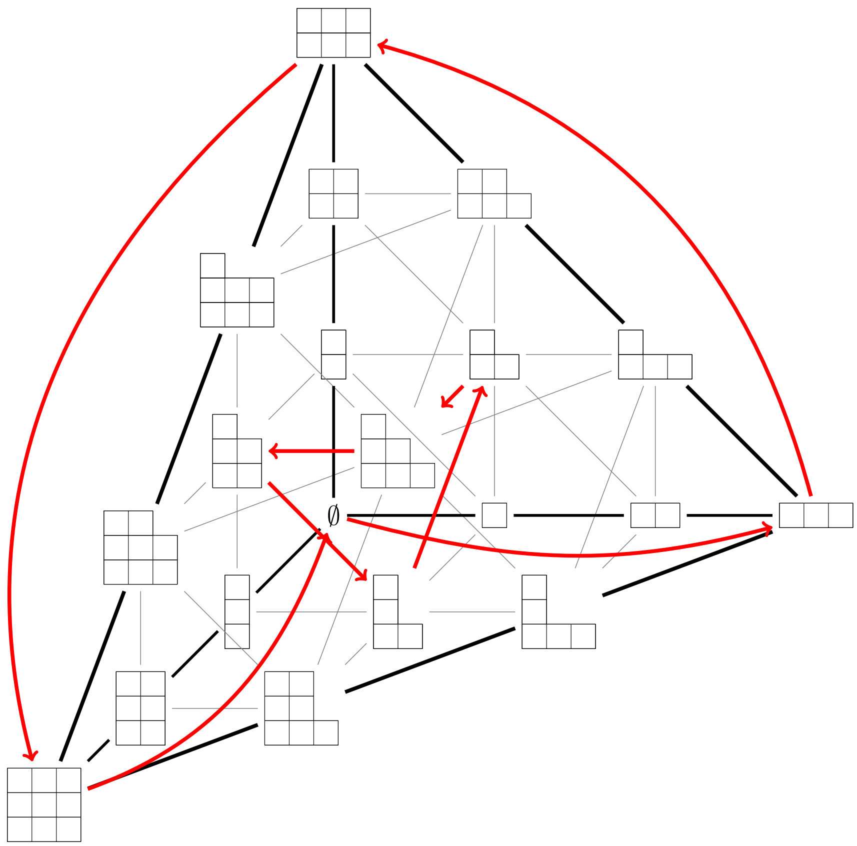
A Tetrahedron.



The 20 Young diagrams that fit in a 3×3 square can be arranged in a tetrahedron. As the symmetric group S_4 acts by rigid motions on the tetrahedron, there is already an interesting action of S_4 on these Young diagrams.

The cyclic shift (row version) corresponds to tensoring with the shape [3] in the $SL(3)_3$ quantum group category ([3] is an *invertible* object, meaning its tensor product with any other simple is again simple. Such an object is to be thought of as "1 dimensional", like the 1-dim'l reps in finite group theory.)

Below in red are drawn 2 of the 5 orbits (5 being the 3rd Catalan number). In this case the permutation on the tetrahedron corresponds to a rotation. The question about the cyclic lemma concerns whether $U(3)$ is a complete set of orbit representatives, which is easily verified here.

It is very curious that the permutation on Young diagrams coming from transposition does not come from a rigid motion of the tetrahedron. It would be very interesting to determine the subgroup of S_{20} generated by the rigid motions (S_4), along with the action by transposition. For instance, this subgroup contains the cyclic shift by a column (another example of a permutation on the Young diagrams which does not come from a rigid motion).

I expect these considerations are true in higher dimension (the 2-dim picture is also nice), but I haven't yet verified that tensoring with the column always results in a rigid motion.

The picture is inspired by the diagrams in ”Knot polynomial identities and quantum group coincidences” by Morrison, Peters and Snyder.