# Probability, etc.

CS 4277 Deep Learning

Kennesaw State University

# Probability[1]

Probability theory: quantification and manipulation of uncertainty.

- Epistemic, a.k.a. systematic uncertainty: we only see data sets of finite size
- Aleatoric, a.k.a. intrinsic, stochastic uncertainty: noise – we only observe partial information

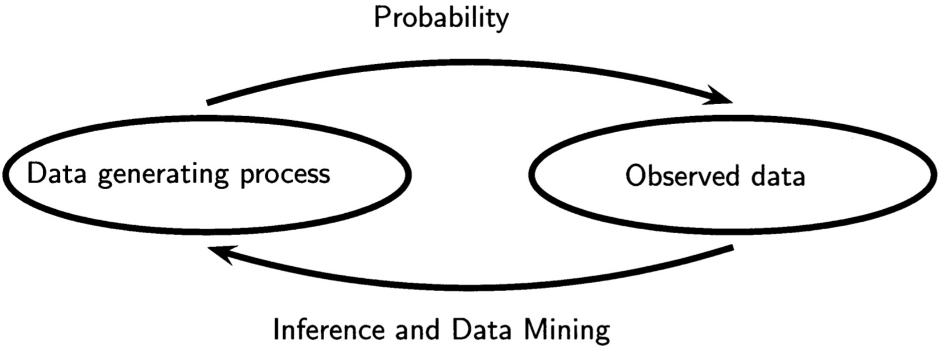
60%

40%

[1] Follows Chapter 2 of Deep Learning Foundations and Concepts

# Probability in Machine Learning

We observe data generated by a random process.



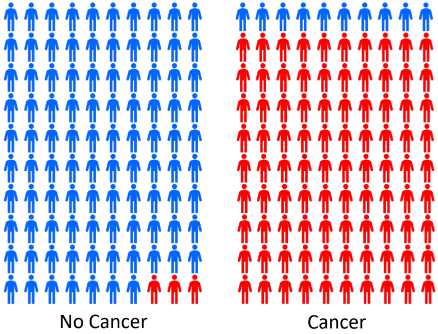

[2]

We make some assumptions about the data generating function and infer its parameters using samples from the process (training data).

[2] All of Statistics

# A Medical Screening Example

A cancer with occurence rate of 1% (.01) has a "90% accurate" test, and:




False positive rate: .03, False negative rate: 0.10

Questions:

- If we screen someone, what is the probability that they test positive?
- If someone tests positive, what is the probability that they have cancer?

We'll return to these questions after we develop some analysis tools.

# Joint Probability

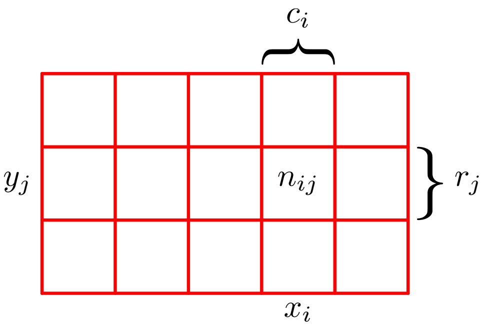
Let $X$ and $Y$ be *random* (a.k.a. *stochastic*) variables and

- $\{x_i\}_{i=1}^{L}$
- $\{y_j\}_{j=1}^{M}$
- $N$ trials in which we sample $X$ and $Y$
- $n_{ij}$ is number of trials in which $X = x_i$ and $Y = y_j$
- $c_i$ is the number of trials in which $X = x_i$, for all $y$s
- $r_j$ is the number of trials in which $Y = y_j$, for all $x$s

Then the joint probability of observing $x_i$ and $y_j$ is

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

We can visualize this event with the grid diagram on the right. Note that we're always observing events where both random variables have values, e.g., when we screen a person for cancer we're observing a joint event of two random variables: the test result and the actual existence of cancer.



# The Sum Rule

$$p(X = x_i) = \frac{c_i}{N}$$

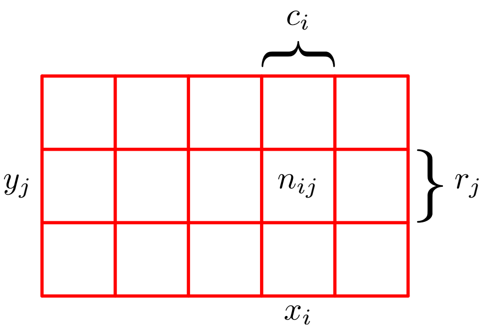
Notice that the number of instances in column $i$, $c_i$, is the sum of instances in each cell having $n_{ij}$ instances, so $c_i = \sum_j n_{ij}$. Recalling that

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

we have

$$p(X = x_i) = \sum_{j=1}^{M} p(X = x_i, Y = y_j)$$

This is the *sum rule*, which is also called the marginal probability becuase we sum over the other variable and write the sum in the margin of the table.



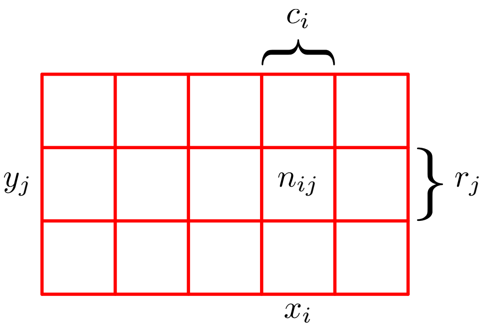
# Conditional Probability

If we consider trials in which $X = x_i$, the fraction of those trials in which $Y = y_j$ is written

$$p(Y = y_j | X = x_i)$$

We call this the *conditional probability* of $Y = y_j$ given $X = x_i$, which is the fraction of points in column $i$ that fall in cell $i, j$ so:

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$



# The Product Rule

Given the previous definitions for conditional probabilities and marginal probabilities, we can derive a formula for joint probabilities:

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i}\frac{c_i}{N}$$
$$p(X = x_i, Y = y_j) = p(Y = y_j | X = x_i)p(X = x_i)$$

This is the *product rule*.

We can summarize the sum and product rules with a more compact notation:

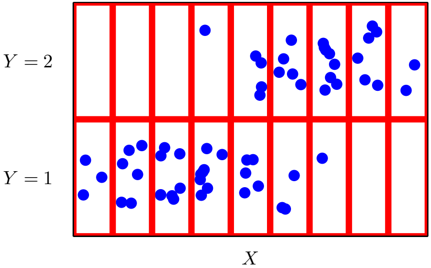
$$\text{Sum rule: } p(X) = \sum_Y p(X, Y)$$

$$\text{Product rule: } p(X, Y) = p(Y|X)p(X)$$

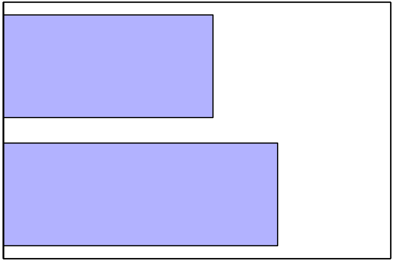
These two rules underlie all the probabilistic machinery we'll use in this course.

# Visualizing Joint Distributions

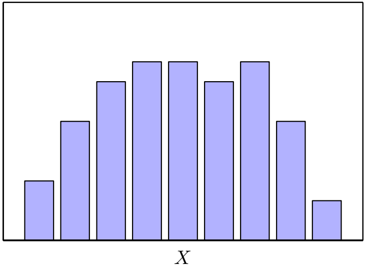
$p(X, Y)$



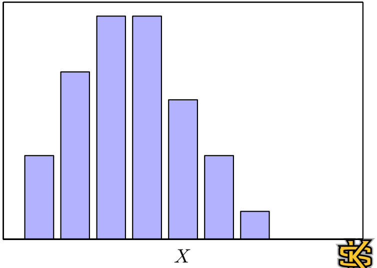

$p(Y)$



$p(X)$




$p(X|Y = 1)$



# Bayes' Theorem

Using the symmetry $p(x, y) = p(Y, X)$ and the product rule:

$$
\begin{aligned}
p(X, Y) &= p(Y, X) \\
p(Y|X)p(X) &= p(X|Y)p(Y) \\
p(Y|X) &= \frac{p(X|Y)p(Y)}{p(X)}
\end{aligned}
$$

This is called *Bayes' Theorem* or *Bayes' Rule.*

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# Analysis of Medical Screening Example

With our probabilistic machinery we can now analyze our cancer screening example.

$$
\begin{aligned}
p(C=1) &= \frac{1}{100} \\
p(C=0) &= \frac{99}{100} \\
p(T=1|C=1) &= \frac{90}{100} \\
p(T=0|C=1) &= \frac{10}{100} \\
p(T=1|C=0) &= \frac{3}{100} \\
p(T=0|C=0) &= \frac{97}{100}
\end{aligned}
$$

If we screen someone, probability that they test positive:

$$
\begin{aligned}
p(T=1) &= p(T=1|C=0)p(C=0) + p(T=1|C=1)p(C=1) \\
&= \frac{3}{100} \times \frac{99}{100} + \frac{90}{100} \times \frac{1}{100} \\
&= \frac{387}{10,000} \\
&= .0387
\end{aligned}
$$

If someone tests positive, probability they have cancer:

$$
\begin{aligned}
p(C=1|T=1) &= \frac{p(T=1|C=1)p(C=1)p(C=1)}{p(T=1)} \\
&= \frac{90}{100} \times \frac{1}{100} \times \frac{10,000}{387} \\
&= \frac{90}{387} \\
&\approx 0.23
\end{aligned}
$$

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# Prior and Posterior Probabilities

Bayes' Theorem updates our belief about someone's cancer.

- Before we run the test, the *prior probability* that someone has cancer is $p(C)$
- After we run the test, we use Bayes' Theorem to calculate the *posterior probability* $p(C|T)$

# Independent Variables

If the joint distribution factorizes into the product of the marginals:

$$p(X, Y) = p(X)p(Y)$$

Then we say that $X$ and $Y$ are *independent*. So

$$P(Y|X) = p(Y)$$

and

$$P(X|Y) = p(X)$$

Question: in our cancer screening example, is the probability of a positive test independent of whether a person has cancer?

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