Artificial Intelligence

Probabilistic Inference

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Exact Inference in Bayesian Networks (AIMA 13.3)

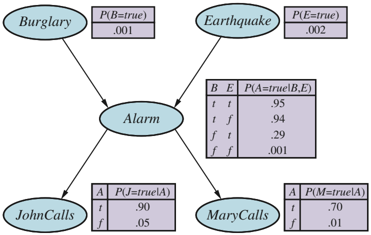
Most common task in probabilistic inference: compute the *posterior probability* of a set of **query variables** given some **event** represented as a set of **evidence variables**.

Notation:

- Query variable: X
- Set of evidence variables: $\boldsymbol{E} = \{E_1, \ldots, E_m\}$
- Particular observed event: $\boldsymbol{e}$
- Hidden (nonevidence, nonquery) variables: $\boldsymbol{Y} = \{Y_1, \ldots, Y_l\}$
- Typical query: $Pr(X \mid e)$

Example:

- X is the boolean random variable $Burglary$
- $\boldsymbol{E} = \{JohnCalls, MaryCalls\}$
- $\boldsymbol{e} = \{JohnCalls = true, MaryCalls = true\}$
- $\boldsymbol{Y} = \{EarthQuake, Alarm\}$



$$Pr(Buglary \mid JohnCalls = true, MaryCalls = true) = < 0.284, 0.716 > .$$

Inference by Enumeration

Recall that we can use the full joint distribution to answer any query:

$$Pr(X|\boldsymbol{e}) = \alpha Pr(X, \boldsymbol{e}) = \alpha \sum_{y} Pr(X, \boldsymbol{e}, \boldsymbol{y}) \tag{12.9}$$

And that a Bayes net completely represents the full joint distribution, so we can reduce the computation of a joint to:

$$Pr(x_1, \ldots, x_n) = \prod_{i=1}^{n} Pr(x_i | parents(X_i)) \tag{13.2}$$

Using these two equations we can enumerate the appropriate probabilities to calculate the answer to any probabilistic query.

- In particular, we can get the answer by computing sums of products of conditional probabilities from a Bayes net.

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Example: $Pr(Burglary \mid JohnCalls = true, MaryCalls = true)$.

Using abbreviations and substituting into Eq 12.9 above (e and a are hidden):

$$Pr(B \mid j, m) = \alpha Pr(B, j, m) = \alpha \sum_e \sum_a Pr(B, j, m, e, a)$$

Then we substitute Eq 13.2 for $Pr(B, j, m, e, a)$ to get (onyly showing Burglary=true):

$$\begin{aligned} Pr(b \mid j, m) &= \alpha \sum_e \sum_a Pr(b)Pr(e)Pr(a \mid b, e)Pr(j \mid a)Pr(m \mid a) && (1) \\ &= \alpha Pr(b) \sum_e \sum_a Pr(e)Pr(a \mid b, e)Pr(j \mid a)Pr(m \mid a) && (2) \\ &= \alpha Pr(b) \sum_e Pr(e) \sum_a Pr(a \mid b, e)Pr(j \mid a)Pr(m \mid a) && (3) \end{aligned}$$

1. Substitute Eq 13.2 for $Pr(B, j, m, e, a)$
2. Pull out $Pr(b)$ from summations because it doesn't depend on the other variable and is thus a constant in all the summation terms.
3. Pull out $Pr(e)$ from the summation over the a values because each value of e doesn't depend on the other variables in the summation over the a values and is thus a constant in the summation terms over the values of a.

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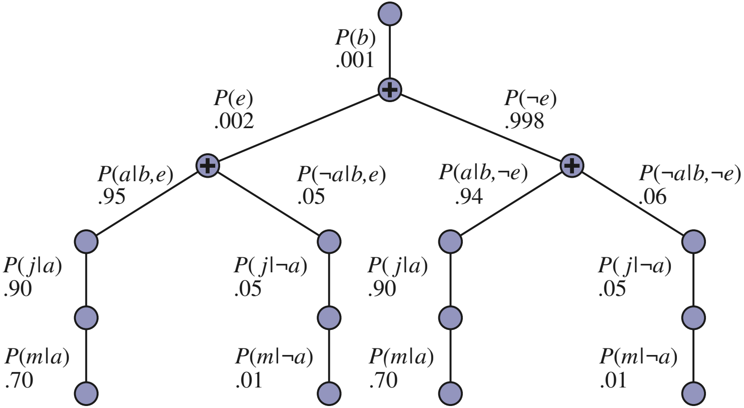
Steps 2 and 3 above reduce the complexity of the computation from $O(n2^n)$ to $O(2^n)$.

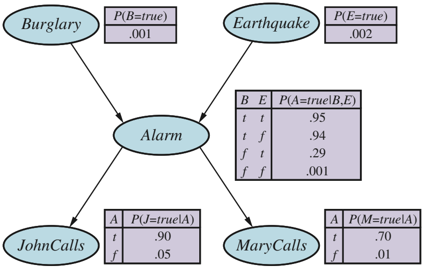
Caclulation of $Pr(b \mid j, m)$

Substiting the values from the CPTs in the Bayes net into

$$\alpha Pr(b) \sum_e Pr(e) \sum_a Pr(a \mid b, e) Pr(j \mid a) Pr(m \mid a)$$

we get the expression tree:





Enumeration Algorithm

The ENUMERATION-ASK algorithm evaluates these expression trees using depth-first, left-to-right recursion.

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayes net with variables $vars$

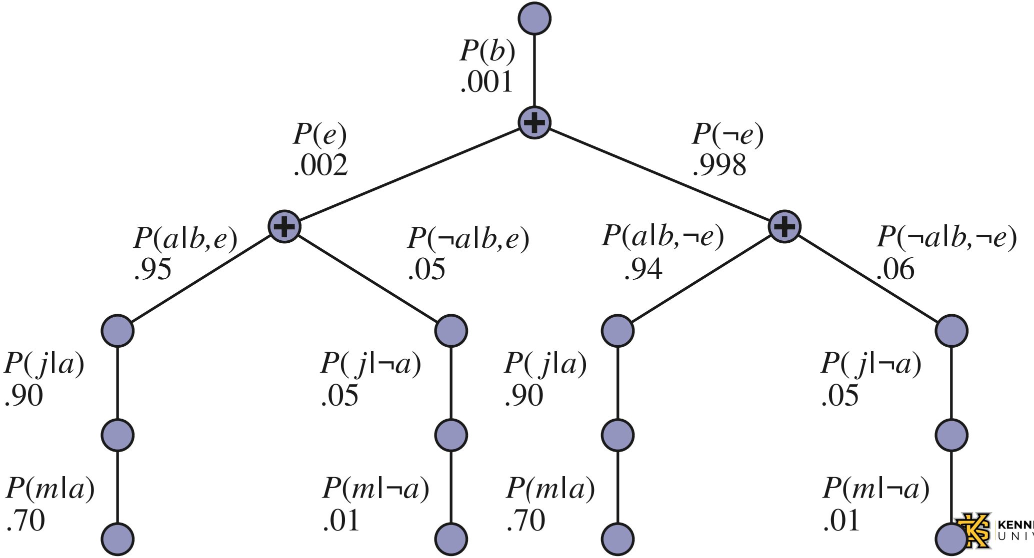
  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL($vars$, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is $\mathbf{e}$ extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $V \leftarrow$ FIRST($vars$)
  if V is an evidence variable with value v in $\mathbf{e}$
    then return $P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\sum_v P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_v$)
      where $\mathbf{e}_v$ is $\mathbf{e}$ extended with $V = v$

Unfortunately, its time complexity is $O(s^n)$. But we can improve it ...

Repeated Calculations

Notice that the subexpressions for the products $Pr(j \mid a)Pr(m \mid a)$ and $Pr(j \mid \neg a)Pr(m \mid \neg a)$ are computed twice, once for each value of E.



Variable Elimination

The enumeration algorithm can be improved substantially by eliminating repeated calculations.

- Idea: do the calculation once and save the results for later use.
- This is a form of dynamic programming.
- Several versions of this approach; variable elimination algorithm is simplest.

Variable elimination works by evaluating expressions such as

$$Pr(b \mid j, m) = \alpha Pr(b) \sum_{e} Pr(e) \sum_{a} Pr(a \mid b, e) Pr(j \mid a) Pr(m \mid a) \tag{13.5}$$

in right-to-left order (that is, bottom up in the expression tree), storing intermediate results, and only doing summations for portions of the expression that depend on the variable.

Example: Variable Elimination in Burglary Network

First, annotate the **factor**s in the expression for the network:

$$Pr(B \mid j, m) = \alpha \underbrace{Pr(B)}_{\boldsymbol{f}_1(B)} \sum_e \underbrace{Pr(e)}_{\boldsymbol{f}_2(E)} \sum_a \underbrace{Pr(a \mid B, e)}_{\boldsymbol{f}_3(A,B,E)} \underbrace{Pr(j \mid a)}_{\boldsymbol{f}_4(A)} \underbrace{Pr(m \mid a)}_{\boldsymbol{f}_5(A)}$$

- Each factor is a matrix indexed by the values of its argument variables.
- Notice that the factors for $Pr(j \mid a)$ and $Pr(m|a)$ do not include j and m. This is because the values of j and m ($JohnCalls = true$ and $MaryCalls - true$) are fixed by the query.

So the factors are:

$$\boldsymbol{f}_1(B) = \begin{bmatrix} Pr(b) \\ Pr(\neg b) \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.999 \end{bmatrix} \qquad \boldsymbol{f}_2(E) = \begin{bmatrix} Pr(e) \\ Pr(\neg e) \end{bmatrix} = \begin{bmatrix} 0.002 \\ 0.998 \end{bmatrix}$$

$$\boldsymbol{f}_4(A) = \begin{bmatrix} Pr(j \mid a) \\ Pr(j \mid \neg a) \end{bmatrix} = \begin{bmatrix} 0.090 \\ 0.05 \end{bmatrix} \qquad \boldsymbol{f}_5(A) = \begin{bmatrix} Pr(m \mid a) \\ Pr(m \mid \neg a) \end{bmatrix} = \begin{bmatrix} 0.070 \\ 0.01 \end{bmatrix}$$

$\boldsymbol{f}_3(A, B, E)$ is a little more complicated ...

$\boldsymbol{f}_3(A, B, E)$

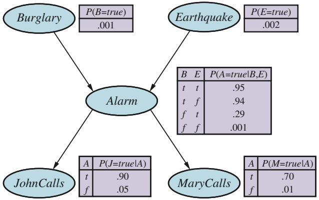
$$Pr(B \mid j, m) = \alpha \underbrace{Pr(B)}_{\boldsymbol{f}_1(B)} \sum_e \underbrace{Pr(e)}_{\boldsymbol{f}_2(E)} \sum_a \underbrace{Pr(a \mid B, e)}_{\boldsymbol{f}_3(A,B,E)} \underbrace{Pr(j \mid a)}_{\boldsymbol{f}_4(A)} \underbrace{Pr(m \mid a)}_{\boldsymbol{f}_5(A)}$$

$\boldsymbol{f}_3(A, B, E)$ is a $2 \times 2 \times 2$ matrix (or a rank-3 tensor). Here's one way to think about it:

- First index with A, yielding two 2×2 submatrices (one for each of the two values of A).
- Rows of each submatrix is indexed by B and columns by E.
- The entries in the submatrices are the values of $Pr(A \mid B, E)$

$$\boldsymbol{f}_3^{(a)}(B, E) = \begin{bmatrix} Pr(a \mid b, e) & Pr(a \mid b, \neg e) \\ Pr(a \mid \neg b, e) & Pr(a \mid \neg b, \neg e) \end{bmatrix} = \begin{bmatrix} 0.95 & 0.94 \\ 0.29 & 0.001 \end{bmatrix}$$

$$\boldsymbol{f}_3^{(\neg a)}(B, E) = \begin{bmatrix} Pr(\neg a \mid b, e) & Pr(\neg a \mid b, \neg e) \\ Pr(\neg a \mid \neg b, e) & Pr(\neg a \mid \neg b, \neg e) \end{bmatrix} = \begin{bmatrix} 0.05 & 0.06 \\ 0.71 & 0.999 \end{bmatrix}$$



Factorized Query

From our original query:

$$Pr(b \mid j, m) = \alpha Pr(b) \sum_{e} Pr(e) \sum_{a} Pr(a \mid b, e) Pr(j \mid a) Pr(m \mid a) \tag{13.5}$$

We annotated the factors:

$$Pr(B \mid j, m) = \alpha \underbrace{Pr(B)}_{\boldsymbol{f}_1(B)} \sum_{e} \underbrace{Pr(e)}_{\boldsymbol{f}_2(E)} \sum_{a} \underbrace{Pr(a \mid B, e)}_{\boldsymbol{f}_3(A,B,E)} \underbrace{Pr(j \mid a)}_{\boldsymbol{f}_4(A)} \underbrace{Pr(m \mid a)}_{\boldsymbol{f}_5(A)}$$

And now we substitute the factor expressions for the original expresions so we can manipulate the factors using the **pointwise product** operation, denoted with $\times$ here:

$$Pr(B \mid j, m) = \alpha \boldsymbol{f}_1(B) \times \sum_{e} \boldsymbol{f}_2(E) \times \sum_{a} \boldsymbol{f}_3(A, B, E) \times \boldsymbol{f}_4(A) \times \boldsymbol{f}_5(A)$$

Now we are ready to evaluate the expression ...

Expression Evaluation

First, sum out A from the pointwise product of $\boldsymbol{f}_3(A, B, E)$, $\boldsymbol{f}_4(A)$, and $\boldsymbol{f}_5(A)$ yielding a new 2×2 factor, $\boldsymbol{f}_6(B, E)$:

$$\begin{aligned}\boldsymbol{f}_6(B, E) &= \sum_a \boldsymbol{f}_3(A, B, E) \times \boldsymbol{f}_4(A) \times \boldsymbol{f}_5(A) \\ &= (\boldsymbol{f}_3(a, B, E) \times \boldsymbol{f}_4(a) \times \boldsymbol{f}_5(a)) + (\boldsymbol{f}_3(\neg a, B, E) \times \boldsymbol{f}_4(\neg a) \times \boldsymbol{f}_5(\neg a))\end{aligned}$$

Now the query expression is $Pr(B \mid j, m) = \alpha \boldsymbol{f}_1(B) \times \sum_e \boldsymbol{f}_2(E) \times \boldsymbol{f}_6(B, E)$

Next, sum out E from the product of $\boldsymbol{f}_2(E)$ and $\boldsymbol{f}_6(B, E)$, yielding a new factor $\boldsymbol{f}_7(B)$:

$$\begin{aligned}\boldsymbol{f}_7(B) &= \sum_e \boldsymbol{f}_2(E) \times \boldsymbol{f}_6(B, E) \\ &= \boldsymbol{f}_2(e) \times \boldsymbol{f}_6(B, e) + \boldsymbol{f}_2(\neg e) \times \boldsymbol{f}_6(B, \neg e)\end{aligned}$$

Which leaves our final form of the query: $Pr(B \mid j, m) = \alpha \boldsymbol{f}_1(B) \times \boldsymbol{f}_7(B)$

This expression can be evaluated by taking the pointwise product and normalizing the result.

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Operations on Factors

Two basic operations in variable elimination:

1. the pointwise product operation, and
2. summing out hidden variables from products of factors.

Pointwise Product Example

The pointwise product of two factors $\boldsymbol{f}$ and $\boldsymbol{g}$ yields a new factor $\boldsymbol{h}$ whose variables are the union of the variables in $\boldsymbol{f}$ and $\boldsymbol{g}$ and whose elements are given by the product of the corresponding elements in the two factors.

If we have X, Y, Z boolean variables, then here's the result of pointwise product $\boldsymbol{f}(X,Y) \times \boldsymbol{g}(Y,Z) = \boldsymbol{h}(X,Y,Z)$:

X	Y	$\mathbf{f}(X,Y)$	Y	Z	$\mathbf{g}(Y,Z)$	X	Y	Z	$\mathbf{h}(X,Y,Z)$
t	t	$.3$	t	t	$.2$	t	t	t	$.3 \times .2 = .06$
t	f	$.7$	t	f	$.8$	t	t	f	$.3 \times .8 = .24$
f	t	$.9$	f	t	$.6$	t	f	t	$.7 \times .6 = .42$
f	f	$.1$	f	f	$.4$	t	f	f	$.7 \times .4 = .28$
						f	t	t	$.9 \times .2 = .18$
						f	t	f	$.9 \times .8 = .72$
						f	f	t	$.1 \times .6 = .06$
						f	f	f	$.1 \times .4 = .04$

Summing out Variables

Summing out a variable from a product of factors is done by adding up the submatrices formed by fixing the variable to each of its values in turn. For example, to sum out X from $h(X, Y, Z)$, we write

$$\begin{aligned}
\boldsymbol{h}_2(Y, Z) &= \sum_x \boldsymbol{h}(X, Y, Z) \\
&= \boldsymbol{h}(x, Y, Z) + \boldsymbol{h}(\neg x, Y, Z) \\
&= \begin{bmatrix} .06 & .24 \\ .42 & .28 \end{bmatrix} + \begin{bmatrix} .18 & .72 \\ .06 & .04 \end{bmatrix} \\
&= \begin{bmatrix} .24 & .96 \\ .48 & .32 \end{bmatrix}
\end{aligned}$$

Variable Elimination Algorithm

With these two basic operations, we can implement the variable elimination algorithm:

function ELIMINATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
        $\mathbf{e}$, observed values for variables $\mathbf{E}$
        bn, a Bayesian network with variables $vars$

  $factors \leftarrow [\,]$
  for each V **in** ORDER($vars$) **do**
    $factors \leftarrow$ [MAKE-FACTOR(V, $\mathbf{e}$)] + $factors$
    if V is a hidden variable **then** $factors \leftarrow$ SUM-OUT(V, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Notes about the `order` function:

- Any ordering works, some orderings lead to more efficient algorithms.
- No tractable algorithm for determining optimal ordering.
- One heuristic: eliminate whichever variable minimizes the size of the next factor to be contructed.
- General rule: every variable that is not an ancestor of a query variable or evidence variable is irrelevant to the query.