Self-Organized Criticality in Sandpile Models

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Outline

Motivation

Why SOC?

- A mechanism to produce power-law distribution, that can be applied to many complex systems.
- Simplicity: simple rules in cellula automation models
- Attractive ideas: self-organization, criticality, fractal structures. However, controversial!

We focus on the classical paradigm of SOC, which is the **abelian sandpile model**. We want to

- implement sandpile model in a cellula automation model and introduce some realistic effects.
- understand why power-law distributions (and deviations) are obtained
- characterize SOC systems

Abelian sandpile model as a cellula automata

Consider a lattice (dimension d) with discrete value of an energy field $E(v)$ at each lattice site v. If $E(v) \geq E_c$ (critical value), then it topples and its $2d$ nearest neighbours receive the energy grain, i.e.

- $E(v) \longrightarrow E(v) - 2d$
- $E(v_n) \longrightarrow E(v_n) + 1$

Abelian means the order of which neighbour site receive the enery grain first does not matter. The rules can be slightly modified, for example:

- random energy grains size, between 0 and 1
- dissipation during propagation
- other neighbours

The boundary conditions can be:

- periodic boundary
- finite boundary: closed or open (energy lost outside the lattice).

Example of avalanche in 5×5 lattice, with $E_c = 4$

3	3	3	3	3
3	3	3	3	3
3	3	4	3	3
3	3	3	3	3
3	3	3	3	3

$\rightarrow$

3	3	3	3	3
3	3	4	3	3
3	4	0	4	3
3	3	4	3	3
3	3	3	3	3

$\rightarrow$

3	3	4	3	3
3	5	0	5	3
4	0	4	0	4
3	5	0	5	3
3	3	4	3	3

$\rightarrow$

3	5	0	5	3
5	1	4	1	5
0	4	0	4	0
5	1	4	1	5
3	5	0	5	3

$\rightarrow$

5	1	0	1	5
1	5	0	5	1
0	0	4	0	0
1	5	0	5	1
5	1	0	1	5

$\rightarrow$

1	2	0	2	1
2	1	3	1	2
0	3	0	3	0
2	1	3	1	2
1	2	0	2	1

The **avalanche size** is 35 (the largest) and the **avalanche lifetime** is 5 (not the longest). The total energy before avalanche was 76 and after it, 36. This is an example of what we call *catastrophic* avalanche.

Dynamical evolution

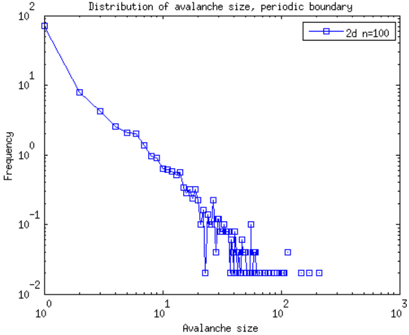
The system is drived externally by adding energy value to a random site. The schematic procedure is:

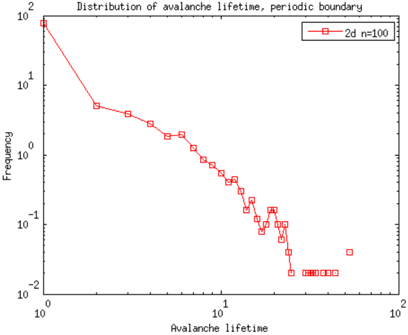
1. Stable initial configuration: can be random, or a uniform energy field.
2. Add one energy grain to a random site.
3. If the site becomes critical, stop adding more grains and apply the toppling rules to the critical site, until the avalanche finishes. Then, continue the driving.
   If the site is not critical, continue driving.

The driving time and the avalanche time are completely decoupled!
Most of time, no avalanche happens during the driving time. As our plots are log-log plots, in some of them, we added 1 to avalanche size, so to consider them in the statistics.

Periodic boundary

The added energy grains stay in the system, so the system will become **unstable** some time. This situation is unphysical, so we avoid it by introducing large lattice or short driving time. It is an example of **non-conservative** (the energy of the system increasing) sandpile model.



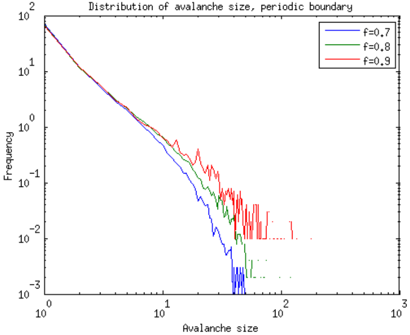


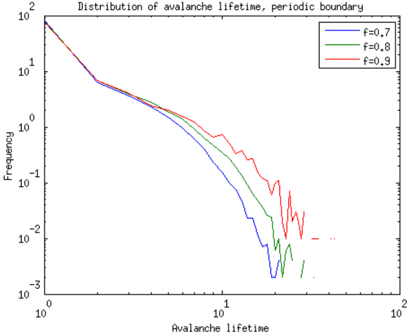
Periodic boundary and friction

Dissipation during the avalanche propagation, i.e. the energy after toppling is

$$E'(v) = f(E(v) - 2d);\ f < 1$$

Results for 3d lattices (similar results for other dimensions):

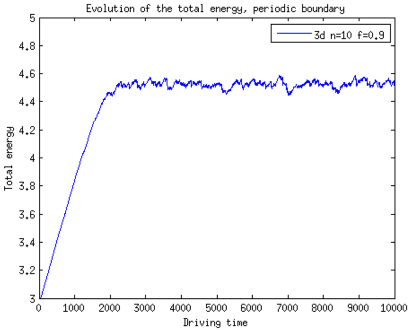


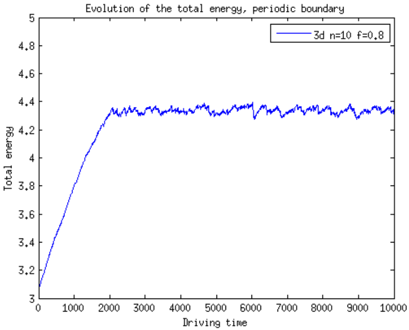


Average total energy is constant, as a dynamical equilibrium between:

- energy addition during driving time
- energy dissipation during avalanche time

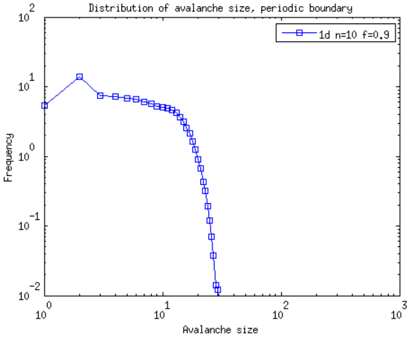
It is an example of **conservative** sandpile model.
(The same behaviour for open boundary case, but the dissipation occurs only at the boundary if no friction parameter is introduced)

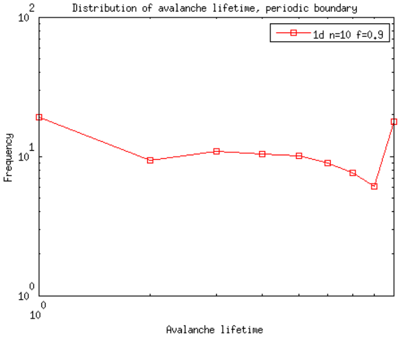




1-dimensional case

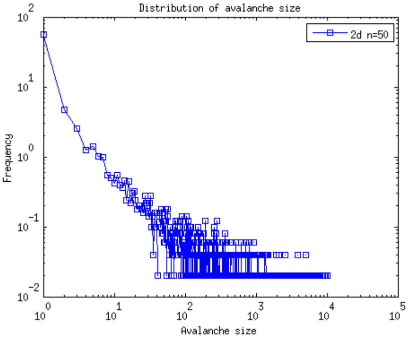
Apparently very different from higher dimensions (similar results for open boundary case).
Many authors, including Bak himself, argues no criticality for this case.

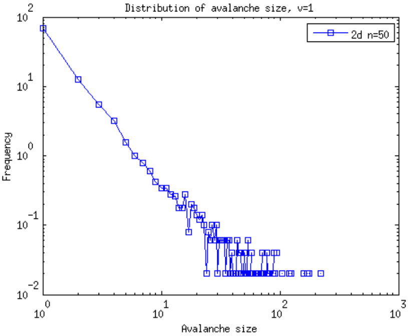




Boundary effects

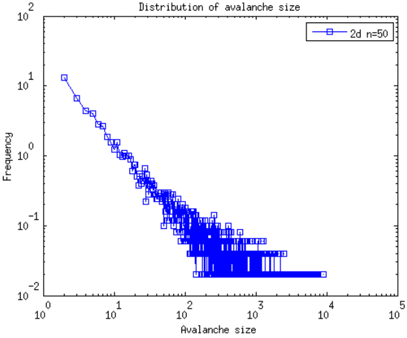
Different behaviour for boundary avalanches and bulk avalanches (i.e. triggered by a perturbation in a fix boundary or bulk site, respectively).

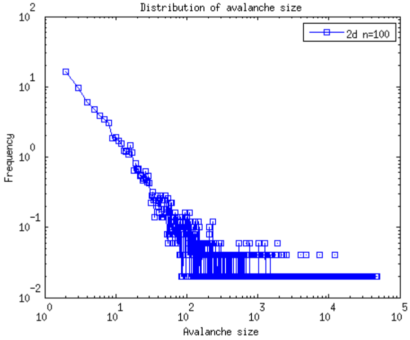




Size effects

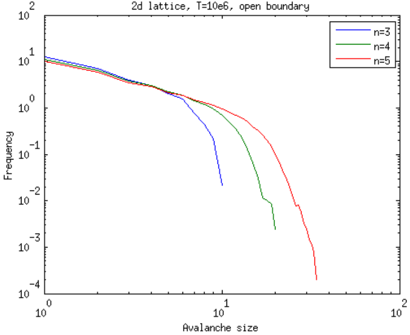
The larger the size, the larger avalanche might occur. (More statistics are needed, so longer driving time)

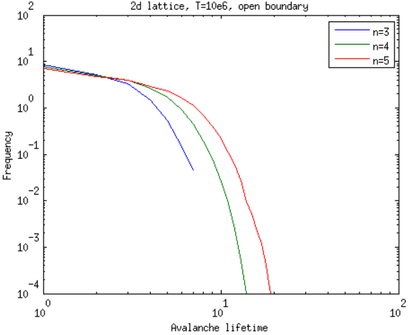




Size effects in small 2 dimensional lattices

Exponential-like distributions, but power-law-like behaviour for small and medium-large avalanches!

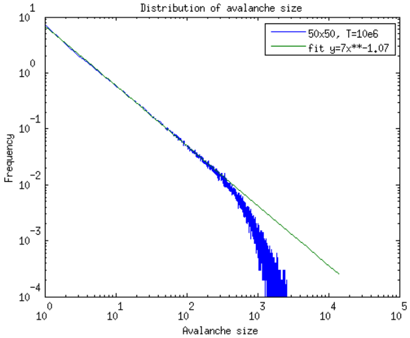




More statistics for a larger 2 dimensional lattices

Indeed,

- nice power-law behaviour for small and medium-large ($s < n^d$) avalanches
- large (or catastrophic, i.e. $s \geq n^d$) avalanches supressed



Relationships

As cellula automata models are discrete, there is a finite number of all possible configurations.

Definitions:

- A={config. with 1 site critical (E_c) and the rest of states subcritical}
- B={config. with all sites subcritical}

Relationships:

- Avalanche process: $a \in A \longrightarrow b \in B$
- External driving process: $b \in B \longrightarrow a' \in A$

Different time-scales: avalanche lifetime and driving time.

The system evolves to a dynamical equilibrium between avalanche and driving process (see energy evolution), this implies only a subset of A actually contributes to our statistics.

Counting the distribution for avalanche sizes

Counting the number of configurations from set A which will give avalanche sizes s once relaxed, the main contribution in a rough approximation is

$$D(s) \sim c^{N-s} \sim e^{-s}; \quad N = n^d$$

where c is a related with the number of possible values a site can have. Example: 2d lattice, $E_c = 4$, ignore finite boundary (or large N). For $s = 2$,

$$D(s = 2) \sim (4 \cdot N \cdot 3^{N-2} + N \cdot 3^6 \cdot 4^{N-6-2})$$

The first contribution is associated with $\begin{array}{|c|c|}\hline 4 & 3 \\ \hline \end{array}$, where 3 can be in any of the 4 nearest neighbours. The other contribution corresponds to $\begin{array}{cccc} \cline{2-3} & \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{x} & \\ \hline \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{x} \\ \hline & \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{x} & \\ \cline{2-3} \end{array}$,

x $\in \{0, 1, 2\}$, it is the dominant one for large N.

The situation gets complicated for larger s, with more possible patterns, but we can see an exponential dependence. The number of surrounding sites, 6, also depend on s, but note the factor $(3/4)^6 < 1$, so the exponential distribution should be a good first approximation.
A more general method should be developed for a real analytic approach.

As the average dynamical equilibrium restrict us to a subset of A, namely the catastrophic avalanches are forbidden,

$$D(s) \sim e^{-s} \sim \frac{1}{1+s} \sim \frac{1}{s}$$

A result coherent with the fit for 2-dimensional case!

For $d \geq 2$, the exponent is slightly more negative, due to effective effects from the contributions we neglected.

Achievements

- We wrote a MATLAB/Octave program for cellula automata model of abelian sandpile, with many parameters (dimensions, boundary conditions...) to be varied.
- We proposed a simple explanation to understand our results, but a more general way in counting should be developed still.
- We did not find fractal structure (maybe related with the subconfigurations used for the counting).
- We did not see the criticality in the thermodynamic sense (no second order phase transition).
- We did not see self-organization in its usual sense (we have relaxation and external driving).
- The type of SOC systems studied here should be characterized as **slowly driven systems** that evolve to a critical configuration, which relaxes itself quickly through the **avalanche phenomena**.