



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Lecture with Computer Exercises: Modelling and Simulating Social Systems with MATLAB

Project Report

Self-Organized Criticality in Sandpile Models



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Abstract

This paper describes the principles of self-organized criticality and their validity in cellular automation models, in particular the sandpile model. The model, its diversity, its implementation in MATLAB/Octave with different parameters and its analysis are presented in detail. RESULTS ASDF?!?

Acknowledgements

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Chapter 1

Introduction and Motivations

...

Chapter 2

The Sandpile Model

2.1 Bak-Tang-Wiesenfeld Model

The classical sandpile model represents a cellular automation describing a dynamical system following certain rules that can be described as follows.

The field/lattice, which is chosen to be two-dimensional, represents a sandpile. Each site on the lattice has a certain value z that intuitively represents the height or slope of the sandpile at certain position described with the coordinates x and y . At each time step, a number of grains of sand is placed on top of a random site, which increases its value by a given value, e.g. one. If the value of the site exceeds a critical value z_c (e.g. three), the site collapses/topples and its grains are evenly distributed to its neighbours.

In certain cases some of the adjacent sites will exceed the critical value too and the toppling process will continue until an equilibrium state is again reached. This series of collapsing sites is classically described as an avalanche. The next grain is not placed until the equilibrium state is reached, meaning that the time scale of the random grain placement and of the development of avalanches are decoupled.

The classical model description can mathematically be represented as follows.

Initially, the lattice is empty:

$$z(x, y) = 0 \quad \forall x, y$$

Then, the value of a random site x, y is increased:

$$z(x, y) \rightarrow z(x_r, y_r) + 1$$

If its value exceeds the critical value $z_c = 3$, then it topples and distributes its grains to its neighbours:

$$\begin{aligned} z(x, y) &\stackrel{?}{>} 3 \Rightarrow z(x, y) \rightarrow z(x, y) - 4 \\ z(x \pm 1, y) &\rightarrow z(x \pm 1, y) + 1 \\ z(x, y \pm 1) &\rightarrow z(x, y \pm 1) + 1 \end{aligned}$$

Clearly, many variations of the described model can be considered and can produce different results. The classical sandpile model, as originally described by Per Bak, Chao Tang and Kurt Wiesenfeld, represents the starting point of any further investigations considered in this paper.

2.2 Parameters

The behavior of the model is analysed dependent on different parameters such as:

- lattice size
- number of dimensions of lattice
- mass conservation, i.e. if the number of grains removed from a collapsed site is equal to the sum of grains its neighbour sites received
- boundary conditions, see below
- etc.

Different types of boundary conditions can be thought of:

- open: If a site near the border topples, some of its grains leave the system (mass is lost).
- closed: Near-border site does not fully collapse, but keeps the grains that would fall off in an open case.
- periodic: The system has no boundaries, i.e. toppling near the border is “wrapped over”.
- mixed: E.g. the lattice is periodic in one dimension and has open boundaries in another dimension.

Figure 2.1 illustrates the first three basic types of boundary conditions.

2.3 Abelian Model

One important property which can be used to categorize different sandpile models is whether they behave in a commutative or *abelian* way. In particular, this can be applied to the development of avalanches in the model described above: The question posed here is, whether an equilibrium state resulting from an avalanche depends on the way the avalanche is calculated. More precisely, it can be shown that any avalanche, being a sequence of topplings, always results in the same equilibrium state i.e. does not depend

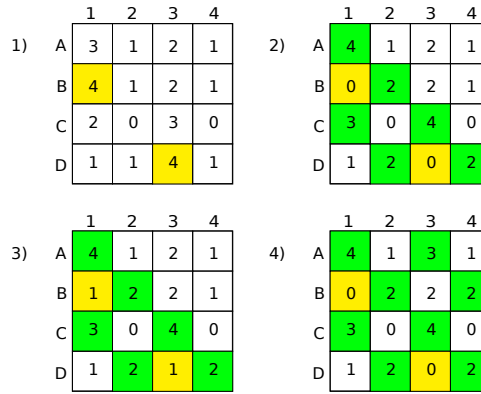


Figure 2.1: Effect of different types of boundary conditions on a sample lattice (1): open (2), closed (3) and periodic (4)

on the order, in which the topplings occur. The mathematical proof of this hypothesis is nicely presented in [7].

To illustrate this practical but not necessarily obvious fact, a sample 4x4-field with one active site is considered (see figure 2.2). At step (2), two different sites simultaneously become active, therefore creating a “choice”, which site to topple first. Depending on such choices, different sequences of topplings occur, but all lead to the same equilibrium state.

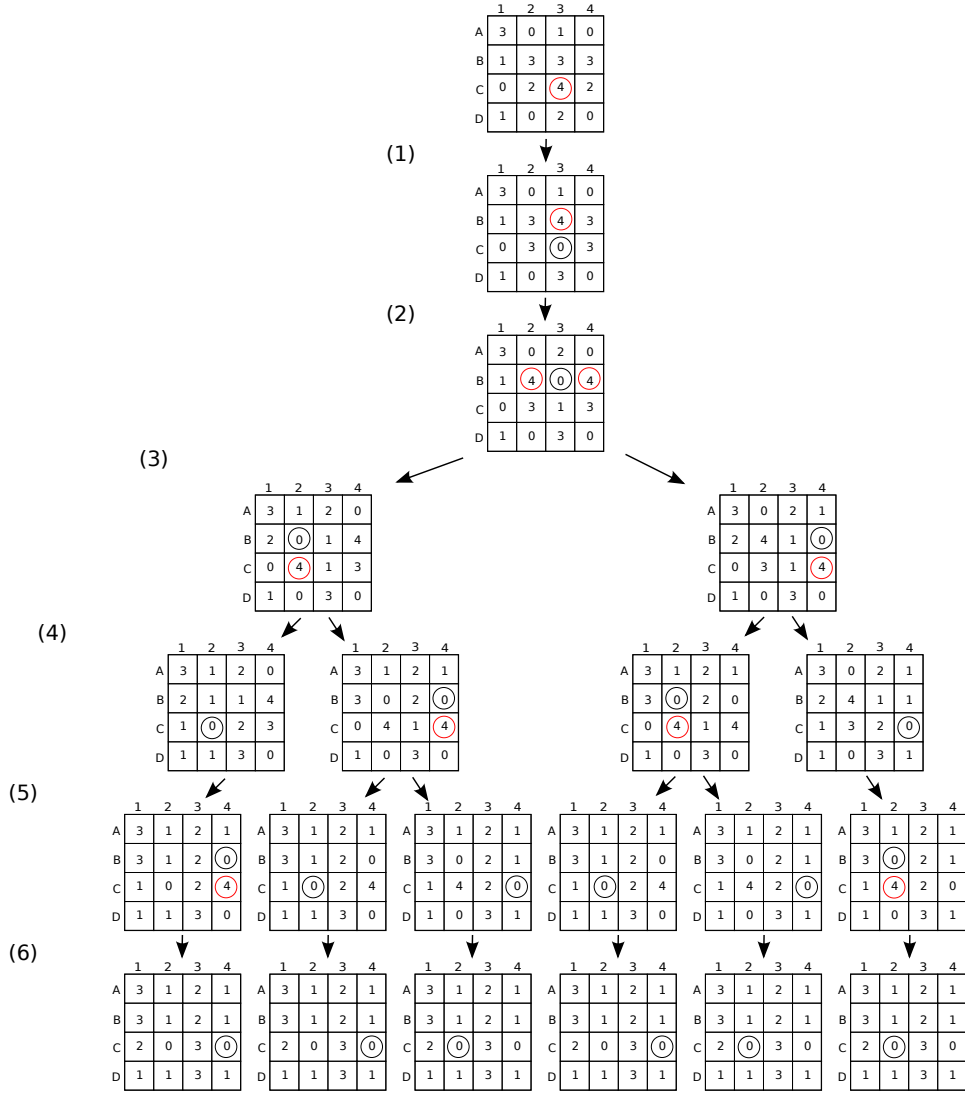


Figure 2.2: Demonstration of the abelian property: six different orders of topplings all lead to the same equilibrium state. The sites circled red indicate sites that have just become active, those circled black have just collapsed. Here, continuous boundary conditions have been used.

Chapter 3

Model Implementation in MATLAB/Octave

Note on Code and Programming Sustainability

In order to produce “sustainable” code and to share the spirit of independency coming from the open-source community, the coded routines were tested in MATLAB and in Octave (one of its open-source clones). The source code can be found in Appendix A.

3.1 Basic Sandpile Code

First, a lattice/field is generated using uniformly distributed random numbers from 0 to z_c (`critical_state`). This is done in order to start with a potentially critical field and not to place single grains of sand until a site gets critical.

```
f = floor(unifrnd(0,critical_state+1,height,width));
```

Another interesting starting point is a *uniform* critical field, where every site is either 0 or z_c .

```
f = floor(unifrnd(0,2,height,width))*critical_state;
```

When the field is ready, a global loop runs through a defined number of timesteps, placing a grain on a random site, checking if the site becomes active and if so, computing the resulting avalanche.

```
for t=1:timesteps
    % choose random site
    y=floor(unifrnd(1,height));
    x=floor(unifrnd(1,width));

    % place grain
    f(y,x) = f(y,x) + 1;

    % check if overcritical/active
    if (f(y,x) > critical_state)
        % avalanche code here
    end
end
```

```

% ...
end
end

```

3.2 Simple Avalanche Code

...ASDF...

3.3 Optimization of Avalanche Code

The simple avalanche code checks the whole field including the fields, that cannot possibly be affected by the avalanche. It can therefore be optimized, for example using a LIFO data structure – a *stack*. The coordinates of very site that needs to be checked are placed on the stack, so that the computation of the avalanche consists of working through the stack and toppling all the active sites in it. During their toppling, their neighbours are again put on the stack, which makes the procedure dynamical and not easily comprehensive. The algorithm is illustrated in figure 3.1.

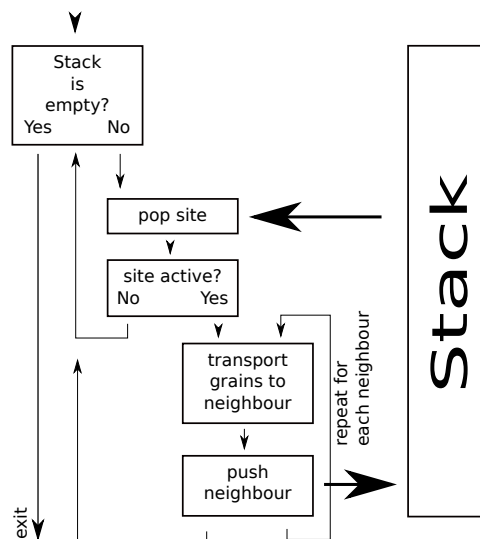


Figure 3.1: using a stack for avalanche calculation

Considering the example from figure 2.2, the stack algorithm results in the following sequence:

0. push C3
1. pop C3, topple, push its neighbours (C2,C4,B3 and D3) to stack
2. pop B3, topple, push B2, B4, A3 and C3 to stack

3. pop B2, ...
4. pop C2, ...
5. pop B4, ...
6. pop C4, ...

Figure 3.2 shows the states of the stack after each of these steps. To avoid confusion, only the active sites are shown here.

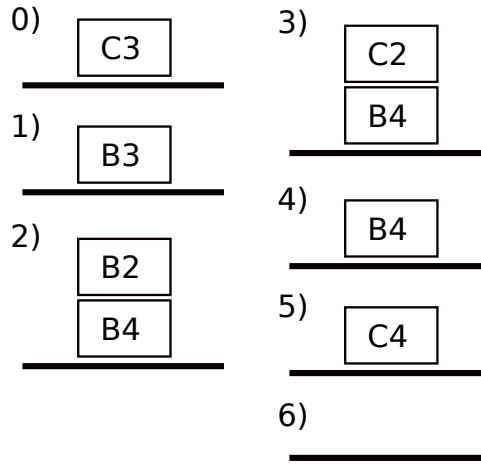


Figure 3.2: A sample sequence of stack states. The algorithm proceeds until the stack is empty.

The main loop including the stack feature looks like this:

```
for t=1:timesteps
    % choose random site
    % ...

    % place grain
    % ...

    % push site to stack
    stack_n = 1;
    stack_x(1) = x;
    stack_y(1) = y;

    % avalanche — work through stack
    while (stack_n > 0)

        % pop from stack
        x = stack_x(stack_n);
        y = stack_y(stack_n);
        stack_n = stack_n - 1;

        % check if overcritical/active
        if (f(y,x) > critical.state)
            % collapse/topple
            f(y,x) = f(y,x) - neighbours * collapse;

            % look at every neighbour
            for n=1:neighbours
                % add/transport grain to neighbour
                f(y+neighbour_offset_y(n),x+neighbour_offset_x(n)) = ...
                    f(y+neighbour_offset_y(n),x+neighbour_offset_x(n)) + collapse;
            end
        end
    end
end
```



```

        % push neighbour to stack
        stack_n = stack_n + 1;
        stack_x(stack_n) = x + neighbour_offset_x(n);
        stack_y(stack_n) = y + neighbour_offset_y(n);
    end
end
end
end

```

3.4 Statistics

Many different variables may be of interest for the statistical analysis of sandpile models. The easiest to implement is avalanche size:

```

...
% check if overcritical/active
if (f(y,x) > critical_state)

    % collapse/topple
    f(y,x) = f(y,x) - neighbours * collapse;

    % record statistics
    avalanche_sizes(t) = avalanche_sizes(t) + 1;

...

```

Here, the number of avalanches is recorded at every time step by increasing the counter after each toppling that happens during the avalanche. After the main loop, the data is sorted and the distribution is fitted into a power-law distribution given by

$$P(s) = a \cdot s^b$$

where P is the number of avalanches of size s . The coefficients a and b are determined using a simple solver that minimizes $a \cdot s^b - P(s)$.

```

% count avalanche sizes - calculate distribution
for s=1:max(avalanche_sizes)
    avalanche_count(s) = size(avalanche_sizes(avalanche_sizes==s),2);
end

% filter zero values
s = [1:max(avalanche_sizes)];
P = avalanche_count(1:end);
s = s(P>0);
P = P(P>0);

% fit into power-law
[c,fval,info,output]=fsolve(@(c)((c(1).*s.^c(2))-P),[100,1]);
a = c(1);
b = c(2);

```

In order to implement statistics of avalanche lifetime in the stack code, the number of additional (i.e. more than one) topplings per timestep must be counted. The reason for this is that the stack algorithm does not follow a timescale and therefore the number of timesteps taken by an avalanche cannot be counted directly. Therefore, the following equation is used:

$$s = \sum_t n = \underbrace{\sum_t (n-1)}_a + \sum_t 1 \Rightarrow \sum_t 1 = s - a$$

where s is the avalanche size, t is the avalanche lifetime, n is the number of topplings per timestep and a is the total number of additional topplings. The neighbour-checking part of the stack loop looks like this:

```
% count future topplings to be caused by this toppling
future_topplings = 0;

% look at every neighbour
for n=1:neighbours

    % add/transport grain to neighbour
    % ...

    % push neighbour to stack
    % ...

    % count future topplings to be caused by this toppling
    if (f(y+neighbour_offset_y(n),x+neighbour_offset_x(n)) == (critical_state+1))
        % i.e. if neighbour site becomes active
        future_topplings = future_topplings + 1;
    end
end

% calculate additional topplings caused
if (future_topplings > 0)
    avalanche_add(t) = avalanche_add(t) + future_topplings - 1;
end
```

For each toppling, the number of further topplings is counted, summed up and subtracted by one. Then, the avalanche lifetime is calculated according to the formula $\sum_t = s - a$ as follows:

```
% return avalanche lifetimes
at = avalanche_sizes - avalanche_add;
```

Chapter 4

Simulation Results

Chapter 5

Summary and Outlook

Bibliography

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Appendix A

MATLAB/Octave-Code

A.1 critical field

```
function f = critical.field(width,height,critical_state,uniform)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PARAMETERS:
% width, height = size of lattice/field to be created
% critical_state = maximum/critical state of a site, usually = 3
% uniform = true will generate a field of e.g. 0's and 3's only
% uniform = false will generate a field e.g. with numbers 0 to 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% define field using uniform distribution
if (uniform)
    f = floor(unifrnd(0,2,height,width))*critical_state;
else
    f = floor(unifrnd(0,critical_state+1,height,width));
end
end
```

A.2 sandpile

```
function [as,nc,at,final] = sandpile(f, neighbour, critical_state, ...
    collapse_per_neighbour, timesteps, boundary_type, make_pictures, ...
    silent, driving_plane_reduction, var_grain)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% sandpile simulation using stack algorithm for avalanche generation
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% INPUTS
% f          field matrix
% neighbour   2xN matrix with x & y offsets of neighbours
% critical_state  critical/max. number of grains before collapse
% collapse_per_neighbour  number of grains to collapse
% timesteps   simulation duration in steps (excl. avalanches)
% boundary_type  type of boundary condition
%              1 - infinite/continuous, like pac-man
%              2 - energy loss at boundaries, table-like
%              3 - ...
% make_pictures  draw and export all frames or not
% silent         produces no output (except time progress) if true
```

```

% driving-plane-reduction percentage of field close to the boundary
% not to be affected by driving (putting grains)
% = 0 => use whole field (default)
% = 0.2 => put grains at least 0.2*width
% and 0.2*height far away from boundary
% > 0.5 => invalid [!]
% var-grain true/false - use random grain size [0...1]

% OUTPUTS
% as avalanche sizes (topplings count) for each timestep
% nc size at avalanche-starting-site for each t
% at avalanche lifetime for each t
% final final field

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% translate parameters

width = size(f,2);
height = size(f,1);
neighbours = size(neighbour,2); % number of neighbours to collapse to
neighbour.offset_x = neighbour(1,:);
neighbour.offset_y = neighbour(2,:);
collapse = collapse_per_neighbour;
boundary = boundary_type;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% define stack for avalanches
stack_x = 0;
stack_y = 0;
stack_n = 0;

% avalanche statistics
avalanche_sizes = zeros(1, timesteps);
av_begin_t = zeros(1, timesteps);
avalanche_add = zeros(1, timesteps); % = av_size - av_ltime

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% show starting field
if (silent==false)
    disp('starting from this field:');
    disp(f);
end

for t=1:timesteps
    % display time progress
    disp(['time: ' num2str(t) ' / ' num2str(timesteps)]);

    % choose random site
    y=floor(unifrnd(1,height*(1-2*driving_plane_reduction)) + ...
        height*driving_plane_reduction);
    x=floor(unifrnd(1,width*(1-2*driving_plane_reduction)) + ...
        width*driving_plane_reduction); % uniform distribution rnd

    % place grain
    f(y,x) = f(y,x) + 1;

    % communicate
    if (silent==false)
        disp(['random grain on x' num2str(x) ',y' num2str(y)]);
    end

    % save picture of field before collapsing (with active field)
    if (make_pictures)
        draw_field(f,2);
        print(['field' num2str(t) '.png'], '-dpng');
    end

    % push site to stack
    stack_n = 1;
    stack_x(1) = x;
    stack_y(1) = y;

    % save avalanche starting site
    av_begin_x = x;
    av_begin_y = y;
    av_begin_t(t) = 0; % # topplings at avalanche starting site

    % avalanche - work through stack
    while (stack_n > 0)
        % pop from stack
        x = stack_x(stack_n);
        y = stack_y(stack_n);
    end
end

```



```

stack.n = stack.n - 1;
% display current site
if (silent==false)
    disp(['current site: x ' num2str(x) ' ; y ' num2str(y)]);
end
% check if overcritical/active
if (f(y,x) > critical.state)
    % communicate collapsing
    if (silent==false)
        disp('collapse!');
    end
    % save avalanche size for statistics
    avalanche.sizes(t) = avalanche.sizes(t) + 1;
    if ((x==av.begin_x) && (y==av.begin_y))
        % save # topplings at av starting site
        av.begin_t(t) = av.begin_t(t) + 1;
    end
    % collapse/topple
    f(y,x) = f(y,x) - neighbours * collapse;
    % count future topplings to be caused by this toppling
    future.topplings = 0;
    % look at every neighbour
    for n=1:neighbours
        % communicate
        if (silent==false)
            disp(['neighbour ' num2str(n)]);
        end
        % check boundary
        % 1) no-boundary conditions (continuous field, pack-man style)
        if (boundary == 1)
            % modify neighbour offsets
            if (y+neighbour_offset.y(n) < 1)
                neighbour_offset.y(n) = neighbour_offset.y(n) + height;
            end
            if (y+neighbour_offset.y(n) > height)
                neighbour_offset.y(n) = neighbour_offset.y(n) - height;
            end
            if (x+neighbour_offset.x(n) < 1)
                neighbour_offset.x(n) = neighbour_offset.x(n) + width;
            end
            if (x+neighbour_offset.x(n) > width)
                neighbour_offset.x(n) = neighbour_offset.x(n) - width;
            end
            % add/transport grain to neighbour
            f(y+neighbour_offset.y(n),x+neighbour_offset.x(n)) = ...
                f(y+neighbour_offset.y(n),x+neighbour_offset.x(n)) + collapse;
            % push neighbour to stack
            stack.n = stack.n + 1;
            stack.x(stack.n) = x + neighbour_offset.x(n);
            stack.y(stack.n) = y + neighbour_offset.y(n);
            % count future topplings to be caused by this toppling
            if (f(y+neighbour_offset.y(n),x+neighbour_offset.x(n)) == (critical.state+1))
                future.topplings = future.topplings + 1;
            end
        % 2) energy loss at boundary (table style)
        elseif (boundary == 2)
            % keep offsets, but check if outside of boundary
            if ((y+neighbour_offset.y(n) < 1) || ...
                (y+neighbour_offset.y(n) > height) || ...
                (x+neighbour_offset.x(n) < 1) || ...
                (x+neighbour_offset.x(n) > width))
                % outside of boundary...do nothing =)
            else
                % add/transport grain to neighbour
                f(y+neighbour_offset.y(n),x+neighbour_offset.x(n)) = ...
                    f(y+neighbour_offset.y(n),x+neighbour_offset.x(n)) + collapse;
                % push neighbour's neighbours to stack
                stack.n = stack.n + 1;
                stack.x(stack.n) = x + neighbour_offset.x(n);
            end
        end
    end
end

```

```

        stack_y(stack_n) = y + neighbour_offset_y(n);
        % count future topplings to be caused by this toppling
        if (f(y+neighbour_offset_y(n),x+neighbour_offset_x(n)) == (critical_state+1))
            future_topplings = future_topplings + 1;
        end
    end
end

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
end

% calculate additional topplings caused
if (future_topplings > 0)
    avalanche_add(t) = avalanche_add(t) + future_topplings - 1;

    % communicate additional topplings to come
    if (silent==false)
        disp(['this collapse generates ' ...
            num2str(future_topplings - 1) ...
            ' additional toppling(s)']);
    end
else
    % communicate additional topplings to come
    if (silent==false)
        disp(['this collapse generates no additional topplings']);
    end
end

end
end

% display field after collapsing
if (silent==false)
    disp(f);
    disp('');
end
end

% return avalanche sizes
as = avalanche_sizes;

% return number of topplings at avalanche starting site
nc = av_begin_t;

% return final state
final = f;

% return avalanche lifetimes
at = avalanche_sizes - avalanche_add;
end

```

A.3 avalanche distribution analysis

```

function [a,b,a2,b2] = avalanche_distribution_analysis(avalanche_sizes,avalanche_lifetimes)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% analysis avalanche distribution and fits it to a power-law
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% INPUTS
% avalanche_sizes array of avalanche size for each timestep
% avalanche_lifetimes same for av. lifetime

% OUTPUTS
% a,b coefficients of power law  $P(s) = a*s^b$ 
% a2,b2 coefficients of power law  $P(t) = a2*t^{b2}$ 

% count avalanche sizes/lifetimes
avalanche_count = zeros(1,max(avalanche_sizes)); % init
for s=1:max(avalanche_sizes)
    avalanche_count(s) = size(avalanche_sizes(avalanche_sizes==s),2);

```

A.4 test sandpile

[illegible]

```
f = critical.field(20,20,3,true);
[s,nc,ts,f] = sandpile(f, [-1 +1 0 0; 0 0 -1 +1], 3, 1, 100, 2, true, false, 0.2, false);
[a,b,c,d] = avalanche.distribution.analysis(s,ts)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% to do: continuous grain placing (0...1) Å la grain size
%
%      h parameter
%
% movie compilation with ffmpeg:
% ffmpeg -r 30 -b 1800 -i field%02d.png movie.avi
```
