Probability
Lecture 8

Lecture objectives

- Find the probability of compound events, using the multiplication rules.
- Define the term conditional probability.
- Calculate probabilities using Bayes' theorem.

The Multiplication Rules

Two events A and B are **independent events** if the fact that A occurs does not affect the probability of B occurring.

1. When two events are independent, the probability of both occurring is

$$P(A\, and\, B) = P(A) * P(B)$$

Example 1. A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

using the multiplication rule
$P(head\, and\, 4) = P(head) * P(4) = \frac{1}{2} * \frac{1}{6} = \frac{1}{12}$
we can get the result using the classical probability by getting the sample space for this experiment
H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6
so the probability of getting the head and 4 outcome is $\frac{1}{12}$

Example 2. A card is drawn from a deck and replaced; then a second card is drawn. Find the probability of getting a queen and then an ace.

Solution

The probability of getting a queen is $\frac{4}{52}$, and since the card is replaced, the probability of getting an ace is $\frac{4}{52}$. Hence, the probability of getting a queen and an ace is
$P(queen\, and\, ace) = P(queen) * P(ace) = \frac{4}{52} * \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$

the first Multiplication rule can be extended to three or more independent events by using the formula
$P(A\, and\, B\, and\, C\, and \ldots and\, K) = P(A) * P(B) * P(C) \ldots P(K)$

When a small sample is selected from a large population and the subjects are not replaced, the probability of the event occurring changes so slightly that for the most part, it is considered to remain the same

Example 3. A poll found that 46% of people say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Solution

if S denote the person with stress complain,then
$P(S \; and \; S \; and \; S) = P(S) * P(S) * P(S) = (0.46) * (0.46) * (0.46) = 0.097$

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent events.**

2. When two events are dependent, the probability of both occurring is
   $P(A \; and \; B) = P(A) * P(B|A)$

Example 4. At one of the neighbourhoods , there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004.

Solution

In this case, the events are dependent since the researcher wishes to investigate two distinct cases. the first case is selected and not replaced.
$P(C_1 \; and \; C_2) = P(C_1) * P(C_2|C_1) = \frac{16}{53} * \frac{15}{52} = 0.0871$

Example 5. Three cards are drawn from an ordinary deck and not replaced. Find the probability of these events.

1. Getting 3 jacks
2. Getting an ace, a king, and a queen in order
3. Getting a club, a spade, and a heart in order
4. Getting 3 clubs

Solution

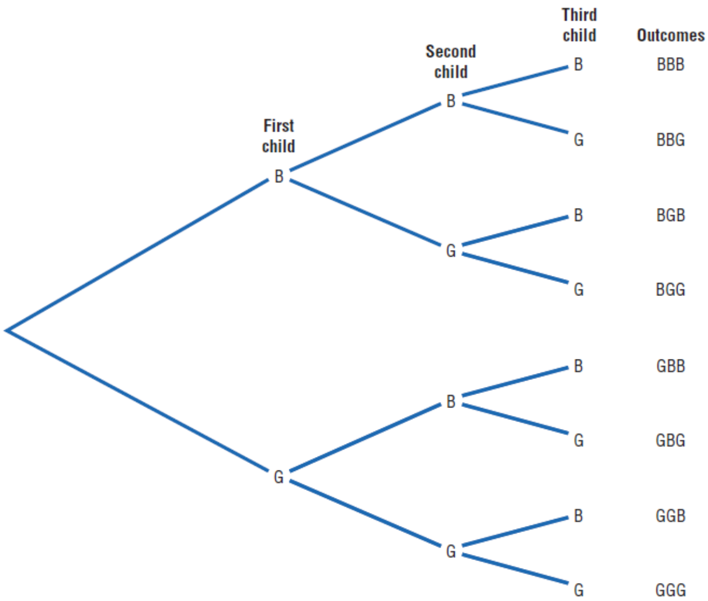
1. $P(3 \; jacks) = \frac{4}{52} * \frac{3}{51} * \frac{2}{50} = \frac{1}{5525} = 0.0002$
2. $P(ace \; and \; king \; and \; queen) = \frac{4}{52} * \frac{4}{51} * \frac{4}{50} = \frac{8}{16575} = 0.0005$
3. $P(club \; and \; spade \; and \; heart) = \frac{13}{52} * \frac{13}{51} * \frac{13}{50} = 0.01657$

Tree Diagrams

A tree diagram is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment

Example 6. Use the tree diagram to find the sample space for the gender of the children if a family has three children.

Solution



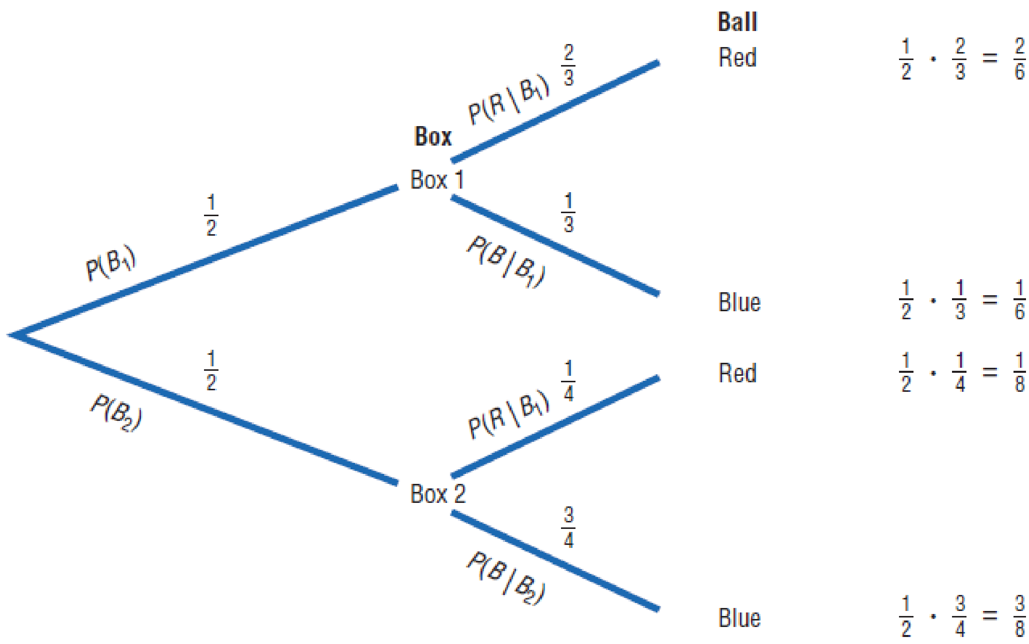
An event consists of a set of outcomes of a probability experiment.*A simple event* is An event with one outcome.*compound event* is the event consists of two or more outcomes or simple events.

Example 7. Box 1 contains 2 red balls and 1 blue ball. Box 2 contains 3 blue balls and 1 red ball. A coin is tossed. If it falls heads up, box 1 is selected and a ball is drawn. If it falls tails up, box 2 is selected and a ball is drawn. Find the probability of selecting a red ball.

Solution

Tree diagrams can be used as an aid to finding the solution
using the rule $P(A\, and\, B) = P(A) * P(B|A)$ the probability of selecting box 1 and selecting a red ball is$\frac{1}{2} * \frac{2}{3} = \frac{2}{6}$ and so on
find the rest of the probabilities ?

a red ball can be selected from either box 1 or box 2 so
$P(red) = \frac{2}{6} + \frac{1}{8} = 0.4583$



Conditional Probability

The conditional probability of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred. The notation for conditional probability is $P(B|A)$.
The conditional probability can be found by dividing the probability that both events occurred by the probability that the first event has occurred. The formula is

$$P(B|A) = \frac{P(A\, and\, B)}{P(A)}$$

Example 8. A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip and a white chip is $\frac{15}{56}$, and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip.

Solution

Let
B≡selecting a black chip
W≡selecting a white chip
Then $P(W|B) = \frac{P(W\, and\, B)}{P(B)} = \frac{15/56}{3/8} = 0.7143$
the probability of selecting a white chip on the second draw given that the first chip selected was black is 0.7143.

Example 9. A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find these probabilities.

1. The respondent answered yes, given that the respondent was a female.
2. The respondent was a male, given that the respondent answered no.

Solution

Let
M = respondent was a male
Y = respondent answered yes
F= respondent was a female
N= respondent answered no

1. The problem is to find $P(Y|F)$. The rule states
   $P(Y|F) = \frac{P(F\, and\, Y)}{P(F)}$
   The probability $P(F\, and\, Y)$is the number of females who responded yes, divided by the total number of respondents$P(F\, and\, Y) = \frac{8}{100}$.
   The probability $P(F)$ is the probability of selecting a female $P(F) = \frac{50}{100}$
   $P(Y|F) = \frac{P(F\, and\, Y)}{P(F)} = \frac{(8/100)}{50/100} = \frac{4}{25} = 0.16$
2. The problem is to find $P(M|N)$
   $P(M|N) = \frac{P(M\, and\, N)}{P(N)} = \frac{(18/100)}{60/100} = \frac{3}{10} = 0.3$

Example 10. A game is played by drawing 4 cards from an ordinary deck and replacing each card after it is drawn. Find the probability that at least 1 ace is drawn

Solution

Let E=at least 1 ace is drawn and $\overline{E}$ =no aces drawn. Then
$P(\overline{E}) = \frac{48}{52} * \frac{48}{52} * \frac{48}{52} * \frac{48}{52} = 0.726$
hence
$P(E) = 1 - p(\overline{E}) = 1 - 0.726 = 0.274$

Example 11. 86% of households had cable TV Choose 3 households at random. Find the probability that.

1. None of the 3 households had cable TV
2. All 3 households had cable TV
3. At least 1 of the 3 households doesn't has cable TV
4. At least 1 of the 3 households had cable TV

Solution

let C=houshould with a cable TV ,N=houshould without cable TV

1. $P(N) = 1 - P(C) = 1 - 0.86 = 0.14$
   $P(N\,N\,N) = P(N) * P(N) * P(N) = 0.14 * 0.14 * 0.14 = 0.003$

2. $P(C\,C\,C) = P(C) * P(C) * P(C) = 0.86 * 0.86 * 0.86 = 0.636$

Bayes' Theorem

Let $A_1, A_2, ...A_n$ be a partition of a sample space S Let B be some fixed event. Then

$$P(A_i|B) = \frac{P(A_i)(B|Ai)}{\sum_{i=1}^{n} P(A_i)(B|Ai)}$$

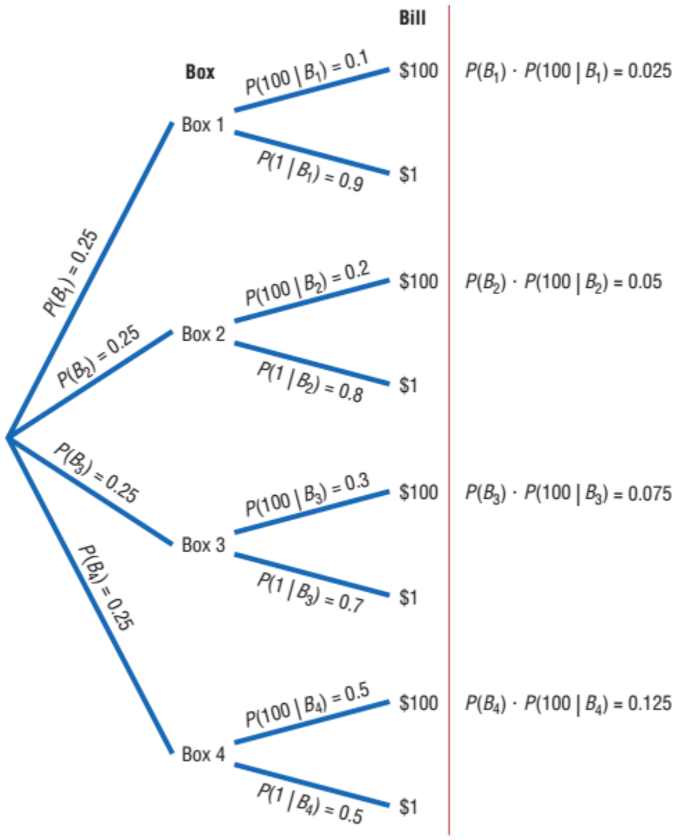
Example 12. On a game show, a contestant can select one of four boxes. Box 1 contains one 100$ bill and nine 1$ bills. Box 2 contains two 100$ bills and eight 1$ bills. Box 3 contains three 100$ bills and seven 1$ bills. Box 4 contains five 100$ bills and five 1$ bills. The contestant selects a box at random and selects a bill from the box at random. If a 100$ bill is selected, find the probability that it came from box 4

Solution

- Select the proper notation. Let B1, B2, B3, and B4 represent the boxes and 100 and 1 represent the values of the bills in the boxes.
- Draw a tree diagram and find the corresponding probabilities. The probability of selecting each box is 0.25.The probabilities of selecting the 100$ bill from each box, respectively, are 0.1, 0.2, 0.3, and 0.5.
- Using Bayes' theorem, write the corresponding formula.

$$p(B4 \mid 100) = \frac{P(B4)*P(100|B4)}{P(B1)*P(100|B1)+P(B2)*P(100|B2)+P(B3)*P(100|B3)+P(B4)*P(100|B4)}$$

$$p(B4 \mid 100) = \frac{0.125}{0.025+0.05+0.075+0.0125} = 0.455$$


Bill
Box
$P(B_1) = 0.25$
$P(B_2) = 0.25$
$P(B_3) = 0.25$
$P(B_4) = 0.25$
Box 1
Box 2
Box 3
Box 4
$P(100 \mid B_1) = 0.1$
$P(1 \mid B_1) = 0.9$
$P(100 \mid B_2) = 0.2$
$P(1 \mid B_2) = 0.8$
$P(100 \mid B_3) = 0.3$
$P(1 \mid B_3) = 0.7$
$P(100 \mid B_4) = 0.5$
$P(1 \mid B_4) = 0.5$
\$100
\$1
\$100
\$1
\$100
\$1
\$100
\$1
$P(B_1) \cdot P(100 \mid B_1) = 0.025$
$P(B_2) \cdot P(100 \mid B_2) = 0.05$
$P(B_3) \cdot P(100 \mid B_3) = 0.075$
$P(B_4) \cdot P(100 \mid B_4) = 0.125$