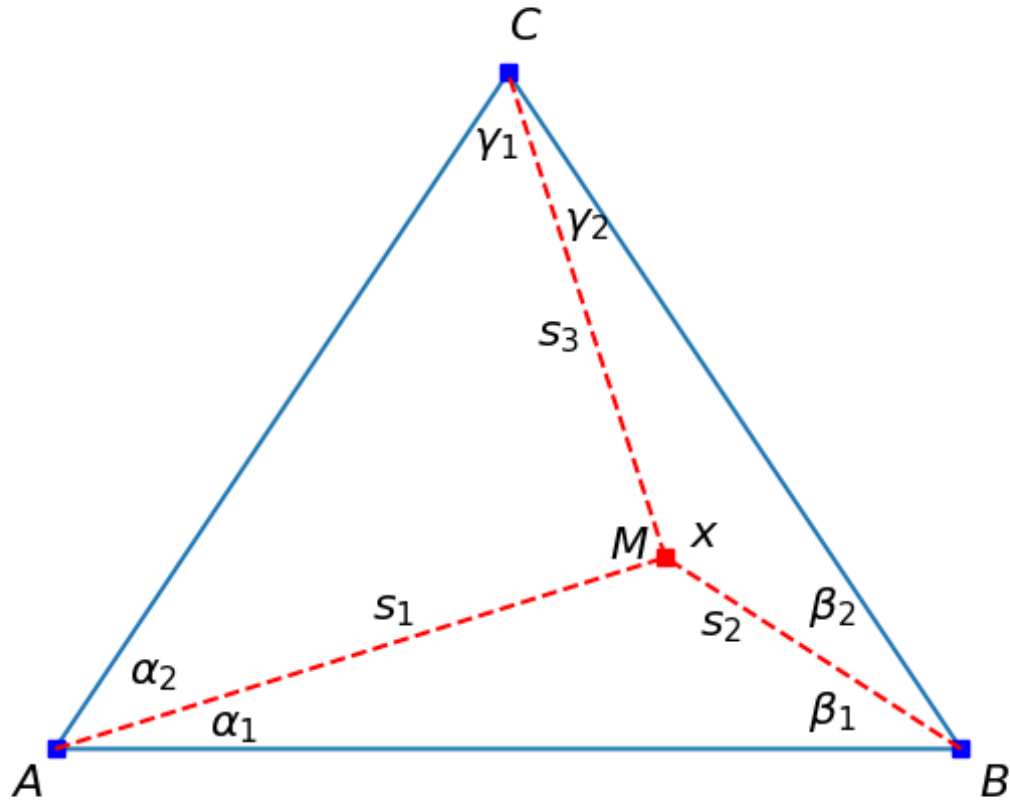


Triangle problem

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1 The problem



- A , B and C denote vertices
- s_1 , s_2 and s_3 denote the lengths of the line segments AM , BM and CM , respectively
- α_k , β_k and γ_k for $k = 1, 2$ are angles
- given are α_k , β_k ($k = 1, 2$)
- x (i.e., $\angle BMC$) is the unknown

All angles are assumed to be in radians.

2 Solution

From triangle ABM we have

$$\frac{s_1}{\sin \beta_1} = \frac{s_2}{\sin \alpha_1} \Rightarrow \frac{s_2}{s_1} = \frac{\sin \alpha_1}{\sin \beta_1}. \quad (1)$$

From triangle AMC we have

$$\frac{s_3}{\sin \alpha_2} = \frac{s_1}{\sin \gamma_1} \Rightarrow s_3 = s_1 \frac{\sin \alpha_2}{\sin \gamma_1}. \quad (2)$$

From triangle BMC we have

$$\frac{s_3}{\sin \beta_2} = \frac{s_2}{\sin \gamma_2} \Rightarrow s_3 = s_2 \frac{\sin \beta_2}{\sin \gamma_2}. \quad (3)$$

Combining (2) and (3) leads to

$$\frac{\sin \gamma_2}{\sin \gamma_1} = \frac{s_2 \sin \beta_2}{s_1 \sin \alpha_2}. \quad (4)$$

Let

$$z = \frac{s_2 \sin \beta_2}{s_1 \sin \alpha_2} = \frac{\sin \alpha_1 \sin \beta_2}{\sin \beta_1 \sin \alpha_2},$$

where we have used s_2/s_1 from (1). Then (4) can be expressed as

$$\sin \gamma_2 = z \sin \gamma_1, \quad (5)$$

and since $\gamma_1 + \gamma_2 = v := \pi - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)$, we have

$$\sin \gamma_2 = z \sin \gamma_1 \quad (6)$$

$$= z \sin(v - \gamma_2) \quad (7)$$

$$= \underbrace{z \sin v}_{c_1} \cos \gamma_2 - \underbrace{z \cos v}_{c_2} \sin \gamma_2. \quad (8)$$

Hence

$$\frac{\sin \gamma_2}{\cos \gamma_2} = \tan \gamma_2 = \frac{c_1}{1 + c_2}, \quad (9)$$

and we obtain

$$\gamma_2 = \arctan \left(\frac{c_1}{1 + c_2} \right). \quad (10)$$

Finally, $x = \pi - (\beta_2 + \gamma_2)$.

References