Triangle problem

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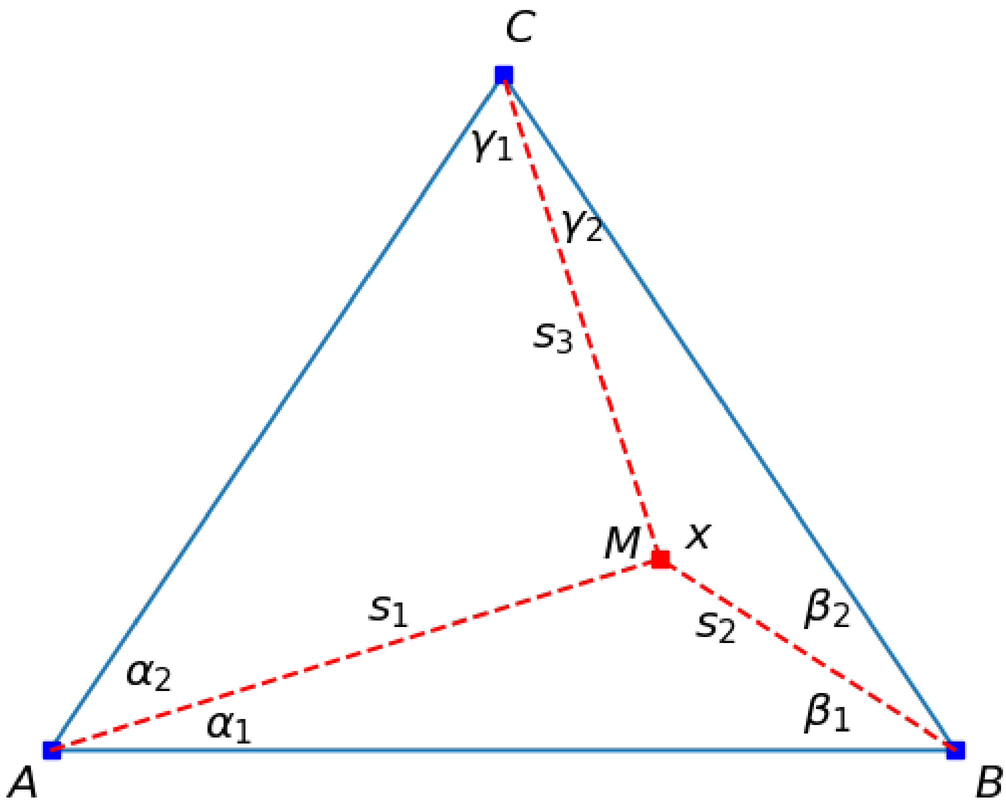
1 The problem



- A, B and C denote vertices
- s_1, s_2 and s_3 denote the lengths of the line segments AM, BM and CM, respectively
- α_k, β_k and γ_k for $k = 1, 2$ are angles
- given are α_k, β_k $(k = 1, 2)$
- x (*i.e.*, $\measuredangle BMC$) is the unknown

All angles are assumed to be in radians.

2 Solution

From triangle ABM we have

$$\frac{s_1}{\sin\beta_1} = \frac{s_2}{\sin\alpha_1} \Rightarrow \frac{s_2}{s_1} = \frac{\sin\alpha_1}{\sin\beta_1}. \tag{1}$$

From triangle AMC we have

$$\frac{s_3}{\sin\alpha_2} = \frac{s_1}{\sin\gamma_1} \Rightarrow s_3 = s_1\frac{\sin\alpha_2}{\sin\gamma_1}. \tag{2}$$

From triangle BMC we have

$$\frac{s_3}{\sin\beta_2} = \frac{s_2}{\sin\gamma_2} \Rightarrow s_3 = s_2\frac{\sin\beta_2}{\sin\gamma_2}. \tag{3}$$

Combining (2) and (3) leads to

$$\frac{\sin\gamma_2}{\sin\gamma_1} = \frac{s_2}{s_1}\frac{\sin\beta_2}{\sin\alpha_2}. \tag{4}$$

Let

$$z = \frac{s_2}{s_1}\frac{\sin\beta_2}{\sin\alpha_2} = \frac{\sin\alpha_1}{\sin\beta_1}\frac{\sin\beta_2}{\sin\alpha_2},$$

where we have used s_2/s_1 from (1). Then (5) can be expressed as

$$\sin\gamma_2 = z\sin\gamma_1, \tag{5}$$

and since $\gamma_1 + \gamma_2 = v \coloneqq \pi - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)$, we have

$$\begin{aligned} \sin\gamma_2 &= z\sin\gamma_1 && (6)\\ &= z\sin(v - \gamma_2) && (7)\\ &= \underbrace{z\sin v}_{c_1}\cos\gamma_2 - \underbrace{z\cos v}_{c2}\sin\gamma_2. && (8) \end{aligned}$$

Hence

$$\frac{\sin\gamma_2}{\cos\gamma_2} = \tan\gamma_2 = \frac{c_1}{1 + c_2}, \tag{9}$$

and we obtain

$$\gamma_2 = \arctan\left(\frac{c_1}{1 + c_2}\right). \tag{10}$$

Finally, $x = \pi - (\beta_2 + \gamma_2)$.

References