Homework #1
SOLUTION

1. In this problem, you will determine the finest detail that the human eye can discern at a set distance. Assume the pinhole camera model in figure 2.3 of the text where the distance between the pinhole and the retina along the visual axis is 17mm. Assume that the density of cones in the fovea is 150,000 elements per mm^2 and that the cones are arranged in a grid with no spacing between them.

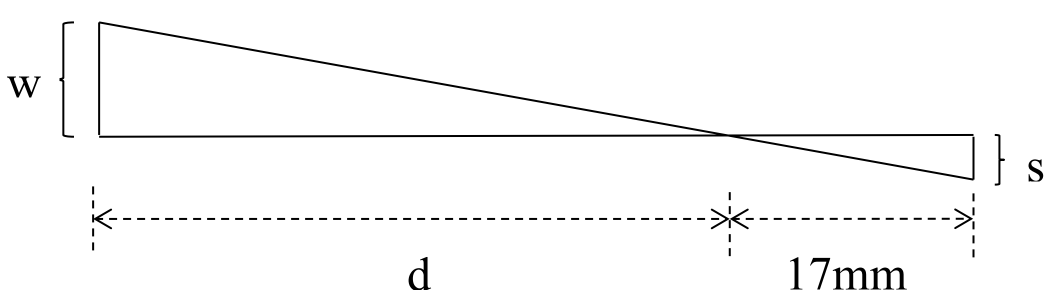
   Suppose you are looking at a scene with alternating black and white lines of equal width. Assume that you are able to discern the individual lines up to the point where the image of a line on your retina is smaller than a single cone. That is, when the image of a line is smaller than a cone, you can no longer tell it apart from an adjacent line.

   Calculate the width of the smallest line you can discern when the scene is:

   a) 0.2 meters from your eye (from the pinhole). (Note, due some simplifying assumptions, you will probably get a result which seems smaller than you expect.)

   SOLUTION:
   We first compute how large a single cone is. A density of 150,000 cones per mm^2 corresponds to a 1mm by 1mm grid of $\sqrt{150{,}000}$ or 387 cones per side. Thus, each cone measures approximately $1mm/387$ or 2.58×10^{-6}m along each dimension.

   Figure 2.3 models the human eye as the following pinhole camera (the diagram below is a bit different from that in figure 2.3 but they are really the same):



   Where *d* is the distance of an object, *w* is the size of the object, and *s* is the size of the image the object makes on the retina. From similar triangles,

$$\frac{w}{d} = \frac{s}{1.7 \times 10^{-2} m}$$

In this first case, d=0.2m. In order to see an individual line, we need

$$s \geq 2.58 \times 10^{-6} m.$$

Substituting, we get

$$w \geq 3.04 \times 10^{-5} m.$$

That is, the lines must be larger than approximately 0.03mm in width to be discerned at 0.2m.

b) 100 meters from your eye.

SOLUTION:
Now, d=100m and we must have

$$w \geq 1.52 \times 10^{-2} m.$$

That is, the lines must be larger than approximately 1.52cm in width to be discerned at 100m.

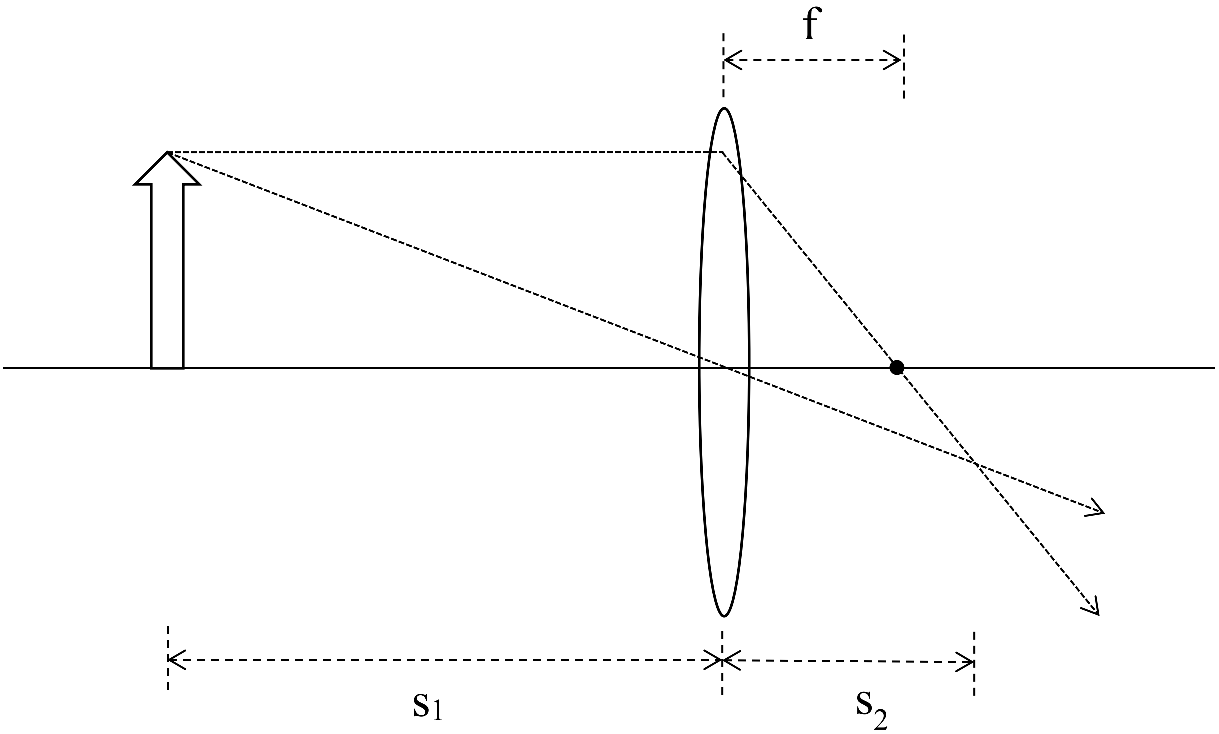
2. In this problem, you will determine the correct distance the imaging plane should be from the lens for an object at a certain distance to be in focus.

   Assume the thin lens model discussed in lecture for a camera with a focal length of 35mm. An object in the scene will be in focus when all the rays from the object that pass through the lens converge at the same point on the imaging plane.

   What should be the distance from the lens to the imaging plane for an object that is:

   a) 1m from the camera?

   SOLUTION:



Shown above is the geometry of a thin lens model. As derived in class, the relation between the focal length f, the distance of the object s_1, and the distance of where the rays of light converge s_2 is

$$\frac{1}{f} = \frac{1}{s_1} + \frac{1}{s_2}$$

We have f=35mm. For this first case, s_1=1m so from

$$s_2 = (f^{-1} - s_1^{-1})^{-1}$$

we have s_2=0.0362m or 36.2mm.

b) 100m from the camera?

SOLUTION:
Now, s_1=100m and so s_2=0.0350m or 35.0mm. Not much difference from the first case.

3. In this problem, you will calculate the cost of transmitting various forms of multimedia across a mobile data channel.

   Assume that your mobile data plan costs you $30 for 2GB (where 1GB is 1,024MB, 1MB is 1,024 KB, and 1KB is 1,024 bytes). Assume, also, that you pay proportionally. That is, if your usage is not an increment of 2GB then you pay only for the fraction that you use (whether it is less or more than 2GB).

   a) Calculate how much it would cost for you to download all of Shakespeare's plays where the total word count of all his plays is 835,997. Assume the average word length is five letters and that you represent each letter using a byte (we are ignoring spaces, punctuation, and formatting).

   SOLUTION:
   We can compute the cost per byte as

$$\frac{\$30}{2GB} \times \frac{1GB}{1{,}024MB} \times \frac{1MB}{1{,}024KB} \times \frac{1KB}{1{,}024bytes} = \$1.40 \times 10^{-8}\ per\ byte$$

   At an average of five letters per word, there are 4,179,985 letters in all his plays.

   So, the cost to download all of Shakespeare's plays is approximately

$$\$5.85 \times 10^{-2}$$

   or, about six cents.

   b) Calculate how much it would cost you to download a single image from a Canon EOS 5D Mark III DSLR camera without any compression.

   This camera produces 22.3 mega-pixel images where

$$1\ mega - pixel = 1{,}024 \times 1{,}024\ pixels$$

   and each pixel is represented using three bytes, one for each of the red, green, and blue color channels.

   SOLUTION:
   We first compute how many bytes are required to represent each image:

$$\frac{22.3\ mega-pixels}{1\ image} \times \frac{1{,}024 \times 1{,}024\ pixels}{1\ mega-pixel} \times \frac{3\ bytes}{1\ pixel} = 7.01 \times 10^{7}\ bytes/im$$

Now, multiplying by the cost per byte, the cost for downloading a single image is approximately

$$\$9.8 \times 10^{-1}$$

or, about one dollar.

c) Calculate how much it would cost to download a one-hour ultra high-definition 4k video without compression.

A 4k video has 60 frames per second where each frame measures 3,840 by 2,150 pixels. Again, assume each pixel is represented using three bytes.

SOLUTION:
We first compute how many bytes are required to represent the video:

$$\frac{60 \times 60\ seconds}{1\ video} \times \frac{60\ frames}{1\ second} \times \frac{3{,}840 \times 2{,}150\ pixels}{1\ frame} \times \frac{3\ bytes}{1\ pixel} = 5.35 \times 10^{12}\ bytes/video$$

Now, multiplying by the cost per byte, the cost for downloading the video is approximately

$$\$7.49 \times 10^{4}$$

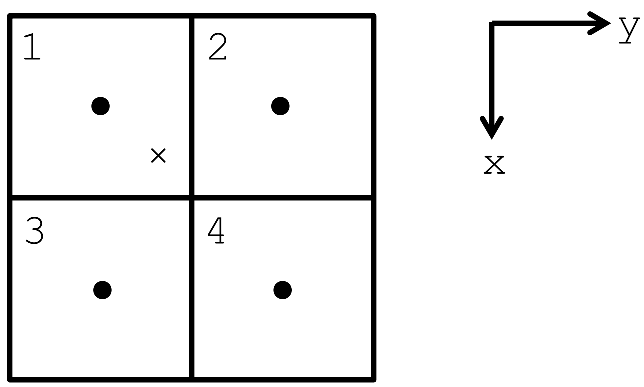
or, about \$75,000. It's a good thing we have compression!

Homework #2
SOLUTION

PROBLEM 1:

This problem investigates nearest neighbor and bilinear interpolation. For simplicity, we will focus on estimating the image intensity at a single location. Interpolation is used when transforming an image through resizing, rotating, etc. in which case, the image intensity will need to be estimated at a number of locations.

Consider the diagram below of four pixels



where the dots (●) represent the locations where we know the image intensity and the × represents the location where we would like to estimate the image intensity. By convention, the vertical axis is the x-axis and the horizontal axis is the y-axis.

Suppose the four pixels are at the following locations (indicated by (x,y)) and have the following intensity values (indicated by p)

$$
\begin{aligned}
(x_1, y_1) &= (4,10) \qquad p_1 = 100 \\
(x_2, y_2) &= (4,11) \qquad p_2 = 107 \\
(x_3, y_3) &= (5,10) \qquad p_3 = 120 \\
(x_4, y_4) &= (5,11) \qquad p_4 = 130
\end{aligned}
$$

and that we would like to estimate the image intensity at a fifth location

$$(x_5, y_5) = (4.3,10.4)$$

That is, we want to estimate p_5.

a) Provide an estimate for p_5 using nearest neighbor interpolation.

SOLUTION:

Nearest neighbor interpolation assumes the digital image is a sampled version of a continuous image that has constant intensity from a pixel center to the boundaries of the adjacent pixels. Therefore, to estimate the value at a particular location, we first determine which pixel boundary it falls in and then assign the value from that pixel.

To do this, we simply use the value of the nearest pixel (and thus the name of the method). In our case, the nearest pixel to location × is pixel 1 which has a value of 100. So,

$$p_5 = 100.$$

b) Provide an estimate for p_5 using bilinear interpolation. Round your value to the nearest integer. You can use either of the two methods discussed in lecture. (You might want to use both methods to check your answer.)

SOLUTION:

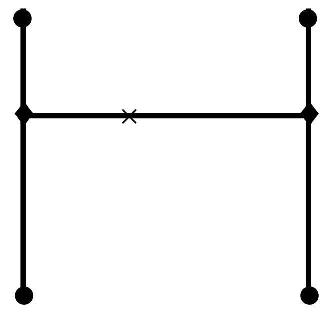
Nearest neighbor interpolation assumes the digital image is a sampled version of a continuous image with a bilinear surface. That is, the surface varies linearly along each of the x and y axes, or, equivalently, has the parametric form:

$$f(x, y) = ax + by + cxy + d \quad .$$

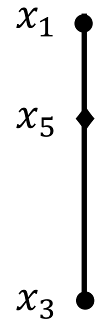
Method 1:

The first method presented in lecture performs linear interpolation in stages, first along one dimension and then the other. The order does not matter.

We will first interpolate along the x-axis to estimate the values at the locations indicated by the diamonds (♦) in the figure below and then interpolate along the y-axis to estimate the value at the location ×, which is what we want.

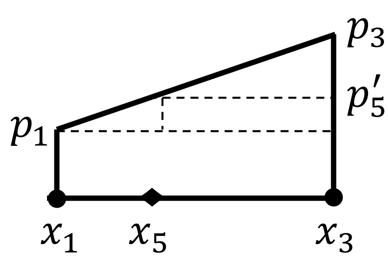


We have the following linear situation for pixels 1 and 3



where location x_1 has value p_1, location x_3 has value p_3, and we want to estimate the value at location x_5 which we will denote as p_5' .

Looking at this "sideways", where height is the image intensity, we have



From similar triangles, we have

$$\frac{p_5' - p_1}{x_5 - x_1} = \frac{p_3 - p_1}{x_3 - x_1}$$

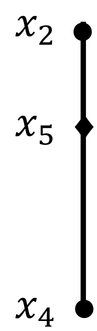
or,

$$p_5' = (p_3 - p_1)\frac{(x_5 - x_1)}{(x_3 - x_1)} + p_1 \,.$$

Substituting for the known values, $x_1 = 4$, $x_3 = 5$, $x_5 = 4.3$, $p_1 = 100$, and $p_3 = 120$, we get

$$p_5' = 106 \,.$$

Repeating for pixels 2 and 4



where location x_2 has value p_2, location x_4 has value p_4, and we want to again estimate the value at location x_5 which we will denote as p_5'' .

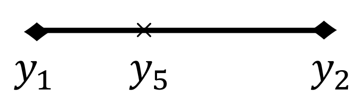
Using similar triangles, we end up with

$$p_5'' = (p_4 - p_2)\frac{(x_5 - x_2)}{(x_4 - x_2)} + p_{21} \,.$$

Substituting for the known values, $x_2 = 4$, $x_4 = 5$, $x_5 = 4.3$, $p_2 = 107$, and $p_4 = 130$, we get

$$p_5'' = 113.9 \,.$$

We now have the following linear situation for the two locations that we just estimated the value at (represented by diamonds):



Using similar triangles, we end up with

$$p_5 = (p_5'' - p_5')\frac{(y_5 - y_1)}{(y_2 - y_1)} + p_5' \,.$$

Substituting for the known values, $y_1 (= y_3) = 10$, $y_2 (= y_4) = 11$, $y_5 = 10.4$, $p_5' = 106$, and $p_5'' = 113.9$, we get

$$p_5 = 109.16$$

or, rounding to the nearest integer

$$p_5 = 109 \,.$$

Method 2:

Here, we first determine the parameters *a*, *b*, *c*, and *d* in the bilinear equation

$$f(x, y) = ax + by + cxy + d \quad .$$

We then can simply use this equation to estimate the value at any location.

Assuming the surface specified by the bilinear equation goes through the four locations at which we know the intensity values, we have:

$$\begin{aligned} p_1 &= ax_1 + by_1 + cx_1y_1 + d \\ p_2 &= ax_2 + by_2 + cx_2y_2 + d \\ p_3 &= ax_3 + by_3 + cx_3y_3 + d \\ p_4 &= ax_4 + by_4 + cx_4y_4 + d \end{aligned}$$

All the values in these equations are known except for the parameters *a*, *b*, *c*, and *d*. This is a system of four equations and four unknowns.

A simple way to solve for the unknowns is to write this system in matrix multiplication form:

$$\begin{bmatrix} x_1 & y_1 & x_1y_1 & 1 \\ x_2 & y_2 & x_2y_2 & 1 \\ x_3 & y_3 & x_3y_3 & 1 \\ x_4 & y_4 & x_4y_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

or

$$Ag = h \,.$$

We can solve for g

$$g = A^{-1}h\,.$$

In our case,

$$A = \begin{bmatrix} 4 & 10 & 40 & 1 \\ 4 & 11 & 44 & 1 \\ 5 & 10 & 50 & 1 \\ 5 & 11 & 55 & 1 \end{bmatrix}$$

so,

$$A^{-1} = \begin{bmatrix} -11 & 10 & 11 & -10 \\ -5 & 5 & 5 & -4 \\ 1 & -1 & -1 & 1 \\ 55 & -50 & -44 & 40 \end{bmatrix} \quad .$$

(You can compute this matrix inverse in Matlab.)

And,

$$h = \begin{bmatrix} 100 \\ 107 \\ 120 \\ 130 \end{bmatrix}$$

so,

$$g = \begin{bmatrix} -10 \\ -5 \\ 3 \\ 70 \end{bmatrix}$$

or, $a = -10$, $b = -5$, $c = 3$, and $d = 70$.

We now know the bilinear equation

$$f(x, y) = -10x + (-5)y + 3xy + 70$$

and can estimate the intensity at $(x_5, y_5) = (4.3,10.4)$ as

$$p_5 = 109.16$$

or, rounding to the nearest integer

$$p_5 = 109 .$$

PROBLEM 2:

SOLUTION:

Let p and q be as shown in Fig. P2.11. Then, (a) S_1 and S_2 are not 4-connected because q is not in the set $N_4(p)$; (b) S_1 and S_2 are 8-connected because q is in the set $N_8(p)$; (c) S_1 and S_2 are m-connected because (i) q is in $N_D(p)$, and (ii) the set $N_4(p) \cap N_4(q)$ is empty.

		S_1				S_2			
0	0	0	0	0	0	0	1	1	0
1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	1 q	1	0	0	0
0	0	1	1	p 1	0	0	0	0	0
0	0	1	1	1	0	0	1	1	1

Figure P2.11

PROBLEM 3:

SOLUTION:

(a) When $V = \{0,1\}$, 4-path does not exist between p and q because it is impossible to get from p to q by traveling along points that are both 4-adjacent and also have values from V. Figure P2.15(a) shows this condition; it is not possible to get to q. The shortest 8-path is shown in Fig. P2.15(b); its length is 4. The length of the shortest m- path (shown dashed) is 5. Both of these shortest paths are unique in this case.

(b) One possibility for the shortest 4-path when $V = \{1,2\}$ is shown in Fig. P2.15(c); its length is 6. It is easily verified that another 4-path of the same length exists between p and q. One possibility for the shortest 8-path (it is not unique) is shown in Fig. P2.15(d); its length is 4. The length of a shortest m-path (shown dashed) is 6. This path is not unique.

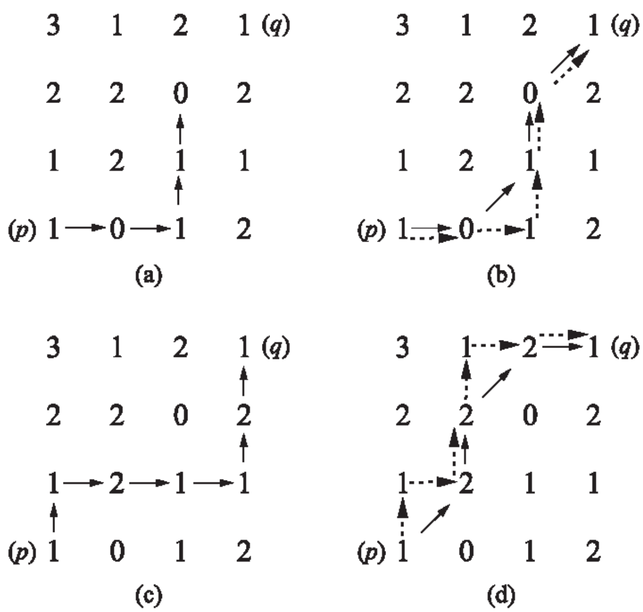


Figure P.2.15

PROBLEM 4:

This problem will help with your project on image resizing.

a) Find the linear mapping from a location m′ in the transformed image to the location x in the original image. That is, derive a function that given m′, computes x. The transformed image measures M′ pixels and the original image measures M pixels. Note that M′ can be greater than or less than M.

Your mapping should have the following form

$$x = t(m') = \frac{A}{B}\left(m' - C\right) + D$$

where the constants A, B, C, D are determined from the following constraints:

- The left boundary of the transformed image should map to the left boundary of the original image.
- The right boundary of the transformed image should map to the right boundary of the original image.
- Locations in between should be mapped linearly (proportionally).

Note that the linear equation above really only has two constants and could be written as

$$x = t(m') = Am' + B .$$

I provided the four-constant version above to help you think about how to incorporate the constraints (that form helps me, at least). You can use either form in your answer.

Note that x typically won't be integer valued (that is, it won't fall on a pixel location in the original image) even if m′ is integer valued.

SOLUTION:

$$x = t(m') = \frac{M}{M'}\left(m' - 0.5\right) + 0.5$$

so that

$$t(0.5) = 0.5$$

and

$$t(M' + 0.5) = M + 0.5 .$$

$t(\cdot)$ can also be written as

$$x = t(m') = \left(\frac{M}{M'}\right)m' + \left(0.5 - \frac{M}{M'}\left(0.5\right)\right)$$

b) Derive the logic and computation to determine the closest pixel m to a location x in the original image. That is, given x, determine m where m $\in$[1,…,M]. You can assume

$$0.5 \le x \le M + 0.5 .$$

SOLUTION:

We can simply set $m = \text{round}(x)$. A special case is when $x = M + 0.5$ where we need to set $x = M$.

c) Derive the logic and computation to determine the two closest pixels m1 and m2 to a location x in the original image. Some special cases you might need to consider:

- x is integer valued
- x is less than 1.0
- x is greater than M
- x is equal to 1.0
- x is equal to M

SOLUTION:

```
If x is integer valued
    m1 = x
    m2 = x
Else
    If x < 1
            m1 = 1
            m2 = 2
    Else if x > m
            m1 = M-1
            m2 = M
    Else
            m1 = floor(x)                // largest integer smaller than x
            m2 = ceiling(x)              // smallest integer larger than x
    End
End
```

d) Now, think about the 2D case. Both dimensions will need to be mapped from the transformed image to the original image. Nearest neighbor interpolation still only requires the closest pixel. Bilinear interpolation now requires the four closest pixels.

Suppose mapping a pixel in the transformed image along the m dimension results in the locations m1 and m2 being the m-coordinates of the closest points, and that mapping a pixel in the transformed image along the n dimension results in the locations n1 and n2 being the n-coordinates of the closest points. List the coordinates of the four closest points in 2D. (This should be fairly straightforward but I want to get you thinking about the problem in 2D for your project).

SOLUTION:

The four closest pixels are simply

$(m1, n1)$
$(m1, n2)$
$(m2, n1)$
$(m2, n2)$

Homework #3
SOLUTION

1. A simple example shows that computing the median value of an image is not a linear operator. Let 1x3 image $F1 = [1, -2, 3]$, 1x3 image $F2 = [4,5,6]$, and $a = b = 1$. Let H be the median operator. We then have $H(F1 + F2) = median([5,3,9]) = 5$. We also have $H(F1) = median([1, -2, 3]) = 1$ and $H(F2) = median([4,5,6]) = 5$. Then, because $H(aF1 + bF2) \neq aH(F1) + bH(F2)$, the median operator is not linear.

2. In general, do affine transformations commute?

   SOLUTION
   Affine transformations, in general, do not commute. We just need to find a counterexample to prove this.

   Consider a rotation transformation and a shear transformation and their respective matrices:

   Rotation matrix = $T_R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   Shear matrix = $T_S = \begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   Case 1: First rotate and then shear. That is, find where the pixel at (v, w) is mapped to by the combination of these transformations.

   First, the rotation:

$$[x' \quad y' \quad 1] = [v \quad w \quad 1] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta \quad v\sin\theta + w\cos\theta \quad 1]$$

   Now, the shear:

$$[x \quad y \quad 1] = [x' \quad y' \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta \quad v\sin\theta + w\cos\theta \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta + s_v(v\sin\theta + w\cos\theta) \quad v\sin\theta + w\cos\theta \quad 1]$$

   So, $(x, y) = (v\cos\theta - w\sin\theta + s_v(v\sin\theta + w\cos\theta),\ v\sin\theta + w\cos\theta)$

   Case 2: First shear and then rotate. That is, find where the pixel at (v, w) is mapped to by the combination of these transformations.

First, the shear:

$$[x' \quad y' \quad 1] = [v \quad w \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v + s_v w \quad w \quad 1]$$

Now, the rotation:

$$[x \quad y \quad 1] = [x' \quad y' \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v + s_v w \quad w \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(v + s_v w)\cos\theta - w\sin\theta \quad (v + s_v w)\cos\theta - w\sin\theta \quad 1]$$

So, $(x, y) = \big((v + s_v w)\cos\theta - w\sin\theta\,,\ (v + s_v w)\cos\theta - w\sin\theta\big)$

This is not equal to (x, y) in case 1 above.

So, these two transformations do not commute and we have found our counterexample.

We can then conclude that affine transformations do not, in general, commute.

3. An affine transformation can be completely specified by its action on three points. What is the matrix for the transformation that maps the points A=(0,0) B=(1,0) and C=(0,1) to the points A'=(2,2) B'=(1,2) and C'=(2,1) respectively? (You can use Matlab.)

   Draw these points before and after the transformation. What is this transformation doing (in terms of scaling, rotating, shearing, translating, etc.)?

   SOLUTION
   Setup a system of equations representing the known mappings to solve for the six unknowns in the transformation matrix.

   Each mapping has the form

$$[x \quad y \quad 1] = [v \quad w \quad 1]\begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix}$$

   For example, the fact that $A = (0,0)$ is mapped to $A' = (2,2)$ gives

$$[2 \quad 2 \quad 1] = [0 \quad 0 \quad 1]\begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix}$$

   Now, the three mappings can be combined together in the following equation:

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix}$$

Writing this as $G = FT$, we can solve for T as

$$T = F^{-1}G$$

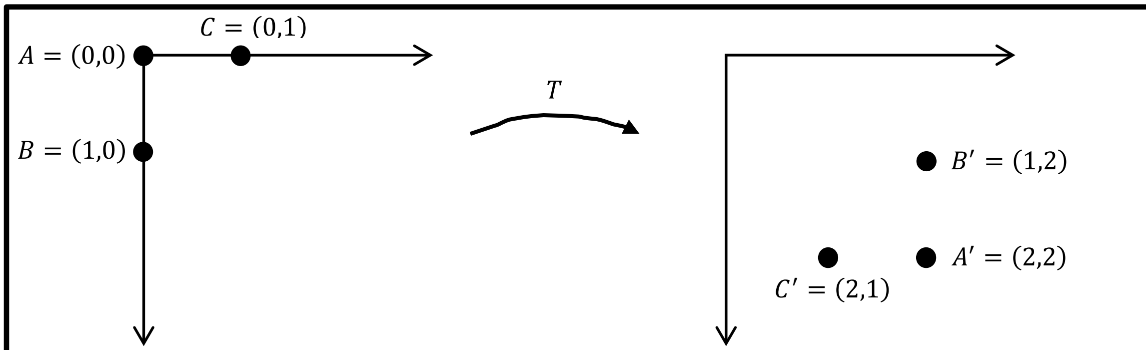
Using Matlab, we compute

$$F^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and then we can solve for T

$$\boxed{T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}}$$

We now plot the points before and after the transformation:



We observe this transformation is a rotation by 180° followed by a translation by (2,2).

4. Suppose images $f(x,y)$ and $g(x,y)$ have histograms h_f and h_g. Suppose you form image $k(x,y)$ as the sum of these two images:

$$k(x,y) = f(x,y) + g(x,y).$$

(a) Under what conditions (on images f and g) can you determine the histogram h_k of image k in terms of the histograms h_f and h_g?

(That is, under what conditions can you determine h_k if you are only given h_f and h_g? Hint: this turns out to be fairly restrictive.)

(b) Explain how you would derive h_k in this case.

SOLUTION

The problem here is that you need to know the spatial correspondences between the intensity values in images f and g in order to determine the pixel intensities and counts in image k. Histograms do not convey information about the spatial locations of the pixel intensities.

One case in which you can determine h_k from h_f and h_g is when one or both of images f and g is constant (that is, has only one intensity value).

(b) Explain how you would derive h_k in this case.

SOLUTION

Without loss of generality, suppose image f is constant. That is,

$$f(x, y) = c$$

for some constant c.

Then, the number of pixels with value u_i in image k is the same as the number of pixels with value $u_i - c$ in image g.

Thus, the histogram of image k can be computed as

$$\boxed{h_k\ (u_i) = h_g\ (u_i - c)}$$

Note that this is the more general case of the situation where all the pixels in one of the images have value 0.

ALTERNATE SOLUTION

(a) Another case in which you can determine h_k from h_f and h_g is when images f and g are the same; i.e, $f(x, y) = g(x, y)$.

(b) Now, all the pixels in image k have twice the value of the corresponding pixels in image f (equivalently, image g). So, the number of pixels with value u_i in image k is the same as the number of pixels with value $\frac{u_i}{2}$ in image f.

Thus, the histogram of image k can be computed as

$$\boxed{h_k\ (u_i) = h_f\ \left(\frac{u_i}{2}\right)}$$