MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

A NON-RESIDUAL BASED ADAPTIVE SCHEME FOR BOUNDARY INTEGRAL METHODS

DARREN READING

ABSTRACT. An adaptive boundary element procedure is proposed for the boundary value problem for Laplace's equation that is based only on non-residual a-posteriori error estimates. The error estimates are used to compute information about the distribution of the error around the boundary, identifying where the error is large and appropriately refining the mesh, steering the computational process until a prescribed tolerance is reached.

These adaptive schemes are cheap and easy to compute, offering a considerable saving over the existing residual based methods. We present some numerical examples for this procedure and compare this procedure with residual based adaptive schemes as well as with uniform and graded meshes.

1. Introduction

Adaptive algorithms for finite element methods have been studied in numerous papers [2, 3, 11, 12, 14] and have proven to be an efficient numerical procedure for a wide range of problems. In contrast there has been, relatively few results on adaptive boundary element methods. This torpor can be attributed to the lack of theoretical foundation of a-posteriori error estimates and the fact that the error is less obviously local. However this situation has started to be rectified with adaptive boundary element methods receiving more attention.

Most papers investigating adaptive schemes for boundary element methods deal with Galerkin methods [4, 5, 6, 7, 8, 9, 10, 13, 18, 20, 22]. One the first rigorous mathematical analysis was given by Rank [15], Yu [13, 22] and Wendland and Yu [20]. This approach was further refined by Wendland and Stephen [18] and then by Carstensen, Estep and Stephan [4, 5, 6, 7, 8, 9, 10].

For the collocation method there exists only a few papers which deal adaptive algorithms. Rank [15, 16] showed the feasibility of such schemes for boundary element methods, albeit non-rigorously. While Wendland and Yu [20], presented a rigorous analysis, which exploited the Arnold- Wendland lemma to utilise the equivalence between the nodal collocation method with odd degree smoothest splines and a non-standard Galerkin method. Unfortunately such analysis cannot be extended to cover the even degree case, which includes the important case of piece-wise constant elements.

The next two sections, Sections 2 and 3, briefly review the existing adaptive procedures for boundary element methods and describe the adaptive algorithms refinement process. These adaptive procedures are based upon residual type a-posteriori error estimates and as such take up a substantial part of the overall

2010 *Mathematics Subject Classification.* Primary 65N15, 65N35.

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

computational process. Section 4 is devoted to exploring some adaptive schemes based only on information about the jumps in the approximation. These adaptive schemes are cheap and easy to compute, offering a considerable saving over residual based methods described in Section 2. In the final section, we present some numerical examples for this procedure and compare this procedure with residual based adaptive schemes as well as with uniform and graded meshes.

2. Residual Based Adaptive Schemes

In this section we briefly describe the a-posteriori error estimates upon which the adaptive boundary element schemes are based. The literature on adaptive boundary element methods follows three different approaches. We begin by looking at one of the first approaches at constructing adaptive boundary element methods.

2.1. **Wendland and Yu.** The reason for the remarkable efficiency of adaptive finite element methods is the fact that differential operators are local operators and therefore the local error corresponding to the differential equation influences the residual only locally, too. However boundary integral operators are non-local and this causes additional difficulties for designing adaptive schemes. This non-locality is reflected in the stiffness matrix for the boundary element matrix being fully populated despite the local support of the trial functions. However for certain integral operators a pseudo-local property [19, 20] can be defined which states the influence of a local disturbance decays sufficiently fast. This is the starting point of Wendland and Yu's papers [13, 20].

Consider an integral equation on the sufficiently smooth boundary Γ of some given bounded domain:

$$Av(s) \; := \; \int_\Gamma a(s,\sigma)v(\sigma)\,d\sigma \; = \; f(s) \tag{2.1}$$

where $A : H^\gamma(\Gamma) \to H^{-\gamma}(\Gamma)$ is a pseudodifferential operator of order 2γ. Further assume that (2.1) has a unique solution and A is strongly elliptic. The latter implies that

$$A \; = \; D + K$$

where D maps $H^\gamma(\Gamma)$ continuously onto $H^{-\gamma}(\Gamma)$, K maps $H^\gamma(\Gamma)$ continuously into $H^{-\gamma+\delta}(\Gamma)$ for some $\delta > 0$ and D is strongly coercive

$$(\,Dv, v\,) \; \geq \; C\,||\,v\,||^2_{H^\gamma(\Gamma)}, \qquad \forall v \in H^\gamma(\Gamma)$$

where C is a positive constant. Let us denote the corresponding bilinear forms by

$$A(v,u) \; = \; (\,Av, u\,) \; = \; \int_\Gamma u\,Av$$

$$D(v,u) \; = \; (\,Dv, u\,) \; = \; \int_\Gamma u\,Dv.$$

Since D is strongly coercive the norm $||\cdot||_D$ defined by $D(\cdot,\cdot)$ on $H^\gamma(\Gamma)$ and the norm $||\cdot||_{H^\gamma(\Gamma)}$ are equivalent.

For the boundary element method we divide Γ up into n elements Γ_j. Let $\{L_Q(s)\}_{Q\in J}$ be a basis of the function space $S^p(\Gamma)$ of piecewise polynomials of degree p associated with the partition $\{\Gamma_j\}$ satisfying $S^p(\Gamma) \subset H^\gamma(\Gamma)$. Here J

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

denotes the set of given nodes on Γ associated with $S^p(\Gamma)$. Then the Galerkin formulation of (2.1) is:

$$\textit{Find } v^n \in S^p(\Gamma) \textit{ such that } A(v^n, u^n) \;=\; \int_\Gamma f\, u^n \;\forall\; u^n \in S^p(\Gamma).$$

Let $r = Av^n - f$ be the residual, which is computable when v^n is known.

Assumption 2.1. *The set J can be partitioned into subsets $J_1, \ldots, J_k$ such that for any $l = 1, 2, \ldots, k$ and for all $P \neq Q \in J_l$*

$$D(v_P, v_Q) \;\leq\; \frac{1}{|J_l|} D(v_P, v_P)^{1/2} D(v_Q, v_Q)^{1/2}$$

for all $v_P \in H_P$ and $v_Q \in H_Q$ where $|J_l|$ denotes the number of nodes in J_l.

$$H_Q \;=\; \{v \in H^\gamma(\Gamma) : v = 0 \textit{ on } \Gamma \setminus \Gamma_Q\} \qquad \Gamma_Q \;=\; \operatorname{supp} L_Q(s)$$

This assumption is the requirement for a pseudo-local property. Such partitions always exists, for one can take k to be the number of interpolation nodes and $|J_l| = 1$. In addition k is obviously less than $|J|$ and greater than 1 so there will be a minimum of these k, called the influence index ρ. The stronger the singularity of D the smaller the influence index ρ will be. Yu [13] showed that for a uniform partition and piecewise constant or linear functions if the kernel of D behaves like $\log|x-y|$ then $\rho \sim \alpha n$, $0 \leq \alpha \leq 1$ and if it behaves like $|x-y|^{-1}$ then $\rho \sim n^{1/2}$.

The main result of [13, 20, 22] is an a-posteriori error estimate for the Galerkin method. Before we state this result, first define ξ_Q to be the solution of

$$D(\xi_Q, u_Q) \;=\; \int_\Gamma r u_Q$$

for all $u_Q \in H_Q$. If $A = D$ then ξ_Q is just the projection of the error $v^n - v$ onto H_Q with respect to $D(\cdot,\cdot)$.

Theorem 2.2. *If Assumption 2.1 is satisfied and $S^p(\Gamma)$ satisfies certain approximation properties then there exists positive constants C_1, C_2 and h_0 such that for $0 < h \leq h_0$ there holds*

$$C_1 \sum_{Q\in J} ||\xi_Q||_D^2 \;\leq\; ||u_h - u||_D^2 \leq C_2 \sum_{Q \in J} ||\xi_Q||_D^2 \tag{2.2}$$

where $C_1 = 1/2\rho$.

The abstract estimate (2.2) is not all useful in its present form but it is possible to estimate $||\xi_Q||_D$ using the residual r. Let r_Q be the restriction of the residual r to Γ_Q. Then r_Q defines a computable indicator for the local error since

$$D(\xi_Q, \xi_Q) = \int_\Gamma r\xi_Q = \int_\Gamma r_Q \xi_Q \leq ||r_Q||_{-\gamma} ||\xi_Q||_\gamma \leq C||r_Q||_{-\gamma}||\xi_Q||_D$$

so that

$$||\xi_Q||_D \;\leq\; C||r_Q||_{-\gamma}.$$

In addition if $\gamma \leq 0$ we also have a lower estimator

$$||r_Q||^2_{L^2(\Gamma)} = \int_\Gamma r_Q r_Q = \int_\Gamma r r_Q = D(\xi_Q, r_Q) \leq C||\xi_Q||_D ||r_Q||_D$$

that is

$$C||r_Q||^2_{L^2(\Gamma)} \,/\, ||r_Q||_D \;\leq\; ||\xi_Q||_D$$

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

Wendland and Yu [20] generalized the a-posteriori estimate in (2.2) to spline collocation methods with smoothest odd degree splines using the Arnold Wendland lemma [1] to analyze the collocation method as a modified Galerkin method. But this analysis cannot be extended to even degree spline collocation method.

2.2. **Saranen and Wendland.** Wendland and Yu [21] investigated a Galerkin method for general strongly elliptic boundary integral equation showing that the residual can serve as a local error estimator assuming only that the mesh possesses a bounded local mesh ratio. Saranen and Wendland [18] improved those results and removed various restrictions present in [21].

The mesh $\{\Gamma_i\}$ has a bounded local mesh ratio if there exists a constant C such that

$$1/C \;\le\; h_{i+1}/h_i \;\le\; C$$

for all i where $h_i = |\Gamma_i|$ be the length of the arc Γ_i. The set of all meshes which possess this property will be denoted by $\mathfrak{B}$. In addition a mesh is β- regular if for $\beta \ge 1$ if there exists C such that

$$\min h_i \ge C(\max h_i)^{\beta}.$$

The set of all regular meshes will be denoted by $\mathfrak{R}$. These are weak restrictions that are satisfied by graded meshes of the form $x_i = (i/n)^q$ for some integer q and by meshes generated by successive subdivision of an initial coarse mesh. Although for adaptively generated meshes, one needs to ensure appropriate safeguards are incorporated into the program to guarantee that the mesh satisfies the β-regularity condition.

Theorem 2.3. *Suppose that the mesh belongs to $\mathfrak{B}$ and that the closed subarcs $I_0 \subset\subset I_1 \subset\subset I_2$ of Γ are given. Then the Galerkin solution $v^n \in S^p(\Gamma)$ of (2.1) satisfies the following local error estimates:*
(i) Assume $0 \le \gamma \le 1$ and $p \le 2\gamma$. If $v \in H^{2\gamma}(I_2) \cap H^{\gamma}(\Gamma)$ then there holds

$$|| \, v^n - v \, ||_{H^0(I_0)} \;\le\; C \, (\, h^{p+1-\gamma} ||v||_{H^{\gamma}(\Gamma)} + h^{2\gamma} ||r||_{H^0(I_1)} \,) \tag{2.3}$$

(ii) Assume $\gamma < 0$ and the mesh belongs to $\mathfrak{B} \cup \mathfrak{R}$. If $v \in H^0(\Gamma)$ then

$$|| \, v^n - v \, ||_{H^0(I_0)} \;\le\; C \, (\, h^{p+1-2\gamma} || \, v \, ||_{H^0(\Gamma)} + || \, r \, ||_{H^{-2\gamma}(I_1)} \,) \tag{2.4}$$

From (2.3) and (2.4) we can see that the local error can be bounded by a sum of the local residual together with some global term which is negligible, although for negative γ, β-regular meshes are required.

2.3. **Carstensen and Stephan.** Carstensen et al [4, 5, 10] follow a different approach to Yu, Wendland and Stephan [18, 21] in deriving a posteriori estimates for the Galerkin method for boundary integral equations. The results by Saranen and Wendland give a local a posteriori error estimates whereas the approach in [4, 5, 10] gives an upper bound of the global error in energy norms which can be evaluated locally. In addition the adaptive procedures in [4, 5, 10] are not restricted to closed smooth curves in two dimensions allowing for open arcs, corners and problems in any dimension. Finally no restriction at all is placed on the mesh unlike [18] where the analysis required a bounded local mesh ratio and/or β-regularity.

For simplicity we concentrate on Symm's integral equation although the analysis in [4, 5, 6, 7, 8, 9, 10] covers a wide range of problems. For Symm's integral equation

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

(2.1) becomes

$$Vv(s) := -\frac{1}{\pi}\int_\Gamma v(\sigma)\log|s-\sigma|dl_\sigma = f(s) \tag{2.5}$$

where V is a pseudo-differential operator of order -1. It is well known that such integral operators arise when one reformulates the boundary value problem for Laplace's equation in plane using the boundary integral method.

The main result from [4, 5, 10] is the following estimate for the the error in the Galerkin method.

Theorem 2.4. *Let a_j be the local contribution of the derivative of the residual r given by*

$$a_j = ||\partial r/\partial s||_{L^2(\Gamma_j)}$$

where $\partial/\partial s$ denotes differentiation with respect to arc length. Then the following a posteriori estimates hold

$$|| v^n - v ||_{H^{-1/2}(\Gamma)} \quad \leq \quad C\Big(\sum_{j=1}^n a_j^2\Big)^{1/4}\Big(\sum_{j=1}^n h_j^2 a_j^2\Big)^{1/4} \tag{2.6}$$

$$|| v^n - v ||_{H^{-1/2}(\Gamma)} \quad \leq \quad C\sum_{j=1}^n a_j h_j^{1/2}. \tag{2.7}$$

Although Carstensen et al [4, 5, 10] do not consider collocation methods we can argue along similar lines to obtain error estimates for this method.

The mapping properties of V give the following inequality

$$||v||_{H^{s-1}(\Gamma)} \ \leq \ C||Vv||_{H^s(\Gamma)}$$

for all real s and $v \in H^{s-1}(\Gamma)$. Using this inequality and the definition of the residual one can easily see

$$|| v^n - v ||_{H^{-1}(\Gamma)} \ \leq \ C \ ||r||_{L^2(\Gamma)}.$$

Standard arguments can then be applied to derive a-posteriori estimates provided that the solution is sufficiently smooth.

By definition $r(\xi_i)$ equals zero for $i = 1, \dots, n$ and it follows for $\sigma \in [x_{j-1}, x_j]$ that

$$|r(\sigma)| \ = \ |r(\sigma) - r(\xi_i)| \ \leq \ \int_{\Gamma_j} |\partial r/\partial s|$$

so that

$$\int_{\Gamma_j} |\, r\,|^2 \ \leq \ h_j^2 \int_{\Gamma_j} |\partial r/\partial s|^2.$$

Recalling the definition of a_j then gives

$$|| v^n - v ||_{H^{-1}(\Gamma)} \ \leq \ C \ \Big(\sum_{j=1}^n h_j^2 a_j^2\Big)^{1/2}. \tag{2.8}$$

One can see that this approach leads in a simple way to a-posteriori error estimates in terms of the residual r for both the Galerkin and Collocation method for solving the equation (2.5).

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

3. Steering the Adaptive Algorithm

Once a posteriori error estimate are derived it is simple matter to build adaptive schemes. Although if we are given a tolerance $TOL > 0$ we cannot determine whether or not the norm of the error is smaller than TOL because the constants in the a posteriori estimates are unknown. Without computing upper bounds for these constants we have only some relative error control, guaranteeing that the error will be less constant times TOL. The basic problem for an adaptive procedure can be formulated as follows:

Basic Problem [5] *Given a tolerance $TOL > 0$ find a partition of Γ which minimizes the computational cost for assembling and solving the related system of equations under the side condition that the a posteriori error estimate is less than TOL.*

The computational cost for assembling and solving the related Galerkin equations can be estimated approximately by n the number of degrees of freedom that is the number of elements Γ_j..

Adaptive Algorithm [5] *Start with an initial coarse mesh Γ_1,...,Γ_n and repeat (1) to (3) until termination.*

(1) *Solve the Galerkin or Collocation equations using the partition $\Gamma_1, \ldots, \Gamma_n$.*
(2) *Compute the error estimate and if it is less than TOL stop else continue with (3).*
(3) *For $j = 1, \ldots, n$ refine Γ_j by the* **Refinement Rule** *and continue with the new mesh in (1).*

Usually an adaptive algorithm has the aim to equidistribute all members of the sum which bound the error in order to yield some optimal mesh. Although this does not necessarily minimise the sum. Let λ_j be the local contribution to the a-posteriori estimate from Γ_j. Then a possible refinement rule would be;

Refinement Rule [5] *Halve Γ_j if and only if $\lambda_j \geq \theta \max \lambda_j$, where θ is some parameter between 0 and 1.*

Choosing $\theta = 0$ gives uniform meshes while increasing θ decreases the the number of elements refined. Other refinement rules are also considered in [4, 5, 10] with the various justifications for their choice, however the one described is the simplest.

Consider the a-posteriori estimates of Carstensen and Stephan. For the a-posteriori estimate in (2.7) the local contribution to the estimate from Γ_j is just $a_j h_j^{1/2}$, so for this estimate

$$\lambda_j := a_j h_j^{1/2}.$$

However for the a-posteriori estimates in (2.6) there are two sums and it is not obvious which of them and how they should be equidistributed. The estimate (2.7) is therefore more convenient for steering an adaptive algorithm since any element Γ_j contributes to the sum only once. We can still use (2.6) to steer the adaptive algorithm. Assuming that the fist term is 'well-behaved', we just take the λ_j to be the local contribution to the second term of estimate in (2.7) so

$$\lambda_j := a_j h_j.$$

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

If the first term $(\sum a_j^2)^{1/2}$ blows up it is possible to modify the algorithm and include steps of uniform refinement so that this term becomes small.

Up to now all the a-posteriori error estimates and hence all the adaptive procedures based on them, have relied upon bounds in terms of the residual r. Unfortunately the computation of the residual is quite expensive. In the next section we shall look at some non-residual based a-posteriori error estimates for steering adaptive schemes for the differenced collocation method, a modified form of the collocation method investigated in [17]. As we shall see this will offer a cheap alternative to the residual based methods.

Before we finish this section, we shall construct an adaptive scheme for computing the approximation to the solution of the boundary value problem at some test point z in the interior of the domain using the residual based error estimate in (2.8). Using the definition of the approximation U^n and the duality paring property of Sobolev norms we have for z in the interior of Ω the following estimate for the error in approximation at the point z

$$|\,U^n(z) - U(z)\,| \;\leq\; C\,||\,v^n - v\,||_{H^{-1}(\Gamma)} \;\leq\; C\,\Big(\sum_{j=1}^{n} h_j^2 a_j^2\Big)^{1/2}. \tag{3.1}$$

This a-posteriori estimate then leads us to define

$$\lambda_j^{I*} := h_j^2 a_j^2.$$

This quantity is just the local contribution to the a-posteriori error estimate in (3.1) from Γ_j. We can then equidistribute this quantity to steer the adaptive scheme. We shall develop a non-residual based estimate for this problem in the next section and this will then enable us to compare the two approaches, residual and non-residual. Similarly one can construct adaptive algorithms for the other a posteriori estimates given in this section.

4. Non-Residual Based Adaptive Schemes

Section 2 dealt with the existing adaptive procedures for Galerkin methods. All of these schemes were based upon residual type a-posteriori error estimates. In this section we shall show that it possible to construct adaptive procedures for the differenced collocation method for the boundary value problem described in Chapter 3 of [17], which only use information about the jump in the approximation to the solution to steer the adaptive scheme. The advantage of this type of adaptive procedure is that it avoids the costly computation of the residual. We shall restrict our attention to the case where Γ is a circle or radius r:

$$\Gamma = \{\; r(\mathrm{cis}(s)) : s \in [0, 2\pi]\;\}.$$

For this special case $V = L$ where

$$Lv(s) \;:=\; -\frac{1}{2\pi}\int_0^{2\pi} \log|\,2r\,\sin((s-\sigma)/2)\,|\,v(\sigma)\,d\sigma. \tag{4.1}$$

In addition to using the usual Sobolev norms to measure the error in our approximations, we also use a discrete norm $|||\cdot|||$ on piecewise constant elements $\phi \in S^0(\Delta)$ defined by

$$|||\phi||| \;:=\; \Big(\sum_{j=1}^{n} h_j^2\,\phi(t_j)^2\Big)^{1/2}$$

where $\{t_j\}$ are the midpoints of the partition Δ. Let $i^* = \max\{j \in \mathbb{Z} : x_{i-1} - x_j > \pi\}$, so that $[x_{i-1}, x_i]$ is in the middle of the interval $[x_{i^*}, x_{i^*+n}]$ of length 2π. and

$$\Lambda_i = \{j \in \{i^* + 1, \ldots, i^* + n\} : j < i - 1 \text{ or } j > i + 1\}$$

and

$$\Lambda_i^c = \{i \pm 1, i\}.$$

The basis for the adaptive algorithm is the coercivity result from Theorem 3.3.2 [17]. Using this we can prove the following theorem:

Theorem 4.1. *Let Γ be a circle and $\Delta \in \mathfrak{B}$. For all h sufficiently small, the differenced collocation solution v^n satisfies for some point $\eta_j \in [x_{j-1}, x_j]$*

$$|||\, v^n - P_\Delta v \,|||^2 \ \leq \ C\, \Phi^2 \sum_{j=1}^{n} h_j^4 \, Dv(\eta_j)^2$$

where $P_\Delta u$ is the orthogonal projection of $L^2(\Gamma)$ onto $S^0(\Gamma)$,

$$\Phi^2 \ := \ \log \underline{h} \ \ \max_i \sum_{j \in \Lambda_i} \frac{h_i}{|x_j - \xi_i|}$$

and $C = C(\mu_1, \mu_3)$ is some constant.

Proof. Since v^n is the differenced collocation solution we have from Theorem 3.5.4 [17] the following inequality

$$|||\, v^n - P_\Delta v \,|||^2 \ \leq \ C \sum_{i=1}^{n} |\, L(v - P_\Delta v)(\xi_i)\,|^2.$$

The kernel of the integral operator L, $l(s, \sigma)$, can be split up into a log function $l_1(s, \sigma) \ = \ \log|s - \sigma|$ plus a smooth term $l_2(s, \sigma)$, so that

$$\begin{aligned}
L(v - P_\Delta v)(\xi_i) \ &= \ \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (v(\sigma) - P_\Delta v(\sigma))\, l(\xi_i, \sigma)\, d\sigma \\
&= \ \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (v(\sigma) - P_\Delta v(\sigma))\, l_1(\xi_i, \sigma)\, d\sigma \\
&\quad + \ \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (v(\sigma) - P_\Delta v(\sigma))\, l_2(\xi_i, \sigma)\, d\sigma \\
&=: \ T_1(i) \ + \ T_2(i).
\end{aligned}$$

We shall consider the term that contains the logarithmic singularity first, as the other term since it possesses a smooth kernel will be somewhat simpler. For $T_1(i)$ we have from the definition of the orthogonal projection P_Δ, that for any constants l_j

$$T_1(i) \ = \ \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (v(\sigma) - P_\Delta v(\sigma))\, (l_1(\xi_i, \sigma) - l_j)\, d\sigma. \tag{4.2}$$

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

We want to chose the constants l_j in such a way to make the above sum small. This can be accomplished by choosing

$$l_j = \begin{cases} h_j^{-1} \displaystyle\int_{x_{j-1}}^{x_j} l_1(\xi_i, \sigma)\, d\sigma & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

With the constants l_j chosen we then split $T_1(i)$ into two sums $T_{11}(i)$ and $T_{12}(i)$ where $T_{11}(i)$ ($T_{12}(i)$) is the sum corresponding to the right hand side of (4.2) where the summation is over Λ_i (Λ_j^c) instead of 1 to n.

Using elementary arguments we can bound T_{12} to obtain

$$T_{12}(i) \;\leq\; C \, \log \underline{h} \sum_{j \in \Lambda_i^c} h_j^2 \, Dv(\eta_j) \tag{4.3}$$

where η_j is some point in the interval $[x_{j-1}, x_j]$. While for $T_{11}(i)$, from Theorem 3.4.3 [17] we have

$$|\, l_1(\xi_i, \sigma) - l_j \,| \;\leq\; C \, \frac{h_j}{|\, \xi_i - x_j \,|}$$

so that

$$T_{11}(i) \;\leq\; C \sum_{j \in \Lambda_i} h_j^2 \, Dv(\eta_j) \frac{h_j}{|\, \xi_i - x_j \,|}.$$

Squaring and then summing the above inequality over i yields on applying the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{i=1}^{n} T_{11}(i)^2 \;&\leq\; C \sum_{i=1}^{n} \Big(\sum_{j \in \Lambda_i} h_j^2 \, Dv(\eta_j) \frac{h_j}{|\, \xi_i - x_j \,|} \Big)^2 \\ &\leq\; C \sum_{i=1}^{n} \sum_{j \in \Lambda_i} h_j^4 Dv(\eta_j)^2 \frac{h_j}{|\, \xi_i - x_j \,|} \sum_{j \in \Lambda_i} Dv(\eta_j) \frac{h_j}{|\, \xi_i - x_j \,|}. \end{aligned}$$

Reversing the order of integration and bounding the last sum by a Riemann sum we then obtain

$$\sum_{i=1}^{n} T_{11}(i)^2 \;\leq\; C\, \Phi^2 \sum_{j=1}^{n} h_j^4 \, Dv(\eta_j)^2. \tag{4.4}$$

Since the mesh possesses a bounded local mesh ratio and $|\Lambda_i| \leq 3$ we see from inequality (4.3) that

$$\sum_{i=1}^{n} T_{12}(i)^2 \;\leq\; C\, \Phi^2 \sum_{j=1}^{n} h_j^4 \, Dv(\eta_j)^2. \tag{4.5}$$

The last two inequalities (4.4) and (4.5) combined together, then give the desired bound for T_1. A similar argument will then give the same bound for T_2 completing the proof.

□

If we assume that the derivative of v behaves like the 'derivative' of v^n then we can construct adaptive procedures for the collocation method. Let us define $[Dv^n]_i$

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

to be an approximation to the 'derivative' of $v^n = \sum \alpha_i \chi_i$

$$[Dv^n]_i \;:=\; \frac{\alpha_{i+1} - \alpha_i}{h_{i+1} + h_i} + \frac{\alpha_i - \alpha_{i-1}}{h_i + h_{i-1}}$$

where $\alpha_0 = \alpha_n$ and $\alpha_{n+1} = \alpha_1$ for $i = 1, \ldots, n$. $[Dv^n]_i$ is the average of the slopes of the lines joining α_{i-1} to α_i and α_i to α_{i+1} at the midpoints of the intervals respectively, see Figure 1.

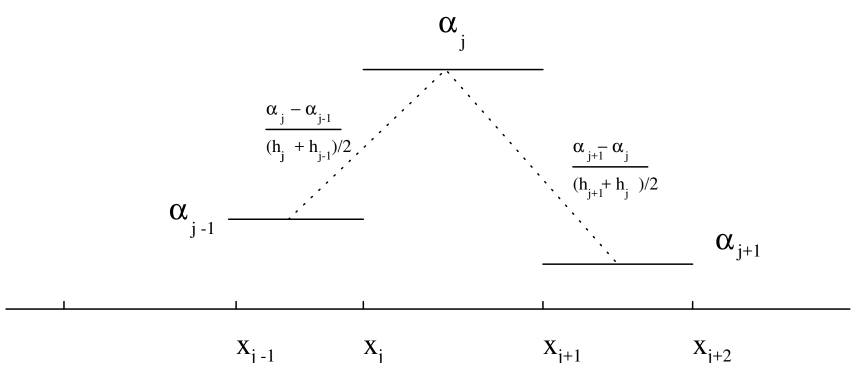


FIGURE 1. The construction of $[Dv^n]_j$

Assumption 4.2. *$Dv(\sigma) \;\sim\; [Dv^n]_i$ for $\sigma \in \Gamma_i$.*

Combining Assumption 4.2 with Theorem 4.1, we can replace $Dv(\eta_j)$ by $[Dv^n]_j$ to obtain an a-posteriori estimate. The local contribution from Γ_j to this estimate would be

$$\lambda_j^D := \Phi\, h_j^2\, [Dv^n]_j.$$

With this assumption, it is also possible to derive a-posteriori error estimates for approximation in the interior of the domain. Let z be some point in the interior of the domain. Using the inequality

$$|U^n(z) - U(z)| \;\leq\; C\ \Phi\ h^{1/2}\ |||v - v^n|||$$

from the proof of Theorem 3.4.3 [17] we see that

$$\lambda_j^I := \Phi^2\ \ h^{1/2}\ \ h_j^2\ \ [Dv^n]_j$$

provides an estimate of the local contribution from the j-th interval to the error in $U^n(z)$.

The discrete norm is an unusual way to measure error, a more traditional way is to use weighted L^2 norms. If we assume that the solution v behaves like $s^{-1/2}$ at some point a then

$$||v^n - v||^2_{L^2_w(\Gamma)} \;:=\; \int_\Gamma \;|\, v^n(\sigma) - v(\sigma)\,|^2\, |\sigma - a|\, d\sigma$$

Since

$$||P_\Delta v - v^n||^2_{L^2_w(\Gamma)} \;\leq\; C \sum_{j=1}^n h_j (v^n(\xi_j) - P_\Delta v(\xi_j))^2 |x_j - a|$$

we can take

$$\lambda_j^W \;:=\; \Phi\ h_j^2\ [Dv_h]_j\ \max(|x_j - a|/h_j)^{1/2}$$

as the local contribution from the j-th interval for the error in the weighted L^2 norm.

In the **Refinement Rule** those terms independent of j, that is Φ, $h^{1/2}$ and $\max(|x_j - a|/h_j)^{1/2}$, plays no role in refinement process and as such we only to refine those to elements for which $h_j^2\,[Dv^n]_j \geq \theta \max h_j^2\,[Dv^n]_j$, thereby reducing the computational effort required in steering the algorithm.

5. Numerical Examples

For our numerical examples, we consider the region Ω bounded by the ellipse Γ parameterized by

$$\gamma(s) \;=\; (\cos(s), \exp(-1/4)\sin(s)) \qquad 0 \leq s \leq 2\pi.$$

This region is illustrated in Figure 2. The boundary data f is constructed so that the solution is given by

$$U(x,y) = \Re((x+1)+iy)^{1/2}.$$

For this example the solution u behaves like $(s-\pi)^{-1/2}$ near the point $(-1,0)$.

The singularity present in the solution will cause sub-optimal convergence rates unless special precautions are taken. In particular if we consider the error in the solution U at a point in the interior of the domain and naively used the collocation method with some arbitrarily chosen mesh we would only expect $\mathcal{O}(n^{-1})$ rates of convergence. However if one were to use graded meshes which were sufficiently graded at the singularity we could overcome the poor convergence rates. The drawback to this approach is that we require some a-priori knowledge of the solution. Difficulties arise when we have no information about the singularities since we do not know how to correctly graded the mesh. Also it is not clear that a mesh grading strategy would be the most efficient means of solving the problem. Adaptive scheme overcome these shortcomings.

These numerical examples provide a numerical justification of the non-residual based adaptive procedure presented in the previous section. These demonstrate that the non-residual based method compares favorable with the residual based adaptive schemes and with graded meshes. These experiments also give imputeus

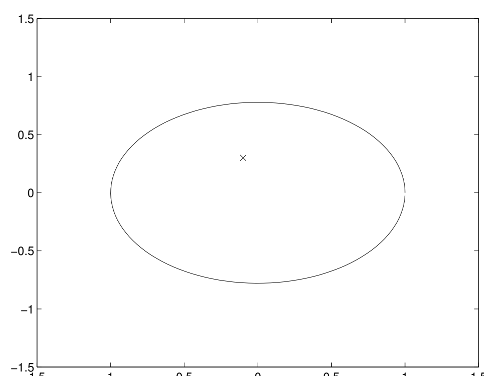


Figure 2. Region Ω for the numerical examples with the test point $z = (-1/10, 1/3)$ $(\times)$.

into investigating the theoretical aspects of adaptive schemes for collocation methods that have largely been untouched.

5.1. **Non-Residual Based Adaptive Schemes.** For the first numerical experiment, we look at the non-residual based adaptive scheme presented in the previous section. The boundary integral equation in (2.5) is solved using the differenced collocation method [17] with the underlying mesh being generated by the refinement rule and adaptive algorithm given in Section 4. We start the adaptive procedure with a coarse mesh with 8 equal partitions. In addition to using the adaptive scheme, we also solve the (2.5) using an uniform mesh to illustrate the poor convergence rates from a naive strategy. The error in the discrete norm, weighted $L^2(\Gamma)$ norm and error at the test point $z = (-.1, .3)$ in the interior of the domain were recorded and are presented in Tables 1 to 3.

From Section 4, for circular regions we would expect the error in the discrete norm to behave like $\mathcal{O}(n^{-1/2})$ on the uniform mesh and the best rate of convergence we would predict is $\mathcal{O}(n^{-3/2})$, which we would obtain on a sufficiently graded mesh. We expect these rates of convergence to carry over to general smooth boundaries. Indeed the results in Table 1 show that for a uniform mesh we get $\mathcal{O}(n^{-1/2})$ rate of convergence. One can clearly see the improvement in Table 1 when one switches to the adaptive scheme, with the error in the discrete norm converging with order $\mathcal{O}(n^{-3/2})$. The value for the parameter θ in the refinement rule is chosen to be 0.25. We shall investigate the effect of changing this parameter in the next subsection. The situation is illustrated in Figure 5A, in which the error in the discrete norm for both the uniform mesh and the adaptive mesh is plotted versus n the number of intervals.

A similar situation is observed for the error in the weighted $L^2(\Gamma)$ norm and for the solution at the test point $z = (-1/10, 1/3)$ in the interior of the domain Ω. The results for these are given Table 2 and 3 and plots of the respective errors versus n are given Figures 5B and 5C. For the weighted $L^2(\Gamma)$ norm the error appears to be converging with orders $\mathcal{O}(n^{-1/2})$ and $\mathcal{O}(n^{-1})$ for the uniform and adaptive cases respectively. We note that from Section 3.4.1 [17], if the solution is smooth the rate of convergence for the error in the $L^2(\Gamma)$ on quasi-uniform meshes is $\mathcal{O}(n^{-1})$. The absolute error in the solution in the interior of the domain appears to converge with order $\mathcal{O}(n^{-3/2})$, while for the adaptive mesh it appears to be at least $\mathcal{O}(n^{-3})$. The error for the adaptive scheme in Table 3 seems to be somewhat unpredictable however underlying convergence rate can easily seen in the plot of the error in Figure 5B.

From these results we see that for the uniform mesh the rates of convergence are quite slow, a consequence of the singularity in u. However for the adaptive schemes one can see the improvement with the rates of convergence being restored to what would be expected if we were dealing with smooth solutions.

It is also interesting to see the meshes generated by the adaptive procedure and the local error contributions λ_j to the a-posteriori estimate that steer the adaptive procedure. To illustrate this we consider the adaptive procedure for the computing the approximation to the solution at the test point $z = (-1/10, 1/3)$. The refinement parameter θ for the adaptive scheme is taken to be 0.30.

We plot the mesh points on the ellipse, starting with the original coarse uniform mesh with 8 partitions and then the next four adaptively generated meshes. These are given in Figure 3. Looking at the plot of the mesh points, one can see that the adaptive procedure picks up the fact that the singularly is at the point $(-1, 0)$ and concentrates the mesh points around it.

In Figure 4 we have the corresponding local contributions λ_j to the a-posteriori estimate from the j-th partition on the adaptively generated meshes. The x-axis is the parameterization variable so that π is the point corresponding to the singularity. Superimposed upon these plots is the dotted line at $0.3 \max \lambda_j$. The intervals whose corresponding local contribution is above this line are halved while the remaining intervals are untouched. We can see that the adaptive procedure to trying equidistribute the local contributions at each refinement step.

5.2. **Comparison between Adaptive Schemes.** In this section we compare the non-residual scheme used in the previous example with the residual based a-posteriori estimate in inequality (3.1). The same problem was used as in Section 5.1 except this time we only consider the interior potential at the test point $z = (-.1, .3)$. The adaptive schemes were steered using the non-residual and residual error indicators λ_j^I and λ_j^{I*}, respectively.

Two different values for θ_n and θ_r, the refinement parameter for the non-residual and the residual based adaptive scheme were investigated and the results are listed

TABLE 1. Adaptive scheme for the discrete norm for §5.1.

Uniform Mesh		
n	$\|\|\| v^n - P_\triangle v \|\|\|$	
8	3.357e-02	
16	1.711e-02	(-0.97)
32	1.051e-02	(-0.70)
64	6.965e-03	(-0.59)
128	4.770e-03	(-0.55)

Adaptive Mesh with $\theta = 0.25$		
n	$\|\|\| v^n - P_\triangle v \|\|\|$	
8	3.357e-02	
12	1.815e-02	(-1.52)
18	1.129e-02	(-1.17)
26	7.785e-03	(-1.01)
32	5.831e-03	(-1.39)
44	3.636e-03	(-1.48)
56	2.616e-03	(-1.37)
68	1.840e-03	(-1.81)
84	1.333e-03	(-1.53)
104	9.783e-04	(-1.45)
122	7.538e-04	(-1.63)

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

TABLE 2. Adaptive scheme for the weighted L^2 norm for §5.1.

Uniform Mesh		
n	$\|\| v - v^n \|\|_{L^2_w(\Gamma)}$	
8	2.978e-01	
16	1.909e-01	(-0.64)
32	1.270e-01	(-0.59)
64	8.668e-02	(-0.55)
128	6.006e-02	(-0.53)

Adaptive Mesh with $\theta = 0.3$		
n	$\|\| v - v^n \|\|_{L^2_w(\Gamma)}$	
8	2.978e-01	
12	1.968e-01	(-1.02)
16	1.423e-01	(-1.13)
22	1.023e-01	(-1.04)
28	7.814e-02	(-1.12)
34	6.242e-02	(-1.16)
46	4.717e-02	(-0.93)
56	3.641e-02	(-1.32)
68	2.983e-02	(-1.03)
84	2.336e-02	(-1.16)
100	1.922e-02	(-1.12)
118	1.648e-02	(-0.93)

in Table 4. For the non-residual based adaptive scheme, $\theta_n = 0.15$ and $\theta_n = 0.3$ were used. While for the residual based scheme, $\theta_r = 0.05$ and $\theta_r = 0.15$ were used. In addition, Table 4 includes the results for a uniform mesh and a mesh graded towards π like $\min h^4$. The graded mesh will be sufficient to overcome the singularity in the solution and restore optimal convergence rates.

Figure 6 a plots the results from the adaptive schemes for $\theta_r = 0.05$ and $\theta_n = 0.15$ including the results on the uniform mesh. We can see that both adaptive schemes exhibit the same behaviour. Similarly Figure 6 b plots the results from the adaptive schemes and the uniform mesh, although this time for $\theta_r = 0.15$ and $\theta_n = 0.30$. It is clear from these plots that both adaptive scheme are similar, converging with order $\mathcal{O}(n^{-3})$.

In Figure 6 c a plot of the error in graded mesh and the non-residual method with $\theta_n = 0.30$ is given. Both approaches for solving the problem seem to converge with the same order although for the adaptive scheme we need no information about the singularity in the solution to obtain optimal converge result.

We finish off the examples by looking at the effect of changing the refinement parameter θ for both adaptive methods. For both the residual and non-residual

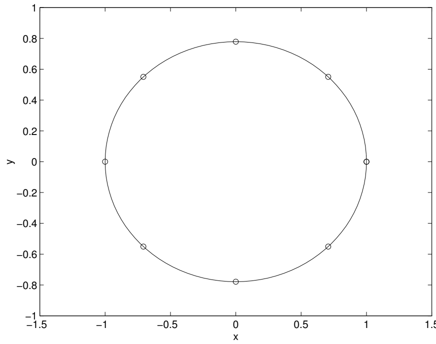
6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

TABLE 3. Adaptive scheme for $U^n(z)$ for $z = (-0.1, 0.3)$ for §5.1.

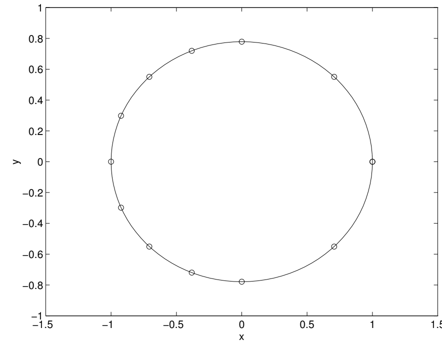
Uniform Mesh		
n	$\mid U(z) - U^n(z) \mid$	
8	2.941e-03	
16	7.619e-04	(-1.95)
32	2.411e-04	(-1.66)
64	8.172e-05	(-1.56)
128	2.843e-05	(-1.52)

Adaptive Mesh with $\theta = 0.3$		
n	$\mid U(z) - U^n(z) \mid$	
8	2.941e-03	
12	5.514e-04	(-4.13)
18	2.484e-04	(-1.97)
26	5.525e-05	(-4.09)
36	2.445e-05	(-2.51)
46	1.126e-05	(-3.16)
58	6.955e-07	(-12.01)
72	6.991e-07	(0.02)
86	2.472e-07	(-5.85)
104	5.770e-07	(4.46)
128	1.068e-07	(-8.13)

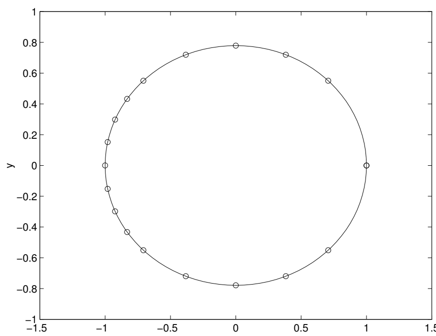
adaptive schemes we plotted the results for θ_r equal 0, 0.05 and 0.15 and for θ_n equal 0, 0.15 and 0.30 in Figures 7 a and 7 b respectively. We note that $\theta = 0$ is the same as using a uniform mesh. From both plots we can see the same salient features. As θ increases the plots seems to become less smooth, jumping around more, although the overall convergence seems to better. In addition as θ increases the number of new intervals generated at each step in the adaptive algorithm decreases.



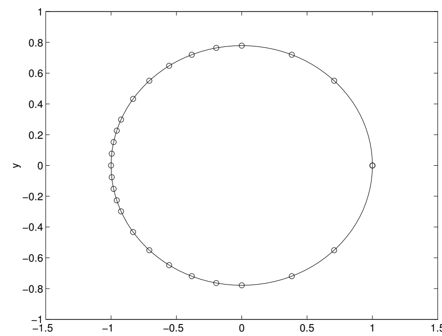
(A) Initial coarse mesh $n = 8$



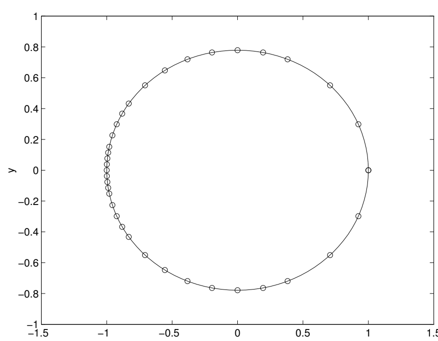
(B) First adaptive mesh $n = 12$.



(C) Second adaptive mesh $n = 18$.

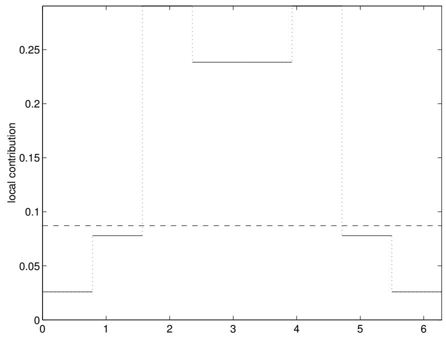


(D) Third adaptive mesh $n = 26$.

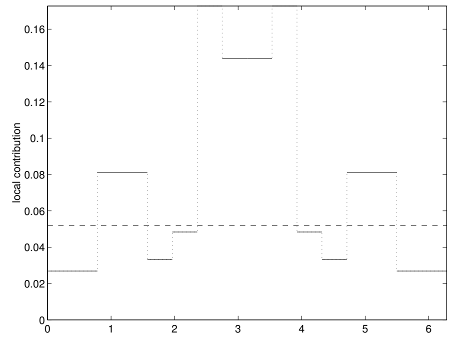


(E) Fourth adaptive mesh $n = 36$.

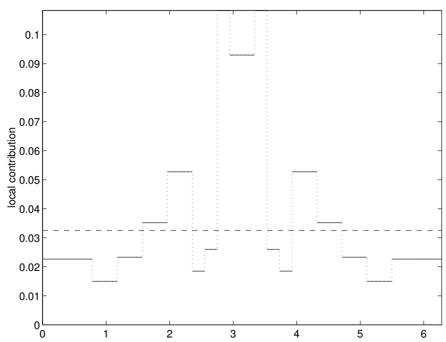
FIGURE 3. The adaptively generated mesh points for §5.1



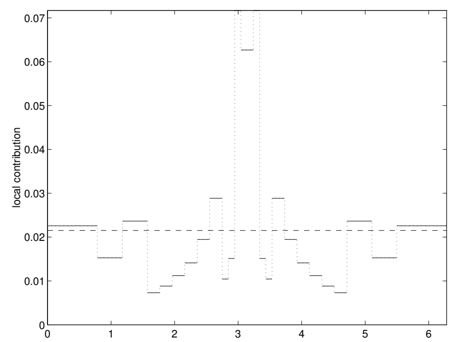
(A) Initial coarse mesh $n = 8$



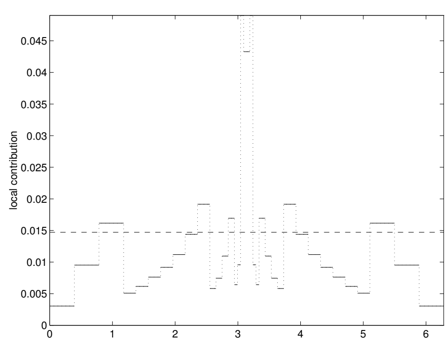
(B) First adaptive mesh $n = 12$.



(C) Second adaptive mesh $n = 18$.

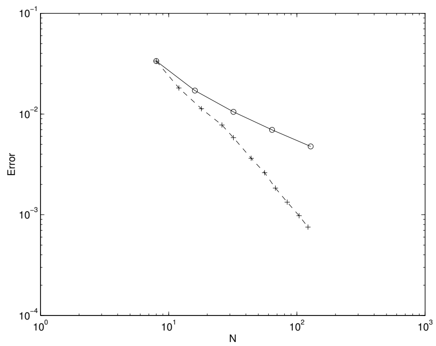


(D) Third adaptive mesh $n = 26$.

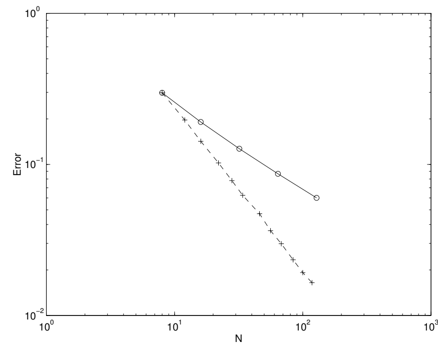


(E) Fourth adaptive mesh $n = 36$.

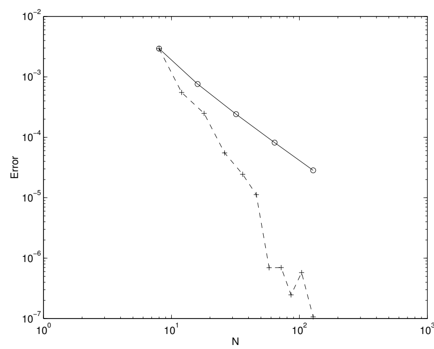
FIGURE 4. The local contributions Λ_j from the j-th interval



(A) Plot of $|||\, v - v^n \,|||$ versus n for the non-residual adaptive scheme (dashed) with $\theta = 0.25$ and for a uniform mesh (solid).



(B) Plot of $||\, v - v^n \,||_{L^2_w(\Gamma)}$ versus n for the non-residual adaptive scheme (dashed) with $\theta = 0.5$ and for a uniform mesh.



(C) Plot of the error in $U^n(z)$ versus n for the non-residual adaptive scheme (dashed) with $\theta = 0.3$ and for a uniform mesh (solid).

FIGURE 5. Comparisons of non-residual adaptive schemes and uniform meshes.

TABLE 4. Comparison between the residual and non-residual adaptive schemes for §5.2.

Residual $\theta = 0.05$		
n	$\lvert U(z) - U^n(z) \rvert$	
8	2.941e-03	
10	8.761e-04	(-5.43)
16	2.805e-04	(-2.42)
26	6.472e-05	(-3.02)
36	3.238e-05	(-2.13)
50	9.313e-06	(-3.79)
68	3.398e-06	(-3.28)
90	1.562e-06	(-2.77)
120	2.795e-07	(-5.98)

Non-Residual $\theta = 0.15$		
n	$\lvert U(z) - U^n(z) \rvert$	
8	2.941e-03	
14	7.559e-04	(-2.43)
22	2.091e-04	(-2.84)
36	8.197e-05	(-1.90)
48	2.542e-05	(-4.07)
60	8.148e-06	(-5.10)
76	3.444e-06	(-3.64)
94	1.352e-06	(-4.40)
116	6.265e-07	(-3.66)

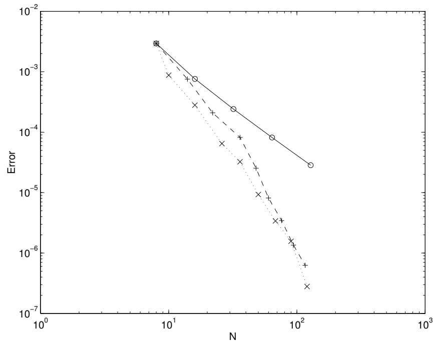
Residual $\theta = 0.15$		
n	$\lvert U(z) - U^n(z) \rvert$	
8	2.941e-03	
10	8.761e-04	(-5.43)
14	7.776e-05	(-7.20)
20	9.765e-05	(0.64)
28	1.481e-05	(-5.61)
36	6.981e-06	(-2.99)
44	5.300e-06	(-1.37)
60	2.589e-07	(-9.73)
80	6.791e-07	(3.35)
108	1.200e-07	(-5.77)
142	5.818e-08	(-2.65)

Non-Residual $\theta = 0.30$		
n	$\lvert U(z) - U^n(z) \rvert$	
8	2.941e-03	
12	5.514e-04	(-4.13)
18	2.484e-04	(-1.97)
26	5.525e-05	(-4.09)
36	2.445e-05	(-2.51)
46	1.126e-05	(-3.16)
58	6.955e-07	(-12.01)
72	6.991e-07	(0.02)
86	2.472e-07	(-5.85)
104	5.770e-07	(4.46)
128	1.068e-07	(-8.13)

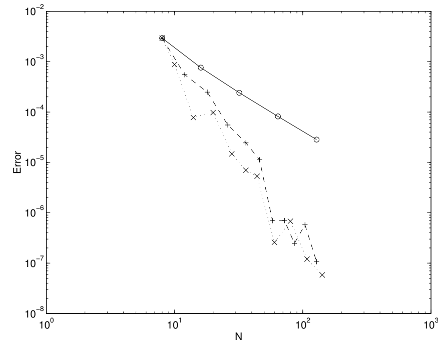
Uniform Mesh		
n	$\lvert U(z) - U^n(z) \rvert$	
8	2.941e-03	
16	7.619e-04	(-1.95)
32	2.411e-04	(-1.66)
64	8.172e-05	(-1.56)
128	2.843e-05	(-1.52)

Graded Mesh		
n	$\lvert U(z) - U^n(z) \rvert$	
8	7.322e-3	
16	2.674e-4	(-4.78)
32	4.335e-5	(-2.62)
64	4.961e-6	(-3.13)
128	5.017e-7	(-3.31)

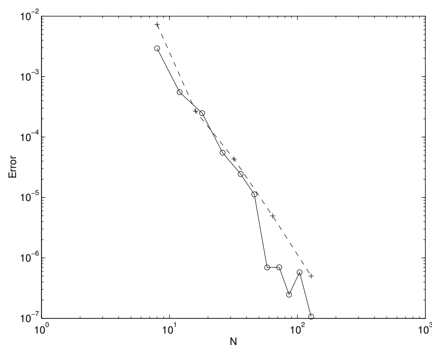
6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.



(A) Plots of the error in $U^n(z)$ for the uniform mesh (solid), adaptive residual $\theta_r = 0.05$ (dotted) and adaptive non-residual $\theta_n = 0.15$ (dashed).

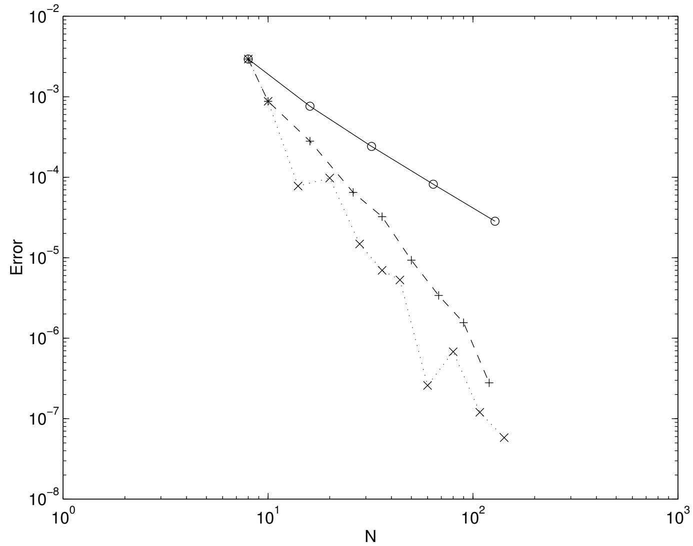


(B) Plots of the error in $U^n(z)$ for the uniform mesh (solid), adaptive residual $\theta_r = 0.15$ (dotted) and adaptive non-residual $\theta_n = 0.30$ (dashed).

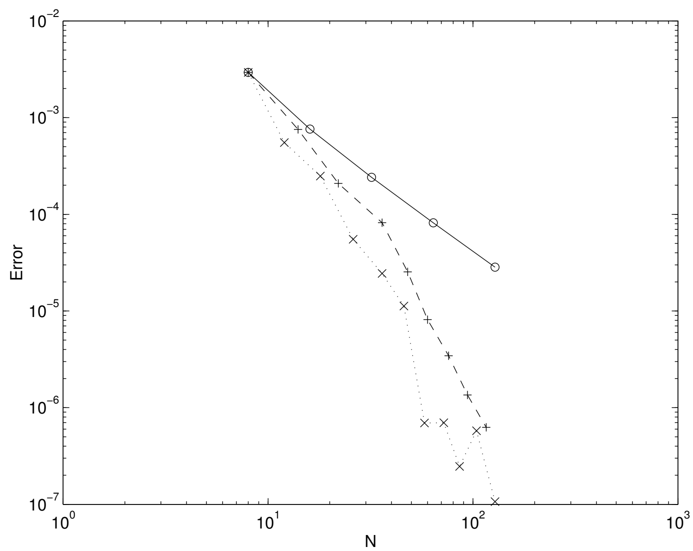


(C) Plots of the error in $U^n(z)$ for the graded mesh with $q = 4$ (dashed) and the adaptive non-residual $\theta_n = 0.30$ (solid).

FIGURE 6. Comparisons of residual and non residual methods for $U^n(z)$



(A) Plots of the error in $U^n(z)$ for $\theta = 0$ (solid), $\theta = 0.05$ (dashed) and $\theta = 0.15$ (dotted) for the residual adaptive scheme.



(B) Plots of the error in $U^n(z)$ for $\theta = 0$ (solid), $\theta = 0.15$ (dashed) and $\theta = 0.30$ (dotted) for the non-residual adaptive scheme.

FIGURE 7. The effect of changing the refinement parameter θ for the adaptive scheme

6 Jan 2019 11:52:27 EST
Version 1 - Submitted to Math. Comp.

6. Conclusion

We have presented an adaptive procedure for a modified form of the collocation method, which only relies upon information about the jump in the approximation. This approach offers a quick and cheap alternative to the residual based methods. Numerical examples demonstrate the practicability of this non-residual based adaptive scheme, showing optimal convergence for solutions with singularities. Further this adaptive method compared favourable with the residual based approach as well as with graded meshes.

References

1. D. Arnold and W. Wendland, *On the asymptotic convergence of collocation methods*, Math. Comp. **41** (1983), 349–381.
2. I. Babuska and W. Rheinboldt, *Error estimates for adaptive finite element methods*, SIAM J. Numer. Anal. **15** (1978), 736–754.
3. ______, *A-posteriori error analysis of finite element solutions for one-dimensional problems*, SIAM J. Numer. Anal. **18** (1981), 565–589.
4. C. Carstensen, *Adaptive boundary element methods and adaptive finite element and boundary element coupling*, Preprint.
5. C. Carstensen, D. Estep, and E. Stephan, *h-adaptive boundary element schemes*, Preprint.
6. C. Carstensen, S. Funken, and E. Stephan, *On the adaptive coupling of fem and bem in 2d-elasticity*, In preparation.
7. C. Carstensen and E. Stephan, *Adaptive boundary element methods*, Preprint.
8. ______, *Adaptive boundary element methods for transmission problems*, Preprint.
9. ______, *Adaptive coupling of boundary elements and finite elements*, Preprint.
10. C.Carstensen and E. Stephan, *A posteriori error estimates for boundary element methods*, Accepted by Math. Comp. 1994.
11. K. Eriksson and C. Johnson, *An adaptive finite element method for linear elliptic problems*, Math. Comp. **50** (1988), 361–383.
12. ______, *Adaptive finite element methods for parabolic problems I. a linear problem*, SIAM J, Numer. Anal. **28** (1991), 43–77.
13. De hao Yu, *Self-adaptive boundary element methods*, Z. Angew. Math, Mech. **68** (1988), 435–437.
14. C. Johnson and P. Hamsbo, *Adaptive finite element method in computational mechanics*, Comp. Meth. Appl. Mech. Eng. **101** (1992), 143–181.
15. E. Rank, *A-posteriori error estimates and adaptive refinement for some boundary integral methods*, Proc. Int. Conf. on Accuracy Estimates and Adaptive Refinements in Finite Element Computatios ARFEC (Lisbon), 1984.
16. ______, *Adaptive h-, p- and hp-versions for boundary integral element methods*, Int. J. Num. Meth. Eng. **28** (1989), 1335–1349.
17. D. Reading, *The differenced collocation method: A new technique for proving the stability and convergence of the collocaton method*, Ph.D. thesis, University of Queensland, Jan 1998.
18. J. Saranen and W. Wendland, *Local residual-type error estimates for adaptive boundary element methods on closed curves*, Appl. Anal. **48** (1993), 37–50.
19. M. Taylor, *Pseudo-differential operators*, Princeton Univ. Press, Princeton NJ, 1987.
20. W. Wendland and De hao Yu, *Adaptive boundary element methods for strongly elliptic integral equations*, Numer. Math. **53** (1988), 539–558.
21. ______, *A-posteriori local error estimates of boundary element methods with some pseudo-differential equations on closed curves*, J. Comp. Math. **10** (1992), 273–289.
22. De-Hao Yu, *A-posteriori error estimates and adaptive refinement for some boundary element methods*, Proc. 9th International Conference on BEM (Stuttgart), 1987.

PO Box 61, Essendon North, VIC 3041 Australia
E-mail address: darren.reading@infinit.com.au