MATH 3235 - Probability Theory
Georgia Institute of Technology
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Test Assignment

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§ 1 Test exercise

Claim 1.A

Hello, World!

Proof. I just said hello, what more do you want from me? △

Claim 1.B

Hello again, World!

Solution. Ugh.

§ 2 Markov's + Chebyshev's inequalities

Claim 2.A

Supppose that X and Y are discrete random variables on the same probability space, both with finite expectations $\mathbb{E}[X] < \infty$, $\mathbb{E}[Y] < \infty$. Prove that, if $X \leq Y$ as functions, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Proof. Let X and Y be discrete random variables on the same probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and with finite expectation such that $X \leq Y$. Then for any $\omega \in \Omega$ we have that $X(\omega) \leq Y(\omega)$. Writing $X(\omega) = k_1$ and $Y(\omega) = k_2$, we can see that $\mathbb{P}(X = k_1) = \mathbb{P}(Y = k_2)$ for any pair of elements k_1 and k_2 with the same preimage. Then it is the case that $k_1\mathbb{P}(X = k_1) \leq \mathbb{P}(Y = k_2)$ for all of these pairs, and so we may sum across all ω to find that

$$\mathbb{E}[X] = \sum_{k \in \mathrm{Im}(X)} k\mathbb{P}(X = k) \leq \sum_{k \in \mathrm{Im}(Y)} k\mathbb{P}(Y = k) = \mathbb{E}[Y].$$

△

Claim 2.B

Prove **Markov's inequality:** If X is a nonnegative random variable, then for any $t > 0$ we have

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

Proof. Let X be a nonnegative discrete random variable on some probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and t some positive constant. Begin by noting that, if $X(\omega) < t$ for all $\omega \in \Omega$ then the inequality holds by the nonnegativity of X and t. Suppose $X(\omega) = k \geq t$ for some $\omega \in \Omega$, $k \in \mathbb{R}$. Note that by the nonnegativity of X we have that

$$\sum_{k \in \mathrm{Im}(X), k<t} k\mathbb{P}(X = k) \geq 0.$$

Then we have

$$\begin{aligned}\mathbb{E}[X] - t\mathbb{P}(X \geq t) &= \sum_{k \in \mathrm{Im}(X)} k\mathbb{P}(X = k) - \sum_{k \in \mathrm{Im}(\omega), k \geq t} t\mathbb{P}(X = k) \\ &= \sum_{k \in \mathrm{Im}(X), k<t} k\mathbb{P}(X = k) \geq 0.\end{aligned}$$

Therefore $\mathbb{E}[X] - t\mathbb{P}(X \geq t) \geq 0$, and dividing across by t we see that the result follows. △

Claim 2.C

Use Markov's inequality to give a short proof of **Chebyshev's inequality:** If X is a random variable with finite expectation and finite, positive variance, then for any $a > 0$ we have

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq a\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{a^2}.$$

Proof. Let X be a discrete random variable with finite expectation and finite, positive variance. Let a be a positive constant. Note that

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq a\sqrt{\mathrm{Var}(X)}\right) = \mathbb{P}\left((X - \mathbb{E}[X])^2 \geq a^2\mathrm{Var}(X)\right).$$

By Markov's inequality we can see that

$$\mathbb{P}\left((X - \mathbb{E}[X])^2 \geq a^2\mathrm{Var}(X)\right) \leq \frac{\mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\right]}{a^2\mathrm{Var}(X)}$$

$$= \frac{\mathbb{E}\left[X^2\right] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2}{a^2\mathrm{Var}(X)} = \frac{\mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2}{a^2\mathrm{Var}(X)} = \frac{1}{a^2}.$$

Therefore we have that $\mathbb{P}\left(|X - \mathbb{E}[X]| \geq a\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{a^2}$. $\triangle$