

MATH 3406

Georgia Institute of Technology

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Homework 9

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Problem 1:

(6A.1) Prove or give a counterexample: If $v_1, \dots, v_m \in V$, then

$$\sum_{j=1}^m \sum_{k=1}^m \langle v_j, v_k \rangle \geq 0.$$

Proof. Note that by right and left additivity respectively we find that

$$\sum_{j=1}^m \sum_{k=1}^m \langle v_j, v_k \rangle = \sum_{j=1}^m \langle v_j, \sum_{k=1}^m v_k \rangle = \langle \sum_{j=1}^m v_j, \sum_{k=1}^m v_k \rangle$$

and because $\sum_{j=1}^m v_j = \sum_{k=1}^m v_k$ we have that

$$\sum_{j=1}^m \sum_{k=1}^m \langle v_j, v_k \rangle = \langle \sum_{j=1}^m v_j, \sum_{k=1}^m v_k \rangle \geq 0.$$

△

Problem 2:

(6A.3)

1. Show that the function taking an ordered pair $((x_1, x_2), (y_1, y_2))$ of elements in \mathbb{R}^2 to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on \mathbb{R}^2 .

Solution. Note that the function given does not satisfy homogeneity in the first slot for negative numbers.

2. Show that the function taking an ordered pair $((x_1, x_2, x_3), (y_1, y_2, y_3))$ to $x_1 y_1 + x_3 y_3$ is not an inner product on \mathbb{R}^3 .

Solution. Note that this function does not satisfy that $\langle v, v \rangle = 0$ if and only if $v = 0$, as $v = (0, 1, 0)$ satisfies the equality.

Problem 3:

(6A.6) Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.

Proof. First the forward direction. By the fact that norms are positive semidefinite we have that showing $\|u\| \leq \|u + av\|$ is equivalent to showing that $\|u\|^2 \leq \|u + av\|^2$. We may utilize the left and right additivity (and homogeneity) of the inner product to see that

$$\langle u + av, u + av \rangle = \langle u, u \rangle + \langle u, av \rangle + \langle av, u \rangle + \langle av, av \rangle = \langle u, u \rangle + a^2 \langle v, v \rangle = \|u\|^2 + a^2 \|v\|^2.$$

Therefore $\|u\|^2 \leq \|u + av\|^2$.

Now the backward direction. We prove the contrapositive. Suppose $\langle u, v \rangle \neq 0$. Then

$$\|u + av\|^2 = \|v\|^2 + \|u\|^2 + a\langle u, v \rangle + a\langle v, u \rangle > \|u\|^2.$$

Therefore the result holds. \triangle

Problem 4:

(6A.17) Prove that

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \left(\sum_{k=1}^n k a_k^2 \right) \left(\sum_{k=1}^n \frac{b_k^2}{k} \right)$$

for all real numbers a_1, \dots, a_n and b_1, \dots, b_n .

Proof. Begin by noting that

$$\left(\sum_{k=1}^n k a_k^2 \right) \left(\sum_{k=1}^n \frac{b_k^2}{k} \right) = \sum_{i=1}^n \sum_{k=1}^n \frac{i}{k} a_i^2 b_k^2 = \sum_{k=1}^n a_k^2 b_k^2 + \sum_{k < i} \frac{i}{k} a_i^2 b_k^2 + \sum_{k > i} \frac{i}{k} a_i^2 b_k^2.$$

Now let $u = (a_1, \dots, a_n)$ and $v = (b_1, \dots, b_n)$ so that

$$\begin{aligned} \|u\|^2 \|v\|^2 &= \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) = \sum_{i=1}^n \sum_{k=1}^n a_i^2 b_k^2 = \sum_{i=1}^n a_i^2 b_i^2 + \sum_{k \neq i} a_i^2 b_k^2 \\ &\leq \sum_{k=1}^n a_k^2 b_k^2 + \sum_{k < i} \frac{i}{k} a_i^2 b_k^2 + \sum_{k > i} \frac{i}{k} a_i^2 b_k^2. \end{aligned}$$

Therefore

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = |\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2 \leq \left(\sum_{k=1}^n k a_k^2 \right) \left(\sum_{k=1}^n \frac{b_k^2}{k} \right).$$

\triangle

Problem 5:

(6A.19) Suppose v_1, \dots, v_n is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T , then

$$|\lambda|^2 \leq \sum_{j=1}^n \sum_{k=1}^n |\mathcal{M}(T)_{j,k}|^2.$$

Proof. Suppose v is the eigenvector of T corresponding to λ . Then $|\lambda|^2 \|v\|^2 = \|\lambda v\|^2 = \|Tv\|^2$. Let a_j denote the j -th row of $\mathcal{M}(T)$. Then $\|Tv\|^2 = \sum_{j=1}^n |\langle v, a_j \rangle|^2$ and therefore $\|Tv\|^2 \leq \sum_{j=1}^n \|v\|^2 \|a_j\|^2$. Succinctly,

$$|\lambda|^2 \|v\|^2 \leq \sum_{j=1}^n \|v\|^2 \|a_j\|^2$$

and we may divide both sides by $\|v\|^2$ to see that

$$|\lambda|^2 \leq \sum_{j=1}^n \|a_j\|^2 = \sum_{j=1}^n \sum_{k=1}^n |a_{jk}|^2$$

where a_{jk} is the k -th element of row j . Equivalently,

$$|\lambda|^2 \leq \sum_{j=1}^n \sum_{k=1}^n |\mathcal{M}(T)_{j,k}|^2.$$

△

Problem 6:

(6A.21) Suppose $u, v \in V$ are such that

$$\|u\| = 3, \quad \|u + v\| = 4, \quad \|u - v\| = 6.$$

What number does $\|v\|$ equal?

Solution. By the parallelogram equality,

$$16 + 36 = 2(9 + \|v\|^2), \quad 17 = \|v\|^2, \quad \|v\| = \sqrt{17}.$$

Problem 7:

(6A.22) Show that if $u, v \in V$, then

$$\|u + v\| \|u - v\| \leq \|u\|^2 + \|v\|^2.$$

Proof. By the parallelogram equality,

$$\|u + v\|^2 = 2(\|v\|^2 + \|u\|^2) - \|u - v\|^2, \quad \|u - v\|^2 = 2(\|v\|^2 + \|u\|^2) - \|u + v\|^2.$$

Now by the triangle inequality we have the following

$$-(\|u\|^2 + \|v\|^2 + 2\|u\|\|v\|) \leq -\|u + v\|^2, -\|u - v\|^2.$$

Now we can write that

$$-\|u + v\|^2 \leq \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \leq \|u\|^2 + \|v\|^2$$

for both $\|u + v\|^2$ and $\|u - v\|^2$. Therefore

$$\|u + v\|^2 \|u - v\|^2 \leq (\|u\|^2 + \|v\|^2)^2, \quad \|u + v\| \|u - v\| \leq \|u\|^2 + \|v\|^2.$$

△

Problem 8:

(6A.26) Suppose V is a real inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}$$

for all $u, v \in V$.

Proof. Note that

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle v, v \rangle + 2\langle u, v \rangle,$$

and

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \langle u, u \rangle + \langle v, v \rangle - 2\langle u, v \rangle.$$

Therefore

$$\frac{\|u + v\|^2 - \|u - v\|^2}{4} = \frac{4\langle u, v \rangle}{4} = \langle u, v \rangle$$

△

Problem 9:

(6A.27) Suppose V is a complex inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 - \|u - iv\|^2}{4}i$$

for all $u, v \in V$.

Proof. Note that if $\langle u, v \rangle = a + bi$, then $\langle v, u \rangle = a - bi$ and

$$\|u + v\|^2 = \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle = \|u\|^2 + \|v\|^2 + 2a,$$

$$\|u - v\|^2 = \langle u, u \rangle + \langle v, v \rangle - \langle u, v \rangle - \langle v, u \rangle = \|u\|^2 + \|v\|^2 - 2a,$$

$$\|u + iv\|^2 = \langle u, u \rangle - \langle v, v \rangle + i\langle u, v \rangle + i\langle v, u \rangle = \|u\|^2 - \|v\|^2 + 2b,$$

$$\|u - iv\|^2 = \langle u, u \rangle + \langle v, v \rangle - i\langle u, v \rangle - i\langle v, u \rangle = \|u\|^2 + \|v\|^2 - 2b.$$

Therefore

$$\frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 - \|u - iv\|^2}{4} = \frac{4a + 4bi}{4} = a + bi = \langle u, v \rangle.$$

△

Problem 10:

(6A.30) Suppose V is a real inner product space. For $u, v, w, x \in V$, define

$$\langle u + iv, w + ix \rangle_C = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

1. Show that $\langle \cdot, \cdot \rangle_C$ makes V_C into a complex inner product space.

Proof. First we show that this proposed inner product is positive semi-definite for $\langle a + bi, a + bi \rangle_C$:

$$\langle a + bi, a + bi \rangle_C = \langle a, a \rangle + \langle b, b \rangle + (\langle b, a \rangle - \langle a, b \rangle)i = \|a\|^2 + \|b\|^2.$$

Now we must show that $\langle 0, 0 \rangle_C = 0$, which follows from the above argument:

$$\langle 0, 0 \rangle_C = \|0\|^2 + \|0\|^2 = 0.$$

For left additivity note that

$$\begin{aligned} \langle (a + bi) + (c + di), x + yi \rangle_C &= \langle a + c + (b + d)i, x + yi \rangle \\ &= \langle a + c, x \rangle + \langle b + d, x \rangle + (\langle b + d, x \rangle + \langle a + c, y \rangle)i \\ &= \langle a, x \rangle + \langle c, x \rangle + \langle b, y \rangle + \langle d, y \rangle + (\langle b, x \rangle + \langle a, y \rangle)i + (\langle d, x \rangle + \langle c, y \rangle)i \\ &= \langle a + bi, x + yi \rangle + \langle c + di, x + yi \rangle. \end{aligned}$$

To show that the product satisfies left homogeneity, we write

$$\langle \lambda(a + bi), x + yi \rangle_C = \langle \lambda a, x \rangle + \langle \lambda b, y \rangle + (\langle \lambda b, x \rangle + \langle \lambda a, y \rangle)i = \lambda(\langle a + bi, x + yi \rangle_C).$$

What remains is to show that the product obeys conjugate symmetry. To see this, note that it follows from the fact that if

$$\langle a + bi, c + di \rangle_C = \langle a, c \rangle + \langle b, d \rangle + (\langle b, c \rangle - \langle a, d \rangle)i,$$

then the imaginary component of $\langle c + di, a + bi \rangle_C$ takes the form

$$(\langle a, d \rangle - \langle b, c \rangle)i = -(\langle b, c \rangle - \langle a, d \rangle)i$$

and so the two are indeed conjugates. △

2. Show that if $u, v \in V$, then

$$\langle u, v \rangle_C = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_C^2 = \|u\|^2 + \|v\|^2.$$

Proof. Note that if $u, v \in V$ then $\langle 0, v \rangle - \langle u, 0 \rangle = 0$ and so $\langle u, v \rangle_C = \langle u, v \rangle$. Further,

$$\|u + iv\|_C^2 = \langle u + iv, u + iv \rangle_C = \langle u, u \rangle + \langle v, v \rangle + (\langle v, u \rangle - \langle u, v \rangle)i = \|u\|^2 + \|v\|^2.$$

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