MATH 3406
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Homework 9

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Problem 0.1:

(6A.1) Prove or give a counterexample: If $v_1, \dots, v_m \in V$, then

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j, v_k\rangle \geq 0.$$

Proof. Note that by right and left additivity respectively we find that

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j, v_k\rangle = \sum_{j=1}^{m}\langle v_j, \sum_{k]1}^{m} v_k\rangle = \langle \sum_{j=1}^{m} v_j, \sum_{k=1}^{m} v_k\rangle$$

and because $\sum_{j=1}^{m} v_j = \sum_{k=1}^{m} v_k$ we have that

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j, v_k\rangle = \langle \sum_{j=1}^{m} v_j, \sum_{k=1}^{m} v_k\rangle \geq 0.$$

$\triangle$

Problem 0.2:

(6A.3)

1. Show that the function taking an ordered pair $((x_1, x_2), (y_1, y_2))$ of elements in $\mathbb{R}^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on $\mathbb{R}^2$.

   Solution. Note that the function given does not satisfy homogeneity in the first slot for negative numbers.

2. Show that the function taking an ordered pair $((x_1, x_2, x_3), (y_1, y_2, y_3))$ to $x_1y_1 + x_3y_3$ is not an inner product on $\mathbb{R}^3$.

   Solution. Note that this function does not satisfy that $\langle v, v\rangle = 0$ if and only if $v = 0$, as $v = (0, 1, 0)$ satisfies the equality.

Problem 0.3:

(6A.6) Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if $||u|| \leq ||u + av||$ for all $a \in \mathbb{F}$.

Proof. First the forward direction. By the fact that norms are positive semidefinite we have that showing $||u|| \leq ||u+av||$ is equivalent to showing that $||u||^2 \leq ||u+av||^2$. We may utilize the left and right additivity (and homogeneity) of the inner product to see that

$$\langle u+av, u+av\rangle = \langle u, u\rangle + \langle u, av\rangle + \langle av, u\rangle + \langle av, av\rangle = \langle u, u\rangle + a^2\langle v, v\rangle = ||u||^2 + a^2||v||^2.$$

Therefore $||u||^2 \leq ||u+av||^2$.
Now the backward direction. We prove the contrapositive. Suppose $\langle u, v\rangle \neq 0$. Then

$$||u+av||^2 = ||v||^2 + ||u||^2 + a\langle u, v\rangle + a\langle v, u\rangle > ||u||^2.$$

Therefore the result holds. △

Problem 0.4:

(6A.17) Prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right)$$

for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$.

Proof. Begin by noting that

$$\left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{i}{k} a_i^2 b_k^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k<i}^{n} \frac{i}{k} a_i^2 b_k^2 + \sum_{k>i}^{n} \frac{i}{k} a_i^2 b_k^2.$$

Now let $u = (a_1, \dots, a_n)$ and $v = (b_1, \dots, b_n)$ so that

$$||u||^2||v||^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_i^2 b_k^2 = \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{k\neq i}^{n} a_i^2 b_k^2$$

$$\leq \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k<i}^{n} \frac{i}{k} a_i^2 b_k^2 + \sum_{k>i}^{n} \frac{i}{k} a_i^2 b_k^2.$$

Therefore

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = |\langle u, v\rangle| \leq ||u||^2||v||^2 \leq \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right).$$

△

Problem 0.5:

(6A.19) Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda|^2 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}|^2.$$

Proof. Suppose v is the eigenvector of T corresponding to λ. Then $|\lambda|^2||v||^2 = ||\lambda v||^2 = ||Tv||^2$. Let a_j denote the j-th row of $\mathcal{M}(T)$. Then $||Tv||^2 = \sum_{j=1}^{n} |\langle v, a_j \rangle|^2$ and therefore $||Tv||^2 \leq \sum_{j=1}^{n} ||v||^2||a_j||^2$. Succinctly,

$$|\lambda|^2||v||^2 \leq \sum_{j=1}^{n} ||v||^2||a_j||^2$$

and we may divide both sides by $||v||^2$ to see that

$$|\lambda|^2 \leq \sum_{j=1}^{n} ||a_j||^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}|^2$$

where a_{jk} is the k-th element of row j. Equivalently,

$$|\lambda|^2 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}|^2.$$

△

Problem 0.6:

(6A.21) Suppose $u, v \in V$ are such that

$$||u|| = 3, \quad ||u + v|| = 4, \quad ||u - v|| = 6.$$

What number does $||v||$ equal?

Solution. By the parallelogram equality,

$$16 + 36 = 2(9 + ||v||^2), \quad 17 = ||v||^2, \quad ||v|| = \sqrt{17}.$$

Problem 0.7:

(6A.22) Show that if $u, v \in V$, then

$$||u+v||||u-v|| \leq ||u||^2 + ||v||^2.$$

Proof. By the parallelogram equality,

$$||u+v||^2 = 2(||v||^2 + ||u||^2) - ||u-v||^2, \quad ||u-v||^2 = 2(||v||^2 + ||u||^2) - ||u+v||^2.$$

Now by the triangle inequality we have the following

$$-\left(||u||^2 + ||v||^2 + 2||u||||v||\right) \leq -||u+v||^2, -||u-v||^2.$$

Now we can write that

$$-||u+v||^2 \leq ||u||^2 + ||v||^2 - 2||u||||v|| \leq ||u||^2 + ||v||^2$$

for both $||u+v||^2$ and $||u-v||^2$. Therefore

$$||u+v||^2||u-v||^2 \leq (||u||^2 + ||v||^2)^2, \quad ||u+v||||u-v|| \leq ||u||^2 + ||v||^2.$$

△

Problem 0.8:

(6A.26) Suppose V is a real inner product space. Prove that

$$\langle u, v \rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for all $u, v \in V$.

Proof. Note that

$$||u+v||^2 = \langle u+v, u+v \rangle = \langle u, u \rangle + \langle v, v \rangle + 2\langle u, v \rangle,$$

and

$$||u-v||^2 = \langle u-v, u-v \rangle = \langle u, u \rangle + \langle v, v \rangle - 2\langle u, v \rangle.$$

Therefore

$$\frac{||u+v||^2 - ||u-v||^2}{4} = \frac{4\langle u, v \rangle}{4} = \langle u, v \rangle$$

△

Problem 0.9:

(6A.27) Suppose V is a complex inner product space. Prove that

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}$$

for all $u, v \in V$.

Proof. Note that if $\langle u, v\rangle = a + bi$, then $\langle v, u\rangle = a - bi$ and

$$||u+v||^2 = \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle = ||u||^2 + ||v||^2 + 2a,$$

$$||u-v||^2 = \langle u,u\rangle + \langle v,v\rangle - \langle u,v\rangle - \langle v,u\rangle = ||u||^2 + ||v||^2 - 2a,$$

$$||u+iv|| = \langle u,u\rangle - \langle v,v\rangle + i\langle u,v\rangle + i\langle v,u\rangle = ||u||^2 - ||v||^2 + 2b,$$

$$||u-iv|| = \langle u,u\rangle + \langle v,v\rangle - i\langle u,v\rangle - i\langle v,u\rangle = ||u||^2 + ||v||^2 - 2b.$$

Therefore

$$\frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4} = \frac{4a+4bi}{4} = a + bi = \langle u, v\rangle.$$

$\triangle$

Problem 0.10:

(6A.30) Suppose V is a real inner product space. For $u, v, w, x \in V$, define

$$\langle u + iv, w + ix\rangle_C = \langle u, w\rangle + \langle v, x\rangle + (\langle v, w\rangle - \langle u, x\rangle)i.$$

1. Show that $\langle \cdot, \cdot\rangle_C$ makes V_C into a complex inner product space.

Proof. First we show that this proposed inner product is positive semi-definite for $\langle a + bi, a + bi\rangle_C$:

$$\langle a + bi, a + bi\rangle_C = \langle a, a\rangle + \langle b, b\rangle + (\langle b, a\rangle - \langle a, b\rangle)\, i = ||a||^2 + ||b||^2.$$

Now we must show that $\langle 0, 0\rangle_C = 0$, which follows from the above argument:

$$\langle 0, 0\rangle_C = ||0||^2 + ||0||^2 = 0.$$

For left additivity note that

$$\langle (a + bi) + (c + di), x + yi\rangle_C = \langle a + c + (b + d)i, x + iy\rangle$$

$$= \langle a + c, x\rangle + \langle b + d, x\rangle + (\langle b + d, x\rangle + \langle a + c, y\rangle)i$$

$$= \langle a, x\rangle + \langle c, x\rangle + \langle b, y\rangle + \langle d, y\rangle + (\langle b, x\rangle + \langle a, y\rangle)i + (\langle d, x\rangle + \langle c, y\rangle)i$$

$$= \langle a + bi, x + yi\rangle + \langle c + di, x + yi\rangle.$$

To show that the product satisfies left homogeneity, we write

$$\langle \lambda(a + bi), x + yi\rangle_C = \langle \lambda a, x\rangle + \langle \lambda b, y\rangle + (\langle \lambda b, x\rangle + \langle \lambda a, y\rangle)i = \lambda(\langle a + bi, x + yi\rangle_C).$$

What remains is to show that the product obeys conjugate symmetry. To see this, note that it follows from the fact that if

$$\langle a + bi, c + di\rangle_C = \langle a, c\rangle + \langle b, d\rangle + (\langle b, c\rangle - \langle a, d\rangle)i,$$

then the imaginary component of $\langle c + di, a + bi\rangle_C$ takes the form

$$(\langle a, d\rangle - \langle b, c\rangle)i = -(\langle b, c\rangle - \langle a, d\rangle)i$$

and so the two are indeed conjugates. △

2. Show that if $u, v \in V$, then

$$\langle u, v\rangle_C = \langle u, v\rangle \quad \text{and} \quad ||u + iv||_C^2 = ||u||^2 + ||v||^2.$$

Proof. Note that if $u, v \in V$ then $\langle 0, v\rangle - \langle u, 0\rangle = 0$ and so $\langle u, v\rangle_C = \langle u, v\rangle$. Further,

$$||u + iv||_C^2 = \langle u + iv, u + iv\rangle_C = \langle u, u\rangle + \langle v, v\rangle + (\langle v, u\rangle - \langle u, v\rangle)i = ||u||^2 + ||v||^2.$$

△