

微积分

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基本积分表及其补充:

$$1. \int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C (\alpha \in \mathbb{R})$$

$$2. \int \frac{1}{x} dx = \ln|x| + C$$

$$3. \int \sin x dx = -\cos x + C$$

$$4. \int \cos x dx = \sin x + C$$

$$5. \int a^x dx = a^x \frac{1}{\ln a} + C (a \in \mathbb{R}) \quad \int e^x dx = e^x + C$$

$$6. \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C = -\arccos x + C$$

$$7. \int \frac{1}{1+x^2} dx = \arctan x + C = -\operatorname{arccot} x + C$$

$$8. \int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d(\cos x)$$

$$= -\ln|\cos x| + C$$

$$9. \int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d(\sin x)$$

$$= \ln|\sin x| + C$$

$$10. \int \sec^2 x dx = \tan x + C$$

$$11. \int \csc^2 x dx = -\cot x + C$$

$$12. \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$13. \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$14. \int \csc x dx = \ln|\csc x - \cot x| + C$$

$$15. \int \sec x \tan x dx = \sec x + C$$

$$16. \int \csc x \cot x dx = -\csc x + C$$

$$17. \int \ln x dx = x \ln x - x + C$$

$$18. \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$$

额外知识补充:

★ 1. 三角关系:

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \begin{cases} 1 + \cot^2 x = \csc^2 x & \text{除 } \sin^2 x \\ 1 + \tan^2 x = \sec^2 x & \text{除 } \cos^2 x \end{cases}$$

2. 和差化积: [补]: $\sin \alpha + \cos \alpha = \sqrt{2} \sin(\alpha + \frac{\pi}{4})$ 辅助角!

$$\begin{cases} \sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\ \sin \alpha - \sin \beta = 2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} \\ \cos \alpha + \cos \beta = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\ \cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} \end{cases}$$

积化和差:

$$\begin{cases} \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha+\beta) + \sin(\alpha-\beta)] \\ \cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha+\beta) - \sin(\alpha-\beta)] \\ \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha+\beta) + \cos(\alpha-\beta)] \\ \sin \alpha \sin \beta = -\frac{1}{2} [\cos(\alpha+\beta) - \cos(\alpha-\beta)] \end{cases}$$

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3. 万能公式 (均用 $\tan \frac{\alpha}{2}$ 表示)

$$\cos \alpha = \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2} = \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}} = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}} = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$\tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

★ 4. 对于多项式常见处理技巧:

1° 分母有理化 2° 分离常量 3° 因式分解 4° 裂项

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$\frac{1}{(x+1)(x-1)} = \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

$$\frac{1}{(x+1)^2(x^2+x+1)^3} = \left[\frac{A}{(x+1)^1} + \frac{B}{(x+1)^2} \right] +$$

$$\left[\frac{C}{(x^2+x+1)^1} + \frac{D}{(x^2+x+1)^2} + \frac{E}{(x^2+x+1)^3} \right]$$

一. 基础积分

1. $\int (x^2+1)^2 dx =$

2. $\int \frac{(1-x)^2}{\sqrt{x}} dx =$

3. $\int \frac{x}{x^2+1} dx =$

$\int \frac{1}{x^2+1} dx = ?$

4. $\int \sec x (\sec x - \tan x) dx =$

5. $\int \frac{2 \cdot 3^x - 5 \cdot 2^x}{3^x} dx =$

6. $\int \cot^2 \theta d\theta =$

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二. 第一类换元积分

7. $\int \cos 2x dx =$

8. $\int \frac{1}{x(2+\ln x)} dx =$

9. $\int \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} dx =$

10. $\int \frac{1}{a^2+x^2} dx =$

11. $\int \frac{1}{\sqrt{a^2-x^2}} dx =$

12. $\int \frac{e^{\sqrt[3]{x}}}{\sqrt{x}} dx =$

13. $\int \sin^3 x dx =$

★ 14. $\int \sin^2 x \cdot \cos^5 x dx =$

15. $\int \sec^6 x dx =$

16. $\int \tan x dx =$

17. $\int \cot x dx =$

★ 18. $\int \csc x =$

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$$19. \int \sec x dx =$$

$$20. \int \cos 3x \cos 2x dx =$$

三. 第2步换元积分

着力解决关于“ $\sqrt{\dots}$ ”的问题!

$$21. \int \sqrt{a^2 - x^2} dx =$$

$$22. \int \sqrt{a^2 + x^2} dx =$$

$$23. \int \sqrt{x^2 - a^2} dx =$$

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$$24. \int \frac{\sqrt{a^2 - x^2}}{x^4} dx =$$

$$25. \int \frac{dx}{\sqrt{4x^2 + 9}} =$$

$$26. \int \frac{dx}{\sqrt{-x^2 + x + 1}} =$$

三. 分部积分 $\int u dv = uv - \int v du \Rightarrow \int u dv + \int v du = uv$

拿进 dx 的优先级: 1° e^x 2° $\frac{\sin x}{\cos x}$ 3° x^n 4° $\ln x$ ★

$$27. \int \ln x dx =$$

$$28. \int e^x \cos x dx =$$

$$29. \int x / \ln x dx =$$

$$30. \int \sec^3 x dx =$$

$$31. \int e^{-2x} \sin \frac{x}{2} dx =$$

$$32. \int x \cos x dx =$$

$$33. \int x^2 e^x dx =$$

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四. 有理函数的积分

(一). 假分式 $\xrightarrow{\text{除法/因式分解/分离常数/裂项}}$ 真分式

(二). 齐次式 $\xrightarrow{\text{分离常数/因式分解}}$ 真分式

(三). 真分式的积分:

34. $\int \frac{1}{2x+1} dx$

0次
2次

35. $\int \frac{1}{x^2-a^2} dx =$

36. $\int \frac{1}{x^2-3x+2} dx =$

37. $\int \frac{1}{x^2-2x+4} dx =$

38. $\int \frac{1}{x^2-2x-1} dx =$

1次
2次

39. $\int \frac{x}{x^2+2x+2} dx =$

0次
高次

40. $\int \frac{1}{(1+2x)(1+x^2)} dx =$

$$\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$$

$$\begin{array}{r} x^2 - x + 1 \\ x+1 \overline{) x^3 } \\ \underline{x^3 + x^2 } \\ -x^2 - x \\ \underline{-x^2 - x } \\ x \\ \underline{x+1} \\ -1 \end{array}$$

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$$41. \int \frac{x^2+1}{(x+2)(x+1)^2} dx =$$

$$42. \int \frac{x^2+1}{(x+1)^2(x-1)} dx =$$

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能转化成有理分式的情况

$$\sqrt[n]{ax+b} / \sqrt[n]{\frac{ax+b}{cx+d}}$$

换元

$$43. \int \frac{\sqrt{x-1}}{x} dx$$

$$44. \int \frac{dx}{1 + \sqrt[3]{x+2}} =$$

$$45. \int \frac{dx}{(1 + \sqrt[3]{x})\sqrt{x}} =$$

$$46. \int \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1} dx =$$

$$\star 47. \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$$

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(四). 有理三角函数 万能代换 → 有时反而更复杂!

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \quad \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \quad \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$$

$$48. \int \frac{1}{1 + \sin x + \cos x} dx =$$

$$49. \int \frac{dx}{3 + \cos x} =$$

$$50. \int \frac{dx}{2 + \sin x} =$$

$$51. \int \frac{1}{\sin^4 x} dx =$$

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五. 有关数列递推公式的积分:

52. $I_n = \int \sin^n x \, dx \, (n \geq 1)$, 求递推公式

53. $\int \frac{1}{(x^2 + a^2)^n} \, dx$, 求递推公式

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*54. Γ 函数 $\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx (s > 0)$

$$1) \Gamma(s) = -e^{-x} x^{s-1} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-x} d x^{s-1} = \underbrace{-e^{-x} x^{s-1} \Big|_0^{+\infty}}_{=0} + (s-1) \underbrace{\int_0^{+\infty} e^{-x} x^{s-2} dx}_{\Gamma(s-1)} - \left(\lim_{x \rightarrow +\infty} \frac{x^{s-1}}{e^x} - 0 \right)$$

$\Gamma(s) = (s-1) \Gamma(s-1)$ 递推公式

$$2) \Gamma(s+1) = s \Gamma(s)$$

$$s=1 \text{ 时, } \Gamma(1) = \int_0^{+\infty} e^{-x} dx = -e^{-x} \Big|_0^{+\infty} = 0 - (-1) = 1$$

$$s=2 \text{ 时, } \Gamma(2) = 1 \Gamma(1) = 1$$

$$s=3 \text{ 时, } \Gamma(3) = 2 \Gamma(2) = 2 \times 1 \Gamma(1) = 2 = 2!$$

$$s=4 \text{ 时, } \Gamma(4) = 3 \Gamma(3) = 3 \times 2 \times 1 \Gamma(1) = 6 = 3!$$

$$\Gamma(n+1) = n!$$

[总结]

代数变形 (换元, 倒代换, 分母有理化, $\sqrt{x(1+x)} \rightarrow \frac{1}{1+x} \sqrt{\frac{1+x}{x}}$, 裂项, 三角恒等变换)

1° 逐项积分, 基本积分公式.

2° 换元积分

- 第一类换元: $\int f(\Delta) d\Delta$ $\Delta = \Delta$ $d\Delta$
- 第二类换元: $\sqrt{a^2+x^2}$ $\sqrt{a^2-x^2}$ $\sqrt{x^2-a^2}$... $\tan t$ $\cot t$ $\sec t$ $\csc t$... \triangle $x \rightarrow x$

3° 分部积分: $\int u dv = uv - \int v du$ 1° e^x 2° $\frac{\sin x}{\cos x}$ 3° x^n 4° $\ln x$

轮换重复 \Rightarrow 移项

4° 有理函数积分

(i) 假分式 $\xrightarrow{\text{除/因/公/常/量/裂}} \text{真分式}$

(ii) 齐次式 $\xrightarrow{\text{因/分离}} \text{真分式}$

(iii) 真分式: $\frac{0/1x}{1/1x} / \frac{0/2x}{2/1x} / \frac{1/2x}{2/2x} / \frac{0/2x}{1/2x} / \frac{1/2x}{1/2x}$

$\xrightarrow{\text{待定系数法}} \text{分母次数高, 令 } x = \dots$
裂项, 倒代换, 分母有理化

$\sqrt[n]{ax+b} / \sqrt[n]{cx+d} \Rightarrow \text{整体换元!}$ 有理三角函数万能代换

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5° 数列通项公式 $\begin{cases} \int \sin^n x dx, \int \cos^n x dx \\ \int \frac{1}{(u^2+a^2)^n} du \\ P(s) = \int_0^{\infty} e^{-x} x^{s-1} dx \end{cases}$

6° 三角函数处理: 二倍角, 升角降幂 $\cos^2 x \left\{ \frac{1-\sin^2 x}{1+\cos 2x} \cdot 1+\cos x \left\{ \frac{2\cos^2 \frac{x}{2}}{1+\frac{1-t^2}{1+t^2}} \right. \right.$

$\begin{cases} \sin^2 x + \cos^2 x = 1 \\ \tan^2 x + 1 = \sec^2 x & \text{和差化积} \\ \cot^2 x + 1 = \csc^2 x & \text{万能公式代换} \end{cases}$

六. 综合练习:

55. $\int \frac{1}{x^2+2x+5} dx =$

56. $\int \frac{dx}{\sqrt{x^2-4x+5}} =$

57. $\int \frac{dx}{e^x - e^{-x}} =$

58. $\int \frac{x}{(1-x)^3} dx =$

59. $\int \frac{\ln(\ln x)}{x} dx =$

60. $\int \frac{\sin x \cos x}{1+\sin^4 x} dx =$

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$$61. \int \sin 3x \sin 5x dx =$$

$$62. \int \frac{x}{\sqrt{3x+1} + \sqrt{2x+1}} dx =$$

$$63. \int \frac{dx}{\sqrt{x(1+x)}} =$$

$$64. \int x^2 \cos x dx =$$

$$65. \int \frac{dx}{\sqrt{1+e^x}}$$

$$66. \int \frac{dx}{x^2 \sqrt{x^2-1}} =$$

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$$67. \int \sqrt{x} \sin \sqrt{x} dx =$$

$$68. \int \ln(1+x^2) dx = \quad \text{模型: } \int \ln(\odot) dx \text{ 直接分部积分!}$$

$$69. \int \frac{\sin^2 x}{\cos^3 x} dx =$$

$$72. \int \frac{x}{(1+x^8)^2} dx =$$

$$73. \int \frac{dx}{(x^2+4)^2} =$$

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$$74. \int \frac{1}{(x^2 + a^2)^2} dx =$$

$$75. \int \frac{x''}{x^8 + 3x^4 + 2} dx =$$

$$76. \int \frac{1 + \cos x}{x + \sin x} dx =$$

$$77. \int \frac{x + \sin x}{1 + \cos x} dx$$

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$$78. \int \frac{\sqrt[3]{x}}{x(\sqrt{x} + \sqrt[3]{x})} dx =$$

$$79. \int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1} dx =$$

$$80. \int \frac{x e^x}{(e^x + 1)^2} dx$$

$$81. \int \sqrt{1-x^2} \arcsin x dx =$$

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