幂级数求和
专题练习

1. $\sum_{n=1}^{\infty}\frac{x^n}{n}$

$S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$

$S'(x)=\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)'=\sum_{n=1}^{\infty}x^{n-1}=\frac{1}{1-x}$

$\therefore S(x)-S(0)=\int_0^x S'(x)dx=\int_0^x\frac{1}{1-x}dx=-\ln(1-x)$

$\therefore S(x)=-\ln(1-x)$

2. $\sum_{n=1}^{\infty}n\cdot x^n$

原式 $=x\sum_{n=1}^{\infty}n\cdot x^{n-1}=x\,S(x)$

$S(x)=\sum_{n=1}^{\infty}n\cdot x^{n-1}$　$\therefore\int_0^x S(x)dx=\int_0^x\sum_{n=1}^{\infty}n\cdot x^{n-1}=\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}$

$\Rightarrow S(x)=\left(\int_0^x S(x)dx\right)'=\left(\frac{x}{(1-x)}\right)'=\frac{1}{(1-x)^2}$

$\therefore$原式 $=\frac{x}{(1-x)^2}$

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n}=x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cdot x^{2n-1}=x\cdot S(x)$

$(S(x))'=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}=\sum_{n=1}^{\infty}(-x^2)^{n-1}=\frac{1}{1+x^2}$

$S(x)-S(0)=\int_0^x(S(x))'dx=\int_0^x\frac{1}{1+x^2}dx=\arctan x$　而 $S(0)=0$

$\Rightarrow S(x)=\arctan x$

$\therefore$原式 $=x\arctan x$

4. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n = s(x)$

$\Rightarrow x\,S(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}$

$(xS(x))' = \sum_{n=1}^{\infty} \frac{1}{n} x^n$

$(xS(x))'' = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$　　　　$\ln(1-x)\,d(-x+1)$

$\therefore (xS(x))' = \int_0^x (xS(x))'' dx = \int_0^x \frac{1}{1-x} dx = -\ln(1-x)$

$xS(x) = \int_0^x (xS(x))' dx = (1-x)\ln(1-x) + x$

$$\Rightarrow S(x) = \begin{cases} \frac{x+(1-x)\ln(1-x)}{x}, & [-1,0)\cup(0,1) \\ 0, & x=0 \\ 1, & x=1 \end{cases}$$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1} x^{2n+1} = s(x)$

$\int_0^x s(x)dx = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+1}{2n+1} \cdot x^{2n+2} \cdot \frac{1}{2n+2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \cdot \frac{x^{2n+2}}{2n+1} = x \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \cdot \frac{x^{2n+1}}{2n+1}$

设 $A(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \cdot \frac{x^{2n+1}}{2n+1}$

则: $A'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2} x^{2n} = \frac{1}{2} \sum_{n=1}^{\infty} (-x^2)^n = \frac{1}{2} \cdot \frac{-x^2}{1+x^2}$

$\therefore A(x) = \int_0^x (A'(x))\,dx = \frac{1}{2}\int_0^x \frac{-x^2}{1+x^2} dx = \frac{1}{2}\int_0^x \left(-1 + \frac{1}{1+x^2}\right) dx = (-x + \arctan x)\frac{1}{2}$

$\therefore \int_0^x s(x)\,dx = xA(x) = (-x^2 + x\arctan x)\frac{1}{2}$

$S(x) = \left(\int_0^x s(x)dx\right)' = (-x^2 + x\arctan x)' \cdot \frac{1}{2} = \frac{1}{2}\left(-2x + \arctan x + \frac{x}{1+x^2}\right)$

$= -x + \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)}$

6. $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-1\right)x^{2n}=\sum_{n=1}^{\infty}\frac{x^{2n}}{2n+1}-\sum_{n=1}^{\infty}x^{2n}$

$=\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{2n+1}x^{2n+1}-\sum_{n=1}^{\infty}x^{2n}$

设 $s(x)=\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}$ $s(0)=0$

$s'(x)=\sum_{n=1}^{\infty}x^{2n}=\frac{x^2}{1-x^2}$

$s(x)=\int_0^x s'(x)dx=\int_0^x\frac{x^2}{1-x^2}dx=\int_0^x\left(-1-\frac{1}{x^2-1}\right)dx=-x-\ln\left|\frac{x-1}{x+1}\right|$

$\Rightarrow$ 原式 $=\frac{1}{x}\left(-x-\ln\left|\frac{x-1}{x+1}\right|\right)-\frac{x^2}{1-x^2}$

7. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n+1}}{n(2n-1)}=x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}$

$s_1(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}$

则: $s_1'(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot x^{2n-1}\cdot 2n\cdot 2}{n(2n-1)}=\sum_{n=1}^{\infty}\frac{4(-1)^{n-1}x^{2n-1}}{2n-1}$

$s_1''(x)=\sum_{n=1}^{\infty}4(-1)^{n-1}\cdot x^{2(n-1)}=\sum_{n=1}^{\infty}4(-x^2)^{n-1}=4\frac{1}{1+x^2}$

$\therefore s_1'(x)=\int_0^x s_1''(x)=4\arctan x$

$s_1(x)=\int_0^x s_1'(x)dx=4\int_0^x\arctan x\,dx=4\left(x\arctan x-\int_0^x\frac{x}{1+x^2}dx\right)=4x\arctan x-2\ln(1+x^2)$

$\therefore$ 原式 $=x\left[4x\arctan x-2\ln(1+x^2)\right]$

[小结]:

① 先求导, 再求积分　　② 先积分, 再求导

$\frac{1}{n}x^n$　　$n\cdot x^{n-1}$

③ 提 $x^1 \cdot x^2 \cdot x^{-1} \cdot x^{-2}\ldots$　　④ 求2次导, 求两次积分

⑤ 拆项

日期:　　　/