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STOCHASTIC MODELLING FOR THE FINANCIAL MARKETS
PART 1. PROBABILISTIC TOOLS

Lectures notes
for the courses "Stochastic Modelling" and
"Theory of the Random Processes"
taken by most Mathematics students and Economics students
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The main goal of these lectures notes is to give the basic notions of the stochastic calculus such that conditional expectations, predictable processes, martingales, stochastic integrals and Ito's formula. The notes are intended for students of the Mathematics and Economical Faculties.

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1 Introduction

1.1 Probability space

Definition 1.1. *The measurable space* $(\Omega, \mathcal{F}, \mathbf{P})$ *is called the probability space, where* Ω *is any fixed universal set,* $\mathcal{F}$ *is* σ *- field and* $\mathbf{P}$ *is a probability measure.*

It should be noted that if the set Ω is finite or countable then the field (or σ - field) $\mathcal{F}$ is defined as all subsets of the set Ω, i.e. $\mathcal{F} = \{A : A \subseteq \Omega\}$. Moreover, in this case the probability is defined as

$$\mathbf{P}(A) = \sum_{\omega \in A} \mathbf{P}(\{\omega\}), \tag{1.1}$$

where $\mathbf{P}(\{\omega\})$ is defined for every ω from Ω.

Examples

1. *The Bernoulli space.*

   The set $\Omega = \{0, 1\}$ and $\mathcal{F} = \{\Omega, \emptyset, \{0\}, \{1\}\}$. The probability is defined as $\mathbf{P}(\{0\}) = p$ and $\mathbf{P}(\{1\}) = 1 - p$ for some fixed $0 < p < 1$. Note that, if $p = 1/2$, then we obtain the "throw a coin" model.

2. *The binomial space.*

The set $\Omega = \{0, 1, \dots, n\}$ and $\mathcal{F} = \{A : A \subseteq \Omega\}$. In this case for any $0 \le k \le n$ the probability is defined as

$$\mathbf{P}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} . \tag{1.2}$$

3. *The finite power of the Bernoulli spaces.*

The set $\Omega = \{0, 1\}^n = \{\omega_l\}_{1 \le l \le 2^n}$, where ω_l are n-dimensional vectors, i.e. $\omega_l = (\omega_{l,1}, \dots, \omega_{l,n})$ and $\omega_{l,j} \in \{0, 1\}$. The field $\mathcal{F} = \{A : A \subseteq \Omega\}$ and

$$\mathbf{P}(\omega_l) = p^{\nu_l} (1-p)^{n-\nu_l} , \tag{1.3}$$

where $\nu_l = \sum_{j=1}^n \omega_{l,j}$.

4. *The infinite power of the Bernoulli spaces.*

The set $\Omega = \{0, 1\}^\infty = \{\omega\}$. In this case $\omega = (\omega_l)_{l \ge 1}$, $\omega_l \in \{0, 1\}$ and the set Ω is note countable, moreover, this set is isomorphes to interval $[0, 1]$ by the natural representation

$$x = \sum_{l \ge 1} \omega_l \, 2^{-l} \in [0, 1] . \tag{1.4}$$

So, for such set Ω the σ - field $\mathcal{F}$ is Borel σ - field generated by the intervals from $[0,1]$ i.e. $\mathcal{F} = \mathcal{B}([0,1])$. The probability is the Lebesgue measure on the interval $[0,1]$.

1.2 Random variables, vectors and mappings

We remind, that any measurable $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ function ξ is called a *random variable* and $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called a *random vector*. Generally, for any measurable space $((\mathcal{X}, \mathcal{B}(\mathcal{X}))$ a measurable $(\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ function is called a *random mapping*.

For any nonnegative random variable ξ we can define the Lebesgue integral as

$$\mathbf{E}\,\xi = \int_\Omega \xi(\omega)\,\mathrm{d}P$$

which is called *expectation*. Note that any random variable $\xi = \xi_+ - \xi_-$, where $\xi_+ = \max(\xi, 0)$ and $\xi_- = -\min(\xi, 0)$. So, if $\mathbf{E}\min(\xi_+\,, \xi_-) < \infty$, then we can define the expectation in general case as

$$\mathbf{E}\,\xi = \mathbf{E}\,\xi_+ - \mathbf{E}\,\xi_-\,.$$

The function defined as $F(x) = \mathbf{P}(\xi \le x)$ is called *distribution function*.

Examples

1. **Construction of the probability space for fixed random variables with values in $\mathbb{R}$.**

   Let F be a distribution function on $\mathbb{R}$. Now through Carathéodory's extension theorem we obtain the probability measure μ on the Borel σ - field $\mathcal{B}(\mathbb{R})$ for which $\mu(b,a) = F(b) - F(a)$ for any interval (a,b) with $a < b$. To define the probability space we set $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $\mathbf{P} = \mu$. In this case the random variable $\xi(x) = x$ has the distribution function F.

2. **Construction of the probability space for fixed random variables with values in $\mathbb{R}^m$.**

   Let μ be a probability measure on $\mathbb{R}^m$. Similarly to the previous example we set $\Omega = \mathbb{R}^m$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^m)$ and $\mathbf{P} = \mu$. In this case the random variable $\xi(x) = x$ has the distribution μ.

1.3 Conditional expectations and conditional probabilities

Let now $(\Omega, \mathcal{F}, \mathbf{P})$ be some probability space. Moreover, let now ξ be some integrated random variable with values in $\mathbb{R}$ and $\mathcal{G}$ a some σ - field in the probability space, i.e. $\mathcal{G} \subseteq \mathcal{F}$.

Definition 1.2. *The random variable* $\mathbf{E}(\xi|\mathcal{G})$ *is called the conditional expectation if the following conditions hold:*

1. $\mathbf{E}(\xi|\mathcal{G})$ *is* $\mathcal{G}$ *measurable random variable;*

2. *For any bounded* $\mathcal{G}$ *measurable random variable* α

$$\mathbf{E}\alpha\xi = \mathbf{E}(\alpha\mathbf{E}(\xi|\mathcal{G})) \ . \tag{1.5}$$

Note that this definition is correct, i.e. if there exists a $\mathcal{G}$ - measurable random variable $\check{\xi}$ satisfying the property (1.5), then it equals to the conditional expatiation. Indeed, if we set

$$\alpha = \operatorname{sign}(\check{\xi} - \mathbf{E}(\xi|\mathcal{G}))\,,$$

then the equality (1.5) implies that $\mathbf{E}\,|\check{\xi} - \mathbf{E}(\xi|\mathcal{G})| = 0$, i.e. $\check{\xi} = \mathbf{E}(\xi|\mathcal{G})$ a.s.

We can use the another definition for the conditional expectation also.

Definition 1.3. *The random variable* $\mathbf{E}(\xi|\mathcal{G})$ *is called the conditional expectation if the following conditions hold:*

1. $\mathbf{E}(\xi|\mathcal{G})$ *is* $\mathcal{G}$ *- measurable random variable;*

2. For any $A \in \mathcal{G}$

$$\mathbf{E}\mathbf{1}_A\xi = \mathbf{E}\left(\mathbf{1}_A\mathbf{E}(\xi|\mathcal{G})\right) \tag{1.6}$$

Note that to show the existence of the condition expectations we use the Radon – Nikodym theorem. Indeed, for any $A \in \mathcal{G}$ we introduce the measure ν as

$$\nu(A) = \int_A \xi \, \mathrm{d}\mathbf{P}\,. \tag{1.7}$$

It is clear that the measure ν is finite and, moreover, $\nu \ll \mathbf{P}$. So, through the Radon – Nikodym theorem, there exists a $\mathcal{G}$ - measurable unique random variable ρ such that

$$\nu(A) = \int_A \rho \, \mathrm{d}\mathbf{P}\,.$$

We can do the same construction for any positive random variable ξ, not necessary integrable. So, we can define the conditional expectation for any positive random variable ξ. For a general random variable ξ we can define the conditional expectation if, $\mathbf{E}\,\xi_- < \infty$. In this case we set

$$\mathbf{E}(\xi|\mathcal{G}) = \mathbf{E}(\xi_+|\mathcal{G}) - \mathbf{E}(\xi_-|\mathcal{G})\,.$$

Definition 1.4. *Let η be a some random variable. We define the conditional expectation with respect to the random variable η as*

$$\mathbf{E}\left(\xi|\eta\right) = \mathbf{E}\left(\xi|\mathcal{G}_\eta\right) \,,$$

where $\mathcal{G}_\eta = \sigma\left\{\eta\right\}$ is σ - field generated by random variable η.

From the definition of the conditional expectation $\mathbf{E}(\xi|\eta)$ it follows that there exists a some Borel $\mathbb{R} \to \mathbb{R}$ function $\mathbf{m}$ such that

$$\mathbf{E}\left(\xi|\eta\right) = \mathbf{m}(\eta) \,.$$

This function is called the *conditional expectation* with respect to the fixed values of η, i.e. for any $y \in \mathbb{R}$

$$\mathbf{E}(\xi|\eta = y) = \mathbf{m}(y) \,. \tag{1.8}$$

Properties of the condition expectations.

1. If η is a constant, then $\mathbf{E}(\xi|\eta) = \mathbf{E}\xi$.

2. Let ξ and η be two random variables such that the conditional expectations $\mathbf{E}(\xi|\mathcal{G})$ and $\mathbf{E}(\eta|\mathcal{G})$ exist and $\xi \leq \eta$ a.s. Then $\mathbf{E}(\xi|\mathcal{G}) \leq \mathbf{E}(\eta|\mathcal{G})$ a.s.

3. If ξ and η are independents, then $\mathbf{E}(\xi|\eta) = \mathbf{E}\xi$.

4. If the σ - field generated by the random variable ξ is more small than the σ - field generated by the random variable η, i.e.

$$\sigma\{\xi\} \subseteq \sigma\{\eta\},$$

then $\mathbf{E}(\xi|\eta) = \xi$.

5. Let ξ and η the random variables such that $\sigma\{\xi\} \subseteq \sigma\{\eta\}$. Then for any integrable random variable γ

$$\mathbf{E}(\gamma\xi|\eta) = \xi\,\mathbf{E}(\gamma|\eta).$$

6. Let $\mathcal{A}$ and $\mathcal{B}$ two σ - fields such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{F}$. Then

$$\mathbf{E}(\xi|\mathcal{A}) = \mathbf{E}(\mathbf{E}(\xi|\mathcal{B})|\mathcal{A}).$$

7. Let ξ be a square integrated random variable, i.e. $\mathbf{E}\,\xi^2 < \infty$. The conditional expectation $\mathbf{E}(\xi|\mathcal{G})$ is the projection in $\mathbf{L}_2(\Omega, \mathcal{F}, \mathbf{P})$ into the subspace $\mathbf{L}_2(\Omega, \mathcal{G}, \mathbf{P})$, i.e. for any $\eta \in \mathbf{L}_2(\Omega, \mathcal{G}, \mathbf{P})$

$$\mathbf{E}\left(\xi - \mathbf{E}(\xi|\mathcal{G})\right)^2 \leq \mathbf{E}\left(\xi - \eta\right)^2.$$

Let now ξ and η be two random variables with the density

$f_{\xi,\eta}(\cdot,\cdot)$, i.e. for any Borel set $B \subseteq \mathbb{R}^2$

$$\mathbf{P}\left((\xi,\eta) \in B\right) = \int_B f_{\xi,\eta}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Note that, on this case the one-dimensional densities can be represented as

$$f_\xi(x) = \int_{\mathbb{R}} f_{\xi,\eta}(x,y)\,\mathrm{d}y \quad \text{and} \quad f_\eta(y) = \int_{\mathbb{R}} f_{\xi,\eta}(x,y)\,\mathrm{d}x\,.$$

In this case the conditional density is defined as

$$f_{\xi|\eta}(x|y) = \frac{f_{\xi,\eta}(x,y)}{f_\eta(y)}\,\mathbf{1}_{\{f_\eta(y)>0\}}\,. \tag{1.9}$$

Proposition 1.1. *Let g be a some measurable $\mathbb{R} \to \mathbb{R}$ function for which the expectation $\mathbf{E}\,|g(\xi)| < \infty$. Then for any $y \in \mathbb{R}$*

$$\mathbf{E}(g(\xi)|\eta = y) = \int_{\mathbb{R}} g(x)\,f_{\xi|\eta}(x|y)\,\mathrm{d}x\,.$$

This proposition means that the conditional density may be used to calculate the corresponding conditional expectations.

1.4 Stochastic basis

The family of the σ - fields $(\mathcal{F}_t)_{t\ge 0}$ is called *filtration* if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for any $0 \le s \le t$. Moreover, for all $t \ge 0$ we set

$$\mathcal{F}_{t-} = \sigma\{\cup_{s<t}\mathcal{F}_s\} \quad \text{and} \quad \mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s\,.$$

We set $\mathcal{F}_{0-} = \mathcal{F}_0$. The filtration is called *left continuous* if $\mathcal{F}_{t-} = \mathcal{F}_t$, *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$ and *continuous* if $\mathcal{F}_{t-} = \mathcal{F}_{t+}$ for any $t \ge 0$. The probability space with a filtration

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P}\right) \tag{1.10}$$

is called *stochastic basis.*

Exercises

1. Show that the definitions 1.2 and 1.3 are equivalents.

2. Show the properties 1 – 7 of the conditional expectations.

3. Let ξ and η be two independent random variables with $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 = 1$. Calculate

$$\mathbf{E}\left(\eta\xi^2|\eta\right) =? \quad \text{and} \quad \mathbf{E}\left(\frac{\eta}{1+\eta^2}\,\xi|\eta\right) =?$$

4. Let ξ and η be two random variables sucht that $\mathbf{E}(\xi|\eta) = 1$ and $\mathbf{E}\eta = 2$. Calculate $\mathbf{E}\,\xi\eta = ?$

5. Let ξ and η be two independent random variables and ξ be uniform on the interval $[0,1]$.

   (a) Calculate
   $$\mathbf{E}\left(\frac{\eta}{1+\eta^2}\,\sin(2\pi\xi)|\eta\right) =?$$

   (b) Show that
   $$\mathbf{E}\left(\cos(\eta\xi)\,|\,\eta\right) = \frac{\sin(\eta)}{\eta}\,.$$

6. Let ξ and η be two random variables such that their joint density function is

$$f_{\xi,\eta}(x,y) = \frac{1}{\pi\sqrt{2}}\,e^{-\left(x^2+y^2-\sqrt{2}xy\right)}\,.$$

   (a) Find the densities $f_\xi(x)$ and $f_\eta(y)$.

   (b) Find the conditional density $f_{\xi|\eta}(x|y)$.

   (c) Calculate

$$\mathbf{E}\left(\xi\,|\,\eta\right) =? \quad \text{and} \quad \mathbf{E}\left(\xi^2\,|\,\eta\right) =?$$

7. Show Proposition 1.1.

8. Let $(\xi_j)_{j\ge 1}$ be a sequence of random variables and $\mathcal{F}_t = \sigma\{\xi_1, \ldots, \xi_{[t]}\}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

   (a) Show that the filtration $(\mathcal{F}_t)_{t\ge 0}$ is right continuous.

   (b) Check is this filtration left continuous or not ?

2 Markovian moments

Let $\big(\Omega, \mathcal{F}\,, (\mathcal{F}_t)_{t\ge 0}\,, \mathbf{P}\big)$ be a fixed stochastic basis.

Definition 2.1. *Random variable $\tau \in \mathbb{R}_+$ is called markovian moment, if*

$$\{\tau \le t\} \in \mathcal{F}_t \qquad \forall t \ge 0\,.$$

If $\mathbf{P}(\tau < \infty) = 1$, then τ is stopping time.

We denote by $\mathcal{M}$ the set of all markovian moments and by $\check{\mathcal{M}}$ the set of all stopping times.

Definition 2.2. *Let $\tau \in \mathcal{M}$. We set*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}\,:\, A \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall t \ge 0\}$$

and $\mathcal{F}_{\tau-}$ is σ- field generated by $\mathcal{F}_0$ and the set of the form $A \cap \{t < \tau\}$ with $A \in \mathcal{F}_t$.

Definition 2.3. *A subset $A \subseteq \Omega \times \mathbb{R}_+$ is called random set.*

It is clear that if A is a random set, then

$$\pi_A = \left\{\omega \in \Omega \,:\, \exists t \in \mathbb{R}_+ \quad (\omega, t) \in A\right\} \tag{2.1}$$

is its Ω - projection.

Now for any random set A we define

$$D_A = D_A(\omega) = \inf\{t \geq 0 \,:\, (\omega, t) \in A\}. \tag{2.2}$$

We set $D_A = \infty$ if $\{t \geq 0 \,:\, (\omega, t) \in A\} = \emptyset$. This function is called the debut of A.

Let σ and τ be two random variables with values in $\mathbb{R}$.

Definition 2.4. *We call stochastic intervals the following random sets*

$$[\![\sigma, \tau]\!] = \{(\omega, t) \,:\, \sigma(\omega) \leq t \leq \tau(\omega)\},$$

$$[\![\sigma, \tau[\![= \{(\omega, t) \,:\, \sigma(\omega) \leq t < \tau(\omega)\},$$

$$]\!]\sigma, \tau]\!] = \{(\omega, t) \,:\, \sigma(\omega) < t \leq \tau(\omega)\},$$

$$]\!]\sigma, \tau[\![= \{(\omega, t) \,:\, \sigma(\omega) < t < \tau(\omega)\}.$$

Moreover, the set $[\![\tau]\!] = [\![\tau, \tau]\!]$ *is called the graphics of the random variable* τ.

Definition 2.5. *A set is called thin if there exits a sequence of random variables* $(\tau_n)_{n\geq 1}$ *for which* $[\![\tau_n]\!] \cap [\![\tau_m]\!] = \emptyset$ *for* $n \neq m$ *such that*

$$A = \cup_{n\geq 1}[\![\tau_n]\!]\,.$$

Definition 2.6. *A random set* A *is called thin if there exits a sequence of random variables* $(\tau_n)_{n\geq 1}$ *for which* $[\![\tau_n]\!] \cap [\![\tau_m]\!] = \emptyset$ *for* $n \neq m$ *such that*

$$A = \cup_{n\geq 1}[\![\tau_n]\!]\,.$$

A random set A *is called negligible if* $\mathbf{P}(\pi_A) = 0$.

Exercises

1. Show that if the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$, then

$$\tau \in \mathcal{M} \quad \Longleftrightarrow \quad \{\tau < t\} \in \mathcal{F}_t \quad \forall t \geq 0\,.$$

2. Show, that the constants (i.e. $\tau(\cdot) \equiv t_0 \geq 0$) are stopping moments.

3. Show, that τ is measurable with respect to $\mathcal{F}_{\tau-}$ for any $\tau \in \mathcal{M}$.

4. Show, that $\mathcal{F}_\tau$ is σ - field for any $\tau \in \mathcal{M}$.

5. If $\tau(\cdot) \equiv t_0 \geq 0$, then $\mathcal{F}_\tau = \mathcal{F}_{t_0}$.

6. Show, that $\mathcal{F}_{\tau-} \subseteq \mathcal{F}_\tau$ for any $\tau \in \mathcal{M}$.

7. Show, that the random variable $\tau + \mathbf{c} \in \mathcal{M}$ for any $\tau \in \mathcal{M}$ and any constant $\mathbf{c} > 0$.

8. Show, that for any τ and σ from $\mathcal{M}$ such that $\tau(\omega) \leq \sigma(\omega)$ we have
$$\mathcal{F}_{\tau-} \subseteq \mathcal{F}_{\sigma-} \quad \text{and} \quad \mathcal{F}_\tau \subseteq \mathcal{F}_\sigma\,.$$

9. Let $(\tau_n)_{n\geq 1} \in \mathcal{M}$. Show that
$$\sigma = \inf_{n\geq 1} \tau_n \in \mathcal{M}\,, \quad \tau = \sup_{n\geq 1} \tau_n \in \mathcal{M} \quad \text{and} \quad \mathcal{F}_\sigma = \cap_{n\geq 1} \mathcal{F}_{\tau_n}\,.$$

10. Show, that for any τ and σ from $\mathcal{M}$ and $A \in \mathcal{F}_\sigma$, the intersection
$$A \cap \{\sigma < \tau\} \in \mathcal{F}_{\tau-}\,.$$

11. Show, that for any $A \in \mathcal{F}_\infty = \sigma\{\cup_{t\geq 0}\mathcal{F}_t\}$ and $\tau \in \mathcal{M}$ the intersection $A \cap \{\tau = \infty\} \in \mathcal{F}_{\tau-}$.

3 Stochastic processes

Let X be a stochastic process, i.e. X is a family of the random variables $X = (X_t)_{t\geq 0}$. We recall that for any fixed $\omega \in \Omega$ the $\mathbb{R}_+ \to \mathbb{R}$ function $(X_t(\omega))_{t\geq 0}$ is called trajectory of the stochastic process X.

Definition 3.1. *A stochastic process X is called adapted if the random variable X_t is $\mathcal{F}_t$ measurable for each $t \in \mathbb{R}_+$. The stochastic process X is called measurable if the mapping*

$$X \,:\, \big(\Omega \times \mathbb{R}_+\,,\, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\big) \to (\mathbb{R}\,,\, \mathcal{B}(\mathbb{R}))$$

is measurable.

Definition 3.2. *A stochastic process X is called progressively measurable if for any $t \in \mathbb{R}_+$ the mapping*

$$X \,:\, (\Omega \times [0,t]\,,\, \mathcal{F}_t \otimes \mathcal{B}([0,t])) \to (\mathbb{R}\,,\, \mathcal{B}(\mathbb{R}))$$

is measurable.

In the sequel we denote by $\mathbf{C}$ the space of the continuous $\mathbb{R}_+ \to \mathbb{R}$ functions and by $\mathbf{D}$ the Skorokhod space, i.e. the space of $\mathbb{R}_+ \to \mathbb{R}$ functions which are right-continuous and have left-hand limits. Such function are called *cadlag*. For any stochastic process X with the trajectories from $\mathbf{D}$ we set:

$$X_- = (X_{t-})_{t\geq 0} \quad \text{and} \quad \Delta X = (\Delta X_t)_{t\geq 0}\,,$$

where $\Delta X_t = X_t - X_{t-}$. Moreover, for any markovian moment $\tau \in \mathcal{M}$ we set

$$X^\tau = (X_{t\wedge\tau})_{t\geq 0} \quad \text{and} \quad \Delta X_\tau = \big(X_\tau - X_{\tau-}\big)\, \mathbf{1}_{\{\tau<\infty\}}\,,$$

where $a \wedge b = \min(a, b)$. The process X^τ is called stopped.

Exercises

1. Show that if X is right or left continuous, then it is progressively measurable.

2. Show that the debut D_A defined in (2.2) is markovian moment for any progressively measurable set A.

3. For any random moment τ and $A \in \mathcal{F}$ we set

$$\tau_A = \begin{cases} \tau\,, & \text{if} \quad \omega \in A\,; \\ +\infty\,, & \text{if} \quad \omega \in A^c\,. \end{cases} \tag{3.1}$$

   Show that τ is markovian moment for any $A \in \mathcal{F}_\tau$ and $\tau \in \mathcal{M}$.

4. Let τ and σ be two markovian moments, i.e. τ and σ from $\mathcal{M}$. Show that

$$\{\tau = \sigma\} \in \mathcal{F}_{\tau} \cap \mathcal{F}_{\sigma}\,, \quad \{\tau \le \sigma\} \in \mathcal{F}_{\tau} \cap \mathcal{F}_{\sigma}\,,$$

$$\{\sigma < \tau\} \in \mathcal{F}_{\sigma-}\,, \quad \{\sigma < \infty\} \in \mathcal{F}_{\sigma-}\,.$$

5. Let $X = (X_t)_{t\ge 0}$ be progressively measurable process and $\tau \in \mathcal{M}$. Show that $X_{\tau}\mathbf{1}_{\{\tau<\infty\}}$ is measurable with respect to $\mathcal{F}_{\tau}$.

6. Let $X = (X_t)_{t\ge 0}$ be progressively measurable process and $\tau \in \mathcal{M}$. Show that the stopped process $X^{\tau} = (X_{\tau\wedge t})_{t\ge 0}$ is adapted with respect to $(\mathcal{F}_t)_{t\ge 0}$.

4 Optional and Predictable σ - fields

Let $\big(\Omega, \mathcal{F}\,, (\mathcal{F}_t)_{t\ge 0}\,, \mathbf{P}\big)$ be a fixed stochastic basis.

Definition 4.1. *A σ - field $\mathcal{D} \subset \mathcal{F} \times \mathcal{B}(\mathbb{R}_+)$ is called optional if it is generated by the stochatsic intervals $[\![0\,,\,\tau]\!]$, $\tau \in \mathcal{M}$.*

A σ - field $\mathcal{P} \subset \mathcal{F} \times \mathcal{B}(\mathbb{R}_+)$ is called predictable if it is generated by the set $[\![0_A]\!] = \{(\omega, t)\,:\,\omega \in A\,,\, t = 0\}$ with $A \in \mathcal{F}_0$ and the stochatsic intervals $[\![0\,,\,\tau]\!]$ with $\tau \in \mathcal{M}$.

Note that $\mathcal{P} \subseteq \mathcal{D}$. We set

$$\tau_A^0 = \begin{cases} 0\,, & \text{if} \quad \omega \in A\,; \\ \\ +\infty\,, & \text{if} \quad \omega \in A^c\,. \end{cases}$$

As it is shown τ_A^0 is markovian moment for any $A \in \mathcal{F}_0$. Taking this into account we can represent the set $[\![0_A]\!]$ as

$$[\![0_A]\!] = \cap_{n \geq 1} [\![\tau_A^0\,,\, \tau_A^0 + 1/n[\![\quad \in \quad \mathcal{D}\,.$$

Moreover, for any $\tau \in \mathcal{M}$ we have

$$[\![0\,,\, \tau]\!] = \cap [\![0\,,\, \tau + 1/n[\![\quad \in \quad \mathcal{D}\,.$$

Therefore, $\mathcal{P} \subseteq \mathcal{D}$.

Let us now $\mathcal{P}_1$ be a σ - field generated by the adapted left continuous processes, and

$$\mathcal{P}_2 = \sigma\left\{[\![0_A]\!]\,, \quad A \in \mathcal{F}_0\,, \quad A \times [s,t]\,, \quad A \in \mathcal{F}_s\right\} \tag{4.1}$$

Proposition 4.1. *The σ - fields $\mathcal{P}_1$ and $\mathcal{P}_2$ are predictable, i.e.*

$$\mathcal{P} = \mathcal{P}_1 = \mathcal{P}_2\,. \tag{4.2}$$

Exercises

1. Show that $\mathcal{P}$ is generated by the adapted continuous processes.

2. Show that any set A from $\mathcal{P}$ is progressively measurable.

Definition 4.2. *A process X is called predictable if it is measurable with respect to $\mathcal{P}$.*

Proposition 4.2. *Let X be a predictable process and τ a markovian moment, i.e. $\tau \in \mathcal{M}$. Then*

1. *the random variable $X_\tau \mathbf{1}_{\tau<\infty}$ is measurable with respect to $\mathcal{F}_{\tau-}$;*

2. *the stopped process X^τ is predictable.*

Definition 4.3. *A random variable τ with values in $\mathbb{R}_+$ is called predictable moment if $[\![\tau]\!] \in \mathcal{P}$.*

We denote by $\mathcal{M}_p$ the set of all predictable moments.

Proposition 4.3. *A moment τ is predictable, i.e. $\tau \in \mathcal{M}_p$ if and only if $[\![0, \tau[\![\in \mathcal{P}$.*

Exercises

1. Let X be a predictable increasing process. Now, for some fixed $c \in \mathbb{R}$ we set

$$\tau = \inf\{t \geq 0 \,:\, X_t \geq c\}$$

and $\tau = +\infty$ if this set is empty, i.e. $\{t \geq 0 \,:\, X_t \geq c\} = \emptyset$. Show that $\tau \in \mathcal{M}_p$.

2. Let X be a predictable cadlag process, i.e. with values in the Skorokhod space $\mathbf{D}$. We set

$$\tau = \inf\{t \geq 0 \,:\, |\Delta X_t| > 0\}$$

and $\tau = +\infty$ if this set is empty, i.e. $\{t \geq 0 \,:\, |\Delta X_t| > 0\} = \emptyset$. Show that $\tau \in \mathcal{M}_p$.

3. Let τ and σ be two predictable moments, i.e. τ and σ from $\mathcal{M}_p$. Show that $\sigma \wedge \tau \in \mathcal{M}_p$ and $\sigma \vee \tau \in \mathcal{M}_p$.

4. Let a sequence $(\tau_n)_{n\geq 1}$ be from $\mathcal{M}_p$. Show that $\sup_{n\geq 1} \tau_n \in \mathcal{M}_p$.

Theorem 4.1. *Let τ be a predictable moment (i.e. $\tau \in \mathcal{M}_p$) and A be a set from $\mathcal{F}_{\tau-}$. Then τ_A is predictable moment, i.e. $\tau_A \in \mathcal{M}_p$.*

Corollary 4.1. *Let $\sigma \in \mathcal{M}_p$ and $\tau \in \mathcal{M}$. Then for any set A from $\mathcal{F}_{\sigma-}$ the set $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\tau-}$.*

Proposition 4.4. *Let σ, τ be markovian moments (i.e. σ, τ from $\mathcal{M}$) and Y be a some random variable.*

1. *If $\tau \in \mathcal{M}_p$ and $Y \in \mathcal{F}_\sigma$ then the process*

$$X = Y\,\mathbf{1}_{]\!]\sigma,\tau[\![}$$

is predictable;

2. *If $\sigma \in \mathcal{M}_p$ and $Y \in \mathcal{F}_{\sigma-}$ then the process*

$$X = Y\,\mathbf{1}_{[\![\sigma,\tau]\!]}\,\mathbf{1}_{\{\tau<\infty\}}$$

is predictable;

3. *If $\sigma, \tau \in \mathcal{M}_p$ and $Y \in \mathcal{F}_{\sigma-}$ then the process*

$$X = Y\,\mathbf{1}_{[\![\sigma,\tau[\![}\,\mathbf{1}_{\{\tau<\infty\}}$$

is predictable;

Definition 4.4. *An increasing sequence of markov moments $(\tau_n)_{n\geq 1}$ (i.e. $\tau_n \leq \tau_{n+1}$ and $(\tau_n)_{n\geq 1} \subset \mathcal{M}$) is called foreshadowing sequence for some random moment τ if the following condtions hold:*

1. for any $\omega \in \Omega$

$$\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)\,;$$

2. for any $n \geq 1$ *and* $\omega \in \{\omega \in \Omega \,:\, \tau(\omega) > 0\}$

$$\tau_n(\omega) < \tau(\omega)\,.$$

Theorem 4.2. *A random moment* $\tau \in \mathcal{M}_p$ *if and only if there exists a foreshadowing sequence of markovian moments* $(\tau_n)_{n\geq 1}$ *for* τ.

Theorem 4.3. *Any predictable process* X *can be represented as*

$$X = Y + \sum_{k\geq 1} \Delta\, X_{\tau_k}\, \mathbf{1}_{[\![\tau_k]\!]}\,,$$

where the sequence stopping times $\tau_k \in \mathcal{M}_p$.

Exercises

1. Show that any non random constant $t \geq 0$ belongs to $\tau + t \in \mathcal{M}_p$
2. Show that for any $\tau \in \mathcal{M}$ and any fixed $t > 0$ the moment $\tau + t \in \mathcal{M}_p$.

5 Martingales

Definition 5.1. *A stochastic process $M = (M_t)_{t\geq 0}$ is called* martingale, submartingale *or* supermartingale *if*

1. *M is adaptive;*

2. *M is integrated, i.e. $\mathbf{E}\,|X_t| < \infty$ for any $t \geq 0$;*

3. *for any $0 \leq s \leq t$*

$$\mathbf{E}\,(M_t|\mathcal{F}_s) = M_s;$$

$$\mathbf{E}\,(M_t|\mathcal{F}_s) \geq M_s;$$

$$\mathbf{E}\,(M_t|\mathcal{F}_s) \leq M_s\,.$$

Definition 5.2. *A stochastic process $M = (M_t)_{t\geq 0}$ is called* local martingale *if there exists increasing sequence τ_n with $\tau_n \to \infty$ as $n \to \infty$ such that the stopping process M^{τ_n} is martingale for any $n \geq 1$.*

Definition 5.3. *A martingale $M = (M_t)_{t\geq 0}$ is called* square integrated *if for any $t > 0$*

$$\mathbf{E}\,M_t^2 \,<\, \infty\,.$$

It should be noted that for any square integrated martingale M there exists

$$M_\infty = \lim_{t\to\infty} \mathbf{M}_t \qquad \text{a.s.},$$

such that $\mathbf{E}\, M_\infty^2 < \infty$ and $M_t = \mathbf{E}\,(M_\infty|\mathcal{F}_t)$ for any $t \geq 0$.

Definition 5.4. *A process* $X = (X_t)_{t\geq 0}$ *is called* process of the Dirichlet class *if the family* $(X_\tau)_{\tau\in\check{\mathcal{M}}}$ *is uniformly integrated, i.e.*

$$\lim_{a\to+\infty} \sup_{\tau\in\check{\mathcal{M}}} \mathbf{E}|X_\tau|\, \mathbf{1}_{\{|X_\tau|>a\}} = 0\,.$$

In the stochastic analysis the Doob – Meyer decomposition theorem plays the key role for the construction of the integrals.

Theorem 5.1. *Let X be a submartingale of the Dirichlet class. Then there exists unique predictable integrated process $A = (A_t)_{t\geq 0}$ with $A_0 = 0$ such that*

$$X_t = A_t + M_t$$

where $M = (M_t)_{t\geq 0}$ is an uniformly integrated martingale with $M_0 = X_0$, i.e.

$$\lim_{a\to+\infty} \sup_{t\geq 0} \mathbf{E}\,|M_t|\, \mathbf{1}_{\{|M_t|>a\}} = 0\,.$$

Let now $X = (X_t)_{t\geq 0}$ be a square integrated martingale. Then by the Doob – Meyer theorem

$$X_t^2 = A_t + M_t\,, \tag{5.1}$$

where A is a predictable process and M is a martingale. The predictable process A is called *quadratic characteristic or predictable variation* which is denoted as $< X, X >$ or $< X >$. Let now X and Y be two square integrated martingales. Then the *joint quadratic characteristic or predictable quadratic covariation* $< X, Y >$ is called the predictable process defined as

$$< X, Y >_t = \frac{1}{4}\left(< X + Y >_t - < X - Y >_t\right)\,. \tag{5.2}$$

Definition 5.5. *A process $W = (W_t)_{t\geq 0}$ is called brownian motion if the following properties hold:*

1. *the process W is a.s. continuous and $W_0 = 0$;*
2. *the process W is a process with the independent increments, i.e. for any $m \geq 2$ and $0 = t_0 < t_1 < \ldots < t_m$ the random variables*

   $$W_{t_1}\,, W_{t_2} - W_{t_1}\,, \ldots, W_{t_m} - W_{t_{m-1}}$$

are jointly independent;

3. *for any* $0 \le s \le t$ *the random variable* $W_t - W_s$ *is gaussian with the parameters* $(0\,,\,t-s)$.

Note that the existence of such process can be shown by the Kolmogorov theorem 7.3. Indeed, to this end it suffices to define the family (7.3) as the family of the $(0, K_m)$ gaussian distributions in $\mathbb{R}^m$ with the correlation matrix

$$K_m = \big(\min(t_i, t_j)\big)_{1\le i,j\le m}\,,$$

i.e. for any $0 \le t_1 < \ldots < t_m$ the distributions $\Phi_{t_1,\ldots,t_m}$ is m dimensional gaussian with the parameters $(0, K_m)$.

Definition 5.6. *A process* $N = (N_t)_{t\ge 0}$ *is called homogeneous Poisson process of the intensity* $\lambda > 0$ *if for any* $t > 0$

$$N_t = \sum_{k\ge 1} \mathbf{1}_{\{\check{\epsilon}_1+\ldots+\check{\epsilon}_k\le t\}}\,, \tag{5.3}$$

where $(\check{\epsilon}_k)_{k\ge 1}$ *i.i.d.* λ *exponential random variables.*

In the sequel we denote by the $(\mathcal{F}^W_t)_{t\ge 0}$ and $(\mathcal{F}^N_t)_{t\ge 0}$ the corresponding filtrations with $\mathcal{F}^W_t = \sigma\{W_u\,,\, u\le t\}$ and $\mathcal{F}^N_t = \sigma\{N_u\,,\, u\le t\}$.

Exercises

1. Let $M = (M_t)_{t\geq 0}$ be a square integrated martingale. Show that $X_t = M_t^2$ is submartingale;

2. Let ξ be a some integrated random variable. Show that $X_t = \mathbf{E}(\xi|\mathcal{F}_t)$ is martingale.

3. Show the Doob - Meyer decomposition theorem for the discrete time.

4. Let $W = (W_t)_{t\geq 0}$ be brownian motion.

   (a) Show that the processes W_t with respect to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$.

   (b) Show that the process $X_t = W_t^2 - t$ is martingale with respect to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$.

   (c) Calculate the quadratic characteristics $< W >$ and $< X >$.

5. Let N be a homogeneous Poisson process of the intensity $\lambda > 0$. Show that the process $N_t - \lambda t$ is martingale with respect to the filtration $(\mathcal{F}_t^N)_{t\geq 0}$.

6. Let N be a homogeneous Poisson process of the intensity $\lambda > 0$ and $(\xi_k)_{k\geq 1}$ be i.i.d. sequence of the gaussian $(0,1)$ random

variables. We set

$$Z_t = \sum_{j=1}^{N_t} \xi_j \,. \tag{5.4}$$

(a) Show that the process Z is a martinagale with respect to the filtration $(\mathcal{F}_t^Z)_{t\ge 0}$ with $\mathcal{F}_t^Z = \sigma\{Z_u\,,\, u \le t\}$.

(b) Calculate the quadratic characteristic $< Z >$.

6 Stochastic integral

Let now $M = (M_t)_{t\ge 0}$ be a square integrated martingale. Let $H = (H_t)_{t\ge 0}$ be a simple predictable process, i.e.

$$H_t = \alpha_1\, \mathbf{1}_{[0,t_1]} + \sum_{j=2}^{n} \alpha_j\, \mathbf{1}_{\{]t_{j-1},t_j]\}} \,, \tag{6.1}$$

where $0 = t_0 < t_1 < \ldots < t_n$ are fixed nonrandom moments and the random variable α_j is measurable with respect to $\mathcal{F}_{t_{j-1}}$. Moreover, we assume that

$$\mathbf{E} \int_0^T \, H_t^2 \mathrm{d} < M >_t < \,\infty \,. \tag{6.2}$$

In this case the stochastic integral with respect to the martingale

M is defied as

$$\int_0^T H_t \mathrm{d}M_t = \sum_{j=1}^n \alpha_j \left(M_{t_j} - M_{t_{j-1}}\right). \tag{6.3}$$

It should be noted (see, for example, [5], p.82) that for any predictable process $H = (H_t)_{t\geq 0}$ which satisfies the condition (6.2) there exists a sequence of simple square integrated predictable processes $H^n = (H^n_t)_{t\geq 0}$, i.e. processes which satisfies the conditions (6.1) and (6.2) such that

$$\lim_{n\to\infty} \mathbf{E} \int_0^T \left(H_t - H^n_t\right)^2 \mathrm{d}t = 0\,.$$

In this case the stochastic integral with respect to the martingale M is defined as

$$\int_0^T H_t \mathrm{d}M_t = l.i.m. \int_0^T H^n_t \mathrm{d}M_t\,, \tag{6.4}$$

i.e.

$$\lim_{n\to\infty} \mathbf{E} \left(\int_0^T H_t \mathrm{d}M_t - \int_0^T H^n_t \mathrm{d}M_t\right)^2 = 0\,.$$

One can show (see, for example, [5], p.82) that the limit in (6.4) is the same for any approximate sequence $(H^n)_{n\geq 1}$, i.e. for any predictable square integrated process the stochastic integral with the martingale M is correctly defined by the limit (6.4).

For any integrated predictable process $H = (H_t)_{t\geq 0}$ we define the process $X = (X_t)_{t\geq 0}$ as

$$X_t = \int_0^t H_s \mathrm{d} M_s \,. \tag{6.5}$$

Properties of the stochastic integrals.

1.

$$\mathbf{E}\, X_t = 0 \,.$$

2.

$$\mathbf{E}\, X_t^2 = \mathbf{E} \int_0^t H_s^2 \mathrm{d} < M >_s \;.$$

3.

$$< X >_t = \int_0^t H_s^2 \mathrm{d} < M >_s \;.$$

4. For any $t > 0$

$$\Delta X_t = H_t \Delta M_t \,.$$

5. The process $X = (X_t)_{t\geq 0}$ is the martingale.

Let f be a twice continuous differentiable $\mathbb{R} \to \mathbb{R}$ function. Let

$M = (M_t)_{t\geq 0}$ be some square integrated martingale such that

$$M_t = M_t^c + M_t^d\,, \tag{6.6}$$

where $M^c = (M_t^c)_{t\geq 0}$ is continuous martingale and $M^d = (M_t^d)_{t\geq 0}$ is the pure discret martingale defined as

$$M_t^d = M_0^d + \sum_{0\leq s\leq t} \Delta M_s - A_t^d\,, \tag{6.7}$$

where $A^d = (A_t^d)_{t\geq 0}$ is some predictable process with the finite variation on the finite interval $[0,T]$ (see, for example, [5], Theorem 4, p. 44) .

Theorem 6.1. *Assume that*

$$\mathbf{E}\int_0^T \left(f'(M_{t-})\right)^2 \mathrm{d} < M >_t < \infty$$

and

$$\int_0^T \left|f''(M_{t-})\right| \mathrm{d} < M^c >_t < \infty \quad a.s..$$

Then

$$f(M_T) = f(M_0) + \int_0^T f'(M_{t-})\,\mathrm{d}M_t + \frac{1}{2}\int_0^T f''(M_{t-})\mathrm{d} < M^c >_t \tag{6.8}$$

$$+ \sum_{0 \le t \le T} \left(f(M_t) - f(M_{t-}) - f'(M_{t-})\,\Delta M_t \right) \,,$$

where $< M^c >$ *is the quadratic characteristic of the continuous martingale* M^c *defined in* (6.6).

Exercises

1. Show the properties 1)–5) of stochastic integrals.

2. We set
$$X_t = \int_0^t N_{s-}\mathrm{d}Z_s \,, \tag{6.9}$$
where $Z = (Z_t)_{t \ge 0}$ is the martingale defined in (5.4) and $N = (N_t)_{t \ge 0}$ is the Poisson process. Calculate $< X >$.

3. Write the Ito formula for the function $f(x) = x^4$ for the process (6.9).

4. Let now
$$M_t = W_t + X_t \,, \tag{6.10}$$

where $W = (W_t)_{t\geq 0}$ is the Wiener process and $X = (X_t)_{t\geq 0}$ is the process (6.9).

(a) Find the martingales M^c and M^d in the decomposition (6.7).

(b) Write the Ito formula for the martingale (6.10) for the functions $f(x) = x^2$ and $f(x) = x^4$.

7 Appendix

7.1 Carathéodory's extension theorem

In this section we consider some fixed universal set Ω.

Definition 7.1. *For a given Ω the family $\mathcal{S}$ of the subsets of Ω is called semi-ring if the following properties hold*

- $\emptyset \in \mathcal{S}$*;*
- *for all $A, B \in \mathcal{S}$ the intersection $A \cap A \in \mathcal{S}$ (closed under pairwise intersections);*
- *for all $A, B \in \mathcal{S}$ there exist disjoints sets $(D_i)_{1 \leq i \leq n}$ from $\mathcal{S}$ such that $A \backslash B = \cup_{j=1}^{n} D_i$ (relative complements can be written as finite disjoint unions).*

Definition 7.2. *For a given Ω the family $\mathcal{R}$ of the subsets of Ω is called ring if the following properties hold*

- $\emptyset \in \mathcal{R}$*;*
- *for all $A, B \in \mathcal{R}$ the intersection $A \cup B \in \mathcal{R}$ (closed under pairwise unions);*
- *for all $A, B \in \mathcal{R}$ we have $A \setminus B \in \mathcal{R}$ (closed under relative complements).*

Examples

1. Let $(\mathcal{X}_j\,,\,\mathcal{B}_j)_{1\le j\le n}$ be n measurable spaces, i.e. $\mathcal{X}_i$ be some sets with σ - fields $\mathcal{B}_i$.

   In this case the family $\mathcal{S}=\{A=D_1\times\ldots\times D_n, D_i\in\mathcal{B}_i\}$ is the semi-ring in $\Omega=\mathcal{X}_1\times\ldots\times\mathcal{X}_n$.

2. Let $\mathcal{S}$ is a semi - ring in Ω. Then family

$$\mathcal{R}=\Big\{A\,:\,A=\cup_{j=1}^n D_j\quad\text{with the}\quad D_j\in\mathcal{S}\Big\}$$

   is the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$.

3. Let $(\mathcal{F}_j)_{1\le j\le n}$ be σ - fields in Ω. In this case

$$\mathcal{R}=\cup_{j=1}^n\mathcal{F}_j$$

   is the ring.

Definition 7.3. *Let $\mathcal{R}$ be a ring and let $\nu\,:\,\mathcal{R}\to[0,+\infty]$ be a set - function. This function is called pre-measure if*

- $\nu(\emptyset)=0$;
- *for any countable (finite) sequence $(A_j)_{j\in J}$ (with $A_j\subseteq\mathcal{R}$ for*

$j \in J$) pairwise disjoint sets whose union lies in $\mathcal{R}$

$$\nu\left(\cup_{j\in J} A_j\right) = \sum_{j\in J} \nu\left(A_j\right) \, .$$

Theorem 7.1. *(Carathéodory's extension theorem) Let $\mathcal{R}$ be a ring on some set Ω and μ be a pre-measure on $\mathcal{R}$. Then there exists a measure $\breve{\mu}$ on the σ - field generated by the ring $\mathcal{R}$ such that $\breve{\mu}(A) = \mu(A)$ for any $A \in \mathcal{R}$.*

7.2 Radon – Nikodym theorem

In this subsection we consider the measurable space $(\Omega, \mathcal{F})$, where Ω is some set and $\mathcal{F}$ is a σ - field of the subsets of Ω.

Definition 7.4. *Un positive measure ν on $\mathcal{F}$ is called σ - finite if there exists a sequence of disjoint sets $(A_n)_{n\geq 1}$ from $\mathcal{F}$ such that*

$$\Omega = \cup_{n\geq 1} A_n \quad \textit{and} \quad \nu(A_n) < \infty$$

for any $n \geq 1$.

Definition 7.5. *Let ν and μ be two positive measures on $\mathcal{F}$. The measure ν is called absolutely continuous with respect to the measure μ if for any $A \in \mathcal{F}$ for which $\mu(A) = 0$ the measure $\nu(A) = 0$. We write in this case $\nu \ll \mu$. The, measures ν and μ are called* *equivalents if simultaneously $\nu \ll \mu$ and $\mu \ll \nu$. In this case we write $\nu \sim \mu$.*

Theorem 7.2. *(Radon – Nikodym theorem) Let ν and μ be two positive σ - finite measures on $\mathcal{F}$ such that $\nu \ll \mu$. Then there exists a measurable $\Omega \to [0, +\infty[$ function f such that for any $A \in \mathcal{F}$*

$$\nu(A) = \int_A f \mathrm{d}\mu \,. \tag{7.1}$$

The function f in (7.1) is called the Radon - Nikodym derivative and denoted as

$$f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,.$$

Note that, if ν is a probability measure and μ is the Lebesgue measure in $\mathbb{R}^n$ then the function f is called the *probability density.*

7.3 Kolmogorov theorem

Let now $\mathcal{X}$ be the set of all $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+) \to (\mathbb{R}\,, \mathcal{B}(\mathbb{R}))$ functions, where $\mathcal{B}(\mathbb{R}_+)$ and $\mathcal{B}(\mathbb{R})$ are the Borel σ - fields on $\mathbb{R}_+$ and on $\mathbb{R}$ correspondently. For any $0 \le t_1 < \ldots < t_m$ and any Borel sets $\Gamma_1 \in \mathcal{B}(\mathbb{R})\,, \ldots, \Gamma_m \in \mathcal{B}(\mathbb{R})$ the set in $\mathcal{X}$

$$C_{t_1,\ldots t_m}(\Gamma_1) = \{x \in \mathcal{X} \,:\, x(t_1) \in \Gamma_1\,, \ldots, x(t_m) \in \Gamma_m\} \tag{7.2}$$

is called *cylinder set.* We denote by $\mathcal{C}$ the family of all cylinder sets in $\mathcal{X}$ and by $\mathcal{F}$ the σ - field generated by this family, i.e. $\mathcal{F} = \sigma\{\mathcal{C}\}$. Let now

$$\left(\left(\Phi_{t_1,\ldots,t_m}\right)_{(t_1,\ldots,t_m)\in\mathbb{R}_+^m}\right)_{m\geq 1} \tag{7.3}$$

be a family of the finite dimensional distributions $\Phi_{t_1,\ldots,t_m}$ on $\mathcal{B}(\mathbb{R}^m)$.

$\mathbf{C}_1$) *Assume that for any $0 \leq t_1 < \ldots < t_m$ and any Borel sets $\Gamma_1 \in \mathcal{B}(\mathbb{R}), \ldots, \Gamma_m \in \mathcal{B}(\mathbb{R})$*

$$\Phi_{t_1,\ldots,t_m}(\Gamma_1 \times \ldots \times \Gamma_m) = \Phi_{t_{j_1},\ldots,t_{j_m}}(\Gamma_{j_1} \times \ldots \times \Gamma_{j_m}).$$

$\mathbf{C}_2$) *Assume that for any $0 \leq t_1 < \ldots << t_m < t_{m+1}$ and any Borel sets $\Gamma_1 \in \mathcal{B}(\mathbb{R}), \ldots, \Gamma_m \in \mathcal{B}(\mathbb{R})$*

$$\Phi_{t_1,\ldots,t_m,t_{m+1}}(\Gamma_1 \times \ldots \times \Gamma_m \times \mathbb{R}) = \Phi_{t_1,\ldots,t_m}(\Gamma_1 \times \ldots \times \Gamma_m).$$

Theorem 7.3. *(Kolmogorov theorem) Assume that the family of the finite dimensional distributions* (7.3) *satisfies the conditions* $\mathbf{C}_1$) *and* $\mathbf{C}_2$). *Then there exists an unique measure μ on the σ - field $\mathcal{F}$ such that for any $0 \leq t_1 < \ldots < t_m$ and any Borel sets $\Gamma_1 \in$*

$\mathcal{B}(\mathbb{R}), \ldots, \Gamma_m \in \mathcal{B}(\mathbb{R})$

$$\mu(\Gamma_1 \times \ldots \times \Gamma_m) = \Phi_{t_1,\ldots,t_m}(\Gamma_1 \times \ldots \times \Gamma_m)\,.$$

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