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# Non-crossing convex quantile regression

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A R T I C L E I N F O

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A B S T R A C T

Quantile crossing is a common phenomenon in shape constrained nonparametric quantile regression. A direct approach to address this problem is to impose non-crossing constraints to convex quantile regression. However, the non-crossing constraints may violate an intrinsic quantile property. This paper proposes a penalized convex quantile regression approach that can circumvent quantile crossing while maintaining the quantile property. A Monte Carlo study demonstrates the superiority of the proposed penalized approach in addressing the quantile crossing problem.

## 1. Introduction

Quantile estimation has been widely applied in various fields of economics and econometrics (see, e.g., Wang et al., 2014; Jradi et al., 2019; Tsionas et al., 2020; Kuosmanen and Zhou, 2021; Zhao, 2021). However, when multiple quantiles are separately estimated to obtain a family of conditional quantile functions, two or more quantile curves may cross on the condition that the distribution functions and their associated inverse functions are not monotone increasing (He, 1997). Such quantile crossing is a longstanding problem in quantile regression.

There are at least three commonly used approaches to avoid quantile crossing: post-processing, stepwise estimation, and simultaneous estimation. In the post-processing procedure, a non-crossing assumption is usually enforced via a sorting or monotonic rearrangement of the original estimated non-monotone functions (e.g., Dette and Volgushev, 2008; Chernozhukov et al., 2010). While this indirect approach is effective in estimating the conditional quantile, it lacks the ability to quantify the effects of the predictors (Bondell et al., 2010). The stepwise procedure prevents an estimated quantile function from crossing the previously estimated one by adding an extra set of non-crossing constraints iteratively to the regression model (e.g., Liu and Wu, 2009). However, this approach is subject to path dependence; the results may change depending on which quantile is estimated first. In the simultaneous estimation, non-crossing constraints are imposed to ensure that the estimated conditional quantile functions are monotone nondecreasing, with all quantiles being estimated simultaneously (e.g., Takeuchi et al., 2006; Bondell et al., 2010). More recently, Wang et al. (2014) extend this simultaneous estimation technique to convex quantile regression (sCQR). However, the non-crossing constraints may violate the quantile property (Takeuchi et al., 2006).

This paper proposes a new regularization approach to avoid quantile crossing, which is guaranteed to satisfy the quantile property. For brevity, we focus on the case of nonparametric quantile regression, but the general approach readily extends to other nonparametric and parametric quantile regression techniques. The main advantage compared to the existing sCQR approach (Wang et al., 2014) is that our proposed penalized convex quantile regression (pCQR) approach is guaranteed to satisfy the intrinsic quantile property. Furthermore, the proposed pCQR approach performs better than sCQR in Monte Carlo simulations.

## 2. Penalized convex quantile regression

Consider a general nonparametric regression model with observations $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ satisfying

$$y_i = f(\boldsymbol{x}_i) + \varepsilon_i, \qquad \text{for } i = 1, \dots, n, \tag{1}$$

where $y_i \in \mathbb{R}$ and $\boldsymbol{x}_i \in \mathbb{R}^d$ are output and inputs variables, and $\varepsilon_i$ is an error term with zero mean. Accordingly, for a given quantile $\tau \in (0, 1)$, the nonparametric quantile function $Q_y(\tau \mid \boldsymbol{x})$ is defined as

$$Q_{y_i}(\tau \mid \boldsymbol{x}_i) = f(\boldsymbol{x}_i) + F_{\varepsilon_i}^{-1}(\tau), \tag{2}$$

where $F_{\varepsilon_i}$ is the distribution function of the error term $\varepsilon_i$.

To estimate quantiles empirically, we resort to convex quantile regression (CQR) that does not require any assumptions about the functional form of the regression function $f$ or its smoothness, but imposes the shape constraints such as monotonicity and concavity. Specifically, CQR estimates the quantile function (2) by solving the

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following linear programming problem (Wang et al., 2014)

$$\min_{\alpha,\beta,\varepsilon^+,\varepsilon^-} \tau \sum_{i=1}^{n} \varepsilon_i^+ + (1-\tau)\sum_{i=1}^{n} \varepsilon_i^- \tag{3}$$
$$\begin{aligned} s.t.\quad & y_i = \alpha_i + \beta_i' x_i + \varepsilon_i^+ - \varepsilon_i^- && \forall i \\ & \alpha_i + \beta_i' x_i \le \alpha_h + \beta_h' x_i && \forall i, h \\ & \beta_i \ge \mathbf{0} && \forall i \\ & \varepsilon_i^+ \ge 0,\ \varepsilon_i^- \ge 0 && \forall i \end{aligned}$$

where the first set of constraints can be interpreted as a multivariate regression equation, the second set of constraints imposes concavity on the quantile function, the third set of constraints (i.e., a system of Afriat inequalities) guarantees monotonicity, and the last refers to sign constraints of the error terms. Note that there exists an intrinsic quantile property in terms of the optimal solutions to problem (3), $\hat{\varepsilon}_i^+$ and $\hat{\varepsilon}_i^-$.

**Theorem 1.** *For any $\tau \in (0,1)$, the number of strict positive residuals ($\hat{\varepsilon}_i^+ > 0$) by $n_\tau^+$ and the number of strict negative residuals ($\hat{\varepsilon}_i^- > 0$) by $n_\tau^-$ always satisfy the inequalities:*

$$\frac{n_\tau^+}{n} \le 1-\tau \quad \text{and} \quad \frac{n_\tau^-}{n} \le \tau.$$

**Proof.** See proofs in Wang et al. (2014) and Kuosmanen and Zhou (2021).

Compared with the conventional full frontier estimation, the quantile function estimation is more robust to random noise, heteroscedasticity, and the choice of direction vectors. However, when separately estimating each conditional quantile function $Q_y(\tau \mid x)$, CQR is likely to violate the assumption that the distribution functions and their associated inverse functions should be monotone nondecreasing; see Fig. B.1 for an example of the quantile crossing problem detected in our empirical application of CQR.

We notice that the quantile crossing problem could be addressed by simultaneous estimation, which imposes an extra set of linear non-crossing constraints in the CQR approach (see, e.g., Takeuchi et al., 2006; Wang et al., 2014). Following Wang et al. (2014), the simultaneous convex quantile regression (sCQR) estimator of $j$ conditional quantile functions at $0 < \tau_1 < \tau_2 < \cdots < \tau_j < 1$ is formulated as

$$\min_{\alpha,\beta,\varepsilon^+,\varepsilon^-,C} \sum_{j=1}^{\mathcal{J}} \Big( \tau_j \sum_{i=1}^{n} \varepsilon_{i,j}^+ + (1-\tau_j)\sum_{i=1}^{n} \varepsilon_{i,j}^- \Big) \tag{4}$$
$$\begin{aligned} \text{s.t.}\quad & y_{i,j} = \alpha_{i,j} + \beta_{i,j}' x_i + \varepsilon_{i,j}^+ - \varepsilon_{i,j}^- && \forall i, j \\ & \alpha_{i,j} + \beta_{i,j}' x_i \le \alpha_{h,j} + \beta_{h,j}' x_i && \forall i, h, j \\ & \alpha_{i,j} + \beta_{i,j}' x_i + C_{i,j} \le \alpha_{i,j+1} + \beta_{i,j+1}' x_i && \forall i, 1 \le j \le \mathcal{J}-1 \\ & \beta_{i,j} \ge \mathbf{0} && \forall i, j \\ & \varepsilon_{i,j}^+ \ge 0,\ \varepsilon_{i,j}^- \ge 0 && \forall i, j \end{aligned}$$

where $C_{i,j} \ge 0$ are small nonnegative constants for quantiles, which are introduced in sCQR to ensure that $Q_y(\tau_j \mid x) \le Q_y(\tau_{j+1} \mid x), \forall i$ and $j \in \mathcal{J}$. For the purpose of non-crossing, $C$ can simply be given by zero; that is, there may exist touching rather than crossing between two neighboring quantiles (see Fig. 1(a) for an illustration). In practice, however, after enforcing the non-crossing constraints, sCQR may violate the quantile property (Theorem 1) due to the fact that the approach simultaneously optimizes for both the quantile property and the non-crossing property (Takeuchi et al., 2006).

This paper proposes an alternative to sCQR to address the quantile crossing problem. By using the $L_2$-norm regularization on subgradients $\beta_i$, we formulate penalized convex quantile regression (pCQR) as

$$\min_{\alpha,\beta,\varepsilon^+,\varepsilon^-} \tau \sum_{i=1}^{n} \varepsilon_i^+ + (1-\tau)\sum_{i=1}^{n} \varepsilon_i^- + \gamma \sum_{i=1}^{n} \|\beta_i\|_2^2 \tag{5}$$
$$\begin{aligned} s.t.\quad & y_i = \alpha_i + \beta_i' x_i + \varepsilon_i^+ - \varepsilon_i^- && \forall i \\ & \alpha_i + \beta_i' x_i \le \alpha_h + \beta_h' x_i && \forall i, h \\ & \beta_i \ge \mathbf{0} && \forall i \\ & \varepsilon_i^+ \ge 0,\ \varepsilon_i^- \ge 0 && \forall i \end{aligned}$$

where $\gamma \ge 0$ is the tuning parameter and $\|\cdot\|_2$ denotes the standard Euclidean norm. The $L_2$-norm regularization in pCQR can effectively help restrict extreme $\beta_i$, which will eventually eliminate quantile crossings. As $\gamma \to +\infty$, the regularization will dominate the minimization and then all estimated subgradients $\beta_i$ "flatten out" to 0. In such a case, the estimated quantile function will be a horizontal line (for $d = 1$) or hyperplane ($d > 1$). Therefore, when estimating multiple quantiles, the estimated quantile functions could be in parallel without crossings given a prespecified $\gamma$. Further, the optimal solutions to problem (5) also satisfy the quantile property.

**Theorem 2.** *For any prespecified $\gamma > 0$, the minimizer of (5) satisfies,*

$$\frac{n_\tau^+}{n} \le 1-\tau \quad \text{and} \quad \frac{n_\tau^-}{n} \le \tau.$$

**Proof.** See Appendix A.

While introducing $L_2$-norm in pCQR can to some extent avoid quantile crossings thanks to the uniqueness of subgradients $\hat{\beta}_i$ (cf. Waltrup et al., 2015), it is not always immune to the quantile crossing issue. We thus design the following Algorithm 1 to find the smallest $\gamma^*$ for which no quantile crossings occur.

**Algorithm 1:** Searching the minimal tuning parameter $\gamma^*$.

**Data:** $\{x_i, y_i\}_{i=1}^n \in \mathbb{R}^d \times \mathbb{R}$, $\tau_1$ and $\tau_2$ ($\tau_1 < \tau_2$)
```
1  out = 0 and γ = 0;
2  while out = 0 do
3      Solve problem (5) with quantiles τ1 and τ2, separately, to
       calculate Q̂_y(τ1 | x_i) and Q̂_y(τ2 | x_i);
4      if Σ_{i=1}^{n} 1{Q̂_y(τ1 | x_i) − Q̂_y(τ2 | x_i) ≤ 0} ≠ n then
5          Re-solve problem (5) using the updated γ;
6      else
7          out = 1;
8      γ = γ + 0.01;
```
**Result:** $\gamma^*$

The main idea of Algorithm 1 is to let $\gamma$ vary from zero to positive infinity and check if there are any crossings between two neighboring quantiles in each iteration. If yes, then add a fixed step to $\gamma$; otherwise, the algorithm stops. As $\gamma$ increases, the estimated two quantile functions will have the equivalent subgradient and then be in parallel at the point where the crossing occurs. Consider an extreme example: as $\gamma$ reaches positive infinity, all subgradients become zero; that is, the estimated quantile functions will be parallel lines or hyperplanes. Note that in most cases a small $\gamma$ suffices for avoiding crossings (see, e.g., Figs. 2 and 1(b)).

However, as $\gamma$ increases, there could be multiple possible $\gamma$ (i.e., $\{\gamma_1, \gamma_2, \dots\}$) that eliminate quantile crossings. Algorithm 1 finds the minimum of all these possible $\gamma$, that is, $\gamma^* = \min\{\gamma_1, \gamma_2, \dots\}$. Note that it is possible that $\gamma^*$ equals zero. For sufficiently small $\gamma$, the optimal solutions to (3) are also the optimal solutions to (5) due to the exact regularization property in convex quadratic programming problems (see Friedlander and Tseng, 2008).

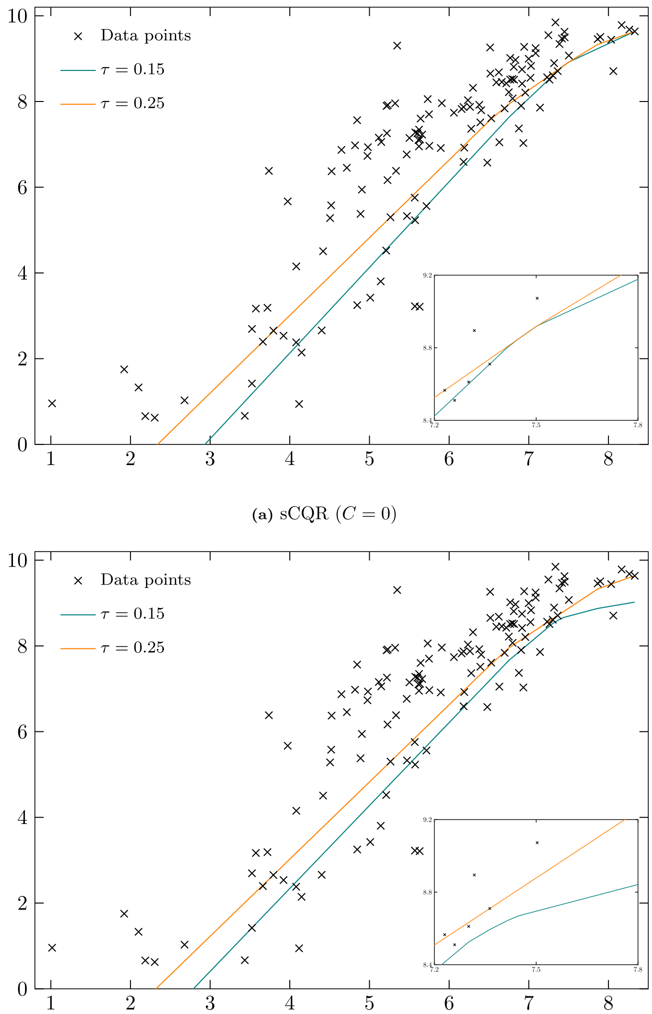
Besides $L_2$-norm, other generic norms can also be integrated into CQR to restrict the extreme $\beta_i$ and eliminate quantile crossings by following Algorithm 1. For example, one could adapt the $L_1$-norm based pCQR formulation by Dai (2023) to the present setting by replacing the objective function of (5) by

$$\min_{\alpha,\beta,\varepsilon^{+},\varepsilon^{-}} \tau \sum_{i=1}^{n} \varepsilon_i^{+} + (1-\tau) \sum_{i=1}^{n} \varepsilon_i^{-} + \gamma \sum_{i=1}^{n} \|\boldsymbol{\beta}_i\|_1 \tag{6}$$

Note that in the context of convex regression, $L_1$-norm can make certain $\hat{\beta}_{i,h}$ in (6) very small but cannot reduce them exactly to zero due to the existence of Afriat inequalities, even if an optimal $\gamma$ is prespecified (see, e.g., Xu et al., 2016; Dai, 2023). Here, $L_1$-norm is not necessarily superior to $L_2$-norm in terms of forcing $\hat{\boldsymbol{\beta}}_i$ to approach zeros as it is in the context of general regression. A holistic performance comparison among different norms in reducing crossings would be an interesting avenue for future research. In this paper we focus on $L_2$-norm, which can make pCQR invariant with respect to orthogonal transformations and proves sufficient for our purposes.

We proceed to illustrate how non-crossing quantile functions look like with a real dataset used in Kuosmanen and Zhou (2021) and Dai et al. (2023). It contains plant-level data on 130 U.S. electric power plants operating in 2014; see Kuosmanen and Zhou (2021) for a more detailed description of the data. For the sake of demonstration, we simply consider a univariate case of one input and one output. The input is the total cost involved in electricity production and the output is the net electricity generation of each power plant. Both variables are in natural logarithm.

An application of CQR to the empirical data finds that the 15th quantile curve crosses the 25th quantile curve twice (see Fig. B.1). We then demonstrate how the sCQR and pCQR approaches can address this problem. It is evident from Fig. 1 that both approaches manage to circumvent the quantile crossing problem; that is, we observe that $\hat{Q}_y(0.25 \mid \boldsymbol{x}_i)$ is greater than or equal to $\hat{Q}_y(0.15 \mid \boldsymbol{x}_i)$ in both approaches. However, the shapes of the estimated quantile functions in Figs. 1(a) and 1(b) (see particularly the upper right corner) are slightly different. As mentioned earlier, this difference arises because sCQR tries to simultaneously optimize for the non-crossing property, the quantile property, and the production axioms, whereas the pCQR approach independently estimates the quantile production functions. Further, the difference affects which approach can better retain the quantile property.




(a) sCQR ($C = 0$)

(b) pCQR ($\gamma^* = 0.01$)

**Fig. 1.** Empirical illustration of estimated non-crossing quantile functions using the U.S. power plant data.

## 3. Monte Carlo study

We perform a Monte Carlo study to examine whether pCQR or sCQR can better satisfy the quantile property while addressing quantile crossing. Consider the following data generating process (Dai, 2023)

$$y_i = \prod_{j=1}^{d} \boldsymbol{x}_{j,i}^{\frac{0.8}{d}} + v_i,$$

where the input matrix $\boldsymbol{x}_i \in \mathbb{R}^{n\times d}$ is generated independently from $U[1,10]$, random noise $v_i$ is drawn independently from $N(0,\sigma_v^2)$, and $d$ is the number of covariates.

We consider 54 scenarios with $n \in \{99,199,499\}$, $d \in \{2,3,4\}$, $\tau \in \{0.85,0.90,0.95\}$, and $\sigma \in \{0.5,1,2\}$. Each scenario is replicated 500 times using the pyStoNED package (Dai et al., 2021) on Python with the standard solver Mosek (9.3). We then compute the ramp loss (RL $= |\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{Q_i<\hat{Q}_y(\tau|x_i)} - \tau|$) (Takeuchi et al., 2006) to examine the quantile property and the mean squared error (MSE) to evaluate the finite-sample performance. Replications for which no quantile crossings happen (i.e., $\gamma^* = 0$) are excluded from the calculations of RL and MSE. Note that the smaller the ramp loss, the better the quantile performance.

Tables 1 and B.1 present the estimated ramp loss and MSE statistics across different scenarios. The results clearly show that compared with sCQR, pCQR has lower ramp loss in virtually all the scenarios and lower MSE in all the scenarios. This is because the minimal tuning parameter $\gamma^*$ used in the pCQR simulations mainly locates in the interval $[0.00, 0.15]$ that is closer to zero (see Fig. 2), suggesting that pCQR can better fit the true quantile functions. Several other findings are summarized as follows:

- The higher the number of covariates, the lower the ramp loss. As $d$ increases, the data space becomes more sparse, thereby indicating that the probability of crossing between two neighboring quantiles is relatively small.
- The differences in the ramp loss among quantiles in pCQR are smaller than those in sCQR due to the different estimation strategies, i.e., independent and simultaneous estimation, respectively.
- For both approaches, the MSE increases as more inputs are included and decreases as the sample size gets larger.
- The performance of both approaches in terms of MSE becomes worse as the variance of noise increases.

Overall, the pCQR approach can better fit the true quantile functions and satisfy the quantile property while at the same time addressing the quantile crossing problem. Regularizing the quantile function, instead of imposing an extra set of non-crossing constraints, proves a better remedy to quantile crossing according to our simulations.

## 4. Conclusions

In this paper, a penalized convex quantile regression approach has been developed to address the quantile crossing problem. The proposed

**Table 1**
Ramp loss and MSE of two neighboring quantiles with $n = 499$.

| $d$ | $\sigma$ | $\tau_1$ | $\tau_2$ | $RL_{\tau_1}$ | | $MSE_{\tau_1}$ | | $RL_{\tau_2}$ | | $MSE_{\tau_2}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | pCQR | sCQR | pCQR | sCQR | pCQR | sCQR | pCQR | sCQR |
| 2 | 0.5 | 0.85 | 0.90 | 0.324 | 0.355 | 0.012 | 0.019 | 0.370 | 0.394 | 0.015 | 0.024 |
| | | 0.90 | 0.95 | 0.371 | 0.399 | 0.014 | 0.023 | 0.397 | 0.414 | 0.021 | 0.035 |
| | 1 | 0.85 | 0.90 | 0.355 | 0.385 | 0.033 | 0.057 | 0.393 | 0.425 | 0.039 | 0.074 |
| | | 0.90 | 0.95 | 0.394 | 0.425 | 0.040 | 0.072 | 0.409 | 0.431 | 0.059 | 0.113 |
| | 2 | 0.85 | 0.90 | 0.420 | 0.440 | 0.105 | 0.165 | 0.456 | 0.479 | 0.115 | 0.213 |
| | | 0.90 | 0.95 | 0.465 | 0.482 | 0.111 | 0.204 | 0.473 | 0.497 | 0.155 | 0.341 |
| 3 | 0.5 | 0.85 | 0.90 | 0.259 | 0.307 | 0.021 | 0.038 | 0.294 | 0.319 | 0.026 | 0.050 |
| | | 0.90 | 0.95 | 0.255 | 0.297 | 0.028 | 0.053 | 0.255 | 0.267 | 0.036 | 0.081 |
| | 1 | 0.85 | 0.90 | 0.304 | 0.360 | 0.050 | 0.113 | 0.324 | 0.379 | 0.057 | 0.148 |
| | | 0.90 | 0.95 | 0.314 | 0.361 | 0.074 | 0.151 | 0.293 | 0.340 | 0.092 | 0.233 |
| | 2 | 0.85 | 0.90 | 0.374 | 0.428 | 0.131 | 0.293 | 0.375 | 0.439 | 0.150 | 0.394 |
| | | 0.90 | 0.95 | 0.395 | 0.457 | 0.187 | 0.418 | 0.369 | 0.428 | 0.240 | 0.663 |
| 4 | 0.5 | 0.85 | 0.90 | 0.213 | 0.269 | 0.031 | 0.069 | 0.238 | 0.249 | 0.037 | 0.092 |
| | | 0.90 | 0.95 | 0.181 | 0.230 | 0.041 | 0.089 | 0.179 | 0.160 | 0.054 | 0.139 |
| | 1 | 0.85 | 0.90 | 0.212 | 0.290 | 0.087 | 0.208 | 0.216 | 0.291 | 0.100 | 0.274 |
| | | 0.90 | 0.95 | 0.269 | 0.343 | 0.117 | 0.263 | 0.227 | 0.283 | 0.151 | 0.412 |
| | 2 | 0.85 | 0.90 | 0.335 | 0.418 | 0.217 | 0.517 | 0.312 | 0.413 | 0.249 | 0.712 |
| | | 0.90 | 0.95 | 0.380 | 0.443 | 0.305 | 0.698 | 0.319 | 0.400 | 0.391 | 1.137 |

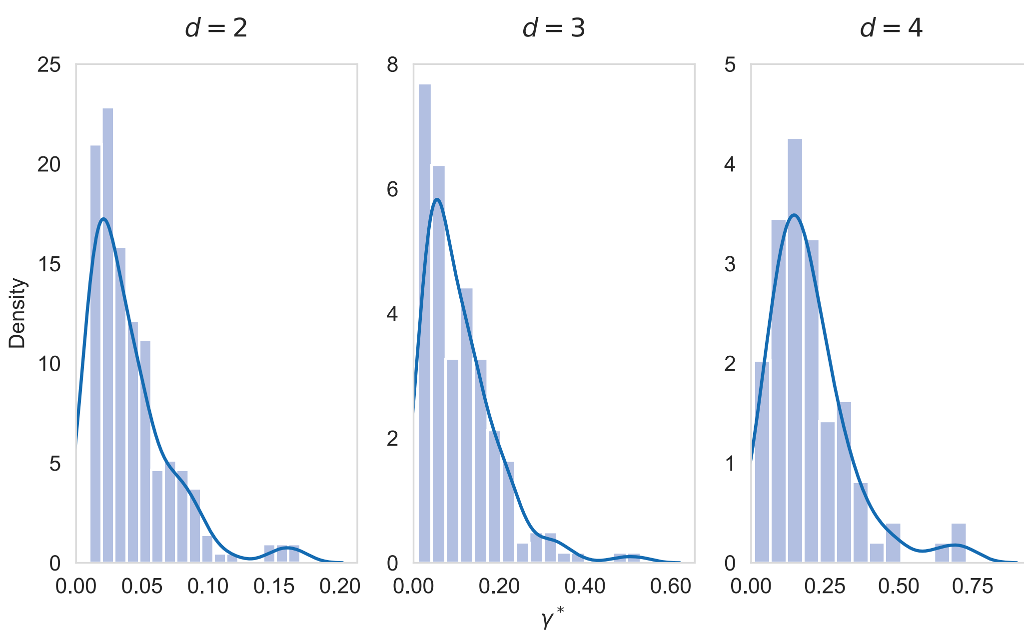



**Fig. 2.** Empirical distribution of $\gamma^*$ with $n = 499$, $\sigma = 0.5$, $\tau_1 = 0.85$, and $\tau_2 = 0.90$.

algorithm can search the minimal tuning parameter such that the occurrences of quantile crossing are avoided and the quantile property is ensured as well as possible. A Monte Carlo study confirms the superiority of the proposed approach compared to the existing sCQR approach in addressing the quantile crossing problem. We believe the regularization approach can be readily introduced to other nonparametric and parametric quantile regression approaches in addressing quantile crossings. Furthermore, extending the relevant statistical theory to CQR would also be useful. We leave those extensions as fascinating avenues for future research.

## Data availability

Data will be made available on request.

## Acknowledgments

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## Appendix A. Proof of theorem 2

The key for pCQR to satisfy the quantile property is that we do not impose the regularization on the constant $\alpha$.

Specifically, the objective function in problem (5) can be rewritten as (Koenker and Bassett, 1978)

$$\tau \sum_{n=1}^{n} \rho(y_i - \alpha_i - \boldsymbol{\beta}_i \boldsymbol{x}_i) + \gamma \sum_{i=1}^{n} \|\boldsymbol{\beta}_i\|_2^2$$

Let the objective function in problem (5) be $Z_{reg}[f] = \tau \sum_{n=1}^{n} \rho(y_i - f(\boldsymbol{x}_i)) + \gamma \sum_{i=1}^{n} \|g\|_2^2$, where $f = g + \alpha$ and $\alpha \in \mathbb{R}$. Assume that $f^*$ is the minimum of $Z_{reg}[f]$ with $f^* = g^* + \alpha^*$. If and only if $\alpha = \alpha^*$, $Z_{reg}[g^* + \alpha]$ can be minimized. Therefore, with respect to $\alpha$, minimizing the objective function $Z_{reg}[f]$ is equivalent to finding the quantile $\tau$ in terms of $y_i - g(\boldsymbol{x}_i)$. This is now the same case as in the CQR approach, and we can then follow Wang et al. (2014) to prove the statements in Theorem 2.

## Appendix B. Supplementary tables and figures

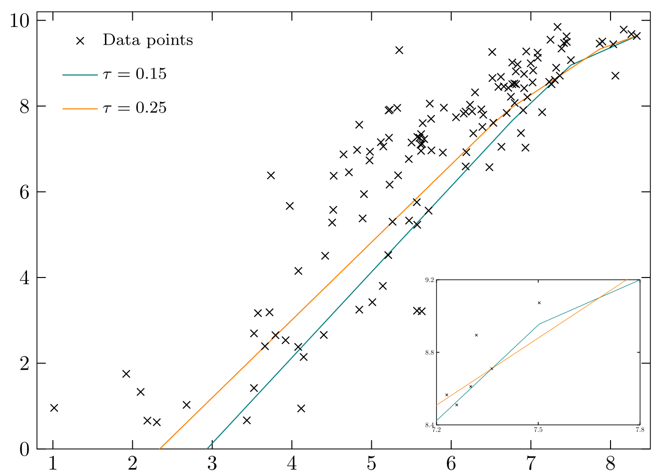


**Fig. B.1.** Illustration of the quantile crossing problem in CQR estimation.

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**Table B.1**
Ramp loss and MSE of two neighboring quantiles with $n = 99$ and 199.

| $n$ | $d$ | $\sigma$ | $\tau_1$ | $\tau_2$ | $RL_{\tau_1}$ | | $MSE_{\tau_1}$ | | $RL_{\tau_2}$ | | $MSE_{\tau_2}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | pCQR | sCQR | pCQR | sCQR | pCQR | sCQR | pCQR | sCQR |
| 99 | 2 | 0.5 | 0.85 | 0.90 | 0.308 | 0.324 | 0.045 | 0.063 | 0.323 | 0.334 | 0.055 | 0.081 |
| | | | 0.90 | 0.95 | 0.315 | 0.336 | 0.055 | 0.087 | 0.331 | 0.267 | 0.079 | 0.127 |
| | | 1 | 0.85 | 0.90 | 0.305 | 0.330 | 0.128 | 0.201 | 0.319 | 0.332 | 0.144 | 0.254 |
| | | | 0.90 | 0.95 | 0.319 | 0.354 | 0.158 | 0.256 | 0.297 | 0.326 | 0.204 | 0.386 |
| | | 2 | 0.85 | 0.90 | 0.381 | 0.415 | 0.340 | 0.559 | 0.381 | 0.414 | 0.386 | 0.723 |
| | | | 0.90 | 0.95 | 0.409 | 0.422 | 0.491 | 0.781 | 0.376 | 0.391 | 0.613 | 1.167 |
| | 3 | 0.5 | 0.85 | 0.90 | 0.220 | 0.228 | 0.072 | 0.114 | 0.196 | 0.230 | 0.090 | 0.152 |
| | | | 0.90 | 0.95 | 0.211 | 0.192 | 0.104 | 0.152 | 0.097 | 0.181 | 0.154 | 0.243 |
| | | 1 | 0.85 | 0.90 | 0.220 | 0.271 | 0.184 | 0.337 | 0.237 | 0.270 | 0.219 | 0.451 |
| | | | 0.90 | 0.95 | 0.224 | 0.259 | 0.235 | 0.448 | 0.170 | 0.157 | 0.322 | 0.716 |
| | | 2 | 0.85 | 0.90 | 0.316 | 0.359 | 0.471 | 0.902 | 0.283 | 0.342 | 0.557 | 1.228 |
| | | | 0.90 | 0.95 | 0.311 | 0.370 | 0.633 | 1.234 | 0.235 | 0.277 | 0.852 | 1.965 |
| | 4 | 0.5 | 0.85 | 0.90 | 0.131 | 0.125 | 0.128 | 0.174 | 0.080 | 0.120 | 0.168 | 0.237 |
| | | | 0.90 | 0.95 | 0.100 | 0.100 | 0.182 | 0.231 | 0.042 | 0.063 | 0.269 | 0.366 |
| | | 1 | 0.85 | 0.90 | 0.188 | 0.225 | 0.280 | 0.533 | 0.167 | 0.182 | 0.340 | 0.697 |
| | | | 0.90 | 0.95 | 0.173 | 0.203 | 0.399 | 0.704 | 0.125 | 0.105 | 0.548 | 1.095 |
| | | 2 | 0.85 | 0.90 | 0.258 | 0.318 | 0.752 | 1.440 | 0.221 | 0.283 | 0.947 | 1.964 |
| | | | 0.90 | 0.95 | 0.237 | 0.296 | 1.041 | 2.048 | 0.165 | 0.192 | 1.385 | 3.172 |
| 199 | 2 | 0.5 | 0.85 | 0.90 | 0.299 | 0.322 | 0.026 | 0.038 | 0.345 | 0.347 | 0.033 | 0.048 |
| | | | 0.90 | 0.95 | 0.324 | 0.339 | 0.031 | 0.047 | 0.327 | 0.329 | 0.042 | 0.071 |
| | | 1 | 0.85 | 0.90 | 0.325 | 0.353 | 0.072 | 0.117 | 0.362 | 0.387 | 0.082 | 0.148 |
| | | | 0.90 | 0.95 | 0.372 | 0.403 | 0.082 | 0.146 | 0.350 | 0.380 | 0.107 | 0.219 |
| | | 2 | 0.85 | 0.90 | 0.376 | 0.409 | 0.194 | 0.318 | 0.376 | 0.416 | 0.216 | 0.408 |
| | | | 0.90 | 0.95 | 0.417 | 0.449 | 0.268 | 0.458 | 0.377 | 0.426 | 0.323 | 0.684 |
| | 3 | 0.5 | 0.85 | 0.90 | 0.255 | 0.283 | 0.041 | 0.071 | 0.262 | 0.290 | 0.051 | 0.092 |
| | | | 0.90 | 0.95 | 0.237 | 0.250 | 0.053 | 0.094 | 0.181 | 0.234 | 0.078 | 0.149 |
| | | 1 | 0.85 | 0.90 | 0.255 | 0.316 | 0.105 | 0.219 | 0.272 | 0.313 | 0.125 | 0.288 |
| | | | 0.90 | 0.95 | 0.290 | 0.342 | 0.150 | 0.306 | 0.241 | 0.259 | 0.198 | 0.470 |
| | | 2 | 0.85 | 0.90 | 0.333 | 0.390 | 0.270 | 0.555 | 0.334 | 0.392 | 0.296 | 0.740 |
| | | | 0.90 | 0.95 | 0.363 | 0.415 | 0.398 | 0.743 | 0.322 | 0.384 | 0.484 | 1.182 |
| | 4 | 0.5 | 0.85 | 0.90 | 0.177 | 0.201 | 0.057 | 0.112 | 0.159 | 0.201 | 0.069 | 0.151 |
| | | | 0.90 | 0.95 | 0.153 | 0.178 | 0.082 | 0.166 | 0.094 | 0.150 | 0.103 | 0.260 |
| | | 1 | 0.85 | 0.90 | 0.194 | 0.265 | 0.178 | 0.355 | 0.203 | 0.253 | 0.199 | 0.461 |
| | | | 0.90 | 0.95 | 0.209 | 0.261 | 0.222 | 0.464 | 0.153 | 0.162 | 0.295 | 0.728 |
| | | 2 | 0.85 | 0.90 | 0.293 | 0.375 | 0.399 | 0.935 | 0.264 | 0.349 | 0.467 | 1.271 |
| | | | 0.90 | 0.95 | 0.298 | 0.377 | 0.566 | 1.273 | 0.214 | 0.290 | 0.754 | 2.055 |