DEFINITE INTEGRALS

The definite integral of a real-valued function f(x) with respect to a real variable x on an interval [a, b] is expressed as:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Here,

$\int$ = Integration symbol

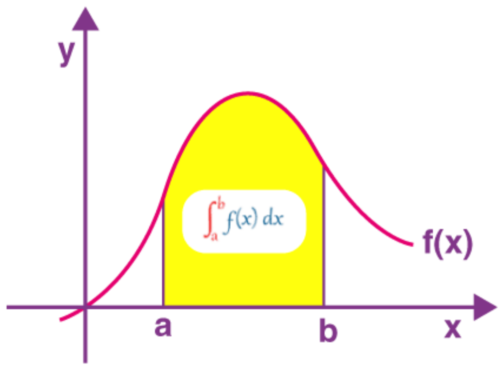
a = Lower limit

b = Upper limit

f(x) = Integrand

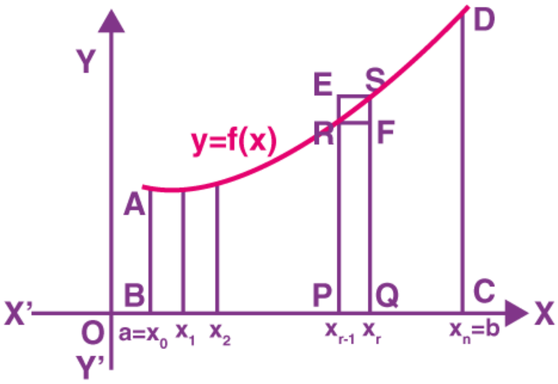
dx = Integrating agent

Thus, $\int_a^b f(x)\, dx$ is read as the definite integral of f(x) with respect to dx from a to b.



Definite Integral as Limit of Sum

The definite integral of any function can be expressed either as the limit of a sum or if there exists an antiderivative F for the interval [a, b], then the definite integral of the function is the difference of the values at points a and b. Let us discuss definite integrals as a limit of a sum. Consider a continuous function f in x defined in the closed interval [a, b]. Assuming that f(x) > 0, the following graph depicts f in x.



The integral of f(x) is the area of the region bounded by the curve y = f(x). This area is represented by the region ABCD as shown in the above figure. This entire region lying between [a, b] is divided into n equal subintervals given by $[x_0, x_1]$, $[x_1, x_2]$, …… $[x_{r-1}, x_r]$, $[x_{n-1}, x_n]$.

Let us consider the width of each subinterval as h such that $h \to 0$, $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$,…..,$x_r = a + rh$, $x_n = b = a + nh$

and n = (b – a)/h

Also, $n\to\infty$ in the above representation.

Now, from the above figure, we write the areas of particular regions and intervals as:

Area of rectangle PQFR < area of the region PQSRP < area of rectangle PQSE ….(1)

Since, $h\to 0$, therefore $x_r - x_{r-1} \to 0$. Following sums can be established as;

$$s_n = h\,[f(x_0) + \dots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r)$$

$$S_n = h[f(x_1) + f(x_2) + \dots + f(x_n)] = h\sum_{r=1}^{n} f(x_r)$$

From the first inequality, considering any arbitrary subinterval $[x_{r-1}, x_r]$ where r = 1, 2, 3….n, it can be said that, $s_n <$ area of the region ABCD $< S_n$

Since, $n\to\infty$, the rectangular strips are very narrow, it can be assumed that the limiting values of s_n and S_n are equal and the common limiting value gives us the area under the curve, i.e.,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{Area of the region ABCD} = \int_a^b f(x)dx$$

From this, it can be said that this area is also the limiting value of an area lying between the rectangles below and above the curve. Therefore,

$$\int_a^b f(x)dx = \lim_{h \to 0} h[f(a) + f(a+h) + ... + f(a+(n-1)h]$$

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + ... + f(a+(n-1)h]$$

where,

$$h = \frac{b-a}{n} \to 0 \; as \; n \to \infty$$

This is known as the definition of definite integral as the limit of sum.

Example 1: Evaluate the value of $\int_2^3 x^2\, dx$.

Solution:

Let I = $\int_2^3 x^2\, dx$

Now, $\int x^2\, dx = (x^3)/3$

Now, I = $\int_2^3 x^2\, dx = [(x^3)/3]_2^3$

= $(3^3)/3 – (2^3)/3$

= (27/3) – (8/3)

= (27 – 8)/3

= 19/3

Therefore, $\int_2^3 x^2\, dx = 19/3$

Example 2: Calculate: $\int_0^{\pi/4} \sin 2x\, dx$

Solution:

Let I = $\int_0^{\pi/4} \sin 2x\, dx$

Now, $\int \sin 2x\, dx = -(½) \cos 2x$

I = $\int_0^{\pi/4} \sin 2x\, dx$

= $[-(½) \cos 2x]_0^{\pi/4}$

= -(½) cos 2(π/4) – {-(½) cos 2(0)}

= -(½) cos π/2 + (½) cos 0

= -(½) (0) + (½)

= 1/2

Therefore, $\int_0^{\pi/4} \sin 2x\, dx = 1/2$

Properties of Definite Integrals

Properties	Description
Property 1	${}_p\int^q f(a)\, da = {}_p\int^q f(t)\, dt$
Property 2	${}_p\int^q f(a)\, d(a) = -\,{}_q\int^p f(a)\, d(a)$, Also ${}_p\int^p f(a)\, d(a) = 0$
Property 3	${}_p\int^q f(a)\, d(a) = {}_p\int^r f(a)\, d(a) + {}_r\int^q f(a)\, d(a)$
Property 4	${}_p\int^q f(a)\, d(a) = {}_p\int^q f(p + q - a)\, d(a)$
Property 5	${}_o\int^p f(a)\, d(a) = {}_o\int^p f(p - a)\, d(a)$
Property 6	$\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_0^p f(2p-a)da$ … if $f(2p-a) = f(a)$
Property 7	2 Parts • $\int_0^2 f(a)da = 2\int_0^a f(a)\, da$ … if $f(2p-a) = f(a)$ • $\int_0^2 p\, f(a)da = 0$ … if $f(2p-a) = -f(a)$
Property 8	2 Parts • $\int_{-p}^p f(a)da = 2\int_0^p f(a)\, da$ … if $f(-a) = f(a)$ or it's an even function • $\int_{-p}^p f(a)da = 0$ … if $f(2p-a) = -f(a)$ or it's an odd function

Properties of Definite Integrals Proofs

Property 1: ${}_p\int^q f(a)\, da = {}_p\int^q f(t)\, dt$

This is the simplest property as only a is to be substituted by t, and the desired result is obtained.

Property 2: ${}_p\int^q f(a)\, d(a) = -\,{}_q\int^p f(a)\, d(a)$, Also ${}_p\int^p f(a)\, d(a) = 0$

Suppose $I = {}_p\int^q f(a)\, d(a)$

If f' is the anti-derivative of f, then use the second fundamental theorem of calculus, to get $I = f'(q)-f'(p) = -[f'(p) - f'(q)] = -\,{}_q\int^p(a)da$.

Also, if p = q, then $I= f'(q)-f'(p) = f'(p) -f'(p) = 0$. Hence, ${}_a\int^a f(a)da = 0$.

Property 3: ${}_p\int^q f(a)\, d(a) = {}_p\int^r f(a)\, d(a) + {}_r\int^q f(a)\, d(a)$

If f' is the anti-derivative of f, then use the second fundamental theorem of calculus, to get;

${}_p\int^q f(a)da = f'(q)-f'(p)$… (1)

${}_p\int^r f(a)da = f'(r) - f'(p)$… (2)

${}_r\int^q f(a)da = f'(q) - f'(r)$ … (3)

Let's add equations (2) and (3), to get

$_p\int^r f(a)daf(a)da + {}_r\int^q f(a)daf(a)da = f'(r) - f'(p) + f'(q)$

$= f'(q) - f'(p) = {}_p\int^q f(a)da$

Property 4: $_p\int^q$ f(a) d(a) = $_p\int^q$ f(p + q – a) d(a)

Let, t = (p+q-a), or a = (p+q – t), so that dt = – da … (4)

Also, note that when a = p, t = q and when a = q, t = p. So, $_p\int^q$ wil be replaced by $_q\int^p$ when we replace a by t. Therefore,

$_p\int^q f(a)da = -{}_q\int^p f(p+q-t)dt$ … from equation (4)

From property 2, we know that $_p\int^q f(a)da = -{}_q\int^p f(a)da$. Use this property, to get

$_p\int^q f(a)da = {}_p\int^q f(p+q-t)da$

Now use property 1 to get

$_p\int^q f(a)da = {}_p\int^q f(p + q - a)da$

Property 5: $\int_0^p f(a)da = \int_0^p f(p-a)da$

Let, t = (p-a) or a = (p – t), so that dt = – da …(5)

Also, observe that when a = 0, t =p and when a = p, t = 0. $\int_0^p$ So, will be $\int_0^p$ replaced by when we replace a by t. Therefore,

$\int_0^p f(a)da = -\int_p^0 f(p-t)da$ … from equation (5)

From Property 2, we know that $\int_p^q f(a)da = -\int_q^p f(a)da$. Using this property, we get

$\int_0^p f(a)da = \int_0^p f(p-t)dt$

Next, using Property 1, we get

$\int_0^a f(a)da = \int_0^p f(p-a)da$

Property 6: $\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_0^p f(2p-a))da$

From property 3, we know that

$\int_p^q f(a)da = \int_p^r f(a)da + \int_r^q f(a)da$

Therefore, $\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_p^{2p} f(a)da = I_1 + I_2 \ldots (6)$

Where, $I_1 = \int_0^p f(a)da$ and $I_2 = \int_p^{2p} f(a)da$

Let, t = (2p – a) or a = (2p – t), so that dt = -da …(7)

Also, note that when a = p, t = p, and when a =2p, t= 0. $\int_a^0$ Hence, when we replace a by t. Therefore,

$I_2 = \int_p^{2p} f(a)da = -\int_p^0 f(2p\text{-}0)da\ldots$ from equation (7)

From Property 2, we know that $\int_p^q f(a)da = -\int_q^p f(a)da$. Using this property, we get I_2 = ∫p0f(2p-t)dt

Next, using Property 1, we get

$I_2 = \int_0^a f(a)da + \int_0^a f(2p\text{-}a)da$

Replacing the value of I_2 in equation (6), we get

$\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_0^p f(2p - a)da$

Property 7: $\int_0^{2a} f(a)da = 2\int_0^a f(a)da$ **… if f(2p – a) = f(a) and**

$\int_0^{2a} f(a)da = 0$ **… if f(2p- a) = -f(a)**

we know that

$\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_0^p f(2p - a)da \ldots (8)$

Now, if f(2p – a) = f(a), then equation (8) becomes

$\int_0^{2p} f(a)da = \int_0^p f(a)da + \int_0^p f(a)da$

$=2\int_{0}^{p} f(a)da$

And, if f(2p – a) = – f(a), then equation (8) becomes

$\int_{0}^{2p} f(a)da = \int_{0}^{p} f(a)da - \int_{0}^{p} f(a)da = 0$

Property 8: $\int_{-p}^{p} f(a)da = 2\int_{0}^{p} f(a)da$ … if f(-a) =f(a) or it is an even function and $\int_{-a}^{a} f(a)da = 0$, ….. if f(-a) = -f(a) or it is an odd function.

Using Property 3, we have

$\int_{-p}^{p} f(a)da = \int_{-a}^{0} f(a)da + \int_{0}^{p} f(a)da = I_1 + I_{,2}$ …(9)

Where, $I_1 = \int_{-a}^{0} f(a)da$, $I_2 = \int_{0}^{p} f(a)da$

Consider I_1

Let, t = -a or a = -t, so that dt = -dx … (10)

Also, observe that when a = -p, t = p, when a = 0, t = 0. $\int_{-a}^{0}$ Hence, will be $\int_{a}^{0}$ replaced by when we replace a by t. Therefore,

$I_1 = \int_{-a}^{0} f(a)da = -\int_{a}^{0} f(-a)da$ … from equation (10)

From Property 2, we know that $\int_{p}^{q} f(a)da = -\int_{q}^{p} f(a)da$, use this property to get,

$I_1 = \int_{-p}^{0} f(a)da = \int_{0}^{p} f(-a)da$

Next, using Property 1, we get

$I_1 = \int_{-p}^{0} f(a)da = \int_{0}^{p} f(-a)da$

Replacing the value of I_2 in equation (9), we get

$\int_{-p}^{p} f(a)da = I_1 + I_2 = \int_{0}^{p} f(-a)da + \int_{0}^{p} f(a)da = 2\int_{0}^{p} f(a)da$ … (11)

Now, if ‘f’ is an even function, then f(– a) = f(a). Therefore, equation (11) becomes

$$\int_{-p}^{p} f(\text{a})\text{da} = \int_{0}^{p} f(\text{a})\text{da} + \int_{0}^{p} f(\text{a})\text{da} = 2\int_{0}^{p} f(\text{a})\text{da}$$

And, if 'f' is an odd function, then f(–a) = – f(a). Therefore, equation (11) becomes

$$\int_{-p}^{p} f(\text{a})\text{da} = -\int_{0}^{a} f(\text{a})\text{da} + \int_{0}^{p} f(\text{a})\text{da} = 0$$

Example 1: Evaluate $\int_{-1}^{2} f(\mathbf{a^3 - a})\mathbf{da}$

Solution: Observe that, $(a^3 – a) \geq 0$ on [– 1, 0], $(a^3 – a) \leq 0$ on [0, 1] and $(a^3 – a) \geq 0$ on [1, 2]

Hence, using Property 3, we can write

$\int_{-1}^{2} f(a^3 - a)da = \int_{-1}^{0} f(a^3 - a)da + \int_{0}^{1} f\text{-}(a^3 - a)da + \int_{1}^{2} f(a^3 - a)da = \int_{-1}^{0} f(a^3 - a)da + \int_{0}^{1} f(a - a^3)da + \int_{1}^{2} f$

$(a^3 – a)da$

∫0−1f$(a^3 – a)$da +∫10f$(a – a^3)$da +∫21f$(a^3 – a)$da

Solving the integrals, we get

$\int_{-1}^{2} f(a^3–a)da$ = x4/4–(x2/2)] −10 + [(x2/2 – (x4/4))01 + [x4/4−(x2/2)]12

= – [1/4 – 1/2] + [– 1/4] + [4 – 2] - [1/4 -1/2] = 11/4

Example 2: Prove that $_0\int^{\pi/2}$ (2log sinx – log sin 2x)dx = – (π/2) log 2 using the properties of definite integral

Solution:

To prove: $_0\int^{\pi/2}$ (2log sinx – log sin 2x)dx = – (π/2) log 2

Proof:

Let take I = $_0\int^{\pi/2}$ (2log sinx – log sin 2x)dx …(1)

By using the property of definite integral

$_0\int^{a}$ f(x) dx = $_0\int^{a}$ f(a-x) dx

Now, apply the property in (1), we get

I = $_0\int^{\pi/2}$ 2log sin[(π/2)-x] – log sin 2[(π/2)-x])dx

The above expression can be written as

I = $_0\int^{\pi/2}$ [2log cosx- log sin(π-2x)]dx (Since, sin (90-θ = cos θ)

I = $_0\int^{\pi/2}$ [2log cosx- log sin2x]dx ..(2)

Now, add the equation (1) and (2), we get

$I + I = \int_0^{\pi/2} [(2\log \sin x - \log \sin 2x) + (2\log \cos x - \log \sin 2x)]dx$

$2I = \int_0^{\pi/2} [2\log \sin x - 2\log 2\sin x + 2\log \cos x]dx$

$2I = 2 \int_0^{\pi/2} [\log \sin x - \log 2\sin x + \log \cos x]dx$

Now, cancel out 2 on both the sides, we get

$I = \int_0^{\pi/2} [\log \sin x + \log \cos x - \log 2\sin x]dx$

Now, apply the logarithm property, we get

$I = \int_0^{\pi/2} \log[(\sin x . \cos x)/\sin 2x]dx$

We know that $\sin 2x = 2 \sin x \cos x$)

Now, the integral expression can be written as

$I = \int_0^{\pi/2} \log[(\sin x . \cos x)/(2 \sin x \cos x)]dx$

Cancel the terms which are common in both numerator and denominator, then we get

$I = \int_0^{\pi/2} \log(1/2)dx$

It can be written as

$I = \int_0^{\pi/2} (\log 1 - \log 2)dx$ [Since, $\log (a/b) = \log a - \log b$]

$I = \int_0^{\pi/2} -\log 2 \, dx$ (value of $\log 1 = 0$)

Now, take the constant $-\log 2$ outside the integral,

$I = -\log 2 \int_0^{\pi/2} dx$

Now, integrate the function

$I = -\log 2 \, [x]_0^{\pi/2}$

Now, substitute the limits

$I = -\log 2 \, [(\pi/2) - 0]$

$I = -\log 2 \, (\pi/2)$

$I = -(\pi/2) \log 2$ = R.H.S

Therefore, L.H. S = R.H.S

Hence. $\int_0^{\pi/2} (2\log \sin x - \log \sin 2x)dx = -(\pi/2) \log 2$ is proved.

Definite Integrals Rational or Irrational Expression

- $\int_a^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$
- $\int_a^\infty \frac{x^m dx}{x^n+a^n} = \frac{\pi a^{m-n+1}}{n \sin\left(\frac{(m+1)\pi}{n}\right)}, 0 < m+1 < n$
- $\int_a^\infty \frac{x^{p-1} dx}{1+x} = \frac{\pi}{\sin(p\pi)}, 0 < p < 1$
- $\int_a^\infty \frac{x^m dx}{1+2x \cos\beta + x^2} = \frac{\pi \sin(m\beta)}{\sin(m\pi) \sin\beta}$
- $\int_a^\infty \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$
- $\int_a^\infty \sqrt{a^2-x^2} dx = \frac{\pi a^2}{4}$

Definite integrals of Trigonometric Functions

- $\int_0^\pi \sin(mx)\sin(nx)dx = \begin{cases} 0 & if\ m \neq n \\ \frac{\pi}{2} & if\ m = n \end{cases}$ $m, n\ positive\ integers$
- $\int_0^\pi \cos(mx)\cos(nx)dx = \begin{cases} 0 & if\ m \neq n \\ \frac{\pi}{2} & if\ m = n \end{cases}$ $m, n\ positive\ integers$
- $\int_0^\pi \sin(mx)\cos(nx)dx = \begin{cases} 0 & if\ m+n\ even \\ \frac{2m}{m^2-n^2} & if\ m+n\ odd \end{cases}$ $m, n\ integers$