APPLICATION OF DERIVATIVES

There are various applications of derivatives not only in maths and real life but also in other fields like science, engineering, physics, etc.

Derivatives have various important applications in Mathematics such as:

- Rate of Change of a Quantity
- Increasing and Decreasing Functions
- Tangent and Normal to a Curve
- Minimum and Maximum Values
- Newton's Method
- Linear Approximations

Rate of Change of a Quantity

This is the general and most important application of derivative. For example, to check the rate of change of the volume of a cube with respect to its decreasing sides, we can use the derivative form as dy/dx. Where dy represents the rate of change of volume of cube and dx represents the change of sides of the cube.

Consider a function y = f(x), the rate of change of a function is defined as-

dy/dx = f'(x)

Further, if two variables x and y are varying to another variable, say if x = f(t), and y = g(t), then using Chain Rule, we have:

dy/dx = (dy/dt)/(dx/dt)

where dx/dt is not equal to 0.

Increasing and Decreasing Functions

To find that a given function is increasing or decreasing or constant, say in a graph, we use derivatives. If f is a function which is continuous in [p, q] and differentiable in the open interval (p, q), then,

- f is increasing at [p, q] if f'(x) > 0 for each x $\in$ (p, q)
- f is decreasing at [p, q] if f'(x) < 0 for each x $\in$ (p, q)
- f is constant function in [p, q], if f'(x)=0 for each x $\in$ (p, q)

Tangent and Normal to a Curve

Tangent is the line that touches the curve at a point and doesn't cross it, whereas normal is the perpendicular to that tangent.

Let the tangent meet the curve at P(x_1, y_1)
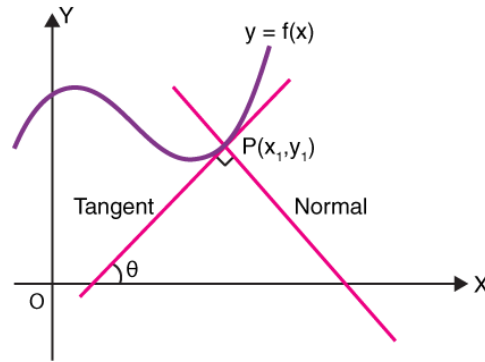
Now the straight-line equation which passes through a point having slope m could be written as;

$y - y_1 = m(x - x_1)$

We can see from the above equation, the slope of the tangent to the curve y = f(x) and at the point $P(x_1, y_1)$, it is given as dy/dx at $P(x_1, y_1) = f'(x)$. Therefore,

Equation of the tangent to the curve at $P(x_1, y_1)$ can be written as:

$y - y_1 = f'(x_1)(x - x_1)$

Equation of normal to the curve is given by;

$y - y_1 = [-1/ f'(x_1)] (x - x_1)$

Or

$(y - y_1) f'(x_1) + (x-x_1) = 0$

Maxima and Minima

To calculate the highest and lowest point of the curve in a graph or to know its turning point, the derivative function is used.

- When x = a, if f(x) ≤ f(a) for every x in the domain, then f(x) has an Absolute Maximum value and the point a is the point of the maximum value of f.
- When x = a, if f(x) ≤ f(a) for every x in some open interval (p, q) then f(x) has a Relative Maximum value.
- When x= a, if f(x) ≥ f(a) for every x in the domain then f(x) has an Absolute Minimum value and the point a is the point of the minimum value of f.
- When x = a, if f(x) ≥ f(a) for every x in some open interval (p, q) then f(x) has a Relative Minimum value.
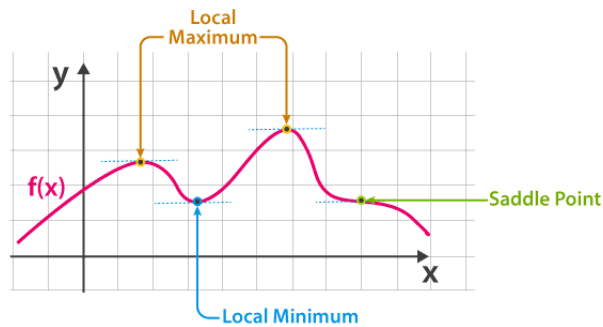
Monotonicity

Functions are said to be monotonic if they are either increasing or decreasing in their entire domain. $f(x) = e^x$, $f(x) = n^x$, $f(x) = 2x + 3$ are some examples.

Functions which are increasing and decreasing in their domain are said to be non-monotonic

For example: $f(x) = \sin x$, $f(x) = x^2$

Monotonicity Of A function At A Point

A function is said to be monotonically decreasing at $x = a$ if $f(x)$ satisfy;

$f(x + h) < f(a)$ for a small positive h

- $f'(x)$ will be positive if the function is increasing
- $f'(x)$ will be negative if the function is decreasing
- $f'(x)$ will be zero when the function is at its maxima or minima

Approximation or Finding Approximate Value

To find a very small change or variation of a quantity, we can use derivatives to give the approximate value of it. The approximate value is represented by delta Δ.

Suppose change in the value of x, dx = x then,

$dy/dx = \Delta x = x$.

Since the change in x, dx ≈ x therefore, dy ≈ y.

Point of Inflection

For continuous function $f(x)$, if $f'(x_0) = 0$ or $f'''(x_0)$ does not exist at points where $f'(x_0)$ exists and if $f''(x)$ changes sign when passing through $x = x_0$ then x_0 is called the point of inflection.

(a) If $f''(x) < 0$, $x \in (a, b)$ then the curve $y = f(x)$ in concave downward

(b) if $f''(x) > 0$, $x \in (a, b)$ then the curve $y = f(x)$ is concave upwards in (a, b)

For example: f(x) = sin x

Solution: $f'(x) = \cos x$

$f''(x) = \sin x = 0 \quad x = n\pi, n \in z$

Application of Derivatives in Real Life

- To calculate the profit and loss in business using graphs.
- To check the temperature variation.
- To determine the speed or distance covered such as miles per hour, kilometre per hour etc.
- Derivatives are used to derive many equations in Physics.
- In the study of Seismology like to find the range of magnitudes of the earthquake.

Solved Examples

Q1. Show that the function $f(x) = x^3 - 2x^2 + 2x$, $x \in Q$ is increasing on Q.

Solution: $f(x) = x^3 - 2x^2 + 2x$

By differentiating both sides, we get,

$f'(x) = 3x^2 - 4x + 2 > 0$ for every value of x

Therefore, f is increasing on Q.

Q2. The tangent to the curve $y = x^2 - 5x + 5$ parallel to the line $2y = 4x + 1$, also passes through a point. Find the coordinates of the point.

Solution:

$dy/dx = 2x - 5|_{x=x1} = 2$

$2x_1 = 7$

$\Rightarrow x_1 = 7/2$

$y_1 = (49/4) - (35/2) + 5 = (49 - 70 + 20)/4 = -¼$

$y + ¼ = 2(x - 7/2)$

$4y + 1 = 8x - 28$

$\Rightarrow 8x - 4y - 29 = 0$

$x = ⅛$, $y = -7$ satisfies the equation

Q3. The tangent to the curve, $y = x e^{x^2}$ passing through the point (1, e) also passes through another point. Find it.

Solution: $\frac{dy}{dx} = e^{x^2} + x e^{x^2} \cdot 2x$

At x = 1, slope of tangent m = 3e

Equation of tangent:

$y - e = 3e(x-1)$

$\Rightarrow y = 3ex - 2e$

$(4/3, 2e)$ lies on it.