# ARITHMETIC PROGRESSION (A.P.)

Consider the following lists of numbers:

**(i)** 1, 2, 3, 4, . . .

**(ii)** 100, 70, 40, 10, . . .

**(iii)** - 3, -2, -1, 0, . . .

**(iv)** 3, 3, 3, 3, . . .

**(v)** -1.0, -1.5, -2.0, -2.5, . . .

Each of the numbers in the list is called a term.

- In (i), each term is 1 more than the term preceding it.
- In (ii), each term is 30 less than the term preceding it.
- In (iii), each term is obtained by adding 1 to the term preceding it.
- In (iv), all the terms in the list are 3, i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.
- In (v), each term is obtained by adding - 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

## Arithmetic Progression (AP)

- So, an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.
- This fixed number is called the common difference of the AP. Remember that it can be positive, negative or zero.
- Let us denote the first term of an AP by $a_1$, second term by $a_2$, . . ., nth term by $T_n$ and the common difference by d. Then the AP becomes $a_1$, $a_2$, $a_3$, . . ., $T_n$.
- So, $a_2 - a_1 = a_3 - a_2 = . . . = T_n - (T_n - 1) = d$.

## General form of an AP

Let ‘a’ the first term and 'd' the common difference of A.P.' then the successive terms of the A.P. are: a, a + d, a + 2d, a + 3d, . . . represents an arithmetic progression where ‘a’ is the first term and ‘d’ the common difference. This is called the general form of an AP.

**Example: Which of the following list of numbers form an AP? If they form an AP, write the next two terms:**

**(i) 4, 10, 16, 22. . .**

**(ii) 1, - 1, - 3, - 5 . . ..**

**Solution:**

(i) We have $a_2 - a_1 = 10 - 4 = 6$

$a_3 - a_2 = 16 - 10 = 6$

$a_4 - a_3 = 22 - 16 = 6$

common difference is the same every time.

So, the given list of numbers forms an AP with the common difference d = 6.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii) $a_2 - a_1 = -1 - 1 = -2$

$a_3 - a^2 = -3 - (-1) = -3 + 1 = -2$

So, the given list of numbers forms an AP with the common difference d = - 2.

The next two terms are:

$-5 + (-2) = -7$ and $-7 + (-2) = -9$

## Exercise

**1. Write first four terms of the AP, when the first term a and the common difference d are given as follows:**

**(i)** a = -2, d = 0

**(ii)** a = 4, d = - 3

**(iii)** a = - 1.25, d = - 0.25

**$n^{th}$ Term of an A.P.**

The nth term an of the AP with first term a and common difference d is given by:

$$T_n = a + (n - 1) d.$$

$T_n$ is also called the **general term of the AP**. If there are n terms in the AP, then $T_n$ represents the **last term** which is sometimes also denoted by **l**.

$$l = a + (n-1) d$$

## Solved Examples:

**1. Find the 10th term of the AP : 2, 7, 12, . . .**

**Solution:** Here, a = 2, d = 7 - 2 = 5 and n = 10.

We have $T_n = a + (n - 1) d$

So, $T_{10} = 2 + (10 - 1) \times 5 = 2 + 45 = 47$

Therefore, the 10th term of the given AP is 47.

**2. Which term of the AP : 21, 18, 15, . . . is - 81? Also, is any term 0?**

**Solution:** Here, a = 21, d = 18 - 21 = - 3 and $T_n$ = - 81, and we have to find n.

As, $T_n = a + (n - 1) d$,
we have $- 81 = 21 + (n - 1)(- 3)$

$- 81 = 24 - 3n$

$- 105 = - 3n$

So, n = 35

Therefore, the 35th term of the given AP is - 81.

Next, we want to know if there is any n for which $T_n = 0$. If such an n is there, then

$21 + (n - 1) (-3) = 0$,

i.e., $3(n - 1) = 21$

i.e., n = 8

So, the eighth term is 0.

**3. How many two-digit numbers are divisible by 3?**

**Solution:** The list of two-digit numbers divisible by 3 is: 12, 15, 18, . . . , 99

Here, a = 12, d = 3, $T_n$ = 99.

As $T_n = a + (n - 1) d$,
we have $99 = 12 + (n - 1) \times 3$

i.e., $87 = (n - 1) \times 3$

i.e., $n - 1 = 29$

i.e., $n = 29 + 1 = 30$

So, there are 30 two-digit numbers divisible by 3.

**4. Find the 11th term from the last term (towards the first term) of the AP : 10, 7, 4, . . ., - 62.**

**Solution:** Here, a = 10, d = 7 - 10 = - 3, l = - 62,

where $l = a + (n - 1) d$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $- 62 = 10 + (n - 1)(-3)$

i.e., $- 72 = (n - 1)(-3)$

i.e., $n - 1 = 24$

or n = 25

So, there are 25 terms in the given AP.

The 11th term from the last term will be the 15th term. So, $T_{15} = 10 + (15 - 1)(-3) = 10 - 42 = - 32$

i.e., the 11th term from the last term is - 32.

**Alternative Solution:**

If we write the given AP in the reverse order, then a = - 62 and d = 3

So, the question now becomes finding the 11th term with these a and d.

So, $T_{11} = - 62 + (11 - 1) \times 3 = - 62 + 30$

$= - 32$

So, the 11th term, which is now the required term, is - 32.

**5. In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?**

**Solution:** The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are : 23, 21, 19, . . ., 5

Let the number of rows in the flower bed be n.

Then a = 23, d = 21 - 23 = - 2, $T_n = 5$

As, $T_n = a + (n - 1) d$

We have, 5 = 23 + (n - 1)(- 2)

i.e., - 18 = (n - 1)(- 2)

i.e., n = 10

So, there are 10 rows in the flower bed.

## Sum of First n Terms of an A.P.

$S_n = {}_2\,[2a + (n - 1)\,d]$

So, the sum of the first n terms of an AP is given by

$$\mathbf{S_n = \frac{n}{2}\,[2a + (n - 1)\,d]}$$

If the last term of an AP is 1

$$\mathbf{S = \frac{n}{2}\,[a + 1]}$$

**Example: Find the sum of the first 22 terms of the AP : 8, 3, -2,. . . .**

**Solution:** Here a = 8, d = 3 – 8 = -5, n = 22

We know that

$S_n = \frac{n}{2}\,[2a + (n-1)\,d]$

$S_n = 11\,(16-105)$

= 11(-89) = -979

So, the sum of the first 22 terms of the AP is -979.

**Example: If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.**

**Solution:** Here, $S_{14}$ = 1050, n = 14, a = 10.

$S_n = \frac{n}{2}[2a + (n - 1) d]$

1050 = 7[20+13d]

1050= 140 + 91d

i.e., 910 = 91d

or, d = 10

Therefore,

$t_{20}$ = 10 + (19)10

$t_{20}$ = 10 + 190

i.e., 20th term is 200.

## Arithmetic Mean Two Given Numbers a and b

If a, b, c are in AP, then $b = \left(\frac{a + c}{2}\right)$

and b is called the arithmetic mean of a and c.

**Example: Find the arithmetic mean of the following:**

(i) **5 and -7**

(ii) **0.008 and 0.08**

**Solution:**

(i) A.M.= 15+(-7)/2 =4

(ii) A.M.= $\frac{0.008 + 0.08}{2}$

= $\frac{0.088}{2}$ = 0.044

## Properties of Arithmetic Progression

We can verify the following simple properties of an A.P.:

(i) If a constant is added to each term of an A.P., the resulting sequence is also an A.P.

(ii) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.

(iii) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.

(iv) If each term of an A.P. is divided by a non- zero constant then the resulting sequence is also an A.P.

## Geometric Progression (G.P.)

Let us consider the following sequences:

(i) 2,4,8,16,..., (ii) 9, 27, 81, 243... (iii) .01,.0001,.000001,...

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In a geometric progression, a, ar, $ar^2$, $ar^3$,…., where 'a' is called the first term and 'r' is called the common ratio of the G.P.
Common ratio in geometric progression (i), (ii) and (iii) above are 2, 3 and 0.01, respectively.

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

$S_n$ = the sum of first n terms.
General term of a G .P.

Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'

1st term = $a_1 = a$

2nd term = $a_2 = ar$

3rd term = $a_3 = ar^2$

4th term= $a_4 = ar^3$

**then the nth term of a G.P. is given by: $T_n = ar^{n-1}$**
Thus, a, G.P. can be written as a, ar, $ar^2$, $ar^3$, … $ar^{n-1}$.

Consequently, $n^{th}$ term from the end of a
G.P. = $l\,(1/r)^{n-1}$

**Example: Find the $7^{th}$ terms of the sequences -**

**(i)** $2^2, 2^3, 2^4$ ................

**(ii)** $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}$................

**Solution:**

**(i)** $T_n = a.r^{n-1}$

$T_7 = 2^2 . 2^{7-1}$

$= 2^2 . 2^6$

$= 2^8 = 256$

**(ii)** $T_n = a.r^{n-1}$

$a = \sqrt{3}, r = 1/3$

$T_7 = \frac{\sqrt{3}}{3^6}$

$= \frac{\sqrt{3}}{729}$

**Example: Find the 10th term of the G.P. 5, 25,125, ….**

**Solution:** Here a = 5 and r = 5.

Thus, $T10 = 5(5)^{10-1} = 5(5)^9 = 5^{10}$

**Example: In a G.P., the 3rd term is 24 and the 6th term is 192. Find the 10th term.**

**Solution:**

ar2 = 24 ... (1)

and $ar^5$ =192 ... (2)

Dividing (2) by (1), we get r = 2.

Substituting r = 2 in (1)

we get a = 6.

Hence $a_{10} = 6$

$(2)^9 = 3072.$

## Sum to n terms of a G.P.

Let the first term of a G.P. be a and the common ratio be r. Let us denote by $S_n$ the sum to first n terms of G.P. Then

$S_n = a + ar + ar^2 + ... + ar^{n-1}$ ...

$S_n = \frac{a(1 - r^n)}{1 - r}$ For r<1

$S_n = \frac{a(r^n - 1)}{r - 1}$ For r>1

and the sum of infinite G.P. is

$S_\infty = \frac{a}{1 - r}$ , if r < 1

**Example: Find the sum of the given series**
**½ + 1 + 2 + ……... to 8 terms ?**

**Solution:** a = ½ , r = 2 (2>1) , n=8

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1}{2}\,\frac{2^8 - 1}{2-1} = \frac{255}{2} = 127\frac{1}{2}$

**Example: Find the sum of the following series to infinity**

$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}$ …………

**Solution:** a=1, r = - 1/3 where (r<1)

$S_\infty = \frac{a}{1 - r}$

$= \frac{1}{1 - (-1/3)}$

$= \frac{1}{1 + 1/3} = \frac{3}{4}$

**Example: The sum of first three terms of a G.P.13/12 and their product is - 1.Find the common ratio and the terms.**

**Solution:** Let a/r, a, ar be the first three terms of the G.P. Then

a/r+ a+ ar = 13/12 … …….. (1)

and (a/r)a(ar) = -1 ………… (2)

From (2), we get $a^3 = -1$, i.e., a = - 1

(considering only real roots)

Substituting a = -1 in (1), we have

(-1/r)(-1)(r) = 13/12 or 12r2 + 25r + 12= 0

Solving, we get…. $r = \frac{-3}{4}$ or $\frac{-4}{3}$

Thus the term of G.P. are 4/3 , -1 , 3/4 and 3/4,-1, 4/3

**Example: A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.**

**Solution:** Here a = 2, r = 2 and n = 10

Using the sum formula $S_n = a(r^n - 1)/r-1$

We have $S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

## Geometric Mean (G.M.)

The geometric mean of two positive numbers a and b is the number $\pm\sqrt{ab}$.

Therefore, the geometric mean of 2 and 8 is 4

**Example: Find the Geometric mean of**

**(i) 0.2 and 0.002 (ii) 18 and 8**

**Solution:**

(i) 0.2 and 0.002

G.M. = $\sqrt{0.2 \times 0.002}$

= $\sqrt{0.0004}$ = ± 0.02

(ii) 18 and 8

G.M. = $\sqrt{18 \times 8}$ = ± 12

**Example: Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.**

**Solution:** Let $G_1$, $G_2$,$G_3$ be three numbers between 1 and 256 such that 1, $G_1$, $G_2$, $G_3$, 256 is a G.P.
Therefore 256 = $r^4$
giving r = ± 4 (Taking real roots only)
For r = 4, we have $G_1$ = ar = 4
$G_2 = ar^2 = 16$
$G_3 = ar^3 = 64$
Similarly, for r = - 4
numbers are - 4, 16 and - 64.
Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

## Sum to n Terms of Special Series

We shall now find the sum of first n terms of some special series, namely;

(i) 1 + 2 + 3 +… + n (sum of first n natural numbers) $\frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2$ (sum of squares of the first n natural numbers)
$\frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots$ + n3 (sum of cubes of the first n natural numbers)

$$\left\{\frac{n(n+1)}{2}\right\}^2$$

## Important facts about the selection of consecutive terms in G.P.

(i) 3 consecutive terms of a G.P., whose product is given a/r, a ,ar.
(ii) 4 consecutive terms of a G.P., whose product is given ....... $a/r^3$, a/r, ar, $ar^3$

(iii) Usual form as — a, ar, $ar^2$ ...... (If the product of no. is not given)

## Harmonic Progression (H.P.)

Let H be the Harmonic Mean between a and b,

**Harmonic Mean (H.M) =** $\mathbf{\frac{2ab}{a+b}}$

**Example: Find the Harmonic Mean between 4 and 6?**

**Solution:**

$$\text{H.M.} = \frac{2ab}{a+b} = \frac{2 \times 4 \times 6}{4+6} = \frac{48}{10} = 4.8$$

## Relation Between A.M., G.M. and H.M.

Let the A, G, H be respectively the arithmetic, geometric and harmonic mean between two quantities a and b, then-

(I) A> G>H

(II) AH= G2

## Practice Questions:

**Q1. Find the sum of the sequence 2, 3, 5, 9, 8, 15, 11, ... to (2n + 1) terms**

**Solution:** Let S denote the sum.

Then S = 2 + 3 + 5 + 9 + 8 + 15 + 11 + ...

(2n + 1) terms

= [2 + 5 + 8 + 11 + ... (n + 1) terms] + [3 + 9 + 15 + 21 + ... n terms]

$= \frac{n+1}{2}[2 \times 2 + (n + 1 - 1) \times 3] + \frac{n}{2}[2 \times 3 + (n - 1) \times 6]$

$= \frac{n+1}{2}[4 + 3n] + \frac{n}{2}[6 + 6n - 6]$

$= \frac{(n+1)(3n+4)}{4} + 3n^2$

$= ½ [3n^2 + 7n + 4 + 6n^2]$

$= ½ [9n^2 + 7n + 4]$

**Q2. The 35$^{th}$ term of an A. P. is 69. Find the sum of its 69 terms.**

**Solution:** Let a be the first term and d be the common difference of the A. P.

We have

$t_{35} = a + (35 – 1) d = a + 34 d$.

$\therefore$ a + 34 d = 69 ... (i)

Now by the formula $S_n = n/2 [2a + (n – 1) d]$

We have

$S_{69} = \frac{69}{2} [2a + (69 – 1) d]$

= 69 (a + 34d) [using (i)]

= 69 × 69 = 4761

**Q3. The first term of an A. P. is 10, the last term is 50. If the sum of all the terms is 480, find the common difference and the number of terms.**

**Solution:** We have: a = 10, $l = t_n = 50$, $S_n = 480$.

By substituting the values of a, $t_n$ and $S_n$ in the formulae

$S_n = \frac{n}{2} [2a + (n – 1) d]$ and $t_n = a + (n – 1) d$,

we get

$480 = \frac{n}{2} [20 + (n – 1) d]$ …….. (i)

50 = 10 + (n – 1) d ……. (ii)

From (ii), (n – 1) d = 50 – 10 = 40 ……(iii)

From (i), we have

$480 = \frac{n}{2} [20 + 40]$ …….using (i)

or, 60n = 2 × 480

$\therefore n = \frac{2 \times 480}{60} = 16$

From (iii),

$\therefore d = \frac{40}{15} = \frac{8}{3}$ (as n −1 =16 −1 =15)

**Q4. Let the nth term and the sum of n terms of an A. P. be p and q respectively.**
**Prove that its first term is $\left(\frac{2q – pn}{n}\right)$**

**Solution:** In this case, $t_n = p$ and $S_n = q$

Let a be the first term of the A. P.

Now, $S_n = \frac{n}{2} (a + t_n)$

or, $\frac{n}{2} (a + p) = q$

2

or, $a + p = \frac{2q}{n}$

or, $a = \frac{2q}{n} - p$

$\therefore a = \frac{2q - pn}{n}$

**Q5. Find the 10th term of the AP : 2, 7, 12, . . .**

**Solution:** Here, $a = 2$, $d = 7 – 2 = 5$ and $n = 10$.
We have $a_n = a + (n – 1) d$
So, $a_{10} = 2 + (10 – 1) \times 5 = 2 + 45 = 47$
Therefore, the 10th term of the given AP is 47.

**Q6. Which term of the AP : 21, 18, 15, . . . is – 81? Also, is any term 0? Give reason for your answer.**

**Solution:** Here, $a = 21$, $d = 18 – 21 = – 3$ and $a_n = – 81$, and we have to find n.
As $a_n = a + (n – 1) d$,
we have $– 81 = 21 + (n – 1)(– 3)$
$– 81 = 24 – 3n$
$– 105 = – 3n$, So, $n = 35$
Therefore, the 35th term of the given AP is – 81.

Next, we want to know if there is any n for which an = 0. If such an n is there, then
$21 + (n – 1) (–3) = 0$, i.e., $3(n – 1) = 21$
i.e., $n = 8$
So, the eighth term is 0.

**Q7. Determine the AP whose 3rd term is 5 and the 7th term is 9.**

**Solution:** We have
$a_3 = a + (3 – 1) d = a + 2d = 5$ (1)
and $a_7 = a + (7 – 1) d = a + 6d = 9$ (2)
Solving the pair of linear equations (1) and (2), we get
$a = 3$, $d = 1$
Hence, the required AP is 3, 4, 5, 6, 7, . . .

**Q8. How many two-digit numbers are divisible by 3?**

**Solution:** The list of two-digit numbers divisible by 3 is:
12, 15, 18, . . . , 99
Here, $a = 12$, $d = 3$, $a_n = 99$.
As $a_n = a + (n – 1) d$,
we have $99 = 12 + (n – 1) \times 3$

i.e., $87 = (n - 1) \times 3$
i.e., $n - 1 = 87/3 = 29$
i.e., $n = 29 + 1 = 30$
So, there are 30 two-digit numbers divisible by 3.

**Q9. Find the 11th term from the last term (towards the first term) of the AP : 10, 7, 4, . . ., – 62.**

**Solution:** Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,
where $l = a + (n - 1) d$
To find the 11th term from the last term, we will find the total number of terms in the AP.
So, $-62 = 10 + (n - 1)(-3)$
i.e., $-72 = (n - 1)(-3)$, i.e., $n - 1 = 24$
or $n = 25$
So, there are 25 terms in the given AP.
The 11th term from the last term will be the 15th term.
So, $a_{15} = 10 + (15 - 1)(-3) = 10 - 42 = -32$
i.e., the 11th term from the last term is – 32.

**Alternative Solution:**
If we write the given AP in the reverse order, then $a = -62$ and $d = 3$
So, the question now becomes finding the 11th term with these a and d.
So, $a_{11} = -62 + (11 - 1) \times 3 = -62 + 30 = -32$
So, the 11th term, which is now the required term, is – 32.

**Q10. A sum of Rs. 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years making use of this fact.**

**Solution:** We know that the formula to calculate simple interest is given by
Simple Interest = $\frac{P \times R \times T}{100}$
So, the interest at the end of the 1st year
$= \frac{1000 \times 8 \times 1}{100}$ = Rs. 80
The interest at the end of the 2nd year
$= \frac{1000 \times 8 \times 2}{100}$ = Rs. 160
The interest at the end of the 3rd year
$= \frac{1000 \times 8 \times 3}{100}$ = Rs. 240
Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on.
So, the interest (in Rs) at the end of the 1st, 2nd, 3rd, . . . years, respectively are
80, 160, 240, . . .
It is an AP as the difference between the consecutive terms in the list is 80, i.e.,

$d = 80$. Also, $a = 80$.
So, to find the interest at the end of 30 years, we shall find $a_{30}$.
Now, $a_{30} = a + (30 – 1) d$
$= 80 + 29 \times 80 = 2400$
So, the interest at the end of 30 years will be Rs. 2400.