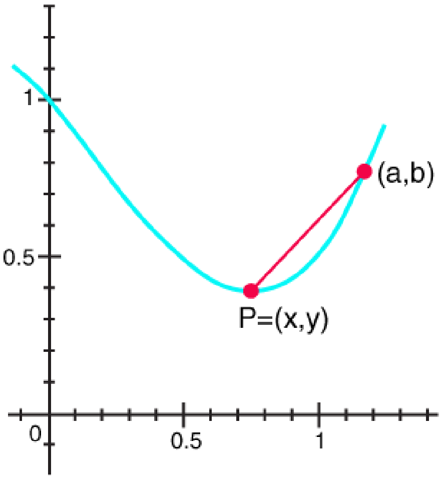
DIFFERENTIATION

What is Differentiation?

Differentiation is the essence of Calculus. A derivative is defined as the instantaneous rate of change in function based on one of its variables. It is similar to finding the slope of a tangent to the function at a point.

Suppose you need to find the slope of the tangent line to a graph at point P. The slope can be approximated by drawing a line through point P and finding the slope by a line that is known as the secant line.



A function f in x is said to be differentiable at the point x = a, if the derivative f'(a) exists at every point in its domain. The derivative of a function f(x) is given by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For a function to be differentiable at any point x = a, in its domain, it must be continuous at that particular point but vice-versa is necessarily not always true. The domain of f'(x) is defined by the existence of its limits.

If y = f(x) is a function in x, then the derivative of f(x) is given as dy/dx. This is known as the derivative of y with respect to x.

Also, the derivative of a function f(x) at x = a, is given by:

$$\frac{d}{dx}(f(x))\Big|_a \ or \ \frac{df}{dx}\Big|_a$$

The derivative of a function f(x) signifies the rate of change of the function f(x) with respect to x at a point ‘a’, lying in its domain.

If the derivative of the function, f', is known which is differentiable in its domain then we can find the function f. In integral calculus, we call f as the anti-derivative or primitive of the function f'. The method of calculating the anti-derivative is known as anti-differentiation or integration.

What is Differentiation in Maths

In Mathematics, Differentiation can be defined as a derivative of a function with respect to an independent variable. Differentiation, in calculus, can be applied to measure the function per unit change in the independent variable.

Let y = f(x) be a function of x. Then, the rate of change of "y" per unit change in "x" is given by:

dy / dx

If the function f(x) undergoes an infinitesimal change of 'h' near to any point 'x', then the derivative of the function is defined as

[latex]\lim\limits_{h \to 0} \frac{f(x+h) – f(x)}{h}[/latex]

Differentiation Formulas

d/dx (a) = 0 where a is constant
d/dx (x) = 1
$d/dx(x^n) = nx^{n-1}$
d/dx sin x = cos x
d/dx cos x = -sin x
$d/dx \tan x = \sec^2 x$
d/dx ln x = 1/x
$d/dx\ e^x = e^x$

Derivative of Function as Limits

If we are given with real valued function (f) and x is a point in its domain of definition, then the derivative of function, f, is given by:

$f'(a) = \lim_{h\to 0}[f(x+h)-f(x)]/h$

provided this limit exists.

Let us see an example here for better understanding.

Example: Find the derivative of f=2x, at x =3.

Solution: By using the above formulas, we can find,

$f'(3) = \lim_{h\to 0}[f(3+h)-f(3]/h = \lim_{h\to 0}[2(3+h)-2(3)]/h$

$f'(3) = \lim_{h\to 0}[6+2h-6]/h$

$f'(3) = \lim_{h\to 0}2h/h$

$f'(3) = \lim_{h\to 0}2 = 2$

Notations

When a function is denoted as y=f(x), the derivative is indicated by the following notations.

- **D(y) or D[f(x)]** is called Euler's notation.
- **dy/dx** is called Leibniz's notation.
- **F'(x)** is called Lagrange's notation.

The meaning of differentiation is the process of determining the derivative of a function at any point.

Linear and Non-Linear Functions

Functions are generally classified in two categories under Calculus, namely:

(i) Linear functions

(ii) Non-linear functions

A linear function varies with a constant rate through its domain. Therefore, the overall rate of change of the function is the same as the rate of change of a function at any point.

However, the rate of change of function varies from point to point in case of non-linear functions. The nature of variation is based on the nature of the function.

The rate of change of a function at a particular point is defined as a **derivative** of that particular function.

Differentiation Rules

Power Rule of Derivatives

This is one of the most common rules of derivatives. If x is a variable and is raised to a power n, then the derivative of x raised to the power is represented by:

$d/dx(x^n) = nx^{n-1}$

Example: Find the derivative of x^5

Solution: As per the power rule, we know;

$d/dx(x^n) = nx^{n-1}$

Hence, $d/dx(x^5) = 5x^{5-1} = 5x^4$

Sum or Difference Rule of Derivatives

If the function is sum or difference of two functions, the derivative of the functions is the sum or difference of the individual functions, i.e.,

If $f(x) = u(x) \pm v(x)$

then, $f'(x)=u'(x) \pm v'(x)$

Example 1: $f(x) = x + x^3$

Solution: By applying sum rule of derivative here, we have:

$f'(x) = u'(x) + v'(x)$

Now, differentiating the given function, we get;

$f'(x) = d/dx(x + x^3)$

$f'(x) = d/dx(x) + d/dx(x^3)$

$f'(x) = 1 + 3x^2$

Example 2: Find the derivative of the function $f(x) = 6x^2 – 4x$.

Solution:

Given function is: $f(x) = 6x^2 – 4x$

This is of the form $f(x) = u(x) – v(x)$

So by applying the difference rule of derivatives, we get,

$f'(x) = d/dx\ (6x^2) – d/dx(4x)$

$= 6(2x) – 4(1)$

$= 12x – 4$

Therefore, $f'(x) = 12x – 4$

Product Rule of Derivatives

As per the product rule, if the function f(x) is product of two functions u(x) and v(x), the derivative of the function is,

If $f(x) = u(x)\times v(x)$, then:

$f'(x) = u'(x) \times v(x) + u(x) \times v'(x)$

Example: Find the derivative of $x^2(x+3)$.

Solution: As per the product rule of derivative, we know;

$f'(x) = u'(x) \times v(x) + u(x) \times v'(x)$

Here,

$u(x) = x^2$ and $v(x) = x+3$

Therefore, on differentiating the given function, we get;

$f'(x) = d/dx[x^2(x+3)]$

$f'(x) = d/dx(x^2)(x+3)+x^2 d/dx(x+3)$

$f'(x) = 2x(x+3)+x^2(1)$

$f'(x) = 2x^2+6x+x^2$

$f'(x) = 3x^2+6x$

$f'(x) = 3x(x+2)$

Quotient rule

If f(x) is a function, which is equal to ratio of two functions u(x) and v(x) such that;

$f(x) = u(x)/v(x)$

Then, as per the quotient rule, the derivative of f(x) is given by;

$$\mathbf{f'(x)} = \frac{\mathbf{u'(x) \times v(x) - u(x) \times v'(x)}}{\mathbf{(v(x))^2}}$$

Example: Differentiate $f(x)=(x+2)^3/\sqrt{x}$

Solution: Given,

$f(x)=(x+2)^3/\sqrt{x}$

$= (x+2)(x^2+4x+4)/\sqrt{x}$

$= [x^3+6x^2+12x+8]/x^{1/2}$

$= x^{-1/2}(x^3+6x^2+12x+8)$

$= x^{5/2}+6x^{3/2}+12x^{1/2}+8x^{-½}$

Now, differentiating the given equation, we get;

$f'(x) = 5/2x^{3/2} + 6(3/2x^{1/2})+12(1/2x^{-1/2})+8(-1/2x^{-3/2})$

$= 5/2x^{3/2} + 9x^{1/2} + 6x^{-½} - 4x^{-3/2}$

Chain Rule

If a function y = f(x) = g(u) and if u = h(x), then the chain rule for differentiation is defined as,

$dy/dx = (dy/du) \times (du/dx)$

This plays a major role in the method of substitution that helps to perform differentiation of composite functions.

Example 1: Differentiate $f(x) = (x^4 – 1)^{50}$

Solution: Given,

$f(x) = (x^4 – 1)^{50}$

Let $g(x) = x^4 – 1$ and $n = 50$

$u(t) = t^{50}$

Thus, $t = g(x) = x^4 – 1$

$f(x) = u(g(x))$

According to chain rule,

$df/dx = (du/dt) \times (dt/dx)$

Here,

$du/dt = d/dt\ (t50) = 50t^{49}$

$dt/dx = d/dx\ g(x)$

$= d/dx\ (x^4 – 1)$

$= 4x^3$

Thus, $df/dx = 50t^{49} \times (4x^3)$

$= 50(x^4 – 1)^{49} \times (4x^3)$

$= 200\ x^3(x^4 – 1)^{49}$

Example 2: Find the derivative of $f(x) = e^{\sin(2x)}$

Solution: Given,

$f(x) = e^{\sin(2x)}$

Let $t = g(x) = \sin 2x$ and $u(t) = e^t$

According to chain rule,

$df/dx = (du/dt) \times (dt/dx)$

Here,

$du/dt = d/dt\ (e^t) = e^t$

$dt/dx = d/dx\ g(x)$

$= d/dx\ (\sin 2x)$

$= 2 \cos 2x$

Thus, $df/dx = e^t \times 2 \cos 2x$

$= e^{\sin(2x)} \times 2 \cos 2x$

$= 2 \cos(2x)\, e^{\sin(2x)}$