EQUATION IN A LINE

General Equation of a Line

The general equation of a line in two variables of the first degree is represented as

$Ax + By + C = 0$,

$A, B \neq 0$ where A, B and C are constants which belong to real numbers.

When we represent the equation geometrically, we always get a straight line.

Below is a representation of straight-line formulas in different forms:

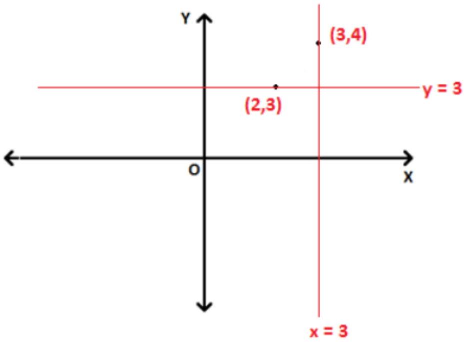
Equations of horizontal and vertical lines

Equation of the lines which are horizontal or parallel to the X-axis is $y = a$, where a is the y – coordinate of the points on the line.

Similarly, equation of a straight line which is vertical or parallel to Y-axis is $x = a$, where a is the x-coordinate of the points on the line.

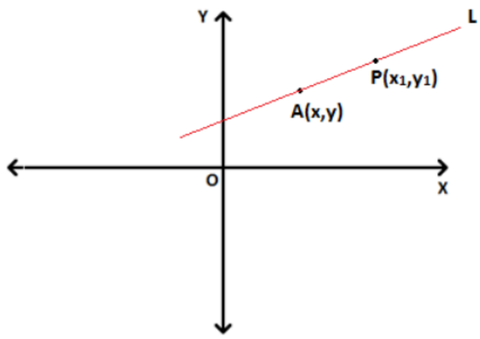
For example, the equation of the line which is parallel to X-axis and contains the point (2,3) is $y = 3$.

Similarly, the equation of the line which is parallel to Y-axis and contains the point (3,4) is $x = 3$.



Point-slope form equation of line

Consider a non-vertical line L whose slope is m, A(x,y) be an arbitrary point on the line and $P(x_1,y_1)$ be the fixed point on the same line.



Slope of the line by the definition is,

$$m = \frac{y - y_1}{x - x_1}$$

$y - y_1 = m(x - x_1)$

For example, equation of the straight line having a slope m = 2 and passes through the point (2,3) is

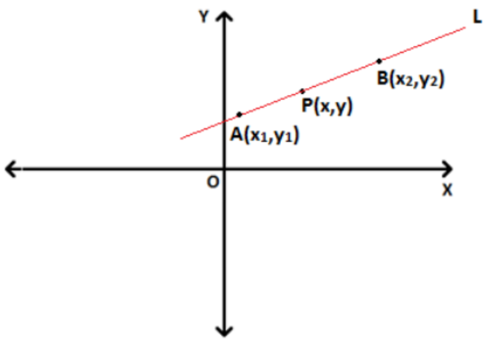
y – 3 = 2(x – 2)

y= 2x-4+3

2x-y-1 = 0

Two-point form equation of line

Let P(x,y) be the general point on the line L which passes through the points $A(x_1,y_1)$ and $B(x_2,y_2)$.



Since the three points are collinear,

slope of PA = slope of AB

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = (y_2 - y_1) \cdot \frac{x - x_1}{x_2 - x_1}$$

Slope-intercept Form

We know that the equation of a straight line in slope-intercept form is given as:

y = mx + c

Where m indicates the slope of the line and c is the y-intercept

When $B \neq 0$ then, the standard equation of first degree $Ax + By + C = 0$ can be rewritten in slope-intercept form as:

$y = (-A/B)\,x - (C/B)$

Thus, $m = -A/B$ and $c = -C/B$

Intercept Form

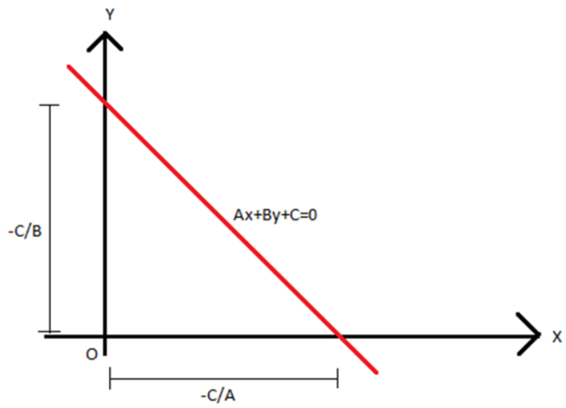
The intercept of a line is the point through which the line crosses the x-axis or y-axis. Suppose a line cuts the x-axis and y-axis at (a, 0) and (0, b), respectively. Then, the equation of a line making intercepts equal to a and b on the x-axis and the y-axis respectively is given by:

$x/a + y/b = 1$

Now in case of the general form of the equation of the straight line, i.e., $Ax + By + C = 0$, if $C \neq 0$, then $Ax + By + C = 0$ can be written as;

$x/(-C/A) + y/(-C/B) = 1$

where $a = -C/A$ and $b = – C/B$



Normal Form

The equation of the line whose length of the perpendicular from the origin is p and the angle made by the perpendicular with the positive x-axis is given by α is given by:

$x \cos \alpha + y \sin \alpha = p$

This is known as the normal form of the line.

In case of the general form of the line $Ax + By + C = 0$ can be represented in normal form as:

$A \cos \alpha = B \sin \alpha = – p$

From this we can say that $\cos \alpha = -p/A$ and $\sin \alpha = -p/B$.

Also it can be inferred that,

$\cos^2\alpha + \sin^2\alpha = (p/A)^2 + (p/B)^2$

$1 = p^2 (A^2 + B^2/A^2 .B^2)$

$\Rightarrow p = \left(\frac{AB}{\sqrt{A^2 + B^2}}\right)$

From the general equation of a straight line $Ax + By + C = 0$, we can conclude the following:

- The slope is given by $-A/B$, given that $B \neq 0$.
- The x-intercept is given by $-C/A$ and the y-intercept is given by $-C/B$.
- It can be seen from the above discussion that:

$p = \pm \frac{AB}{\sqrt{A^2 + B^2}}$, $\cos \alpha = \pm \frac{B}{\sqrt{A^2 + B^2}}$, $\sin \alpha = \pm \frac{A}{\sqrt{A^2 + B^2}}$

- If two points (x_1, y_1) and (x_2, y_2) are said to lie on the same side of the line $Ax + By + C = 0$, then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will have the same sign or else these points would lie on the opposite sides of the line.

Straight Line Formulas

Slope (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2)	$m = (y_2-y_1)/(x_2-x_1)$, $x_1 \neq x_2$
Equation of a horizontal line	$y = a$ or $y=-a$
Equation of a vertical line	$x=b$ or $x=-b$
Equation of the line passing through the points (x_1, y_1) and (x_2, y_2)	$y-y_1= [(y_2-y_1)/(x_2-x_1)] \times (x-x_1)$
Equation of line with slope m and intercept c	$y = mx + c$
Equation of line with slope m makes x-intercept d.	$y = m(x - d)$.
Intercept form of the equation of a line	$(x/a) + (y/b) = 1$
The normal form of the equation of a line	$x \cos \alpha + y \sin \alpha = p$

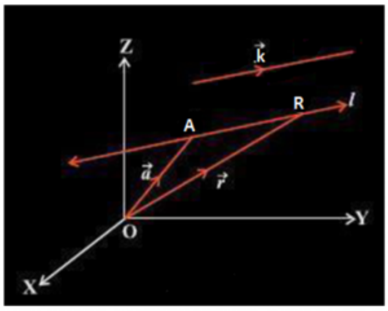
Equation of a Line in Three Dimensions

Equation of a line is defined as $y = mx + c$, where c is the y-intercept and m is the slope. Vectors can be defined as a quantity possessing both direction and magnitude. Position vectors simply denote the position or location of a point in the three-dimensional Cartesian system with respect to a reference origin. It is known that we can uniquely determine a line if:

- It passes through a particular point in a specific direction, or
- It passes through two unique points

Equation of a Line passing through a point and parallel to a vector

Let us consider that the position vector of the given point be $\vec{a}$ with respect to the origin. The line passing through point A is given by *l* and it is parallel to the vector $\vec{k}$ as shown below. Let us choose any random point R on the line *l* and its position vector with respect to origin of the rectangular co-ordinate system is given by $\vec{r}$.



Since the line segment, $\overline{AR}$ is parallel to vector $\vec{k}$, therefore for any real number α,
$\overline{AR}$ = α $\vec{k}$
Also, $\overline{AR}$ = $\overline{OR}$ – $\overline{OA}$
Therefore, α $\vec{r}$ = $\vec{r}$ – $\vec{a}$
From the above equation it can be seen that for different values of α, the above equations give the position of any arbitrary point R lying on the line passing through point A and parallel to vector k. Therefore, the vector equation of a line passing through a given point and parallel to a given vector is given by:
$\vec{r}$ = $\vec{a}$ + α$\vec{k}$
If the three-dimensional co-ordinates of the point 'A' are given as (x_1, y_1, z_1) and the direction cosines of this point is given as a, b, c then considering the rectangular co-ordinates of point R as (x, y, z):

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$$

$$\vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

Substituting these values in the vector equation of a line passing through a given point and parallel to a given vector and equating the coefficients of unit vectors i, j and k, we have,

$$x = x_1 + \alpha \mathrm{a}; y = y_1 + \alpha \mathrm{b}; \mathrm{z} = \mathrm{z}_1 + \alpha \mathrm{c}$$

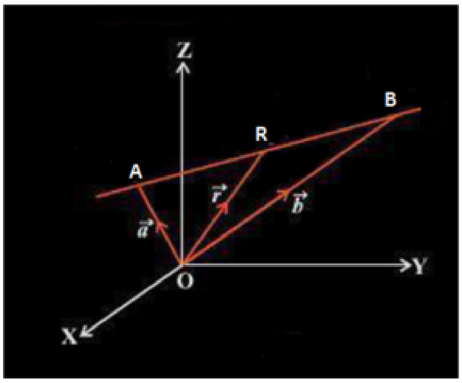
Eliminating α we have:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

This gives us the Cartesian equation of line.

Equation of a Line passing through two given points

Let us consider that the position vector of the two given points A and B be $\vec{a}$ and $\vec{b}$ with respect to the origin. Let us choose any random point R on the line and its position vector with respect to origin of the rectangular co-ordinate system is given by $\vec{r}$.



Point R lies on the line AB if and only if the vectors $\overline{AR}$ and $\overline{AB}$ are collinear. Also,

$\overline{AR} = \vec{r} - \vec{a}$

$\overline{AB} = \vec{b} - \vec{a}$

Thus, R lies on AB only if;

$\vec{r} - \vec{a} = \alpha(\vec{b} - \vec{a})$

Here α is any real number.

From the above equation it can be seen that for different values of α, the above equation gives the position of any arbitrary point R lying on the line passing through point A and B. Therefore, the vector equation of a line passing through two given points is given by:

$\vec{r} = \vec{a} + \alpha(\vec{b} - \vec{a})$

If the three-dimensional coordinates of the points A and B are given as (x_1, y_1, z_1) and (x_2, y_2, z_2) then considering the rectangular co-ordinates of point R as (x, y, z)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Substituting these values in the vector equation of a line passing through two given points and equating the coefficients of unit vectors i, j and k, we have

$$x = x_1 + \alpha(x_2 - x_1);\ y = y_1 + \alpha(y_2 - y_1);\ z = z_1 + \alpha(z_2 - z_1)$$

Eliminating α we have:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

This gives us the Cartesian equation of a line.

Solved Examples

Example 1: The equation of a line is given by, 2x – 6y +3 = 0. Find the slope and both the intercepts.

Solution:

The given equation 2x – 6y + 3 = 0 can be represented in slope-intercept form as:

y = x/3 + 1/2

Comparing it with y = mx + c,

Slope of the line, m = 1/3

Also, the above equation can be re-framed in intercept form as;

x/a + y/b = 1

2x – 6y = -3

x/(-3/2) – y/(-1/2) = 1

Thus, x-intercept is given as a = -3/2 and y-intercept as b = 1/2.

Example 2: The equation of a line is given by, 13x – y + 12 = 0. Find the slope and both the intercepts.

Solution: The given equation 13x – y + 12 = 0 can be represented in slope-intercept form as:

y = 13x + 12

Comparing it with y = mx + c,

Slope of the line, m = 13

Also, the above equation can be re-framed in intercept form as;

$x/a + y/b = 1$

$13x – y = -12$

$x/(-12/13) + y/12 = 0$

Thus, x-intercept is given as $a = -12/13$ and y-intercept as $b = 12$.