MAXIMA NAD MINIMA

Maxima and Minima are one of the most common concepts in differential calculus. A branch of Mathematics called “Calculus of Variations” deals with the maxima and the minima of the functional. The calculus of variations is concerned with the variations in the functional, in which small change in the function leads to the change in the functional value.

The first variation is defined as the linear part of the change in the functional, and the second part of the variation is defined in the quadratic part. Functional is expressed as the definite integrals which involve the functions and their derivatives.

What are the Curves?

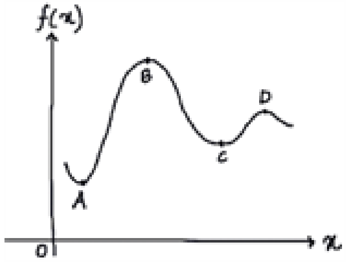


Fig. 1 Curves

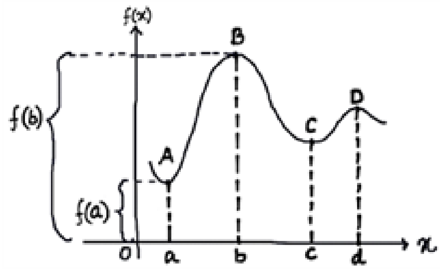


Fig. 2 Value of a Function

A curve is defined as one-dimensional continuum**.** In figure 1, that curve is graph of a function f in x. f(x) represents the value of function at x. The value of f when x=a, will be f(a). Similarly, for B, C and D. You can refer fig. 2 to understand this. From the figure it is quite clear that the value of the given function has its maximum value at x=b, i.e., f(b).

Interval of a function plays a very important role to find extreme values of a function. If the interval for which the function f is defined in R, then we can’t talk about maxima and minima of f. We can understand it logically that though f(b) appears to have the maximum value, we can’t be sure it has the largest value till we have seen the graph for its entire domain.

Local Maxima and Minima

We may not be able to tell whether f(b) is the maximum value of f, but we can give some credit to point. We can do this by declaring B as the local maximum for function f. These are also called relative maxima and minima. These local maxima and minima are defined as:

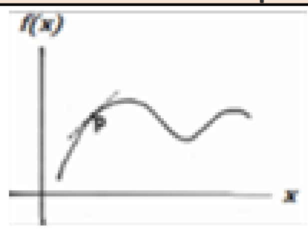
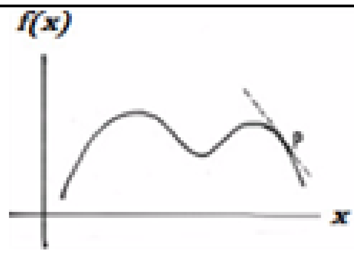
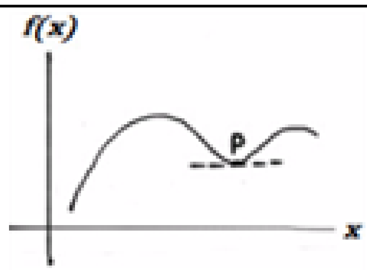
- If f(a)≤f(x) for all x in P′s neighbourhood (within the distance nearby P, where x=a), f is said to have a local minimum at x=a.
- If f(a)≥f(x) for all in P′s neighbourhood (within the distance nearby P, where x=a), f is said to have a local maximum at x=a.

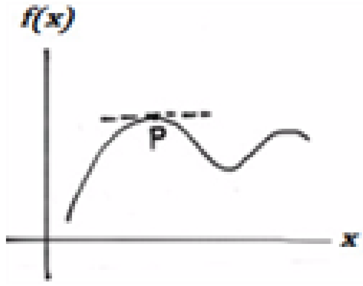
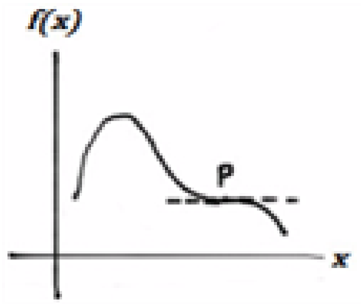
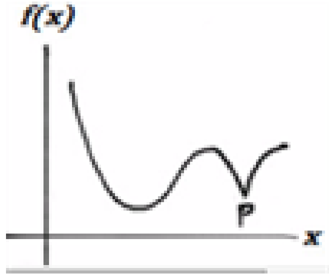
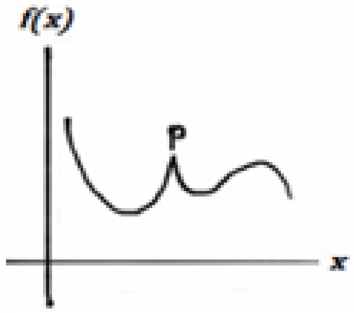
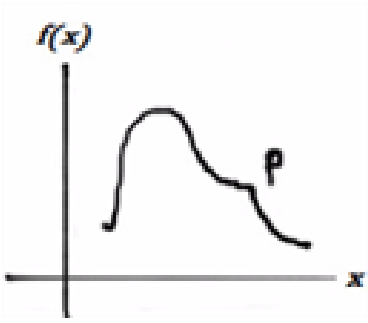
In the above example, B and D are local maxima and A and C are local minima. Local maxima and minima are together referred to as **Local extreme.**

Let us now take a point P, where x=a and try to analyze the nature of the derivatives. There are total of four possibilities:

- If f'(a)=0, the tangent drawn is parallel to x−axis, i.e., slope is zero. There are three possible cases:
    - The value of f, when compared to the value of f at P, increases if you move towards right or left of P (Local minima: look like valleys)
    - The value of f, when compared to the value of f at P, decreases if you move towards right or left of P (Local maxima: look like hills)
    - The value of f, when compared to the value of f at P, increases and decreases as you move towards left and right respectively of P (Neither: looks like a flat land)
- If, the tangent is drawn at a negative slope. The value of f'(a), at p, increases if you move towards left of and decreases if you move towards the right of. So, in this case, also, we can't find any local extrema.
- If, the tangent is drawn at a positive slope. The value of f'(a), at P, increases if you move towards the right of and decreases if you move towards left of. So, in this case, we can't find any local extrema.
- f' doesn't exist at point P, i.e., the function is not differentiable at P. This normally happens when the graph of f has a sharp corner somewhere. All the three cases discussed in the previous point also hold true for this point.

Various Possibilities of Derivatives of a Function

Nature of f'(a)	Nature of Slope	Example	Local Extremum
f'(a) > 0	Positive		*Neither*
f'(a) < 0	Negative		*Neither*
f'(a) = 0	Zero		*Local Minimum* Local Maximum

		 	Neither
Not Defined	Not Defined	  	*Local Minimum* *Local Maximum* Neither

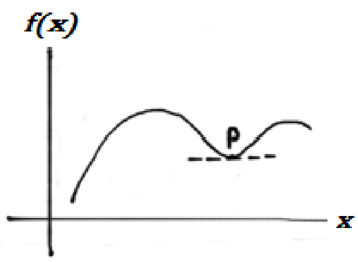
What is Critical Point?

In mathematics, a Critical point of a differential function of a real or complex variable is any value in its domain where its derivative is 0. We can hence infer from here that every local extremum is a critical point but every critical point need not be a local extremum. So, if we have a function which is continuous, it must have maxima and minima or local extrema. This means that every such function will have critical points. In case the given function is **monotonic**, the maximum and minimum values lie at the endpoints of the domain of the definition of that particular function.

First Derivative Test for Maxima and Minima

Let us see the first method i.e., **first derivative test**. This method is based on the basic concept of increasing and decreasing functions.

From the definition of the function, we can determine the critical points by f′ = 0. If at any point on the curve where x = a.f'(a), or is not differentiable at a, then is known as a critical point. If you could find out where the function is increasing and decreasing, we can tell whether the given **critical point** is a local maximum or minimum. But wait, there is a catch. This first derivative test definition holds true only if the function is continuous at the point x = a, where the test is being applied. If it's not continuous there, then you cannot apply this method. Let us try to understand this with an example.



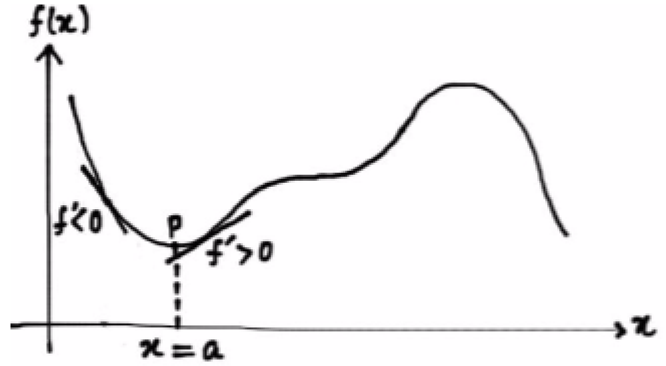
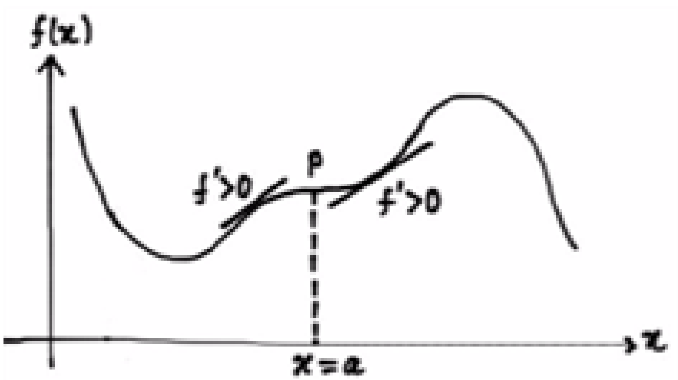
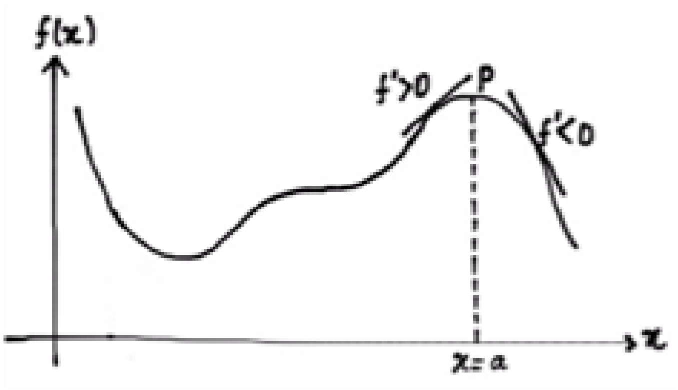
In the above figure, we can see that point P, where x=a,f′(a)=0 . What do you observe here? If you move towards just left of P, the function f is decreasing in nature and towards just right of P, it is increasing in nature. To put it in terms of derivatives, towards just left of P, f′(0)<0 and towards just right of P,f′(0) . This is an important observation. If we go with the converse of the previous observation and find that f′<0 and f′>0 towards just left and right of point P, we can conclude that point P is a local minimum. Now, what do we mean by "just left" and "just right" of point P? When we say f′(x)>0 towards just left of point P, it means that there is a positive width h and interval (a–h,a)for which f′>0 . Similarly, for f′(x)>0 towards just right of P means that there is a positive width and interval (a,a+k) for which f′>0.

Let us assume a function which has a critical point at P(x=a) and is also continuous at x=a. We can hence conclude whatever we have discussed in three points:

- If f′ is negative towards just left of a and it is positive towards just right of a , then the pointP(x=a)is a local minimum for the function f.
- If f′ is positive towards just left of a and it is negative towards just right of a , then the point P(x=a) is a local maximum for the function f.
- If f′ has same sign towards just left and just right of a, then the point P(x=a) is neither a local maximum nor a local minimum for the function f.

To summarize what we have learned, let us put it in a tabular form.

Different cases possible for first derivative method

Pictorial representation of f' towards left and right of point P(In all cases, f'(a) = 0)	**Nature of f' Towards just left**	**Towards just right**	**Whether Local maximum or minimum**
	Negative	Positive	Local Minimum
	Positive	Positive	Neither
	Positive	Negative	Local Maximum

Example:

Let us take an example to understand the application of this test. We will try to find out all the critical points for the function and tell whether they are local maxima or minima.

$f(x)=0.2x^5+1.25x^4+2x^3+2015^{2016}$

Since it is a polynomial function, it is both continuous and differentiable in its entire domain.

$f(x)=0.2x^5+1.25x^4+2x^3+2015^{2016}$

Differentiating both sides w. r. t., we get:

$f'(x)=x^4+5x^3+6x^2+0$
$f'(x)=x^2(x^2+5x+6)$
$f'(x)=x^2(x+3)(x+2)$ —————- (1)
For getting the critical points, we equate to 0,
$f'(x)=0$
$x^2(x+3)(x+2)=0$
$x=0,-2,-3$
So, our critical points are 0, -2 and -3.

From equation 1,
$f'(-2.5) < 0$ $f'(-1) > 0$
The sign of f ' is changing from negative to positive as we vary from left to right of . This means that is local minimum.
$f'(-3.5) > 0$
$f'(-2.5) < 0$ The sign of f ' is changing from positive to negative as we vary from left to right of . This means that is local maximum.
$f'(-1) > 0$
$f'(1) > 0$
The sign of f′ is not changing as we vary from left to right of. This means that is neither a local maximum nor a local minimum.

Second Derivative Test to Find Maxima & Minima

Let us consider a function f defined in the interval I and let $c \in I$. Let the function be twice differentiable at c. Then,

(i) Local Minima: x= c, is a point of local minima, if $f'(c)=0$ and $f''(c)>0$. The value of local minima at the given point is f(c).

(ii) Local Maxima: x= c, is a point of local maxima, where $f'(c)=0$ and $f''(c)<0$. The value of local maxima at the given point is f(c).

(iii) If in case $f'(c)=0$ and $f''(c)=0$, the second derivative test fails. Thus we go back to the first derivative test.

Working rules:

- In the given interval in f, find all the critical points.
- Calculate the value of the functions at all the points found in step (i) and also at the end points.
- From the above step, identify the maximum and minimum value of the function, which are said to be absolute maximum and absolute minimum value of the function.

Point of Inflection

If the value of the function does not change the sign as x increases from c, then c is neither a point of Local Maxima or Minima. This is known as Point of Inflection.

Example: Find all the local maxima and minima of the given function.

$f(x) = \frac{3}{4}x^4 + 8x^3 + \frac{45}{2}x^2 + 250$

Solution: Given $f(x) = \frac{3}{4}x^4 + 8x^3 + \frac{45}{2}x^2 + 250$

$f'(x) = 3x^3 + 24x^2 + 45x$

$= 3x(x^2 + 8x + 15) = 3x(x+3)(x+5)$

$f'(x) = 0$

$\Rightarrow x = 0 \;\; or \;\; x = -3 \;\; or \;\; x = -5$

Critical Points = 0, -3, -5

Now, $f''(x) = 9x^2 + 48x + 45$

$= 3(3x^2 + 16x + 15)$

Now checking the value of functions at all the critical point, we have:

$f''(0) = 45 > 0$, point of local minima.

$f''(-3) = -18$, point of local maxima.

$f''(-5) = 30$,, point of local minima.