EQUATION IN A PLANE

Equation of a Plane in the Normal and Cartesian Form

The vector form of the equation of a plane in normal form is given by:

$\vec{r} \cdot \hat{n} = d$

Where $\vec{r}$ is the position vector of a point in the plane, *n* is the unit normal vector along the normal joining the origin to the plane and *d* is the perpendicular distance of the plane from the origin.

Let P (x, y, z) be any point on the plane and O is the origin. Then, we have,

$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now the direction cosines of $\hat{n}$ as *l*, *m* and *n* are given by:

$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

From the equation $\vec{r} \cdot \hat{n} = d$ we get

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = d$

Thus, the Cartesian form of the equation of a plane in normal form is given by:

lx + my + nz = d

Example 1: A plane is at a distance of 9/√38 from the origin O. From the origin, its normal vector is given by 5i^ + 3j^ – 2k^.

What is the vector equation for the plane?

Solution:

Let the normal vector be:

$\vec{n} = 5\hat{i} + 3\hat{j} - 2\hat{k}$

We now find the unit vector for the normal vector. It can be given by:

$$\vec{n} = \frac{\vec{n}}{|\vec{n}|}$$

$$\vec{n} = \frac{5\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{25+9+4}}$$

$$\vec{n} = \frac{5\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{38}}$$

So, the required equation of the plane can be given by substituting it in the vector equation is:

$\vec{r} \cdot \left(\frac{5}{\sqrt{38}}\hat{i} + \frac{3}{\sqrt{38}}\hat{j} + \frac{-2}{\sqrt{38}}\hat{k}\right) = \frac{9}{\sqrt{38}}$

Example 2: Find the cartesian equations for the following planes.

(a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k} = 2$

(b) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k} = 1$

Solution:

(a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k})=2$ -------(1)

We know that for any arbitrary point, P(x, y, z)on the plane, the position vector is given as:

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, substitute the value of $\vec{r}$ in equation (1), we get

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow x + y - z = 2$

Thus, the cartesian equation of the plane is x + y – z = 2.

(b) $\vec{r} .(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$ ---------(2)

We know that for any arbitrary point, P(x, y, z)on the plane, the position vector is given as:

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, substitute the value of $\vec{r}$ in equation (2), we get

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

$\Rightarrow 2x + 3y - 4z = 1$

Thus, the cartesian equation of the plane is 2x + 3y – 4z = 1.

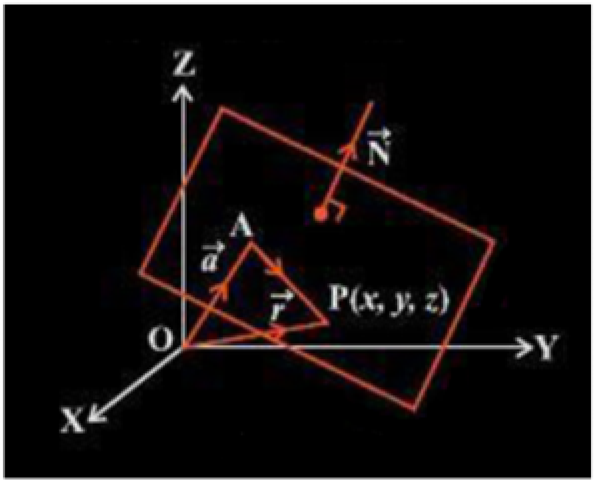
Equation of a Plane in Three Dimensional Space

Generally, the plane can be specified using four different methods. They are:

- Two intersecting lines
- A line and point (not on a line)
- Three non-collinear points (Three points are not on the line)
- Two parallel and the non-coincident line
- The normal vector and the point

There are infinite planes that lie perpendicular to a specific vector. But only one unique plane exists to a specific point which remains perpendicular to the point while going through it

Let us consider a plane passing through a given point A having position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$. Let us consider a point P(x, y, z) lying on this plane and its position vector is given by $\vec{r}$ as shown in the figure given below.



Position vector simply denotes the position or location of a point in the three-dimensional Cartesian system with respect to a reference origin.

For point P to lie on the given plane it must satisfy the following condition:

$\vec{AP}$ is perpendicular to $\vec{N}$, i.e. $\vec{AP}.\vec{N}$ = 0

From the figure given above it can be seen that,

$\vec{AP} = (\vec{r} - \vec{a})$

Substituting this value in $\vec{AP} \cdot \vec{N} = 0$, we have $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

This equation represents the vector equation of a plane.

We will assume that P, Q and R points are regarded as x_1, y_1, z_1 and x_2, y_2, z_2 in respectively to change the equation into the Cartesian system. A, B and C will be the assumed direction ratios. Thus,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = x_{1\hat{i}} + y_{1\hat{j}} + z_{1\hat{k}}$

$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$

Substituting these values in the vector equation of a plane, we have

$(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

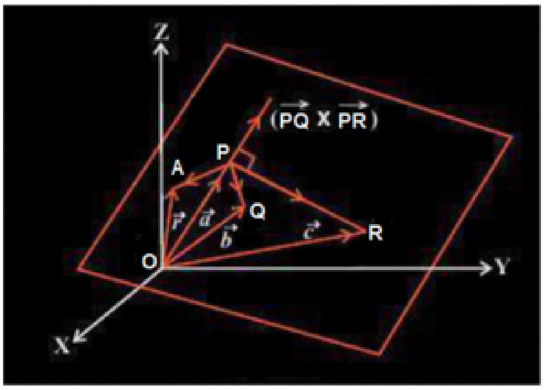
$((x\hat{i} + y\hat{j} + z\hat{k}) - (x_{1\hat{i}} + y_{1\hat{j}} + z_{1\hat{k}})) \cdot A\hat{i} + B\hat{j} + C\hat{k} = 0$

$[(x - x_1)\,\hat{i} + (y - y_1)\,\hat{j} + (z - z_1)\,\hat{k}]$ (Ai^+Bj^+Ck^) = 0

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

Equation of a plane passing through three Non collinear points

Let us consider three non collinear points P, Q, R lying on a plane such that their position vectors are given by $\vec{a}$, $\vec{b}$ and $\vec{c}$ as shown in the figure given below.



The vectors $\vec{PQ}$ and $\vec{PR}$ lie in the same plane. The vector lying perpendicular to plane containing the points P, Q and R is given by $\vec{PQ} \times \vec{PR}$. If it is the position vector of any point A lying in the plane containing P, Q, R then using the vector equation of a plane as mentioned above, the equation of the plane passing through P and perpendicular to the vector $\vec{PQ} \times \vec{PR}$ is given by

$(\vec{r} - \vec{a}) \,.\, (\vec{PQ} \times \vec{PR}) = 0$

Also, from the above figure and substituting these values in the above equation, we have

$(\vec{r} - \vec{a}) \,.\, [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

This represents the equation of a plane in vector form passing through three points which are non-collinear.

To convert this equation in Cartesian system, let us assume that the coordinates of the point P, Q and R are given as (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Also let the coordinates of point A be x, y and z.

$$\overrightarrow{PA} = (x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}$$

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{PR} = (x_3 - x_1)\hat{\imath} + (y_3 - y_1)\hat{\jmath} + (z_3 - z_1)\hat{k}$$

Substituting these values in the equation of a plane in Cartesian form passing through three non-collinear points, we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the equation of a plane in Cartesian form passing through three points which are non-collinear.

Intercept form of the Equation of the Plane

There are infinite number of planes which are perpendicular to a particular vector as we have already discussed in our earlier sections. But when talking of a specific point only one exclusive plane occurs which is perpendicular to the point going through the given area. This can be denoted by this particular vector equation:

$(\vec{r} - \vec{a}) \,.\, \vec{N}$

Here, $\vec{r}$ and $\vec{a}$ denote the position vector

The denotation of this type of plane in a Cartesian equation is the following:

$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$

The direction ratios here are denoted by A, B, and C.

Also the equation of a plane crossing the three non-collinear points in vector form is given as:

$(\vec{r} - \vec{a}) \,.\, [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

The equation of a plane in Cartesian form passing through three non-collinear points is given as:

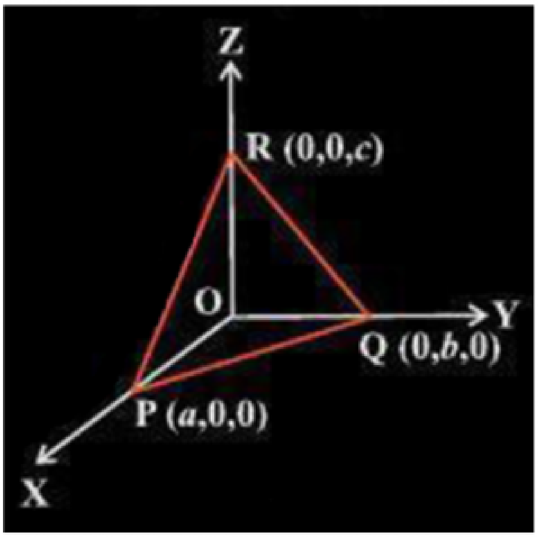
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Let us now discuss the equation of a plane in intercept form.

The general equation of a plane is given as:

$Ax + By + Cz + D = 0 \ (D \neq 0)$

Let us now try to determine the equation of a plane in terms of the intercepts which is formed by the given plane on the respective co-ordinate axes. Let us assume that the plane makes intercepts of a, b and c on the three co-ordinate axes respectively. Thus, the coordinates of the point of intersection of the plane with x, y and z axes are given by (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.



Substituting these values in the general equation of a plane, we have

Aa + D = 0

Bb + D = 0

Cc + D = 0

From the above three equations, we have

$$A = -\frac{D}{a}, B = -\frac{D}{b}, C = -\frac{D}{c}$$

Substituting these values of A, B, c and D in the general equation of the plane, we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This gives us the required equation of a plane in the intercept form.