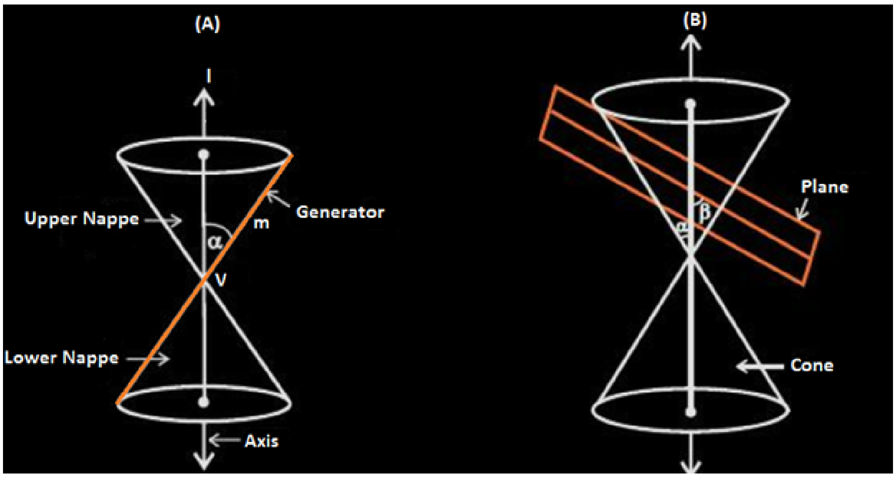
CONIC SECTIONS

What are Conic Sections?

A curve, generated by intersecting a right circular cone with a plane is termed as **'conic'**. It has distinguished properties in Euclidean geometry. The vertex of the cone divides it into two nappes referred to as the upper nappe and the lower nappe.



In figure B, the cone is intersected by a plane and the section so obtained is known as a conic section. Depending upon the position of the plane which intersects the cone and the angle of intersection β, different types of conic sections are obtained. Namely;

- Circle
- Ellipse
- Parabola
- Hyperbola

Conic Section Formulas

Circle	$(x-a)^2 + (y-b)^2 = r^2$	Center is (a,b) Radius is r
Ellipse with the horizontal major axis	$(x-a)^2/h^2 + (y-b)^2/k^2 = 1$	Center is (a, b) Length of the major axis is 2h. Length of the minor axis is 2k. Distance between the centre and either focus is c with $c^2=h^2-k^2$, h>k>0
Ellipse with the vertical major axis	$(x-a)^2/k^2 + (y-b)^2/h^2 = 1$	Center is (a, b) Length of the major axis is 2h. Length of the minor axis is 2k. Distance between the centre and either focus is c with $c^2=h^2-k^2$, h>k>0
Hyperbola with the horizontal transverse axis	$(x-a)^2/h^2-(y-b)^2/k^2=1$	Center is (a,b) Distance between the vertices is 2h

		Distance between the foci is 2k. $c^2=h^2+k^2$
Hyperbola with the vertical transverse axis	$(x-a)^2/k^2-(y-b)^2/h^2=1$	Center is (a,b) Distance between the vertices is 2h Distance between the foci is 2k. $c^2= h^2+k^2$
Parabola with the horizontal axis	$(y-b)^2=4p(x-a),\ p\neq0$	Vertex is (a,b) Focus is (a+p,b) Directrix is the line $x=a-p$ Axis is the line $y=b$
Parabola with vertical axis	$(x-a)^2=4p(y-b),\ p\neq0$	Vertex is (a,b) Focus is (a+p,b) Directrix is the line $x=b-p$ Axis is the line $x=a$

Focus, Eccentricity and Directrix of Conic

A conic section can also be described as the locus of a point P moving in the plane of a fixed point F known as **focus (F)** and a fixed line d known as **directrix** (with the focus not on d) in such a way that the ratio of the distance of point P from focus F to its distance from d is a constant e known as **eccentricity**. Now,

- If eccentricity, $e = 0$, the conic is a circle
- If $0<e<1$, the conic is an ellipse
- If $e=1$, the conic is a parabola
- And if $e>1$, it is a hyperbola

So, eccentricity is a measure of the deviation of the ellipse from being circular. Suppose, the angle formed between the surface of the cone and its axis is β and the angle formed between the cutting plane and the axis is α, the eccentricity is;

$e = \cos\alpha/\cos\beta$

Parameters of Conic

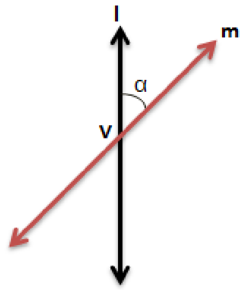
Apart from focus, eccentricity and directrix, there are few more parameters defined under conic sections.

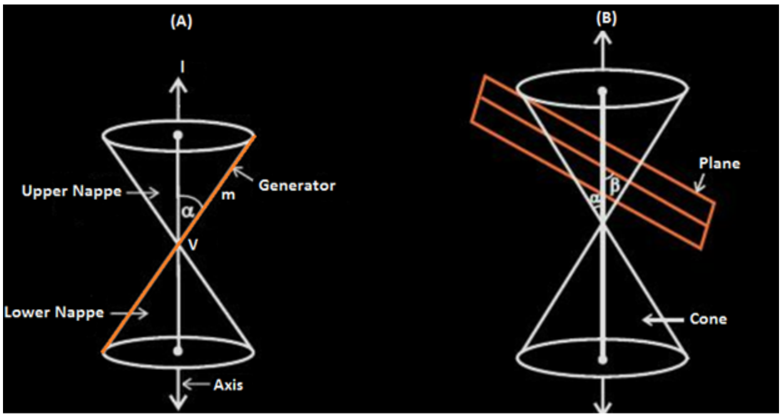
- **Principal Axis:** Line joining the two focal points or foci of ellipse or hyperbola. Its midpoint is the centre of the curve.
- **Linear Eccentricity:** Distance between the focus and centre of a section.
- **Latus Rectum:** A chord of section parallel to directrix, which passes through a focus.
- **Focal Parameter:** Distance from focus to the corresponding directrix.
- **Major axis:** Chord joining the two vertices. It is the longest chord of an ellipse.

- **Minor axis:** Shortest chord of an ellipse.

Sections of the Cone

Consider a fixed vertical line ‘l’ and another line ‘m’ inclined at an angle ‘α’ intersecting ‘l’ at point V as shown below:



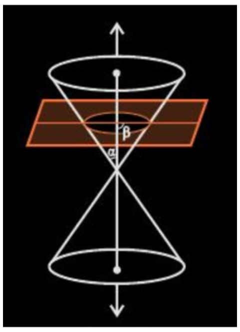


The initials as mentioned in the above figure A carry the following meanings:

1. V is the vertex of the cone
2. l is the axis of the cone
3. m, the rotating line the is a generator of the cone

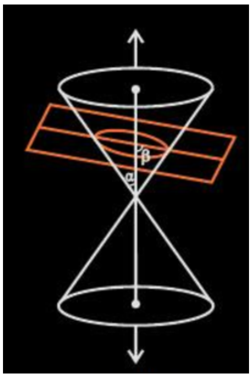
Conic Section Circle

If $\beta=90^{o}$, the conic section formed is a circle as shown below.



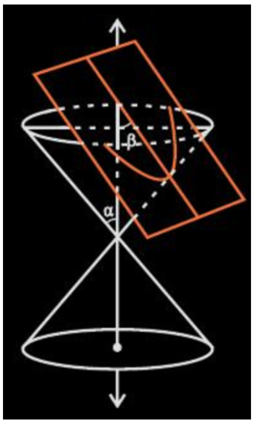
Conic Section Ellipse

If α<β<90°, the conic section so formed is an ellipse as shown in the figure below.



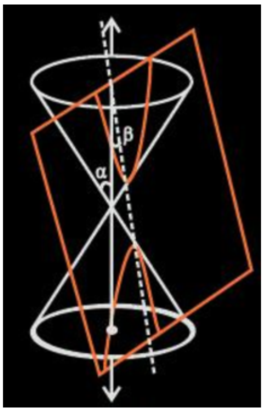
Conic Section Parabola

If α=β, the conic section formed is a parabola (represented by the orange curve) as shown below.



Conic Section Hyperbola

If 0 ≤ β < α, then the plane intersects both nappes and the conic section so formed is known as a hyperbola (represented by the orange curves).



Conic Section Standard Forms

After the introduction of Cartesian coordinates, the focus-directrix property can be utilised to write the equations provided by the points of the conic section. When the coordinates are changed along with the rotation and translation of axes, we can put these equations into standard forms. For ellipses and hyperbolas, the standard form has the x-axis as the principal axis and the origin (0,0) as the centre. The vertices are (±a, 0) and the foci (±c, 0). Define b by the equations $c^2= a^2 - b^2$ for an ellipse and $c^2 = a^2 + b^2$ for a hyperbola.

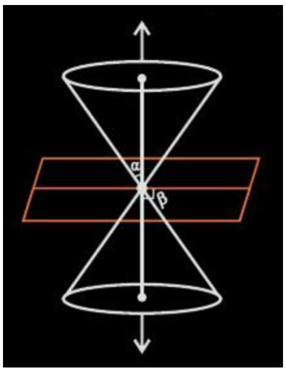
For a circle, c = 0 so $a^2 = b^2$. For the parabola, the standard form has the focus on the x-axis at the point (a, 0) and the directrix is the line with equation $x = -a$. In standard form, the parabola will always pass through the origin.

- Circle: $x^2+y^2=a^2$
- Ellipse: $x^2/a^2 + y^2/b^2 = 1$
- Hyperbola: $x^2/a^2 – y^2/b^2 = 1$
- Parabola: $y^2=4ax$ when $a>0$

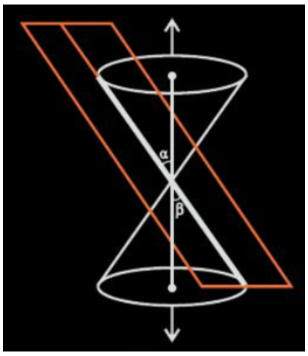
Conic Sections Examples

If the plane intersects exactly at the vertex of the cone, the following cases may arise:

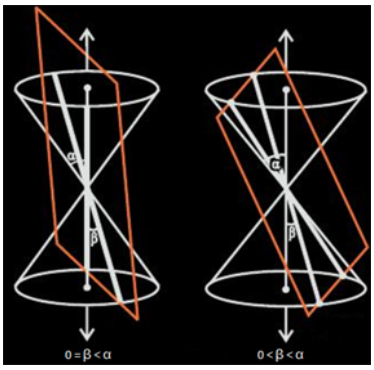
- If $\alpha< \beta\leq 90^\circ$, then the plane intersects the vertex exactly at a point.



- If $\alpha=\beta$, the plane upon an intersection with a cone forms a straight line containing a generator of the cone. This condition is a degenerated form of a parabola.



- If 0≤β<α, the section formed is a pair of intersecting straight lines. This condition is a degenerated form of a hyperbola.



Conic Sections Equations

Conic section Name	Equation when the centre is at the Origin, i.e. (0, 0)	Equation when centre is (h, k)
Circle	$x^2 + y^2 = r^2$; r is the radius	$(x – h)^2 + (y – k)^2 = r^2$; r is the radius
Ellipse	$(x^2/a^2) + (y^2/b^2) = 1$	$(x – h)^2/a^2 + (y – k)^2/b^2 = 1$
Hyperbola	$(x^2/a^2) – (y^2/b^2) = 1$	$(x – h)^2/a^2 – (y – k)^2/b^2 = 1$
Parabola	$y^2 = 4ax$, where a is the distance from the origin to the focus	

Eccentricity of Conic Sections

We know that there are different conics such as a parabola, ellipse, hyperbola and circle. The eccentricity of the conic section is defined as the distance from any point to its focus, divided by the perpendicular distance from that point to its nearest directrix. The eccentricity value is constant for any conics.

For any conic section, there is a locus of a point in which the distances to the point (focus) and the line (directrix) are in the constant ratio. That ratio is known as eccentricity, and the symbol “e denotes it”.

Eccentricity Formula

The formula to find out the eccentricity of any conic section is defined as:

Eccentricity, e = c/a

Where,

c = distance from the centre to the focus

a = distance from the centre to the vertex

For any conic section, the general equation is of the quadratic form:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Eccentricity of Circle

A circle is defined as the set of points in a plane that are equidistant from a fixed point in the plane surface called “centre”. The term “radius” defines the distance from the centre and the point on the circle. If the centre of the circle is at the origin, it will be easy to derive the equation of a circle. The equation of the circle is derived using the below-given conditions.

If “r’ is the radius and C (h, k) be the centre of the circle, by the definition, we get, | CP | = r.

We know that the formula to find the distance is,

$\sqrt{[(x - h)^2 + (y - k)^2]} = r$

Take Square on both the sides, we get

$(x - h)^2 + (y - k)^2 = r^2$

Thus, the equation of the circle with centre C(h, k) and radius “r” is $\mathbf{(x - h)^2 + (y - k)^2 = r^2}$

Also, the eccentricity of the circle is equal 0, i.e., **e = 0.**

Eccentricity of Parabola

A parabola is defined as the set of points P in which the distances from a fixed point F (focus) in the plane are equal to their distances from a fixed-line l(directrix) in the plane. In other words, the distance from the fixed point in a plane bears a constant ratio equal to the distance from the fixed-line in a plane.

Therefore, the eccentricity of the parabola is equal 1, i.e., **e = 1.**

The general equation of a parabola is written as $x^2 = 4ay$ and the eccentricity is given as 1.

Eccentricity of Ellipse

An ellipse is defined as the set of points in a plane in which the sum of distances from two fixed points is constant. In other words, the distance from the fixed point in a plane bears a constant ratio less than the distance from the fixed-line in a plane.

Therefore, the eccentricity of the ellipse is less than 1, i.e., **e < 1.**

The general equation of an ellipse is written as:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the eccentricity formula is written as $\sqrt{1 - \frac{b^2}{a^2}}$

For an ellipse, a and b are the lengths of the semi-major and semi-minor axes respectively.

Eccentricity of Hyperbola

A hyperbola is defined as the set of all points in a plane in which the difference of whose distances from two fixed points is constant. In other words, the distance from the fixed point in a plane bears a constant ratio greater than the distance from the fixed-line in a plane.

Therefore, the eccentricity of the hyperbola is greater than 1, i.e., **e > 1.**

The general equation of a hyperbola is given as

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the eccentricity formula is written as $\sqrt{1 + \frac{b^2}{a^2}}$

For any hyperbola, a and b are the lengths of the semi-major and semi-minor axes respectively.

Example: Find the eccentricity of the ellipse for the given equation $9x^2 + 25y^2 = 225$

Solution:

Given :

$9x^2 + 25y^2 = 225$

The general form of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

To make it in general form, divide both sides by 225, we get

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

So, the value of a = 5 and b = 3

From the formula of the eccentricity of an ellipse, $e = \sqrt{1 - \frac{b^2}{a^2}}$

Substituting a = 5 and b = 3,

$e = \sqrt{1 - \frac{3^2}{5^2}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}}$

$e = 4/5$

Therefore, the eccentricity of the given ellipse is 4/5.

Solved Examples

Example 1: Find an equation of the circle with centre at (0, 0) and radius r.

Solution: Given,

Centre = (h, k) = (0, 0)

Radius = r

Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 2: Find the equation of the circle with centre (–3, 2) and radius 4.

Solution: Given,

Centre = (h, k) = (-3, 2)

Radius = r = 4

Therefore, the equation of the required circle is

$(x - h)^2 + (y - k)^2 = r^2$

$(x + 3)^2 + (y - 2)^2 = 4^2$

$(x + 3)^2 + (y - 2)^2 = 16$

Example 3: Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Solution: The given equation is

$x^2 + y^2 + 8x + 10y - 8 = 0$

$(x^2 + 8x) + (y^2 + 10y) = 8$

By completing the squares within the parenthesis, we get

$(x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$

i.e. $(x + 4)^2 + (y + 5)2 = 49$

i.e. $[x - (-4)]^2 + [y - (-5)]^2 = 7^2$

Comparing with the standard form, h = -4, k = -5 and r = 7

Therefore, the given circle has centre at (– 4, –5) and radius 7.

Example 4: Find the equation of the circle which passes through the points (2, – 2), and (3, 4) and whose centre lies on the line x + y = 2.

Solution: Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Given that the circle passes through the points (2, –2) and (3, 4).

Thus,

$(2 - h)^2 + (-2 - k)^2 = r^2$….(1)

and $(3 - h)^2 + (4 - k)^2 = r^2$….(2)

Also, given that the centre lies on the line x + y = 2.

$\Rightarrow$ h + k = 2 ….(3)

Solving the equations (1), (2) and (3), we get

h = 0.7, k = 1.3 and $r^2 = 12.58$

Hence, the equation of the required circle is

$(x - 0.7)^2 + (y - 1.3)^2 = 12.58$

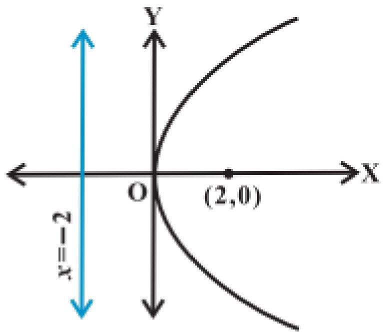
Example 5: Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

Solution: The given equation involves y^2, that means the axis of symmetry is along the x-axis.

The coefficient of x is positive so the parabola opens to the right.

Now by comparing the given equation with $y^2 = 4ax$,

a = 2



Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is x = –2 (see the figure)

Length of the latus rectum = 4a = 4 × 2 = 8

Example 6: Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2).

Solution: Given,

Vertex = (0,0)

Focus = (0,2)

The focus lies on the y-axis.

Thus, the y-axis is the axis of the parabola.

Therefore, the equation of the parabola is of the form $x^2 = 4ay$.

$\Rightarrow x^2 = 4(2)y$

$\Rightarrow x^2 = 8y$

This is the required equation of parabola.

Example 7: Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Solution: Given, $9x^2 + 4y^2 = 36$

The given equation of the ellipse can be written in standard form as:

$(x^2/4) + (y^2/9) = 1$

Here, 9 is greater than 4.

Thus, the major axis is along the y-axis.

By comparing with $(x^2/b^2) + (y^2/a^2) = 1$,

$a^2 = 9$, $b^2 = 4$

$\Rightarrow a = 3, b = 2$

$c = \sqrt{(a^2 - b^2)} = \sqrt{(9 - 4)} = \sqrt{5}$

$e = c/a = \sqrt{5}/3$

Therefore, the foci are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, vertices are $(0, 3)$ and $(0, -3)$, length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is $\sqrt{5}/3$.

Example 8: Find the equation of the hyperbola whose foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Solution: Given foci are $(0, \pm 12)$

That means $c = 12$.

Length of the latus rectum $= 2b^2/a = 36$ or $b^2 = 18a$

Now, $c^2 = a^2 + b^2$

$144 = a^2 + 18a$

i.e., $a^2 + 18a - 144 = 0$

$\Rightarrow a = -24, 6$

The value of a cannot be negative.

Therefore, $a = 6$ and so $b^2 = 108$.

Hence, the equation of the required hyperbola is $(y^2/36) – (x^2/108) = 1$ or $3y^2 – x^2 = 108$.