BINOMIAL DISTRIBUTION

In probability theory and statistics, the **binomial distribution** is the discrete probability distribution that gives only two possible results in an experiment, either **Success or Failure**. For example, if we toss a coin, there could be only two possible outcomes: heads or tails, and if any test is taken, then there could be only two results: pass or fail. This distribution is also called a binomial probability distribution.

There are two parameters n and p used here in a binomial distribution. The variable 'n' states the number of times the experiment runs and the variable 'p' tells the probability of any one outcome. Suppose a die is thrown randomly 10 times, then the probability of getting 2 for anyone throw is ⅙. When you throw the dice 10 times, you have a binomial distribution of n = 10 and p = ⅙.

Binomial Probability Distribution

In binomial probability distribution, the number of 'Success' in a sequence of n experiments, where each time a question is asked for yes-no, then the boolean-valued outcome is represented either with success/yes/true/one (probability p) or failure/no/false/zero (probability q = 1 − p). A single success/failure test is also called a Bernoulli trial or Bernoulli experiment, and a series of outcomes is called a **Bernoulli process**. For n = 1, i.e. a single experiment, the binomial distribution is a **Bernoulli distribution**. The binomial distribution is the base for the famous binomial test of statistical importance.

Bernoulli Trials

Many random experiments that we carry have only two outcomes that are either failure or success. For example, a product can be defective or non-defective, etc. These types of independent trials which have only two possible outcomes are known as Bernoulli trials. For the trials to be categorized as Bernoulli trials it must satisfy these conditions:

- A number of trials should be finite.
- The trials must be independent.
- Each trial should have exactly two outcomes: success or failure.
- The probability of success or failure remains does not change for each trial.

Example of Bernoulli Trials

Eight balls are drawn from a bag containing 10 white and 10 black balls. Predict whether the trials are Bernoulli trials if the ball drawn is replaced and not replaced.

Solution:

(a) For the **first** case, when a ball is drawn with replacement, the probability of success (say, white ball) is p=10/20=1/2 which is same for all eight trials (draws). Hence, the trial involving drawing of balls with replacements are said to be Bernoulli trials.

(b) For the **second** case, when a ball is drawn without replacement, the probability of success (say, white ball) varies with the number of trials. For example, for the first trial, probability of success, p=10/20 for second trial, probability of success, p=9/19 which is not equal the first trial. Hence, the trials involving drawing of balls without replacements are not Bernoulli trials.

Negative Binomial Distribution

In probability theory and statistics, the number of successes in a series of independent and identically distributed Bernoulli trials before a particularised number of failures happens. It is termed as the negative binomial distribution. Here the number of failures is denoted by 'r'. For instance, if we throw a dice and determine the occurrence of 1 as a failure and all non-1's as successes. Now, if we throw a dice frequently until 1 appears the third time, i.e., r = three failures, then the probability distribution of the number of non-1s that arrived would be the negative binomial distribution.

Binomial Distribution Examples

As we already know, binomial distribution gives the possibility of a different set of outcomes. In real life, the concept is used for:

- Finding the quantity of raw and used materials while making a product.
- Taking a survey of positive and negative reviews from the public for any specific product or place.
- By using the YES/ NO survey, we can check whether the number of persons views the particular channel.
- To find the number of male and female employees in an organisation.
- The number of votes collected by a candidate in an election is counted based on 0 or 1 probability.

Binomial Distribution Formula

The binomial distribution formula is for any random variable X, given by;

$$P(x:n,p) = {}^nC_x\, p^x\, (1-p)^{n-x}$$

Or

$$P(x:n,p) = {}^nC_x\, p^x\, (q)^{n-x}$$

Where,

n = the number of experiments

x = 0, 1, 2, 3, 4, …

p = Probability of Success in a single experiment

q = Probability of Failure in a single experiment = 1 – p

The binomial distribution formula can also be written in the form of n-Bernoulli trials, where ${}^nC_x = n!/x!(n-x)!$. Hence,

$$P(x:n,p) = n!/[x!(n-x)!].p^x.(q)^{n-x}$$

Binomial Distribution

Consider three Bernoulli trials for tossing a coin. Let obtaining head, stand for success, S and tails for failure, F. There are three ways in which we can have one success in three trials, {SFF, FSF, FFS}.

Similarly, two successes and one failure will have three ways. The general formula can be seen as ${}^{n}C_{r}$. Where 'n' stands for the number of trials and 'r' stands for number of success or failures.

The number of success for above cases can take four values 0,1,2,3.

Let '**a**' denote the probability of success and '**b**' denote the probability of failure. Random variable X denoting success can be given as:

P(X=0) = P(FFF) = P(F) × P(F) × P(F)

= b × b × b=b^3

And;

P(X=1) = P(SFF, FSF, FFS)

= P(S) × P(F) × P(F) + P(F) × P(S) × P(F) + P(F) × P(F) × P(S)

= a × b × b + b × a × b + b × b × a = $3ab^2$

And;

P(X=2) = P(SSF, SFS, FSS)

= P(S) × P(S) × P(F) + P(S) × P(F) × P(S) + P(F) × P(S) × P(S)

= a × a × b + b × a × b + b × b × a = $3a^2b$

And;

P(X=3) = P(SSS, SSS, SSS) = P(S) × P(S) × P(S)

= a × a × a = a^3

And;

The probability distribution is given as:

X	0	1	2	3
P(X)	b^3	$3ab^2$	$3a^2b$	a^3

We can relate it with binomial expansion of $(a + b)^3$ for determining probability of 0,1,2,3 successes.

As, $(a + b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$

For n trials, number of ways for x successes, S and (n-x) failures, F can be given as:

${}^{n}C_{x} = n! / (n-x)!(x)!$

In each way, the probability of x success and (n-x) failures:

P(S) × P(S) ×….. × P(S) × P(F) × ….. × P(F) × P(F) = $a^x\, b^{(n-x)}$

Thus, the probability of x successes in n-Bernoulli trials:

$$\frac{n!}{(n-x)!x!} \times a^x\, b^{(n-x)} = {}^{n}C_{x}\, a_x b^{(n-x)}$$

Hence, P(x) successes can be given by $(x+1)^{th}$ term in the binomial expansion of $(a + b)^x$

Probability distribution for above can be given as,

X(0, 1, 2, 3….x), P(X) = ${}^nC_0\, a^0\, b^n$

= ${}^nC_1\, a^1\, b^{n-1}$

= ${}^nC_2\, a^2\, b^{n-2}$

= ${}^nC_3\, a^3\, b^{n-3}$

= ${}^nC_x\, a^x\, b^{n-x}$

The above probability distribution is known as binomial distribution.

Binomial Distribution Mean and Variance

For a binomial distribution, the mean, variance and standard deviation for the given number of success are represented using the formulas

Mean, $\mu = np$

Variance, $\sigma^2 = npq$

Standard Deviation $\sigma = \sqrt{(npq)}$

Where p is the probability of success

q is the probability of failure, where q = 1-p

Binomial Distribution Vs Normal Distribution

The main difference between the binomial distribution and the normal distribution is that binomial distribution is discrete, whereas the normal distribution is continuous. It means that the binomial distribution has a finite amount of events, whereas the normal distribution has an infinite number of events. In case, if the sample size for the binomial distribution is very large, then the distribution curve for the binomial distribution is similar to the normal distribution curve.

Properties of Binomial Distribution

The properties of the binomial distribution are:

- There are two possible outcomes: true or false, success or failure, yes or no.
- There is 'n' number of independent trials or a fixed number of n times repeated trials.
- The probability of success or failure varies for each trial.
- Only the number of success is calculated out of n independent trials.
- Every trial is an independent trial, which means the outcome of one trial does not affect the outcome of another trial.

Solved Examples

Example 1: If a coin is tossed 5 times, find the probability of:

(a) Exactly 2 heads

(b) At least 4 heads.

Solution:

(a) The repeated tossing of the coin is an example of a Bernoulli trial. According to the problem:

Number of trials: n=5

Probability of head: p= 1/2 and hence the probability of tail, q =1/2

For exactly two heads:

x=2

$P(x=2) = {}^5C2\ p^2\ q^{5-2} = 5!\ /\ 2!\ 3! \times (½)^2 \times (½)^3$

P(x=2) = 5/16

(b) For at least four heads,

$x \geq 4$, $P(x \geq 4) = P(x = 4) + P(x=5)$

Hence,

$P(x = 4) = {}^5C4\ p^4\ q^{5-4} = 5!/4!\ 1! \times (½)^4 \times (½)^1 = 5/32$

$P(x = 5) = {}^5C5\ p^5\ q^{5-5} = (½)^5 = 1/32$

Therefore,

$P(x \geq 4) = 5/32 + 1/32 = 6/32 = 3/16$

Example 2: For the same question given above, find the probability of:

a) **Getting at least 2 heads**

Solution: P (at most 2 heads) = $P(X \leq 2) = P(X = 0) + P(X = 1)$

$P(X = 0) = (½)^5 = 1/32$

$P(X=1) = {}^5C_1\ (½)^{5.} = 5/32$

Therefore,

$P(X \leq 2) = 1/32 + 5/32 = 3/16$

Example 3: If a fair coin is tossed 8 times, find the probability of:

(a) Exactly 5 heads

(b) At least 5 heads.

Solution:

(a) The repeated tossing of the coin is an example of a Bernoulli trial. According to the problem:

Number of trials: n=8

Probability of head: a= 1/2 and hence the probability of tail, b =1/2

For exactly five heads:

x=5, P(x=5) = $^8C_5\ a^5\ b^{8-5}$ = 8!/3!5! × $(1/2)^5$ × $(1/2)^3$

= 8!/3!5! × $(1/2)^8$ 7/32

= 219/256

(b) For at least five heads,

x ≥ 5, P(x ≥ 5) = P(x = 5) + P(x = 6) + P(x = 7) + P(x=8)

And;

=$^8C_5\ a^5\ b^{8-5}$ **+** $^8C_6\ a^6\ b^{8-6}$ + $^8C_7\ a^7\ b^{8-7}$ **+** $^8C_8\ a^8\ b^{8-8}$

And;

$$= \frac{8!}{3!.5!} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^3 + \frac{8!}{2!.6!} \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^2 + \frac{8!}{1!.7!} \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^1 + \frac{8!}{0!.8!} \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^0$$

= 7/32 + 7/64 + 1/32 + 1/256 = 93/256