BAYES' THEOREM

Bayes' theorem describes the probability of occurrence of an event related to any condition. It is also considered for the case of conditional probability. Bayes theorem is also known as the formula for the probability of "causes". For example: if we have to calculate the probability of taking a blue ball from the second bag out of three different bags of balls, where each bag contains three different colour balls viz. red, blue, black. In this case, the probability of occurrence of an event is calculated depending on other conditions is known as conditional probability.

Conditional probability: Bayes' Theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes Theorem Statement

Let $E_1, E_2, \ldots, E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2, \ldots, E_n$ have nonzero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Bayes theorem,

$$P(E_i \mid A) = \frac{P\left(E_i\right) P\left(E_i\right)}{\sum_{k=1}^{n} P\left(E_k\right) P\left(E_k\right)}$$

for any k = 1, 2, 3, …., n

Bayes Theorem Proof

According to the conditional probability formula,

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Using the multiplication rule of probability,

$$P(E_i \cap A) = P(E_i)P(A \mid E_i) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Using total probability theorem,

$$P(A) = \sum_{k=1}^{n} P\left(E_k\right) P\left(E_k\right) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Putting the values from equations (2) and (3) in equation 1, we get

$$P(E_i | A) = \frac{P(E_i) P(E_i)}{\sum_{k=1}^{n} P(E_k) P(E_k)}$$

Note: The following terminologies are also used when the Bayes theorem is applied:

Hypotheses: The events E_1, E_2,… E_n is called the hypotheses

Priori Probability: The probability $P(E_i)$ is considered as the priori probability of hypothesis E_i

Posteriori Probability: The probability $P(E_i|A)$ is considered as the posteriori probability of hypothesis E_i

Bayes Theorem Formula

If A and B are two events, then the formula for Bayes theorem is given by:

P(A|B) = P(A∩B)/P(B)

Where P(A|B) is the probability of condition when event A is occurring while event B has already occurred.

P(A ∩ B) is the probability of event A and event B

P(B) is the probability of event B

Bayes Theorem Derivation

Bayes Theorem can be derived for events and random variables separately using the definition of conditional probability and density.

From the definition of conditional probability, Bayes theorem can be derived for events as given below:

P(A|B) = P(A ∩ B)/ P(B), where P(B) ≠ 0

P(B|A) = P(B ∩ A)/ P(A), where P(A) ≠ 0

Here, the joint probability P(A ∩ B) of both events A and B being true such that,

P(B ∩ A) = P(A ∩ B)

P(A ∩ B) = P(A | B) P(B) = P(B | A) P(A)

P(A|B) = [P(B|A) P(A)]/ P(B), where P(B) ≠ 0

Similarly, from the definition of conditional density, Bayes theorem can be derived for two continuous random variables namely X and Y as given below:

$$f_{X|Y=y}(x) = \frac{f_{X,Y(x,y)}}{f_Y(y)}$$

$$f_{Y|X=x}(y) = \frac{f_{X,Y(x,y)}}{f_X(x)}$$

Therefore,

$$f_{X|Y=y}(x) = \frac{f_{Y|X=x}(y) f_X(x)}{f_Y(y)}$$

Solved Examples

Example 1: A bag I contain 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags, and it is found to be black. Find the probability that it was drawn from Bag I.

Solution:

Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II, and A be the event of drawing a black ball.

Then, $P(E_1) = P(E_2) = ½$

Also, $P(A|E_1)$ = P(drawing a black ball from Bag I) = 6/10 = 3/5

$P(A|E_2)$ = P(drawing a black ball from Bag II) = 3/7

By using Bayes' theorem, the probability of drawing a black ball from bag I out of two bags,

$$P(E1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{2} x \frac{3}{5}}{\frac{1}{2} x \frac{3}{5} + \frac{1}{2} x \frac{3}{7}} = \frac{7}{12}$$

Example 2: A man is known to speak truth 2 out of 3 times. He throws a die and reports that the number obtained is a four. Find the probability that the number obtained is actually a four.

Solution:

Let A be the event that the man reports that number four is obtained.

Let E_1 be the event that four is obtained and E_2 be its complementary event.

Then, $P(E_1)$ = Probability that four occurs = 1/6

$P(E_2)$ = Probability that four does not occurs = $1 – P(E_1) = 1 − 1/6 = 5/6$

Also, $P(A|E_1)$ = Probability that man reports four and it is actually a four = 2/3

$P(A|E_2)$ = Probability that man reports four and it is not a four = 1/3

By using Bayes' theorem, probability that number obtained is actually a four,

$P(E_1|A) =$ $P(E_1)P(A|E_1)$

$$\overline{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{6} x \frac{2}{3}}{\frac{1}{6} x \frac{2}{3} + \frac{5}{6} x \frac{1}{3}} = \frac{2}{7}$$