

## METHODS OF INTEGRATION

The different methods of integration include:

- Integration by Substitution
- Integration by Parts
- Integration Using Trigonometric Identities
- Integration of Some particular function
- Integration by Partial Fraction

### Integration By Substitution

Sometimes, it is really difficult to find the integration of a function, thus we can find the integration by introducing a new independent variable. This method is called Integration By Substitution. This method is also called u-substitution.

In this method of integration, any given integral is transformed into a simple form of integral by substituting the independent variable by others.

Take for example an equation having independent variable in  $x$ , i.e.  $\int \sin(x^3) \cdot 3x^2 \cdot dx$  —————(i),

In the equation given above the independent variable can be transformed into another variable say  $t$ .

Substituting  $x^3 = t$  —————(ii)

Differentiation of above equation will give-

$3x^2 \cdot dx = dt$  —————(iii)

Substituting the value of (ii) and (iii) in (i), we have

$$\int \sin(x^3) \cdot 3x^2 \cdot dx = \int \sin t \cdot dt$$

Thus, the integration of the above equation will give

$$\int \sin t \cdot dt = -\cos t + c$$

Again putting back the value of  $t$  from equation (ii), we get

$$\int \sin(x^3) \cdot 3x^2 \cdot dx = -\cos(x^3) + c$$

The General Form of integration by substitution is:

$$\int f(g(x)) \cdot g'(x) \cdot dx = \int f(t) \cdot dt,$$

where  $t = g(x)$

Usually the method of integration by substitution is extremely useful when we make a substitution for a function whose derivative is also present in the integrand. Doing so, the function simplifies and then the basic formulas of integration can be used to integrate the function. To understand this concept better let us look into an example.

**Example: Find the integration of  $\int \frac{e^{\tan^{-1} x}}{1+x^2} \cdot dx$**

**Solution:**

Given  $\int \frac{e^{\tan^{-1} x}}{1+x^2} \cdot dx$

Let  $t = \tan^{-1} x \dots\dots\dots(i)$

$$\Rightarrow dt = \frac{1}{1+x^2} \cdot dx$$

$$I = \int e^t \cdot dt$$

$$= e^t + C \dots\dots\dots(ii)$$

Substituting the value of (i) in (ii), we have

$$I = e^{\tan^{-1} x} + C$$

**Example: Integrate  $2x \cos (x^2-5)$  with respect to x.**

**Solution:**

$$I = \int 2x \cos (x^2-5) \cdot Dx$$

Let  $x^2 - 5 = t \dots\dots\dots(i)$

$$\Rightarrow 2x \cdot dx = dt$$

Substituting these values, we have

$$I = \int \cos(t) \cdot dt$$

$$= \sin t + C \dots\dots\dots(ii)$$

Substituting the value of (i) in (ii), we have

$$= \sin (x^2-5) + C$$

This is the required integration for the given function.

## Integration By Parts

Integration by parts requires a special technique for integration of a function, where the integrand function is the multiple of two or more function. This method is also termed as partial integration.

If u and v are any two differentiable functions of a single variable x. Then, by the product rule of differentiation, we have;

$$d/dx(uv) = u(dv/dx) + v(du/dx)$$

By integrating both the sides, we get;

$$uv = \int u(dv/dx)dx + \int v(du/dx)dx$$

or

$$\int u(dv/dx)dx = uv - \int v(du/dx)dx \dots\dots\dots(1)$$

Now let us consider,

$$u=f(x) \text{ and } dv/dx = g(x)$$

Thus, we can write now;

$$du/dx = f'(x) \text{ and } v = \int g(x) dx$$

Therefore, now equation 1 becomes;

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int [\int g(x) dx] f'(x) dx$$

or

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int [f'(x) \int g(x) dx] dx$$

This is the basic formula which is used to integrate products of two functions by parts.

If we consider f as the first function and g as the second function, then this formula may be pronounced as:

**“The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]”.**

## ILATE Rule

Identify the function that comes first on the following list and select it as f(x).

ILATE stands for:

I: Inverse trigonometric functions : arctan x, arcsec x, arcsin x etc.

L: Logarithmic functions : ln x, log<sub>5</sub>(x), etc.

A: Algebraic functions.

T: Trigonometric functions, such as sin x, cos x, tan x etc.

E: Exponential functions.

## Integration by parts uv formula

As derived above, integration by parts uv formula is:

$$\int u \left( \frac{dv}{dx} \right) dx = uv - \int v \left( \frac{du}{dx} \right) dx$$

Here,

u = Function of u(x)

v = Function of v(x)

dv = Derivative of v(x)

$du$  = Derivative of  $u(x)$

## Integration by parts with limits

In calculus, definite integrals are referred to as the integral with limits such as upper and lower limits. It is also possible to derive the formula of integration by parts with limits. Thus, the formula is:

$$\int_a^b du \left( \frac{dv}{dx} \right) dx = [uv]_a^b - \int_a^b v \left( \frac{du}{dx} \right) dx$$

Here,

$a$  = Lower limit

$b$  = Upper limit

**Example: Evaluate  $\int x \cdot e^x dx$**

**Solution:** From ILATE theorem,  $f(x) = x$ , and  $g(x) = e^x$

Thus, using the formula for integration by parts, we have

$$\int f(x) \cdot g(x) dx = f(x) \int g(x) dx - \int f'(x) \cdot \left( \int g(x) dx \right) dx$$

$$\int x \cdot e^x dx = x \cdot \int e^x dx - \int 1 \cdot \left( \int e^x dx \right) dx$$

$$= x \cdot e^x - e^x + c$$

**Example: Evaluate  $\int_0^1 \arctan x \cdot dx$**

**Solution:** Let,  $u = \arctan x$ ,  $dv = dx$

$$du = \frac{1}{1+x^2} \cdot dx \quad v = x$$

Integration by parts-

$$\int_0^1 \arctan x \cdot dx = (x \arctan x)_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \left( \frac{\pi}{4} - 0 \right) - \left( \frac{1}{2} \ln(1+x^2) \right)_0^1$$

$$= \left( \frac{\pi}{4} \right) - \frac{1}{2} \ln 2$$

$$= \left( \frac{\pi}{4} \right) - \ln \sqrt{2}$$

## Integration Using Trigonometric Identities

Trigonometric Identities are useful whenever trigonometric functions are involved in an expression or an equation. Identity inequalities which are true for every value occurring on both sides of an

equation. Geometrically, these identities involve certain functions of one or more angles. There are various distinct identities involving the side length as well as the angle of a triangle. The trigonometric identities hold true only for the right-angle triangle.

The six basic trigonometric ratios are sine, cosine, tangent, cosecant, secant, and cotangent. All these trigonometric ratios are defined using the sides of the right triangle, such as an adjacent side, opposite side, and hypotenuse side. All the fundamental trigonometric identities are derived from the six trigonometric ratios.

### Trigonometric Identities List

There are various identities in trigonometry which are used for many trigonometric problems. Let us see all the fundamental trigonometric identities here.

#### Reciprocal Identities

- $\sin \theta = 1/\csc \theta$  or  $\csc \theta = 1/\sin \theta$
- $\cos \theta = 1/\sec \theta$  or  $\sec \theta = 1/\cos \theta$
- $\tan \theta = 1/\cot \theta$  or  $\cot \theta = 1/\tan \theta$

#### Pythagorean Identities

- $\sin^2 a + \cos^2 a = 1$
- $1 + \tan^2 a = \sec^2 a$
- $\operatorname{cosec}^2 a = 1 + \cot^2 a$

#### Ratio Identities

- $\tan \theta = \sin \theta / \cos \theta$
- $\cot \theta = \cos \theta / \sin \theta$

#### Opposite Angle Identities

- $\sin (-\theta) = -\sin \theta$
- $\cos (-\theta) = \cos \theta$
- $\tan (-\theta) = -\tan \theta$
- $\cot (-\theta) = -\cot \theta$
- $\sec (-\theta) = \sec \theta$
- $\csc (-\theta) = -\csc \theta$

#### Complementary Angles Identities

- $\sin (90 - \theta) = \cos \theta$
- $\cos (90 - \theta) = \sin \theta$

- $\tan(90 - \theta) = \cot \theta$
- $\cot(90 - \theta) = \tan \theta$
- $\sec(90 - \theta) = \csc \theta$
- $\csc(90 - \theta) = \sec \theta$

### Angle Sum and Difference Identities

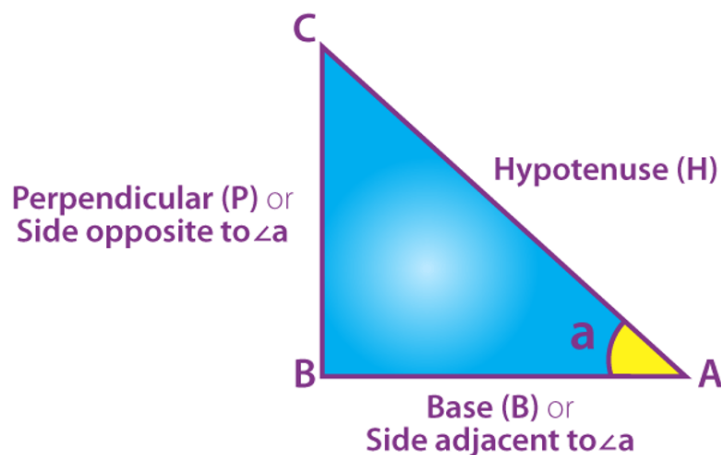
Consider two angles,  $\alpha$  and  $\beta$ , the trigonometric sum and difference identities are as follows:

- $\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$
- $\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$
- $\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$
- $\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$
- $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$
- $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$

### Trigonometric Identities Formula

Similarly, an equation which involves trigonometric ratios of an angle represents a trigonometric identity.

Consider the right angle  $\triangle ABC$  which is right-angled at B as shown in the given figure.



Applying Pythagoras Theorem for the given triangle, we have

$$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{perpendicular})^2$$

$$AC^2 = AB^2 + BC^2 \dots\dots\dots(1)$$

Let us prove the three Pythagoras trigonometric identities, which are commonly used.

## Trigonometric Identity 1

Now, divide each term of equation (1) by  $AC^2$ , we have

$$\begin{aligned}\frac{AC^2}{AC^2} &= \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} \\ \Rightarrow \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} &= 1 \\ \Rightarrow \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 &= 1 \dots\dots\dots(2)\end{aligned}$$

We know that,

$\left(\frac{AB}{AC}\right)^2 = \cos^2 a$  and  $\left(\frac{BC}{AC}\right)^2 = \sin^2 a$ , thus equation (2) can be written as-

$$\sin^2 a + \cos^2 a = 1$$

Identity 1 is valid for angles  $0 \leq a \leq 90$ .

## Trigonometric Identity 2

Now Dividing the equation (1) by  $AB^2$ , we get

$$\begin{aligned}\frac{AC^2}{AB^2} &= \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} \\ \Rightarrow \frac{AC^2}{AB^2} &= 1 + \frac{BC^2}{AB^2} \\ \Rightarrow \left(\frac{AC}{AB}\right)^2 &= 1 + \left(\frac{BC}{AB}\right)^2 \dots\dots\dots(3)\end{aligned}$$

By referring trigonometric ratios, it can be seen that:

$$\frac{AC}{AB} = \frac{\text{hypotenuse}}{\text{side adjacent to angle } a} = \sec a$$

Similarly,

$$\frac{BC}{AB} = \frac{\text{side opposite to angle } a}{\text{side adjacent to angle } a} = \tan a$$

Replacing the values of  $\frac{AC}{AB}$  and  $\frac{BC}{AB}$  in the equation (3) gives,

$$1 + \tan^2 a = \sec^2 a$$

As it is known that  $\tan a$  is not defined for  $a = 90^\circ$  therefore identity 2 obtained above is true for  $0 \leq a < 90$ .

### Trigonometric Identity 3

Dividing the equation (1) by  $BC^2$ , we get

$$\begin{aligned}\frac{AC^2}{BC^2} &= \frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} \\ \Rightarrow \frac{AC^2}{BC^2} &= \frac{AB^2}{BC^2} + 1 \\ \Rightarrow \left(\frac{AC}{BC}\right)^2 &= \left(\frac{AB}{BC}\right)^2 + 1 \dots\dots\dots(iv)\end{aligned}$$

By referring trigonometric ratios, it can be seen that:

$$\frac{AC}{BC} = \frac{\text{hypotenuse}}{\text{side opposite to angle } a} = \text{cosec } a$$

Also,

$$\frac{AB}{BC} = \frac{\text{side adjacent to angle } a}{\text{side opposite to angle } a} = \cot a$$

Replacing the values of  $\frac{AC}{BC}$  and  $\frac{AB}{BC}$  in the equation (4) gives,

$$\text{cosec}^2 a = 1 + \cot^2 a$$

Since cosec  $a$  and cot  $a$  are not defined for  $a = 0^\circ$ , therefore the identity 3 is obtained is true for all the values of 'a' except at  $a = 0^\circ$ . Therefore, the identity is true for all such that,  $0^\circ < a \leq 90^\circ$ .

### Triangle Identities

If the identities or equations are applicable for all the triangles and not just for right triangles, then they are the triangle identities. These identities will include:

- Sine law
- Cosine law
- Tangent law

If A, B and C are the vertices of a triangle and a, b and c are the respective sides, then;  
According to the sine law or sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

According to cosine law,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Or

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



According to tangent law,

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

**Example 1:** Consider a triangle ABC, right-angled at B. The length of the base, AB = 4 cm and length of perpendicular BC = 3 cm. Find the value of sec A.

**Solution:**

As the length of the perpendicular and base is given; it can be concluded that,

$$\tan A = 3/4$$

Now, using the trigonometric identity:  $1 + \tan^2 a = \sec^2 a$

$$\sec^2 A = 1 + (3/4)^2$$

$$\sec^2 A = 25/16$$

$$\sec A = \pm 5/4$$

Since, the ratio of lengths is positive, we can neglect  $\sec A = 5/4$ .

Therefore,  $\sec A = 5/4$

**Example 2:**  $(1 - \sin A)/(1 + \sin A) = (\sec A - \tan A)^2$

**Solution:** Let us take the Left hand side of the equation.

$$\text{L.H.S} = (1 - \sin A)/(1 + \sin A)$$

Multiply both numerator and denominator by  $(1 - \sin A)$

$$= (1 - \sin A)^2 / (1 - \sin A)(1 + \sin A)$$

$$= (1 - \sin A)^2 / (1 - \sin^2 A)$$

$$= (1 - \sin A)^2 / (\cos^2 A), [\text{Since } \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \cos^2 \theta = 1 - \sin^2 \theta]$$

$$= \{(1 - \sin A)/\cos A\}^2$$

$$= (1/\cos A - \sin A/\cos A)^2$$

$$= (\sec A - \tan A)^2$$

$$= \text{R.H.S.}$$

**Example 3:** Prove that:  $1/(\operatorname{cosec} A - \cot A) - 1/\sin A = 1/\sin A - 1/(\operatorname{cosec} A + \cot A)$

**Solution:**  $1/(\operatorname{cosec} A - \cot A) - 1/\sin A = 1/\sin A - 1/(\operatorname{cosec} A + \cot A)$

Now rearrange the following, such that;

$$1/(\operatorname{cosec} A - \cot A) + 1/(\operatorname{cosec} A + \cot A) = 2/\sin A$$

Now let us take the L.H.S.

$$= 1/(\operatorname{cosec} A - \cot A) + 1/(\operatorname{cosec} A + \cot A)$$

$$= (\operatorname{cosec} A + \cot A + \operatorname{cosec} A - \cot A)/(\operatorname{cosec}^2 A - \cot^2 A)$$

$$= (2 \operatorname{cosec} A)/1 \quad [\operatorname{cosec}^2 A = 1 + \cot^2 A \Rightarrow \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$= 2/\sin A \quad [\operatorname{cosec} A = 1/\sin A]$$

Hence, proved.

## Integration of Some particular function

S.No	Integral function	Integral value
1	$\int \frac{dx}{x^2 - a^2}$	$\frac{1}{2a} \log \left  \frac{x-a}{x+a} \right  + C$
2	$\int \frac{dx}{a^2 - x^2}$	$\frac{1}{2a} \log \left  \frac{a+x}{a-x} \right  + C$
3	$\int \frac{dx}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$
4	$\int \frac{dx}{\sqrt{x^2 - a^2}}$	$\log \left  x + \sqrt{x^2 - a^2} \right  + C$
5	$\int \frac{dx}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left( \frac{x}{a} \right) + C$
6	$\int \frac{dx}{\sqrt{x^2 + a^2}}$	$\log \left  x + \sqrt{x^2 + a^2} \right  + C$

## Proofs of Integrals Functions

### Integral of function 1

$$(1) \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

The integral function can be splitted into the sums of partial fraction, i.e.

$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x-a)(x+a)} = \int \frac{A}{(x-a)} \cdot dx + \int \frac{B}{(x+a)} \cdot dx \dots\dots\dots(i)$$

Solving for values of A and B, we have,

$$1 = A(x+a) + B(x-a),$$

Putting  $x = a$  and then  $-a$ , we get the values of A and B to be  $\frac{1}{2a}$  and  $-\frac{1}{2a}$  respectively.

Substituting these values in (i), we have

$$\begin{aligned} \int \frac{dx}{x^2-a^2} &= \int \frac{dx}{2a(x-a)} + \int \frac{-dx}{2a(x+a)} \\ &= \frac{1}{2a} \left[ \int \frac{dx}{(x-a)} - \int \frac{dx}{(x+a)} \right] \\ &= \frac{1}{2a} [\log|x-a| - \log|x+a|] + C \\ &= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \end{aligned}$$

## Integral of function 2

$$(2) \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

Breaking function into the sums of partial fraction, we have

$$\int \frac{dx}{a^2-x^2} = \int \frac{A}{a-x} \cdot dx + \int \frac{B}{a+x} \cdot dx$$

Solving for values of A and B, we have

$$\begin{aligned} \int \frac{dx}{a^2-x^2} &= \int \frac{dx}{2a(a-x)} + \int \frac{dx}{2a(a+x)} \\ &= \frac{1}{2a} \left[ \int \frac{dx}{(a-x)} + \int \frac{dx}{(a+x)} \right] \\ &= \frac{1}{2a} [-\log|a-x| + \log|a+x|] + C \\ &= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C \end{aligned}$$

### Integral of function 3

$$(3) \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$

Substituting  $x = a \tan \theta$ .....(i)

$$dx = a \sec^2 \theta . d\theta$$

$$\int \frac{dx}{x^2+a^2} = \int \frac{a \sec^2 \theta . d\theta}{a^2 \tan^2 \theta + a^2}$$

$$= \frac{1}{a} \int \frac{\sec^2 \theta . d\theta}{\sec^2 \theta} = \frac{1}{a} \int d\theta$$

$$= \frac{1}{a} \theta + C \text{ .....(ii)}$$

From (i), we know  $\theta = \tan^{-1} \frac{x}{a}$ ,

$$\text{therefore } \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

#### Integral of function 4

$$(4) \int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + C$$

Substituting  $x = a \sec \theta$  .....(i)

$$dx = a \sec \theta \tan \theta . d\theta$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta . d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}}$$

$$= \int \frac{a \sec \theta \tan \theta . d\theta}{a \sqrt{\tan^2 \theta}}$$

$$= \int \sec \theta . d\theta$$

$$= \log|\sec \theta + \tan \theta| + C_1 \text{ .....(ii)}$$

$$\text{from (i) we know } \sec \theta = \frac{x}{a} \text{ and } \tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\frac{x^2}{a^2} - 1}$$

Substituting these values in equation (ii), we have

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$

$$= \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C_1$$

$$= \log|x + \sqrt{x^2 - a^2}| - \log|a| + C_1$$

$$= \log|x + \sqrt{x^2 - a^2}| + C, (\text{where } C = C_1 + \log|a|)$$

### Integral of function 5

$$(5) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C$$

Putting  $x = a \sin \theta$

$$dx = a \cos \theta \cdot d\theta$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta \cdot d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}}$$

$$= \int \frac{a \cos \theta \cdot d\theta}{a \sqrt{1 - \sin^2 \theta}}$$

$$= \int \frac{a \cos \theta \cdot d\theta}{a \cos \theta} = \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1} \frac{x}{a} + C$$

### Integral of function 6

$$(6) \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x + \sqrt{x^2+a^2}| + C$$

Putting  $x = a \tan \theta$ .....(i)

$$dx = a \sec^2 \theta d\theta$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a \sec^2 \theta d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{a \sqrt{\tan^2 \theta + 1}} = \int \frac{a \sec^2 \theta d\theta}{a \sec \theta}$$

$$= \int \sec \theta \cdot d\theta$$

$$= \log|\sec \theta + \tan \theta| + c \text{.....(ii)}$$

$$\text{From (i), we have } \tan \theta = \frac{x}{a} \text{ and } \sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\frac{x^2}{a^2} + 1}$$

Putting these value in (ii), we get

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1}\right| + c$$

$$= \log\left|\frac{x + \sqrt{x^2+1}}{a}\right| + c$$

$$= \log|x + \sqrt{x^2+1}| - \log a + c$$

$$= \log|x + \sqrt{x^2+1}| + C \text{ (where } C = c - \log a \text{)}$$

### Integral of function 7

$$(7) \int \frac{dx}{ax^2+bx+c}$$

$$\text{the denominator can be written as } ax^2 + bx + c = a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a \left[ \left( x + \frac{b}{2a} \right)^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

$$\text{Substituting } x + \frac{b}{2a} = t, \text{ so } dx = dt$$

$$\text{Also } \frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$$

Hence the integral becomes,

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a} \int \frac{dt}{t^2 \pm k^2}$$

## Integral of function 8

$$(8) \int \frac{px+q}{ax^2+bx+c} \cdot dx$$

where p,q,a,b,c are constants

$$px + q = A \frac{d}{dx}(ax^2 + bx + c) = A(2ax + b) + B$$

we equate the coefficient of x of both the sides to determine the value of A and B, and hence the integral is reduced to one of the known forms.

**Example:** Find the Integral of the function  $\int \frac{dx}{\sqrt{7x^2-2x}}$ .

**Solution:** The given function can be converted into the standard form

$$\begin{aligned} \int \frac{dx}{\sqrt{7x^2-2x}} &= \int \frac{dx}{\sqrt{7} \cdot \sqrt{x^2 - \frac{2}{7}x}} \\ &= \frac{1}{\sqrt{7}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{7}\right)^2 - \left(\frac{1}{7}\right)^2}} \text{ (completing the squares)} \end{aligned}$$

Substituting  $x - \frac{1}{7} = t$ , so  $dx = dt$

$$\begin{aligned} \text{therefore } \int \frac{dx}{\sqrt{7x^2-2x}} &= \frac{1}{\sqrt{7}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{7}\right)^2}} \\ &= \frac{1}{\sqrt{7}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{7}\right)^2} \right| + C \\ &= \frac{1}{\sqrt{7}} \log \left| x - \frac{1}{7} + \sqrt{\left(x - \frac{1}{7}\right)^2 - \left(\frac{1}{7}\right)^2} \right| + C \\ &= \frac{1}{\sqrt{7}} \log \left| x - \frac{1}{7} + \sqrt{x^2 - \frac{2}{7}x} \right| + C \end{aligned}$$

## Integration by partial fraction

We know that a Rational Number can be expressed in the form of  $p/q$ , where p and q are integers and  $q \neq 0$ . Similarly, a rational function is defined as the ratio of two polynomials which can be expressed in the form of partial fractions:  $P(x)/Q(x)$ , where  $Q(x) \neq 0$ .

There are in general two forms of partial fraction:



- **Proper partial fraction:** When the degree of the numerator is less than the degree of the denominator, then the partial fraction is known as a proper partial fraction.
- **Improper partial fraction:** When the degree of the numerator is greater than the degree of the denominator then the partial fraction is known as an improper partial fraction. Thus, the fraction can be simplified into simpler partial fractions, that can be easily integrated.

### Partial Fraction Definition

An algebraic fraction can be broken down into simpler parts known as “partial fractions”. Consider an algebraic fraction,  $(3x+5)/(2x^2-5x-3)$ . This expression can be split into simple form like  $(2)/(x-3) - (1)/(2x+1)$ .

The simpler parts  $[(2)/(x-3)] - [(1)/(2x+1)]$  are known as partial fractions.

This means that the algebraic expression can be written in the form as given in the figure:

$$\frac{3X + 5}{2X^2 - 5X - 3} = \frac{2}{X - 3} - \frac{1}{2X + 1}$$

**Rational Expression**

**Partial Fractions**

**Note:** The partial fraction decomposition only works for the proper rational expression (the degree of the numerator is less than the degree of the denominator). In case, if the rational expression is in an improper rational expression (the degree of the numerator is greater than the degree of the denominator), first do the division operation to convert into proper rational expression. This can be achieved with the help of a polynomial long division method.

### Partial Fraction Formula

The procedure or the formula for finding the partial fraction decomposition is explained with steps here:

**Step 1:** While decomposing the rational expression into the partial fraction, begin with the proper rational expression.

**Step 2:** Now, factor the denominator of the rational expression into the linear factor or in the form of irreducible quadratic factors (Note: Don't factor the denominators into the complex numbers).

**Step 3:** Write down the partial fraction for each factor obtained, with the variables in the numerators, say A and B.

**Step 4:** To find the variable values of A and B, multiply the whole equation by the denominator.

**Step 5:** Solve for the variables by substituting zero in the factor variable.

**Step 6:** Finally, substitute the values of A and B in the partial fractions.

## Partial Fractions from Rational Functions

Any number which can be easily represented in the form of  $p/q$ , such that  $p$  and  $q$  are integers and  $q \neq 0$  is known as a rational number. Similarly, we can define a rational function as the ratio of two polynomial functions  $P(x)$  and  $Q(x)$ , where  $P$  and  $Q$  are polynomials in  $x$  and  $Q(x) \neq 0$ . A rational function is known as proper if the degree of  $P(x)$  is less than the degree of  $Q(x)$ ; otherwise, it is known as an improper rational function. With the help of the long division process, we can reduce improper rational functions to proper rational functions. Therefore, if  $P(x)/Q(x)$  is improper, then it can be expressed as:

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

Here,  $A(x)$  is a polynomial in  $x$  and  $R(x)/Q(x)$  is a proper rational function.

We know that the integration of a function  $f(x)$  is given by  $F(x)$  and it is represented by:

$$\int f(x) dx = F(x) + C$$

Here R.H.S. of the equation means integral of  $f(x)$  with respect to  $x$  and  $C$  is the constant of integration.

## Partial Fractions Decomposition

In order to integrate a rational function, it is reduced to a proper rational function. The method in which the integrand is expressed as the sum of simpler rational functions is known as decomposition into partial fractions. After splitting the integrand into partial fractions, it is integrated accordingly with the help of traditional integrating techniques. Here the list of Partial fractions formulas is given.

S.No	Rational Function	Partial Function
1	$\frac{p(x)+q}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-b)}$
2	$\frac{p(x)+q}{(x-a)^2}$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2}$
3	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4	$\frac{px^2+q(x)+r}{(x-a)^2(x-b)}$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{(x-b)}$
5	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Here A, B and C are real numbers.

### Partial Fraction of Improper Fraction

An algebraic fraction is improper if the degree of the numerator is greater than or equal to that of the denominator. The degree is the highest power of the polynomial. Suppose, m is the degree of the denominator and n is the degree of the numerator. Then, in addition to the partial fractions arising from factors in the denominator, we must include an additional term: this additional term is a polynomial of degree  $n - m$ .

#### Note:

- A polynomial with zero degree is K, where K is a constant
- A polynomial of degree 1 is  $Px + Q$
- A polynomial of degree 2 is  $Px^2 + Qx + K$

**Example:** Integrate the function  $\frac{1}{(x-3)(x+1)}$  with respect to x.

**Solution:** The given integrand can be expressed in the form of partial fraction as:

$$\frac{1}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

To determine the value of real coefficients A and B, the above equation is rewritten as:

$$1 = A(x+1) + B(x-3)$$

$$\Rightarrow 1 = x(A+B) + A-3B$$

Equating the coefficients of x and the constant, we have

$$A + B = 0$$

$$A - 3B = 1$$

Solving these equations simultaneously, the value of  $A = 1/4$  and  $B = -1/4$ . Substituting these values in the equation 1, we have

$$\frac{1}{(x-3)(x+1)} = \frac{1}{4(x-3)} + \frac{-1}{4(x+1)}$$

Integrating with respect to x we have;

$$\int \frac{1}{(x-3)(x+1)} = \int \frac{1}{4(x-3)} + \int \frac{-1}{4(x+1)}$$

According to the properties of integration, the integral of sum of two functions is equal to the sum of integrals of the given functions, i.e.,

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Therefore,

$$\begin{aligned} &= \frac{1}{4} \int \frac{1}{(x-3)} - \frac{1}{4} \int \frac{1}{(x+1)} \\ &= \frac{1}{4} \ln|x-3| - \frac{1}{4} \ln|x+1| \\ &= \frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| \end{aligned}$$