INVERSE TRIGONOMETRIC FUNCTIONS

What are Inverse Trigonometric Functions?

Inverse trigonometric functions are also called "Arc Functions" since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value. The inverse trigonometric functions perform the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant, and cotangent. We know that trigonometric functions are especially applicable to the right angle triangle. These six important functions are used to find the angle measure in the right triangle when two sides of the triangle measures are known.

Formulas

Inverse Trig Functions	Formulas
Arcsine	$\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1, 1]$
Arccosine	$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, $x \in [-1, 1]$
Arctangent	$\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in R$
Arccotangent	$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, $x \in R$
Arcsecant	$\sec^{-1}(-x) = \pi - \sec^{-1}(x)$, $\|x\| \geq 1$
Arccosecant	$\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}(x)$, $\|x\| \geq 1$

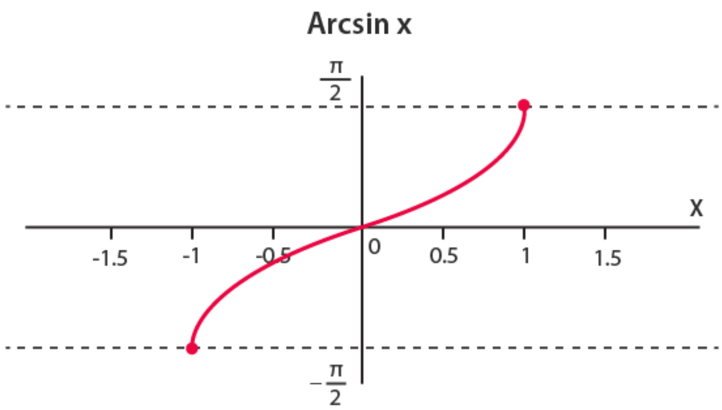
Inverse Trigonometric Functions Graphs

There are particularly six inverse trig functions for each trigonometry ratio. The inverse of six important trigonometric functions are:

- Arcsine
- Arccosine
- Arctangent
- Arccotangent
- Arcsecant
- Arccosecant

Arcsine Function

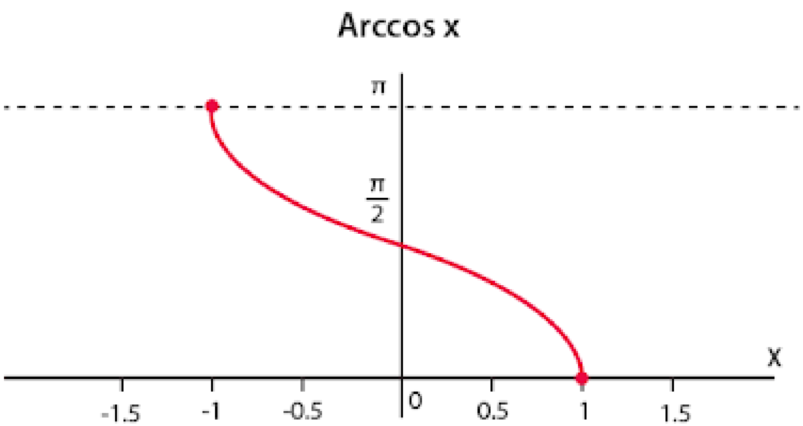
Arcsine function is an inverse of the sine function denoted by $\sin^{-1}x$. It is represented in the graph as shown below:



Domain	-1 ≤ x ≤ 1
Range	-π/2 ≤ y ≤ π/2

Arccosine Function

Arccosine function is the inverse of the cosine function denoted by $\cos^{-1}x$. It is represented in the graph as shown below:



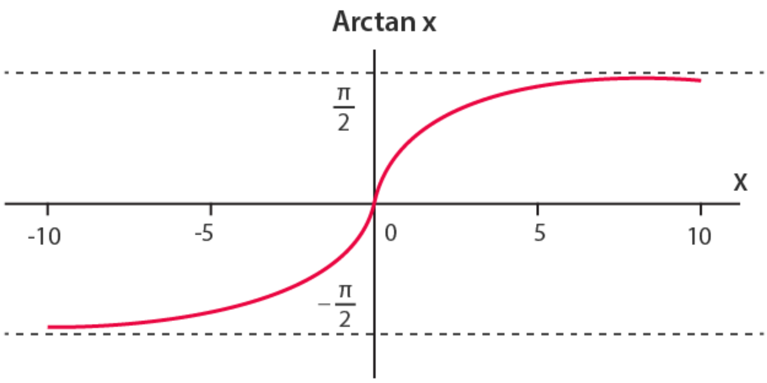
Therefore, the inverse of cos function can be expressed as; **y = $\cos^{-1}$x (arccosine *x*)**

Domain & Range of arcsine function:

Domain	-1≤x≤1
Range	0 ≤ y ≤ π

Arctangent Function

Arctangent function is the inverse of the tangent function denoted by $\tan^{-1}x$. It is represented in the graph as shown below:



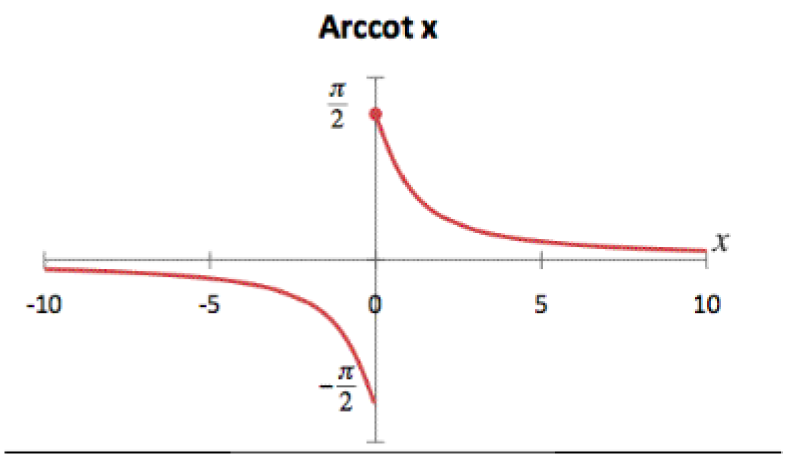
Therefore, the inverse of tangent function can be expressed as; **y = $\tan^{-1}$x (arctangent *x*)**

Domain & Range of Arctangent:

Domain	-∞ < x < ∞
Range	-π/2 < y < π/2

Arccotangent (Arccot) Function

Arccotangent function is the inverse of the cotangent function denoted by $\cot^{-1}x$. It is represented in the graph as shown below:



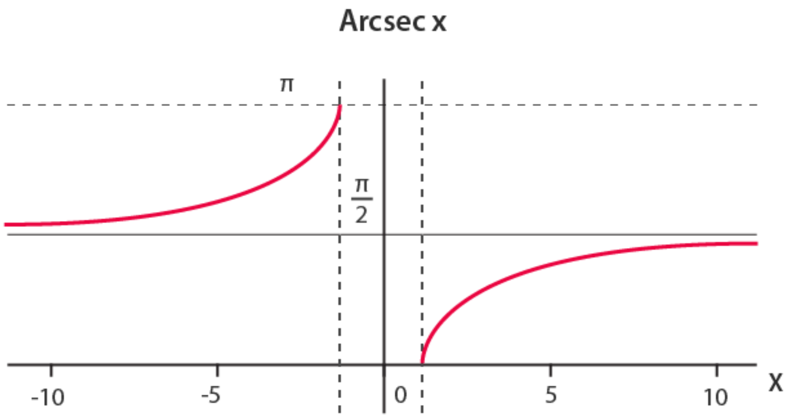
Therefore, the inverse of cotangent function can be expressed as; **y = $\cot^{-1}$x (arccotangent *x*)**

Domain & Range of Arccotangent:

Domain	-∞ < x < ∞
Range	0 < y < π

Arcsecant Function

Arcsecant function is the inverse of the secant function denoted by $\sec^{-1}x$. It is represented in the graph as shown below:



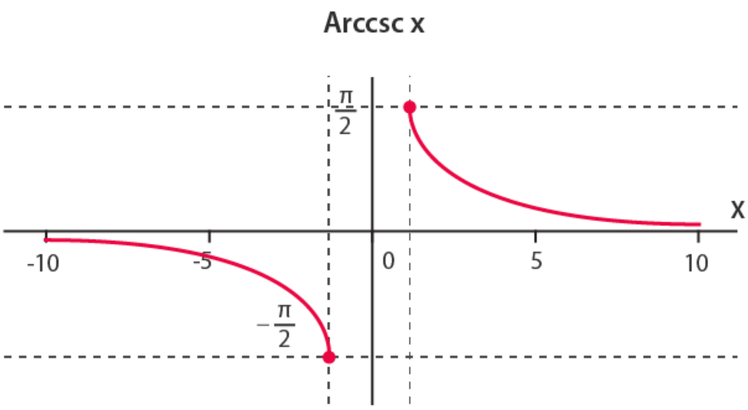
Therefore, the inverse of secant function can be expressed as; **y = $\sec^{-1}$x (arcsecant *x*)**

Domain & Range of Arcsecant:

Domain	$-\infty \le x \le -1$ or $1 \le x \le \infty$
Range	$0 \le y \le \pi$, $y \ne \pi/2$

Arccosecant Function

Arccosecant function is the inverse of the cosecant function denoted by $\text{cosec}^{-1}x$. It is represented in the graph as shown below:



Therefore, the inverse of cosecant function can be expressed as; **y = cosec^{-1}x (arccosecant *x*)**

Domain & Range of Arccosecant is:

Domain	$-\infty \le x \le -1$ or $1 \le x \le \infty$
Range	$-\pi/2 \le y \le \pi/2, y \ne 0$

Inverse Trigonometric Functions Table

Function Name	Notation	Definition	Domain of x	Range
Arcsine or inverse sine	$y = \sin^{-1}(x)$	x=sin y	$-1 \le x \le 1$	• $-\pi/2 \le y \le \pi/2$ • $-90° \le y \le 90°$
Arccosine or inverse cosine	$y=\cos^{-1}(x)$	x=cos y	$-1 \le x \le 1$	• $0 \le y \le \pi$ • $0° \le y \le 180°$
Arctangent or Inverse tangent	$y=\tan^{-1}(x)$	x=tan y	For all real numbers	• $-\pi/2 < y < \pi/2$ • $-90° < y < 90°$
Arccotangent or Inverse Cot	$y=\cot^{-1}(x)$	x=cot y	For all real numbers	• $0 < y < \pi$ • $0° < y < 180°$
Arcsecant or Inverse Secant	$y = \sec^{-1}(x)$	x=sec y	$x \le -1$ or $1 \le x$	• $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$ • $0° \le y < 90°$ or $90° < y \le 180°$
Arccosecant	$y=\csc^{-1}(x)$	x=csc y	$x \le -1$ or $1 \le x$	• $-\pi/2 \le y < 0$ or $0 < y \le \pi/2$ • $-90° \le y < 0°$ or $0° < y \le 90°$

Inverse Trigonometric Functions Derivatives

The derivatives of inverse trigonometric functions are first-order derivatives. Let us check here the derivatives of all the six inverse functions.

Function	Domain	dy/dx
arcsin x	$y = \sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
arccos x	$y = \cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
arctan x	$y = \tan^{-1}(x)$	$1/(1+x^2)$
arccot x	$y = \cot^{-1}(x)$	$-1/(1+x^2)$
arcsec x	$y = \sec^{-1}(x)$	$1/[\lvert x\rvert\sqrt{(x^2-1)}]$
arccsc x	$y = \csc^{-1}(x)$	$-1/[\lvert x\rvert\sqrt{(x^2-1)}]$

Properties of Inverse Trigonometric Functions

Property Set 1:

- $\text{Sin}^{-1}(x) = \text{cosec}^{-1}(1/x)$, $x \in [-1,1]-\{0\}$
- $\text{Cos}^{-1}(x) = \sec^{-1}(1/x)$, $x \in [-1,1]-\{0\}$
- $\text{Tan}^{-1}(x) = \cot^{-1}(1/x)$, if $x > 0$ **(or)** $\cot^{-1}(1/x) - \pi$, if $x < 0$
- $\text{Cot}^{-1}(x) = \tan^{-1}(1/x)$, if $x > 0$ **(or)** $\tan^{-1}(1/x) + \pi$, if $x < 0$

Property Set 2:

- $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}(x)$
- $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x)$
- $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}(x)$
- $\text{Cosec}^{-1}(-x) = -\text{Cosec}^{-1}(x)$
- $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}(x)$
- $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}(x)$

Proofs:

1. $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}(x)$

Let $\sin^{-1}(-x) = y$, i.e.,$-x = \sin y$

$\Rightarrow x = -\sin y$

Thus,

$x = \sin(-y)$

Or,

$\sin^{-1}(x) = -y = -\sin^{-1}(-x)$

Therefore, **$\sin^{-1}(-x) = -\sin^{-1}(x)$**

Similarly, using the same concept following results can be obtained:

- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \geq 1$
- $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in R$

2. $Cos^{-1}(-x) = \pi - Cos^{-1}(x)$

Let $\cos^{-1}(-x) = y$ i.e., $-x = \cos y$

$\Rightarrow x = -\cos y = \cos(\pi - y)$

Thus,

$\cos^{-1}(x) = \pi - y$

Or,

$\cos^{-1}(x) = \pi - \cos^{-1}(-x)$

Therefore, **$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$**

Similarly using the same concept following results can be obtained:

- $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

Property Set 3:

- $Sin^{-1}(1/x) = cosec^{-1}x$, $x \geq 1$ or $x \leq -1$
- $Cos^{-1}(1/x) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$
- $Tan^{-1}(1/x) = -\pi + \cot^{-1}(x)$

Proof: $Sin^{-1}(1/x) = cosec^{-1}x$, $x \geq 1$ or $x \leq -1$

Let $cosec^{-1}x = y$, i.e. $x = cosec\ y$

$\Rightarrow (1/x) = \sin y$

Thus, $\sin^{-1}(1/x) = y$

Or,

$\sin^{-1}(1/x) = cosec^{-1}x$

Similarly using the same concept, the other results can be obtained.

Example:

- $\sin^{-1}(⅓) = cosec^{-1}(3)$
- $\cos^{-1}(¼) = \sec^{-1}(4)$
- $\sin^{-1}(-¾) = cosec^{-1}(-4/3) = \sin^{-1}(3/4)$
- $\tan^{-1}(-3) = \cot^{-1}(-⅓) - \pi$

Property Set 4:

- $Sin^{-1}(\cos\theta) = \pi/2 - \theta$, if $\theta \in [0, \pi]$
- $Cos^{-1}(\sin\theta) = \pi/2 - \theta$, if $\theta \in [-\pi/2, \pi/2]$

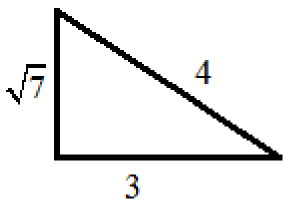
- $\mathrm{Tan}^{-1}(\cot\theta) = \pi/2 - \theta$, $\theta \in [0,\pi]$
- $\mathrm{Cot}^{-1}(\tan\theta) = \pi/2 - \theta$, $\theta \in [-\pi/2, \pi/2]$
- $\mathrm{Sec}^{-1}(\mathrm{cosec}\,\theta) = \pi/2 - \theta$, $\theta \in [-\pi/2, 0] \cup [0, \pi/2]$
- $\mathrm{Cosec}^{-1}(\sec\theta) = \pi/2 - \theta$, $\theta \in [0,\pi] - \{\pi/2\}$
- $\mathrm{Sin}^{-1}(x) = \cos^{-1}[\sqrt{(1-x^2)}]$, $0 \le x \le 1$

  $= -\cos^{-1}[\sqrt{(1-x^2)}]$, $-1 \le x < 0$

Example:

1. Given, $\cos^{-1}(-3/4) = \pi - \sin^{-1}A$. Find A.

Solution: Draw the diagram from the question statement.



So,

$\cos^{-1}(-3/4) = \pi - \sin^{-1}(\sqrt{7}/4)$

Thus, $A = \sqrt{7}/4$

2. $\cos^{-1}(¼) = \sin^{-1}\sqrt{(1-1/16)} = \sin^{-1}(\sqrt{15}/4)$

3. $\sin^{-1}(-½) = -\cos^{-1}\sqrt{(1-¼)} = -\cos^{-1}(\sqrt{3}/2)$

4. $\sin^2(\tan^{-1}(¾)) = \sin^2(\sin^{-1}(⅗)) = (⅗)^2 = 9/25$.

5. $\sin^{-1}(\sin 2\pi/3) = \pi/3$

6. $\cos^{-1}(\cos 4\pi/3) = 2\pi/3$

7. $\sin^{-1}(\cos 33\pi/10) = \sin^{-1}\cos(3\pi + 3\pi/10) = \sin^{-1}(-\sin(\pi/2 - 3\pi/10)) = -(\pi/2 - 3\pi/10) = -\pi/5$

Property Set 5:

- $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \pi/2$
- $\mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}(x) = \pi/2$
- $\mathrm{Sec}^{-1}x + \mathrm{Cosec}^{-1}x = \pi/2$

Proof: $\sin^{-1}(x) + \cos^{-1}(x) = (\pi/2)$, $x \in [-1,1]$

Let $\sin^{-1}(x) = y$, i.e., $x = \sin y = \cos((\pi/2) - y)$

⇒ $\cos^{-1}(x) = (\pi/2) – y = (\pi/2) − \sin^{-1}(x)$

Thus,

$\sin^{-1}(x) + \cos^{-1}(x) = (\pi/2)$

Similarly using the same concept following results can be obtained:

- $\tan^{-1}(x) + \cot^{-1}(x) = (\pi/2)$, $x \in R$
- $\text{cosec}^{-1}(x) + \sec^{-1}(x) = (\pi/2)$, $|x| \geq 1$

Example:

1. $\text{Sec}^{-1}(4) + \text{Cosec}^{-1}(4) = \pi/2$
2. $\text{Tan}^{-1}(3) + \text{Cot}^{-1}(3) = \pi/2$

Property Set 6:

(1) If x, y > 0

$$\text{Tan}^{-1}x + \text{Tan}^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy > 1 \end{cases}$$

(2) If x, y < 0

$$\text{Tan}^{-1}x + \text{Tan}^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy < 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy < 1 \end{cases}$$

(3) $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) xy = \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) \begin{matrix} x > 0 \\ y < 0 \end{matrix} = -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) \begin{matrix} x < 0 \\ y > 0 \end{matrix}$

(4) $\tan^{-1}(x) – \tan^{-1}(y) = \tan^{-1}[(x−y)/ (1+xy)]$, $xy>−1$

(5) $2\tan^{-1}(x) = \tan^{-1}[(2x)/ (1–x^2)]$, $|x|<1$

Proof: $\text{Tan}^{-1}(x) + \tan^{-1}(y) = \tan^{-1}[(x+y)/ (1−xy)]$, $xy<1$

Let $\tan^{-1}(x) = \alpha$ and $\tan^{-1}(y) = \beta$, i.e., $x = \tan(\alpha)$ and $y = \tan(\beta)$

⇒ $\tan(\alpha+\beta) = (\tan \alpha + \tan \beta) / (1 – \tan \alpha \tan \beta)$

Thus,

$(\alpha) + (\beta) = \tan^{-1}[(x+y) / (1−xy)]$

Therefore,

$\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}[(x+y) / (1−xy)]$

Similarly using the same concept following results can be obtained:

- $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}[(x-y)/ (1+xy)]$, $xy>-1$
- $2\tan^{-1}(x) = \tan^{-1}[(2x)/ (1-x^2)]$, $|x|<1$

Example:

1. $\mathrm{Tan}^{-1}(-½) + \mathrm{Tan}^{-1}(-⅓) = \mathrm{Tan}^{-1}[(-½ - ⅓)/ (1- ⅙)]$

$= \tan^{-1}(-1)$

$= -\pi/4$

2. $\mathrm{Tan}^{-1}(-2) + \mathrm{Tan}^{-1}(-3) = \mathrm{Tan}^{-1}[(-2+-3)/ (1-6)]$

$= \mathrm{Tan}^{-1}(-5/ -5) = \mathrm{Tan}^{-1}1$

$= \pi/4$

3. $\mathrm{Tan}^{-1}(-3) + \mathrm{Tan}^{-1}(-⅓) = -(\mathrm{Tan}^{-1}B) + \mathrm{Tan}^{-1}(⅓)$

$= -\pi/2$

4. $\mathrm{Tan}^{-1}(5/3) - \mathrm{Tan}^{-1}(¼) = \mathrm{Tan}^{-1}[(5/3-¼)/ (1+5/12)]$

$= \mathrm{Tan}^{-1}(17/17)$

$= \mathrm{Tan}^{-1}1 = \pi/4$

5. $\mathrm{Tan}^{-1}2x + \mathrm{Tan}^{-1}3x = \pi/4$

$\Rightarrow \mathrm{Tan}^{-1}[(5x)/ (1-6x^2)] = \pi/4$

$\Rightarrow 5x/ (1-6x^2) = 1$

$\Rightarrow 6x^2 - 5x + 1 = 0$

$\Rightarrow x = 1/6$ or -1

$\therefore x = 16$ as, $x = -1$

6. If $\tan^{-1}(4) + \mathrm{Tan}^{-1}(5) = \mathrm{Cot}^{-1}(\lambda)$. Find λ

Here,

$\mathrm{Tan}^{-1}[9/ (1-20)] = \mathrm{Cot}^{-1}\lambda$

$\Rightarrow \mathrm{Tan}^{-1}(-9/19) = \mathrm{Cot}^{-1}(\lambda)$

$\Rightarrow -\mathrm{Tan}^{-1}(9/19) = \mathrm{Cot}^{-1}(\lambda)$

$\Rightarrow -\mathrm{Cot}^{-1}(19/9) = \mathrm{Cot}^{-1}(\lambda)$

Or, $\lambda = -19/9$

Property Set 7:

- $\sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}[x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$
- $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}]$

Example:

1. $\sin^{-1}(4/5) + \sin^{-1}(7/25) = \sin^{-1}(A)$. Find A.

Solution:

$= \sin^{-1}(4/5 \sqrt{\{1-(7/25)^2\}} + \sqrt{\{1-(4/5)^2\}}\ 7/25)$

$= \sin^{-1}(117/125)$

2. Prove that $\sin^{-1}(4/5) + \sin(5/13) + \sin^{-1}(16/65) = \pi/2$

Solution:

$\sin^{-1}(63/65) + \sin^{-1}(16/65)$

$= \cos^{-1}(16/65) + \sin^{-1}(16/65)$

$= \pi/2$

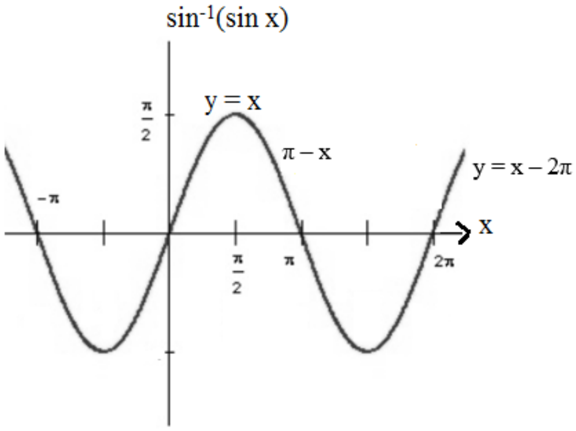
Property Set 8: Corresponding Graphs

$\sin^{-1}(\sin x) = -\pi-\pi$, if $x \in [-3\pi/2, -\pi/2]$

$= x$, if $x \in [-\pi/2, \pi/2]$

$= \pi - x$, if $x \in [\pi/2, 3\pi/2]$

$= -2\pi + x$, if $x \in [3\pi/2, 5\pi/2]$ And so on.



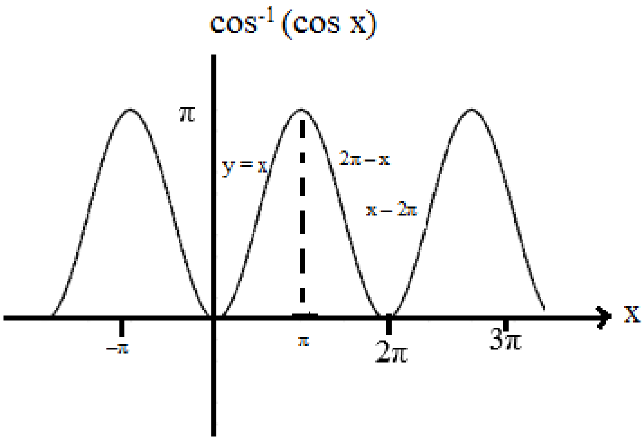
$\cos^{-1}(\cos x) = 2\pi+x$, if $x\in[-2\pi,-\pi]$

$= -x, \in[-\pi,0]$

$= x, \in[0,\pi]$

$= 2\pi-x, \in[\pi,2\pi]$

$=-2\pi+x, \in[2\pi,3\pi]$

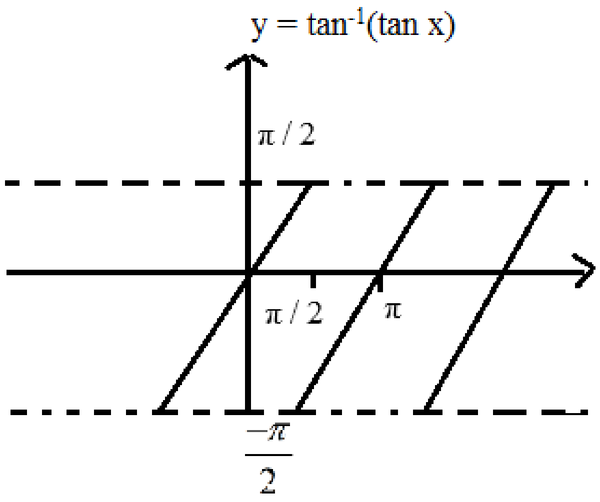


$\tan^{-1}(\tan x) = \pi+x, x\in(-3\pi/2, -\pi/2)$

$= x, (-\pi/2, \pi/2)$

$= x-\pi, (\pi/2, 3\pi/2)$

$= x-2\pi, (3\pi/2, 5\pi/2)$



Examples:

1. $\sin^{-1}(\sin 2\pi/3) = \pi - 2\pi/3 = \pi/3$

2. $\cos^{-1}(\cos(13\pi/6)) = \pi/6$

3. $\sin^{-1}\sin(4) = \pi - 4$

4. $\sin^{-1}\sin(6) = 6 - 2\pi$

5. $\sin^{-1}\sin(12) = 12 - 4\pi$

6. $\cos^{-1}(\cos 3) = 3$

7. $\cos^{-1}(\cos 5) = 2\pi - 5$

8. $\cos^{-1}(\cos 6) = 2\pi - 6$

9. $\tan^{-1}(\tan 3) = 3 - \pi$

Property Set 9:

1. $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$
2. $2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right)$
3. $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
4. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$
5. $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$
6. $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$
7. $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$
8. $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Example:

1. f(x) = $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}x.$

If x > 1 find cos (f(10))

Ans. $f(10) = sin^{-1}\left(\frac{20}{101}\right) + 2\tan^{-1}(10) = \tan^{-1}\left(\frac{20}{99}\right) + 2\tan^{-1}(10)$
$= \pi + \tan^{-1}\left(\frac{20}{99}\right) \pm \tan^{-1}\left(\frac{20}{99}\right)$

= π

2. Find $\tan\left(\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right) = \tan\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

= $\frac{{}^3/_4 + {}^2/_3}{1-\left(\frac{3}{4}\times{}^2/_3\right)} = \frac{17}{6}$

Proof: $2\tan^{-1}x = \sin^{-1}[(2x)/(1+x^2)]$, $|x|<1$

Let, $\tan^{-1}x = y$ i.e., $x = \tan y$

⇒ $\sin^{-1}[(2x)/(1+x^2)] = \sin^{-1}[(2\tan y)/(1+\tan^2 y)]$

Thus,

⇒ $\sin^{-1}[(2\tan y)/(1+\tan^2 y)] = \sin^{-1}(\sin 2y) = 2y = 2\tan^{-1}x$

Similarly using the same concept following results can be concluded:

- $2\tan^{-1}x = \cos^{-1}[(1-x^2)/(1+x^2)$, $x\geq 0$
- $2\tan^{-1}x = \tan^{-1}[(2x)/(1-x^2)$, $-1<x<1$

Solved Examples

Example 1: Find the value of x, for sin(x) = 2.

Solution: Given: sin x = 2

x =$\sin^{-1}(2)$, which is not possible.

Hence, there is no value of x for which sin x = 2; since the domain of $\sin^{-1}x$ is -1 to 1 for the values of x.

Example 2: Find the value of $\sin^{-1}(\sin(\pi/6))$.

Solution: $\sin^{-1}(\sin(\pi/6) = \pi/6$ (Using identity $\sin^{-1}(\sin(x)) = x$)

Example 3: Find sin ($\cos^{-1}$ 3/5).

Solution: Suppose that, $\cos^{-1} 3/5 = x$

So, cos x = 3/5

We know, sin x = $\sqrt{1 - cos^2 x}$

So, sin x = $\sqrt{1 - \frac{9}{25}}$ = 4/5

This implies, sin x = sin ($\cos^{-1}$ 3/5) = 4/5

Example 5: $\sec^{-1}[\sec(-30°)] =$

Solution: $\sec^{-1}[\sec(-30°)] = \sec^{-1}(\sec 30°) = 30°$

Example 5: Determine the principal value of $\cos^{-1}(-1/2)$.

Solution:

Let us assume that, y = $\cos^{-1}(-1/2)$

We can write this as:

cos y =- 1/2

cos y = cos (2π/3).

Thus, the Range of the principal value of $\cos^{-1}$ is [0, π].

Therefore, the principal value of $\cos^{-1}(-1/2)$ is 2π /3.

Example 6: Find the value of cot ($\tan^{-1}$ α + $\cot^{-1}$ α).

Solution:

Given that: cot ($\tan^{-1} \alpha + \cot^{-1} \alpha$)

= cot ($\pi/2$) (since, $\tan^{-1} x + \cot^{-1} x = \pi/2$)

= cot (180°/2) (we know that cot 90° = 0)

= cot (90°)

= 0

Therefore, the value of cot ($\tan^{-1} \alpha + \cot^{-1} \alpha$) is 0.

Example 7: The value of $\tan^{-1} \sqrt{3} - \sec^{-1}(-2)$ is equal to:

(A) π (B) – π/3 (C) π/3 (D) 2π/3

Solution:

Now, solve the first part of the expression: $\tan^{-1} \sqrt{3}$

Let us take $y = \tan^{-1}\sqrt{3}$

This can be written as:

$\tan y = \sqrt{3}$

Now, use the trigonometry table to find the radian value

$\tan y = \tan (\pi/3)$

Thus, the range of principal value of $\tan^{-1}$ is $(-\pi/2, \pi/2)$

Therefore, the principal value of $\tan^{-1}\sqrt{3}$ is π/3.

Now, solve the second part of the expression: $\sec^{-1}(-2)$

Now, assume that $y = \sec^{-1} (-2)$

$\sec y = -2$

$\sec y = \sec (2\pi/3)$

We know that the principal value range of $\sec^{-1}$ is $[0,\pi] - \{\pi/2\}$

Therefore, the principal value of $\sec^{-1} (-2) = 2\pi/3$

Now we have:

$\tan^{-1}(\sqrt{3}) = \pi/3$

$\sec^{-1} (-2) = 2\pi/3$

Now, substitute the values in the given expression:

$= \tan^{-1} \sqrt{3} - \sec^{-1} (-2)$

$= \pi/3 - (2\pi/3)$

$= \pi/3 - 2\pi/3$

$= (\pi - 2\pi)/3$

$= - \pi/3$

Hence, the correct answer is an option (B)

Example 8: Prove that $\sin^{-1}(3/5) - \sin^{-1}(8/17) = \cos^{-1}(84/85)$.

Solution:

Let $\sin^{-1}(3/5) = a$ and $\sin^{-1}(8/17) = b$

Thus, we can write sin a = 3/5 and sin b = 8/17

Now, find the value of cos a and cos b

To find cos a:

$\text{Cos } a = \sqrt{[1 - \sin^2 a]}$

$= \sqrt{[1 - (3/5)^2]}$

$= \sqrt{[1 - (9/25)]}$

$= \sqrt{[(25-9)/25]}$

= 4/5

Thus, the value of cos a = 4/5

To find cos b:

$\text{Cos } b= \sqrt{[1 - \sin^2 b]}$

$= \sqrt{[1 - (8/17)^2]}$

$= \sqrt{[1 - (64/289)]}$

$= \sqrt{[(289-64)/289]}$

= 15/17

Thus, the value of cos b = 15/17

We know that cos (a- b) = cos a cos b + sin a sin b

Now, substitute the values for cos a, cos b, sin a and sin b in the formula, we get:

cos (a – b) = (4/5)x (15/17) + (3/5)x(8/17)

cos (a – b) = (60 + 24)/(17x 5)

cos (a – b) = 84/85

$(a - b) = \cos^{-1}(84/85)$

Substituting the values of a and b $\sin^{-1}(3/5) - \sin^{-1}(8/7) = \cos^{-1}(84/85)$

Hence proved.

Example 9: Find the value of $\cos^{-1}(1/2) + 2\sin^{-1}(1/2)$.

Solution:

First, solve for $\cos^{-1}$ (1/2):

Let us take, $y = \cos^{-1}(1/2)$

This can be written as:

$\cos y = (1/2)$

$\cos y = \cos(\pi/3)$.

Thus, the range of principal value of $\cos^{-1}$ is $[0, \pi]$

Therefore, the principal value of $\cos^{-1}(1/2)$ is $\pi/3$.

Now, solve for $\sin^{-1}$ (1/2):

Let $y = \sin^{-1}(1/2)$

$\sin y = 1/2$

$\sin y = \sin(\pi/6)$

Thus, the range of principal value of $\sin^{-1}$ is $[(-\pi)/2, \pi/2]$

Hence, the principal value of $\sin^{-1}(1/2)$ is $\pi/6$.

Now we have $\cos^{-1}(1/2) = \pi/3$ & $\sin^{-1}(1/2) = \pi/6$

Now, substitute the obtained values in the given formula, we get:

$= \cos^{-1}(1/2) + 2\sin^{-1}(1/2)$

$= \pi/3 + 2(\pi/6)$

$= \pi/3 + \pi/3$

$= (\pi+\pi)/3$

$= 2\pi/3$

Thus, the value of $\cos^{-1}(1/2) + 2\sin^{-1}(1/2)$ is $2\pi/3$.