CONTINUITY AND DIFFERENTIABILITY

Definition of Continuity

The continuity of a real function (f) on a subset of the real numbers is defined when the function exists at point c and is given as-

$$\lim_{x \to c} f(x) = f(c)$$

A real function (f) is said to be continuous if it is continuous at every point in the domain of f.

Consider a function f(x), and the function is said to be continuous at every point in [a, b] including the endpoints a and b.

Continuity of "f" at a means,

$$\lim_{x \to a} f(x) = f(a)$$

Continuity of "f" at b means,

$$\lim_{x \to b} f(x) = f(b)$$

Differentiability Formula

Assume that if f is a real function and c is a point in its domain. The derivative of f at c is defined by 0

The derivative of a function f at c is defined by-

$$\lim_{h \to 0} \frac{f(x+h) - f(c)}{h}$$

Theorem 1: Algebra of continuous functions:

If the two real functions, say f and g, are continuous at a real number c, then

(i) f + g is continuous at x=c.

(ii) f – g is continuous at x=c.

(iii) f. g is continuous at x=c.

(iv)f/g is continuous at x=c, (provided g(c) ≠ 0).

Theorem 2: Suppose f and g are real-valued functions such that (f o g) is defined at c. If g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Theorem 3: If a function f is differentiable at a point c, then it is also continuous at that point.

Theorem 4 (Chain Rule): Let f be a real-valued function which is a composite of two functions u and v; i.e., f = v o u.

Suppose t = u(x) and if both dt/dx and dv/dt exist, we have df/dx = (dv/dt). (dt/dx)

Theorem 5:

(i) The derivative of e^x with respect to x is e^x; i.e., $d/dx(e^x) = e^x$.

(ii) The derivative of log x with respect to x is 1/x.

i.e., d/dx(log x) =1/x.

Theorem 6 (Rolle's Theorem): Let f : [a, b] ⟶ R be continuous on [a, b] and differentiable on (a, b), such that f(a) = f(b), where a and b are some real numbers. Then there exists some c in (a, b) such that f'(c) = 0.

Solved Examples

Example 1: Let [.] denotes the greatest integer function and f(x) = [$\tan^2 x$], then does the limit exist or is the function differentiable or continuous at 0?

Solution: Given f(x) = [$\tan^2 x$]
Now, -45°< x < 45°
tan(-45°)< tanx < tan45°
-tan 45°< tan x < tan 45°
-1< tan x <1
So, 0 <$\tan^2 x$ < 1
[$\tan^2 x$] = 0
So, f(x) is zero for all values of x form x = -45° to 45°.
Hence, f is continuous at x =0 and f is also differentiable at 0 and has a value zero.

Example 2: A function is defined as follows:
f(x)= x^3 , x^2< 1
x , x^2≥ 1
Discuss the differentiability of the function at x=1.

Solution: We have R.H.D. = Rf'(1)
= $\lim_{h\to 0}$ (f(1-h)-f(1))/h
= $\lim_{h\to 0}$ (1+h-1)/h = 1
and L.H.D. = Lf'(1) = $\lim_{h\to 0}$ (f(1-h)-f(1))/(-h)

$= \lim_{h\to 0} ((1-h)^3-1)/(-h)$

$= \lim_{h\to 0} (3-3h+h^2) = 3$

?Rf'(1) ≠ Lf'(1) ⇒ f(x) is not differentiable at x=1.

Example 3: If $y = (\sin^{-1}x)^2 + k \sin^{-1}x$, show that $(1-x^2) (d^2 y)/dx^2 - x\, dy/dx = 2$

Solution: Here $y = (\sin^{-1}x)^2 + k \sin^{-1}x$.

Differentiating both sides with respect to x, we have

$dy/dx = 2(\sin^{-1} x)/\sqrt{(1-x^2)} + k/\sqrt{(1-x^2)}$

$\Rightarrow (1-x^2) (dy/dx)^2 = 4y + k^2$

Differentiating this with respect to x, we get

$(1-x^2)\, 2\, dy/dx.(d^2 y)/(dx^2) - 2x (dy/dx)^2 = 4(dy/dx)$

$\Rightarrow (1-x^2) (d^2 y)/dx^2 -x\, dy/dx = 2$