# DIFFERENTIAL EQUATIONS

A **differential equation** is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable).

**dy/dx = f(x)**

Here "x" is an independent variable and "y" is a dependent variable

For example, dy/dx = 5x

A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity. There are a lot of differential equations formulas to find the solution of the derivatives.

## Differential Equation formula

$$\frac{dy}{dt} + p(t)y = g(t)$$

p(t) & g(t) are the functions which are continuous.

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}$$

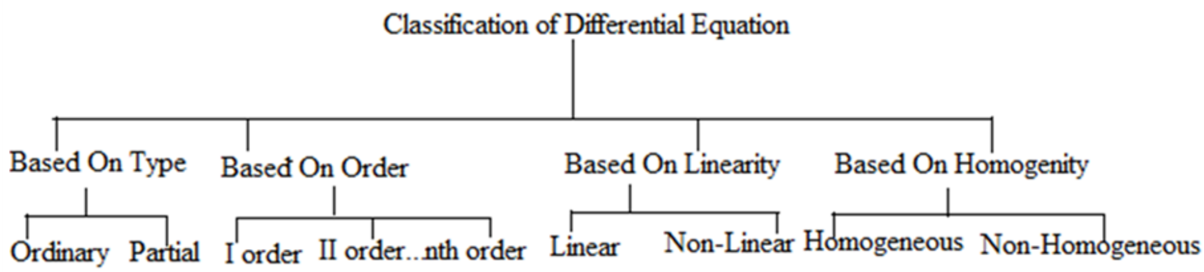
*Where* $\mu(t) = e^{\int p(t)d(t)}$

## Differential Equations Solutions

There exist two methods to find the solution of the differential equation.

1. **Separation of variables** - **Separation of the variable** is done when the differential equation can be written in the form of dy/dx = f(y)g(x) where f is the function of y only and g is the function of x only. Taking an initial condition, rewrite this problem as 1/f(y)dy = g(x)dx and then integrate on both sides.
2. **Integrating factor** – This technique is used when the differential equation is of the form dy/dx + p(x)y = q(x) where p and q are both the functions of x only.

## Classification of Differential Equations




## Differential Equations- Based on Type

### Ordinary Differential Equation

In mathematics, the term "**Ordinary Differential Equations**" also known as **ODE** is an equation that contains only one independent variable and one or more of its derivatives with respect to the variable. In other words, the ODE is represented as the relation having one independent variable x, the real dependent variable y, with some of its derivatives.

$y', y'', \dots y^n, \dots$ with respect to x.

The order of ordinary differential equations is defined to be the order of the highest derivative that occurs in the equation. The general form of n-th order ODE is given as;

$F(x, y, y', \dots, y^n) = 0$

Note that, $y'$ can be either $dy/dx$ or $dy/dt$ and $y^n$ can be either $d^ny/dx^n$ or $d^ny/dt^n$.

An n-th order ordinary differential equations is linear if it can be written in the form;

$a_0(x)y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = r(x)$

The function $a_j(x)$, $0 \le j \le n$ are called the coefficients of the linear equation. The equation is said to be homogeneous if $r(x) = 0$. If $r(x) \neq 0$, it is said to be a non- homogeneous equation.

The ordinary differential equation is further classified into three types. They are:

- **Autonomous ODE** - A differential equation which does not depend on the variable, say x is known as an autonomous differential equation.
- **Linear ODE** - If differential equations can be written as the linear combinations of the derivatives of y, then they are called linear ordinary differential equations. These can be further classified into two types:
    - Homogeneous linear differential equations
    - Non-homogeneous linear differential equations
- **Non-linear ODE** - If the differential equations cannot be written in the form of linear combinations of the derivatives of y, then it is known as a non-linear ordinary differential equation.

### Partial Differential Equation

A **Partial Differential Equation** commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with more than one independent variable. A PDE for a function $u(x_1, \dots x_n)$ is an equation of the form

$$f\left(x_1, \dots . x_n; u, \frac{\partial u}{\partial x_1}, \dots \dots, \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \dots .\right) = 0$$

The PDE is said to be linear if f is a linear function of u and its derivatives. The simple PDE is given by;

**$\partial u/\partial x\ (x,y) = 0$**

The above relation implies that the function u(x,y) is independent of x which is the reduced form of partial differential equation formula stated above. The order of PDE is the order of the highest derivative term of the equation.

In PDEs, we denote the partial derivatives using subscripts, such as;

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

In some cases, like in Physics when we learn about wave equations or sound equation, partial derivative, $\partial$ is also represented by ∇(del or nabla).

**Partial Differential Equation Types**

- **First-Order Partial Differential Equation** - when we speak about the first-order partial differential equation, then the equation has only the first derivative of the unknown function having 'm' variables. It is expressed in the form of:

  **$F(x_1,\dots,x_m, u,u_{x1},\dots,u_{xm})=0$**
- **Linear Partial Differential Equation** - If the dependent variable and all its partial derivatives occur linearly in any PDE then such an equation is called linear PDE otherwise a nonlinear PDE.
- **Quasi-Linear Partial Differential Equation** - A PDE is said to be quasi-linear if all the terms with the highest order derivatives of dependent variables occur linearly, that is the coefficient of those terms are functions of only lower-order derivatives of the dependent variables. However, terms with lower-order derivatives can occur in any manner.
- **Homogeneous Partial Differential Equation** - If all the terms of a PDE contain the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise.

## Differential Equations - Based on Order

The order of the highest differential coefficient (derivative) involved in the differential equation is known as the order of the differential equation.

For Example: $\frac{d^3y}{dx^2} + 5\frac{dy}{dx} + y = \sqrt{x}$

Here, the order = 3 as the order of the highest derivative involved is 3.

For derivatives the use of single quote notation is preferred which is

$y' = \frac{dy}{dx}$

$y'' = \frac{d^2y}{dx^2}$

$y''' = \frac{d^3y}{dx^3}$

and so on

For the higher order derivatives it would become cumbersome to use multiple quotes so for these derivatives we prefer using the notation $y_n$ for the $n^{th}$ order derivative $\frac{d^ny}{dx^n}$

## Differential Equations – Based on Linearity

By linearity, it means that the variable appearing in the equation is raised to the power of one. The graph of linear functions is generally a straight line. For example: $(3x + 5)$ is Linear but $(x^3 + 4x^2)$ is non-linear.

**Linear Differential Equation -** If all the dependent variables and its entire derivatives occur linearly in a given equation, then it represents a linear differential equation.

**Non-Linear Differential Equations -** Any differential equation with non-linear terms is known as a non-linear differential equation.

Consider the following examples for illustration:

Example 1: $\frac{dy}{dx} + xy = 5x$ ......... (i)

$\frac{d^2y}{dx^2} - \ln y = 10$ .......... (ii)

Example 1: $\frac{dy}{dx} + xy = 5x$

It is a linear differential equation as $\frac{dy}{dx}$ and both are linear.

Example 2: $\frac{d^2y}{dx^2} - \ln y = 10$

In y is not linear. Hence, this equation is non-linear.

## Differential Equations – Based on Homogeneity

Consider the following functions:

$f_1(x,y) = y^3 + 2/3xy^2$

$f_2(x,y) = x^3 \div y^2x$

$f_3(x,y) = \tan x + \sec y$

If we replace x and y by αx and αy respectively, where α is any non-zero constant, we get;

$f1(x,y)=(\alpha y)3+23(\alpha x)(\alpha y)2=\alpha 3(y3+23xy)=\alpha 3f1(x,y)$

$f2(x,y)=(\alpha x)3(\alpha y)3(\alpha x)=x3xy2=\alpha\circ f2(x,y)$

$f3(x,y)=\tan(\alpha x)+sex(\alpha y)$

We observe that,

f1, f2 can be written in the form *f(αx,αy) = αnf(x,y)* but this is not applicable to f3(x,y) Therefore, if a function satisfies the condition that *f(αx,xy) = αnf(x,y)* for a non-zero constant α, it is known as homogeneous equation of degree n.

The linear differential equation of the form, *fn(x)yn+…….+f1(x)y'+f0(x)y=g(x)* represents a homogeneous differential equation if the R.H.S is zero i.e., g(x) = 0 , else it represents non-homogeneous differential equation if g(x) ≠ 0.

## Solution Of a Differential Equation

### General Solution of a Differential Equation

When the arbitrary constant of the general solution takes some unique value, then the solution becomes the particular solution of the equation.

By using the boundary conditions (also known as the initial conditions) the particular solution of a differential equation is obtained.

So, to obtain a particular solution, first of all, a general solution is found out and then, by using the given conditions the particular solution is generated.

Suppose,

$dy/dx = e^x + \cos 2x + 2x^3$

Then we know, the general solution is:

$y = e^x + \sin 2x/2 + x^4/2 + C$

Now, x = 0, y = 5 substituting this value in the general solution we get,

$5 = e^0 + \sin(0)/2 + (0)^4/2 + C$

C = 4

Hence, substituting the value of C in the general solution we obtain,

$y = e^x + \sin 2x/2 + x^4/2 + 4$

This represents the particular solution of the given equation.

### General Solution for First Order and Second Order

If we have to solve a first-order differential equation by variable separable method, we need have to mention an arbitrary constant before we start performing integration. Hence, we can see that a solution of the first-order differential equation has at least one fixed arbitrary constant after simplification.

**Variable separable differential Equations:** The differential equations which are represented in terms of (x,y) such as the x-terms and y-terms can be ordered to different sides of the equation (including delta terms). Thus, each variable after separation can be integrated easily to find the solution of the differential equation. The equations can be written as:

**$f(x)dx + g(y)dy = 0$**, where $f(x)$ and $g(y)$ are either constants or functions of x and y respectively.

Similarly, the general solution of a second-order differential equation will consist of two fixed arbitrary constants and so on. The general solution geometrically interprets an m-parameter group of curves.

## Particular Solution of a Differential Equation

A Particular Solution is a solution of a differential equation taken from the General Solution by allocating specific values to the random constants. The requirements for determining the values of the random constants can be presented to us in the form of an Initial-Value Problem, or Boundary Conditions, depending on the query.

### Singular Solution

The Singular Solution of a given differential equation is also a type of Particular Solution but it can't be taken from the General Solution by designating the values of the random constants.

**Example**: **$dy/dx = x^2$**

**Solution:** $dy = x^2\, dx$

Integrating both sides, we get

$\Rightarrow \int dy = \int x^2 dx$

If we solve this equation to figure out the value of y we get

$y = x^3/3 + C$

where C is an arbitrary constant.

In the above-obtained solution, we see that y is a function of x. On substituting this value of y in the given differential equation, both the sides of the differential equation becomes equal.

**Example**: **Find out the particular solution of the differential equation $\ln dy/dx = e^{4y + \ln x}$, given that for $x = 0, y = 0$.**

**Solution**: $dy/dx = e^{4y + \ln x}$

$dy/dx = e^{4y} \times e^{\ln x}$

$dy/dx = e^{4y} \times x$

$1/e^{4y} dy = x\, dx$

$e^{-4y} dy = x\, dx$

Integrating both the sides with respect to y and x respectively we get,

$e^{-4y}/-4 = x^2/2 + C$

This represents the general solution of the differential equation given.

Now, it is also given that y(0) = 0, substituting this value in the above general solution we get,

$e^0/-4 = 0^2/2 + C$

$\Rightarrow C = -1/4$

Hence, the above equation can be rewritten as

$e^{-4y}/-4 = x^2/2 - 14$

$\Rightarrow e-^{4y} = -2x^2 + 1$

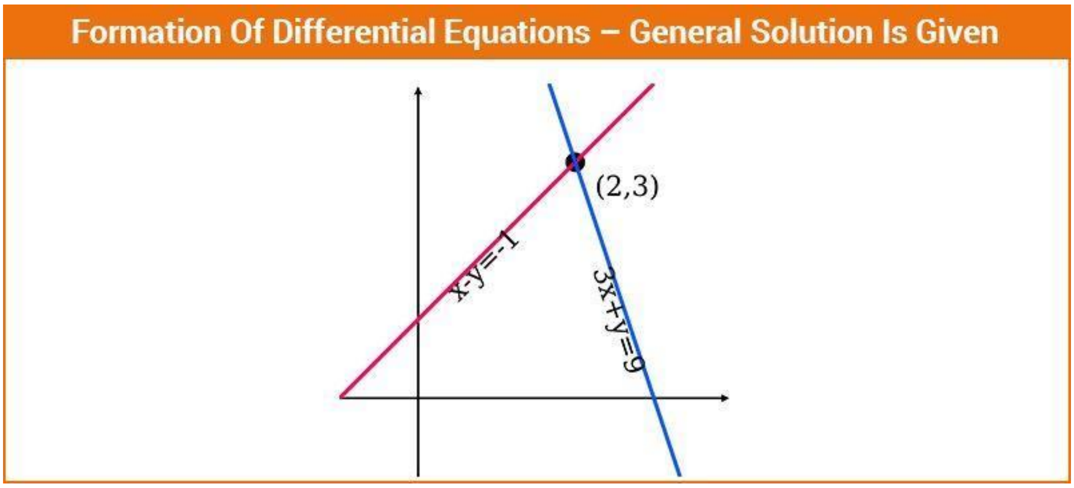
$\Rightarrow \ln(e^{-4y}) = \ln(1-2x^2)$

$\Rightarrow -4y = \ln(1-2x^2)$

$\Rightarrow y = -\ln(1-2x^2)/4$

which is the particular solution of the differential equation given.

## Formation of Differential Equations




To obtain the differential equation from this equation we follow the following steps:

**Step 1:** Differentiate the given function w.r.t to the independent variable present in the equation.

**Step 2:** Keep differentiating times in such a way that (n+1) equations are obtained.

**Step 3:** Using the (n+1) equations obtained, eliminate the constants ($c_1$ , $c_2$ … …. $c_n$).

## The solution of Differential Equations

The general solution of the differential equation is the relation between the variables x and y which is obtained after removing the derivatives (i.e., integration) where the relation contains arbitrary constant to denote the order of an equation. The solution of the first-order differential equations contains one arbitrary constant whereas the second-order differential equation contains two arbitrary constants. If particular values are given to the arbitrary constant, the general solution of the differential equations is obtained. To solve the first-order differential equation of first degree, some standard forms are available to get the general solution. They are:

- Variable separable method

- Reducible into the variable separable method
- Homogeneous differential equations
- Non-homogeneous differential equations
- Linear differential equation
- Reducible into a linear differential equation
- Exact differential equations
- Linear differential equations with constant coefficients

Let us see a few differential equations examples to understand this concept clearly.

**Example 1: Find the differential equation corresponding to the equation** $y = ae^{x} + be^{2x} + ce^{-3x}$ **where a, b, c are arbitrary constants.**

**Solution:** Given: $y = ae^x + be^{2x} + ce^{-3x}$

$y' = ae^x + 2be^{2x} - 3\,ce^{-3x}$

$y'' = ae^x + 4be^{2x} + 9\,ce^{-3x}$

$y''' = ae^x + 8be^{2x} - 27\,ce^{-3x}$

$y' - y = be^{2x} - 4ce^{-3x}$———————— (1)

$y'' - y' = 2be^{2x} + 12ce^{-3x}$———————— (2)

$y''' - y'' = 4be^{2x} - 36ce^{-3x}$———————— (3)

Now, with the help of matrices and by using elimination technique for eliminating b and c we get,

$$\begin{bmatrix} y' - y & 1 & -4 \\ y" - y' & 2 & 12 \\ y"' - y" & 4 & -36 \end{bmatrix} = 0$$

This matrix can be simplified as,

$$\begin{bmatrix} y' - y & 1 & -1 \\ y" - y' & 2 & 3 \\ y"' - y" & 4 & -9 \end{bmatrix} = 0$$

Expanding we get

$(y'-y)(-18-12) -1\,(-9(y''-y') - 3(y''' - y'')) - 1(4(y'' - y') - 2(y''' - y'')) = 0$

$-30(y'-y) + 5(y'' - y') + 5(y''' - y'') = 0$

$(y''' - y'') + (y'' - y') - 6(y' - y) = 0$

$y''' - 7y' + 6y = 0$

$d^3y/dx^3 - 7\,dy/dx + 6y = 0$

This is the required differential equation.

**Example 2: Find the differential equation of all the hyperbolas whose axes are along both the axes.**

**Solution:** The standard equation of a hyperbola is

$x^2/a^2 – y^2/b^2 = 1$, whose transverse and conjugate axes are along the coordinate axes.

As there are two arbitrary constants, to eliminate them we need to differentiate the standard equation of hyperbola twice.

Differentiating the above equation with respect to x, we get,

$2x/a^2 – 2yy'/b^2 = 0$….(1)

$2x/a^2 = 2yy'/b^2$

$b^2/a^2 = yy'/x$…..(2)

Differentiating (1) w. r. t. x again, we get,

$2/a^2 – 2/b^2 ((y')^2 + yy'') = 0$

$2/a^2 = 2/b^2 ((y')^2 + yy'')$

$b^2/a^2 = y'^2 + yy''$ …….. (3)

Equating the values of $b^2/a^2$ from (2) and (3)

$yy'/x = y'^2 + yy''$

Or, $y (dy/dx) = x(dy/dx)^2 + xy (d^2y/dx^2)$

This represents a differential equation of second order obtained by eliminating two parameters.

**Example 3: Find the differential equation of the family of circles of radius 5cm and their centres lying on the x-axis.**

**Solution:** Let the centre of the circle on x-axis be (a,0).

The equation of such a circle can be given as:

$(x-a)^2 + y^2 = 5^2$ ……(1)

$(x-a)^2 + y^2 = 25$

Differentiating this equation with respect to x, we get,

$2(x-a) + 2y\, dy/dx = 0$

Taking 2 as common and eliminating,

$(x-a) = -y (dy/dx)$

Substituting the value of (x – a) in equation (1), we get

$y^2(dy/dx)^2 + y^2 = 25$

This is the required differential equation.

## Solution of Separable Differential Equation

The solution of a differential equation is a function, that represents a relationship between the variables, independent of derivatives. Such as:

Given differential equation: $\frac{dy}{dx} = \cos x$

Solution: $y = \sin x + c$ (thus derivative is eliminated)

The solution of a differential equation is also known as its primitive.

### Variable Separable Differential Equations

The differential equations which are expressed in terms of (x,y) such that, the x-terms and y-terms can be separated to different sides of the equation (including delta terms). Thus, each variable separated can be integrated easily to form the solution of differential equation.

The equations can be written as:

$f(x)dx + g(y)dy = 0$

where *f(x)* and *g(y)* are either constants or functions of *x* and *y* respectively.

In simpler terms all the differential equations in which all the terms involving *x* and *dx* can be written on one side of the equation and the terms involving *y* and *dy* on the other side are known as variable separable differential equations.

**Example: Solve** $\frac{dy}{dx} = \frac{x^3 + 3}{y^2 + 1}$

**Solution:** $(y^2+1)dy = (x^3+3)dx$

Integrating both the sides we get,

$\int(y^2+1)dy = \int(x^3+3)dx$

$\frac{y^3}{3} + y = \frac{x^4}{4} + 3x + c$

$\frac{y^3}{3} + y - \frac{x^4}{4} - 3x = c$

It is the required solution.

**Example: Arjun is riding his bike at an initial velocity of 10 m/s. To reach his home at time he continuously increases his velocity at the rate of** 10 **in what time will his velocity be 2.718 times of what it is now?**

**Solution**: Let the velocity of Arjun be v at any time t. Then

$\underline{dv} = \underline{10} \times V$

$$\frac{dv}{dt} = \frac{V}{10}$$

dt 100

Separating the variables we get

$$\frac{1}{V}\,dv = \frac{1}{10}\,dt$$

Integrating both the sides we get;

$$\ln V = \frac{t}{10} + c$$

Where c is any arbitrary constant;

$V = e^{t/10} \times e^{c}$

$V = Ce^{t/10}$

We know at t = 0, V = 10m/s

So the equation becomes

$V = 10e^{t/10}$

Now V = 2.718 × 10ms

Substituting this value we get,

$27.18 = 10e^{t/10}$

$\Rightarrow e^{t/10} = 2.718$

$\Rightarrow \ln e^{t/10} = \ln 2.718$

$\Rightarrow t/10 = 1$

$\Rightarrow t = 10$ seconds

## Exact Differential Equation

The equation P (x,y) dx + Q (x,y) dy = 0 is an exact differential equation. If there exists a function f of two variables x and y having continuous partial derivatives such that the exact differential equation definition is separated as follows:

$u_x(x, y) = p(x, y)$ and $u_y(x, y) = Q(x, y)$;

Therefore, the general solution of the equation is u(x, y) = C.

Where "C" is an arbitrary constant.

### Testing for Exactness

Assume the functions P(x, y) and Q(x, y) having the continuous partial derivatives in a particular domain D, and the differential equation is exact if and only if it satisfies the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

## Exact Differential Equation Integrating Factor

If the differential equation P (x, y) dx + Q (x, y) dy = 0 is not exact, it is possible to make it exact by multiplying using a relevant factor u(x, y) which is known as integrating factor for the given differential equation.

Consider an example,

2ydx + x dy = 0

Now check it whether the given differential equation is exact using testing for exactness.

The given differential equation is not exact.

In order to convert it into the exact differential equation, multiply by the integrating factor u(x,y) = x, the differential equation becomes,

2 xy dx + $x^2$ dy = 0

The above resultant equation is exact differential equation because the left side of the equation is a total differential of $x^2y$.

Sometimes it is difficult to find the integrating factor. But there are two classes of differential equation whose integrating factor may be found easily. Those equations have the integrating factor having the functions of either x alone or y alone.

When you consider the differential equation P (x,y) dx + Q (x,y) dy = 0, the two cases involved are:

**Case 1:** If $\frac{1}{Q(x,y)}$ [$P_y$(x,y) − $Q_x$(x,y)] = h(x), which is a function of x alone, then $e^{\int h(x)dx}$ is an integrating factor

**Case 2:** If $\frac{1}{P(x,y)}$ [$Q_x$(x,y) − $P_y$(x,y)] = k(y), which is a function of y alone, then $e^{\int k(y)dy}$ is an integrating factor

## How to Solve Exact Differential Equation

The following steps explains how to solve the exact differential equation in a detailed way.

**Step 1:** The first step to solve exact differential equation is that to make sure with the given differential equation is exact using testing for exactness.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

**Step 2:** Write the system of two differential equations that defines the function u(x,y). That is

$$\frac{\partial u}{\partial x} = P(x,y) \quad \frac{\partial u}{\partial y} = Q(x,y)$$

**Step 3:** Integrating the first equation over the variable x, we get

$u(x,y) = \int P(x,y)dx + \varphi(y)$

Instead of an arbitrary constant C, write an unknown function of y.

**Step 4:** Differentiating with respect to y, substitute the function u(x,y) in the second equation

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x,y)dx + \varphi(y)\right] = Q(x,y)$$

From the above expression we get the derivative of the unknown function $\varphi(y)$ and it is given by

$$\varphi(y) = Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)dx\right)$$

**Step 5:** We can find the function $\varphi(y)$ by integrating the last expression, so that the function u(x,y) becomes

$$u(x,y) = \int P(x,y)dx + \varphi(y)$$

**Step 6:** Finally, the general solution of the exact differential equation is given by

u (x,y) = C.

## Exact Differential Equation Examples

Some of the examples of the exact differential equations are as follows:

- $(2xy - 3x^2)\, dx + (x^2 - 2y)\, dy = 0$
- $(xy^2 + x)\, dx + yx^2\, dy = 0$
- $\text{Cos } y\, dx + (y^2 - x \sin y)\, dy = 0$
- $(6x^2 - y + 3)\, dx + (3y^2 - x - 2)\, dy = 0$
- $e^y\, dx + (2y + xe^y)\, dy = 0$

**Example 1: Solve the following $(x^3 + 3xy^2 + 25)dx + (y^3 + 3yx^2 + 25)dy = 0$**

**Solution**:

$$\frac{\partial df(x)}{\partial dx} = 3x^2 + 3y^2$$

$$\frac{\partial dg(y)}{\partial dx} = 3x^2 + 3y^2$$

Since, $\frac{\partial df(x)}{\partial dx} = \frac{\partial dg(y)}{\partial dx}$, it is an exact differential equation.

Its solution is $\int f(x)dx + \int g(y)dy = c$

$\Rightarrow c = \int(x^3 + 3xy^2 + 25)dx + \int(y^3+25)dy$

$\Rightarrow c = x^4/4 + y^4/4 + 3/2x^2y^2 + 25(x+y)$

$\Rightarrow c = x^4 + y^4 + 6x^2y^2 + 100(x+y)$

**Example 2**: **Find the solution for the differential equation $( 2xy - \sin x )\, dx + ( x^2 - \cos y)\, dy = 0$**

**Solution:** Given, $(2xy - \sin x)\, dx + (x^2 - \cos y)\, dy = 0$

First check this equation for exactness,

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - \cos y) = 2x$$

$$\frac{\partial P}{\partial} = \frac{\partial}{}(2xy - \sin x) = 2x$$

$\partial y \quad \partial y$

The equation is exact because it satisfies the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

From the system of two equations, find the functions u ( x, y )

$$\frac{\partial u}{\partial x} = 2xy - \sin x \quad \ldots.(1)$$

$$\frac{\partial u}{\partial y} = x^2 - \cos y \quad \ldots.(2)$$

By integrating the first equation with respect to the variable x, we get

$u(x,y) = \int(2xy - \sin x)dx = x^2 + \cos x + \varphi(y)$

Substituting the above equation in equation (2), it becomes

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2y + \cos x + \varphi(y)] = x^2 - \cos y$$

$\Rightarrow x^2 + \varphi(y) = x^2 - \cos y$

We get, $\Rightarrow \varphi(y) = -\cos y$

Hence, $\varphi(y) = \int(-\cos y)dy = -\sin y$

So the function u ( x , y ) becomes

$u ( x , y ) = x^2y + \cos x - \sin y$

Therefore, the general solution for the given differential equation is

$x^2y + \cos x - \sin y = C$

## First Order Differential Equation

If the function f is a linear expression in y, then the first-order differential equation y' = f (x, y) is a linear equation. That is, the equation is linear and the function f takes the form

**$f(x,y) = p(x)y + q(x)$**

Since the linear equation is ***y = mx+b***

where p and q are continuous functions on some interval I. Differential equations that are not linear are called nonlinear equations.

Consider the first-order differential equation *y' = f (x,y),* is a linear equation and it can be written in the form

**$y' + a(x)y = f(x)$**

where a(x) and f(x) are continuous functions of x

The alternate method to represent the first-order linear equation in a reduced form is

**$(dy/dx) + P(x)y = Q(x)$**

Where P(x) and Q(x) are the functions of x which are the continuous functions. If P(x) or Q(x) is equal to zero, the differential equation is reduced to the variable separable form. It is easy to solve when the differential equations are in variable separable form.

## Types of First Order Differential Equations

There are basically five types of differential equations in the first order. They are:

1. Linear Differential Equations
2. Homogeneous Equations
3. Exact Equations
4. Separable Equations
5. Integrating Factor

## First Order Differential Equations Solutions

Usually, there are two methods considered to solve the linear differential equation of first order.

1. Using Integrating Factor
2. Method of variation of constant

### Integrating Factor

If a linear differential equation is written in the standard form:

**$y' + a(x)y = 0$**

Then, the integrating factor is defined by the formula

$u(x) = \exp(\int a(x)dx)$

Multiplying the integrating factor u(x) on the left side of the equation that converts the left side into the derivative of the product y(x)u(x).

The general solution of the differential equation is expressed as follows:

$$y = \frac{\int u(x)f(x)dx + C}{u(x)}$$

where C is an arbitrary constant.

### Method of Variation of a Constant

This method is similar to the integrating factor method. Finding the general solution of the homogeneous equation is the first necessary step.

**$y' + a(x)y = 0$**

The general solution of the homogeneous equation always contains a constant of integration C. We can replace the constant C with a certain unknown function C(x). When substituting this solution into the non-homogeneous differential equation, we can be able to determine the function C(x).

This approach of the algorithm is called the method of variation of a constant. However, both methods lead to the same solution.

## Properties of First-order Differential Equations

The Linear first-order differential equation possesses the following properties.

- It does not have any transcendental functions like trigonometric functions and logarithmic functions.
- The products of y and any of its derivatives are not present.

**Example 1: Solve the equation $y'-y-xe^x = 0$**

**Solution:** Given, $y'-y-xe^x = 0$

Rewrite the given equation and the equation becomes,

$y'-y = xe^x$

Using the integrating factor, it becomes;

$u(x) = e^{\int(-1)dx} = e^{-\int dx} = e^{-x}$

Therefore, the general solution of the linear equation is

$y(x) = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int e^{-x}xe^{x}dx + C}{e^{-x}}$ $y(x) = \frac{\int xdx + C}{e^{-x}} = e^x\left(\frac{x^2}{2}+C\right)$.

**Example 2: Solve the differential equation $y'+2xy = x$.**

**Solution:** The given equation is already in a standard form, $y' + P(x)y = Q(x)$

Therefore, $P(x) = 2x$ and $Q(x) = x$

Now multiplying both sides by;

$$\mu(x) = e^{\int Pdx} = e^{\int 2xdx} = e^{x^2}$$

$$e^{x^2} + 2xe^{x^2}y = xe^{x^2}$$

$$\frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

Now integrating both the sides, we get;

$$e^{x^2}y = \int xe^{x^2}dx$$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + c$$

$$y = ½ + ce^{-x^2}$$

## Homogeneous Differential Equation

A differential equation of the form f(x,y)dy = g(x,y)dx is said to be **homogeneous differential equation** if the degree of f(x,y) and g(x, y) is same. A function of form F(x,y) which can be written in the form $k^n$ F(x,y) is said to be a homogeneous function of degree n, for k≠0. Hence, f and g are the homogeneous functions of the same degree of x and y. Here, the change of variable y = ux directs to an equation of the form;

dx/x = h(u) du

which could be easily integrated.

Contrarily, a differential equation is homogeneous if it is a similar function of the anonymous function and its derivatives. For linear differential equations, there are no constant terms. The solutions of any linear ordinary differential equation of any degree or order may be calculated by integration from the solution of the homogeneous equation achieved by eliminating the constant term.

Consider the following functions in x and y,

$F_1(x,y) = 2x-8y$

$F_2(x,y) = x^2+8xy+9y^2$

$F_3(x,y) = \sin(x/y)$

$F_4(x,y) = \sin x + \cos y$

If we replace x and y with vx and vy respectively, for non-zero value of v, we get

$F_1(vx,vy) = 2(vx)-8(vy) = v(2x-8y) = vF_1(x,y)$

$F_2(vx,vy) = v^2x^2 + 8(vx)(vy) + 9v^2y^2 = v^2(x^2+8xy+9y^2) = v^2F_2(x,y)$

$F_3(vx,vy) = \sin(vx/vy) = v^0\sin(vx/vy) = v^0F_3(x,y)$

$F_4(vx,vy) = \sin(vx) + \cos(vy) \neq v^nF_4(x,y)$

Hence, functions $F_1$, $F_2$, $F_3$ can be written in the form $v^nF(x,y)$, whereas $F_4$ cannot be written. Thus, first three are homogeneous functions and the last function is not homogeneous.

### Steps to Solve Homogeneous Differential Equation

To solve a homogeneous differential equation following steps are followed:

Given differential equation of the type dy/dx = F(x,y) = g(y/x)

**Step 1-** Substitute y = vx in the given differential equation.

**Step 2 –** Differentiating, we get, dy/dx = v + x(dv/dx). Now substitute the value of and y in the given differential equation, we get

*v + x <u>dv</u> = g(v)*

$$\Rightarrow x\frac{dv}{dx} = g(v) - v$$

**Step 3** – Separating the variables, we get

$$\frac{dv}{g(v)-v} = \frac{dx}{x}$$

**Step 4 –** Integrating both side of equation, we have

$$\int \frac{dv}{g(v)-v}\,dv = \int \frac{dx}{x} + C$$

**Step 5** – After integration we replace v = y/x

## Nonhomogeneous Differential Equation

A linear nonhomogeneous differential equation of second order is represented by;

y" + p(t)y' + q(t)y = g(t)

where g(t) is a non-zero function.

The associated homogeneous equation is;

y" + p(t)y' + q(t)y = 0

which is also known as complementary equation.

**Example 1: Find the equation of the curve passing through the point** (2,π/3) **when the tangent at any point makes an angle** $\tan^{-1}(y/x - \sin^2 y/x)$**.**

**Solution:** $\phi = \tan^{-1}(y/x - \sin^2 y/x)$

Or $\frac{dy}{dx} = \tan\phi = \frac{y}{x} - \sin^2\frac{y}{x}$

Since this equation represents a differential equation of homogeneous type therefore, we substitute y = vx in the above equation.

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \frac{dx}{x} = -\mathrm{cosec}^2 v\,dv$$

Now integrating both the sides w.r.t. to x and v respectively, we get

$$\int\frac{dx}{x} = \int -\mathrm{cosec}^2 v\,dv$$

$$\ln x = \frac{1}{\tan v} + C \text{ .....................(i)}$$

Also as it passes through the point (2,π/3), for (x,y).

We know that v = y/x, thus value of v = π/3 ÷ 2 = π/6

So, substituting the values of x and v in the equation (i), we get

$ln2 = \sqrt{3} + C$

$\Rightarrow C = ln2 - \sqrt{3}$

Or $lnx = \frac{1}{tanv} + ln2 - \sqrt{3}$

Or $lnx = \frac{1}{tan\ y/x} + ln2 - \sqrt{3}$

This is the required solution.

**Example 2: Find the equation of the curve passing through the point (1,-2) when the tangent at any point is given by** y(x+y3)x(y3–x)**.**

**Solution:** The equation of tangent represents the slope of the curve i.e.

This equation is homogeneous in nature.

On cross-multiplication, we get- $(xy^3 - x^2)dy = (xy+y^4)dx$

Solving the equation, we get

$x^2y^3 \frac{(xdy - ydx)}{x^2} - x(xdy - ydx) = 0$

$\Rightarrow x^2y^3 d\ y/x - xd(xy) = 0$

Dividing both the sides by $x^3y^2$ we get,

$\frac{y}{x} d(\frac{y}{x}) - \frac{d(xy)}{x^2y^2} = 0$

Now integrating this equation with respect to y/x and xy we have,

$\int y/x\ d(y/x) = \int \frac{d(xy)}{x^2y^2}$

$½ (y/x)^2 = - 1/xy + C$ .................. (1)

Now substituting the value of the given point in the above equation, we have

$\Rightarrow ½ \times 4 - ½ = C$

$\Rightarrow C = 32$

Put this value of the constant C in equation (1) we get

$½ (y/x)^2 + 1/xy = 3/2$

This is the required solution.

## How To Find The Order Of Differential Equation And Its Degree?

Differential Equations are classified on the basis of the order. Order of a differential equation is the order of the highest derivative (also known as differential coefficient) present in the equation.

Example (i) $\frac{d^3x}{dx^3} + 3x\frac{dy}{dx} = e^y$

In this equation, the order of the highest derivative is 3 hence, this is a third order differential equation.

Example (ii) $-\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} = 3$

This equation represents a second order differential equation.

This way we can have higher order differential equations i.e., nth order differential equations.

### First order differential equation

The order of highest derivative in case of first order differential equations is 1. A linear differential equation has order 1. In case of linear differential equations, the first derivative is the highest order derivative.

$\frac{dy}{dx} + Py = Q$

P and Q are either constants or functions of the independent variable only.

This represents a linear differential equation whose order is 1.

Example: $\frac{dy}{dx} + (x^2 + 5)y = \frac{x}{5}$

This also represents a First order Differential Equation.

### Second Order Differential Equation

When the order of the highest derivative present is 2, then it is a second order differential equation.

Example: $\frac{d^2y}{dx^2} + (x^3 + 3x)y = 9$

In this example, the order of the highest derivative is 2. Therefore, it is a second order differential equation.

### Degree of Differential Equation

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined.

**Example:** $\frac{d^4y}{dx^4} + \left(\frac{d^2y}{dx^2}\right)^2 - 3\frac{dy}{dx} + y = 9$

Here, the exponent of the highest order derivative is one and the given differential equation is a polynomial equation in derivatives. Hence, the degree of this equation is 1.

**Example:** $[d^2y/dx^2 + (dy/dx)^2]^4 = k^2(d^3y/dx^3)^2$

The order of this equation is 3 and the degree is 2 as the highest derivative is of order 3 and the exponent raised to the highest derivative is 2.

**When the Degree of Differential Equation is not Defined?**

It is not possible every time that we can find the degree of given differential equation. The degree of any differential equation can be found when it is in the form a polynomial; otherwise, the degree cannot be defined.

Suppose in a differential equation dy/dx = tan (x + y), the degree is 1, whereas for a differential equation tan (dy/dx) = x + y, the degree is not defined. These type of differential equations can be observed with other trigonometry functions such as sine, cosine and so on.

Let us see some more examples on finding the degree and order of differential equations.

**Example:** $\frac{d^2y}{dx^2} + \cos \frac{d^2y}{dx^2} = 5x$

The given differential equation is not a polynomial equation in derivatives. Hence, the degree of this equation is not defined.

**Example 4:** $\left(\frac{d^3y}{dx^3}\right)^2 + y = 0$

The order of this equation is 3 and the degree is 2.

**Example: Figure out the order and degree of differential equation that can be formed from the equation $\sqrt{1-x^2} + \sqrt{1-y^2} = k(x-y)$.**

**Solution:**

Let x = sinθ, y = sinϕ

So, the given equation can be rewritten as

$\sqrt{1-\sin\theta^2} + \sqrt{1-\sin\phi^2} = k(\sin\theta-\sin\phi)$

$\Rightarrow (\cos\theta + \cos\phi) = k(\sin\theta - \sin\phi)$

$\Rightarrow 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2k\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$

$\cot\frac{\theta-\phi}{2} = k$

$\theta-\phi = 2\cot^{-1}k$

$\sin^{-1}x - \sin^{-1}y = 2\cot^{-1}k$

Differentiating both sides w. r. t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

So, the degree of the differential equation is 1 and it is a first order differential equation.

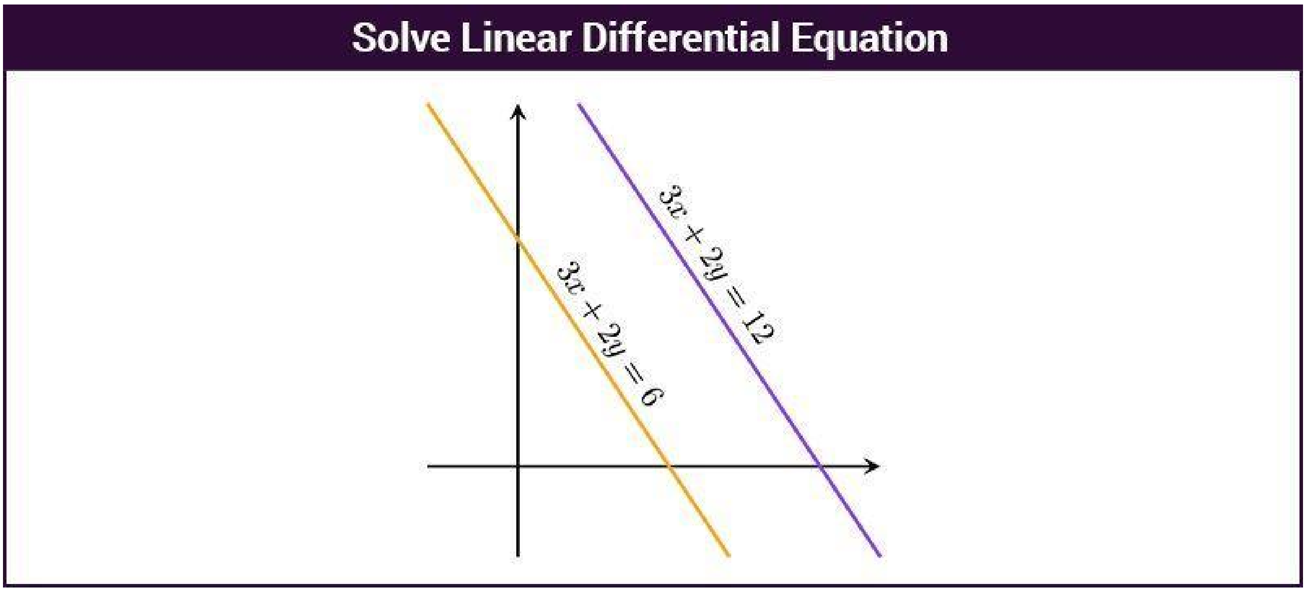
Note: If the DE in which differential coefficient is present inside the parenthesis of any another function as a composite, then first attempt to make it as simple as possible. Now, check whether it in the form of a polynomial in terms of derivatives. If it is a polynomial, the degree can be defined.

## How to Solve Linear Differential Equation

A linear differential equation is defined by the linear polynomial equation, which consists of derivatives of several variables. It is also stated as Linear Partial Differential Equation when the function is dependent on variables and derivatives are partial.

Also, the differential equation of the form, $dy/dx + Py = Q$, is a ***first-order linear differential equation*** where P and Q are either constants or functions of y (independent variable) only.

To find **linear differential equations solution,** we have to derive the general form or representation of the solution.




## Non-Linear Differential Equation

When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

## Solving Linear Differential Equations

For finding the solution of such linear differential equations, we determine a function of the independent variable let us say M(x), which is known as the Integrating factor (I.F).

Multiplying both sides of equation (1) with the integrating factor M(x) we get;

$M(x)dy/dx + M(x)Py = QM(x)$ …..(2)

Now we chose M(x) in such a way that the L.H.S of equation (2) becomes the derivative of y.M(x)

i.e. $d(yM(x))/dx = (M(x))dy/dx + y\,(d(M(x)))dx$ … (Using $d(uv)/dx = v(du/dx) + u(dv/dx)$)

$\Rightarrow M(x)\,/(dy/dx) + M(x)Py = M\,(x)\,dy/dx + y\,d(M(x))/dx$

$\Rightarrow M(x)Py = y\,dM(x)/dx$

$\Rightarrow 1/M'(x) = P.dx$

Integrating both sides with respect to x, we get;

$\log M\,(x) = \int Pdx (As \int f'(x)/f(x)) = \log f(x)$

$\Rightarrow M(x) = e^{\int Pdx}$ I.F

Now, using this value of the integrating factor, we can find out the solution of our first order linear differential equation.

Multiplying both the sides of equation (1) by the I.F. we get

$e^{\int Pdx}\,dy/dx + yPe^{\int Pdx} = Qe^{\int Pdx}$

This could be easily rewritten as:

$d(y.e^{\int Pdx})/dx = Qe^{\int Pdx}$ (Using $d(uv)/dx = v\,du/dx + u\,dv/dx$)

Now integrating both the sides with respect to x, we get:

$\int d(y.e^{\int Pdx}) = \int Qe^{\int Pdx}dx + c$

$y = 1/e^{\int Pdx}(\int Qe^{\int Pdx}\,dx + c)$

where C is some arbitrary constant.

## How to Solve First Order Linear Differential Equation

Learn to solve the first-order differential equation with the help of steps given below.

1. Rearrange the terms of the given equation in the form $dy/dx + Py = Q$
   where P and Q are constants or functions of the independent variable x only.
2. To obtain the integrating factor, integrate P (obtained in step 1) with respect to x and put this integral as a power to e.
   $e^{\int Pdx} = I.F$
3. Multiply both the sides of the linear first-order differential equation with the I.F.
   $e^{\int Pdx}\,dy/dx + yPe^{\int Pdx} = Qe^{\int Pdx}$
4. The L.H.S of the equation is always a derivative of $y \times M\,(x)$
   i.e. L.H.S $= d(y \times I.F)/dx$
   $d(y \times I.F)dx = Q \times I.F$
5. In the last step, we simply integrate both the sides with respect to x and get a constant term C to get the solution.

$\therefore y \times I.F = \int Q \times I.F\,dx + C$ ,

where C is some arbitrary constant

Similarly, we can also solve the other form of linear first-order differential equation dx/dy +Px = Q using the same steps. In this form P and Q are the functions of y. The integrating factor (I.F) comes out to be  and using this we find out the solution which will be

**(x) × (I.F) =** $\int Q \times I.F\,dy + c$

**Example 1: Solve the LDE = dy/dx = $[1/(1+x^3)] – [3x^2/(1 + x^2)]y$**

**Solution:**

The above mentioned equation can be rewritten as $dy/dx + [3x^2/(1 + x^2)]\ y = 1/(1+x^3)$

Comparing it with dy/dx + Py = O, we get

$P = 3x^2/1+x^3$

$Q = 1/1 + x^3$

Let's figure out the integrating factor (I.F.) which is $e\int^{Pdx}$

$$\Rightarrow I.F = e^{\int \frac{3x^2}{1+x^3}\,dx} = e^{ln(1+x^3)}$$

$\Rightarrow$I.F. = $1 + x^3$

Now, we can also rewrite the L.H.S as:

d(y × I.F)/dx,

$\Rightarrow d(y \times (1 + x^3))\ dx = [1/(1 +x^3)] \times (1 + x^3)$

Integrating both the sides w. r. t. x, we get,

$\Rightarrow y \times ( 1 + x^3) = x$

$\Rightarrow y = x/(1 + x^3)$

$\Rightarrow y = [x/(1 + x^3 ) + C$

**Example 2: Solve the following differential equation:**

**dy/dx + (sec x)y = 7**

**Solution:**

Comparing the given equation with dy/dx + Py = Q

We see, P = sec x, Q = 7

Now lets find out the integrating factor using the formula

$e\int^{Pdx}$ = I.F

$\Rightarrow e\int^{secdx}$ = I.F.

$\Rightarrow$ I.F. = $e^{\ln|secx+tanx|}$= secx + tanx

Now we can also rewrite the L.H.S as

d(y × I.F)/dx},

i.e., d(y × (sec x + tan x ))

$\Rightarrow$ d(y × (sec x + tan x ))/dx = 7(sec x + tan x)

Integrating both the sides w. r. t. x, we get,

∫d (y × (secx + tanx)) = ∫7 (secx + tanx)dx

$\Rightarrow$ y × (secx + tanx) = 7(ln|secx + tanx|+log|secx|)

$\Rightarrow$ y = 7(ln|secx + tanx|+ log|secx|(secx + tanx) + c

## Differential Equations Applications

We can describe the differential equations applications in real life in terms of:

### Exponential Growth

For exponential growth, we use the formula;

**$G(t)= G_0 e^{kt}$**

Let $G_0$ is positive and k is constant, then

dG/dt = k

G(t) increases with time

$G_0$ is the value when t=0

G is the exponential growth model.

### Exponential reduction or decay

**$R(t) = R_0 e^{-kt}$**

When $R_0$ is positive and k is constant, R(t) is decreasing with time,

dR/dt = -k

R is the exponential reduction model

Newton's law of cooling, Newton's law of fall of an object, Circuit theory or Resistance and Inductor, RL circuit are also some of the applications of differential equations.