# INTEGRATION

Integration is the calculation of an integral. Integrals in maths are used to find many useful quantities such as areas, volumes, displacement, etc. When we speak about integrals, it is related to usually definite integrals. The indefinite integrals are used for antiderivatives.

## Integration – Inverse Process of Differentiation

We know that differentiation is the process of finding the derivative of the functions and integration is the process of finding the antiderivative of a function. So, these processes are inverse of each other. So, we can say that integration is the inverse process of differentiation or vice versa. The integration is also called the anti-differentiation. In this process, we are provided with the derivative of a function and asked to find out the function (i.e., primitive).

We know that the differentiation of sin x is cos x.

It is mathematically written as:

$(d/dx) \sin x = \cos x$ …(1)

Here, cos x is the derivative of sin x. So, sin x is the antiderivative of the function cos x. Also, any real number "C" is considered as a constant function and the derivative of the constant function is zero.

So, equation (1) can be written as

$(d/dx) (\sin x + C) = \cos x +0$

$(d/dx) (\sin x + C) = \cos x$

Where "C" is the arbitrary constant or constant of integration.

Generally, we can write the function as follow:

$(d/dx) [F(x)+C] = f(x)$, where x belongs to the interval I.

To represent the antiderivative of "f", the integral symbol "∫" symbol is introduced. The antiderivative of the function is represented as $\int f(x)\, dx$. This can also be read as the indefinite integral of the function "f" with respect to x.

Therefore, the symbolic representation of the antiderivative of a function (Integration) is:

$y = \int f(x)\, dx$

$\int f(x)\, dx = F(x) + C$.

## Integrals in Maths

There are two types of integrals in maths:

- Definite Integral
- Indefinite Integral

## Definite Integral

An integral that contains the upper and lower limits then it is a definite integral. On a real line, x is restricted to lie. Riemann Integral is the other name of the Definite Integral.

A definite Integral is represented as:

$\int_{b}^{a} f(x)dx$

## Indefinite Integral

Indefinite integrals are defined without upper and lower limits. It is represented as:

$\int f(x)dx = F(x) + C$

Where C is any constant and the function f(x) is called the integrand.

## Integration Formulas

- $\int 1 \, dx = x + C$
- $\int a \, dx = ax + C$
- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C; \; n \neq -1$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x (\tan x) \, dx = \sec x + C$
- $\int \csc x (\cot x) \, dx = -\csc x + C$
- $\int \frac{1}{x} \, dx = \ln |x| + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x \, dx = \frac{a^x}{\ln a} + C; \; a > 0, a \neq 1$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$
- $\int \sin^n(x) dx = \frac{-1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx$
- $\int \cos^n(x) dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) dx$
- $\int \tan^n(x) dx = \frac{1}{n-1} \tan^{n-1}(x) + \int \tan^{n-2}(x) dx$
- $\int \sec^n(x) dx = \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) dx$
- $\int \csc^n(x) dx = \frac{-1}{n-1} \csc^{n-2}(x) \cot(x) + \frac{n-2}{n-1} \int \csc^{n-2}(x) dx$

## Integral uv Formula

The uv formula of integral is generally used to calculate the integration by parts. This can be expressed as:

∫du(dvdx)dx = uv − ∫v(dudx)dx

Here,

u = Function of u(x)

v = Function of v(x)

dv = Derivative of v(x)

du = Derivative of u(x)

## Integral log x

Let us derive the formula for integral log x here.

∫log x dx

We know that,

∫ uv' = uv – ∫ vu' where u and v are functions of x and prime here indicates derivatives.

Using the ILATE rule, let us take the functions as u(x) and v(x).

u(x)= log x and v(x)= x

Thus, by substituting these functions in the formula we get,

∫log x dx = x(log x) – ∫x.(dx/x) + C

= x(log x) – ∫dx + C

= x(log x) – x + C

Where C is integration constant.

## Integral of tan x

The integral of tan x can be derived using substitution method. Here, we have to assume one part of the function as u and find the derivative of this function to substitute in the integral of the given function. This can be understood in a better way using the derivation given below.

∫tan x dx

Let cos x = u

-sinx dx = du

sin x dx = -du

∫tan x dx

= ∫(sin x/cos x) dx

= -∫du/u

= -ln u + C

=-ln cos x + C

Or

= ln sec x + C

Therefore, ∫tan x dx = -ln (cos x) + C = ln (sec x) + C

## Antiderivative Functions

| Derivatives | Integrals (Anti derivatives) |
|---|---|
| (i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$ ; | $\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\ n\neq -1$ |
| Particularly, we note that $\frac{d}{dx}(x)=1$ ; | $\int dx=x+C$ |
| (ii) $\frac{d}{dx}(\sin x)=\cos x$ ; | $\int \cos x\,dx=\sin x+C$ |
| (iii) $\frac{d}{dx}(-\cos x)=\sin x$ ; | $\int \sin x\,dx=-\cos x+C$ |
| (iv) $\frac{d}{dx}(\tan x)=\sec^2 x$ ; | $\int \sec^2 x\,dx=\tan x+C$ |
| (v) $\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$ ; | $\int \operatorname{cosec}^2 x\,dx=-\cot x+C$ |
| (vi) $\frac{d}{dx}(\sec x)=\sec x\tan x$ ; | $\int \sec x\tan x\,dx=\sec x+C$ |
| (vii) $\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$ ; | $\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+C$ |
| (viii) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$ ; | $\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+C$ |
| (ix) $\frac{d}{dx}(-\cos^{-1} x)=\frac{1}{\sqrt{1-x^2}}$ ; | $\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+C$ |
| (x) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$ ; | $\int \frac{dx}{1+x^2}=\tan^{-1} x+C$ |
| (xi) $\frac{d}{dx}(-\cot^{-1} x)=\frac{1}{1+x^2}$ ; | $\int \frac{dx}{1+x^2}=-\cot^{-1} x+C$ |

## Application of Integrals

### Integrals used to find the area of a curved region

Integrals are the functions which satisfy a given differential equation for finding the area of a curvy region y = f(x), the x-axis and the line x = a and x = b(b>a) is represented through this formula:

Area= $\int_a^b y dx = \int_a^b f(x)dx$

If curvy region is x = Ø(y), y-axis and the line y = c, y = d is represented through this formula:

Area = $\int_a^b x dy = \int_c^d \emptyset(y)dy$

## Area Bounded by Two Curves

If the dimensions of two curves are y = f(x), y = g(x) and lines x = a and x = b is represented by the formula:

$\int_a^b [f(x) - g(x)]dx$ , where f(x) ≥ g(x) in [a,b]

If f(x) ≥ g(x) in [a,c] and f(x) ≤ g(x) in [c,b], a<c<b, then area is

$\int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$

## Solved Examples

**Example 1: Evaluate: ∫ $3ax/(b^2 + c^2x^2)$ dx**

**Solution:**

To evaluate the integral, I = ∫ $3ax/(b^2 + c^2x^2)$ dx

Let us take v = $b^2 + c^2x^2$, then

dv = $2c^2x$ dx

Thus, ∫ $3ax/(b^2 + c^2x^2)$ dx

= $(3ax/2c^2x)\int dv/v$

Now, cancel x on both numerator and denominator, we get

= $(3a/2c^2)\int dv/v$

= $(3a/2c^2)$ log $|b^2 + c^2x^2|$ + C

Where C is an arbitrary constant

**Example 2: Determine ∫$\tan^8 x \sec^4 x$ dx**

**Solution:**

Given: $\int \tan^8 x \sec^4 x \, dx$

Let $I = \int \tan^8 x \sec^4 x \, dx$ — (1)

Now, split $\sec^4 x = (\sec^2 x)(\sec^2 x)$

Now, substitute in (1)

$I = \int \tan^8 x (\sec^2 x)(\sec^2 x) \, dx$

$= \int \tan^8 x (\tan^2 x + 1)(\sec^2 x) \, dx$

It can be written as:

$= \int \tan^{10} x \sec^2 x \, dx + \int \tan^8 x \sec^2 x \, dx$

Now, integrate the terms with respect to x, we get:

$I = (\tan^{11} x / 11) + (\tan^9 x / 9) + C$

Hence, $\int \tan^8 x \sec^4 x \, dx = (\tan^{11} x / 11) + (\tan^9 x / 9) + C$

**Example 3: Write the anti-derivative of the following function: $3x^2+4x^3$**

**Solution:**

Given: $3x^2+4x^3$

The antiderivative of the given function is written as:

$\int 3x^2+4x^3 \, dx = 3(x^3/3) + 4(x^4/4)$

$= x^3 + x^4$

Thus, the antiderivative of $3x^2+4x^3 = x^3 + x^4$

**Example 4: Determine the antiderivative F of "f" , which is defined by f (x) = $4x^3 - 6$, where F (0) = 3**

**Solution:**

Given function: $f(x) = 4x^3 - 6$

Now, integrate the function:

$\int 4x^3 - 6 \, dx = 4(x^4/4) - 6x + C$

$\int 4x^3 - 6 \, dx = x^4 - 6x + C$

Thus, the antiderivative of the function, F is $x^4 - 6x + C$, where C is a constant

Also, given that, $F(0) = 3$,

Now, substitute x = 0 in the obtained antiderivative function, we get:

$(0)^4 - 6(0) + C = 3$

Therefore, C = 3.

Now, substitute C = 3 in antiderivative function

Hence, the required antiderivative function is $x^4 - 6x + 3$.

**Example 5: Integrate the given function using integration by substitution: 2x sin($x^2$+ 1) with respect to x:**

**Solution:**

Given function: 2x sin($x^2$+ 1)

We know that, the derivative of $x^2 + 1$ is 2x.

Now, use the substitution method, we get

$x^2 + 1 = t$, so that 2x dx = dt.

Hence, we get $\int 2x \sin(x^2+1)\, dx = \int \sin t\, dt$

$= -\cos t + C$

$= -\cos(x^2 + 1) + C$

Where C is an arbitrary constant

Therefore, the antiderivative of 2x sin($x^2$+ 1) using integration by substitution method is $= -\cos(x^2 + 1) + C$

**Example 6: Integrate: $\int \sin^3 x \cos^2 x\, dx$**

**Solution:**

Given that, $\int \sin^3 x \cos^2 x\, dx$

This can be written as:

$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x (\sin x)\, dx$

$=\int (1 - \cos^2 x) \cos^2 x (\sin x)\, dx$ —(1)

Now, substitute t = cos x,

Then dt = -sin x dx

Now, equation can be written as:

Thus, $\int \sin^3 x \cos^2 x\, dx = -\int (1-t^2)t^2\, dt$

Now, multiply $t^2$ inside the bracket, we get

$= -\int (t^2-t^4)\, dt$

Now, integrate the above function:

$= -[(t^3/3) - (t^5/5)] + C$ —(2)

Where C is a constant

Now, substitute t = cos x in (2)

$= -(\frac{1}{3})\cos^3 x + (1/5)\cos^5 x + C$

Hence, $\int \sin^3 x \cos^2 x\, dx = -(\frac{1}{3})\cos^3 x + (1/5)\cos^5 x + C$