Adjoint formulation for manifold-reduced system

Seung Whan Chung

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Original governing equation

We refer to the original governing equation as a priori projected system:

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \mathbf{P}(\mathbf{x};\theta)\cdot\mathbf{F}(\mathbf{x};\theta) \\
\mathbf{P}(\mathbf{x};\theta) &= \mathbf{I} - \mathbf{V}_f\mathbf{U}_f \\
\mathbf{J} = \nabla_x\mathbf{F} &= \begin{bmatrix}\mathbf{V}_s & \mathbf{V}_f\end{bmatrix}\begin{bmatrix}\Lambda_{ss} & 0 \\ 0 & \Lambda_{ff}\end{bmatrix}\begin{bmatrix}\mathbf{U}_s \\ \mathbf{U}_f\end{bmatrix}
\end{aligned} \tag{1}$$

where $\theta \in \mathbb{R}^{N_c}$ is the control parameter. For other notations, refer to the original document.

Observable and its sensitivity

We have the observable as a functional of state variable $\mathbf{x}$:

$$\mathcal{O}[\mathbf{x}] = \int_0^T o(\mathbf{x})dt,$$

and we would like to measure its sensitivity to control parameters,

$$\nabla_\theta\mathcal{O} = \int_0^T \nabla_x o\cdot\nabla_\theta\mathbf{x}\,dt = \langle\nabla_x o, \nabla_\theta\mathbf{x}\rangle\,, \tag{2}$$

where the inner product is defined on $\mathbb{R}^N$,

$$\langle\mathbf{a},\mathbf{b}\rangle = \int_0^T a_i b_i\,dt$$

with Einstein notation for indices.

Sensitivity equation

To obtain the sensitivity (2), the sensitivity of the solution $\nabla_\theta\mathbf{x}$ needs to be computed. It can be obtained by solving the linearized equation from (1). From now on the equations are expressed in index notation to avoid the ambiguity.

- i, j, k: the index on $\mathbb{R}^N$ space

- c: the index on control space $\mathbb{R}^{N_c}$
- l: the index on fast manifold space $\mathbb{R}^{N_f}$

$$\begin{aligned}\frac{d}{dt}\partial_c x_i &= \partial_c\left(P_{ij}F_j\right)\\ &= \partial_k\left(P_{ij}F_j\right)\partial_c x_k + \partial_c\left(P_{ij}F_j\right)\\ &= \left[\partial_k P_{ij}\cdot F_j + P_{ij}\cdot\partial_k F_j\right]\partial_c x_k + \partial_c P_{ij}F_j + P_{ij}\partial_c F_j\end{aligned} \tag{3}$$

$$\begin{aligned}\partial_k P_{ij} &= -\partial_k V_{il}U_{lj} - V_{il}\partial_k U_{lj}\\ \partial_c P_{ij} &= -\partial_c V_{il}U_{lj} - V_{il}\partial_c U_{lj}\end{aligned} \tag{4}$$