CALCULUS II

Solutions to Practice Problems

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Preface

Here are the solutions to the practice problems for the Calculus II notes.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Outline

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Integration Techniques – In this chapter we will look at several integration techniques including Integration by Parts, Integrals Involving Trig Functions, Trig Substitutions and Partial Fractions. We will also look at Improper Integrals including using the Comparison Test for convergence/divergence of improper integrals.

> **Integration by Parts** – In this section we will be looking at Integration by Parts. Of all the techniques we'll be looking at in this class this is the technique that students are most likely to run into down the road in other classes. We also give a derivation of the integration by parts formula.
>
> **Integrals Involving Trig Functions** – In this section we look at integrals that involve trig functions. In particular we concentrate integrating products of sines and cosines as well as products of secants and tangents. We will also briefly look at how to modify the work for products of these trig functions for some quotients of trig functions.
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> **Trig Substitutions** – In this section we will look at integrals (both indefinite and definite) that require the use of a substitutions involving trig functions and how they can be used to simplify certain integrals.
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> **Partial Fractions** – In this section we will use partial fractions to rewrite integrands into a form that will allow us to do integrals involving some rational functions.
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> **Integrals Involving Roots** – In this section we will take a look at a substitution that can, on occasion, be used with integrals involving roots.
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> **Integrals Involving Quadratics** – In this section we are going to look at some integrals that involve quadratics for which the previous techniques won't work right away. In some cases, manipulation of the quadratic needs to be done before we can do the integral. We will see several cases where this is needed in this section.
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> **Integration Strategy** – In this section we give a general set of guidelines for determining how to evaluate an integral. The guidelines give here involve a mix of both Calculus I and Calculus II techniques to be as general as possible. Also note that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines.
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> **Improper Integrals** – In this section we will look at integrals with infinite intervals of integration and integrals with discontinuous integrands in this section. Collectively, they are called improper integrals and as we will see they may or may not have a finite (*i.e.* not infinite) value. Determining if they have finite values will, in fact, be one of the major topics of this section.
>
> **Comparison Test for Improper Integrals** – It will not always be possible to evaluate improper integrals and yet we still need to determine if they converge or diverge (*i.e.* if they have a finite value or not). So, in this section we will use the Comparison Test to determine if improper integrals converge or diverge.
>
> **Approximating Definite Integrals** – In this section we will look at several fairly simple methods of approximating the value of a definite integral. It is not possible to evaluate every definite integral (*i.e.* because it is not possible to do the indefinite integral) and yet we may need to know the value of the definite integral anyway. These methods allow us to at least get an approximate value which may be enough in a lot of cases.

Applications of Integrals – In this chapter we'll take a look at a few applications of integrals. We will look at determining the arc length of a curve, the surface area of a solid of revolution, the center of mass of a region bounded by two curves, the hydrostatic force/pressure on a plate submerged in water and a quick look at computing the mean of a probability density function. The applications given here tend to result in integrals that are typically covered in a Calculus II course.

Arc Length – In this section we'll determine the length of a curve over a given interval.

Surface Area – In this section we'll determine the surface area of a solid of revolution, *i.e.* a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.

Center of Mass – In this section we will determine the center of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may the *x* or *y*-axis).

Hydrostatic Pressure and Force – In this section we'll determine the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.

Probability – Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities are fixed values and will depend on a variety of factors. In this section we will look at probability density functions and computing the mean (think average wait in line or average life span of a light blub) of a probability density function.

Parametric Equations and Polar Coordinates – In this chapter we will introduce the ideas of parametric equations and polar coordinates. We will also look at many of the basic Calculus ideas (tangent lines, area, arc length and surface area) in terms of these two ideas.

Parametric Equations and Curves – In this section we will introduce parametric equations and parametric curves (*i.e.* graphs of parametric equations). We will graph several sets of parametric equations and discuss how to eliminate the parameter to get an algebraic equation which will often help with the graphing process.

Tangents with Parametric Equations – In this section we will discuss how to find the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for parametric curves. We will also discuss using these derivative formulas to find the tangent line for parametric curves as well as determining where a parametric curve in increasing/decreasing and concave up/concave down.

Area with Parametric Equations – In this section we will discuss how to find the area between a parametric curve and the *x*-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus I techniques on the resulting algebraic equation).

Arc Length with Parametric Equations – In this section we will discuss how to find the arc length of a parametric curve using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Surface Area with Parametric Equations – In this section we will discuss how to find the surface area of a solid obtained by rotating a parametric curve about the *x* or *y*-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Polar Coordinates – In this section we will introduce polar coordinates an alternative coordinate system to the 'normal' Cartesian/Rectangular coordinate system. We will derive formulas to convert between polar and Cartesian coordinate systems. We will also look at many of the standard polar graphs as well as circles and some equations of lines in terms of polar coordinates.

[Tangents with Polar Coordinates]() – In this section we will discuss how to find the derivative $\frac{dy}{dx}$ for polar curves. We will also discuss using this derivative formula to find the tangent line for polar curves using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).
[Area with Polar Coordinates]() – In this section we will discuss how to the area enclosed by a polar curve. The regions we look at in this section tend (although not always) to be shaped vaguely like a piece of pie or pizza and we are looking for the area of the region from the outer boundary (defined by the polar equation) and the origin/pole. We will also discuss finding the area between two polar curves.
[Arc Length with Polar Coordinates]() – In this section we will discuss how to find the arc length of a polar curve using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).
[Surface Area with Polar Coordinates]() – In this section we will discuss how to find the surface area of a solid obtained by rotating a polar curve about the x or y-axis using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).
[Arc Length and Surface Area Revisited]() – In this section we will summarize all the arc length and surface area formulas we developed over the course of the last two chapters.

[Series and Sequences]() – In this chapter we introduce sequences and series. We discuss whether a sequence converges or diverges, is increasing or decreasing, or if the sequence is bounded. We will then define just what an infinite series is and discuss many of the basic concepts involved with series. We will discuss if a series will converge or diverge, including many of the tests that can be used to determine if a series converges or diverges. We will also discuss using either a power series or a Taylor series to represent a function and how to find the radius and interval of convergence for this series.
[Sequences]() – In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.
[More on Sequences]() – In this section we will continue examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.
[Series – The Basics]() – In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).
[Convergence/Divergence of Series]() – In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.
[Special Series]() – I In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.
[Integral Test]() – In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

Comparison Test/Limit Comparison Test – In this section we will discuss using the Comparison Test and Limit Comparison Tests to determine if an infinite series converges or diverges. In order to use either test the terms of the infinite series must be positive. Proofs for both tests are also given.
Alternating Series Test – In this section we will discuss using the Alternating Series Test to determine if an infinite series converges or diverges. The Alternating Series Test can be used only if the terms of the series alternate in sign. A proof of the Alternating Series Test is also given.
Absolute Convergence – In this section we will have a brief discussion on absolute convergence and conditionally convergent and how they relate to convergence of infinite series.
Ratio Test – In this section we will discuss using the Ratio Test to determine if an infinite series converges absolutely or diverges. The Ratio Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Ratio Test is also given.
Root Test – In this section we will discuss using the Root Test to determine if an infinite series converges absolutely or diverges. The Root Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Root Test is also given.
Strategy for Series – In this section we give a general set of guidelines for determining which test to use in determining if an infinite series will converge or diverge. Note as well that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines. A summary of all the various tests, as well as conditions that must be met to use them, we discussed in this chapter are also given in this section.
Estimating the Value of a Series – In this section we will discuss how the Integral Test, Comparison Test, Alternating Series Test and the Ratio Test can, on occasion, be used to estimating the value of an infinite series.
Power Series – In this section we will give the definition of the power series as well as the definition of the radius of convergence and interval of convergence for a power series. We will also illustrate how the Ratio Test and Root Test can be used to determine the radius and interval of convergence for a power series.
Power Series and Functions – In this section we discuss how the formula for a convergent Geometric Series can be used to represent some functions as power series. To use the Geometric Series formula, the function must be able to be put into a specific form, which is often impossible. However, use of this formula does quickly illustrate how functions can be represented as a power series. We also discuss differentiation and integration of power series.
Taylor Series – In this section we will discuss how to find the Taylor/Maclaurin Series for a function. This will work for a much wider variety of function than the method discussed in the previous section at the expense of some often unpleasant work. We also derive some well known formulas for Taylor series of $\mathbf{e}^x$, $\cos(x)$ and $\sin(x)$ around $x = 0$.
Applications of Series – In this section we will take a quick look at a couple of applications of series. We will illustrate how we can find a series representation for indefinite integrals that cannot be evaluated by any other method. We will also see how we can use the first few terms of a power series to approximate a function.
Binomial Series – In this section we will give the Binomial Theorem and illustrate how it can be used to quickly expand terms in the form $(a+b)^n$ when *n* is an integer. In addition, when *n* is not an integer an extension to the Binomial Theorem can be used to give a power series representation of the term.

Vectors – In this (very brief) chapter we will take a look at the basics of vectors. Included are common notation for vectors, arithmetic of vectors, dot product of vectors (and applications) and cross product of vectors (and applications).

> **Basic Concepts** – In this section we will introduce some common notation for vectors as well as some of the basic concepts about vectors such as the magnitude of a vector and unit vectors. We also illustrate how to find a vector from its starting and end points.
> **Vector Arithmetic** – In this section we will discuss the mathematical and geometric interpretation of the sum and difference of two vectors. We also define and give a geometric interpretation for scalar multiplication. We also give some of the basic properties of vector arithmetic and introduce the common *i*, *j*, *k* notation for vectors.
> **Dot Product** – In this section we will define the dot product of two vectors. We give some of the basic properties of dot products and define orthogonal vectors and show how to use the dot product to determine if two vectors are orthogonal. We also discuss finding vector projections and direction cosines in this section.
> **Cross Product** – In this section we define the cross product of two vectors and give some of the basic facts and properties of cross products.

3-Dimensional Space – In this chapter we will start looking at three dimensional space. This chapter is generally prep work for Calculus III and so we will cover the standard 3D coordinate system as well as a couple of alternative coordinate systems. We will also discuss how to find the equations of lines and planes in three dimensional space. We will look at some standard 3D surfaces and their equations. In addition we will introduce vector functions and some of their applications (tangent and normal vectors, arc length, curvature and velocity and acceleration).

> **The 3-D Coordinate System** – I In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.
> **Equations of Lines** – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.
> **Equations of Planes** – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.
> **Quadric Surfaces** – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.
> **Functions of Several Variables** – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.
> **Vector Functions** – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.
> **Calculus with Vector Functions** – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.
> **Tangent, Normal and Binormal Vectors** – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.
Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.
Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.
Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.
Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system.

Chapter 1 : Integration Techniques

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Section 1-1 : Integration by Parts

1. Evaluate $\int 4x\cos\left(2-3x\right)dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 4x \qquad\qquad dv = \cos\left(2-3x\right)dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 4x & &\rightarrow & du &= 4dx \\ dv &= \cos\left(2-3x\right)dx & &\rightarrow & v &= -\tfrac{1}{3}\sin\left(2-3x\right) \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\begin{aligned} \int 4x\cos\left(2-3x\right)dx &= \left(4x\right)\left(-\tfrac{1}{3}\sin\left(2-3x\right)\right) - \int -\tfrac{4}{3}\sin\left(2-3x\right)dx \\ &= -\tfrac{4}{3}x\sin\left(2-3x\right) + \tfrac{4}{3}\int \sin\left(2-3x\right)dx \end{aligned}$$

Step 4
Okay, the new integral we get is easily doable and so all we need to do to finish this problem out is do the integral.

$$\int 4x\cos\left(2-3x\right)dx = \boxed{-\tfrac{4}{3}x\sin\left(2-3x\right) + \tfrac{4}{9}\cos\left(2-3x\right) + c}$$

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2. Evaluate $\int_{6}^{0}\left(2+5x\right)\mathbf{e}^{\frac{1}{3}x}\,dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Also, don't forget that the limits on the integral won't have any effect on the choices of *u* and *dv*.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 2 + 5x \qquad\qquad dv = \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 2 + 5x & &\rightarrow & du &= 5dx \\ dv &= \mathbf{e}^{\frac{1}{3}x}\,dx & &\rightarrow & v &= 3\mathbf{e}^{\frac{1}{3}x} \end{aligned}$$

Step 3
We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int (2+5x)\mathbf{e}^{\frac{1}{3}x} = (2+5x)\left(3\mathbf{e}^{\frac{1}{3}x}\right) - \int 5\left(3\mathbf{e}^{\frac{1}{3}x}\right)dx = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 15\int \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 4
Okay, the new integral we get is easily doable so let's evaluate it to get,

$$\int (2+5x)\mathbf{e}^{\frac{1}{3}x} = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 45\mathbf{e}^{\frac{1}{3}x} + c = 15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x} + c$$

Step 5
The final step is then to take care of the limits.

$$\int_6^0 (2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx = \left(15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x}\right)\Bigg|_6^0 = \boxed{-39 - 51\mathbf{e}^2 = -415.8419}$$

Do not get excited about the fact that the lower limit is larger than the upper limit. This can happen on occasion and in no way affects how the integral is evaluated.

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3. Evaluate $\int \left(3t + t^2\right)\sin(2t)\,dt$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 3t + t^2 \qquad\qquad dv = \sin\left(2t\right)dt$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 3t + t^2 &\quad\rightarrow\quad& du = \left(3 + 2t\right)dt \\ dv &= \sin\left(2t\right)dt &\quad\rightarrow\quad& v = -\tfrac{1}{2}\cos\left(2t\right) \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int\left(3t + t^2\right)\sin\left(2t\right)dt = -\tfrac{1}{2}\left(3t + t^2\right)\cos\left(2t\right) + \tfrac{1}{2}\int\left(3 + 2t\right)\cos\left(2t\right)dt$$

Step 4
Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the *t*'s have gone down by one we are heading in the right direction.

So, here are the choices for *u* and *dv* for the new integral.

$$\begin{aligned} u &= 3 + 2t &\quad\rightarrow\quad& du = 2dt \\ dv &= \cos\left(2t\right)dt &\quad\rightarrow\quad& v = \tfrac{1}{2}\sin\left(2t\right) \end{aligned}$$

Step 5
Okay, all we need to do now is plug these new choices of *u* and *dv* into the new integral we got in Step 3 and finish the problem out.

$$\begin{aligned} \int\left(3t + t^2\right)\sin\left(2t\right)dt &= -\tfrac{1}{2}\left(3t + t^2\right)\cos\left(2t\right) + \tfrac{1}{2}\left[\tfrac{1}{2}\left(3 + 2t\right)\sin\left(2t\right) - \int\sin\left(2t\right)dt\right] \\ &= -\tfrac{1}{2}\left(3t + t^2\right)\cos\left(2t\right) + \tfrac{1}{2}\left[\tfrac{1}{2}\left(3 + 2t\right)\sin\left(2t\right) + \tfrac{1}{2}\cos\left(2t\right)\right] + c \\ &= \boxed{-\tfrac{1}{2}\left(3t + t^2\right)\cos\left(2t\right) + \tfrac{1}{4}\left(3 + 2t\right)\sin\left(2t\right) + \tfrac{1}{4}\cos\left(2t\right) + c} \end{aligned}$$

4. Evaluate $\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$.

Hint : Be careful with your choices of *u* and *dv* here. If you think about it there is really only one way that the choice can be made here and have the problem be workable.

Step 1
The first step here is to pick *u* and *dv*.

Note that if we choose the inverse tangent for *dv* the only way to get *v* is to integrate *dv* and so we would need to know the answer to get the answer and so that won't work for us. Therefore, the only real choice for the inverse tangent is to let it be *u*.

So, here are our choices for *u* and *dv*.

$$u = 6\tan^{-1}\left(\tfrac{8}{w}\right) \qquad\qquad dv = dw$$

Don't forget the *dw*! The differential *dw* still needs to be put into the *dv* even though there is nothing else left in the integral.

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$u = 6\tan^{-1}\left(\tfrac{8}{w}\right) \qquad \to \qquad du = 6\frac{-\frac{8}{w^2}}{1+\left(\frac{8}{w}\right)^2}dw = 6\frac{-\frac{8}{w^2}}{1+\frac{64}{w^2}}dw$$

$$dv = dw \qquad \to \qquad v = w$$

Step 3
In order to complete this problem we'll need to do some rewrite on *du* as follows,

$$du = \frac{-48}{w^2+64}dw$$

Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int 6\tan^{-1}\left(\tfrac{8}{w}\right)dw = 6w\tan^{-1}\left(\tfrac{8}{w}\right) + 48\int\frac{w}{w^2+64}dw$$

Step 4
Okay, the new integral we get is easily doable (with the substitution $u = 64 + w^2$) and so all we need to do to finish this problem out is do the integral.

$$\int 6\tan^{-1}\left(\tfrac{8}{w}\right)dw = \boxed{6w\tan^{-1}\left(\tfrac{8}{w}\right) + 24\ln\left|w^2+64\right| + c}$$

5. Evaluate $\int \mathbf{e}^{2z} \cos\left(\frac{1}{4}z\right) dz$.

Hint : This is one of the few integration by parts problems where either function can go on *u* and *dv*. Be careful however to not get locked into an endless cycle of integration by parts.

Step 1
The first step here is to pick *u* and *dv*.

In this case we can put the exponential in either the *u* or the *dv* and the cosine in the other. It is one of the few problems where the choice doesn't really matter.

For this problem well use the following choices for *u* and *dv*.

$$u = \cos\left(\tfrac{1}{4}z\right) \qquad\qquad dv = \mathbf{e}^{2z}\,dz$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= \cos\left(\tfrac{1}{4}z\right) & &\rightarrow & du &= -\tfrac{1}{4}\sin\left(\tfrac{1}{4}z\right)dz \\ dv &= \mathbf{e}^{2z}dz & &\rightarrow & v &= \tfrac{1}{2}\mathbf{e}^{2z} \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{8}\int \mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)dz$$

Step 4
We'll now need to do integration by parts again and to do this we'll use the following choices.

$$\begin{aligned} u &= \sin\left(\tfrac{1}{4}z\right) & &\rightarrow & du &= \tfrac{1}{4}\cos\left(\tfrac{1}{4}z\right)dz \\ dv &= \mathbf{e}^{2z}dz & &\rightarrow & v &= \tfrac{1}{2}\mathbf{e}^{2z} \end{aligned}$$

Step 5
Plugging these into the integral from Step 3 gives,

$$\begin{aligned} \int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{8}\left[\tfrac{1}{2}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) - \tfrac{1}{8}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz\right] \\ &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) - \tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz \end{aligned}$$

Step 6
To finish this problem all we need to do is some basic algebraic manipulation to get the identical integrals on the same side of the equal sign.

$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)+\tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)-\tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz$$
$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz+\tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)+\tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)$$
$$\tfrac{65}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)+\tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)$$

Finally, all we need to do is move the coefficient on the integral over to the right side.

$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \boxed{\tfrac{32}{65}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)+\tfrac{4}{65}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)+c}$$

---

6. Evaluate $\int_0^{\pi} x^2\cos\left(4x\right)dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Also, don't forget that the limits on the integral won't have any effect on the choices of *u* and *dv*.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = x^2 \qquad\qquad dv = \cos\left(4x\right)dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= x^2 & &\rightarrow & du &= 2x\,dx \\ dv &= \cos\left(4x\right)dx & &\rightarrow & v &= \tfrac{1}{4}\sin\left(4x\right) \end{aligned}$$

Step 3
We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int x^2\cos\left(4x\right)dx = \tfrac{1}{4}x^2\sin\left(4x\right)-\tfrac{1}{2}\int x\sin\left(4x\right)dx$$

Step 4
Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the *x*'s have gone down by one we are heading in the right direction.

So, here are the choices for *u* and *dv* for the new integral.

$$\begin{aligned} u &= x & &\rightarrow & du &= dx \\ dv &= \sin\left(4x\right)dx & &\rightarrow & v &= -\tfrac{1}{4}\cos\left(4x\right) \end{aligned}$$

Step 5
Okay, all we need to do now is plug these new choices of *u* and *dv* into the new integral we got in Step 3 and evaluate the integral.

$$\begin{aligned} \int x^2\cos\left(4x\right)dx &= \tfrac{1}{4}x^2\sin\left(4x\right) - \tfrac{1}{2}\left[-\tfrac{1}{4}x\cos\left(4x\right) + \tfrac{1}{4}\int\cos\left(4x\right)dx\right] \\ &= \tfrac{1}{4}x^2\sin\left(4x\right) - \tfrac{1}{2}\left[-\tfrac{1}{4}x\cos\left(4x\right) + \tfrac{1}{16}\sin\left(4x\right)\right] + c \\ &= \tfrac{1}{4}x^2\sin\left(4x\right) + \tfrac{1}{8}x\cos\left(4x\right) - \tfrac{1}{32}\sin\left(4x\right) + c \end{aligned}$$

Step 6
The final step is then to take care of the limits.

$$\int_0^{\pi} x^2\cos\left(4x\right)dx = \left(\tfrac{1}{4}x^2\sin\left(4x\right) + \tfrac{1}{8}x\cos\left(4x\right) - \tfrac{1}{32}\sin\left(4x\right)\right)\Big|_0^{\pi} = \boxed{\tfrac{1}{8}\pi}$$

---

7. Evaluate $\int t^7\sin\left(2t^4\right)dt$.

Hint : Be very careful with your choices of *u* and *dv* for this problem. It looks a lot like previous practice problems but it isn't!

Step 1
The first step here is to pick *u* and *dv* and, in this case, we'll need to be careful how we chose them.

If we follow the model of many of the examples/practice problems to this point it is tempting to let *u* be t^7 and to let *dv* be $\sin\left(2t^4\right)$.

However, this will lead to some real problems. To compute *v* we'd have to integrate the sine and because of the t^4 in the argument this is not possible. In order to integrate the sine we would have to have a t^3 in the integrand as well in order to a substitution as shown below,

$$\int t^3\sin\left(2t^4\right)dt = \tfrac{1}{8}\int\sin\left(w\right)dw = -\tfrac{1}{8}\cos\left(2t^4\right) + c \qquad w = 2t^4$$

Now, this may seem like a problem, but in fact it's not a problem for this particular integral. Notice that we actually have 7 *t*'s in the integral and there is no reason that we can't split them up as follows,

$$\int t^7 \sin\left(2t^4\right)dt = \int t^4\, t^3 \sin\left(2t^4\right)dt$$

After doing this we can now choose *u* and *dv* as follows,

$$u = t^4 \qquad\qquad dv = t^3 \sin\left(2t^4\right)dt$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= t^4 && \to && du = 4t^3 dt \\ dv &= t^3 \sin\left(2t^4\right)dt && \to && v = -\tfrac{1}{8}\cos\left(2t^4\right) \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int t^7 \sin\left(2t^4\right)dt = -\tfrac{1}{8}t^4 \cos\left(2t^4\right) + \tfrac{1}{2}\int t^3 \cos\left(2t^4\right)dt$$

Step 4
At this point, notice that the new integral just requires the same Calculus I substitution that we used to find *v*. So, all we need to do is evaluate the new integral and we'll be done.

$$\int t^7 \sin\left(2t^4\right)dt = \boxed{-\tfrac{1}{8}t^4 \cos\left(2t^4\right) + \tfrac{1}{16}\sin\left(2t^4\right) + c}$$

Do not get so locked into patterns for these problems that you end up turning the patterns into "rules" on how certain kinds of problems work. Most of the easily seen patterns are also easily broken (as this problem has shown).

Because we (as instructors) tend to work a lot of "easy" problems initially they also tend to conform to the patterns that can be easily seen. This tends to lead students to the idea that the patterns will always work and then when they run into one where the pattern doesn't work they get in trouble. So, be careful!

Note as well that we're not saying that patterns don't exist and that it isn't useful to recognize them. You just need to be careful and understand that there will, on occasion, be problems where it will look like a pattern you recognize, but in fact will not quite fit the pattern and another approach will be needed to work the problem.

Alternate Solution

Note that there is an alternate solution to this problem. We could use the substitution $w = 2t^4$ as the first step as follows.

$$w = 2t^4 \qquad \to \qquad dw = 8t^3 dt \qquad \& \qquad t^4 = \tfrac{1}{2}w$$
$$\int t^7 \sin\left(2t^4\right) dt = \int t^4\, t^3 \sin\left(2t^4\right) dt = \int \left(\tfrac{1}{2}w\right)\left(\tfrac{1}{8}\right)\sin\left(w\right) dw = \int \tfrac{1}{16} w \sin\left(w\right) dw$$

We won't avoid integration by parts as we can see here, but it is somewhat easier to see it this time. Here is the rest of the work for this problem.

$$u = \tfrac{1}{16}w \qquad \to \qquad du = \tfrac{1}{16}dw$$
$$dv = \sin\left(w\right) dw \qquad \to \qquad v = -\cos\left(w\right)$$

$$\int t^7 \sin\left(2t^4\right) dt = -\tfrac{1}{16} w\cos\left(w\right) + \tfrac{1}{16}\int \cos\left(w\right) dw = -\tfrac{1}{16} w\cos\left(w\right) + \tfrac{1}{16}\sin\left(w\right) + c$$

As the final step we just need to substitution back in for *w*.

$$\int t^7 \sin\left(2t^4\right) dt = -\tfrac{1}{8} t^4 \cos\left(2t^4\right) + \tfrac{1}{16}\sin\left(2t^4\right) + c$$

---

8. Evaluate $\int y^6 \cos\left(3y\right) dy$.

Hint : Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1

Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking *u* and *dv* and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rr|c}
y^6 & \cos\left(3y\right) & + \\
6y^5 & \tfrac{1}{3}\sin\left(3y\right) & - \\
30y^4 & -\tfrac{1}{9}\cos\left(3y\right) & + \\
120y^3 & -\tfrac{1}{27}\sin\left(3y\right) & - \\
360y^2 & \tfrac{1}{81}\cos\left(3y\right) & + \\
720y & \tfrac{1}{243}\sin\left(3y\right) & - \\
720 & -\tfrac{1}{729}\cos\left(3y\right) & + \\
0 & -\tfrac{1}{2187}\sin\left(3y\right) & -
\end{array}$$

Step 2
Here's the integral for this problem,

$$\begin{aligned}\int y^6\cos(3y)\,dy &= \left(y^6\right)\left(\tfrac{1}{3}\sin(3y)\right)-\left(6y^5\right)\left(-\tfrac{1}{9}\cos(3y)\right)+\left(30y^4\right)\left(-\tfrac{1}{27}\sin(3y)\right)\\ &\quad -\left(120y^3\right)\left(\tfrac{1}{81}\cos(3y)\right)+\left(360y^2\right)\left(\tfrac{1}{243}\sin(3y)\right)\\ &\quad -\left(720y\right)\left(-\tfrac{1}{729}\cos(3y)\right)+\left(720\right)\left(-\tfrac{1}{2187}\sin(3y)\right)+c\\ &= \boxed{\begin{array}{c}\tfrac{1}{3}y^6\sin(3y)+\tfrac{2}{3}y^5\cos(3y)-\tfrac{10}{9}y^4\sin(3y)-\tfrac{40}{27}y^3\cos(3y)\\ +\tfrac{40}{27}y^2\sin(3y)+\tfrac{80}{81}y\cos(3y)-\tfrac{80}{243}\sin(3y)+c\end{array}}\end{aligned}$$

---

9. Evaluate $\int\left(4x^3-9x^2+7x+3\right)\mathbf{e}^{-x}\,dx$.

Hint : Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1
Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking *u* and *dv* and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rr|c} 4x^3-9x^2+7x+3 & \mathbf{e}^{-x} & + \\ 12x^2-18x+7 & -\mathbf{e}^{-x} & - \\ 24x-18 & \mathbf{e}^{-x} & + \\ 24 & -\mathbf{e}^{-x} & - \\ 0 & \mathbf{e}^{-x} & + \end{array}$$

Step 2
Here's the integral for this problem,

$$\begin{aligned}\int\left(4x^3-9x^2+7x+3\right)\mathbf{e}^{-x}\,dx &= \left(4x^3-9x^2+7x+3\right)\left(-\mathbf{e}^{-x}\right)-\left(12x^2-18x+7\right)\left(\mathbf{e}^{-x}\right)\\ &\quad +\left(24x-18\right)\left(-\mathbf{e}^{-x}\right)-\left(24\right)\left(\mathbf{e}^{-x}\right)+c\\ &= -\mathbf{e}^{-x}\left(4x^3-9x^2+7x+3\right)-\mathbf{e}^{-x}\left(12x^2-18x+7\right)\\ &\quad -\mathbf{e}^{-x}\left(24x-18\right)-24\mathbf{e}^{-x}+c\\ &= \boxed{-\mathbf{e}^{-x}\left(4x^3+3x^2+13x+16\right)+c}\end{aligned}$$

Section 1-2 : Integrals Involving Trig Functions

1. Evaluate $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the sine is odd and so we can strip one of them out.

$$\int \sin^3\left(\tfrac{2}{3}x\right)\cos^4\left(\tfrac{2}{3}x\right)dx = \int \sin^2\left(\tfrac{2}{3}x\right)\cos^4\left(\tfrac{2}{3}x\right)\sin\left(\tfrac{2}{3}x\right)dx$$

Step 2
Now we can use the trig identity $\sin^2\theta + \cos^2\theta = 1$ to convert the remaining sines to cosines.

$$\int \sin^3\left(\tfrac{2}{3}x\right)\cos^4\left(\tfrac{2}{3}x\right)dx = \int \left(1-\cos^2\left(\tfrac{2}{3}x\right)\right)\cos^4\left(\tfrac{2}{3}x\right)\sin\left(\tfrac{2}{3}x\right)dx$$

Step 3
We can now use the substitution $u = \cos\left(\frac{2}{3}x\right)$ to evaluate the integral.

$$\begin{aligned}\int \sin^3\left(\tfrac{2}{3}x\right)\cos^4\left(\tfrac{2}{3}x\right)dx &= -\tfrac{3}{2}\int \left(1-u^2\right)u^4\,du \\ &= -\tfrac{3}{2}\int u^4 - u^6\,du = -\tfrac{3}{2}\left(\tfrac{1}{5}u^5 - \tfrac{1}{7}u^7\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don't forget to substitute back in for *u*!

$$\int \sin^3\left(\tfrac{2}{3}x\right)\cos^4\left(\tfrac{2}{3}x\right)dx = \boxed{\tfrac{3}{14}\cos^7\left(\tfrac{2}{3}x\right) - \tfrac{3}{10}\cos^5\left(\tfrac{2}{3}x\right)+c}$$

---

2. Evaluate $\int \sin^8\left(3z\right)\cos^5\left(3z\right)dz$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the cosine is odd and so we can strip one of them out.

$$\int \sin^8(3z)\cos^5(3z)\,dz = \int \sin^8(3z)\cos^4(3z)\cos(3z)\,dz$$

Step 2
Now we can use the trig identity $\sin^2\theta + \cos^2\theta = 1$ to convert the remaining cosines to sines.

$$\begin{aligned}\int \sin^8(3z)\cos^5(3z)\,dz &= \int \sin^8(3z)\left[\cos^2(3z)\right]^2\cos(3z)\,dz\\ &= \int \sin^8(3z)\left[1-\sin^2(3z)\right]^2\cos(3z)\,dz\end{aligned}$$

Step 3
We can now use the substitution $u = \sin(3z)$ to evaluate the integral.

$$\begin{aligned}\int \sin^8(3z)\cos^5(3z)\,dz &= \tfrac{1}{3}\int u^8\left[1-u^2\right]^2\,du\\ &= \tfrac{1}{3}\int u^8 - 2u^{10} + u^{12}\,du = \tfrac{1}{3}\left(\tfrac{1}{9}u^9 - \tfrac{2}{11}u^{11} + \tfrac{1}{13}u^{13}\right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don't forget to substitute back in for *u*!

$$\int \sin^8(3z)\cos^5(3z)\,dz = \boxed{\tfrac{1}{27}\sin^9(3z) - \tfrac{2}{33}\sin^{11}(3z) + \tfrac{1}{39}\sin^{13}(3z) + c}$$

---

3. Evaluate $\int \cos^4(2t)\,dt$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that we only have even exponents and so we'll need to use half-angle and double-angle formulas to reduce this integral into one that we can do.

Also, do not get excited about the fact that we don't have any sines in the integrand. Sometimes we will not have both trig functions in the integrand. That doesn't mean that that we can't use the same techniques that we used in this section.

So, let's start this problem off as follows.

$$\int \cos^4(2t)\,dt = \int \left(\cos^2(2t)\right)^2\,dt$$

Step 2
Now we can use the half-angle formula to get,

$$\int \cos^4(2t)\,dt = \int \left[\tfrac{1}{2}\left(1+\cos(4t)\right)\right]^2\,dt = \int \tfrac{1}{4}\left(1+2\cos(4t)+\cos^2(4t)\right)dt$$

Step 3
We'll need to use the half-angle formula one more time on the third term to get,

$$\begin{aligned}\int \cos^4(2t)\,dt &= \tfrac{1}{4}\int 1+2\cos(4t)+\tfrac{1}{2}\left[1+\cos(8t)\right]dt \\ &= \tfrac{1}{4}\int \tfrac{3}{2}+2\cos(4t)+\tfrac{1}{2}\cos(8t)\,dt\end{aligned}$$

Step 4
Now all we have to do is evaluate the integral.

$$\int \cos^4(2t)\,dt = \tfrac{1}{4}\left(\tfrac{3}{2}t+\tfrac{1}{2}\sin(4t)+\tfrac{1}{16}\sin(8t)\right)+c = \boxed{\tfrac{3}{8}t+\tfrac{1}{8}\sin(4t)+\tfrac{1}{64}\sin(8t)+c}$$

---

4. Evaluate $\displaystyle\int_{\pi}^{2\pi} \cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
We have two options for dealing with the limits. We can deal with the limits as we do the integral or we can evaluate the indefinite integral and take care of the limits in the last step. We'll use the latter method of dealing with the limits for this problem.

In this case notice that both exponents are odd. This means that we can either strip out a cosine and convert the rest to sines or strip out a sine and convert the rest to cosines. Either are perfectly acceptable solutions. However, the exponent on the cosine is smaller and so there will be less conversion work if we strip out a cosine and convert the remaining cosines to sines.

Here is that work.

$$\begin{aligned}\int \cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw &= \int \cos^2\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)\cos\left(\tfrac{1}{2}w\right)dw \\ &= \int \left(1-\sin^2\left(\tfrac{1}{2}w\right)\right)\sin^5\left(\tfrac{1}{2}w\right)\cos\left(\tfrac{1}{2}w\right)dw\end{aligned}$$

Step 2

We can now use the substitution $u=\sin\left(\tfrac{1}{2}w\right)$ to evaluate the integral.

$$\begin{aligned}\int \cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw &= 2\int\left(1-u^2\right)u^5\,du \\ &= 2\int u^5-u^7\,du = 2\left(\tfrac{1}{6}u^6-\tfrac{1}{8}u^8\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3

Don't forget to substitute back in for *u*!

$$\int \cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw = \tfrac{1}{3}\sin^6\left(\tfrac{1}{2}w\right)-\tfrac{1}{4}\sin^8\left(\tfrac{1}{2}w\right)+c$$

Step 4

Now all we need to do is deal with the limits.

$$\int_{\pi}^{2\pi}\cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw = \left.\left(\tfrac{1}{3}\sin^6\left(\tfrac{1}{2}w\right)-\tfrac{1}{4}\sin^8\left(\tfrac{1}{2}w\right)\right)\right|_{\pi}^{2\pi} = \boxed{-\tfrac{1}{12}}$$

Alternate Solution

As we noted above we could just have easily stripped out a sine and converted the rest to cosines if we'd wanted to. We'll not put that work in here, but here is the indefinite integral that you should have gotten had you done it that way.

$$\int \cos^3\left(\tfrac{1}{2}w\right)\sin^5\left(\tfrac{1}{2}w\right)dw = -\tfrac{1}{2}\cos^4\left(\tfrac{1}{2}w\right)+\tfrac{2}{3}\cos^6\left(\tfrac{1}{2}w\right)-\tfrac{1}{4}\cos^8\left(\tfrac{1}{2}w\right)+c$$

Note as well that regardless of which approach we use to doing the indefinite integral the value of the definite integral will be the same.

---

5. Evaluate $\int \sec^6\left(3y\right)\tan^2\left(3y\right)dy$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the secant is even and so we can strip two of them out.

$$\int \sec^6(3y)\tan^2(3y)\,dy = \int \sec^4(3y)\tan^2(3y)\sec^2(3y)\,dy$$

Step 2
Now we can use the trig identity $\tan^2\theta + 1 = \sec^2\theta$ to convert the remaining secants to tangents.

$$\begin{aligned}\int \sec^6(3y)\tan^2(3y)\,dy &= \int \left[\sec^2(3y)\right]^2 \tan^2(3y)\sec^2(3y)\,dy \\ &= \int \left[\tan^2(3y)+1\right]^2 \tan^2(3y)\sec^2(3y)\,dy\end{aligned}$$

Step 3
We can now use the substitution $u = \tan(3y)$ to evaluate the integral.

$$\begin{aligned}\int \sec^6(3y)\tan^2(3y)\,dy &= \tfrac{1}{3}\int \left[u^2+1\right]^2 u^2\,du \\ &= \tfrac{1}{3}\int u^6 + 2u^4 + u^2\,du = \tfrac{1}{3}\left(\tfrac{1}{7}u^7 + \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don't forget to substitute back in for *u*!

$$\int \sec^6(3y)\tan^2(3y)\,dy = \boxed{\tfrac{1}{21}\tan^7(3y) + \tfrac{2}{15}\tan^5(3y) + \tfrac{1}{9}\tan^3(3y) + c}$$

---

6. Evaluate $\int \tan^3(6x)\sec^{10}(6x)\,dx$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the tangent is odd and we've got a secant in the problems and so we can strip one of each of them out.

$$\int \tan^3(6x)\sec^{10}(6x)\,dx = \int \tan^2(6x)\sec^9(6x)\ \tan(6x)\sec(6x)\,dx$$

Step 2
Now we can use the trig identity $\tan^2\theta+1=\sec^2\theta$ to convert the remaining tangents to secants.

$$\int\tan^3(6x)\sec^{10}(6x)\,dx=\int\left[\sec^2(6x)-1\right]\sec^9(6x)\;\tan(6x)\sec(6x)\,dx$$

Note that because the exponent on the secant is even we could also have just stripped two of them out and converted the rest of them to tangents. However, that conversion process would have been significantly more work than the path that we chose here.

Step 3
We can now use the substitution $u=\sec(6x)$ to evaluate the integral.

$$\begin{aligned}\int\tan^3(6x)\sec^{10}(6x)\,dx&=\tfrac{1}{6}\int\left[u^2-1\right]u^9\,du\\&=\tfrac{1}{6}\int u^{11}-u^9\,du=\tfrac{1}{6}\left(\tfrac{1}{12}u^{12}-\tfrac{1}{10}u^{10}\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don't forget to substitute back in for *u*!

$$\int\tan^3(6x)\sec^{10}(6x)\,dx=\boxed{\tfrac{1}{72}\sec^{12}(6x)-\tfrac{1}{60}\sec^{10}(6x)+c}$$

---

7. Evaluate $\int_0^{\frac{\pi}{4}}\tan^7(z)\sec^3(z)\,dz$.

Hint : Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
We have two options for dealing with the limits. We can deal with the limits as we do the integral or we can evaluate the indefinite integral and take care of the limits in the last step. We'll use the latter method of dealing with the limits for this problem.

The first thing to notice here is that the exponent on the tangent is odd and we've got a secant in the problems and so we can strip one of each of them out and use the trig identity $\tan^2\theta+1=\sec^2\theta$ to convert the remaining tangents to secants.

$$\begin{aligned}\int \tan^7(z)\sec^3(z)\,dz &= \int \tan^6(z)\sec^2(z)\ \tan(z)\sec(z)\,dz \\ &= \int \left[\tan^2(z)\right]^3 \sec^2(z)\ \tan(z)\sec(z)\,dz \\ &= \int \left[\sec^2(z)-1\right]^3 \sec^2(z)\ \tan(z)\sec(z)\,dz\end{aligned}$$

Step 2

We can now use the substitution $u = \sec(z)$ to evaluate the integral.

$$\begin{aligned}\int \tan^7(z)\sec^3(z)\,dz &= \int \left[u^2-1\right]^3 u^2\,du \\ &= \int u^8 - 3u^6 + 3u^4 - u^2\,du = \tfrac{1}{9}u^9 - \tfrac{3}{7}u^7 + \tfrac{3}{5}u^5 - \tfrac{1}{3}u^3 + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3

Don't forget to substitute back in for *u*!

$$\int \tan^7(z)\sec^3(z)\,dz = \tfrac{1}{9}\sec^9(z) - \tfrac{3}{7}\sec^7(z) + \tfrac{3}{5}\sec^5(z) - \tfrac{1}{3}\sec^3(z) + c$$

Step 4

Now all we need to do is deal with the limits.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz &= \left(\tfrac{1}{9}\sec^9(z) - \tfrac{3}{7}\sec^7(z) + \tfrac{3}{5}\sec^5(z) - \tfrac{1}{3}\sec^3(z)\right)\Big|_0^{\frac{\pi}{4}} \\ &= \boxed{\tfrac{2}{315}\left(8+13\sqrt{2}\right) = 0.1675}\end{aligned}$$

---

8. Evaluate $\int \cos(3t)\sin(8t)\,dt$.

Step 1

There really isn't all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

$$\int \cos(3t)\sin(8t)\,dt = \int \tfrac{1}{2}\left[\sin(8t-3t) + \sin(8t+3t)\right]dt = \tfrac{1}{2}\int \sin(5t) + \sin(11t)\,dt$$

Make sure that you pay attention to the formula! The formula given in this section listed the sine first instead of the cosine. Make sure that you used the formula correctly!

Step 2
Now all we need to do is evaluate the integral.

$$\int \cos(3t)\sin(8t)\,dt = \tfrac{1}{2}\left(-\tfrac{1}{5}\cos(5t) - \tfrac{1}{11}\cos(11t)\right) + c = \boxed{-\tfrac{1}{10}\cos(5t) - \tfrac{1}{22}\cos(11t) + c}$$

---

9. Evaluate $\int_1^3 \sin(8x)\sin(x)\,dx$.

Step 1
There really isn't all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

$$\int_1^3 \sin(8x)\sin(x)\,dx = \int_1^3 \tfrac{1}{2}\left[\cos(8x-x) - \cos(8x+x)\right]dx = \tfrac{1}{2}\int_1^3 \cos(7x) - \cos(9x)\,dx$$

Step 2
Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_1^3 \sin(8x)\sin(x)\,dx &= \tfrac{1}{2}\left[\tfrac{1}{7}\sin(7x) - \tfrac{1}{9}\sin(9x)\right]\Big|_1^3 \\ &= \boxed{\tfrac{1}{14}\sin(21) - \tfrac{1}{18}\sin(27) - \tfrac{1}{14}\sin(7) + \tfrac{1}{18}\sin(9) = -0.0174}\end{aligned}$$

Make sure your calculator is set to radians if you computed a decimal answer!

---

10. Evaluate $\int \cot(10z)\csc^4(10z)\,dz$.

Hint : Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.

Step 1
Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we'll be able to work this on as well.

We can first notice here is that the exponent on the cotangent is odd and we've got a cosecant in the problems and so we can strip the (only) cotangent and one of the secants out.

$$\int \cot(10z)\csc^4(10z)\,dz = \int \csc^3(10z)\quad \cot(10z)\csc(10z)\,dz$$

Step 2
Normally we would use the trig identity $\cot^2\theta + 1 = \csc^2\theta$ to convert the remaining cotangents to cosecants. However, in this case there are no remaining cotangents to convert and so there really isn't anything to do at this point other than to use the substitution $u = \csc(10z)$ to evaluate the integral.

$$\int \cot(10z)\csc^4(10z)\,dz = -\tfrac{1}{10}\int u^3\,du = -\tfrac{1}{40}u^4 + c$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3
Don't forget to substitute back in for *u*!

$$\int \cot(10z)\csc^4(10z)\,dz = \boxed{-\tfrac{1}{40}\csc^4(10z) + c}$$

---

11. Evaluate $\int \csc^6\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)dw$.

Hint : Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.

Step 1
Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we'll be able to work this on as well.

We can first notice here is that the exponent on the cosecant is even and so we can strip out two of them.

$$\int \csc^6\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)dw = \int \csc^4\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)\ \csc^2\left(\tfrac{1}{4}w\right)dw$$

Step 2
Now we can use the trig identity $\cot^2\theta + 1 = \csc^2\theta$ to convert the remaining cosecants to cotangents.

$$\begin{aligned}\int \csc^6\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)dw &= \int \left[\csc^2\left(\tfrac{1}{4}w\right)\right]^2\cot^4\left(\tfrac{1}{4}w\right)\ \csc^2\left(\tfrac{1}{4}w\right)dw \\ &= \int \left[\cot^2\left(\tfrac{1}{4}w\right)+1\right]^2\cot^4\left(\tfrac{1}{4}w\right)\ \csc^2\left(\tfrac{1}{4}w\right)dw\end{aligned}$$

Step 3

Now we can use the substitution $u = \cot\left(\frac{1}{4}w\right)$ to evaluate the integral.

$$\begin{aligned}\int \csc^6\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)dw &= -4\int\left[u^2+1\right]^2 u^4\,du\\ &= -4\int u^8+2u^6+u^4\,du = -4\left(\tfrac{1}{9}u^9+\tfrac{2}{7}u^7+\tfrac{1}{5}u^5\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don't forget to substitute back in for *u*!

$$\int \csc^6\left(\tfrac{1}{4}w\right)\cot^4\left(\tfrac{1}{4}w\right)dw = \boxed{-\tfrac{4}{9}\cot^9\left(\tfrac{1}{4}w\right)-\tfrac{8}{7}\cot^7\left(\tfrac{1}{4}w\right)-\tfrac{4}{5}\cot^5\left(\tfrac{1}{4}w\right)+c}$$

---

12. Evaluate $\displaystyle\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt$.

Hint : How would you do this problem if it were a product?

Step 1
If this were a product of secants and tangents we would know how to do it. The same ideas work here, except that we have to pay attention to only the numerator. We can't strip anything out of the denominator (in general) and expect it to work the same way. We can only strip things out of the numerator.

So, let's notice here is that the exponent on the secant is even and so we can strip out two of them.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \int \frac{\sec^2(2t)}{\tan^9(2t)}\sec^2(2t)\,dt$$

Step 2
Now we can use the trig identity $\tan^2\theta+1=\sec^2\theta$ to convert the remaining secants to tangents.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \int \frac{\tan^2(2t)+1}{\tan^9(2t)}\sec^2(2t)\,dt$$

Step 3
Now we can use the substitution $u=\tan(2t)$ to evaluate the integral.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \tfrac{1}{2}\int \frac{u^2+1}{u^9}\,du = \tfrac{1}{2}\int u^{-7}+u^{-9}\,du = \tfrac{1}{2}\left[-\tfrac{1}{6}u^{-6}-\tfrac{1}{8}u^{-8}\right]+c$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for *u*!

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \boxed{-\tfrac{1}{12}\tfrac{1}{\tan^6(2t)}-\tfrac{1}{16}\tfrac{1}{\tan^8(2t)}+c = -\tfrac{1}{12}\cot^6(2t)-\tfrac{1}{16}\cot^8(2t)+c}$$

---

13. Evaluate $\displaystyle\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz$.

Hint : How would you do this problem if it were a product?

Step 1

Because of the sum in the numerator it makes some sense (hopefully) to maybe split the integrand (and then the integral) up into two as follows.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int \frac{2}{\cos^2(z)}+\frac{7\sin^3(z)}{\cos^2(z)}\,dz = \int \frac{2}{\cos^2(z)}\,dz+\int \frac{7\sin^3(z)}{\cos^2(z)}\,dz$$

Step 2

Now, the first integral looks difficult at first glance, but we can easily rewrite this in terms of secants at which point it becomes a really easy integral.

For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a sine.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int 2\sec^2(z)\,dz+7\int \frac{\sin^2(z)}{\cos^2(z)}\sin(z)\,dz$$

Step 3

As noted above the first integral is now very easy (which we'll do in the next step) and for the second integral we can use the trig identity $\sin^2\theta+\cos^2\theta=1$ to convert the remaining sines in the second integral to cosines.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int 2\sec^2(z)\,dz + 7\int \frac{1-\cos^2(z)}{\cos^2(z)}\sin(z)\,dz$$

Step 4
Now we can use the substitution $u=\cos(z)$ to evaluate the second integral. The first integral doesn't need any extra work.

$$\begin{aligned}\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz &= 2\tan(z)-7\int \frac{1-u^2}{u^2}\,du\\ &= 2\tan(z)-7\int u^{-2}-1\,du = 2\tan(z)-7\left(-u^{-1}-u\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5
Don't forget to substitute back in for *u*!

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \boxed{2\tan(z)+7\tfrac{1}{\cos(z)}+7\cos(z)+c = 2\tan(z)+7\sec(z)+7\cos(z)+c}$$

---

14. Evaluate $\int\left[9\sin^5(3x)-2\cos^3(3x)\right]\csc^4(3x)\,dx$.

Hint : Since this has a mix of trig functions maybe the best option would be to first get it reduced down to just a couple that we know how to deal with.

Step 1
To get started on this problem we should first probably see if we can reduce the integrand down to just sines and cosines. This is easy enough to do simply by recalling the definition of cosecant in terms of sine.

$$\begin{aligned}\int\left[9\sin^5(3x)-2\cos^3(3x)\right]\csc^4(3x)\,dx &= \int\left[9\sin^5(3x)-2\cos^3(3x)\right]\frac{1}{\sin^4(3x)}\,dx\\ &= \int 9\sin(3x)-2\frac{\cos^3(3x)}{\sin^4(3x)}\,dx\end{aligned}$$

Step 2
The first integral is simple enough to do without any extra work.

For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a cosine.

$$\int \left[9\sin^5(3x) - 2\cos^3(3x) \right] \csc^4(3x)\,dx = \int 9\sin(3x) - 2\frac{\cos^2(3x)}{\sin^4(3x)}\cos(3x)\,dx$$

Step 3

For the second integral we can use the trig identity $\sin^2\theta + \cos^2\theta = 1$ to convert the remaining cosines to sines.

$$\int \left[9\sin^5(3x) - 2\cos^3(3x) \right] \csc^4(3x)\,dx = \int 9\sin(3x)\,dx - 2\int \frac{1-\sin^2(3x)}{\sin^4(3x)}\cos(3x)\,dx$$

Step 4

Now we can use the substitution $u = \sin(3x)$ to evaluate the second integral. The first integral doesn't need any extra work.

$$\begin{aligned}\int \left[9\sin^5(3x) - 2\cos^3(3x) \right] \csc^4(3x)\,dx &= \int 9\sin(3x)\,dx - \tfrac{2}{3}\int \frac{1-u^2}{u^4}\,du \\ &= \int 9\sin(3x)\,dx - \tfrac{2}{3}\int u^{-4} - u^{-2}\,du \\ &= -3\cos(3x) - \tfrac{2}{3}\left(-\tfrac{1}{3}u^{-3} + u^{-1} \right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5

Don't forget to substitute back in for *u*!

$$\begin{aligned}\int \left[9\sin^5(3x) - 2\cos^3(3x) \right] \csc^4(3x)\,dx &= -3\cos(3x) + \tfrac{2}{9}\tfrac{1}{\sin^3(3x)} - \tfrac{2}{3}\tfrac{1}{\sin(3x)} + c \\ &= \boxed{-3\cos(3x) + \tfrac{2}{9}\csc^3(3x) - \tfrac{2}{3}\csc(3x) + c}\end{aligned}$$

Section 1-3 : Trig Substitutions

1. Use a trig substitution to eliminate the root in $\sqrt{4-9z^2}$.

Hint : When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1

The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$1-\sin^2(\theta)=\cos^2(\theta)$$

So, it looks like sine is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint : In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2

To get the coefficient on the trig function notice that we need to turn the 9 into a 4 once we've substituted the trig function in for *z* and squared the substitution out. With that in mind it looks like the substitution should be,

$$z=\tfrac{2}{3}\sin(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\sqrt{4-9z^2}&=\sqrt{4-9\left(\tfrac{2}{3}\sin(\theta)\right)^2}=\sqrt{4-9\left(\tfrac{4}{9}\right)\sin^2(\theta)}\\&=\sqrt{4-4\sin^2(\theta)}=2\sqrt{1-\sin^2(\theta)}\\&=2\sqrt{\cos^2(\theta)}=\boxed{2\left|\cos(\theta)\right|}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

---

2. Use a trig substitution to eliminate the root in $\sqrt{13+25x^2}$.

Hint : When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1
The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$1+\tan^2\left(\theta\right)=\sec^2\left(\theta\right)$$

So, it looks like tangent is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint : In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2
To get the coefficient on the trig function notice that we need to turn the 25 into a 13 once we've substituted the trig function in for *x* and squared the substitution out. With that in mind it looks like the substitution should be,

$$x=\tfrac{\sqrt{13}}{5}\tan\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\sqrt{13+25x^2}&=\sqrt{13+25\left(\tfrac{\sqrt{13}}{5}\tan\left(\theta\right)\right)^2}=\sqrt{13+25\left(\tfrac{13}{25}\right)\tan^2\left(\theta\right)}\\&=\sqrt{13+13\tan^2\left(\theta\right)}=\sqrt{13}\sqrt{1+\tan^2\left(\theta\right)}\\&=\sqrt{13}\sqrt{\sec^2\left(\theta\right)}=\boxed{\sqrt{13}\left|\sec\left(\theta\right)\right|}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

---

3. Use a trig substitution to eliminate the root in $\left(7t^2-3\right)^{\frac{5}{2}}$.

Hint : When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1
First, notice that there really is a root here as the term can be written as,

$$\left(7t^2-3\right)^{\frac{5}{2}}=\left[\left(7t^2-3\right)^{\frac{1}{2}}\right]^5=\left[\sqrt{7t^2-3}\right]^5$$

Now, we need to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$\sec^2\left(\theta\right)-1=\tan^2\left(\theta\right)$$

So, it looks like secant is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint : In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2
To get the coefficient on the trig function notice that we need to turn the 7 into a 3 once we've substituted the trig function in for *t* and squared the substitution out. With that in mind it looks like the substitution should be,

$$t=\tfrac{\sqrt{3}}{\sqrt{7}}\sec\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\left(7t^2-3\right)^{\frac{5}{2}}&=\left[\sqrt{7t^2-3}\right]^5\\&=\left[\sqrt{7\left(\tfrac{\sqrt{3}}{\sqrt{7}}\sec\left(\theta\right)\right)^2-3}\right]^5=\left[\sqrt{7\left(\tfrac{3}{7}\right)\sec^2\left(\theta\right)-3}\right]^5\\&=\left[\sqrt{3\sec^2\left(\theta\right)-3}\right]^5=\left[\sqrt{3}\sqrt{\sec^2\left(\theta\right)-1}\right]^5\\&=\left[\sqrt{3}\sqrt{\tan^2\left(\theta\right)}\right]^5=\boxed{3^{\frac{5}{2}}\left|\tan\left(\theta\right)\right|^5}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

---

4. Use a trig substitution to eliminate the root in $\sqrt{\left(w+3\right)^2-100}$.

Hint : Just because this looks a little different from the first couple of problems in this section doesn't mean that it works any differently. The term under the root still looks vaguely like one of three trig identities we need to use to convert the quantity under the root into a single trig function.

Step 1
Okay, first off we need to acknowledge that this does look a little bit different from the first few problems in this section. However, it isn't really all that different. We still have a difference between a squared term with a variable in it and a number. This looks similar to the following trig identity (ignoring the coefficients as usual).

$$\sec^2(\theta)-1=\tan^2(\theta)$$

So, secant is the trig function we'll need to use for the substitution here and we now need to deal with the numbers on the terms and get the substitution set up.

Hint : Dealing with the numbers in this case is no different than the first few problems in this section.

Step 2
Before dealing with the coefficient on the trig function let's notice that we'll be substituting in for $w+3$ in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 100 once we've done the substitution. With that in mind it looks like the substitution should be,

$$w+3=10\sec(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\sqrt{(w+3)^2-100}&=\sqrt{(10\sec(\theta))^2-100}=\sqrt{100\sec^2(\theta)-100}=10\sqrt{\sec^2(\theta)-1}\\&=10\sqrt{\tan^2(\theta)}=\boxed{10\left|\tan(\theta)\right|}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

---

5. Use a trig substitution to eliminate the root in $\sqrt{4(9t-5)^2+1}$.

Hint : Just because this looks a little different from the first couple of problems in this section doesn't mean that it works any differently. The term under the root still looks vaguely like one of three trig identities we need to use to convert the quantity under the root into a single trig function.

Step 1
Okay, first off we need to acknowledge that this does look a little bit different from the first few problems in this section. However, it isn't really all that different. We still have a sum of a squared term with a variable in it and a number. This looks similar to the following trig identity (ignoring the coefficients as usual).

$$\tan^2(\theta)+1=\sec^2(\theta)$$

So, tangent is the trig function we'll need to use for the substitution here and we now need to deal with the numbers on the terms and get the substitution set up.

Hint : Dealing with the numbers in this case is no different than the first few problems in this section.

Step 2
Before dealing with the coefficient on the trig function let's notice that we'll be substituting in for $9t-5$ in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 4 (*i.e.* the coefficient of the squared term) into a 1 once we've done the substitution. With that in mind it looks like the substitution should be,

$$9t-5=\tfrac{1}{2}\tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\sqrt{4(9t-5)^2+1}&=\sqrt{4\left(\tfrac{1}{2}\tan(\theta)\right)^2+1}=\sqrt{4\left(\tfrac{1}{4}\right)\tan^2(\theta)+1}=\sqrt{\tan^2(\theta)+1}\\&=\sqrt{\sec^2(\theta)}=\boxed{\left|\sec(\theta)\right|}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

---

6. Use a trig substitution to eliminate the root in $\sqrt{1-4z-2z^2}$.

Hint : This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1
We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn't mean

that we can't do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.

$$\begin{aligned} 1-4z-2z^2 &= -2\left(z^2+2z-\tfrac{1}{2}\right) \\ &= -2\left(z^2+2z+1-1-\tfrac{1}{2}\right) \\ &= -2\left[\left(z+1\right)^2-\tfrac{3}{2}\right] \\ &= 3-2\left(z+1\right)^2 \end{aligned} \qquad \left[\tfrac{1}{2}\left(2\right)\right]^2 = \left[1\right]^2 = 1$$

So, after completing the square the term can be written as,

$$\sqrt{1-4z-2z^2} = \sqrt{3-2\left(z+1\right)^2}$$

Hint : At this point the problem works in the same manner as the previous problems in this section.

Step 2
So, in this case we see that we have a difference of a number and a squared term with a variable in it. This suggests that sine is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 2 (*i.e.* the coefficient of the squared term) into a 3 once we've done the substitution. With that in mind it looks like the substitution should be,

$$z+1 = \tfrac{\sqrt{3}}{\sqrt{2}}\sin\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned} \sqrt{1-4z-2z^2} &= \sqrt{3-2\left(z+1\right)^2} = \sqrt{3-2\left(\tfrac{\sqrt{3}}{\sqrt{2}}\sin\left(\theta\right)\right)^2} \\ &= \sqrt{3-3\sin^2\left(\theta\right)} = \sqrt{3}\sqrt{\cos^2\left(\theta\right)} = \boxed{\sqrt{3}\left|\cos\left(\theta\right)\right|} \end{aligned}$$

Note that because we don't know the values of θ we can't determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

7. Use a trig substitution to eliminate the root in $\left(x^2-8x+21\right)^{\frac{3}{2}}$.

Hint : This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1
We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn't mean that we can't do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.

$$\begin{aligned} x^2-8x+21 &= x^2-8x+16-16+21 \\ &= \left(x-4\right)^2+5 \end{aligned} \qquad \left[\tfrac{1}{2}\left(-8\right)\right]^2=\left[-4\right]^2=16$$

So, after completing the square the term can be written as,

$$\left(x^2-8x+21\right)^{\frac{3}{2}}=\left(\left(x-4\right)^2+5\right)^{\frac{3}{2}}=\left[\sqrt{\left(x-4\right)^2+5}\right]^3$$

Note that we also explicitly put the root into the problem as well.

Hint : At this point the problem works in the same manner as the previous problems in this section.

Step 2
So, in this case we see that we have a sum of a squared term with a variable in it and a number. This suggests that tangent is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 5 once we've done the substitution. With that in mind it looks like the substitution should be,

$$x-4=\sqrt{5}\tan\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\left(x^2-8x+21\right)^{\frac{3}{2}}=\left[\sqrt{\left(x-4\right)^2+5}\,\right]^3=\left[\sqrt{\left(\sqrt{5}\tan\left(\theta\right)\right)^2+5}\,\right]^3$$
$$=\left[\sqrt{5\tan^2\left(\theta\right)+5}\,\right]^3=\left[\sqrt{5}\sqrt{\tan^2\left(\theta\right)+1}\,\right]^3$$
$$=\left[\sqrt{5}\sqrt{\sec^2\left(\theta\right)}\,\right]^3=\boxed{5^{\frac{3}{2}}\left|\sec\left(\theta\right)\right|^3}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

---

8. Use a trig substitution to eliminate the root in $\sqrt{\mathbf{e}^{8x}-9}$.

Hint : This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1
We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. Even though this doesn't look anything like the "normal" trig substitution problems it is actually pretty close to one. To see this all we need to do is rewrite the term under the root as follows.

$$\sqrt{\mathbf{e}^{8x}-9}=\sqrt{\left(\mathbf{e}^{4x}\right)^2-9}$$

All we did here was take advantage of the basic exponent rules to make it clear that we really do have a difference here of a squared term containing a variable and a number.

Hint : At this point the problem works in the same manner as the previous problems in this section.

Step 2
The form of the quantity under the root suggests that secant is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 9 once we've done the substitution. With that in mind it looks like the substitution should be,

$$\mathbf{e}^{4x}=3\sec\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{\mathbf{e}^{8x}-9} = \sqrt{\left(3\sec\left(\theta\right)\right)^2-9} = \sqrt{9\sec^2\left(\theta\right)-9}$$
$$= 3\sqrt{\sec^2\left(\theta\right)-1} = 3\sqrt{\tan^2\left(\theta\right)} = \boxed{3\left|\tan\left(\theta\right)\right|}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

---

9. Use a trig substitution to evaluate $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$.

Step 1
In this case it looks like we'll need the following trig substitution.

$$x = 4\tan\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let's first use the substitution to eliminate the root.

$$\sqrt{x^2+16} = \sqrt{16\tan^2\left(\theta\right)+16} = 4\sqrt{\sec^2\left(\theta\right)} = 4\left|\sec\left(\theta\right)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{x^2+16} = 4\sec\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dx = 4\sec^2\left(\theta\right)d\theta$$

Step 3
Now let's do the actual substitution.

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \int \frac{4\sec\left(\theta\right)}{\left(4\tan\left(\theta\right)\right)^4}4\sec^2\left(\theta\right)d\theta = \int \frac{\sec^3\left(\theta\right)}{16\tan^4\left(\theta\right)}\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. In this case the integral looks to be a little difficult to do in terms of secants and tangents so let's convert the integrand to sines and cosines and see what we get. Doing this gives,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \frac{1}{16}\int \frac{\cos\left(\theta\right)}{\sin^4\left(\theta\right)}\,d\theta$$

This is a simple integral to evaluate so here is the integral evaluation.

$$\begin{aligned}\int \frac{\sqrt{x^2+16}}{x^4}\,dx &= \frac{1}{16}\int \frac{\cos\left(\theta\right)}{\sin^4\left(\theta\right)}\,d\theta \qquad\qquad u=\sin\left(\theta\right)\\ &= \tfrac{1}{16}\int u^{-4}\,du \\ &= -\tfrac{1}{48}u^{-3}+c = -\tfrac{1}{48}\left[\sin\left(\theta\right)\right]^{-3}+c = -\tfrac{1}{48}\csc^3\left(\theta\right)+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

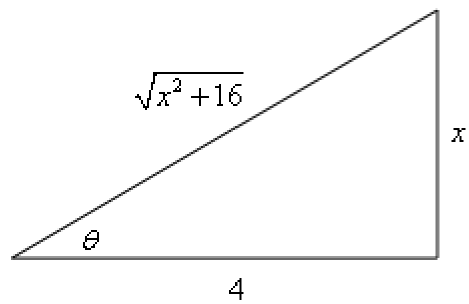
Step 5
As the final step we just need to go back to *x*'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan\left(\theta\right)=\frac{x}{4} \qquad \left(=\frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\csc\left(\theta\right)=\frac{\sqrt{x^2+16}}{x}$$



The integral is then,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{1}{48}\left[\frac{\sqrt{x^2+16}}{x}\right]^3 + c = \boxed{-\frac{\left(x^2+16\right)^{\frac{3}{2}}}{48x^3}+c}$$

---

10. Use a trig substitution to evaluate $\int \sqrt{1-7w^2}\,dw$.

Step 1

In this case it looks like we'll need the following trig substitution.

$$w = \tfrac{1}{\sqrt{7}}\sin\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{1-7w^2} = \sqrt{1-\sin^2\left(\theta\right)} = \sqrt{\cos^2\left(\theta\right)} = \left|\cos\left(\theta\right)\right|$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\sqrt{1-7w^2} = \cos\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dw = \tfrac{1}{\sqrt{7}}\cos\left(\theta\right)d\theta$$

Step 3

Now let's do the actual substitution.

$$\int \sqrt{1-7w^2}\,dw = \int \cos\left(\theta\right)\left(\tfrac{1}{\sqrt{7}}\cos\left(\theta\right)\right)d\theta = \tfrac{1}{\sqrt{7}}\int \cos^2\left(\theta\right)d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

$$\int \sqrt{1-7w^2}\,dw = \tfrac{1}{\sqrt{7}}\int \tfrac{1}{2}\left[1+\cos\left(2\theta\right)\right]d\theta = \tfrac{1}{2\sqrt{7}}\left[\theta + \tfrac{1}{2}\sin\left(2\theta\right)\right]+c$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5
As the final step we just need to go back to *w*'s.

To eliminate the the first term (*i.e.* the θ) we can use any of the inverse trig functions. The easiest is to probably just use the original substitution and get a formula involving inverse sine but any of the six trig functions could be used if we wanted to. Using the substitution gives us,

$$\sin\left(\theta\right) = \sqrt{7}\,w \qquad\qquad \Rightarrow \qquad\qquad \theta = \sin^{-1}\left(\sqrt{7}\,w\right)$$

Eliminating the $\sin\left(2\theta\right)$ requires a little more work. We can't just use a right triangle as we normally would because that would only give trig functions with an argument of θ and we have an argument of 2θ. However, we could use the double angle formula for sine to reduce this to trig functions with arguments of θ. Doing this gives,

$$\int \sqrt{1-7w^2}\,dw = \tfrac{1}{2\sqrt{7}}\left[\theta + \sin\left(\theta\right)\cos\left(\theta\right)\right]+c$$

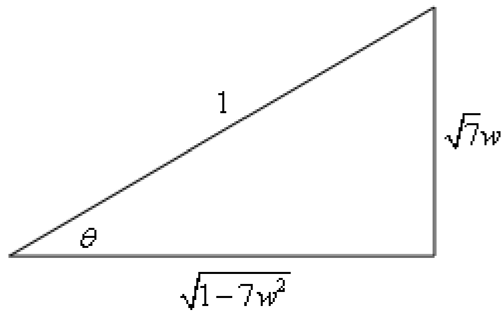
We can now do the right triangle work.

From the substitution we have,

$$\sin\left(\theta\right) = \frac{\sqrt{7}w}{1} \qquad \left(=\frac{\text{opp}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos(\theta) = \sqrt{1-7w^2}$$



The integral is then,

$$\int \sqrt{1-7w^2}\,dw = \tfrac{1}{2\sqrt{7}}\left[\sin^{-1}\left(\sqrt{7}w\right) + \sqrt{7}\,w\sqrt{1-7w^2}\right] + c$$

---

11. Use a trig substitution to evaluate $\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt$.

Step 1

First, do not get excited about the exponent in the integrand. These types of problems work exactly the same as those with just a root (as opposed to this case in which we have a root to a power – you do agree that is what we have right?). So, in this case it looks like we'll need the following trig substitution.

$$t = \tfrac{2}{\sqrt{3}}\sec(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\left(3t^2-4\right)^{\frac{5}{2}} = \left[\sqrt{3t^2-4}\right]^5 = \left[\sqrt{4\sec^2(\theta)-4}\right]^5 = \left[2\sqrt{\tan^2(\theta)}\right]^5 = 32\left|\tan(\theta)\right|^5$$

Next, because we are doing an indefinite integral we will assume that the tangent is positive and so we can drop the absolute value bars to get,

$$\left(3t^2-4\right)^{\frac{5}{2}} = 32\tan^5(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dt = \tfrac{2}{\sqrt{3}}\sec\left(\theta\right)\tan\left(\theta\right)d\theta$$

Step 3

Now let's do the actual substitution.

$$\begin{aligned}\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt &= \int\left(\tfrac{2}{\sqrt{3}}\right)^3\sec^3\left(\theta\right)\left(32\tan^5\left(\theta\right)\right)\left(\tfrac{2}{\sqrt{3}}\sec\left(\theta\right)\tan\left(\theta\right)\right)d\theta\\ &= \tfrac{512}{9}\int\sec^4\left(\theta\right)\tan^6\left(\theta\right)d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt &= \tfrac{512}{9}\int\left(\tan^2\left(\theta\right)+1\right)\tan^6\left(\theta\right)\,\sec^2\left(\theta\right)d\theta \qquad\qquad u=\tan\left(\theta\right)\\ &= \tfrac{512}{9}\int\left(u^2+1\right)u^6\,du = \tfrac{512}{9}\int u^8+u^6\,du\\ &= \tfrac{512}{9}\left[\tfrac{1}{9}\tan^9\left(\theta\right)+\tfrac{1}{7}\tan^7\left(\theta\right)\right]+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5

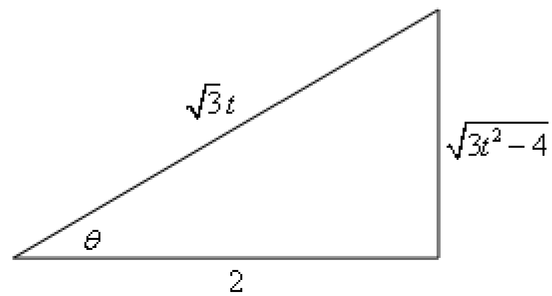
As the final step we just need to go back to *t*'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\sec\left(\theta\right)=\frac{\sqrt{3}\,t}{2}\qquad\left(=\frac{\text{hyp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\tan\left(\theta\right)=\frac{\sqrt{3t^2-4}}{2}$$



The integral is then,

$$\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt = \frac{512}{9}\left[\frac{1}{9}\left(\frac{\sqrt{3t^2-4}}{2}\right)^9+\frac{1}{7}\left(\frac{\sqrt{3t^2-4}}{2}\right)^7\right]+c$$

$$=\boxed{\frac{\left(3t^2-4\right)^{\frac{9}{2}}}{81}+\frac{4\left(3t^2-4\right)^{\frac{7}{2}}}{63}+c}$$

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12. Use a trig substitution to evaluate $\displaystyle\int_{-7}^{-5}\frac{2}{y^4\sqrt{y^2-25}}\,dy$.

Step 1
In this case it looks like we'll need the following trig substitution.

$$y=5\sec\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let's first use the substitution to eliminate the root.

$$\sqrt{y^2-25}=\sqrt{25\sec^2\left(\theta\right)-25}=5\sqrt{\tan^2\left(\theta\right)}=5\left|\tan\left(\theta\right)\right|$$

Step 3
Okay, in this case we have limits on *y* and so we can get limits on θ that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let's get some limits on θ.

$$y = -7:\ -7 = 5\sec\left(\theta\right) \qquad \to \qquad \sec\left(\theta\right) = -\tfrac{7}{5} \qquad \to \qquad \theta = \sec^{-1}\left(-\tfrac{7}{5}\right) = 2.3664$$
$$y = -5:\ -5 = 5\sec\left(\theta\right) \qquad \to \qquad \sec\left(\theta\right) = -1 \qquad \to \qquad \theta = \pi$$

So, θ's for this problem are in the range $2.3664 \le \theta \le \pi$ and these are in the second quadrant. In the second quadrant we know that tangent is negative and so we can drop the absolute value bars provided we add in a minus sign. This gives,

$$\sqrt{y^2 - 25} = -5\tan\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dy = 5\sec\left(\theta\right)\tan\left(\theta\right)d\theta$$

Step 4
Now let's do the actual substitution.

$$\begin{aligned}\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy &= \int_{2.3664}^{\pi} \frac{2}{5^4\sec^4\left(\theta\right)\left(-5\tan\left(\theta\right)\right)}\left(5\sec\left(\theta\right)\tan\left(\theta\right)\right)d\theta \\ &= -\frac{2}{625}\int_{2.3664}^{\pi} \frac{1}{\sec^3\left(\theta\right)}\,d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substation we replaced the *y* limits with the θ limits. This will help with a later step.

Step 5
We now need to evaluate the integral. In terms of secants this integral would be pretty difficult, however we a quick change to cosines we get the following integral.

$$\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy = -\frac{2}{625}\int_{2.3664}^{\pi} \cos^3\left(\theta\right)d\theta$$

This should be relatively simple to do so here is the integration work.

$$\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy = -\frac{2}{625}\int_{2.3664}^{\pi} \left(1-\sin^2\left(\theta\right)\right)\cos\left(\theta\right)d\theta \qquad u=\sin\left(\theta\right)$$

$$= -\tfrac{2}{625}\int_{\sin(2.3664)}^{\sin(\pi)} 1-u^2\,du$$

$$= -\tfrac{2}{625}\left[u-\tfrac{1}{3}u^3\right]\Big|_{0.69986}^{0} = \boxed{0.001874}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the "new" variable we did not need to do any back substitution work on our answer!

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13. Use a trig substitution to evaluate $\int_1^4 2z^5\sqrt{2+9z^2}\,dz$.

Step 1
In this case it looks like we'll need the following trig substitution.

$$z = \tfrac{\sqrt{2}}{3}\tan\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let's first use the substitution to eliminate the root.

$$\sqrt{2+9z^2} = \sqrt{2+2\tan^2\left(\theta\right)} = \sqrt{2}\sqrt{\sec^2\left(\theta\right)} = \sqrt{2}\left|\sec\left(\theta\right)\right|$$

Step 3
Okay, in this case we have limits on *z* and so we can get limits on θ that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let's get some limits on θ.

$$z=1: \quad 1=\tfrac{\sqrt{2}}{3}\tan\left(\theta\right) \quad \to \quad \tan\left(\theta\right)=\tfrac{3}{\sqrt{2}} \quad \to \quad \theta=\tan^{-1}\left(\tfrac{3}{\sqrt{2}}\right)=1.1303$$

$$z=4: \quad 4=\tfrac{\sqrt{2}}{3}\tan\left(\theta\right) \quad \to \quad \tan\left(\theta\right)=\tfrac{12}{\sqrt{2}} \quad \to \quad \theta=\tan^{-1}\left(\tfrac{12}{\sqrt{2}}\right)=1.4535$$

So, θ's for this problem are in the range $1.1303 \le \theta \le 1.4535$ and these are in the first quadrant. In the first quadrant we know that cosine, and hence secant, is positive and so we can just drop the absolute value bars. This gives,

$$\sqrt{2+9z^2} = \sqrt{2}\sec\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dz = \tfrac{\sqrt{2}}{3}\sec^2\left(\theta\right)d\theta$$

Step 4
Now let's do the actual substitution.

$$\begin{aligned}\int_1^4 2z^5\sqrt{2+9z^2}\,dz &= \int_{1.1303}^{1.4535} 2\left(\tfrac{\sqrt{2}}{3}\right)^5 \tan^5\left(\theta\right)\left(\sqrt{2}\sec\left(\theta\right)\right)\left(\tfrac{\sqrt{2}}{3}\sec^2\left(\theta\right)\right)d\theta \\ &= \tfrac{16\sqrt{2}}{729}\int_{1.1303}^{1.4535} \tan^5\left(\theta\right)\sec^3\left(\theta\right)d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substation we replaced the *y* limits with the θ limits. This will help with a later step.

Step 5
We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int_1^4 2z^5\sqrt{2+9z^2}\,dz &= \tfrac{16\sqrt{2}}{729}\int_{1.1303}^{1.4535}\left[\sec^2\left(\theta\right)-1\right]^2\sec^2\left(\theta\right)\,\tan\left(\theta\right)\sec\left(\theta\right)d\theta \\ &= \tfrac{16\sqrt{2}}{729}\int_{\sec(1.1303)}^{\sec(1.4535)}\left[u^2-1\right]^2u^2du \qquad\qquad u=\sec\left(\theta\right) \\ &= \tfrac{16\sqrt{2}}{729}\int_{2.3452}^{8.5444}u^6-2u^4+u^2du \\ &= \tfrac{16\sqrt{2}}{729}\left[\tfrac{1}{7}u^7-\tfrac{2}{5}u^5+\tfrac{1}{3}u^3\right]\Big|_{2.3452}^{8.5444} = \boxed{14182.86074}\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the “new” variable we did not need to do any back substitution work on our answer!

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14. Use a trig substitution to evaluate $\displaystyle\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx$.

Step 1
The first thing we’ll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

$$\begin{aligned} 9x^2 - 36x + 37 &= 9\left(x^2 - 4x + \tfrac{37}{9}\right) = 9\left(x^2 - 4x + 4 - 4 + \tfrac{37}{9}\right) = 9\left[\left(x-2\right)^2 + \tfrac{1}{9}\right] \\ &= 9\left(x-2\right)^2 + 1 \end{aligned}$$

The integral is now,

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \int \frac{1}{\sqrt{9\left(x-2\right)^2 + 1}}\,dx$$

Now we can proceed with the trig substitution.

Step 2
It looks like we’ll need to the following trig substitution.

$$x - 2 = \tfrac{1}{3}\tan\left(\theta\right)$$

Next let’s eliminate the root.

$$\sqrt{9\left(x-2\right)^2 + 1} = \sqrt{\tan\left(\theta\right)^2 + 1} = \sqrt{\sec^2\left(\theta\right)} = \left|\sec\left(\theta\right)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9\left(x-2\right)^2 + 1} = \sec\left(\theta\right)$$

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

$$\left(1\right)dx = \tfrac{1}{3}\sec^2\left(\theta\right)d\theta \qquad \Rightarrow \qquad dx = \tfrac{1}{3}\sec^2\left(\theta\right)d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3
Now let's do the actual substitution.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \int \frac{1}{\sec(\theta)}\left(\frac{1}{3}\sec^2(\theta)\right)d\theta = \frac{1}{3}\int \sec(\theta)\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \frac{1}{3}\ln\left|\sec(\theta) + \tan(\theta)\right| + c$$

Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec(\theta)$ in the previous section and reuse that result here.

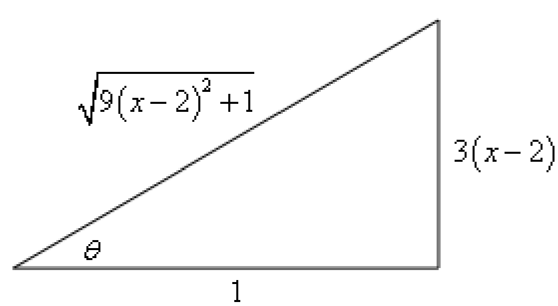
Step 5
As the final step we just need to go back to *x*'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{3(x-2)}{1} \qquad \left(=\frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sec(\theta) = \sqrt{9(x-2)^2 + 1}$$



The integral is then,

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx = \boxed{\frac{1}{3}\ln\left|\sqrt{9(x-2)^2+1}+3(x-2)\right|+c}$$

---

15. Use a trig substitution to evaluate $\displaystyle\int \frac{(z+3)^5}{\left(40-6z-z^2\right)^{\frac{3}{2}}}\,dz$.

Step 1

The first thing we'll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

$$\begin{aligned}40-6z-z^2 &= -\left(z^2+6z-40\right) = -\left(z^2+6z+9-9-40\right) = -\left[(z+3)^2-49\right]\\ &= 49-(z+3)^2\end{aligned}$$

The integral is now,

$$\int \frac{(z+3)^5}{\left(40-6z-z^2\right)^{\frac{3}{2}}}\,dz = \int \frac{(z+3)^5}{\left(49-(z+3)^2\right)^{\frac{3}{2}}}\,dz$$

Now we can proceed with the trig substitution.

Step 2

It looks like we'll need to the following trig substitution.

$$z+3 = 7\sin(\theta)$$

Next let's eliminate the root.

$$\left(49-(z+3)^2\right)^{\frac{3}{2}} = \left[\sqrt{49-(z+3)}\right]^3 = \left[\sqrt{49-49\sin^2(\theta)}\right]^3 = \left[7\sqrt{\cos^2(\theta)}\right]^3 = 343\left|\cos(\theta)\right|^3$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\left(49-(z+3)^2\right)^{\frac{3}{2}} = 343\cos^3(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$\left(1\right)dz = 7\cos\left(\theta\right)d\theta \qquad\qquad \Rightarrow \qquad\qquad dz = 7\cos\left(\theta\right)d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3
Now let's do the actual substitution.

$$\int \frac{\left(z+3\right)^5}{\left(40-6z-z^2\right)^{\frac{3}{2}}}dz = \int \frac{16807\sin^5\left(\theta\right)}{343\cos^3\left(\theta\right)}\left(7\cos\left(\theta\right)\right)d\theta = 343\int \frac{\sin^5\left(\theta\right)}{\cos^2\left(\theta\right)}d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

$$\int \frac{\left(z+3\right)^5}{\left(40-6z-z^2\right)^{\frac{3}{2}}}dz = 343\int \frac{\left[1-\cos^2\left(\theta\right)\right]^2}{\cos^2\left(\theta\right)}\sin\left(\theta\right)d\theta \qquad\qquad u = \cos\left(\theta\right)$$

$$= -343\int \frac{\left[1-u^2\right]^2}{u^2}du = -343\int u^{-2} - 2 + u^2\,du$$

$$= -343\left(-u^{-1} - 2u + \tfrac{1}{3}u^3\right) + c$$

$$= -343\left(-\frac{1}{\cos\left(\theta\right)} - 2\cos\left(\theta\right) + \frac{1}{3}\cos^3\left(\theta\right)\right) + c$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

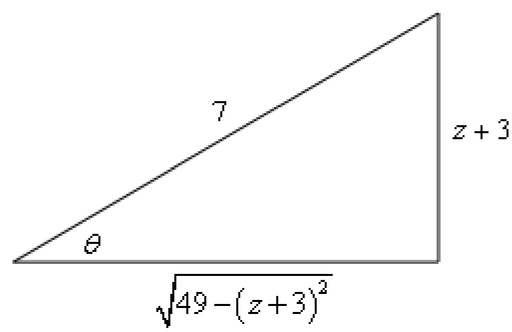
Step 5
As the final step we just need to go back to *z*'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\sin\left(\theta\right)=\frac{z+3}{7}\qquad\left(=\frac{\text{adj}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos\left(\theta\right)=\frac{\sqrt{49-\left(z+3\right)^2}}{7}$$



The integral is then,

$$\int\frac{\left(z+3\right)^5}{\left(40-6z-z^2\right)^{\frac{3}{2}}}\,dz=\boxed{\frac{2401}{\sqrt{49-\left(z+3\right)^2}}+98\sqrt{49-\left(z+3\right)^2}-\frac{\left(49-\left(z+3\right)^2\right)^{\frac{3}{2}}}{3}+c}$$

---

16. Use a trig substitution to evaluate $\int\cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx$.

Step 1

Let's first rewrite the integral a little bit.

$$\int\cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx=\int\cos\left(x\right)\sqrt{9+25\left[\sin\left(x\right)\right]^2}\,dx$$

Step 2

With the integral written as it is in the first step we can now see that we do have a sum of a number and something squared under the root. We know from the problems done previously in this section that looks like a tangent substitution. So, let's use the following substitution.

$$\sin\left(x\right)=\tfrac{3}{5}\tan\left(\theta\right)$$

Do not get excited about the fact that we are substituting one trig function for another. That will happen on occasion with these kinds of problems. Note however, that we need to be careful and make sure that we also change the variable from *x* (*i.e.* the variable in the original trig function) into θ (*i.e.* the variable in the new trig function).

Next let's eliminate the root.

$$\sqrt{9+25\left[\sin\left(x\right)\right]^2} = \sqrt{9+25\left[\tfrac{3}{5}\tan\left(\theta\right)\right]^2} = \sqrt{9+9\tan^2\left(\theta\right)} = 3\sqrt{\sec^2\left(\theta\right)} = 3\left|\sec\left(\theta\right)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9+25\left[\sin\left(x\right)\right]^2} = 3\sec\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$\cos\left(x\right)dx = \tfrac{3}{5}\sec^2\left(\theta\right)d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3
Now let's do the actual substitution.

$$\begin{aligned}\int\cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx &= \int\sqrt{9+25\left[\sin\left(x\right)\right]^2}\;\;\cos\left(x\right)dx \\ &= \int\left(3\sec\left(\theta\right)\right)\left(\tfrac{3}{5}\sec^2\left(\theta\right)\right)d\theta = \tfrac{9}{5}\int\sec^3\left(\theta\right)d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

$$\int\cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx = \tfrac{9}{10}\left[\sec\left(\theta\right)\tan\left(\theta\right)+\ln\left|\sec\left(\theta\right)+\tan\left(\theta\right)\right|\right]+c$$

Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec^3\left(\theta\right)$ in the previous section and reuse that result here.

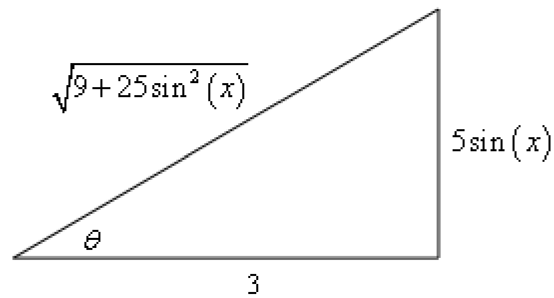
Step 5
As the final step we just need to go back to *x*'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan\left(\theta\right)=\frac{5\sin\left(x\right)}{3}\qquad\left(=\frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sec\left(\theta\right)=\frac{\sqrt{9+25\sin^2\left(x\right)}}{3}$$



The integral is then,

$$\int\cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx=\boxed{\frac{\sin\left(x\right)\sqrt{9+25\sin^2\left(x\right)}}{2}+\frac{9}{10}\ln\left|\frac{5\sin\left(x\right)+\sqrt{9+25\sin^2\left(x\right)}}{3}\right|+c}$$

Section 1-4 : Partial Fractions

1. Evaluate the integral $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$.

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{4}{x^2+5x-14}\,dx = \int \frac{4}{(x+7)(x-2)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{4}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}$$

Step 2
Setting the numerators equal gives,

$$4 = A(x-2) + B(x+7)$$

Step 3
We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{aligned} x=2: &\quad 4=9B \\ x=-7: &\quad 4=-9A \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\tfrac{4}{9} \\ B &= \tfrac{4}{9} \end{aligned}$$

The partial fraction form of the integrand is then,

$$\frac{4}{(x+7)(x-2)} = \frac{-\frac{4}{9}}{x+7} + \frac{\frac{4}{9}}{x-2}$$

Step 4
We can now do the integral.

$$\int \frac{4}{(x+7)(x-2)}\,dx = \int \frac{-\frac{4}{9}}{x+7} + \frac{\frac{4}{9}}{x-2}\,dx = \boxed{\frac{4}{9}\ln\left|x-2\right| - \frac{4}{9}\ln\left|x+7\right| + c}$$

2. Evaluate the integral $\displaystyle\int \frac{8-3t}{10t^2+13t-3}\,dt$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{8-3t}{10t^2+13t-3}\,dt = \int \frac{8-3t}{\left(2t+3\right)\left(5t-1\right)}\,dt$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{8-3t}{10t^2+13t-3} = \frac{A}{2t+3} + \frac{B}{5t-1}$$

Step 2

Setting the numerators equal gives,

$$8-3t = A\left(5t-1\right) + B\left(2t+3\right)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{ll} t=\frac{1}{5}: & \frac{37}{5}=\frac{17}{5}B \\ t=-\frac{3}{2}: & \frac{25}{2}=-\frac{17}{2}A \end{array} \qquad \Rightarrow \qquad \begin{array}{l} A=-\frac{25}{17} \\ B=\frac{37}{17} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8-3t}{10t^2+13t-3} = \frac{-\frac{25}{17}}{2t+3} + \frac{\frac{37}{17}}{5t-1}$$

Step 4

We can now do the integral.

$$\int \frac{8-3t}{10t^2+13t-3}\,dt = \int \frac{-\frac{25}{17}}{2t+3} + \frac{\frac{37}{17}}{5t-1}\,dt = \boxed{\frac{37}{85}\ln\left|5t-1\right| - \frac{25}{34}\ln\left|2t+3\right| + c}$$

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with both of these integrals. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

3. Evaluate the integral $\displaystyle\int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$.

Step 1

In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{w^2+7w}{(w+2)(w-1)(w-4)} = \frac{A}{w+2} + \frac{B}{w-1} + \frac{C}{w-4}$$

Step 2

Setting the numerators equal gives,

$$w^2+7w = A(w-1)(w-4) + B(w+2)(w-4) + C(w+2)(w-1)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{lrcl} w=1: & 8=-9B & & A=-\frac{5}{9} \\ w=4: & 44=18C & \Rightarrow & B=-\frac{8}{9} \\ w=-2: & -10=18A & & C=\frac{22}{9} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{w^2+7w}{(w+2)(w-1)(w-4)} = \frac{-\frac{5}{9}}{w+2} - \frac{\frac{8}{9}}{w-1} + \frac{\frac{22}{9}}{w-4}$$

Step 4

We can now do the integral.

$$\begin{aligned} \int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw &= \int_{-1}^{0} \frac{-\frac{5}{9}}{w+2} - \frac{\frac{8}{9}}{w-1} + \frac{\frac{22}{9}}{w-4}\,dw \\ &= \left(-\tfrac{5}{9}\ln|w+2| - \tfrac{8}{9}\ln|w-1| + \tfrac{22}{9}\ln|w-4|\right)\Big|_{-1}^{0} \\ &= \boxed{\tfrac{22}{9}\ln(4) + \tfrac{3}{9}\ln(2) - \tfrac{22}{9}\ln(5) = \tfrac{47}{9}\ln(2) - \tfrac{22}{9}\ln(5)} \end{aligned}$$

Note that we used a quick logarithm property to combine the first two logarithms into a single logarithm. You should probably review your logarithm properties if you don't recognize the one that we used. These kinds of property applications can really simplify your work on occasion if you know them!

4. Evaluate the integral $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{8}{3x^3+7x^2+4x}\,dx = \int \frac{8}{x\left(3x+4\right)\left(x+1\right)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{8}{x\left(3x+4\right)\left(x+1\right)} = \frac{A}{x} + \frac{B}{3x+4} + \frac{C}{x+1}$$

Step 2

Setting the numerators equal gives,

$$8 = A\left(3x+4\right)\left(x+1\right) + Bx\left(x+1\right) + C\,x\left(3x+4\right)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{lll} x=-\tfrac{4}{3}: & 8=\tfrac{4}{9}B & \\ x=-1: & 8=-C & \Rightarrow \\ x=0: & 8=4A & \end{array} \qquad \begin{array}{l} A=2 \\ B=18 \\ C=-8 \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8}{x\left(3x+4\right)\left(x+1\right)} = \frac{2}{x} + \frac{18}{3x+4} - \frac{8}{x+1}$$

Step 4

We can now do the integral.

$$\int \frac{8}{x\left(3x+4\right)\left(x+1\right)}\,dx = \int \frac{2}{x} + \frac{18}{3x+4} - \frac{8}{x+1}\,dx = \boxed{2\ln\left|x\right| + 6\ln\left|3x+4\right| - 8\ln\left|x+1\right| + c}$$

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with the second integral. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

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5. Evaluate the integral $\displaystyle\int_2^4 \frac{3z^2+1}{(z+1)(z-5)^2}\,dz$.

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{3z^2+1}{(z+1)(z-5)^2} = \frac{A}{z+1} + \frac{B}{z-5} + \frac{C}{(z-5)^2}$$

Step 2
Setting the numerators equal gives,

$$3z^2+1 = A(z-5)^2 + B(z+1)(z-5) + C(z+1)$$

Step 3
We can use the “trick” discussed in the notes to easily get two of the coefficients and then we can just pick another value of *z* to get the third so let’s do that. Here is that work.

$$\begin{array}{lll} z=-1: & 4=36A & \\ z=5: & 76=6C & \Rightarrow \\ z=0: & 1=25A-5B+C=\tfrac{139}{9}-5B & \end{array} \qquad \begin{array}{l} A=\tfrac{1}{9} \\ B=\tfrac{26}{9} \\ C=\tfrac{38}{3} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{3z^2+1}{(z+1)(z-5)^2} = \frac{\frac{1}{9}}{z+1} + \frac{\frac{26}{9}}{z-5} + \frac{\frac{38}{3}}{(z-5)^2}$$

Step 4
We can now do the integral.

$$\begin{aligned}\int_2^4 \frac{3z^2+1}{(z+1)(z-5)^2}\,dz &= \int_2^4 \frac{\frac{1}{9}}{z+1} + \frac{\frac{26}{9}}{z-5} + \frac{\frac{38}{3}}{(z-5)^2}\,dz \\ &= \left.\left(\tfrac{1}{9}\ln|z+1| + \tfrac{26}{9}\ln|z-5| - \frac{\frac{38}{3}}{z-5}\right)\right|_2^4 = \boxed{\tfrac{1}{9}\ln(5) - \tfrac{27}{9}\ln(3) + \tfrac{76}{9}}\end{aligned}$$

6. Evaluate the integral $\displaystyle\int \frac{4x-11}{x^3-9x^2}\,dx$.

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{4x-11}{x^3-9x^2}\,dx = \int \frac{4x-11}{x^2\left(x-9\right)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{4x-11}{x^2\left(x-9\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}$$

Step 2
Setting the numerators equal gives,

$$4x-11 = Ax\left(x-9\right) + B\left(x-9\right) + C\,x^2$$

Step 3
We can use the "trick" discussed in the notes to easily get two of the coefficients and then we can just pick another value of *x* to get the third so let's do that. Here is that work.

$$\begin{aligned} x=0: &\quad -11 = -9B \\ x=9: &\quad 25 = 81C \\ x=1: &\quad -7 = -8A - 8B + C = -8A - \tfrac{767}{81} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\tfrac{25}{81} \\ B &= \tfrac{11}{9} \\ C &= \tfrac{25}{81} \end{aligned}$$

The partial fraction form of the integrand is then,

$$\frac{4x-11}{x^2\left(x-9\right)} = \frac{-\frac{25}{81}}{x} + \frac{\frac{11}{9}}{x^2} + \frac{\frac{25}{81}}{x-9}$$

Step 4
We can now do the integral.

$$\int \frac{4x-11}{x^2\left(x-9\right)}\,dx = \int \frac{-\frac{25}{81}}{x} + \frac{\frac{11}{9}}{x^2} + \frac{\frac{25}{81}}{x-9}\,dx = \boxed{-\tfrac{25}{81}\ln\left|x\right| - \frac{\frac{11}{9}}{x} + \tfrac{25}{81}\ln\left|x-9\right| + c}$$

7. Evaluate the integral $\displaystyle\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz$.

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{z^2+2z+3}{(z-6)(z^2+4)} = \frac{A}{z-6} + \frac{Bz+C}{z^2+4}$$

Step 2
Setting the numerators equal gives,

$$z^2+2z+3 = A(z^2+4) + (Bz+C)(z-6) = (A+B)z^2 + (-6B+C)z + 4A - 6C$$

In this case the "trick" discussed in the notes won't work all that well for us and so we'll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

$$\begin{array}{rrcl} z^2: & A+B=1 & & A=\tfrac{51}{40} \\ z^1: & -6B+C=2 & \Rightarrow & B=-\tfrac{11}{40} \\ z^0: & 4A-6C=3 & & C=\tfrac{7}{20} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{z^2+2z+3}{(z-6)(z^2+4)} = \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z+\frac{7}{20}}{z^2+4}$$

Step 4
We can now do the integral.

$$\begin{aligned} \int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz &= \int \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z+\frac{7}{20}}{z^2+4}\,dz \\ &= \int \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z}{z^2+4} + \frac{\frac{7}{20}}{z^2+4}\,dz \\ &= \boxed{\tfrac{51}{40}\ln\left|z-6\right| - \tfrac{11}{80}\ln\left|z^2+4\right| + \tfrac{7}{40}\tan^{-1}\left(\tfrac{z}{2}\right) + c} \end{aligned}$$

Note that the second integration needed the substitution $u = z^2 + 4$ while the third needed the formula provided in the notes.

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8. Evaluate the integral $\displaystyle\int \frac{8+t+6t^2-12t^3}{\left(3t^2+4\right)\left(t^2+7\right)}\,dt$.

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{8+t+6t^2-12t^3}{\left(3t^2+4\right)\left(t^2+7\right)} = \frac{At+B}{3t^2+4} + \frac{Ct+D}{t^2+7}$$

Step 2
Setting the numerators equal gives,

$$\begin{aligned} 8+t+6t^2-12t^3 &= \left(At+B\right)\left(t^2+7\right) + \left(Ct+D\right)\left(3t^2+4\right) \\ &= \left(A+3C\right)t^3 + \left(B+3D\right)t^2 + \left(7A+4C\right)t + 7B + 4D \end{aligned}$$

In this case the "trick" discussed in the notes won't work all that well for us and so we'll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

$$\begin{array}{ll} t^3: & A+3C=-12 \\ t^2: & B+3D=6 \\ t^1: & 7A+4C=1 \\ t^0: & 7B+4D=8 \end{array} \quad \Rightarrow \quad \begin{array}{l} A=3 \\ B=0 \\ C=-5 \\ D=2 \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8+t+6t^2-12t^3}{\left(3t^2+4\right)\left(t^2+7\right)} = \frac{3t}{3t^2+4} + \frac{-5t+2}{t^2+7}$$

Step 4
We can now do the integral.

$$\begin{aligned}\int \frac{8+t+6t^2-12t^3}{\left(3t^2+4\right)\left(t^2+7\right)}dt &= \int \frac{3t}{3t^2+4}+\frac{-5t+2}{t^2+7}dt \\ &= \int \frac{3t}{3t^2+4}-\frac{5t}{t^2+7}+\frac{2}{t^2+7}dt \\ &= \boxed{\tfrac{1}{2}\ln\left|3t^2+4\right|-\tfrac{5}{2}\ln\left|t^2+7\right|+\tfrac{2}{\sqrt{7}}\tan^{-1}\left(\tfrac{t}{\sqrt{7}}\right)+c}\end{aligned}$$

Note that the first and second integrations needed the substitutions $u=3t^2+4$ and $u=t^2+7$ respectively while the third needed the formula provided in the notes.

---

9. Evaluate the integral $\displaystyle\int \frac{6x^2-3x}{(x-2)(x+4)}dx$.

Hint : Pay attention to the degree of the numerator and denominator!

Step 1

Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case both the numerator and denominator are both degree 2. This can be easily seen if we multiply the denominator out.

$$\frac{6x^2-3x}{(x-2)(x+4)}=\frac{6x^2-3x}{x^2+2x-8}$$

So, the first step is to do long division (we'll leave it up to you to check our Algebra skills for the long division) to get,

$$\frac{6x^2-3x}{(x-2)(x+4)}=6+\frac{48-15x}{(x-2)(x+4)}$$

Step 2

Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.

$$\frac{48-15x}{(x-2)(x+4)}=\frac{A}{x-2}+\frac{B}{x+4}$$

Setting the numerators equal gives,

$$48-15x=A(x+4)+B(x-2)$$

Step 3
The "trick" will work here easily enough so here is that work.

$$\begin{aligned} x=-4: \quad 108 &= -6B \\ x=2: \quad 18 &= 6A \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= 3 \\ B &= -18 \end{aligned}$$

The partial fraction form of the second term is then,

$$\frac{48-15x}{(x-2)(x+4)} = \frac{3}{x-2} - \frac{18}{x+4}$$

Step 4
We can now do the integral.

$$\int \frac{6x^2-3x}{(x-2)(x+4)}\,dx = \int 6 + \frac{3}{x-2} - \frac{18}{x+4}\,dx = \boxed{6x + 3\ln|x-2| - 18\ln|x+4| + c}$$

---

10. Evaluate the integral $\displaystyle\int \frac{2+w^4}{w^3+9w}\,dw$.

Hint : Pay attention to the degree of the numerator and denominator!

Step 1
Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case the degree of the numerator is 4 and the degree of the denominator is 3.

So, the first step is to do long division (we'll leave it up to you to check our Algebra skills for the long division) to get,

$$\frac{2+w^4}{w^3+9w} = w + \frac{2-9w^2}{w\left(w^2+9\right)}$$

Step 2
Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.

$$\frac{2-9w^2}{w\left(w^2+9\right)} = \frac{A}{w} + \frac{Bw+C}{w^2+9}$$

Setting the numerators equal gives,

$$2-9w^2 = A\left(w^2+9\right)+w\left(Bw+C\right)=\left(A+B\right)w^2+Cw+9A$$

In this case the "trick" discussed in the notes won't work all that well for us and so we'll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

$$\begin{array}{rrcl} w^2: & A+B=-9 & & A=\frac{2}{9} \\ w^1: & C=0 & \Rightarrow & B=-\frac{83}{9} \\ w^0: & 9A=2 & & C=0 \end{array}$$

The partial fraction form of the second term is then,

$$\frac{2-9w^2}{w\left(w^2+9\right)}=\frac{\frac{2}{9}}{w}-\frac{\frac{83}{9}w}{w^2+9}$$

Step 4
We can now do the integral.

$$\int\frac{2+w^4}{w^3+9w}dw=\int w+\frac{\frac{2}{9}}{w}-\frac{\frac{83}{9}w}{w^2+9}dw=\boxed{\tfrac{1}{2}w^2+\tfrac{2}{9}\ln\left|w\right|-\tfrac{83}{18}\ln\left|w^2+9\right|+c}$$

Section 1-5 : Integrals Involving Roots

1. Evaluate the integral $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$.

Step 1

The substitution we'll use here is,

$$u = \sqrt{x-4}$$

Step 2

Now we need to get set up for the substitution. In other words, we need so solve for *x* and get *dx*.

$$x = u^2 + 4 \qquad\qquad \Rightarrow \qquad\qquad dx = 2u\,du$$

Step 3

Doing the substitution gives,

$$\int \frac{7}{2+\sqrt{x-4}}\,dx = \int \frac{7}{2+u}(2u)\,du = \int \frac{14u}{2+u}\,du$$

Step 4

This new integral can be done with the substitution $v = u + 2$. Doing this gives,

$$\int \frac{7}{2+\sqrt{x-4}}\,dx = \int \frac{14(v-2)}{v}\,dv = \int 14 - \frac{28}{v}\,dv = 14v - 28\ln|v| + c$$

Step 5

The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{7}{2+\sqrt{x-4}}\,dx = 14(u+2) - 28\ln|u+2| + c = \boxed{14\left(\sqrt{x-4}+2\right) - 28\ln\left|\sqrt{x-4}+2\right| + c}$$

Note that we could have avoided the second substitution if we'd used $u = \sqrt{x-4} + 2$ for the original substitution.

This often doesn't work, but in this case because the only extra term in the denominator was a constant it didn't change the differential work and so would work pretty easily for this problem.

---

2. Evaluate the integral $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$.

Step 1
The substitution we'll use here is,

$$u = \sqrt{1-w}$$

Step 2
Now we need to get set up for the substitution. In other words, we need so solve for *w* and get *dw*.

$$w = 1-u^2 \qquad \Rightarrow \qquad dw = -2u\,du$$

Step 3
Doing the substitution gives,

$$\int \frac{1}{w+2\sqrt{1-w}+2}\,dw = \int \frac{1}{1-u^2+2u+2}\left(-2u\right)du = \int \frac{2u}{u^2-2u-3}\,du$$

Step 4
This integral requires partial fractions to evaluate. Let's start with the form of the partial fraction decomposition.

$$\frac{2u}{\left(u+1\right)\left(u-3\right)} = \frac{A}{u+1} + \frac{B}{u-3}$$

Setting the coefficients equal gives,

$$2u = A\left(u-3\right) + B\left(u+1\right)$$

Using the "trick" to get the coefficients gives,

$$\begin{array}{ll} u=3: & 6=4B \\ u=-1: & -2=-4A \end{array} \qquad \Rightarrow \qquad \begin{array}{l} A=\frac{1}{2} \\ B=\frac{3}{2} \end{array}$$

The integral is then,

$$\int \frac{2u}{\left(u+1\right)\left(u-3\right)}\,du = \int \frac{\frac{1}{2}}{u+1} + \frac{\frac{3}{2}}{u-3}\,du = \frac{1}{2}\ln\left|u+1\right| + \frac{3}{2}\ln\left|u-3\right| + c$$

Step 5
The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{1}{w+2\sqrt{1-w}+2}\,dw = \boxed{\frac{1}{2}\ln\left|\sqrt{1-w}+1\right| + \frac{3}{2}\ln\left|\sqrt{1-w}-3\right| + c}$$

3. Evaluate the integral $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$.

Step 1
The substitution we'll use here is,

$$u = \sqrt{2t-4}$$

Step 2
Now we need to get set up for the substitution. In other words, we need so solve for *t* and get *dt*.

$$t = \tfrac{1}{2}u^2 + 2 \qquad \Rightarrow \qquad dt = u\,du$$

Step 3
Doing the substitution gives,

$$\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt = \int \frac{\tfrac{1}{2}u^2+2-2}{\tfrac{1}{2}u^2+2-3u+2}\left(u\right)du = \int \frac{u^3}{u^2-6u+8}\,du$$

Step 4
This integral requires partial fractions to evaluate.

However, we first need to do long division on the integrand since the degree of the numerator (3) is higher than the degree of the denominator (2). This gives,

$$\frac{u^3}{u^2-6u+8} = u+6+\frac{28u-48}{\left(u-2\right)\left(u-4\right)}$$

The form of the partial fraction decomposition on the third term is,

$$\frac{28u-48}{\left(u-2\right)\left(u-4\right)} = \frac{A}{u-2}+\frac{B}{u-4}$$

Setting the coefficients equal gives,

$$28u-48 = A\left(u-4\right)+B\left(u-2\right)$$

Using the "trick" to get the coefficients gives,

$$\begin{aligned} u=4: &\quad 64=2B \\ u=2: &\quad 8=-2A \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A&=-4 \\ B&=32 \end{aligned}$$

The integral is then,

$$\int \frac{u^3}{u^2-6u+8}\,du = \int u+6-\frac{4}{u-2}+\frac{32}{u-4}\,du = \frac{1}{2}u^2+6u-4\ln\left|u-2\right|+32\ln\left|u-4\right|+c$$

Step 5

The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{u^3}{u^2-6u+8}\,du = \boxed{t-2+6\sqrt{2t-4}-4\ln\left|\sqrt{2t-4}-2\right|+32\ln\left|\sqrt{2t-4}-4\right|+c}$$

Section 1-6 : Integrals Involving Quadratics

1. Evaluate the integral $\displaystyle\int \frac{7}{w^2+3w+3}\,dw$.

Step 1
The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{7}{w^2+3w+3}\,dw = \int \frac{7}{\left(w+\frac{3}{2}\right)^2+\frac{3}{4}}\,dw$$

Step 2
From this we can see that the following substitution should work for us.

$$u = w+\tfrac{3}{2} \qquad \Rightarrow \qquad du = dw$$

Doing the substitution gives,

$$\int \frac{7}{w^2+3w+3}\,dw = \int \frac{7}{u^2+\frac{3}{4}}\,du$$

Step 3
This integral can be done with the formula given at the start of this section.

$$\int \frac{7}{w^2+3w+3}\,dw = \frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+c = \boxed{\frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2w+3}{\sqrt{3}}\right)+c}$$

Don't forget to back substitute in for *u*!

---

2. Evaluate the integral $\displaystyle\int \frac{10x}{4x^2-8x+9}\,dx$.

Step 1
The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{10x}{4x^2-8x+9}\,dx = \int \frac{10x}{4\left(x-1\right)^2+5}\,dx$$

Step 2
From this we can see that the following substitution should work for us.

$$u = x - 1 \qquad \Rightarrow \qquad du = dx \qquad \& \qquad x = u + 1$$

Doing the substitution gives,

$$\int \frac{10x}{4x^2 - 8x + 9}\,dx = \int \frac{10(u+1)}{4u^2 + 5}\,du$$

Step 3
We can quickly do this integral if we split it up as follows,

$$\int \frac{10x}{4x^2 - 8x + 9}\,dx = \int \frac{10u}{4u^2 + 5}\,du + \int \frac{10}{4u^2 + 5}\,du = \int \frac{10u}{4u^2 + 5}\,du + \frac{5}{2}\int \frac{1}{u^2 + \frac{5}{4}}\,du$$

After a quick rewrite of the second integral we can see that we can do the first with the substitution $v = 4u^2 + 5$ and the second is an inverse trig integral we can evaluate using the formula given at the start of the notes for this section.

$$\begin{aligned}\int \frac{10x}{4x^2 - 8x + 9}\,dx &= \frac{5}{4}\ln|v| + \frac{5}{2}\left(\frac{2}{\sqrt{5}}\right)\tan^{-1}\left(\frac{2u}{\sqrt{5}}\right) + c \\ &= \frac{5}{4}\ln\left|4u^2 + 5\right| + \sqrt{5}\tan^{-1}\left(\frac{2u}{\sqrt{5}}\right) + c \\ &= \boxed{\frac{5}{4}\ln\left|4(x-1)^2 + 5\right| + \sqrt{5}\tan^{-1}\left(\frac{2x-2}{\sqrt{5}}\right) + c}\end{aligned}$$

Don't forget to back substitute in for *u*!

---

3. Evaluate the integral $\displaystyle\int \frac{2t+9}{\left(t^2 - 14t + 46\right)^{\frac{5}{2}}}\,dt$.

Step 1
The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{2t+9}{\left(t^2 - 14t + 46\right)^{\frac{5}{2}}}\,dt = \int \frac{2t+9}{\left((t-7)^2 - 3\right)^{\frac{5}{2}}}\,dt$$

Step 2
From this we can see that the following substitution should work for us.

$$u = t - 7 \qquad \Rightarrow \qquad du = dt \qquad \& \qquad t = u + 7$$

Doing the substitution gives,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int \frac{2\left(u+7\right)+9}{\left(u^2-3\right)^{\frac{5}{2}}}\,du = \int \frac{2u+23}{\left(u^2-3\right)^{\frac{5}{2}}}\,du$$

Step 3
Next, we'll need to split the integral up as follows,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int \frac{2u}{\left(u^2-3\right)^{\frac{5}{2}}}\,du + \int \frac{23}{\left(u^2-3\right)^{\frac{5}{2}}}\,du$$

The first integral can be done with the substitution $v = u^2 - 3$ and the second integral will require the trig substitution $u = \sqrt{3}\sec\theta$. Here is the substitution work.

$$\begin{aligned}
\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt &= \int v^{-\frac{5}{2}}\,dv + \int \frac{23}{\left(3\sec^2\theta-3\right)^{\frac{5}{2}}}\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta \\
&= \int v^{-\frac{5}{2}}\,dv + \int \frac{23\sqrt{3}\sec\theta\tan\theta}{\left(3\tan^2\theta\right)^{\frac{5}{2}}}\,d\theta \\
&= \int v^{-\frac{5}{2}}\,dv + \int \frac{23\sec\theta}{9\tan^4\theta}\,d\theta \\
&= \int v^{-\frac{5}{2}}\,dv + \frac{23}{9}\int \frac{\cos^3\theta}{\sin^4\theta}\,d\theta
\end{aligned}$$

Now, for the second integral, don't forget the manipulations we often need to do so we can do these kinds of integrals. If you need some practice on these kinds of integrals go back to the practice problems for the second section of this chapter and work some of them.

Here is the rest of the integration process for this problem.

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int v^{-\frac{5}{2}}\,dv + \frac{23}{9}\int \frac{1-\sin^2\theta}{\sin^4\theta}\cos\theta\,d\theta \qquad w=\sin\theta$$

$$= \int v^{-\frac{5}{2}}\,dv + \tfrac{23}{9}\int w^{-4} - w^{-2}\,dw$$

$$= -\tfrac{2}{3}v^{-\frac{3}{2}} + \tfrac{23}{9}\left[-\tfrac{1}{3}\left(\sin\theta\right)^{-3} + \left(\sin\theta\right)^{-1}\right] + c$$

Step 4

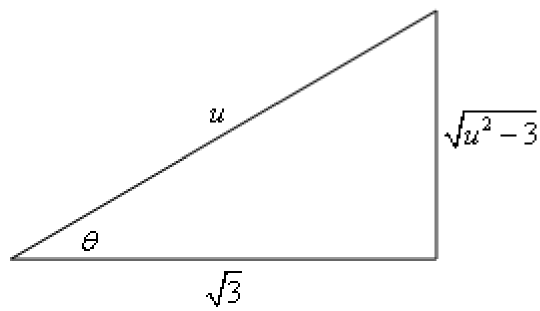
We now need to do quite a bit of back substitution to get the answer back into *t*'s. Let's start with the result of the second integration. Converting the θ's back to *u*'s will require a quick right triangle.

From the substitution we have,

$$\sec\theta = \frac{u}{\sqrt{3}} \qquad \left(=\frac{\text{hyp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sin\theta = \frac{\sqrt{u^2-3}}{u}$$



Plugging this into the integral above gives,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = -\frac{2}{3\left(u^2-3\right)^{\frac{3}{2}}} - \frac{23u^3}{27\left(u^2-3\right)^{\frac{3}{2}}} + \frac{23u}{9\sqrt{u^2-3}} + c$$

Note that we also back substituted for the *v* in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the *u*.

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = -\frac{2}{3\left(\left(t-7\right)^2-3\right)^{\frac{3}{2}}} - \frac{23\left(t-7\right)^3}{27\left(\left(t-7\right)^2-3\right)^{\frac{3}{2}}} + \frac{23\left(t-7\right)}{9\sqrt{\left(t-7\right)^2-3}} + c$$

$$= \boxed{\frac{23\left(t-7\right)}{9\sqrt{\left(t-7\right)^2-3}} - \frac{18+23\left(t-7\right)^3}{27\left(\left(t-7\right)^2-3\right)^{\frac{3}{2}}} + c}$$

We'll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don't get into too much of a hurry when working them.

4. Evaluate the integral $\displaystyle\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz$.

Step 1

The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3z}{\left(3-2\left(z+1\right)^2\right)^2}\,dz$$

Step 2

From this we can see that the following substitution should work for us.

$$u = z+1 \qquad\qquad \Rightarrow \qquad\qquad du = dz \qquad \& \qquad z = u-1$$

Doing the substitution gives,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3\left(u-1\right)}{\left(3-2u^2\right)^2}\,du = \int \frac{3u-3}{\left(3-2u^2\right)^2}\,du$$

Step 3

Next, we'll need to split the integral up as follows,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3u}{\left(3-2u^2\right)^2}\,du - \int \frac{3}{\left(3-2u^2\right)^2}\,du$$

The first integral can be done with the substitution $v = 3-2u^2$ and the second integral will require the trig substitution $u = \frac{\sqrt{3}}{\sqrt{2}}\sin\theta$. Here is the substitution work.

$$\begin{aligned}
\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz &= -\frac{3}{4}\int v^{-2}\,dv - \int \frac{3}{\left(3-3\sin^2\theta\right)^2}\left(\tfrac{\sqrt{3}}{\sqrt{2}}\cos\theta\right)d\theta \\
&= -\frac{3}{4}\int v^{-2}\,dv - \int \frac{3}{\left(3\cos^2\theta\right)^2}\left(\tfrac{\sqrt{3}}{\sqrt{2}}\cos\theta\right)d\theta \\
&= -\tfrac{3}{4}\int v^{-2}\,dv - \tfrac{1}{\sqrt{6}}\int \sec^3\theta\,d\theta
\end{aligned}$$

The second integral for this problem comes down to an integral that was done in the notes for the second section of this chapter and so we'll just use the formula derived in that section to do this integral.

Here is the rest of the integration process for this problem.

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \frac{3}{4}v^{-1} - \frac{1}{2\sqrt{6}}\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]+c$$

Step 4

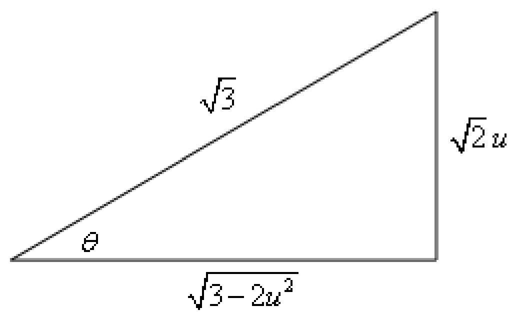
We now need to do quite a bit of back substitution to get the answer back into *z*'s. Let's start with the result of the second integration. Converting the θ's back to *u*'s will require a quick right triangle.

From the substitution we have,

$$\sin\theta = \frac{\sqrt{2}\,u}{\sqrt{3}} \qquad \left(=\frac{\text{opp}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\tan\theta = \frac{\sqrt{2}\,u}{\sqrt{3-2u^2}} \qquad \& \qquad \sec\theta = \frac{\sqrt{3}}{\sqrt{3-2u^2}}$$



Plugging this into the integral above gives,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \frac{3}{4\left(3-2u^2\right)} - \frac{1}{2\sqrt{6}}\left[\frac{\sqrt{6}\,u}{3-2u^2} + \ln\left|\frac{\sqrt{3}+\sqrt{2}\,u}{\sqrt{3-2u^2}}\right|\right]+c$$

Note that we also back substituted for the *v* in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the *u*.

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \boxed{\frac{3}{4\left(3-2\left(z+1\right)^2\right)} - \frac{z+1}{6-4\left(z+1\right)^2} - \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{3}+\sqrt{2}\left(z+1\right)}{\sqrt{3-2\left(z+1\right)^2}}\right|+c}$$

We'll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don't get into too much of a hurry when working them.

Section 1-7 : Integration Strategy

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

Section 1-8 : Improper Integrals

1. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_0^\infty (1+2x)\mathbf{e}^{-x}\,dx$$

Hint : Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_0^\infty (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\int_0^t (1+2x)\mathbf{e}^{-x}\,dx$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

$$\begin{aligned} u &= 1+2x & \to \quad & du = 2dx \\ dv &= \mathbf{e}^{-x}\,dx & \to \quad & v = -\mathbf{e}^{-x} \end{aligned}$$

$$\int (1+2x)\mathbf{e}^{-x}\,dx = -(1+2x)\mathbf{e}^{-x} + 2\int \mathbf{e}^{-x}\,dx = -(1+2x)\mathbf{e}^{-x} - 2\mathbf{e}^{-x} + c = -(3+2x)\mathbf{e}^{-x} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int_0^\infty (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\left(-(3+2x)\mathbf{e}^{-x}\right)\Big|_0^t = \lim_{t\to\infty}\left(3-(3+2t)\mathbf{e}^{-t}\right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step and note that, in this case, we really can't just "evaluate" at infinity! We need to do the limiting process here to make sure we get the correct answer.

We will need to do a quick L'Hospital's Rule on the second term to properly evaluate it. Here is the limit work.

$$\int_0^\infty (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to\infty} 3 - \lim_{t\to\infty}\frac{3+2t}{\mathbf{e}^t} = 3 - \lim_{t\to\infty}\frac{2}{\mathbf{e}^t} = 3 - 0 = 3$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is **3**.

---

2. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^0 (1+2x)\mathbf{e}^{-x}\,dx$$

Hint : Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_{-\infty}^0 (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to-\infty}\int_t^0 (1+2x)\mathbf{e}^{-x}\,dx$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

$$\begin{aligned} u &= 1+2x & &\rightarrow & du &= 2dx \\ dv &= \mathbf{e}^{-x}\,dx & &\rightarrow & v &= -\mathbf{e}^{-x} \end{aligned}$$

$$\int (1+2x)\mathbf{e}^{-x}\,dx = -(1+2x)\mathbf{e}^{-x} + 2\int \mathbf{e}^{-x}\,dx = -(1+2x)\mathbf{e}^{-x} - 2\mathbf{e}^{-x} + c = -(3+2x)\mathbf{e}^{-x} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int_{-\infty}^{0} (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to-\infty}\left(-(3+2x)\mathbf{e}^{-x}\right)\Big|_{t}^{0} = \lim_{t\to-\infty}\left((3+2t)\mathbf{e}^{-t} - 3\right)$$

Step 4
We now need to evaluate the limit in our answer from the previous step. In this case we can see that the first term will go to negative infinity since it is just a product of one factor that goes to negative infinity and another factor that goes to infinity. Therefore, the full limit will also be negative infinity since the constant second term won't affect the final value of the limit.

$$\int_{-\infty}^{0} (1+2x)\mathbf{e}^{-x}\,dx = \lim_{t\to-\infty}(3+2t)\mathbf{e}^{-t} - \lim_{t\to\infty} 3 = (-\infty)(\infty) - 3 = -\infty - 3 = -\infty$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was negative infinity. Therefore, the integral **diverges**.

---

3. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz$$

Hint : Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $z = -5$ and this is the lower limit of integration. We know that as long as that discontinuity is there we can't do the integral. Therefore, we'll need to eliminate the discontinuity first as follows,

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t \to -5^{+}} \int_{t}^{1} \frac{1}{10+2z}\,dz$$

Don't forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is $\left[-5,1\right]$ we are only interested in the values of *z* that are greater than -5 and so we must use a right-hand limit to reflect that fact.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{1}{10+2z}\,dz = \frac{1}{2}\ln\left|10+2z\right| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t \to -5^{+}} \left(\frac{1}{2}\ln\left|10+2z\right| \right)\Bigg|_{t}^{1} = \lim_{t \to -5^{+}} \left(\frac{1}{2}\ln\left|12\right| - \frac{1}{2}\ln\left|10+2t\right| \right)$$

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t \to -5^{+}} \left(\frac{1}{2}\ln\left|12\right| - \frac{1}{2}\ln\left|10+2t\right| \right) = \frac{1}{2}\ln\left|12\right| + \infty = \infty$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral **diverges**.

---

4. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw$$

Hint : Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $w = 2$ and this is the upper limit of integration. We know that as long as that discontinuity is there we can't do the integral. Therefore, we'll need to eliminate the discontinuity first as follows,

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw = \lim_{t \to 2^-} \int_{1}^{t} \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw$$

Don't forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is $\left[1, 2\right]$ we are only interested in the values of *t* that are less than 2 and so we must use a left-hand limit to reflect that fact.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw = 3\left(w^2 - 4\right)^{\frac{2}{3}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw = \lim_{t \to 2^-} \left(3\left(w^2 - 4\right)^{\frac{2}{3}} \right)\Bigg|_{1}^{t} = \lim_{t \to 2^-} \left(3\left(t^2 - 4\right)^{\frac{2}{3}} - 3\left(-3\right)^{\frac{2}{3}} \right)$$

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}}\, dw = \lim_{t \to 2^-} \left(3\left(t^2 - 4\right)^{\frac{2}{3}} - 3\left(-3\right)^{\frac{2}{3}} \right) = -3\left(-3\right)^{\frac{2}{3}} = \left(-3\right)^{\frac{5}{3}}$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is $\left(-3\right)^{\frac{5}{3}}$.

---

5. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{1} \sqrt{6-y}\,dy$$

Hint : Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_{-\infty}^{1} \sqrt{6-y}\,dy = \lim_{t\to-\infty}\int_{t}^{1} \sqrt{6-y}\,dy$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \sqrt{6-y}\,dy = -\tfrac{2}{3}\left(6-y\right)^{\frac{3}{2}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int \sqrt{6-y}\,dy = \lim_{t\to -\infty}\left(-\tfrac{2}{3}\left(6-y\right)^{\frac{3}{2}}\right)\Bigg|_t^1 = \lim_{t\to -\infty}\left(-\tfrac{2}{3}\left(5\right)^{\frac{3}{2}} + \tfrac{2}{3}\left(6-t\right)^{\frac{3}{2}}\right)$$

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int \sqrt{6-y}\,dy = \lim_{t\to -\infty}\left(-\tfrac{2}{3}\left(5\right)^{\frac{3}{2}} + \tfrac{2}{3}\left(6-t\right)^{\frac{3}{2}}\right) = -\tfrac{2}{3}\left(5\right)^{\frac{3}{2}} + \infty = \infty$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral **diverges**.

---

6. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_2^\infty \frac{9}{\left(1-3z\right)^4}\,dz$$

Hint : Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_2^\infty \frac{9}{\left(1-3z\right)^4}\,dz = \lim_{t\to\infty}\int_2^t \frac{9}{\left(1-3z\right)^4}\,dz$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{9}{\left(1-3z\right)^4}\,dz = \frac{1}{\left(1-3z\right)^3}+c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let's take care of the limits on the integral.

$$\int_2^\infty \frac{9}{\left(1-3z\right)^4}\,dz = \lim_{t\to\infty}\left(\left.\frac{1}{\left(1-3z\right)^3}\right|_2^t\right) = \lim_{t\to\infty}\left(\frac{1}{\left(1-3t\right)^3}-\left(-\frac{1}{125}\right)\right)$$

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work

$$\int_2^\infty \frac{9}{\left(1-3z\right)^4}\,dz = \lim_{t\to\infty}\left(\frac{1}{\left(1-3t\right)^3}+\frac{1}{125}\right)=\frac{1}{125}$$

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is $\frac{1}{125}$.

---

7. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_0^4 \frac{x}{x^2-9}\,dx$$

Hint : Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $x=3$ and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can't do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we'll need to break up the integral at $x=3$.

$$\int_0^4 \frac{x}{x^2-9}\,dx = \int_0^3 \frac{x}{x^2-9}\,dx + \int_3^4 \frac{x}{x^2-9}\,dx$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem.

We can eliminate the discontinuity in each as follows,

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}\,dx + \lim_{s\to 3^+}\int_s^4 \frac{x}{x^2-9}\,dx$$

Don't forget that the limits on these kinds of integrals must be one-sided limits.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{x}{x^2-9}\,dx = \frac{1}{2}\ln\left|x^2-9\right| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_0^4 \frac{x}{x^2-9}\,dx &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Big|_0^t + \lim_{s\to 3^+}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Big|_s^4 \\ &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln\left(9\right)\right) + \lim_{s\to 3^+}\left(\frac{1}{2}\ln\left(7\right) - \frac{1}{2}\ln\left|s^2-9\right|\right)\end{aligned}$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work.

$$\begin{aligned}\int_0^4 \frac{x}{x^2-9}\,dx &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right|-\frac{1}{2}\ln(9)\right)+\lim_{s\to 3^+}\left(\frac{1}{2}\ln(7)-\frac{1}{2}\ln\left|s^2-9\right|\right)\\ &= \qquad\left[-\infty-\frac{1}{2}\ln(9)\right]\qquad + \qquad\left[\frac{1}{2}\ln(7)+\infty\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_0^3 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right|-\frac{1}{2}\ln(9)\right)=\left[-\infty-\frac{1}{2}\ln(9)\right]=-\infty$$

$$\int_3^4 \frac{x}{x^2-9}\,dx = \lim_{s\to 3^+}\left(\frac{1}{2}\ln(7)-\frac{1}{2}\ln\left|s^2-9\right|\right)=\left[\frac{1}{2}\ln(7)+\infty\right]=\infty$$

Therefore, each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral **diverges**.

---

8. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$$

Hint : Don't forget that we can't do the integral as long as there is an infinity in one of the limits! Also, don't forget that infinities in both limits need as extra step to get set up.

Step 1
First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Note as well that in this case we have infinities in both limits and so we'll need to split up the integral.

The integral can be split up at any point in this case and $w=0$ seems like a good point to use for the split point. Splitting up the integral gives,

$$\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw = \int_{-\infty}^{0} \frac{6w^3}{\left(w^4+1\right)^2}\,dw + \int_{0}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem.

Now, we can eliminate the infinities as follows,

$$\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw = \lim_{t\to-\infty}\int_{t}^{0} \frac{6w^3}{\left(w^4+1\right)^2}\,dw + \lim_{s\to\infty}\int_{0}^{s} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$$

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{6w^3}{\left(w^4+1\right)^2}\,dw = -\frac{3}{2}\frac{1}{w^4+1}+c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3
Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw &= \lim_{t\to-\infty}\left(-\frac{3}{2}\frac{1}{w^4+1}\right)\Bigg|_{t}^{0} + \lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{w^4+1}\right)\Bigg|_{0}^{s}\\ &= \lim_{t\to-\infty}\left(-\frac{3}{2}+\frac{3}{2}\frac{1}{t^4+1}\right)+\lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{s^4+1}+\frac{3}{2}\right)\end{aligned}$$

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw &= \lim_{t\to-\infty}\left(-\frac{3}{2}+\frac{3}{2}\frac{1}{t^4+1}\right)+\lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{s^4+1}+\frac{3}{2}\right)\\ &= \qquad \left[-\frac{3}{2}\right] \qquad + \qquad \left[\frac{3}{2}\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_{-\infty}^{0} \frac{6w^3}{\left(w^4+1\right)^2}\,dw = \lim_{t\to-\infty}\left(-\frac{3}{2}+\frac{3}{2}\frac{1}{t^4+1}\right) = -\frac{3}{2}$$

$$\int_{0}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw = \lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{s^4+1}+\frac{3}{2}\right) = \frac{3}{2}$$

Therefore, each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw &= \int_{-\infty}^{0} \frac{6w^3}{\left(w^4+1\right)^2}\,dw + \int_{0}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw\\ &= \qquad -\frac{3}{2} \qquad + \qquad \frac{3}{2}\\ &= 0\end{aligned}$$

Therefore, the integral **converges** and its value is **0**.

---

9. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{1}^{4} \frac{1}{x^2+x-6}\,dx$$

Hint : Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $x = 2$ and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can't do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we'll need to break up the integral at $x = 2$.

$$\int_1^4 \frac{1}{x^2+x-6}\,dx = \int_1^2 \frac{1}{(x+3)(x-2)}\,dx + \int_2^4 \frac{1}{(x+3)(x-2)}\,dx$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem.

We can eliminate the discontinuity in each as follows,

$$\int_1^4 \frac{1}{x^2+x-6}\,dx = \lim_{t\to 2^-}\int_1^t \frac{1}{(x+3)(x-2)}\,dx + \lim_{s\to 2^+}\int_s^4 \frac{1}{(x+3)(x-2)}\,dx$$

Don't forget that the limits on these kinds of integrals must be one-sided limits.

Step 2
Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we will need to do some partial fractions in order to the integral. Here is the partial fraction work.

$$\frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \qquad \Rightarrow \qquad 1 = A(x-2) + B(x+3)$$

$$\begin{array}{ll} x = 2: & 1 = 5B \\ x = -3: & 1 = -5A \end{array} \qquad \to \qquad \begin{array}{l} A = -\frac{1}{5} \\ B = \frac{1}{5} \end{array}$$

The integration work is then,

$$\int \frac{1}{(x+3)(x-2)}\,dx = \int \frac{\frac{1}{5}}{x-2} - \frac{\frac{1}{5}}{x+3}\,dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3
Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_1^4 \frac{1}{x^2+x-6}\,dx &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln\left|x-2\right|-\frac{1}{5}\ln\left|x+3\right|\right)\Bigg|_1^t + \lim_{s\to 2^+}\left(\frac{1}{5}\ln\left|x-2\right|-\frac{1}{5}\ln\left|x+3\right|\right)\Bigg|_s^4 \\ &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln\left|t-2\right|-\frac{1}{5}\ln\left|t+3\right|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) \\ &\qquad + \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln\left|s-2\right|-\frac{1}{5}\ln\left|s+3\right|\right)\right)\end{aligned}$$

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work.

$$\begin{aligned}\int_1^4 \frac{1}{x^2+x-6}\,dx &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln\left|t-2\right|-\frac{1}{5}\ln\left|t+3\right|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) \\ &\qquad + \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln\left|s-2\right|-\frac{1}{5}\ln\left|s+3\right|\right)\right) \\ &= \left[-\infty-\frac{1}{5}\ln(5)+\frac{1}{5}\ln(4)\right] + \left[\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)+\frac{1}{5}\ln(5)+\infty\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_1^2 \frac{1}{(x+3)(x-2)}\,dx = \lim_{t\to 2^-}\left(\frac{1}{5}\ln\left|t-2\right|-\frac{1}{5}\ln\left|t+3\right|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) = -\infty$$

$$\int_2^4 \frac{1}{(x+3)(x-2)}\,dx = \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln\left|s-2\right|-\frac{1}{5}\ln\left|s+3\right|\right)\right) = \infty$$

Therefore, each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral **diverges**.

---

10. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Hint : Be very careful with this problem as it is nothing like what we did in the notes. However, you should be able to take the material from the notes and use that to figure out how to do this problem.

Step 1

Now there is clearly an infinite limit here, but also notice that there is a discontinuity at $x = 0$ that we'll need to deal with.

Based on the material in the notes it should make sense that, provided both integrals converge, we should be able to split up the integral at any point. In this case let's split the integral up at $x = -1$. Doing this gives,

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Keep in mind that splitting up the integral like this can only be done if both of the integrals converge! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not let's proceed with the problem.

Now, we can eliminate the problems as follows,

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \lim_{s \to 0^-} \int_{-1}^{s} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = -\mathbf{e}^{\frac{1}{x}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to-\infty}\left(-\mathbf{e}^{\frac{1}{x}}\right)\Bigg|_{t}^{-1} + \lim_{s\to0^-}\left(-\mathbf{e}^{\frac{1}{x}}\right)\Bigg|_{-1}^{s} = \lim_{t\to-\infty}\left(-\mathbf{e}^{-1}+\mathbf{e}^{\frac{1}{t}}\right) + \lim_{s\to0^-}\left(-\mathbf{e}^{\frac{1}{s}}+\mathbf{e}^{-1}\right)$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work

$$\begin{aligned}\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx &= \lim_{t\to-\infty}\left(-\mathbf{e}^{-1}+\mathbf{e}^{\frac{1}{t}}\right) + \lim_{s\to0^-}\left(-\mathbf{e}^{\frac{1}{s}}+\mathbf{e}^{-1}\right)\\ &= \left[-\mathbf{e}^{-1}+\mathbf{e}^{0}\right] + \left[0+\mathbf{e}^{-1}\right]\end{aligned}$$

Note that,

$$\lim_{s\to0^-}\frac{1}{s} = -\infty$$

since we are doing a left-hand limit and so *s* will be negative. This in turn means that,

$$\lim_{s\to0^-}\left(-\mathbf{e}^{\frac{1}{s}}\right) = 0$$

Step 5

The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = -\mathbf{e}^{-1}+1 \qquad\qquad \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \mathbf{e}^{-1}$$

Therefore, each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

$$\begin{aligned}\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx &= \int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx \\ &= -\mathbf{e}^{-1} + 1 + \mathbf{e}^{-1} \\ &= 1\end{aligned}$$

Therefore, the integral **converges** and its value is **1**.

Section 1-9 : Comparison Test for Improper Integrals

1. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_{1}^{\infty} \frac{1}{x^3+1}\,dx$$

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The "+1" in the denominator does not really change the size of the denominator as *x* gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

$$\int_{1}^{\infty} \frac{1}{x^3}\,dx$$

Then, by the fact from the previous section, we know that this integral converges since $p = 3 > 1$.

Therefore, we can guess that the integral,

$$\int_{1}^{\infty} \frac{1}{x^3+1}\,dx$$

will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint : Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3
Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. Also note that for $x > 1$ (which we can assume from the limits on the integral) we have,

$$x^3 + 1 > x^3$$

Therefore, we have,

$$\frac{1}{x^3+1} < \frac{1}{x^3}$$

since we replaced the denominator with something that we know is smaller.

Step 4
Finally, we know that,

$$\int_{1}^{\infty} \frac{1}{x^3}\,dx$$

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_{1}^{\infty} \frac{1}{x^3+1}\,dx$$

must also **converge**.

---

2. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_{3}^{\infty} \frac{z^2}{z^3-1}\,dz$$

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The "-1" in the denominator does not really change the size of the denominator as *z* gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

$$\int_{3}^{\infty} \frac{z^2}{z^3}\,dz \int_{3}^{\infty} \frac{1}{z}\,dz$$

Then, by the fact from the previous section, we know that this integral diverges since $p = 1 \le 1$.

Therefore, we can guess that the integral,

$$\int_{3}^{\infty} \frac{z^2}{z^3-1}\,dz$$

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral diverges we now know that it diverges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint : Now that we've guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have an finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.

Therefore we need to find a smaller function that we know diverges.

Step 3
Okay, now that we know we need to find a smaller function that we know diverges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the denominator larger. Also note that for $z > 3$ (which we can assume from the limits on the integral) we have,

$$z^3 - 1 < z^3$$

Therefore, we have,

$$\frac{z^2}{z^3-1} > \frac{z^2}{z^3} = \frac{1}{z}$$

since we replaced the denominator with something that we know is larger.

Step 4
Finally, we know that,

$$\int_3^\infty \frac{1}{z}\,dz$$

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

$$\int_3^\infty \frac{z^2}{z^3-1}\,dz$$

must also **diverge**.

---

3. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_4^\infty \frac{\mathbf{e}^{-y}}{y}\,dy$$

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

We need to be a little careful with the guess for this problem. We might be tempted to use the fact from the previous section to guess diverge since the exponent on the *y* in the denominator is $p = 1 \le 1$.

That would be incorrect however. Recall that the fact requires a constant in the numerator and we clearly do not have that in this case. In fact what we have in the numerator is $\mathbf{e}^{-y}$ and this goes to zero very fast as $y \to \infty$ and so there is a pretty good chance that this integral will in fact converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint : Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3
Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. From the limits on the integral we can see that,

$$y > 4$$

Therefore, we have,

$$\frac{\mathbf{e}^{-y}}{y} < \frac{\mathbf{e}^{-y}}{4}$$

since we replaced the denominator with something that we know is smaller.

Step 4
Finally, we will need to prove that,

$$\int_4^\infty \mathbf{e}^{-y}\,dy$$

converges. However, after the previous section that shouldn't be too difficult. Here is that work.

$$\int_4^\infty \tfrac{1}{4}\mathbf{e}^{-y}\,dy = \lim_{t\to\infty}\int_4^t \tfrac{1}{4}\mathbf{e}^{-y}\,dy = \lim_{t\to\infty}\left(-\tfrac{1}{4}\mathbf{e}^{-y}\right)\Big|_4^t = \lim_{t\to\infty}\left(-\tfrac{1}{4}\mathbf{e}^{-t}+\tfrac{1}{4}\mathbf{e}^{-4}\right) = \tfrac{1}{4}\mathbf{e}^{-4}$$

The limit existed and was finite and so we know that,

$$\int_4^\infty \tfrac{1}{4}\mathbf{e}^{-y}\,dy$$

converges.

Therefore, because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_4^\infty \frac{\mathbf{e}^{-y}}{y}\,dy$$

must also **converge**.

---

4. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$$

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

Both the numerator and denominator of this function are polynomials and we know that as $z \to \infty$ the behavior of each of the polynomials will be the same as the behavior of the largest power of *z*. Therefore, it looks like this integral should behave like,

$$\int_1^\infty \frac{z}{z^4}\,dz = \int_1^\infty \frac{1}{z^3}\,dz$$

Then, by the fact from the previous section, we know that this integral converges since $p = 3 > 1$.

Therefore, we can guess that the integral,

$$\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$$

will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint : Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3
Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making numerator larger or the denominator smaller.

Note that for $z > 1$ (which we can assume from the limits on the integral) we have,

$$z - 1 < z$$

Therefore, we have,

$$\frac{z-1}{z^4 + 2z^2} < \frac{z}{z^4 + 2z^2} = \frac{1}{z^3 + 2z}$$

since we replaced the numerator with something that we know is larger.

Step 4
It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must converge (since that is what we want after all) and move on to the next problem.

Recall however that we must **know** that the new function converges and we've not gotten to a function yet that we know converges. To get to a function that we know converges we need to do one more manipulation of the function.

Again, note that for $z > 1$ we have,

$$z^3 + 2z > z^3$$

Therefore, we have,

$$\frac{1}{z^3 + 2z} < \frac{1}{z^3}$$

since we replaced the denominator with something that we know is smaller.

Step 5
Finally, putting the results of Steps 3 & 4 together we have,

$$\frac{z-1}{z^4 + 2z^2} < \frac{1}{z^3}$$

and we know that,

$$\int_1^\infty \frac{1}{z^3}\,dz$$

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_1^\infty \frac{z-1}{z^4 + 2z^2}\,dz$$

must also **converge**.

---

5. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_6^\infty \frac{w^2 + 1}{w^3\left(\cos^2\left(w\right) + 1\right)}\,dw$$

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The numerator of this function is a polynomial and we know that as $w \to \infty$ the behavior of polynomials will be the same as the behavior of the largest power of *w*. Also the cosine term in the denominator is bounded and never gets too large or small.

Therefore, it looks like this integral should behave like,

$$\int_6^\infty \frac{w^2}{w^3}\,dw = \int_6^\infty \frac{1}{w}\,dw$$

Then, by the fact from the previous section, we know that this integral diverges since $p = 1 \le 1$.

Therefore, we can guess that the integral,

$$\int_6^\infty \frac{w^2+1}{w^3\left(\cos^2\left(w\right)+1\right)}\,dw$$

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral diverges we now know that it diverges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint : Now that we've guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have an finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.

Therefore we need to find a smaller function that we know diverges.

Step 3
Okay, now that we know we need to find a smaller function that we know diverges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the numerator smaller or the denominator larger. Also note that for $w > 6$ (which we can assume from the limits on the integral) we have,

$$w^2 + 1 > w^2$$

Therefore, we have,

$$\frac{w^2+1}{w^3\left(\cos^2(w)+1\right)} > \frac{w^2}{w^3\left(\cos^2(w)+1\right)} = \frac{1}{w\left(\cos^2(w)+1\right)}$$

since we replaced the numerator with something that we know is smaller.

Step 4
It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must diverge (since that is what we want after all) and move on to the next problem.

Recall however that we must **know** that the new function diverges and we've not gotten to a function yet that we know diverges. To get to a function that we know diverges we need to do one more manipulation of the function.

For this step we know that $0 \le \cos^2(w) \le 1$ and so we will have,

$$\cos^2(w) + 1 < 1 + 1 = 2$$

Therefore, we have,

$$\frac{1}{w\left(\cos^2(w)+1\right)} > \frac{1}{w(2)} = \frac{1}{2w}$$

since we replaced the denominator with something that we know is larger.

Step 5
Finally, putting the results of Steps 3 & 4 together we have,

$$\frac{w^2+1}{w^3\left(\cos^2\left(w\right)+1\right)} > \frac{1}{2w}$$

and we know that,

$$\int_6^\infty \frac{1}{2w}\,dw = \frac{1}{2}\int_6^\infty \frac{1}{w}\,dw$$

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

$$\int_6^\infty \frac{w^2+1}{w^3\left(\cos^2\left(w\right)+1\right)}\,dw$$

must also **diverge**.

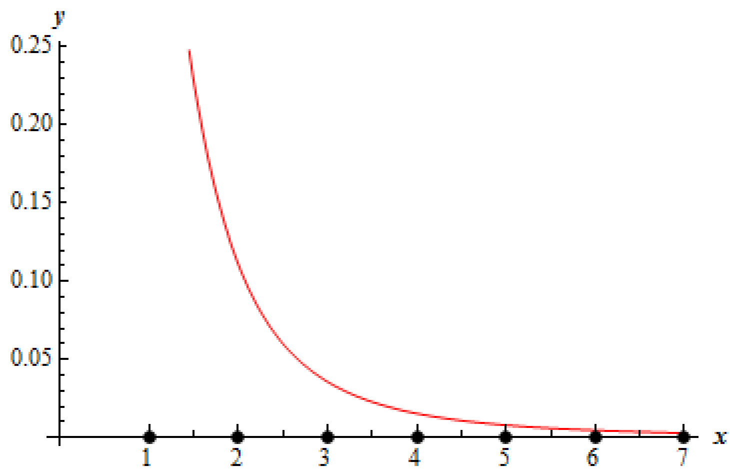
Section 1-10 : Approximating Definite Integrals

1. Using $n = 6$ approximate the value of $\displaystyle\int_1^7 \frac{1}{x^3+1}\,dx$ using

(a) the Midpoint Rule,
(b) the Trapezoid Rule, and
(c) Simpson's Rule

Use at least 6 decimal places of accuracy for your work.

(a) Midpoint Rule
While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $[1,7]$ into 6 subintervals each with width,

$$\Delta x = \frac{7-1}{6} = 1$$

The endpoints of each of these subintervals are represented by the dots on the *x* axis on the graph above.

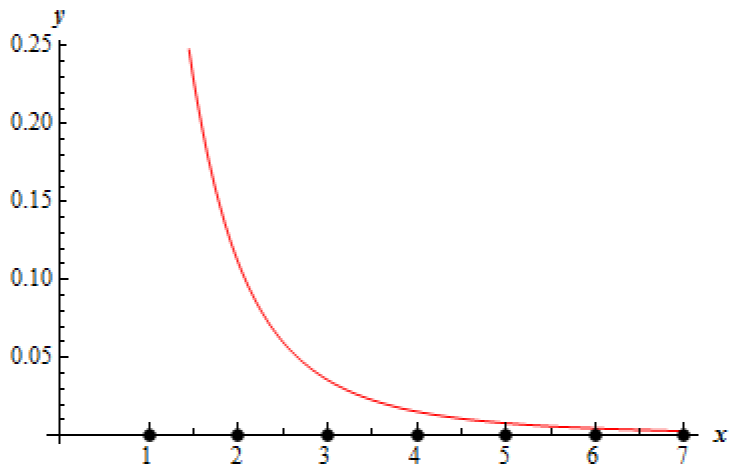
The tick marks between each dot represents the midpoint of each of the subintervals. The *x*-values of the midpoints for each of the subintervals are then,

$$\frac{3}{2},\quad \frac{5}{2},\quad \frac{7}{2},\quad \frac{9}{2},\quad \frac{11}{2},\quad \frac{13}{2}$$

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_{1}^{7} \frac{1}{x^3+1}\,dx \approx (1)\left[f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right) + f\left(\frac{11}{2}\right) + f\left(\frac{13}{2}\right)\right]$$
$$= \boxed{0.33197137}$$

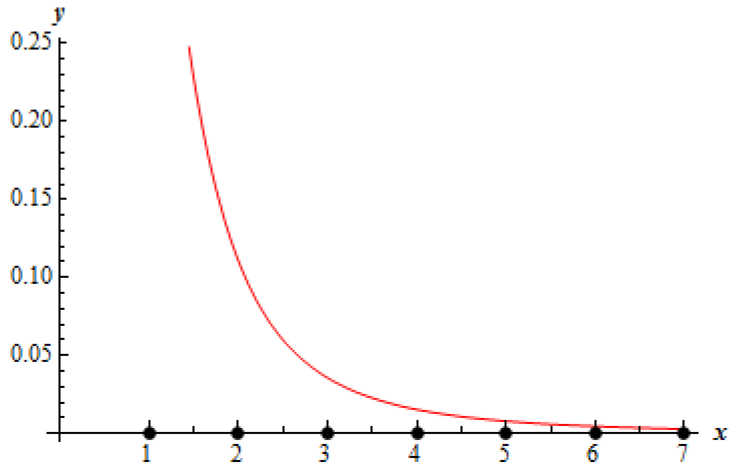
(b) Trapezoid Rule
From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = 1$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_{1}^{7} \frac{1}{x^3+1}\,dx \approx \left(\frac{1}{2}\right)\left[f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + 2f(6) + f(7)\right]$$
$$= \boxed{0.42620830}$$

(c) Simpson's Rule
From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = 1$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\begin{aligned}\int_1^7 \frac{1}{x^3+1}\,dx &\approx \left(\frac{1}{3}\right)\left[f(1)+4f(2)+2f(3)+4f(4)+2f(5)+4f(6)+f(7)\right]\\ &= \boxed{0.37154155}\end{aligned}$$

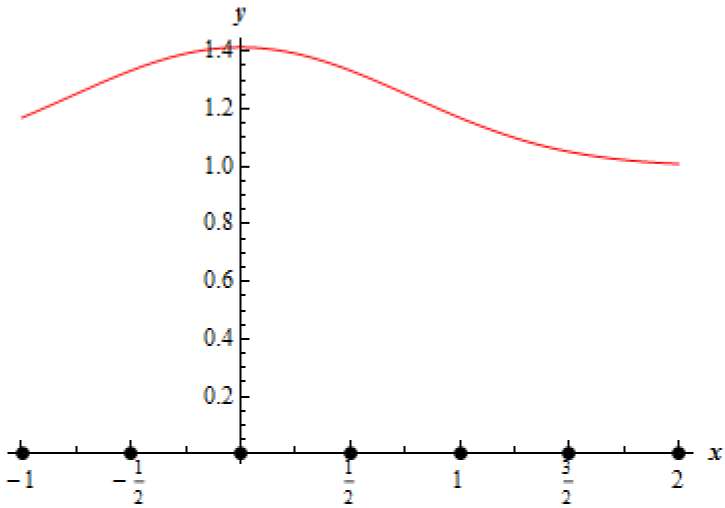
---

2. Using $n=6$ approximate the value of $\int_{-1}^{2}\sqrt{\mathbf{e}^{-x^2}+1}\,dx$ using

(a) the Midpoint Rule,
(b) the Trapezoid Rule, and
(c) Simpson's Rule

Use at least 6 decimal places of accuracy for your work.

(a) Midpoint Rule
While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $\left[-1,2\right]$ into 6 subintervals each with width,

$$\Delta x = \frac{2-\left(-1\right)}{6} = \frac{1}{2}$$

The endpoints of each of these subintervals are represented by the dots on the *x* axis on the graph above.

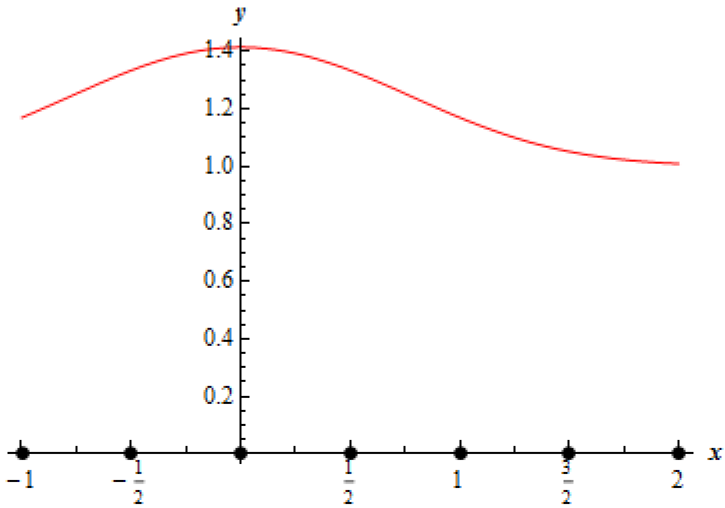
The tick marks between each dot represents the midpoint of each of the subintervals. The *x*-values of the midpoints for each of the subintervals are then,

$$-\frac{3}{4}, \quad -\frac{1}{4}, \quad \frac{1}{4}, \quad \frac{3}{4}, \quad \frac{5}{4}, \quad \frac{7}{4}$$

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned}\int_{-1}^{2}\sqrt{\mathbf{e}^{-x^2}+1}\,dx &\approx \left(\frac{1}{2}\right)\left[f\left(-\frac{3}{4}\right)+f\left(-\frac{1}{4}\right)+f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right]\\ &= \boxed{3.70700857}\end{aligned}$$

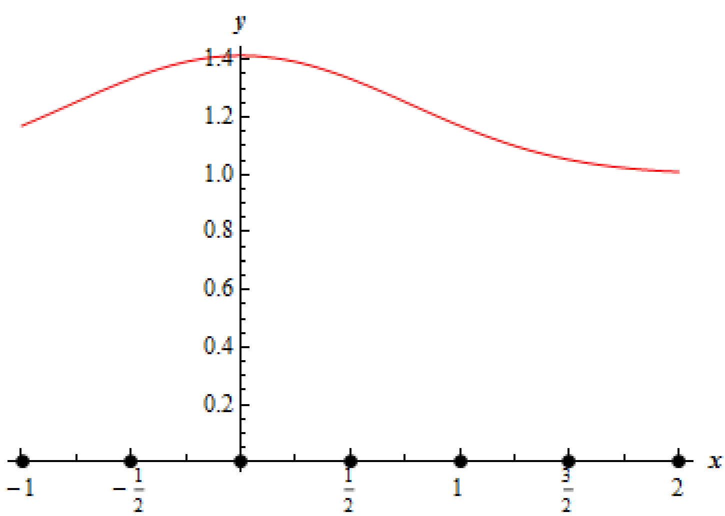
(b) Trapezoid Rule
From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_{-1}^{2}\sqrt{\mathbf{e}^{-x^2}+1}\,dx \approx \left(\frac{½}{2}\right)\left[f\left(-1\right)+2f\left(-\frac{1}{2}\right)+2f\left(0\right)+2f\left(\frac{1}{2}\right)+2f\left(1\right)+2f\left(\frac{3}{2}\right)+f\left(2\right)\right]$$
$$=\boxed{3.69596543}$$

(c) Simpson's Rule
From the Midpoint Rule work we know that the width of each subinterval is $\Delta x=\frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\int_{-1}^{2}\sqrt{\mathbf{e}^{-x^2}+1}\,dx \approx \left(\frac{1/2}{3}\right)\left[f(-1)+4f\left(-\frac{1}{2}\right)+2f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+f(2)\right]$$
$$=\boxed{3.70358145}$$

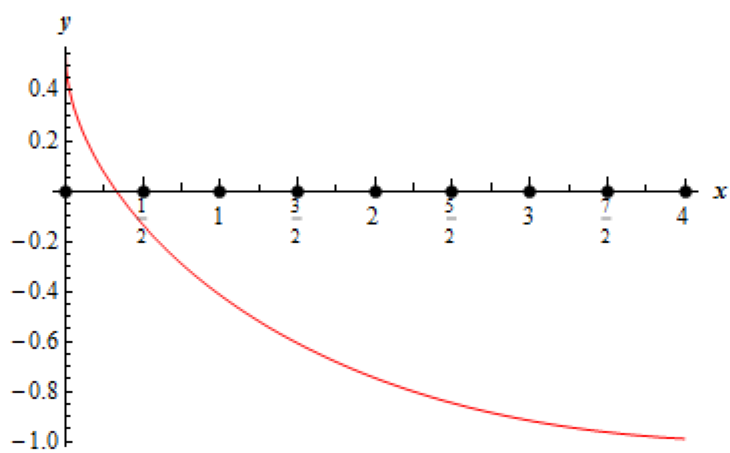
---

3. Using $n=8$ approximate the value of $\int_0^4 \cos\left(1+\sqrt{x}\right)dx$ using

(a) the Midpoint Rule,
(b) the Trapezoid Rule, and
(c) Simpson's Rule

Use at least 6 decimal places of accuracy for your work.

(a) Midpoint Rule
While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $\left[0,4\right]$ into 8 subintervals each with width,

$$\Delta x = \frac{4-0}{8} = \frac{1}{2}$$

The endpoints of each of these subintervals are represented by the dots on the *x* axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The *x*-values of the midpoints for each of the subintervals are then,

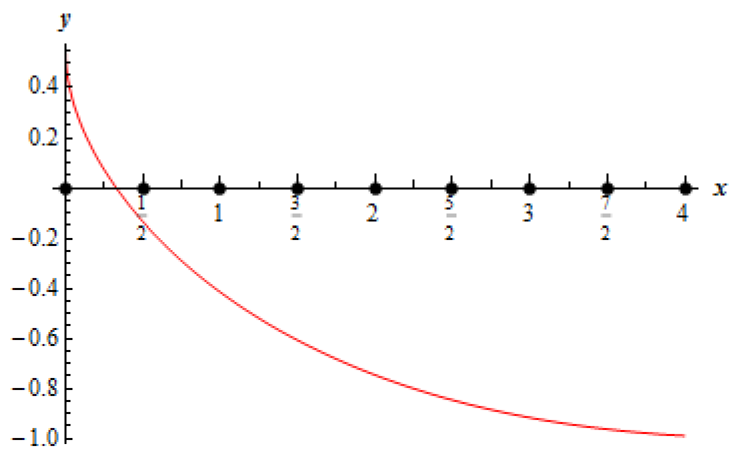
$$\frac{1}{4},\quad \frac{3}{4},\quad \frac{5}{4},\quad \frac{7}{4},\quad \frac{9}{4},\quad \frac{11}{4},\quad \frac{13}{4},\quad \frac{15}{4}$$

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned}\int_0^4 \cos\left(1+\sqrt{x}\right)dx &\approx \left(\frac{1}{2}\right)\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)+\right.\\ &\qquad\qquad \left. f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)\right]\\ &= \boxed{-2.51625938}\end{aligned}$$

(b) Trapezoid Rule

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.

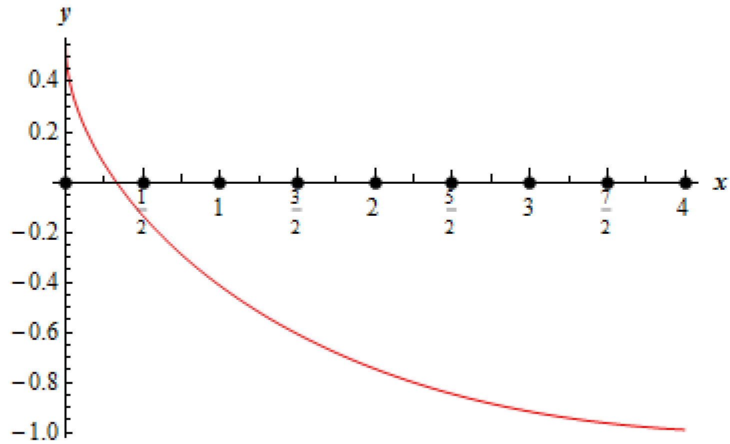


So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned}\int_0^4 \cos\left(1+\sqrt{x}\right)dx &\approx \left(\frac{1/2}{2}\right)\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+\right.\\ &\qquad\qquad \left. 2f(3)+2f\left(\frac{7}{2}\right)+f(4)\right]\\ &= \boxed{-2.43000475}\end{aligned}$$

(c) Simpson's Rule

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\int_0^4 \cos\left(1+\sqrt{x}\right) dx \approx \left(\frac{1/2}{3}\right)\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right]$$

$$= \boxed{-2.47160136}$$

Chapter 2 : Applications of Integrals

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Arc Length – In this section we'll determine the length of a curve over a given interval.

Surface Area – In this section we'll determine the surface area of a solid of revolution, *i.e.* a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.

Center of Mass – In this section we will determine the center of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may the x or y-axis).

Hydrostatic Pressure and Force – In this section we'll determine the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.

Probability – Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities are fixed values and will depend on a variety of factors. In this section we will look at probability density functions and computing the mean (think average wait in line or average life span of a light blub) of a probability density function.

Section 2-1 : Arc Length

1. Set up, but do not evaluate, an integral for the length of $y = \sqrt{x+2}$, $1 \le x \le 7$ using,

(a) $ds = \sqrt{1 + \left[\dfrac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\dfrac{dx}{dy}\right]^2}\, dy$

(a) $ds = \sqrt{1 + \left[\dfrac{dy}{dx}\right]^2}\, dx$

Step 1

We'll need the derivative of the function first.

$$\frac{dy}{dx} = \frac{1}{2}\left(x+2\right)^{-\frac{1}{2}} = \frac{1}{2\left(x+2\right)^{\frac{1}{2}}}$$

Step 2

Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + \left[\frac{1}{2\left(x+2\right)^{\frac{1}{2}}}\right]^2}\, dx = \sqrt{1 + \frac{1}{4\left(x+2\right)}}\, dx = \sqrt{\frac{4x+9}{4\left(x+2\right)}}\, dx$$

Step 3

All we need to do now is set up the integral for the arc length. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$L = \int ds = \boxed{\int_1^7 \sqrt{\frac{4x+9}{4x+8}}\, dx}$$

(b) $ds = \sqrt{1 + \left[\dfrac{dx}{dy}\right]^2}\, dy$

Step 1

In this case we first need to solve the function for *x* so we can compute the derivative in the *ds*.

$$y = \sqrt{x+2} \qquad \to \qquad x = y^2 - 2$$

The derivative of this is,

$$\frac{dx}{dy} = 2y$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1 + \left[2y\right]^2}\,dy = \sqrt{1 + 4y^2}\,dy$$

Step 3
Next, note that the *ds* has a *dy* in it and so we'll need *y* limits of integration.

We are only given *x* limits in the problem statement. However, we can plug these into the function we were given in the problem statement to convert them to *y* limits. Doing this gives,

$$x = 1 : y = \sqrt{3} \qquad\qquad x = 7 : y = \sqrt{9} = 3$$

So, the corresponding *y* limits are : $\sqrt{3} \le y \le 3$.

Step 4
Finally, all we need to do is set up the integral.

$$L = \int ds = \boxed{\int_{\sqrt{3}}^{3} \sqrt{1 + 4y^2}\,dy}$$

---

2. Set up, but do not evaluate, an integral for the length of $x = \cos(y)$, $0 \le x \le \frac{1}{2}$ using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$

Step 1

In this case we first need to solve the function for *y* so we can compute the derivative in the *ds*.

$$x = \cos\left(y\right) \qquad \to \qquad y = \cos^{-1}\left(x\right) = \arccos\left(x\right)$$

Which notation you use for the inverse tangent is not important since it will be "disappearing" once we take the derivative.

Speaking of which, here is the derivative of the function.

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx = \sqrt{1+\left[-\frac{1}{\sqrt{1-x^2}}\right]^2}\,dx = \sqrt{1+\frac{1}{1-x^2}}\,dx = \sqrt{\frac{2-x^2}{1-x^2}}\,dx$$

Step 3
All we need to do now is set up the integral for the arc length. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$L = \int ds = \boxed{\int_0^{\frac{1}{2}} \sqrt{\frac{2-x^2}{1-x^2}}\,dx}$$

(b) $ds = \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

Step 1
We'll need the derivative of the function first.

$$\frac{dx}{dy} = -\sin\left(y\right)$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1+\left[-\sin\left(y\right)\right]^2}\,dy = \sqrt{1+\sin^2\left(y\right)}\,dy$$

Step 3

Next, note that the *ds* has a *dy* in it and so we'll need *y* limits of integration.

We are only given *x* limits in the problem statement. However, in part **(a)** we solved the function for *y* to get,

$$y = \cos^{-1}\left(x\right) = \arccos\left(x\right)$$

and all we need to do is plug *x* limits we were given into this to convert them to *y* limits. Doing this gives,

$$\begin{aligned} x = 0: &\qquad y = \cos^{-1}\left(0\right) = \arccos\left(0\right) = \frac{\pi}{2} \\ x = \frac{1}{2}: &\qquad y = \cos^{-1}\left(\frac{1}{2}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \end{aligned}$$

So, the corresponding *y* limits are : $\frac{\pi}{3} \le y \le \frac{\pi}{2}$.

Note that we used both notations for the inverse cosine here but you only need to use the one you are comfortable with. Also, recall that we know that the range of the inverse cosine function is,

$$0 \le \cos^{-1}\left(x\right) \le \pi$$

Therefore, there is only one possible value of *y* that we can get out of each value of *x*.

Step 4
Finally, all we need to do is set up the integral.

$$L = \int ds = \boxed{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1 + \sin^2\left(y\right)}\, dy}$$

---

3. Determine the length of $y = 7\left(6 + x\right)^{\frac{3}{2}}$, $189 \le y \le 875$.

Step 1
Since we are not told which *ds* to use we will have to decide which one to use. In this case the function is set up to use the *ds* in terms of *x*. Note as well that if we solve the function for *x* (which we'd need to do in order to use the *ds* that is in terms of *y*) we would still have a fractional exponent and the derivative will not work out as nice once we plug it into the *ds* formula.

So, let's take the derivative of the given function and plug into the *ds* formula.

$$\frac{dy}{dx} = \frac{21}{2}\left(6+x\right)^{\frac{1}{2}}$$

$$\begin{aligned} ds &= \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx = \sqrt{1+\left[\frac{21}{2}\left(6+x\right)^{\frac{1}{2}}\right]^2}\,dx = \sqrt{1+\frac{441}{4}\left(6+x\right)}\,dx \\ &= \sqrt{\frac{2650}{4}+\frac{441}{4}x}\,dx = \frac{1}{2}\sqrt{2650+441x}\,dx \end{aligned}$$

We did a little simplification that may or may not make the integration easier. That will probably depend upon the person doing the integration and just what they find the easiest to deal with. The point is there are several forms of the *ds* that we could use here. All will give the same answer.

Step 2
Next, we need to deal with the limits for the integral. The *ds* that we choose to use in the first step has a *dx* in it and that means that we'll need *x* limits for our integral. We, however, were given *y* limits in the problem statement. This means we'll need to convert those to *x*'s before proceeding with the integral.

To do convert these all we need to do is plug them into the function we were given in the problem statement and solve for the corresponding *x*. Doing this gives,

$$\begin{aligned} y = 189: &\qquad 189 = 7\left(6+x\right)^{\frac{3}{2}} \quad\to\quad 6+x = 27^{\frac{2}{3}} = 9 \quad\to\quad x = 3 \\ y = 875: &\qquad 875 = 7\left(6+x\right)^{\frac{3}{2}} \quad\to\quad 6+x = 125^{\frac{2}{3}} = 25 \quad\to\quad x = 19 \end{aligned}$$

So, the corresponding ranges of *x*'s is : $3 \le x \le 19$.

Step 3
The integral giving the arc length is then,

$$L = \int ds = \int_3^{19} \tfrac{1}{2}\sqrt{2650+441x}\,dx$$

Step 4
Finally, all we need to do is evaluate the integral. In this case all we need to do is use a quick Calc I substitution. We'll leave most of the integration details to you to verify.

The arc length of the curve is,

$$L = \int_3^{19} \tfrac{1}{2}\sqrt{2650+441x}\,dx = \tfrac{1}{1323}\left(2650+441x\right)^{\frac{3}{2}}\Big|_3^{19} = \boxed{\tfrac{1}{1323}\left(11029^{\frac{3}{2}} - 3973^{\frac{3}{2}}\right) = 686.1904}$$

4. Determine the length of $x = 4\left(3+y\right)^2$, $1 \le y \le 4$.

Step 1
Since we are not told which *ds* to use we will have to decide which one to use. In this case the function is set up to use the *ds* in terms of *y*.

If we were to solve the function for *y* (which we'd need to do in order to use the *ds* that is in terms of *x*) we would put a square root into the function and those can be difficult to deal with in arc length problems.

So, let's take the derivative of the given function and plug into the *ds* formula.

$$\frac{dx}{dy} = 8\left(3+y\right)$$

$$ds = \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1+\left[8\left(3+y\right)\right]^2}\,dy = \sqrt{1+64\left(3+y\right)^2}\,dy$$

Note that we did not square out the term under the root. Doing that would greatly complicate the integration process so we'll need to leave it as it is.

Step 2
In this case we don't need to anything special to get the limits for the integral. Our choice of *ds* contains a *dy* which means we need *y* limits for the integral and nicely enough that is what we were given in the problem statement.

So, the integral giving the arc length is,

$$L = \int ds = \int_1^4 \sqrt{1+64\left(3+y\right)^2}\,dy$$

Step 3
Finally, all we need to do is evaluate the integral. In this case all we need to do is use a trig substitution. We'll not be putting a lot of explanation into the integration work so if you need a little refresher on trig substitutions you should go back to that section and work a few practice problems.

The substitution we'll need is,

$$3+y = \tfrac{1}{8}\tan\theta \qquad \to \qquad dy = \tfrac{1}{8}\sec^2\theta\,d\theta$$

In order to properly deal with the square root we'll need to convert the *y* limits to θ limits. Here is that work.

$$y = 1: \quad 4 = \tfrac{1}{8}\tan\theta \quad \to \quad \tan\theta = 32 \quad \to \quad \theta = \tan^{-1}\left(32\right) = 1.5396$$
$$y = 4: \quad 7 = \tfrac{1}{8}\tan\theta \quad \to \quad \tan\theta = 56 \quad \to \quad \theta = \tan^{-1}\left(56\right) = 1.5529$$

Now let's deal with the square root.

$$\sqrt{1 + 64\left(3 + y\right)^2} = \sqrt{1 + 64\left(\tfrac{1}{8}\tan\theta\right)^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right|$$

From the work above we know that θ is in the range $1.5396 \le \theta \le 1.5529$. This is in the first and fourth quadrants and cosine (and hence secant) is positive in this range. So,

$$\sqrt{1 + 64\left(3 + y\right)^2} = \sec\theta$$

Putting all of this together gives,

$$L = \int_1^4 \sqrt{1 + 64\left(3 + y\right)^2}\, dy = \tfrac{1}{8}\int_{1.5396}^{1.5529} \sec^3\theta \, d\theta$$

Evaluating the integral gives,

$$L = \int_1^4 \sqrt{1 + 64\left(3 + y\right)^2}\, dy = \tfrac{1}{16}\left(\tan\theta\sec\theta + \ln\left|\tan\theta + \sec\theta\right|\right)\Big|_{1.5396}^{1.5529} = \boxed{130.9570}$$

Note that if you used more decimal places than four here (the standard number of decimal places that we tend to use for these problems) you may have gotten a slightly different answer. Using a computer to get an "exact" answer gives 132.03497085.

These kinds of different answers can be a real issues with these kinds of problems and illustrates the potential problems if you round numbers too much.

Of course, there is also the problem of often not knowing just how many decimal places are needed to get an "accurate" answer. In many cases 4 decimal places is sufficient but there are cases (such as this one) in which that is not enough. Often the best bet is to simply use as many decimal places as you can to have the best chance of getting an "accurate" answer.

Section 2-2 : Surface Area

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $x = \sqrt{y+5}$, $\sqrt{5} \le x \le 3$ about the *y*-axis using,

(a) $ds = \sqrt{1 + \left[\dfrac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\dfrac{dx}{dy}\right]^2}\, dy$

(a) $ds = \sqrt{1 + \left[\dfrac{dy}{dx}\right]^2}\, dx$

Step 1
In this case we first need to solve the function for *y* so we can compute the derivative in the *ds*.

$$x = \sqrt{y+5} \qquad \to \qquad y = x^2 - 5$$

The derivative of this is,

$$\frac{dy}{dx} = 2x$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + \left[2x\right]^2}\, dx = \sqrt{1 + 4x^2}\, dx$$

Step 3
Finally, all we need to do is set up the integral. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi x\, ds = \boxed{\int_{\sqrt{5}}^{3} 2\pi x \sqrt{1 + 4x^2}\, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *y*-axis and so we needed an *x* in the integral.

As an aside, note that the *ds* we chose to use here is technically immaterial. Realistically however, one *ds* may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

(b) $ds = \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

Step 1
We'll need the derivative of the function first.

$$\frac{dx}{dy} = \frac{1}{2}\left(y+5\right)^{-\frac{1}{2}} = \frac{1}{2\left(y+5\right)^{\frac{1}{2}}}$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1+\left[\frac{1}{2\left(y+5\right)^{\frac{1}{2}}}\right]^2}\,dy = \sqrt{1+\frac{1}{4\left(y+5\right)}}\,dy = \sqrt{\frac{4y+21}{4\left(y+5\right)}}\,dy$$

Step 3
Next, note that the *ds* has a *dy* in it and so we'll need *y* limits of integration.

We are only given *x* limits in the problem statement. However, we can plug these into the function we derived in Step 1 of the first part to convert them to *y* limits. Doing this gives,

$$x = \sqrt{5} : y = 0 \qquad\qquad x = 3 : y = 4$$

So, the corresponding *y* limits are : $0 \le y \le 4$.

Step 4
Finally, all we need to do is set up the integral.

$$SA = \int 2\pi x\,ds = \int_0^4 2\pi x\sqrt{\frac{4y+21}{4\left(y+5\right)}}\,dy = \int_0^4 2\pi\sqrt{y+5}\sqrt{\frac{4y+21}{4\left(y+5\right)}}\,dy$$
$$= \int_0^4 2\pi\sqrt{y+5}\,\frac{\sqrt{4y+21}}{2\sqrt{y+5}}\,dy = \boxed{\int_0^4 \pi\sqrt{4y+21}\,dy}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *y*-axis and so we needed an *x* in the integral.

Note that with the *ds* we were told to use for this part we had a *dy* in the final integral and that means that all the variables in the integral need to be *y*'s. This means that the *x* from the formula needs to be converted into *y*'s as well. Luckily this is easy enough to do since we were given the formula for *x* in terms of *y* in the problem statement.

Finally, make sure you simplify these as much as possible as we did here. Had we not taken the square root of the numerator and denominator of the rational expression we would not have seen the cancelation that can happen there. Without that cancelation the integral would be much more difficult to do!

As an aside, note that the *ds* we chose to use here is technically immaterial. Realistically however, one *ds* may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

---

2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the *x*-axis using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

Step 1
We'll need the derivative of the function first.

$$\frac{dy}{dx} = 2\cos(2x)$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + \left[2\cos(2x)\right]^2}\, dx = \sqrt{1 + 4\cos^2(2x)}\, dx$$

Step 3
Finally, all we need to do is set up the integral. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi y\, ds = \int_0^{\frac{\pi}{8}} 2\pi y \sqrt{1 + 4\cos^2(2x)}\, dx = \boxed{\int_0^{\frac{\pi}{8}} 2\pi \sin(2x)\sqrt{1 + 4\cos^2(2x)}\, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *x*-axis and so we needed a *y* in the integral.

Note that with the *ds* we were told to use for this part we had a *dx* in the final integral and that means that all the variables in the integral need to be *x*'s. This means that the *y* from the formula needs to be converted into *x*'s as well. Luckily this is easy enough to do since we were given the formula for *y* in terms of *x* in the problem statement.

As an aside, note that the *ds* we chose to use here is technically immaterial. Realistically however, one *ds* may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

Step 1
In this case we first need to solve the function for *x* so we can compute the derivative in the *ds*.

$$y = \sin\left(2x\right) \qquad \to \qquad x = \tfrac{1}{2}\sin^{-1}\left(y\right)$$

The derivative of this is,

$$\frac{dx}{dy} = \frac{1}{2}\frac{1}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}$$

Step 2
Plugging this into the formula for *ds* gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy = \sqrt{1 + \left[\frac{1}{2\sqrt{1-y^2}}\right]^2}\, dy = \sqrt{1 + \frac{1}{4\left(1-y^2\right)}}\, dy = \sqrt{\frac{5-4y^2}{4\left(1-y^2\right)}}\, dy$$

Step 3
Next, note that the *ds* has a *dy* in it and so we'll need *y* limits of integration.

We are only given *x* limits in the problem statement. However, we can plug these into the function we were given in the problem statement to convert them to *y* limits. Doing this gives,

$$x = 0 : y = \sin\left(0\right) = 0 \qquad\qquad x = \tfrac{\pi}{8} : y = \sin\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2}$$

So, the corresponding *y* limits are : $0 \le y \le \frac{\sqrt{2}}{2}$.

Step 4
Finally, all we need to do is set up the integral.

$$SA = \int 2\pi y\,ds = \boxed{\int_0^{\frac{\sqrt{2}}{2}} 2\pi y\sqrt{\frac{5-4y^2}{4-4y^2}}\,dy}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *x*-axis and so we needed an *y* in the integral.

Also note that the *ds* we chose to use is technically immaterial. Realistically one *ds* may be easier than the other to work with but technically either could be used.

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3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = x^3 + 4$, $1 \le x \le 5$ about the given axis. You can use either *ds*.

(a) x-axis

(b) *y*-axis

(a) *x*-axis

Step 1
We are told that we can use either *ds* here and the function seems to be set up to use the following *ds*.

$$ds = \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$$

Note that we could use the other *ds* if we wanted to. However, that would require us to solve the equation for *x* in terms of *y*. That would, in turn, would give us fractional exponents that would make the derivatives and hence the integral potentially messier.

Therefore, we'll go with our first choice of *ds*.

Step 2
Now we'll need the derivative of the function.

$$\frac{dy}{dx} = 3x^2$$

Plugging this into the formula for our choice of *ds* gives,

$$ds = \sqrt{1+\left[3x^2\right]^2}\,dx = \sqrt{1+9x^4}\,dx$$

Step 3

Finally, all we need to do is set up the integral. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi y\, ds = \int_1^5 2\pi y \sqrt{1+9x^4}\, dx = \boxed{\int_1^5 2\pi \left(x^3+4\right)\sqrt{1+9x^4}\, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *x*-axis and so we needed a *y* in the integral.

Finally, with the *ds* we choose to use for this part we had a *dx* in the final integral and that means that all the variables in the integral need to be *x*'s. This means that the *y* from the formula needs to be converted into *x*'s as well. Luckily this is easy enough to do since we were given the formula for *y* in terms of *x* in the problem statement.

(b) *y*-axis

Step 1
We are told that we can use either *ds* here and the function seems to be set up to use the following *ds* for the same reasons we choose it in the first part.

$$ds = \sqrt{1+\left[\frac{dy}{dx}\right]^2}\, dx$$

Step 2
Now, as with the first part of this problem we'll need the derivative of the function and the *ds*. Here is that work again for reference purposes.

$$\frac{dy}{dx} = 3x^2 \qquad\qquad ds = \sqrt{1+\left[3x^2\right]^2}\, dx = \sqrt{1+9x^4}\, dx$$

Step 3
Finally, all we need to do is set up the integral. Also note that we have a *dx* in the formula for *ds* and so we know that we need *x* limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi x\, ds = \boxed{\int_1^5 2\pi x \sqrt{1+9x^4}\, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the *y*-axis and so we needed an *x* in the integral.

In this part, unlike the first part, we do not do any substitution for the *x* in front of the root. Our choice of *ds* for this part put a *dx* into the integral and this means we need *x*'s the integral. Since the variable in front of the root was an *x* we don't need to do any substitution for the variable.

4. Find the surface area of the object obtained by rotating $y = 4 + 3x^2$, $1 \le x \le 2$ about the *y*-axis.

Step 1
The first step here is to decide on a *ds* to use for the problem. We can use either one, however the function is set up for,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Using the other *ds* will put fractional exponents into the function and make the *ds* and integral potentially messier so we'll stick with this *ds*.

Step 2
Let's now set up the *ds*.

$$\frac{dy}{dx} = 6x \qquad \Rightarrow \qquad ds = \sqrt{1 + [6x]^2}\, dx = \sqrt{1 + 36x^2}\, dx$$

Step 3
The integral for the surface area is,

$$SA = \int 2\pi x\, ds = \int_1^2 2\pi x \sqrt{1 + 36x^2}\, dx$$

Note that because we are rotating the function about the *y*-axis for this problem we need an *x* in front of the root. Also note that because our choice of *ds* puts a *dx* in the integral we need *x* limits of integration which we were given in the problem statement.

Step 4
Finally, all we need to do is evaluate the integral. That requires a quick Calc I substitution. We'll leave most of the integration details to you to verify since you should be pretty good at Calc I substitutions by this point.

$$SA = \int_1^2 2\pi x \sqrt{1 + 36x^2}\, dx = \tfrac{\pi}{54}\left(1 + 36x^2\right)^{\frac{3}{2}}\Big|_1^2 = \boxed{\tfrac{\pi}{54}\left(145^{\frac{3}{2}} - 37^{\frac{3}{2}}\right) = 88.4864}$$

---

5. Find the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the *x*-axis.

Step 1
Note that we actually set this problem up in Part **(a)** of Problem 2. So, we'll just summarize the steps of the set up part of the problem here. If you need to see all the details please check out the work in Problem 2.

Here is *ds* for this problem.

$$\frac{dy}{dx} = 2\cos\left(2x\right) \qquad \Rightarrow \qquad ds = \sqrt{1+4\cos^2\left(2x\right)}\,dx$$

The integral for the surface area is,

$$SA = \int_0^{\frac{\pi}{8}} 2\pi\sin\left(2x\right)\sqrt{1+4\cos^2\left(2x\right)}\,dx$$

Step 2

In order to evaluate this integral we'll need the following trig substitution.

$$\cos\left(2x\right) = \tfrac{1}{2}\tan\left(\theta\right) \qquad \rightarrow \qquad -2\sin\left(2x\right)dx = \tfrac{1}{2}\sec^2\left(\theta\right)d\theta$$
$$\sqrt{1+4\cos^2\left(2x\right)} = \sqrt{1+\tan^2\left(\theta\right)} = \sqrt{\sec^2\left(\theta\right)} = \left|\sec\left(\theta\right)\right|$$

In order to deal with the absolute value bars we'll need to convert the *x* limits to θ limits. Here's that work.

$$x = 0: \ \cos\left(0\right) = 1 = \tfrac{1}{2}\tan\left(\theta\right) \qquad \rightarrow \qquad \theta = \tan^{-1}\left(2\right) = 1.1071$$
$$x = \tfrac{\pi}{8}: \ \cos\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\tan\left(\theta\right) \qquad \rightarrow \qquad \theta = \tan^{-1}\left(\sqrt{2}\right) = 0.9553$$

The corresponding range of θ is $0.9553 \le \theta \le 1.1071$. This is in the first quadrant and secant is positive there. Therefore, we can drop the absolute value bars on the secant.

Step 3

Putting all the work from the previous step together gives,

$$SA = \int_0^{\frac{\pi}{8}} 2\pi\sin\left(2x\right)\sqrt{1+4\cos^2\left(2x\right)}\,dx = -\tfrac{\pi}{2}\int_{1.1071}^{0.9553}\sec^3\left(\theta\right)d\theta$$

Step 4

Using the formula for the integral of $\sec^3\left(\theta\right)$ we derived in the Integrals Involving Trig Functions we get,

$$SA = -\tfrac{\pi}{2}\int_{1.1071}^{0.9553}\sec^3\left(\theta\right)d\theta = -\tfrac{\pi}{4}\left[\sec\left(\theta\right)\tan\left(\theta\right) + \ln\left|\sec\left(\theta\right)+\tan\left(\theta\right)\right|\right]\Bigg|_{1.1071}^{0.9553} = \boxed{1.8215}$$

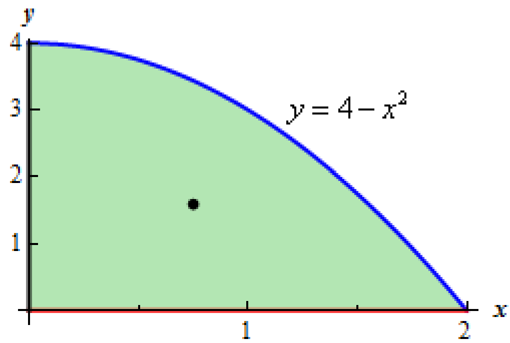
Note that depending upon the number of decimal places you used your answer may be slightly different from that give here. The "exact" answer, obtained by computer, is 1.8222.

Section 2-3 : Center of Mass

1. Find the center of mass for the region bounded by $y = 4 - x^2$ that is in the first quadrant.

Step 1

Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step.....).



We'll also need the area of this region so let's find that first.

$$A = \int_0^2 4 - x^2 \, dx = \left(4x - \tfrac{1}{3}x^3\right)\Big|_0^2 = \tfrac{16}{3}$$

Step 2

Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_0^2 \tfrac{1}{2}\left(4 - x^2\right)^2 dx = \int_0^2 \tfrac{1}{2}\left(16 - 8x^2 + x^4\right) dx = \tfrac{1}{2}\left(16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_0^2 = \tfrac{128}{15}$$

$$M_y = \int_0^2 x\left(4 - x^2\right) dx = \int_0^2 4x - x^3 \, dx = \left(2x^2 - \tfrac{1}{4}x^4\right)\Big|_0^2 = 4$$

Step 3

Finally, the coordinates of the center of mass is,

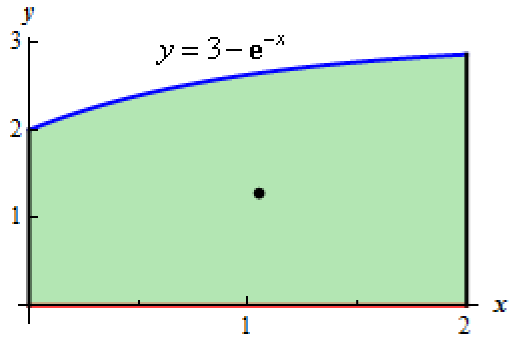
$$\overline{x} = \frac{M_y}{M} = \frac{\rho(4)}{\rho\left(\frac{16}{3}\right)} = \frac{3}{4} \qquad \overline{y} = \frac{M_x}{M} = \frac{\rho\left(\frac{128}{15}\right)}{\rho\left(\frac{16}{3}\right)} = \frac{8}{5}$$

The center of mass is then : $\boxed{\left(\frac{3}{4}, \frac{8}{5}\right)}$.

---

2. Find the center of mass for the region bounded by $y = 3 - \mathbf{e}^{-x}$, the *x*-axis, $x = 2$ and the *y*-axis.

Step 1
Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step…..).



We'll also need the area of this region so let's find that first.

$$A = \int_0^2 3 - \mathbf{e}^{-x}\,dx = \left(3x + \mathbf{e}^{-x}\right)\Big|_0^2 = 5 + \mathbf{e}^{-2}$$

Step 2
Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_0^2 \tfrac{1}{2}\left(3 - \mathbf{e}^{-x}\right)^2 dx = \int_0^2 \tfrac{1}{2}\left(9 - 6\mathbf{e}^{-x} + \mathbf{e}^{-2x}\right)dx = \tfrac{1}{2}\left(9x + 6\mathbf{e}^{-x} - \tfrac{1}{2}\mathbf{e}^{-2x}\right)\Big|_0^2 = \tfrac{25}{4} + 3\mathbf{e}^{-2} - \tfrac{1}{4}\mathbf{e}^{-4}$$
$$M_y = \int_0^2 x\left(3 - \mathbf{e}^{-x}\right)dx = \int_0^2 3x - x\,\mathbf{e}^{-x}\,dx = \left(\tfrac{3}{2}x^2 + x\mathbf{e}^{-x} + \mathbf{e}^{-x}\right)\Big|_0^2 = 5 + 3\mathbf{e}^{-2}$$

For the second term in the M_y integration we used the following integration by parts.

$$\int x\,\mathbf{e}^{-x}\,dx \qquad\qquad u = x \quad du = dx \qquad\qquad dv = \mathbf{e}^{-x}dx \quad v = -\mathbf{e}^{-x}$$
$$\int x\,\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} = -\left(x\mathbf{e}^{-x} + \mathbf{e}^{-x}\right)$$

The minus sign here canceled with the minus sign that was in front of the term in the full integral.

Make sure you don't forget integration by parts! It is a fairly common integration technique for these kinds of problems.

Step 3
Finally, the coordinates of the center of mass is,

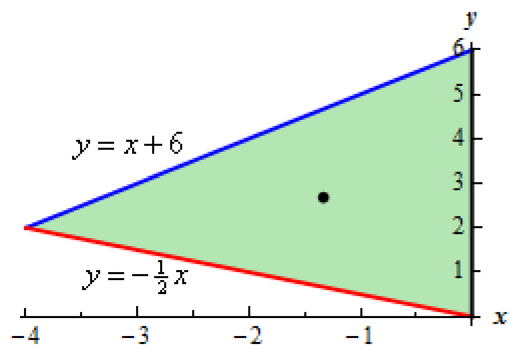
$$\bar{x} = \frac{M_y}{M} = \frac{\rho\left(5 + 3\mathbf{e}^{-2}\right)}{\rho\left(5 + \mathbf{e}^{-2}\right)} = 1.05271 \qquad\qquad \bar{y} = \frac{M_x}{M} = \frac{\rho\left(\tfrac{25}{4} + 3\mathbf{e}^{-2} - \tfrac{1}{4}\mathbf{e}^{-4}\right)}{\rho\left(5 + \mathbf{e}^{-2}\right)} = 1.29523$$

The center of mass is then : $\boxed{(1.05271,\ 1.29523)}$.

---

3. Find the center of mass for the triangle with vertices (0, 0), (-4, 2) and (0,6).

Step 1
Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step…..).



We'll leave it to you verify the equations of the upper and lower leg of the triangle.

We'll also need the area of this region so let's find that first.

$$A = \int_{-4}^{0} \left(x+6\right) - \left(-\tfrac{1}{2}x\right)dx = \int_{-4}^{0} \tfrac{3}{2}x + 6\,dx = \left(\tfrac{3}{4}x^2 + 6x\right)\Big|_{-4}^{0} = 12$$

Step 2
Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_{-4}^{0} \tfrac{1}{2}\left[\left(x+6\right)^2 - \left(-\tfrac{1}{2}x\right)^2\right]dx = \int_{-4}^{0} \tfrac{3}{8}x^2 + 6x + 18\,dx = \left(\tfrac{1}{8}x^3 + 3x^2 + 18x\right)\Big|_{-4}^{0} = 32$$
$$M_y = \int_{-4}^{0} x\left(\left(x+6\right) - \left(-\tfrac{1}{2}x\right)\right)dx = \int_{-4}^{0} \tfrac{3}{2}x^2 + 6x\,dx = \left(\tfrac{1}{2}x^3 + 3x^2\right)\Big|_{-4}^{0} = -16$$

Step 3
Finally, the coordinates of the center of mass is,

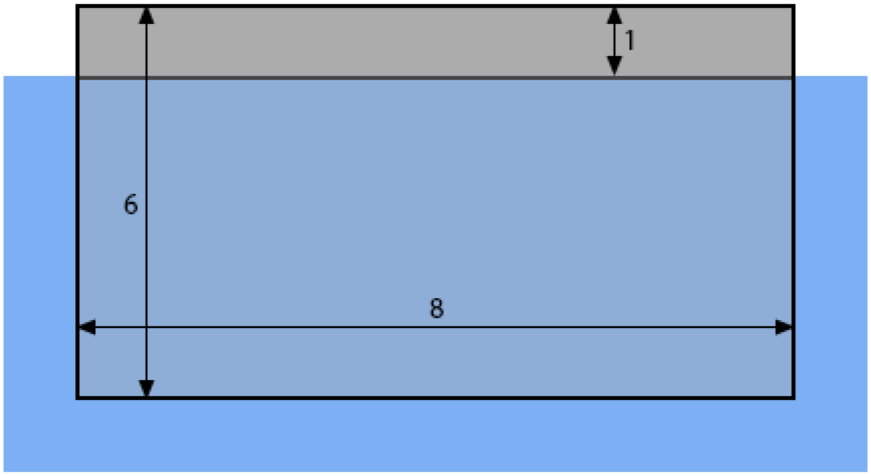
$$\overline{x} = \frac{M_y}{M} = \frac{\rho\left(-16\right)}{\rho\left(12\right)} = -\frac{4}{3} \qquad \overline{y} = \frac{M_x}{M} = \frac{\rho\left(32\right)}{\rho\left(12\right)} = \frac{8}{3}$$

The center of mass is then : $\boxed{\left(-\frac{4}{3}, \frac{8}{3}\right)}$.

Section 2-4 : Hydrostatic Pressure

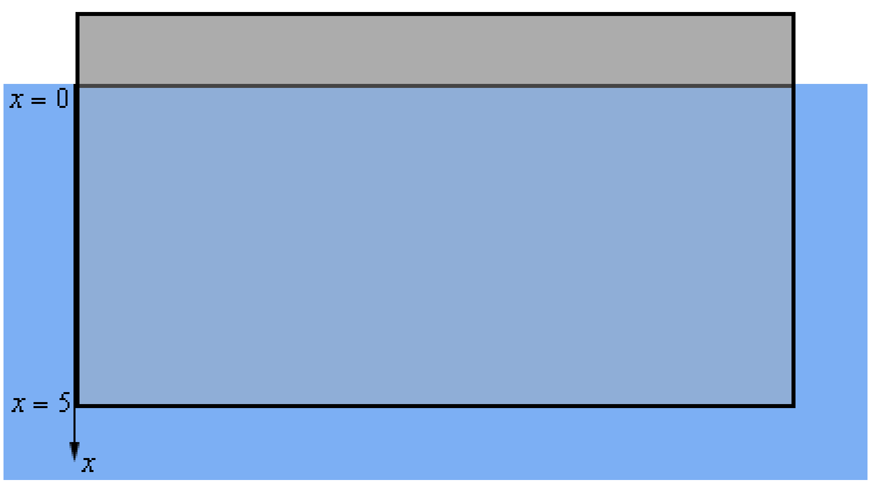
1. Find the hydrostatic force on the plate submerged in water as shown in the image below.

Consider the top of the blue “box” to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.



Hint : Start off by defining an “axis system” for the figure.

Step 1
The first thing we should do is define an axis system for the portion of the plate that is below the water.



Note that we started the *x*-axis at the surface of the water and by doing this *x* will give the depth of any point on the plate below the surface of the water. This in turn means that the bottom of the plate will be defined by $x = 5$.

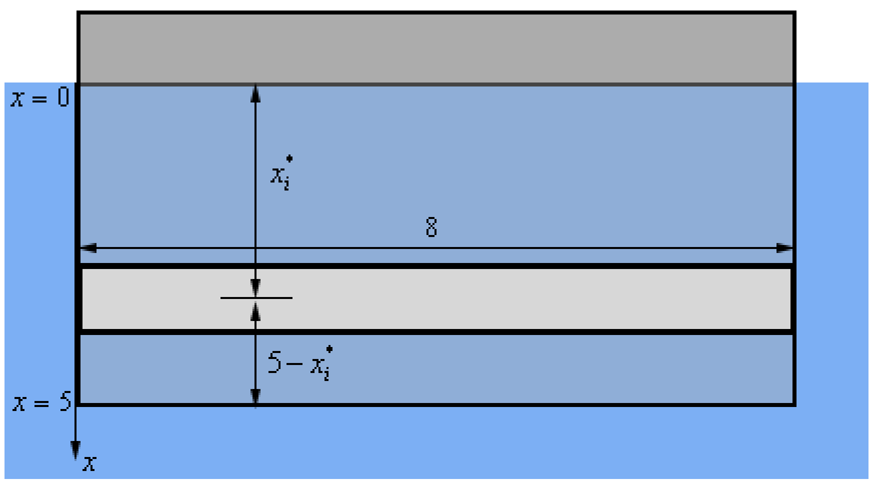
It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

Hint : At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

Step 2

As we did in the notes we'll break up the portion of the plate that is below the surface of the water into *n* horizontal strips of width Δx and we'll let each strip be defined by the interval $\left[x_{i-1}, x_i\right]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let x_i^* be any point that is in the interval and hence will be some point on the strip.

Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



Now x_i^* is a point from the interval defining the strip and so, for sufficiently thin strips, it is safe to assume that the strip will be at the point x_i^* below the surface of the water as shown in the figure above. In other words, the strip is a distance of x_i^* below the surface of the water.

Also, because our plate is a rectangle we know that each strip will have a width of 8.

Hint : What is the hydrostatic pressure and force on the representative strip?

Step 3
We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = (1000)(9.81)x_i^* = 9810x_i^*$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$F_i = P_i A_i = \left(9810x_i^*\right)\left[(8)(\Delta x)\right] = 78480x_i^*\Delta x$$

Hint : How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4
We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 78480x_i^*\Delta x$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting *n* go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 78480x_i^*\Delta x$$

Hint : You do recall the definition of the definite integral don't you?

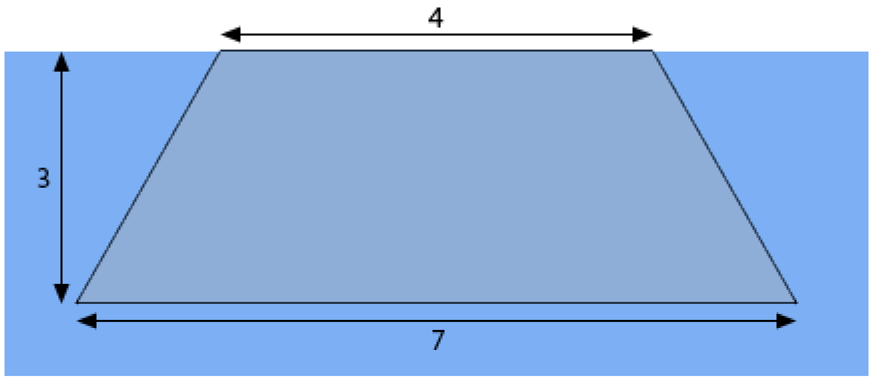
Step 5
Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$F = \int_0^5 78480x\,dx = 39240x^2\Big|_0^5 = \boxed{981{,}000N}$$

---

2. Find the hydrostatic force on the plate submerged in water as shown in the image below.

Consider the top of the blue "box" to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.



Hint : Start off by defining an "axis system" for the figure.

Step 1
The first thing we should do is define an axis system for the portion of the plate that is below the water.

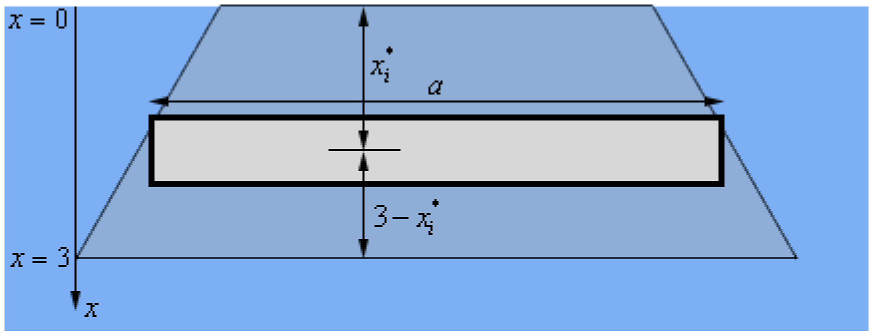
Note that we started the *x*-axis at the surface of the water and by doing this *x* will give the depth of any point on the plate below the surface of the water. This in turn means that the bottom of the plate will be defined by $x = 3$.

It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

Hint : At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

Step 2
As we did in the notes we'll break up the plate into *n* horizontal strips of width Δx and we'll let each strip be defined by the interval $\left[x_{i-1}, x_i\right]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let x_i^* be any point that is in the interval and hence will be some point on the strip.

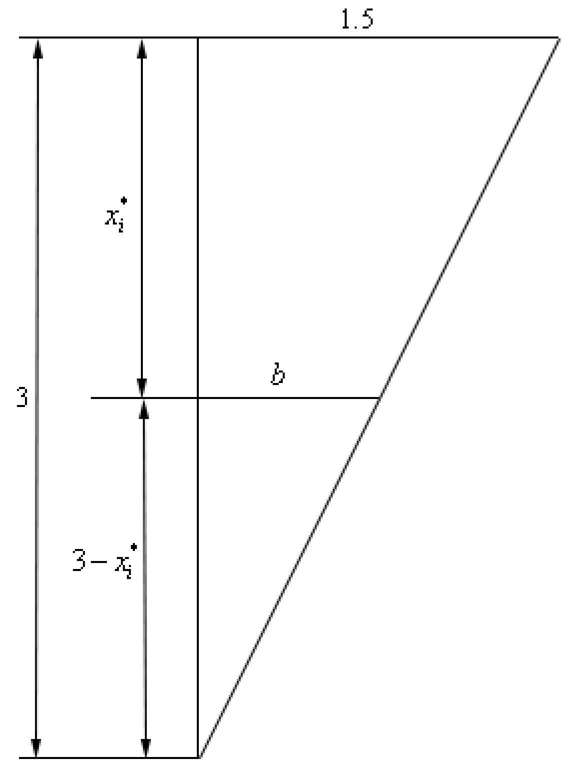
Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



Now x_i^* is a point from the interval defining the strip and so, for sufficiently thin strips, it is safe to assume that the strip will be at the point x_i^* below the surface of the water as shown in the figure above. In other words, the strip is a distance of x_i^* below the surface of the water.

The width of each of the strips will be dependent on the depth of the strip and so temporarily let's just call the width *a*.

To determine the value of *a* for each strip let's consider the following set of similar triangles.



This is the triangle of "empty" space to the left of the plate. The overall height of the larger triangle is the same as the plate, namely 3. The overall width of the larger triangle is 1.5. We arrived at this number by noticing that the top of the plate was 3 meters shorter than the bottom and if we assume the top was perfectly centered over the bottom there must be 1.5 meters of "empty" space to either side of the top.

The top of the smaller triangle corresponds to the strip on the plate. We'll call the width of the smaller triangle *b*. and the height of the smaller triangle must be $3-x_i^*$ for each strip.

Because the two triangles are similar triangles we have the following equation.

$$\frac{b}{3-x_i^*}=\frac{1.5}{3} \qquad\qquad b=\frac{1}{2}\left(3-x_i^*\right)$$

Note that while we looked only at the empty space to the left of the plate we'd get an almost identical triangle for the empty space to the right of the plate. The only exception would be that it would be a mirror image of this triangle.

Now, let's get back to the width of the strip in our picture of the plate. Assuming that the top is centered over the bottom of the plate we can see that we have to have,

$$a=7-2b=3-2\left(\tfrac{1}{2}\right)\left(3-x_i^*\right)=4+x_i^*$$

Hint : What is the hydrostatic pressure and force on the representative strip?

Step 3
We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = (1000)(9.81)x_i^* = 9810x_i^*$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$F_i = P_i A_i = \left(9810x_i^*\right)\left[\left(4 + x_i^*\right)(\Delta x)\right] = 9810\left[4x_i^* + \left(x_i^*\right)^2\right]\Delta x$$

Hint : How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4
We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 9810\left[4x_i^* + \left(x_i^*\right)^2\right]\Delta x$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting *n* go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 9810\left[4x_i^* + \left(x_i^*\right)^2\right]\Delta x$$

Hint : You do recall the definition of the definite integral don't you?

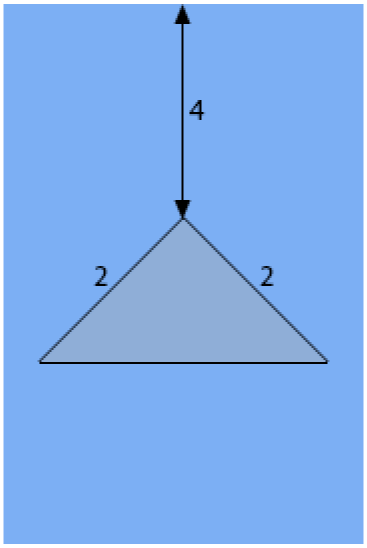
Step 5
Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$F = \int_0^3 9810\left(4x + x^2\right)dx = 9810\left(2x^2 + \tfrac{1}{3}x^3\right)\Big|_0^3 = \boxed{264{,}870N}$$

---

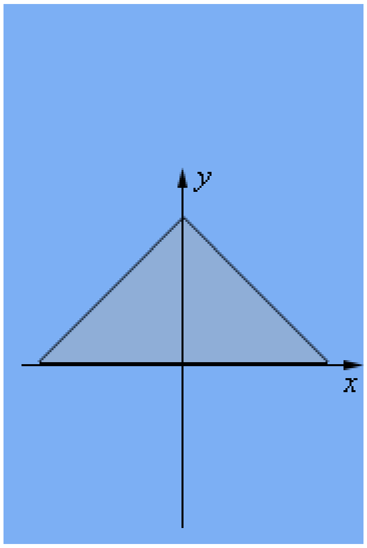
3. Find the hydrostatic force on the plate submerged in water as shown in the image below. The plate in this case is the top half of a diamond formed from a square whose sides have a length of 2.

Consider the top of the blue "box" to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.



Hint : Start off by defining an "axis system" for the figure.

Step 1
The first thing we should do is define an axis system for the portion of the plate that is below the water.



In this case since we had the top half a diamond formed from a square it seemed convenient to center the axis system in the middle of the diamond.

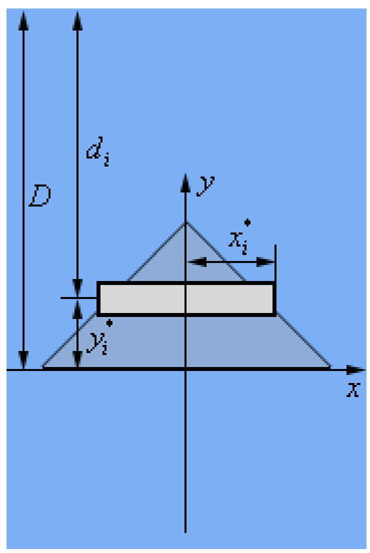
It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

Hint : At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

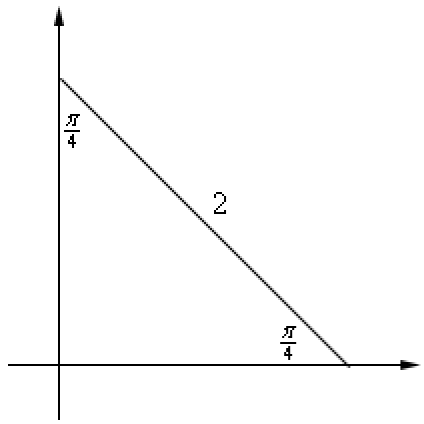
Step 2

As we did in the notes we'll break up the plate into *n* horizontal strips of width Δy and we'll let each strip be defined by the interval $\left[y_{i-1}, y_i\right]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let y_i^* be any point that is in the interval and hence will be some point on the strip.

Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



We've got a few quantities to determine at this point. To do that it would be convenient to have an equation for one of the sides of the plate. Let's take a look at the side in the first quadrant. Here is a quick sketch of that portion of the plate.



Now, as noted above we have the top half of a diamond formed from a square and so we can see that the "triangle" formed in the first quadrant by the plate must be an isosceles right triangle whose hypotenuse is 2 and the two interior angles other than the right angle must be $\frac{\pi}{4}$. Therefore, the bottom/left side of the triangle must be,

$$\text{side} = 2\cos\left(\tfrac{\pi}{4}\right) = \sqrt{2}$$

This means we know that the *x* and *y*-intercepts are $\left(\sqrt{2},0\right)$ and $\left(0,\sqrt{2}\right)$ respectively and so the equation for the line representing the hypotenuse must be,

$$y = \sqrt{2} - x$$

Okay, let's get the various quantities in the figure.

We'll start with x_i^*. This we can get directly from the equation above by acknowledging that if we are at x_i^* then the *y* value must be y_i^*. In other words, plugging these into the equation and solving gives,

$$y_i^* = \sqrt{2} - x_i^* \qquad \rightarrow \qquad x_i^* = \sqrt{2} - y_i^*$$

Notice as well that the width of each strip in terms of y_i^* is then,

$$2x_i^* = 2\left(\sqrt{2} - y_i^*\right)$$

Next, let's get *D*. First we can see that *D* is the distance from the surface of the water to the *x*-axis in our figure. We know that the distance from the surface of the water to the top of the plate is 4 meters. Also, we found above that the top point of the plate is a distance of $\sqrt{2}$ above the *x*-axis. So, we then have,

$$D = 4 + \sqrt{2}$$

Finally, the depth of each strip below the surface of the water is,

$$d_i = D - y_i^* = 4 + \sqrt{2} - y_i^*$$

Hint : What is the hydrostatic pressure and force on the representative strip?

Step 3
We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = \left(1000\right)\left(9.81\right)\left(4 + \sqrt{2} - y_i^*\right) = 9810\left(4 + \sqrt{2} - y_i^*\right)$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$\begin{aligned} F_i = P_i A_i &= \left[9810\left(4 + \sqrt{2} - y_i^*\right)\right]\left[\left(2x_i^*\right)\left(\Delta\, y\right)\right] \\ &= \left[9810\left(4 + \sqrt{2} - y_i^*\right)\right]\left[2\left(\sqrt{2} - y_i^*\right)\left(\Delta\, y\right)\right] \\ &= 19620\left(4 + \sqrt{2} - y_i^*\right)\left(\sqrt{2} - y_i^*\right)\left(\Delta\, y\right) \end{aligned}$$

Hint : How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4
We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 19620\left(4 + \sqrt{2} - y_i^*\right)\left(\sqrt{2} - y_i^*\right)\left(\Delta\, y\right)$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting *n* go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 19620\left(4 + \sqrt{2} - y_i^*\right)\left(\sqrt{2} - y_i^*\right)\left(\Delta\, y\right)$$

Hint : You do recall the definition of the definite integral don't you?

Step 5
Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$
\begin{aligned}
F &= \int_0^{\sqrt{2}} 19620\left(4+\sqrt{2}-y\right)\left(\sqrt{2}-y\right)dy \\
&= \int_0^{\sqrt{2}} 19620\left(y^2-\left(4+2\sqrt{2}\right)y+2+4\sqrt{2}\right)dy \\
&= 19620\left(\tfrac{1}{3}y^3-\left(2+\sqrt{2}\right)y^2+\left(2+4\sqrt{2}\right)y\right)\Big|_0^{\sqrt{2}} = \boxed{96,977.9N}
\end{aligned}
$$

Section 2-5 : Probability

1. Let,

$$f(x) = \begin{cases} \dfrac{3}{37760} x^2 (20 - x) & \text{if } 2 \le x \le 18 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(X \le 7)$.

(c) Find $P(X \ge 7)$.

(d) Find $P(3 \le X \le 14)$.

(e) Determine the mean value of X.

(a) Show that $f(x)$ is a probability density function.

Okay, to show that this function is a probability density function we can first notice that in the range $2 \le x \le 18$ the function is positive and will be zero everywhere else and so the first condition is satisfied.

The main thing that we need to do here is show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{2}^{18} \tfrac{3}{37760} x^2 (20 - x)\,dx \\ &= \tfrac{3}{37760} \int_{2}^{18} 20x^2 - x^3\,dx = \tfrac{3}{37760} \left(\tfrac{20}{3} x^3 - \tfrac{1}{4} x^4 \right) \Big|_{2}^{18} = 1 \end{aligned}$$

The integral is one and so this is in fact a probability density function.

(b) Find $P(X \le 7)$.

First note that because of our limits on *x* for which the function is not zero this is equivalent to $P(2 \le X \le 7)$. Here is the work for this problem.

$$P(X \le 7) = P(2 \le X \le 7) = \int_{2}^{7} \tfrac{3}{37760} x^2 (20 - x)\,dx = \tfrac{3}{37760} \left(\tfrac{20}{3} x^3 - \tfrac{1}{4} x^4 \right) \Big|_{2}^{7} = \boxed{0.130065}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(c) Find $P(X \ge 7)$.

First note that because of our limits on *x* for which the function is not zero this is equivalent to $P\left(7 \le X \le 18\right)$. Here is the work for this problem.

$$P\left(X \ge 7\right) = P\left(7 \le X \le 18\right) = \int_{7}^{18} \tfrac{3}{37760} x^2 \left(20 - x\right) dx = \tfrac{3}{37760}\left(\tfrac{20}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_{7}^{18} = \boxed{0.869935}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(d) Find $P\left(X \ge 7\right)$.

Not much to do here other than compute the integral.

$$P\left(3 \le X \le 14\right) = \int_{3}^{14} \tfrac{3}{37760} x^2 \left(20 - x\right) dx = \tfrac{3}{37760}\left(\tfrac{20}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_{3}^{14} = \boxed{0.677668}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(e) Determine the mean value of *X*.

For this part all we need to do is compute the following integral.

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x\, f\left(x\right) dx = \int_{2}^{18} x\left[\tfrac{3}{37760} x^2 \left(20 - x\right)\right] dx \\ &= \tfrac{3}{37760}\int_{2}^{18} 20x^3 - x^4\, dx = \tfrac{3}{37760}\left(5x^4 - \tfrac{1}{5}x^5\right)\Big|_{2}^{18} = 11.6705\end{aligned}$$

The mean value of *X* is then 11.6705.

---

2. For a brand of light bulb the probability density function of the life span of the light bulb is given by the function below, where *t* is in months.

$$f\left(t\right) = \begin{cases} 0.04\mathbf{e}^{-\frac{t}{25}} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

(a) Verify that $f\left(t\right)$ is a probability density function.
(b) What is the probability that a light bulb will have a life span less than 8 months?
(c) What is the probability that a light bulb will have a life span more than 20 months?
(d) What is the probability that a light bulb will have a life span between 14 and 30 months?
(e) Determine the mean value of the life span of the light bulbs.

(a) Show that $f\left(t\right)$ is a probability density function.

Okay, to show that this function is a probability density function we can first notice that the exponential portion is always positive regardless of the value of *t* we plug in and the remainder of the function is always zero and so the first condition is satisfied.

The main thing that we need to do here is show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_{0}^{n} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt$$
$$= \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{t}{25}}\right)\Bigg|_{0}^{n} = \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{n}{25}}+1\right) = 0+1 = 1$$

The integral is one and so this is in fact a probability density function.

For this integral do not forget to properly deal with the infinite limit! If you don't recall how to deal with these kinds of integrals go back to the [Improper Integral section] and do a quick review!

(b) What is the probability that a light bulb will have a life span less than 8 months?

What this problem is really asking us to compute is $P(X \le 8)$. Also, because of our limits on *t* for which the function is not zero this is equivalent to $P(0 \le X \le 8)$. Here is the work for this problem.

$$P(X \le 8) = P(0 \le X \le 8) = \int_{0}^{8} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = -\mathbf{e}^{-\frac{t}{25}}\Bigg|_{0}^{8} = \boxed{0.273851}$$

(c) What is the probability that a light bulb will have a life span more than 20 months?

What this problem is really asking us to compute is $P(X \ge 20)$. Here is the work for this problem.

$$P(X \ge 20) = \int_{20}^{\infty} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_{20}^{n} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt$$
$$= \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{t}{25}}\right)\Bigg|_{20}^{n} = \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{n}{25}}+\mathbf{e}^{-\frac{20}{25}}\right) = \boxed{0.449329}$$

For this integral do not forget to properly deal with the infinite limit! If you don't recall how to deal with these kinds of integrals go back to the [Improper Integral section] and do a quick review!

(d) What is the probability that a light bulb will have a life span between 14 and 30 months?

What this problem is really asking us to compute is $P(14 \le X \le 30)$. Here is the work for this problem.

$$P\left(14 \le X \le 30\right) = \int_{14}^{30} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = -\mathbf{e}^{-\frac{t}{25}}\Big|_{14}^{30} = \boxed{0.270015}$$

(e) Determine the mean value of the life span of the light bulbs.
For this part all we need to do is compute the following integral.

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} t\,f\left(t\right)dt = \int_{0}^{\infty} 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_{0}^{n} 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt \\ &= \lim_{n\to\infty}\left[-t\mathbf{e}^{-\frac{t}{25}} - 25\mathbf{e}^{-\frac{t}{25}}\right]\Bigg|_{0}^{n} = \lim_{n\to\infty}\left[-n\mathbf{e}^{-\frac{n}{25}} - 25\mathbf{e}^{-\frac{n}{25}} - \left(-25\right)\right] \\ &= \lim_{n\to\infty}\left[-\frac{n}{\mathbf{e}^{\frac{n}{25}}} - 25\mathbf{e}^{-\frac{n}{25}} + 25\right] = \lim_{n\to\infty}\left[-\frac{1}{\frac{1}{25}\mathbf{e}^{\frac{n}{25}}}\right] - 25\left(0\right) + 25 = \boxed{25}\end{aligned}$$

The mean value of the life span of the light bulbs is then 25 months.

We had to use integration by parts to do the integral. Here is that work if you need to see it.

$$\int 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt \qquad\qquad u = 0.04t \quad du = 0.04dt \qquad dv = \mathbf{e}^{-\frac{t}{25}}dt \quad v = -25\mathbf{e}^{-\frac{t}{25}}$$

$$\int 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt = -t\mathbf{e}^{-\frac{t}{25}} + \int \mathbf{e}^{-\frac{t}{25}}\,dt = -t\mathbf{e}^{-\frac{t}{25}} - 25\mathbf{e}^{-\frac{t}{25}}$$

Also, for the limit of the first term we used L'Hospital's Rule to do the limit.

---

3. Determine the value of *c* for which the function below will be a probability density function.

$$f\left(x\right) = \begin{cases} c\left(8x^3 - x^4\right) & \text{if } 0 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

Solution
This problem is actually easier than it might look like at first glance.

First, in order for the function to be a probability density function we know that the function must be positive or zero for all *x*. We can see that for $0 \le x \le 8$ we have $8x^3 - x^4 \ge 0$. Therefore, we need to require that whatever *c* is it must be a positive number.

To find *c* we'll use the fact that we must also have $\int_{-\infty}^{\infty} f\left(x\right)dx = 1$. So, let's compute this integral (with the *c* in the function) and see what we get.

$$\int_{-\infty}^{\infty} f\left(x\right) dx = \int_{0}^{8} c\left(8x^3 - x^4\right) dx = c\left(2x^4 - \tfrac{1}{5}x^5\right)\Big|_{0}^{8} = \tfrac{8192}{5}c$$

So, we can see that in order for this integral to have a value of 1 (as it must in order for the function to be a probability density function) we must have,

$$\boxed{c = \frac{5}{8192}}$$

and note that this is also a positive number as we determined earlier was required.

Chapter 3 : Parametric Equations and Polar Coordinates

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Parametric Equations and Curves – In this section we will introduce parametric equations and parametric curves (*i.e.* graphs of parametric equations). We will graph several sets of parametric equations and discuss how to eliminate the parameter to get an algebraic equation which will often help with the graphing process.

Tangents with Parametric Equations – In this section we will discuss how to find the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for parametric curves. We will also discuss using these derivative formulas to find the tangent line for parametric curves as well as determining where a parametric curve in increasing/decreasing and concave up/concave down.

Area with Parametric Equations – In this section we will discuss how to find the area between a parametric curve and the *x*-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus I techniques on the resulting algebraic equation).

Arc Length with Parametric Equations – In this section we will discuss how to find the arc length of a parametric curve using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Surface Area with Parametric Equations – In this section we will discuss how to find the surface area of a solid obtained by rotating a parametric curve about the *x* or *y*-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Polar Coordinates – In this section we will introduce polar coordinates an alternative coordinate system to the 'normal' Cartesian/Rectangular coordinate system. We will derive formulas to convert between polar and Cartesian coordinate systems. We will also look at many of the standard polar graphs as well as circles and some equations of lines in terms of polar coordinates.

Tangents with Polar Coordinates – In this section we will discuss how to find the derivative $\frac{dy}{dx}$ for polar curves. We will also discuss using this derivative formula to find the tangent line for polar curves using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Area with Polar Coordinates – In this section we will discuss how to the area enclosed by a polar curve. The regions we look at in this section tend (although not always) to be shaped vaguely like a piece of pie or pizza and we are looking for the area of the region from the outer boundary (defined by the polar equation) and the origin/pole. We will also discuss finding the area between two polar curves.

Arc Length with Polar Coordinates – In this section we will discuss how to find the arc length of a polar curve using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Surface Area with Polar Coordinates – In this section we will discuss how to find the surface area of a solid obtained by rotating a polar curve about the *x* or *y*-axis using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Arc Length and Surface Area Revisited – In this section we will summarize all the arc length and surface area formulas we developed over the course of the last two chapters.

Section 3-1 : Parametric Equations and Curves

1. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = 4 - 2t \qquad y = 3 + 6t - 4t^2$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we can do that by solving the *x* equation for *t* and plugging that into the *y* equation.

Doing that gives (we'll leave it to you to verify all the algebra bits…),

$$t = \tfrac{1}{2}\left(4 - x\right) \qquad \to \qquad y = 3 + 6\left[\tfrac{1}{2}\left(4 - x\right)\right] - 4\left[\tfrac{1}{2}\left(4 - x\right)\right]^2 = -x^2 + 5x - 1$$

Step 2

Okay, from this it looks like we have a parabola that opens downward. To sketch the graph of this we'll need the *x*-intercepts, *y*-intercept and most importantly the vertex.

For notational purposes let's define $f\left(x\right) = -x^2 + 5x - 1$.

The *x*-intercepts are then found by solving $f\left(x\right) = 0$. Doing this gives,

$$-x^2 + 5x - 1 = 0 \qquad \to \qquad x = \frac{-5 \pm \sqrt{\left(5\right)^2 - 4\left(-1\right)\left(-1\right)}}{2\left(-1\right)} = \frac{5 \pm \sqrt{21}}{2} = 0.2087,\ 4.7913$$

The *y*-intercept is : $\left(0, f\left(0\right)\right) = \left(0, -1\right)$.

Finally, the vertex is,

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{-5}{2\left(-1\right)}, f\left(\frac{5}{2}\right)\right) = \left(\frac{5}{2}, \frac{21}{4}\right)$$

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of *t* into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations.

t	x	y
-1	6	-7
0	4	3
$\frac{3}{4}$	$\frac{5}{2}$	$\frac{21}{4}$
1	2	5
2	0	-1
3	-2	-15

Note that $t = \frac{3}{4}$ is the value of *t* that give the vertex of the parabola and is not an obvious value of *t* to use! In fact, this is a good example of why just using values of *t* to sketch the graph is such a bad way of getting the sketch of a parametric curve. It is often very difficult to determine a good set of *t*'s to use.

For this table we first found the vertex *t* by using the fact that we actually knew the coordinates of the vertex (the *x*-coordinate for this example was the important one) as follows,

$$x = \frac{5}{2} \ : \quad \frac{5}{2} = 4 - 2t \qquad \to \qquad t = \frac{3}{4}$$

Once this value of *t* was found we chose several values of *t* to either side for a good representation of *t* for our sketch.

Note that, for this case, we used the *x*-coordinates to find the value of the *t* that corresponds to the vertex because this equation was a linear equation and there would be only one solution for *t*. Had we used the *y*-coordinate we would have had to solve a quadratic (not hard to do of course) that would have resulted in two *t*'s. The problem is that only one *t* gives the vertex for this problem and so we'd need to then check them in the *x* equation to determine the correct one. So, in this case we might as well just go with the *x* equation from the start.

Also note that there is an easier way (probably – it will depend on you of course) to determine direction of motion. Take a quick look at the *x* equation.

$$x = 4 - 2t$$

Because of the minus sign in front of the *t* we can see that as *t* increases *x* must decrease (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from right to left as the table of values above in the table also indicates.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of *t* for a table.

Step 4
We could sketch the graph at this point, but let's first get any limits on *x* and *y* that might exist.

Because we have a parabola that opens downward and we've not restricted *t*'s in any way we know that we'll get the whole parabola. This in turn means that we won't have any limits at all on *x* but *y* must satisfy $y \le \frac{21}{4}$ (remember the *y*-coordinate of the vertex?).

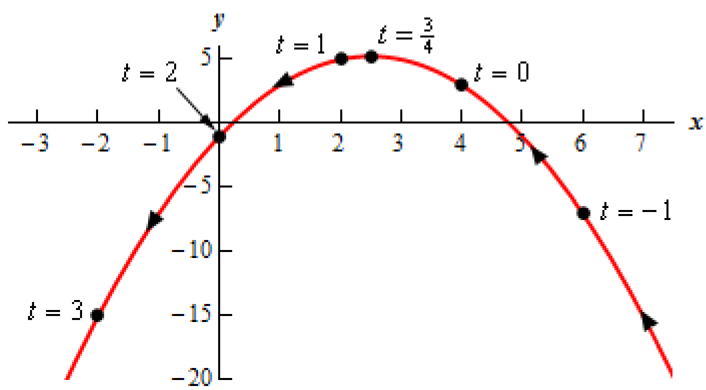
So, formally here are the limits on *x* and *y*.

$$-\infty < x < \infty \qquad\qquad y \le \tfrac{21}{4}$$

Note that having the limits on *x* and *y* will often help with the actual graphing step so it's often best to get them prior to sketching the graph. In this case they don't really help as we can sketch the graph of a parabola without these limits, but it's just good habit to be in so we did them first anyway.

Step 5

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

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2. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = 4 - 2t \qquad y = 3 + 6t - 4t^2 \qquad 0 \le t \le 3$$

Step 1

Before we get started on this problem we should acknowledge that this problem is really just a restriction on the first problem (*i.e.* it is the same problem except we restricted the values of *t* to use). As such we could just go back to the first problem and modify the sketch to match the restricted values of *t* to get a quick solution and in general that is how a problem like this would work.

However, we're going to approach this solution as if this was its own problem because we won't always have the more general problem worked ahead of time. So, let's proceed with the problem assuming we haven't worked the first problem in this section.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we can do that by solving the *x* equation for *t* and plugging that into the *y* equation.

Doing that gives (we'll leave it to you to verify all the algebra bits…),

$$t = \tfrac{1}{2}(4-x) \qquad \to \qquad y = 3 + 6\left[\tfrac{1}{2}(4-x)\right] - 4\left[\tfrac{1}{2}(4-x)\right]^2 = -x^2 + 5x - 1$$

Step 2

Okay, from this it looks like we have a parabola that opens downward. To sketch the graph of this we'll need the *x*-intercepts, *y*-intercept and most importantly the vertex.

For notational purposes let's define $f(x) = -x^2 + 5x - 1$.

The *x*-intercepts are then found by solving $f(x) = 0$. Doing this gives,

$$-x^2 + 5x - 1 = 0 \qquad \to \qquad x = \frac{-5 \pm \sqrt{(5)^2 - 4(-1)(-1)}}{2(-1)} = \frac{5 \pm \sqrt{21}}{2} = 0.2087,\ 4.7913$$

The *y*-intercept is : $(0, f(0)) = (0, -1)$.

Finally, the vertex is,

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{-5}{2(-1)}, f\left(\frac{5}{2}\right)\right) = \left(\frac{5}{2}, \frac{21}{4}\right)$$

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of *t* into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations. Also note that because we've restricted the value of *t* for this problem we need to keep that in mind as we chose values of *t* to use.

t	*x*	*y*
0	4	3
$\frac{3}{4}$	$\frac{5}{2}$	$\frac{21}{4}$
1	2	5
2	0	-1
3	-2	-15

Note that $t = \frac{3}{4}$ is the value of *t* that give the vertex of the parabola and is not an obvious value of *t* to use! In fact, this is a good example of why just using values of *t* to sketch the graph is such a bad way of getting the sketch of a parametric curve. It is often very difficult to determine a good set of *t*'s to use.

For this table we first found the vertex *t* by using the fact that we actually knew the coordinates of the vertex (the *x*-coordinate for this example was the important one) as follows,

$$x = \tfrac{5}{2} \quad : \quad \tfrac{5}{2} = 4 - 2t \qquad \to \qquad t = \tfrac{3}{4}$$

Once this value of *t* was found we chose several values of *t* to either side for a good representation of *t* for our sketch.

Note that, for this case, we used the *x*-coordinates to find the value of the *t* that corresponds to the vertex because this equation was a linear equation and there would be only one solution for *t*. Had we used the *y*-coordinate we would have had to solve a quadratic (not hard to do of course) that would have resulted in two *t*'s. The problem is that only one *t* gives the vertex for this problem and so we'd need to then check them in the *x* equation to determine the correct one. So, in this case we might as well just go with the *x* equation from the start.

Also note that there is an easier way (probably – it will depend on you of course) to determine direction of motion. Take a quick look at the *x* equation.

$$x = 4 - 2t$$

Because of the minus sign in front of the *t* we can see that as *t* increases *x* must decrease (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from right to left as the table of values above in the table also indicates.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of *t* for a table.

Step 4
Let's now get the limits on *x* and *y* and note that we really do need these before we start sketching the curve!

In this case we have a parabola that opens downward and we could use that to get a general set of limits on *x* and *y*. However, for this problem we've also restricted the values of *t* that we're using and that will in turn restrict the values of *x* and *y* that we can use for the sketch of the graph.

As we discussed above we know that the graph will sketch out from right to left and so the rightmost value of *x* will come from $t = 0$, which is $x = 4$. Likewise, the leftmost value of *y* will come from $t = 3$, which is $x = -2$. So, from this we can see the limits on *x* must be $-2 \le x \le 4$.

For the limits on the *y* we've got be a little more careful. First, we know that the vertex occurs in the given range of *t*'s and because the parabola opens downward the largest value of *y* we will have is

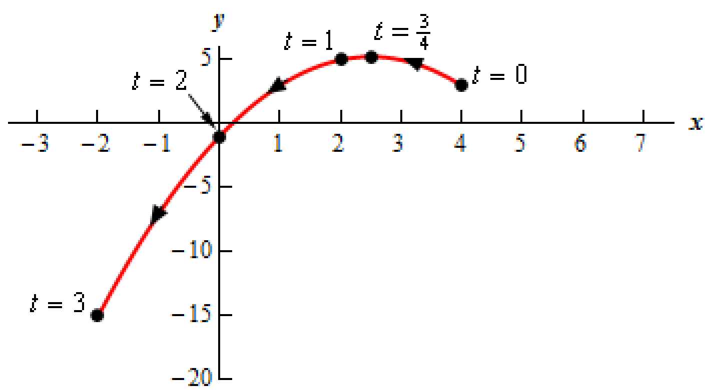
$y = \frac{21}{4}$, *i.e.* the *y*-coordinate of the vertex. Also, because the parabola opens downward we know that the smallest value of *y* will have to be at one of the endpoints. So, for $t = 0$ we have $y = 3$ and for $t = 3$ we have $y = -15$. Therefore, the limits on *y* must be $-15 \le y \le \frac{21}{4}$.

So, putting all this together here are the limits on *x* and *y*.

$$-2 < x < 4 \qquad\qquad -15 \le y \le \tfrac{21}{4}$$

Note that for this problem we must have these limits prior to the sketching step. Because we've restricted the values of *t* to use we will have limits on *x* and *y* (as we just discussed) and so we will only have a portion of the graph of the full parabola. Having these limits will allow us to get the sketch of the parametric curve.

Step 5
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

Also note that it is vitally important that we not extend the graph past the $t = 0$ and $t = 3$ points. If we extend the graph past these points we are implying that the graph will extend past them and of course it doesn't!

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3. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = \sqrt{t+1} \qquad y = \frac{1}{t+1} \qquad\qquad t > -1$$

Step 1
First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations that is actually really easy to do if we notice the following.

$$x = \sqrt{t+1} \qquad \Rightarrow \qquad x^2 = t+1$$

With this we can quickly convert the *y* equation to,

$$y = \frac{1}{x^2}$$

Step 2
At this point we can get limits on *x* and *y* pretty quickly so let's do that.

First, we know that square roots always return positive values (or zero of course) and so from the *x* equation we see that we must have $x > 0$. Note as well that this must be a strict inequality because the inequality restricting the range of *t*'s is also a strict inequality. In other words, because we aren't allowing $t = -1$ we will never get $x = 0$.

Speaking of which, you do see why we've restricted the *t*'s don't you?

Now, from our restriction on *t* we know that $t+1 > 0$ and so from the *y* parametric equation we can see that we also must have $y > 0$. This matches what we see from the equation without the parameter we found in Step 1.

So, putting all this together here are the limits on *x* and *y*.

$$x > 0 \qquad y > 0$$

Note that for this problem these limits are important (or at least the *x* limits are important). Because of the *x* limit we get from the parametric equation we can see that we won't have the full graph of the equation we found in the first step. All we will have is the portion that corresponds to $x > 0$.

Step 3
Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of *t* into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations.

t	*x*	*y*
-0.95	0.2236	20
-0.75	0.5	4
0	1	1
2	$\sqrt{3}$	$\frac{1}{3}$

Note that there is an easier way (probably – it will depend on you of course) to determine direction of motion. Take a quick look at the *x* equation.

$$x = \sqrt{t+1}$$

Increasing the value of *t* will also cause *t* + 1 to increase and the square root will also increase (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from left to right as the table of values above in the table supports.

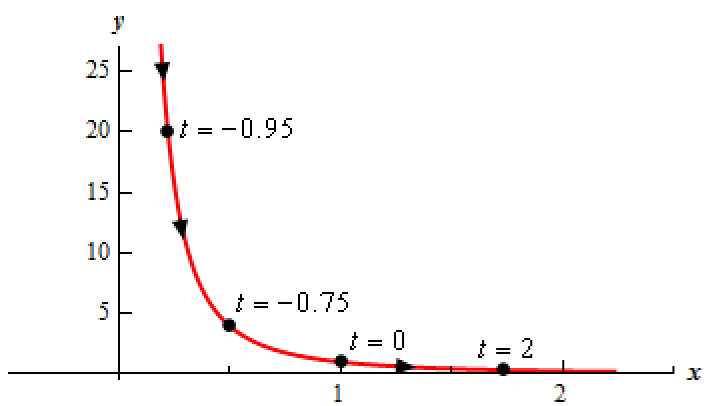
Likewise, we could use the *y* equation.

$$y = \frac{1}{t+1}$$

Again, we know that as *t* increases so does *t* + 1. Because the *t* + 1 is in the denominator we can further see that increasing this will cause the fraction, and hence *y*, to decrease. This means that the graph must be tracing out from top to bottom as both the *x* equation and table of values supports.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of *t* for a table.

Step 4
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

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4. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = 3\sin(t) \qquad y = -4\cos(t) \qquad\qquad 0 \le t \le 2\pi$$

Step 1
First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin(t) = \frac{x}{3} \qquad\qquad \cos(t) = -\frac{y}{4}$$

Plugging these into the trig identity gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad\qquad \Rightarrow \qquad\qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only *x* and *y* may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only *x* and *y*. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only *x* and *y* more than once.

The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2
At this point we can get a good idea on what the limits on *x* and *y* are going to be so let's do that. Note that often we won't get the actual limits on *x* and *y* in this step. All we are really finding here is the largest possible range of limits for *x* and *y*. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to get the following inequalities. Note as discussed above however that these may not be the limits on *x* and *y* we are after.

$$\begin{aligned} -1 &\le \sin\left(t\right) \le 1 & \qquad -1 &\le \cos\left(t\right) \le 1 \\ -3 &\le 3\sin\left(t\right) \le 3 & \qquad 4 &\ge -4\cos\left(t\right) \ge -4 \\ -3 &\le x \le 3 & \qquad -4 &\le y \le 4 \end{aligned}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for *x* and *y*. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on *x* and *y*. This set of inequalities for the limits on *x* and *y* assume that the parametric curve will be completely traced out at least once for the range of *t*'s we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of *t*'s. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on *x* and *y*.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on *t*.

Second, if we can't completely determine the actual limits on *x* and *y* at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on *x* and *y*. Also, there are times where knowing the largest possible limits on *x* and/or *y* will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of *t* into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations. In this case we were also given a range of *t*'s and we need to restrict the *t*'s in our table to that range.

t	x	y
0	0	-4
$\frac{\pi}{2}$	3	0
π	0	4
$\frac{3\pi}{2}$	-3	0
2π	0	-4

Now, this table seems to suggest that the parametric equation will follow the ellipse in a counter clockwise rotation. It also seems to suggest that the ellipse will be traced out exactly once.

However, tables of values for parametric equations involving sine and/or cosine equations can be deceptive.

Because sine and cosine oscillate it is possible to choose "bad" values of *t* that suggest a single trace when in fact the curve is tracing out faster than we realize and it is in fact tracing out more than once. We'll need to do some extra analysis to verify if the ellipse traces out once or more than once.

Also, just because the table suggests a particular direction doesn't actually mean it is going in that direction. It could be moving in the opposite direction at a speed that just happens to match the points you got in the table. Go back to the notes and check out Example 5. Plug in the points we used in our table above and you'll get a set of points that suggest the curve is tracing out clockwise when in fact it is tracing out counter clockwise!

Note that because this is such a "bad" way of getting the direction of motion we put it in its own step so we could discuss it in detail. The actual method we'll be using is in the next step and we'll not be doing table work again unless it is absolutely required for some other part of the problem.

Step 4

As suggested in the previous step the table of values is not a good way to get direction of motion for parametric curves involving trig function so let's go through a much better way of determining the direction of motion. This method takes a little time to think things through but it will always get the correct direction if you take the time.

First, let's think about what happens if we start at $t = 0$ and increase *t* to $t = \pi$.

As we cover this range of *t*'s we know that cosine starts at 1, decreasing through zero and finally stops at -1. So, that means that *y* will start at $y = -4$ (*i.e.* where cosine is 1), go through the *x*-axis (*i.e.* where cosine is zero) and finally stop at $y = 4$ (*i.e.* where cosine is -1). Now, this doesn't give us a direction of motion as all it really tells us that *y* increases and it could do this following the right side of the ellipse (*i.e.* counter clockwise) or it could do this following the left side of the ellipse (*i.e.* clockwise).

So, let's see what the behavior of sine in this range tells us. Starting at $t = 0$ we know that sine will be zero and so *x* will also be zero. As *t* increases to $t = \frac{\pi}{2}$ we know that sine increases from zero to one and so *x* will increase from zero to three. Finally, as we further increase *t* to $t = \pi$ sine will decrease from one back to zero and so *x* will also decrease from three to zero.

So, taking the *x* and *y* analysis above together we can see that at $t = 0$ the curve will start at the point $\left(0,-4\right)$. As we increase *t* to $t = \frac{\pi}{2}$ the curve will have to follow the ellipse with increasing *x* and *y* until it hits the point $\left(3,0\right)$. The only way we can reach this second point and have the correct increasing behavior for both *x* and *y* is to move in a counter clockwise direction along the right half of the ellipse.

If we further increase *t* from $t = \frac{\pi}{2}$ to $t = \pi$ we can see that *y* must continue to increase but *x* now decreases until we get to the point $\left(0,4\right)$ and again the only way we can reach this third point and have the required increasing/decreasing information for *y*/*x* respectively is to be moving in a counter clockwise direction along the right half.

We can do a similar analysis increasing *t* from $t = \pi$ to $t = 2\pi$ to see that we must still move in a counter clockwise direction that takes us through the point $\left(-3,0\right)$ and then finally ending at the point $\left(0,-4\right)$.

So, from this analysis we can see that the curve must be tracing out in a counter clockwise direction.

This analysis seems complicated and maybe not so easy to do the first few times you see it. However, once you do it a couple of times you'll see that it's not quite as bad as it initially seems to be. Also, it really is the only way to guarantee that you've got the correct direction of motion for the curve when dealing with parametric equations involving sine and/or cosine.

If you had trouble visualizing how sine and cosine changed as we increased *t* you might want to do a quick sketch of the graphs of sine and cosine and you'll see right away that we were correct in our analysis of their behavior as we increased *t*.

Step 5
Okay, in the last step notice that we also showed that the curve will trace the ellipse out exactly once in the given range of *t*'s. However, let's assume that we hadn't done the direction analysis yet and see if we can determine this without the direction analysis.

This is actually pretty simple to do, or at least simpler than the direction analysis. All it requires is that you know where sine and cosine are zero, 1 and -1. If you recall your unit circle it's always easy to know where sine and cosine have these values. We'll also be able to verify the ranges of *x* and *y* found in Step 2 were in fact the actual ranges for *x* and *y*.

Let's start with the "initial" point on the curve, *i.e.* the point at the left end of our range of *t*'s, $t = 0$ in this case. Where you start this analysis is really dependent upon the set of parametric equations, the parametric curve and/or if there is a range of *t*'s given. Good starting points are the "initial" point, one of the end points of the curve itself (if the curve does have endpoints) or $t = 0$. Sometimes one option will be better than the others and other times it won't matter.

In this case two of the options are the same point so it seems like a good point to use.

So, at $t = 0$ we are at the point $\left(0,-4\right)$. We know that the parametric curve is some or all of the ellipse we found in the first step. So, at this point let's assume it is the full ellipse and ask ourselves the following question. When do we get back to this point? Or, in other words, what is the next value of *t* after $t = 0$ (since that is the point we choose to start off with) are we back at the point $\left(0,-4\right)$?

Before doing this let's quickly note that if the parametric curve doesn't get back to this point we'll determine that in the following analysis and that will be useful in helping us to determine how much of the ellipse will get traced out by the parametric curve.

Okay let's back to the analysis. In order to be at the point $\left(0,-4\right)$ we know we must have $\sin\left(t\right) = 0$ (only way to get $x = 0$!) and we must have $\cos\left(t\right) = 1$ (only way to get $y = -4$!). For $t > 0$ we know that $\sin\left(t\right) = 0$ at $t = \pi, 2\pi, 3\pi, \ldots$ and likewise we know that $\cos\left(t\right) = 1$ at $t = 2\pi, 4\pi, 6\pi, \ldots$. The first value of *t* that is in both lists is $t = 2\pi$ and so this is the next value of *t* that will put us at that point.

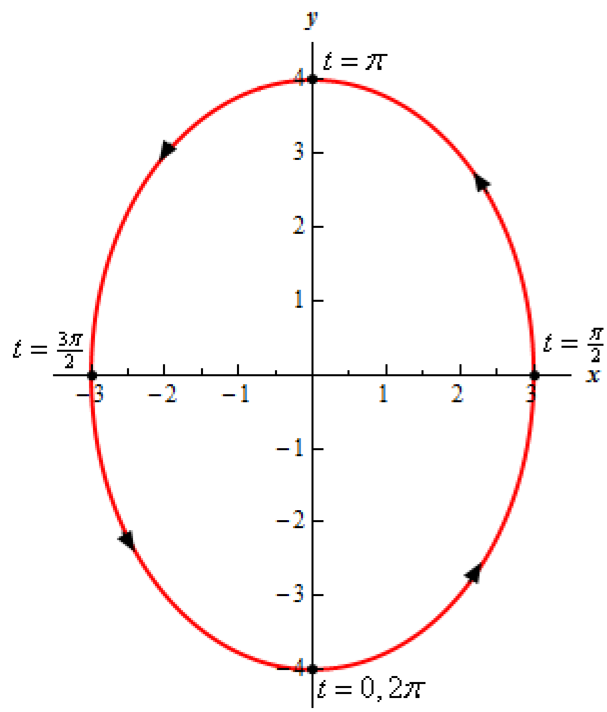
This tells us several things. First, we found that the parametric equation will get back to the initial point and so it is possible for the parametric equation to trace out the full ellipse.

Secondly, we got back to the point $\left(0,-4\right)$ at the very last *t* from the range of *t*'s we were given in the problem statement and so the parametric curve will trace out the ellipse exactly once for the given range of *t*'s.

Finally, from this analysis we found the parametric curve traced out the full ellipse in the range of *t*'s given in the problem statement and so we know now that the limits of *x* and *y* we found in Step 2 are in fact the actual limits on *x* and *y* for this curve.

As a final comment from this step let's note that this analysis in this step was a little easier than normal because the argument of the trig functions was just a *t* as opposed to say 2*t* or $\frac{1}{3}t$ which does make the analysis a tiny bit more complicated. We'll see how to deal with these kinds of arguments in the next couple of problems.

Step 6
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

Also, because the problem asked for it here are the formal limits on *x* and *y* for this parametric curve.

$$-3 \le x \le 3 \qquad\qquad -4 \le y \le 4$$

As a final set of thoughts for this problem you really should go back and make sure you understand the processes we went through in Step 4 and Step 5. Those are often the best way of getting at the information we found in those steps. The processes can seem a little mysterious at first but once you've done a couple you'll find it isn't as bad as they might have first appeared.

Also, for the rest of the problems in this section we'll build a table of *t* values only if it is absolutely necessary for the problem. In other words, the process we used in Step 4 and 5 will be the processes we'll be using to get direction of motion for the parametric curve and to determine if the curve is traced out more than once or not.

You should also take a look at problems 5 and 6 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

5. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = 3\sin\left(2t\right) \qquad y = -4\cos\left(2t\right) \qquad\qquad 0 \le t \le 2\pi$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2\left(\theta\right) + \sin^2\left(\theta\right) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin\left(2t\right) = \frac{x}{3} \qquad\qquad \cos\left(2t\right) = -\frac{y}{4}$$

Plugging these into the trig identity (remember the identity holds as long as the argument of both trig functions, 2*t* in this case, is the same) gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad\qquad \Rightarrow \qquad\qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only *x* and *y* may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only *x* and *y*. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only *x* and *y* more than once.

This observation is especially important for this problem. The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2

At this point we can get a good idea on what the limits on *x* and *y* are going to be so let's do that. Note that often we won't get the actual limits on *x* and *y* in this step. All we are really finding here is the largest possible range of limits for *x* and *y*. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to determine the limits on *x* and *y* as follows,

$$\begin{array}{lcl} -1 \le \sin\left(2t\right) \le 1 & \qquad\qquad & -1 \le \cos\left(2t\right) \le 1 \\ -3 \le 3\sin\left(2t\right) \le 3 & & 4 \ge -4\cos\left(2t\right) \ge -4 \\ -3 \le x \le 3 & & -4 \le y \le 4 \end{array}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for *x* and *y*. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on *x* and *y*. This set of inequalities for the limits on *x* and *y* assume that the parametric curve will be completely traced out at least once for the range of *t*'s we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of *t*'s. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on *x* and *y*.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on *t*.

Second, if we can't completely determine the actual limits on *x* and *y* at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on *x* and *y*. Also, there are times where knowing the largest possible limits on *x* and/or *y* will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

In previous problems one method we looked at was to build a table of values for a sampling of *t*'s in the range provided. However, as we discussed in Problem 4 of this section tables of values for parametric equations involving trig functions they can be deceptive and so we aren't going to use them to determine the direction of motion for this problem.

Also, as noted in the discussion in Problem 4 it also might help to have the graph of sine and cosine handy to look at since we'll be talking a lot about the behavior of sine/cosine as we increase the argument.

So for this problem we'll just do the analysis of the behavior of sine and cosine in the range of *t*'s we were provided to determine the direction of motion. We'll be doing a quicker version of the analysis here than we did in Problem 4 so you might want to go back and check that problem out if you have trouble following everything we're going here.

Let's start at $t = 0$ since that is the first value of *t* in the range of *t*'s we were given in the problem. This means we'll be starting the parametric curve at the point $\left(0,-4\right)$.

Now, what happens if we start to increase *t*? First, if we increase *t* then we also increase 2*t*, the argument of the trig functions in the parametric equations. So, what does this mean for $\sin\left(2t\right)$ and $\cos\left(2t\right)$? Well initially, we know that $\sin\left(2t\right)$ will increase from zero to one and at the same time $\cos\left(2t\right)$ will also have to decrease from one to zero.

So, this means that *x* (given by $x = 3\sin\left(2t\right)$) will have to increase from 0 to 3. Likewise, it means that *y* (given by $y = -4\cos\left(2t\right)$) will have to increase from -4 to 0. For the *y* equation note that while the cosine is decreasing the minus sign on the coefficient means that *y* itself will actually be increasing.

Because this behavior for *x* and *y* must be happening at simultaneously we can see that the only possibility is for the parametric curve to start at $\left(0,-4\right)$ and as we increase the value of *t* we must move to the right in the counter clockwise direction until we reach the point $\left(3,0\right)$.

Okay, we're now at the point $\left(3,0\right)$, so $\sin\left(2t\right) = 1$ and $\cos\left(2t\right) = 0$. Let's continue to increase *t*. A further increase of *t* will force $\sin\left(2t\right)$ to decrease from 1 to 0 and at the same time $\cos\left(2t\right)$ will decrease from 0 to -1.

In terms of *x* and *y* this means that, at the same time, *x* will now decrease from 3 to 0 while *y* will continue to increase from 0 to 4 (again the minus sign on the *y* equation means *y* must increase as the cosine decreases from 0 to -1). So, we must be continuing to move in a counter clockwise direction until we reach the point $\left(0,4\right)$.

For the remainder we'll go a little quicker in the analysis and just discuss the behavior of *x* and *y* and skip the discussion of the behavior of the sine and cosine.

Another increase in *t* will force *x* to decrease from 0 to -3 and at the same time *y* will have to also decrease from 4 to 0. The only way for this to happen simultaneously is to move along the ellipse starting that $\left(0,4\right)$ in a counter clockwise motion until we reach $\left(-3,0\right)$.

Continuing to increase *t* and we can see that, at the same time, *x* will increase from -3 to 0 and *y* will decrease from 0 to -4. Or, in other words we're moving along the ellipse in a counter clockwise motion from $\left(-3,0\right)$ to $\left(0,-4\right)$.

At this point we've gotten back to the starting point and we got back to that point by always going in a counter clockwise direction and did not retrace any portion of the graph and so we can now safely say that the direction of motion for this curve will always counter clockwise.

We have to be very careful here to continue the analysis until we get back to the starting point and see just how we got back there. It is possible, as we'll see in later problems, for us to get back there by retracing back over the curve. This will have an effect on the direction of motion for the curve (*i.e.* the direction will change!). In this case however since we got back to the starting point without retracing any portion of the curve we know the direction will remain counter clockwise.

Step 4
Let's now think about how much of the ellipse is actually traced out or if the ellipse is traced out more than once for the range of *t*'s we were given in the problem. We'll also be able to verify if the ranges of *x* and *y* we found in Step 2 are the correct ones or if we need to modify them (and we'll also determine just how to modify them if we need to).

Be careful to not draw any conclusions about how much of the ellipse is traced out from the analysis in the previous step. If we follow that analysis we see a full single trace of the ellipse. However, we didn't ever really mention any values of *t* with the exception of the starting value. Because of that we can't really use the analysis in the previous step to determine anything about how much of the ellipse we trace out or how many times we trace the ellipse out.

Let's go ahead and start this portion out at the same value of *t* we started with in the previous step. So, at $t=0$ we are at the point $\left(0,-4\right)$. Now, when do we get back to this point? Or, in other words, what is the next value of *t* after $t=0$ (since that is the point we choose to start off with) are we at the point $\left(0,-4\right)$?

In order to be at this point we know we must have $\sin\left(2t\right)=0$ (only way to get $x=0$!) and we must have $\cos\left(2t\right)=1$ (only way to get $y=-4$!). Note the arguments of the sine and cosine! That is very important for this step.

Now, for $t>0$ we know that $\sin\left(2t\right)=0$ at $2t=\pi,2\pi,3\pi,\ldots$ and likewise we know that $\cos\left(2t\right)=1$ at $2t=2\pi,4\pi,6\pi,\ldots$. Again, note the arguments of sine and cosine here! Because we want $\sin\left(2t\right)$ and $\cos\left(2t\right)$ to have certain values we need to determine the values of 2*t* we need to achieve the values of sine and cosine that we are looking for.

The first value of 2*t* that is in both lists is $2t=2\pi$. This now tells us the value of *t* we need to get back to the starting point. We just need to solve this for *t*!

$$2t=2\pi \qquad \Rightarrow \qquad t=\pi$$

So, we will get back to the starting point, without retracing any portion of the ellipse, important in some later problems, when we reach $t = \pi$.

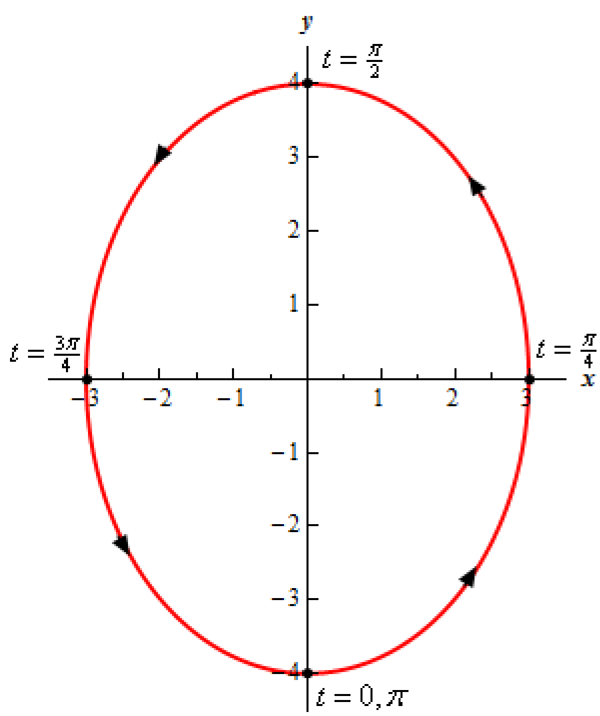
But this is in the middle of the range of *t*'s we were given! So, just what does this mean for us? Well first of all, provided the argument of the sine/cosine is only in terms of *t*, as opposed to t^2 or $\sqrt{t}$ for example, the "net" range of *t*'s for one trace will always be the same. So, we got one trace in the range of $0 \le t \le \pi$ and so the "net" range of *t*'s here is $\pi - 0 = \pi$ and so any range of *t*'s that span π will trace out the ellipse exactly once.

This means that the ellipse will also trace out exactly once in the range $\pi \le t \le 2\pi$. So, in this case, it looks like the ellipse will be traced out twice in the range $0 \le t \le 2\pi$.

This analysis also has shown us that the parametric curve traces out the full ellipse in the range of *t*'s given in the problem statement (more than once in fact!) and so we know now that the limits of *x* and *y* we found in Step 2 are in fact the actual limits on *x* and *y* for this curve.

Before we leave this step we should note that once you get pretty good at the direction analysis we did in Step 3 you can combine the analysis Steps 3 and 4 into a single step to get both the direction and portion of the curve that is traced out. Initially however you might find them a little easier to do them separately.

Step 5
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of *t*'s to illustrate a handful of points and their corresponding values of *t*'s. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of *t* for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice of the process of the Step 4 analysis might be useful to you.

Also, because the problem asked for it here are the formal limits on *x* and *y* for this parametric curve.

$$-3 \le x \le 3 \qquad\qquad -4 \le y \le 4$$

You should also take a look at problems 4 and 6 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

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6. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on *x* and *y*.

$$x = 3\sin\left(\tfrac{1}{3}t\right) \qquad y = -4\cos\left(\tfrac{1}{3}t\right) \qquad\qquad 0 \le t \le 2\pi$$

Step 1
First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin\left(\tfrac{1}{3}t\right) = \frac{x}{3} \qquad\qquad \cos\left(\tfrac{1}{3}t\right) = -\frac{y}{4}$$

Plugging these into the trig identity (remember the identity holds as long as the argument of both trig functions, $\frac{1}{3}t$ in this case, is the same) gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad\qquad \Rightarrow \qquad\qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only *x* and *y* may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only *x* and *y*. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only *x* and *y* more than once.

This observation is especially important for this problem. The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2

At this point we can get a good idea on what the limits on *x* and *y* are going to be so let's do that. Note that often we won't get the actual limits on *x* and *y* in this step. All we are really finding here is the largest possible range of limits for *x* and *y*. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to determine the limits on *x* and *y* as follows,

$$\begin{aligned} -1 &\le \sin\left(\tfrac{1}{3}t\right) \le 1 \\ -3 &\le 3\sin\left(\tfrac{1}{3}t\right) \le 3 \\ -3 &\le x \le 3 \end{aligned} \qquad\qquad \begin{aligned} -1 &\le \cos\left(\tfrac{1}{3}t\right) \le 1 \\ 4 &\ge -4\cos\left(\tfrac{1}{3}t\right) \ge -4 \\ -4 &\le y \le 4 \end{aligned}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for *x* and *y*. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on *x* and *y*. This set of inequalities for the limits on *x* and *y* assume that the parametric curve will be completely traced out at least once for the range of *t*'s we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of *t*'s. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on *x* and *y*.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on *t*. This is especially important for this problem!

Second, if we can't completely determine the actual limits on *x* and *y* at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on *x* and *y*. Also, there are times where knowing the largest possible limits on *x* and/or *y* will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3
Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, *t* in this case.

In previous problems one method we looked at was to build a table of values for a sampling of *t*'s in the range provided. However, as we discussed in Problem 4 of this section tables of values for parametric equations involving trig functions they can be deceptive and so we aren't going to use them to determine the direction of motion for this problem.

Also, as noted in the discussion in Problem 4 it also might help to have the graph of sine and cosine handy to look at since we'll be talking a lot about the behavior of sine/cosine as we increase the argument.

So for this problem we'll just do the analysis of the behavior of sine and cosine in the range of *t*'s we were provided to determine the direction of motion. We'll be doing a quicker version of the analysis here than we did in Problem 4 so you might want to go back and check that problem out if you have trouble following everything we're going here.

Let's start at $t = 0$ since that is the first value of *t* in the range of *t*'s we were given in the problem. This means we'll be starting the parametric curve at the point $\left(0, -4\right)$.

Now, what happens if we start to increase *t*? First, if we increase *t* then we also increase $\frac{1}{3}t$, the argument of the trig functions in the parametric equations. So, what does this mean for $\sin\left(\frac{1}{3}t\right)$ and $\cos\left(\frac{1}{3}t\right)$? Well initially, we know that $\sin\left(\frac{1}{3}t\right)$ will increase from zero to one and at the same time $\cos\left(\frac{1}{3}t\right)$ will also have to decrease from one to zero.

So, this means that *x* (given by $x = 3\sin\left(\frac{1}{3}t\right)$) will have to increase from 0 to 3. Likewise, it means that *y* (given by $y = -4\cos\left(\frac{1}{3}t\right)$) will have to increase from -4 to 0. For the *y* equation note that while the cosine is decreasing the minus sign on the coefficient means that *y* itself will actually be increasing.

Because this behavior for the *x* and *y* must be happening at simultaneously we can see that the only possibility is for the parametric curve to start at $\left(0, -4\right)$ and as we increase the value of *t* we must move to the right in the counter clockwise direction until we reach the point $\left(3, 0\right)$.

Okay, we're now at the point $\left(3,0\right)$, so $\sin\left(\frac{1}{3}t\right)=1$ and $\cos\left(\frac{1}{3}t\right)=0$. Let's continue to increase *t*. A further increase of *t* will force $\sin\left(\frac{1}{3}t\right)$ to decrease from 1 to 0 and at the same time $\cos\left(\frac{1}{3}t\right)$ will decrease from 0 to -1.

In terms of *x* and *y* this means that, at the same time, *x* will now decrease from 3 to 0 while *y* will continue to increase from 0 to 4 (again the minus sign on the *y* equation means *y* must increase as the cosine decreases from 0 to -1). So, we must be continuing to move in a counter clockwise direction until we reach the point $\left(0,4\right)$.

For the remainder we'll go a little quicker in the analysis and just discuss the behavior of *x* and *y* and skip the discussion of the behavior of the sine and cosine.

Another increase in *t* will force *x* to decrease from 0 to -3 and at the same time *y* will have to also decrease from 4 to 0. The only way for this to happen simultaneously is to move along the ellipse starting that $\left(0,4\right)$ in a counter clockwise motion until we reach $\left(-3,0\right)$.

Continuing to increase *t* and we can see that, at the same time, *x* will increase from -3 to 0 and *y* will decrease from 0 to -4. Or, in other words we're moving along the ellipse in a counter clockwise motion from $\left(-3,0\right)$ to $\left(0,-4\right)$.

At this point we've gotten back to the starting point and we got back to that point by always going in a counter clockwise direction and did not retrace any portion of the graph and so we can now safely say that the direction of motion for this curve will always counter clockwise.

We have to be very careful here to continue the analysis until we get back to the starting point and see just how we got back there. It is possible, as we'll see in later problems, for us to get back there by retracing back over the curve. This will have an effect on the direction of motion for the curve (*i.e.* the direction will change!). In this case however since we got back to the starting point without retracing any portion of the curve we know the direction will remain counter clockwise.

Step 4
Let's now think about how much of the ellipse is actually traced out or if the ellipse is traced out more than once for the range of *t*'s we were given in the problem. We'll also be able to verify if the ranges of *x* and *y* we found in Step 2 are the correct ones or if we need to modify them (and we'll also determine just how to modify them if we need to).

Be careful to not draw any conclusions about how much of the ellipse is traced out from the analysis in the previous step. If we follow that analysis we see a full single trace of the ellipse. However, we didn't ever really mention any values of *t* with the exception of the starting value. Because of that we can't really use the analysis in the previous step to determine anything about how much of the ellipse we trace out or how many times we trace the ellipse out.

Let's go ahead and start this portion out at the same value of *t* we started with in the previous step. So, at $t=0$ we are at the point $\left(0,-4\right)$. Now, when do we get back to this point? Or, in other words, what is the next value of *t* after $t = 0$ (since that is the point we choose to start off with) are we at the point $(0, -4)$?

In order to be at this point we know we must have $\sin\left(\frac{1}{3}t\right) = 0$ (only way to get $x = 0$!) and we must have $\cos\left(\frac{1}{3}t\right) = 1$ (only way to get $y = -4$!). Note the arguments of the sine and cosine! That is very important for this step.

Now, for $t > 0$ we know that $\sin\left(\frac{1}{3}t\right) = 0$ at $\frac{1}{3}t = \pi, 2\pi, 3\pi, \ldots$ and likewise we know that $\cos\left(\frac{1}{3}t\right) = 1$ at $\frac{1}{3}t = 2\pi, 4\pi, 6\pi, \ldots$. Again, note the arguments of sine and cosine here! Because we want $\sin\left(\frac{1}{3}t\right)$ and $\cos\left(\frac{1}{3}t\right)$ to have certain values we need to determine the values of $\frac{1}{3}t$ we need to achieve the values of sine and cosine that we are looking for.

The first value of $\frac{1}{3}t$ that is in both lists is $\frac{1}{3}t = 2\pi$. This now tells us the value of *t* we need to get back to the starting point. We just need to solve this for *t*!

$$\frac{1}{3}t = 2\pi \qquad \Rightarrow \qquad t = 6\pi$$

So, we will get back to the starting point, without retracing any portion of the ellipse, important in some later problems, when we reach $t = 6\pi$.

At this point we have a problem that we didn't have in the previous two problems. We get back to the point $(0, -4)$ at $t = 6\pi$ and this is outside the range of *t*'s given in the problem statement, $0 \le t \le 2\pi$!

What this means for us is that the parametric curve will not trace out a full trace for the range of *t*'s we were given for this problem. It also means that the range of limits for *x* and *y* from Step 2 are not the correct limits for *x* and *y*.

We know from the Step 3 analysis that the parametric curve will trace out in a counter clockwise direction and from the analysis in this step it won't trace out a full trace.

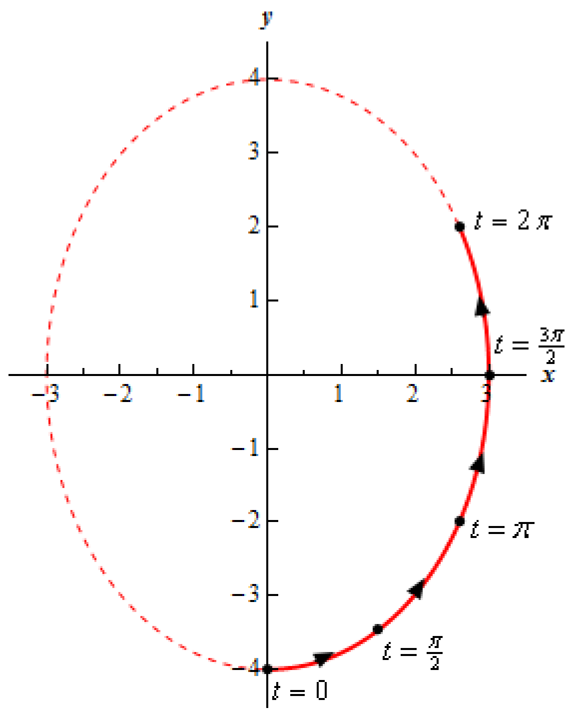
So, we know the parametric curve will start when $t = 0$ at $(0, -4)$ and will trace out in a counter clockwise direction until $t = 2\pi$ at which we will be at the point,

$$\left(3\sin\left(\tfrac{2\pi}{3}\right), -4\cos\left(\tfrac{2\pi}{3}\right)\right) = \left(\tfrac{3\sqrt{3}}{2}, 2\right)$$

This "ending" point is in the first quadrant and so we know that the curve has to have passed through $(3, 0)$. This means that the limits on *x* are $0 \le x \le 3$. The limits on the *y* are simply those we get from the points $-4 \le y \le 2$.

Before we leave this step we should note that once you get pretty good at the direction analysis we did in Step 3 you can combine the analysis Steps 3 and 4 into a single step to get both the direction and portion of the curve that is traced out. Initially however you might find them a little easier to do them separately.

Step 5
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of *t*'s to illustrate a handful of points and their corresponding values of *t*'s. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of *t* for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice would of the Step 4 analysis might be useful to you.

Note as well that we included the full sketch of the ellipse as a dashed graph to help illustrate the portion of the ellipse that the parametric curve is actually covering.

Also, because the problem asked for it here are the formal limits on x and y for this parametric curve.

$$0 \le x \le 3 \qquad\qquad -4 \le y \le 2$$

You should also take a look at problems 4 and 5 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

7.The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on *x* and *y*.

(*iii*) A range of *t*'s for a single trace of the parametric curve.

(*iv*) The number of traces of the curve the particle makes if an overall range of *t*'s is provided in the problem.

$$x = 3 - 2\cos\left(3t\right) \qquad y = 1 + 4\sin\left(3t\right)$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 – 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2\left(\theta\right) + \sin^2\left(\theta\right) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\cos\left(3t\right) = \frac{x-3}{-2} \qquad \sin\left(3t\right) = \frac{y-1}{4}$$

Plugging these into the trig identity gives,

$$\left(\frac{x-3}{-2}\right)^2 + \left(\frac{y-1}{4}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{\left(x-3\right)^2}{4} + \frac{\left(y-1\right)^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the graph of this ellipse.

Step 2

At this point let's get our first guess as to the limits on *x* and *y*. As noted in previous problems what we're really finding here is the largest possible ranges for *x* and *y*. In later steps we'll determine if this the actual set of limits on *x* and *y* or if we have smaller ranges.

We can use our knowledge of sine and cosine to determine the limits on *x* and *y* as follows,

$$\begin{aligned} -1 &\le \cos(3t) \le 1 \\ 2 &\ge -2\cos(3t) \ge -2 \\ 5 &\ge 3-2\cos(3t) \ge 1 \\ 1 &\le x \le 5 \end{aligned} \qquad\qquad \begin{aligned} -1 &\le \sin(3t) \le 1 \\ -4 &\le 4\sin(3t) \le 4 \\ -3 &\le 1+4\sin(3t) \le 5 \\ -3 &\le y \le 5 \end{aligned}$$

Remember that all we need to do is start with the appropriate trig function and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for *x* and *y*.

This problem does not have a range of *t*'s that might restrict how much of the parametric curve gets sketched out. This means that the parametric curve will be fully traced out.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full ellipse we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the ellipse that is described by the set of parametric curves will be completely traced out.

However, for this problem let's also note as well that the ranges for *x* and *y* we found above also correspond the maximum ranges for *x* and *y* we get from the equation of the ellipse we found in Step 1. This means that, for this problem, the ellipse will get fully traced out at least once by the parametric curve and so these are the full limits on *x* and *y*.

Step 3
Let's next get the direction of motion for the parametric curve.

Let's use $t = 0$ as a "starting" point for this analysis. At $t = 0$ we are at the point $(1,1)$. If we increase *t* we can see that both *x* and *y* must increase until we get to the point $(3,5)$. Increasing *t* further from this point will force *x* to continue to increase, but *y* will now start to decrease until we reach the point $(5,1)$. Next, when we increase *t* further both *x* and *y* will decrease until we reach the point $(3,-3)$. Finally, increasing *t* even more we get *x* continuing to decrease while *y* starts to increase until we get back to $(1,1)$, the point we "started" the analysis at.

We didn't put a lot of "explanation into this but if you think about the parametric equations and how sine/cosine behave as you increase *t* you should see what's going on. In the *x* equation we see that the coefficient of the cosine is negative and so if cosine increases *x* must decrease and if cosine decreases *x* must increase. For the *y* equation the coefficient of the sine is positive and so both *y* and sine will increase or decrease at the same time.

Okay, in all of the analysis above we must be moving in a clockwise direction. Also, note that because of the oscillating nature of sine and cosine once we reach back to the "starting" point the behavior will simply repeat itself. This in turn tells us that once we arrive back at the "starting" point we will continue to trace out the parametric curve in a clockwise direction.

Step 4
From the analysis in the last step we saw that without any range of *t*'s restricting the parametric curve, which we don't have here, the parametric curve will completely trace out the ellipse that we found in Step 1.

Therefore, the next thing we should do is determine a range of *t*'s that it will take to complete one trace of the parametric curve. Note that one trace of the parametric curve means that no portion of the parametric curve will ever be retraced. For this problem that means we trace out the ellipse exactly once.

So, as with the last step let's "start" at the point $\left(1,1\right)$, which corresponds to $t=0$. So, the next question to ask is what value of $t>0$ will we reach this point again.

In order to be at the point $\left(1,1\right)$ we need to require that $\cos\left(3t\right)=1$ and $\sin\left(3t\right)=0$. So, for $t>0$ we know we'll have $\cos\left(3t\right)=1$ if $3t=2\pi,4\pi,6\pi,\ldots$ and we'll have $\sin\left(3t\right)=0$ if $3t=\pi,2\pi,3\pi,\ldots$.

The first value of *t* that is in both of these lists is $3t=2\pi$. So, we'll get back to the "starting" point at,

$$3t=2\pi \qquad \Rightarrow \qquad t=\tfrac{2\pi}{3}$$

Therefore, one trace will be completed in the range,

$$0\le t\le \tfrac{2\pi}{3}$$

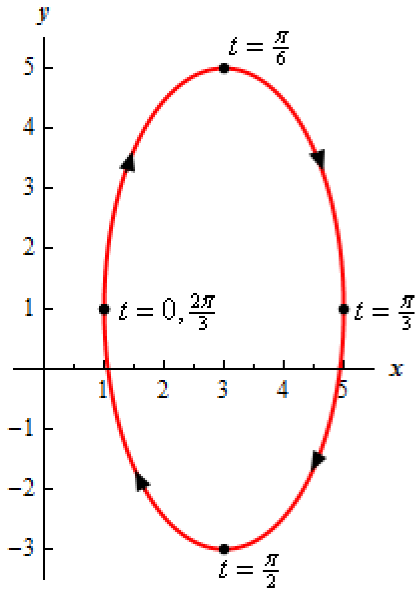
Note that this is only one possible answer here. Any range of *t*'s with a "net" range of $\frac{2\pi}{3}$ *t*'s, with the endpoints of the *t* range corresponding to start/end points of the parametric equation, will work. So, for example, any of the following ranges of *t*'s would also work.

$$-\tfrac{2\pi}{3}\le t\le 0 \qquad\qquad \tfrac{2\pi}{3}\le t\le \tfrac{4\pi}{3} \qquad\qquad -\tfrac{\pi}{3}\le t\le \tfrac{\pi}{3}$$

There are of course many other possible ranges of *t*'s for a one trace. Note however, as the last example above shows, because the full ellipse is traced out, each range doesn't all need to start/end at the same place. The range we originally arrived at as well as the first two ranges above all start/end at $\left(1,1\right)$ while the third range above starts/ends at $\left(5,1\right)$.

Step 5
Now that we have a range of *t*'s for one full trace of the parametric curve we could determine the number of traces the particle makes. However, because we weren't given an overall range of *t*'s we can't do that for this problem.

Step 6
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of *t*'s to illustrate where the particle is at while tracing out of the curve. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of *t* for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice would of the Step 4 analysis might be useful to you.

Here is also the formal answers for all the rest of the information that problem asked for.

$$\begin{array}{ll} \text{Range of } x : & 1 \le x \le 5 \\ \text{Range of } y : & -3 \le y \le 5 \\ \text{Range of } t \text{ for one trace :} & 0 \le t \le \frac{2\pi}{3} \\ \text{Total number of traces :} & \text{n/a} \end{array}$$

8.The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on *x* and *y*.

(*iii*) A range of *t*'s for a single trace of the parametric curve.

(*iv*) The number of traces of the curve the particle makes if an overall range of *t*'s is provided in the problem.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 – 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2\left(\theta\right)+\sin^2\left(\theta\right)=1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin\left(\tfrac{1}{4}t\right)=\frac{x}{4} \qquad\qquad \cos^2\left(\tfrac{1}{4}t\right)=\frac{y-1}{-2}$$

Plugging these into the trig identity gives,

$$\frac{y-1}{-2}+\left(\frac{x}{4}\right)^2=1 \qquad\qquad \Rightarrow \qquad\qquad y=\frac{x^2}{8}-1$$

Therefore, with a little algebraic manipulation, we see that the parametric curve will be some or all of the parabola above. Note that while many parametric equations involving sines and cosines are some or all of an ellipse they won't all be as this problem shows. Do not get so locked into ellipses when seeing sines/cosines that you always just assume the curve will be an ellipse.

Step 2

At this point let's get our first guess as to the limits on *x* and *y*. As noted in previous problems what we're really finding here is the largest possible ranges for *x* and *y*. In later steps we'll determine if this the actual set of limits on *x* and *y* or if we have smaller ranges.

We can use our knowledge of sine and cosine to determine the limits on *x* and *y* as follows,

$$\begin{gathered}-1\le\sin\left(\tfrac{1}{4}t\right)\le1\\ -4\le4\sin\left(\tfrac{1}{4}t\right)\le4\\ -4\le x\le4\end{gathered} \qquad\qquad \begin{gathered}-1\le\cos\left(\tfrac{1}{4}t\right)\le1\\ 0\le\cos^2\left(\tfrac{1}{4}t\right)\le1\\ 0\ge-2\cos^2\left(\tfrac{1}{4}t\right)\ge-2\\ 1\ge1-2\cos^2\left(\tfrac{1}{4}t\right)\ge-1\\ -1\le y\le1\end{gathered}$$

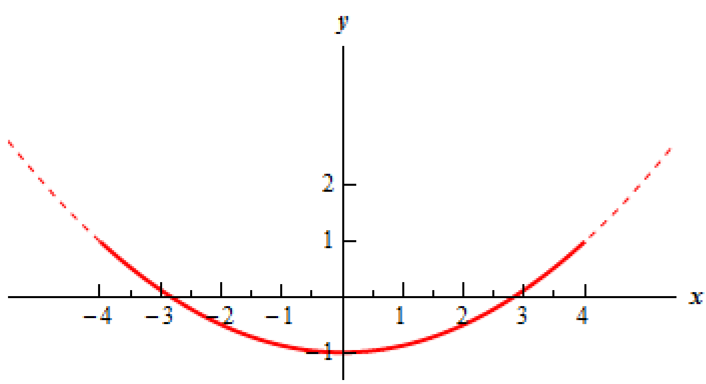
Remember that all we need to do is start with the appropriate trig function and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for *x* and *y*.

Now, at this point we need to be a little careful. As noted above what we've actually found here are the largest possible ranges for the limits on *x* and *y*. This set of inequalities for the limits on *x* and *y* assume that the parametric curve will be fully traced out at least once for the range of *t*'s we were given in the problem statement. It is always possible that the parametric curve will not trace out a full trace in the given range of *t*'s. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on *x* and *y*.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full parabola we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the parabola that is described by the set of parametric curves will be completely traced out.

In fact, for this problem, we can see that the parabola from Step 1 will not get fully traced out by the particle regardless of any range of *t*'s. The largest possible portion of the parabola that can be traced out by the particle is the portion that lies in the range of *x* and *y* given above. In a later step we'll determine if the largest possible portion of the parabola does get traced out or if the particle only traces out part of it.

Step 3
Let's next get the direction of motion for the parametric curve. For this analysis it might be useful to have a quick sketch of the largest possible parametric curve. So, here is a quick sketch of that.



The dashed line is the graph of the full parabola from Step 1 and the solid line is the portion that falls into our largest possible range of *x* and *y* we found in Step 2. As an aside here note that the two ranges are complimentary. In other words, if we sketch the graph only for the range of *x* we automatically get the range for *y*. Likewise, if we sketch the graph only for the range of *y* we automatically get the range for *x*. This is a good check for your graph. The *x* and *y* ranges should always match up!

When our parametric curve was an ellipse (the previous problem for example) no matter what point we started the analysis at the curve would eventually trace out around the ellipse and end up back at the starting point without ever going back over any portion of itself. The main issue we faced with the ellipse problem was we could rotate around the ellipse in a clockwise or a counter clockwise motion to do this and a careful analysis of the behavior of both the *x* and *y* parametric equations was required to determine just which direction we were going.

With a parabola for our parametric curve things work a lot differently. Let's suppose that we "started" at the right end point (this is just randomly picked for no other reason that I'm right handed so don't think there is anything special about this point!) and it doesn't matter what *t* we use to get to that point.

At this point we know that we are at $x = 4$ and in order for *x* to have that value we must also have $\sin\left(\frac{1}{4}t\right) = 1$. Now, as we increase *t* from this point (again it doesn't matter just what the value of *t* is) the only option for sine is for it to decrease until it has the value $\sin\left(\frac{1}{4}t\right) = -1$. This in turn means that if we start at the right end point we have no option but to proceed along the curve going to the left.

However, we don't just reach the left end point and then stop! Once we are at $\sin\left(\frac{1}{4}t\right) = -1$ if we further increase *t* we know that sine will also increase until it has the value $\sin\left(\frac{1}{4}t\right) = 1$ and so we must move back along the curve to the right until we are back at the right end point.

Unlike the ellipse however, the only way for this to happen is for the particle to go back over the parabola moving in a rightward direction. Remember that the particle moves to the right or left it must trace out a portion of the parabola that we found in Step 1! Any particle traveling along the path given by the set of parametric equations must follow the graph of the parabola and never leave it.

In other words, if we don't put any restrictions on *t* a particle on this parametric curve will simply oscillate left and right along the portion of the parabola sketched out above. In this case however we do have a range of *t*'s so we'll need to determine a range of *t*'s for one trace to fully know the direction of motion information of the particle on this path and we'll do that in the next step. With a restriction on the range of *t*'s it is possible that the particle won't make a full trace or it might retrace some or all of the curve so we can't say anything definite about the direction of motion for the particle over the full range of *t*'s until the next step when we determine a range of *t*'s for one full trace of the curve.

Before we move on to the next step there is a quick topic we should address. We only used the *x* equation to do this analysis and never addressed the *y*-equation anywhere in the analysis. It doesn't really matter which one we use as both will give the same information.

Step 4
Now we need to determine a range of *t*'s for one full trace of the parametric curve. It is important for this step to remember that one full trace of the parametric curve means that no portion of the parametric curve can be retraced.

Note that one full trace does not mean that we get back to the "starting" point. When we dealt with an ellipse in the previous problem that was one trace because we did not need to retrace any portion of the ellipse to get back to the starting point. However, as we saw in the previous step that for our parabola here we would have to retrace the full curve to get back to the starting point.

So, one full trace of the parametric curve means we move from the right end point to the left end point only or visa-versa and move from the left end point to the right end point. Which direction we move doesn't really matter here so let's get a range of *t*'s that take us from the left end point to the right end point.

In all the previous problems we've used $t = 0$ as our "starting" point but that won't work for this problem because that actually corresponds to the vertex of the parabola. We want to start at the left end point so the first part of this process is actually determine a *t* that will put us at the left end point.

In order to be at the left end point, $\left(-4,1\right)$, we need to require that $\sin\left(\frac{1}{4}t\right) = -1$ which occurs if $\frac{1}{4}t = \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$. We also need to require that $\cos\left(\frac{1}{4}t\right) = 0$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. There are going to be many numbers that are in both lists here so all we need to do is pick one and proceed. From the numbers that we've listed here we could use either $\frac{1}{4}t = -\frac{\pi}{2}$ or $\frac{1}{4}t = \frac{3\pi}{2}$. We'll use $\frac{1}{4}t = -\frac{\pi}{2}$, *i.e.* when $t = -2\pi$, simply because it is the first one that occurs in both lists. Therefore, we will be at the left end point when $t = -2\pi$.

Let's now move to the right end point, $\left(4,1\right)$. In order to get the range of *t*'s for one trace this means we'll need the next *t* with $t > -2\pi$ (which corresponds to $\frac{1}{4}t > -\frac{\pi}{2}$). To do this we need to require that $\sin\left(\frac{1}{4}t\right) = 1$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ and we need to that $\cos\left(\frac{1}{4}t\right) = 0$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

The first *t* that is in both of these lists with $\frac{1}{4}t > -\frac{\pi}{2}$ is then $\frac{1}{4}t = \frac{\pi}{2}$, *i.e.* when $t = 2\pi$. So, the first *t* after $t = -2\pi$ that puts us at the right end point is $t = 2\pi$. This means that a range of *t*'s for one full trace of the parametric curve is then,

$$-2\pi \le t \le 2\pi$$

Note that this is only one possible answer here. Any range of *t*'s with a "net" range of $2\pi - \left(-2\pi\right) = 4\pi$ *t*'s, with the endpoints of the *t* range corresponding to start/end points of the parametric curve, will work. So, for example, any of the following ranges of *t*'s would also work.

$$-6\pi \le t \le -2\pi \qquad\qquad 2\pi \le t \le 6\pi \qquad\qquad 6\pi \le t \le 10\pi$$

The direction of motion for each may be different range of *t*'s of course. Some will trace out the curve moving from left to right while others will trace out the curve moving from right to left. Because the problem did not specify a particular direction any would work.

Note as well that the range $-2\pi \le t \le 2\pi$ falls completely inside the given range of *t*'s specified in the problem and so we know that the particle will trace out the curve more than once over the full range of *t*'s. Determining just how many times it traces over the curve will be determined in the next step.

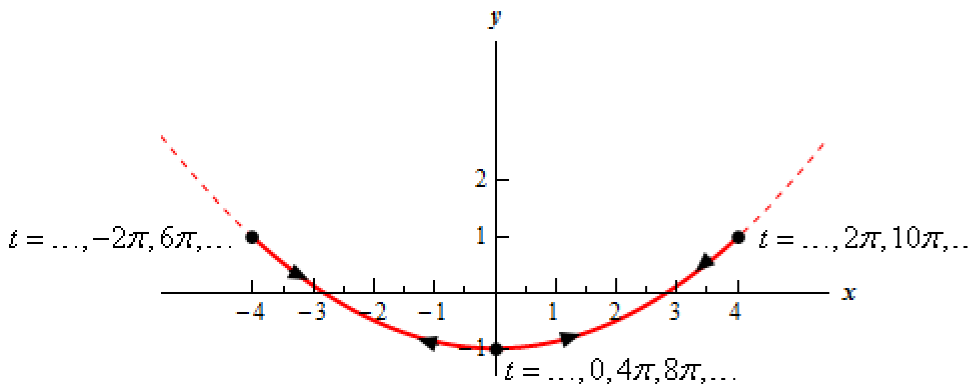
Step 5
Now that we have a range of *t*'s for one full trace of the parametric curve we can determine the number of traces the particle makes.

This is a really easy step. We know the total time the particle was traveling and we know how long it takes for a single trace. Therefore,

$$\text{Number Traces} = \frac{\text{Total Time Traveled}}{\text{Time for One Trace}} = \frac{34\pi - (-52\pi)}{2\pi - (-2\pi)} = \frac{86\pi}{4\pi} = \frac{43}{2} = 21.5 \text{ traces}$$

Step 6

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we indicated the direction of motion by putting arrow heads going both directions in places on the curve. We also included a set of *t*'s for a couple of points to illustrate where the particle is at while tracing out of the curve. The dashed line is the continuation of the parabola from Step 1 to illustrate that our parametric curve is only a part of the parabola.

Here is also the formal answers for all the rest of the information that problem asked for.

$$\begin{array}{ll} \text{Range of } x: & -4 \le x \le 4 \\ \text{Range of } y: & -1 \le y \le 1 \\ \text{Range of } t \text{ for one trace}: & -2\pi \le t \le 2\pi \\ \text{Total number of traces}: & 21.5 \end{array}$$

9.The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on *x* and *y*.

(*iii*) A range of *t*'s for a single trace of the parametric curve.

(*iv*) The number of traces of the curve the particle makes if an overall range of *t*'s is provided in the problem.

$$x = \sqrt{4 + \cos\left(\tfrac{5}{2}t\right)} \qquad y = 1 + \tfrac{1}{3}\cos\left(\tfrac{5}{2}t\right) \qquad -48\pi \le t \le 2\pi$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 – 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations notice that we can quickly and easily eliminate the parameter simply by solving the *y* equation for cosine as follows,

$$\cos\left(\tfrac{5}{2}t\right) = 3y - 3$$

Plugging this into the cosine in the *x* equation gives,

$$x = \sqrt{4 + (3y - 3)} \qquad \Rightarrow \qquad x = \sqrt{1 + 3y}$$

So, the parametric curve will be some or all of the graph of this square root function.

Step 2

At this point let's get our first guess as to the limits on *x* and *y*. As noted in previous problems what we're really finding here is the largest possible ranges for *x* and *y*. In later steps we'll determine if this the actual set of limits on *x* and *y* or if we have smaller ranges.

We can use our knowledge of cosine to determine the limits on *x* and *y* as follows,

$$\begin{aligned} -1 &\le \cos\left(\tfrac{5}{2}t\right) \le 1 & -1 &\le \cos\left(\tfrac{5}{2}t\right) \le 1 \\ 3 &\le 4 + \cos\left(\tfrac{5}{2}t\right) \le 5 & -\tfrac{1}{3} &\le \tfrac{1}{3}\cos\left(\tfrac{5}{2}t\right) \le \tfrac{1}{3} \\ \sqrt{3} &\le \sqrt{4 + \cos\left(\tfrac{5}{2}t\right)} \le \sqrt{5} & \tfrac{2}{3} &\le 1 + \tfrac{1}{3}\cos\left(\tfrac{5}{2}t\right) \le \tfrac{4}{3} \\ \sqrt{3} &\le x \le \sqrt{5} & \tfrac{2}{3} &\le y \le \tfrac{4}{3} \end{aligned}$$

Remember that all we need to do is start with the cosine and then build up the equation for *x* and *y* by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for *x* and *y*.

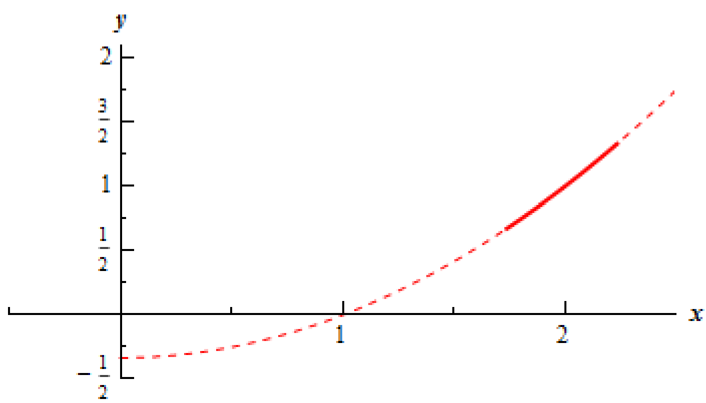
Now, at this point we need to be a little careful. As noted above what we've actually found here are the largest possible ranges for the limits on *x* and *y*. This set of inequalities for the limits on *x* and *y* assume that the parametric curve will be fully traced out at least once for the range of *t*'s we were given in the problem statement. It is always possible that the parametric curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full square root graph we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the square root graph that is described by the set of parametric curves will be completely traced out.

In fact, for this problem, we can see that the square root from Step 1 will not get fully traced out by the particle regardless of any range of t's. The largest possible portion of the square root graph that can be traced out by the particle is the portion that lies in the range of x and y given above. In a later step we'll determine if the largest possible portion of the square root graph does get traced out or if the particle only traces out part of it.

Step 3

Let's next get the direction of motion for the parametric curve. For this analysis it might be useful to have a quick sketch of the largest possible parametric curve. So, here is a quick sketch of that.



The dashed line is the graph of the full square root from Step 1 and the solid line is the portion that falls into our largest possible range of x and y we found in Step 2. As an aside here note that the two ranges are complimentary. In other words, if we sketch the graph only for the range of x we automatically get the range for y. Likewise, if we sketch the graph only for the range of y we automatically get the range for x. This is a good check for your graph. The x and y ranges should always match up!

Before moving on let's address the fact that is doesn't look like square root graphs that most of us are used to seeing. Keep in mind that the typical square root function that we're used to working at is in the form $y = \sqrt{x}$. Our equation for this problem however is in the form $x = \sqrt{y}$. If you think about it the graph of $x = \sqrt{y}$ is nothing more than the portion of the graph of $y = x^2$ corresponding to $x \ge 0$ (recall square roots only return positive or zero values!). Of course the function for this problem is not $x = \sqrt{y}$ but it is similar enough that the ideas discussed here are still valid just for a slightly different function.

Okay, let's get back to the problem.

This problem is going to be a lot like the previous problem in terms of direction of motion. First note that if we start at the lower left hand point we need to require that $\cos\left(\frac{5}{2}t\right) = -1$ since that is the only way for both *x* and *y* to have their minimal values (which puts us at the lower left-hand point)! It also doesn't matter what value of *t* we use at this point. All that matters is that we are at the lower left hand point.

If we now increase *t* (from whatever "starting" value we had) we know that cosine will need to increase from $\cos\left(\frac{5}{2}t\right) = -1$ until it reaches a value of $\cos\left(\frac{5}{2}t\right) = 1$. By looking at the parametric equations we can see that this will also force both *x* and *y* to increase until it reaches the upper right-hand point.

Now, the graph won't just stop here. Once cosine reaches a value of $\cos\left(\frac{5}{2}t\right) = 1$ we know that continuing to increase *t* will now cause cosine to decrease it reaches a value of $\cos\left(\frac{5}{2}t\right) = -1$. This in turn forces both *x* and *y* to decrease until it once again reaches the lower left-hand point.

In other words, if we don't put any restrictions on *t* a particle on this parametric curve will simply oscillate left and right along the portion of the square root sketched out above. In this case however we do have a range of *t*'s so we'll need to determine a range of *t*'s for one trace to fully know the direction of motion information of the particle on this path and we'll do that in the next step. With a restriction on the range of *t*'s it is possible that the particle won't make a full trace or it might retrace some or all of the curve so we can't say anything definite about the direction of motion for the particle over the full range of *t*'s until the next step when we determine a range of *t*'s for one full trace of the curve.

Step 4
Now we need to determine a range of *t*'s for one full trace of the parametric curve. It is important for this step to remember that one full trace of the parametric curve means that no portion of the parametric curve can be retraced.

Note that one full trace does not mean that we get back to the "starting" point. When we dealt with an ellipse in a previous problem that was one trace because we did not need to retrace any portion of the ellipse to get back to the starting point. However, as we saw in the previous step that for our square root here we would have to retrace the full curve to get back to the starting point.

So, one full trace of the parametric curve means we move from the right end point to the left end point only or visa-versa and move from the left end point to the right end point. Which direction we move doesn't really matter here so let's get a range of *t*'s that take us from the left end point to the right end point.

In order to be at the left end point we need to require that $\cos\left(\frac{5}{2}t\right) = -1$ which occurs if $\frac{5}{2}t = \ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$. Note as well that unlike the previous problems, which had both sine and cosine, this set of parametric equations has only cosine and so all we need to do here is look at this. Also, in order to be at the right end point we need to require that $\cos\left(\frac{5}{2}t\right) = 1$ which occurs if $\frac{5}{2}t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$.

So, if we want to move from the left to right all we need to do is chose one from the list of *t*'s corresponding to the left end point and then first *t* that comes that from the list corresponding to the right end point and we'll have a range of *t*'s for one trace. To move from the right to left we just go the opposite direction, *i.e.* chose a *t* from the right end point list and then take the first *t* after that from the left end point list.

So, for this problem, since we said we were going to move from left to right, we'll use $\frac{5}{2}t = \pi$, which corresponds to $t = \frac{2}{5}\pi$, for the left end point. That in turn means that we'll need to use $\frac{5}{2}t = 2\pi$, which corresponds to $t = \frac{4}{5}\pi$, for the right end point. That means the range of *t*'s for one trace is,

$$\frac{2}{5}\pi \le t \le \frac{4}{5}\pi$$

This is only one possible answer here. Any range of *t*'s with a "net" range of $\frac{4}{5}\pi - \left(\frac{2}{5}\pi\right) = \frac{2}{5}\pi$ *t*'s, with the endpoints of the *t* range corresponding to start/end points of the parametric curve, will work. So, for example, any of the following ranges of *t*'s would also work.

$$-\frac{2}{5}\pi \le t \le 0 \qquad\qquad 0 \le t \le \frac{2}{5}\pi \qquad\qquad \frac{4}{5}\pi \le t \le \frac{6}{5}\pi$$

The direction of motion for each may be different range of *t*'s of course. Some will trace out the curve moving from left to right while others will trace out the curve moving from right to left. Because the problem did not specify a particular direction any would work.

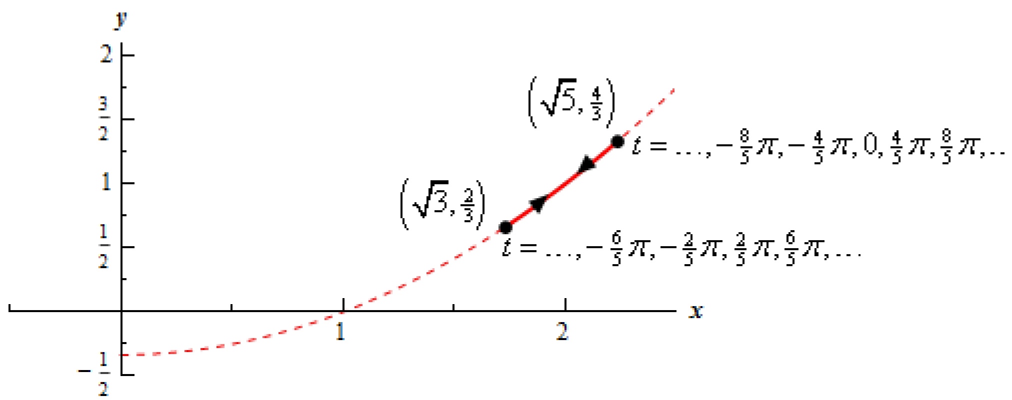
Note as well that the range $\frac{2}{5}\pi \le t \le \frac{4}{5}\pi$ falls completely inside the given range of *t*'s specified in the problem and so we know that the particle will trace out the curve more than once over the full range of *t*'s. Determining just how many times it traces over the curve will be determined in the next step.

Step 5
Now that we have a range of *t*'s for one full trace of the parametric curve we can determine the number of traces the particle makes.

This is a really easy step. We know the total time the particle was traveling and we know how long it takes for a single trace. Therefore,

$$\text{Number Traces} = \frac{\text{Total Time Traveled}}{\text{Time for One Trace}} = \frac{2\pi - \left(-48\pi\right)}{\frac{4}{5}\pi - \left(\frac{2}{5}\pi\right)} = \frac{50\pi}{\frac{2}{5}\pi} = 125 \text{ traces}$$

Step 6
Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we indicated the direction of motion by putting arrow heads going both directions in places on the curve. We also included a set of *t*'s for a couple of points to illustrate where the particle is at while tracing out of the curve as well as coordinates for the end points since they aren't "nice" points.. The dashed line is the continuation of the square root from Step 1 to illustrate that our parametric curve is only a part of the square root.

Here is also the formal answers for all the rest of the information that problem asked for.

$$\begin{array}{ll} \text{Range of } x: & \sqrt{3} \le x \le \sqrt{5} \\ \text{Range of } y: & \frac{2}{3} \le y \le \frac{4}{3} \\ \text{Range of } t \text{ for one trace}: & \frac{2}{5}\pi \le t \le \frac{4}{5}\pi \\ \text{Total number of traces}: & 125 \end{array}$$

10.The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on *x* and *y*.

(*iii*) A range of *t*'s for a single trace of the parametric curve.

(*iv*) The number of traces of the curve the particle makes if an overall range of *t*'s is provided in the problem.

$$x = 2\mathbf{e}^{t} \qquad y = \cos\left(1+\mathbf{e}^{3t}\right) \qquad 0 \le t \le \frac{3}{4}$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 – 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations let's first notice that we can solve the *x* equation for the exponential function as follows,

$$\mathbf{e}^{t} = \tfrac{1}{2}x$$

Now, just recall that $\mathbf{e}^{3t} = \left(\mathbf{e}^{t}\right)^{3}$ and so we can plug the above equation into the exponential in the *y* equation to get,

$$y = \cos\left(1 + \mathbf{e}^{3t}\right) = \cos\left(1 + \left(\mathbf{e}^{t}\right)^{3}\right) = \cos\left(1 + \left(\tfrac{1}{2}x\right)^{3}\right) = \cos\left(1 + \tfrac{1}{8}x^{3}\right)$$

So, the parametric curve will be some or all of the graph of this cosine function.

Step 2
At this point let's work on the limits for *x* and *y*. In this case, unlike most of the previous problems, things will work a little differently.

Let's start by noting that unlike sine and cosine functions we know $\mathbf{e}^{t}$ is always an increasing function (you can do some quick Calculus I work to verify this right?).

Why do we care about this? Well first the *x* equation is just a constant times $\mathbf{e}^{t}$ and we are given a range of *t*'s for the problem. Next, the fact that $\mathbf{e}^{t}$ is an increasing function means that the *x* equation, $x = 2\mathbf{e}^{t}$, is also an increasing function (because the 2 is positive). Therefore, the smallest value of *x* will occur at the smallest value of *t* in the range of *t*'s. Likewise, the largest value of *x* will occur at the largest value of *t* in the range of *t*'s.

Therefore, the range of *x* for our parametric curve is,

$$2\mathbf{e}^{0} \le x \le 2\mathbf{e}^{\frac{3}{4}} \qquad \Rightarrow \qquad 2 \le x \le 2\mathbf{e}^{\frac{3}{4}}$$

Unlike the previous problems where we usually needed to do a little more verification work we know at this point that this is the range of *x*'s.

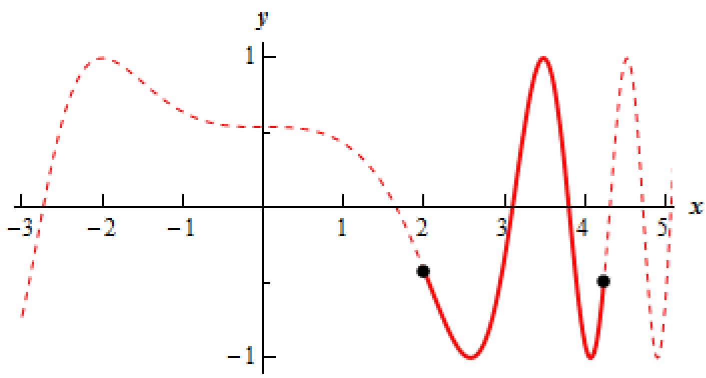
For the range of *y*'s we will need to do a little work to get the correct range of *y*'s but it won't be as much extra work as in previous problems and we can do it all in this step. First let's notice that because the *y* equation is in the form of $y = \cos\left(\cdots\right)$. The argument of the cosine doesn't matter for the first part of the work and so wasn't included here.

From the behavior of cosine we then know that the largest possible range of *y* would then be,

$$-1 \le y \le 1$$

Now, depending on just what values the argument of the cosine in the *y* equation takes over the give range of *t*'s we may or may not cover this full range of values. We could do some work analyzing the argument of the cosine to figure that out if it does cover this full range. However, there is a really easy way to figure that if the full range is covered in this case.

Let's just sketch the graph and see what we get. Here is a quick sketch of the graph.



Given the "messy" nature of the argument of the cosine it's probably best to use some form of computational aid to get the graph. The dotted portion of the graph is full graph of the function on $-3 \le x \le 5$ without regards to the actual restriction on *t*. The solid portion of the graph is the portion that corresponds to the range of *t*'s we were given in the problem.

From this graph we can see that the range of *y*'s is in fact $-1 \le y \le 1$.

Before proceeding with the direction of motion let's note that we could also have just graphed the curve in many of the previous problems to determine if the work in this step was the actual range or not. We didn't do that because we could determine if these ranges were correct or not when we did the direction of motion and range of *t*'s for one trace analysis (which we had to do anyway) and so didn't need to bother with a graph in this step for those problems.

Step 3
We now need to do the direction of motion for this curve but note that we actually found the direction of motion in the previous step.

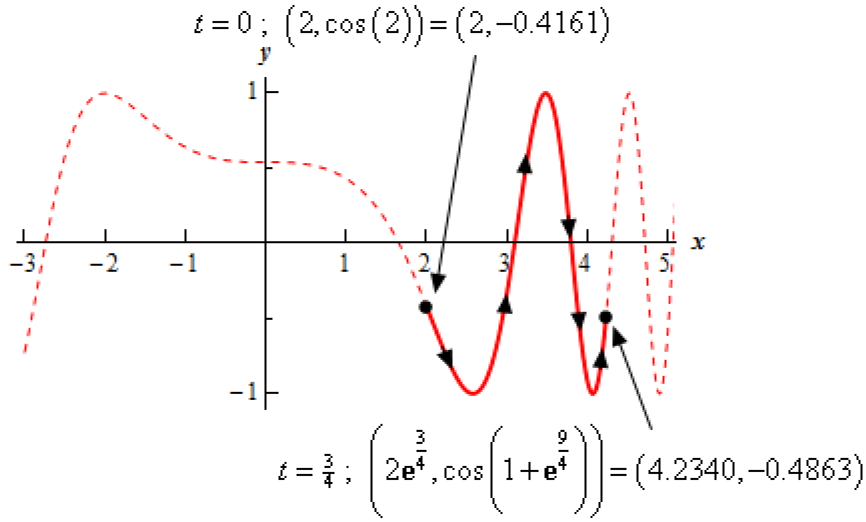
As noted in the previous step we know that $x = 2\mathbf{e}^{t}$ is an increasing function and so the *x*'s must be increasing as *t* increases. Therefore, the equation must be moving from left to right as the curve is traced out over the given range of *t*'s.

Also note that unlike the previous problems we know that no portion of the graph will be retraced. Again, we know the *x* equation is an increasing equation. If the curve were to retrace any portion we can see that the only way to do that would be to move back from right to left which would require *x* to decrease and that can't happen.

This means that we now know as well that the graph will trace out exactly once for the given range of *t*'s, which in turn tells us that the given range of *t*'s is also the range of *t*'s for a single trace.

Step 4

Now that we have all the needed information we can do a formal sketch of the graph.



As with the graph above the dotted portion of the graph is full graph of the function on $-3 \le t \le 5$ without regards to the actual restriction on *t*. The solid portion of the graph is the portion that corresponds to the range of *t*'s we were given in the problem. We also included the *t* value and coordinates of each end point for clarity although these are often not required for many problems.

Here is also the formal answers for all the rest of the information that problem asked for.

$$\begin{aligned} &\text{Range of } x: && 2 \le x \le 2\mathbf{e}^{\frac{3}{4}} \\ &\text{Range of } y: && -1 \le y \le 1 \\ &\text{Range of } t \text{ for one trace}: && 0 \le t \le \tfrac{3}{4} \\ &\text{Total number of traces}: && 1 \end{aligned}$$

11.The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on *x* and *y*.

(*iii*) A range of *t*'s for a single trace of the parametric curve.

(*iv*) The number of traces of the curve the particle makes if an overall range of *t*'s is provided in the problem.

$$x = \tfrac{1}{2}\mathbf{e}^{-3t} \qquad y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8$$

Step 1
There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 – 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations let's first notice that we can solve the *x* equation for the exponential function as follows,

$$\mathbf{e}^{-3t} = 2x$$

Now, just recall that $\mathbf{e}^{-6t} = \left(\mathbf{e}^{-3t}\right)^2$ and so we can plug the above equation into the exponential in the *y* equation to get,

$$y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8 = \left(\mathbf{e}^{-3t}\right)^2 + 2\mathbf{e}^{-3t} - 8 = \left(2x\right)^2 + 2\left(2x\right) - 8 = 4x^2 + 4x - 8$$

So, the parametric curve will be some or all of the graph of this quadratic function.

Step 2
At this point let's work on the limits for *x* and *y*. In this case, unlike most of the previous problems, things will work a little differently.

Let's start by noting that unlike sine and cosine functions we know $\mathbf{e}^{-3t}$ is always a decreasing function as *t* increases (you can do some quick Calculus I work to verify this right?).

Why do we care about this? Well first the *x* equation is just a constant times $\mathbf{e}^{-3t}$ and so the fact that $\mathbf{e}^{-3t}$ is a decreasing function means that the *x* equation, $x = \tfrac{1}{2}\mathbf{e}^{-3t}$, is also a decreasing function (because the coefficient is positive).

Next, we aren't given a range of *t*'s for this problem and so we can assume the largest possible range of *t*'s. Therefore, we are safe in assuming a range of $-\infty < t < \infty$ for the *t*'s.

Now, as we've already noted the know that the *x* equation is decreasing and so the largest value of *x* will occur at the left "end point" of the range. Likewise, the smallest value of *x* will occur at the right "end point" of the range. For this problem both "end points" of our range are in fact infinities so we can't just plug in as we did in the previous problem. We can however take the following two limits.

$$\lim_{t \to -\infty} \left(\tfrac{1}{2}\mathbf{e}^{-3t} \right) = \infty \qquad\qquad \lim_{t \to \infty} \left(\tfrac{1}{2}\mathbf{e}^{-3t} \right) = 0$$

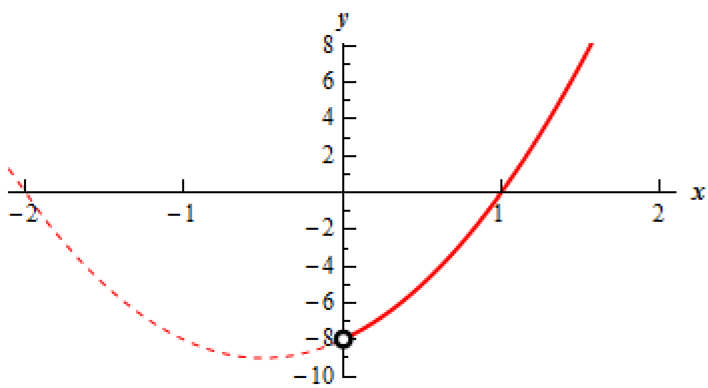
From this we can see that as we approach the left end point of the *t* range the value of *x* is going to infinity and as we approach the right end point of the *t* range the *x* value is going to zero. Note however that *x* can never actually be zero because *x* is still defined in terms of an exponential function (which can't be zero). All the limit is telling us is that as we let $t \to \infty$ we will get $x \to 0$.

The range of *x* for our parametric curve is therefore,

$$0 < x < \infty$$

Again, be careful with the inequalities here! We know that *x* can be neither zero nor infinity so we must use strict inequalities for this range. This is something that we always need to be on the lookout for with variable ranges of parametric equations. Depending on the parametric equations sometimes the end points of the ranges will be strict inequalities (as with this problem) and for others they include the end points (as with the previous problems).

For the range of *y*'s we will need to do a little work to get the correct range of *y*'s but it won't be as much extra work as in previous problems and we can do it all in this step. Let's just sketch the graph and see what we get. Here is a quick sketch of the graph.



The dotted portion of the graph is full graph of the function on $-2 \le t \le 2$ without regards to the actual restriction on *x*. The solid portion of the graph is the portion of the graph that corresponds to the restriction on *x* that we found earlier in this step.

Note the "open dot" on the *y*-axis for the left end of the graph. This needs to be here to acknowledge that $x \neq 0$. We can also see that the *y* value at this point is $y = -8$ and again we can see that for the parametric curve we have $y \neq -8$.

Also, keep in mind that we know that $x \to \infty$ and so we also know that the *y* portion of the graph must also continue up to infinity to match the *x* behavior.

So, from this quick analysis of the graph we can see that the *y* range for the parametric curve must be,

$$-8 < y < \infty$$

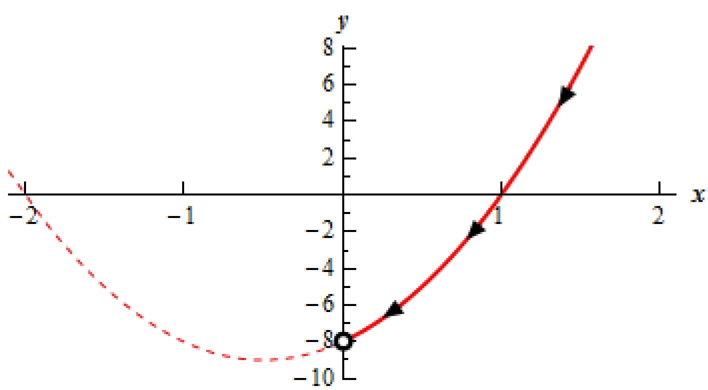
Step 3
We now need to do the direction of motion for this curve but note that we actually found the direction of motion in the previous step.

As noted in the previous step we know that $x = \frac{1}{2}\mathbf{e}^{-3t}$ is a decreasing function and so the *x*'s must be decreasing as *t* increases. Therefore, the equation must be moving from right to left as the curve is traced out.

Also note that unlike most of the previous problems we know that no portion of the graph will be retraced. Again, we know the *x* equation is a decreasing equation. If the curve were to retrace any portion we can see that the only way to do that would be to move back from left to right which would require *x* to increase and that can't happen.

This means that we now know as well that the graph will trace out exactly once.

Step 4
Now that we have all the needed information we can do a formal sketch of the graph.



As with the graph above the dotted portion of the graph is full graph of the function on $-2 \le t \le 2$ and the solid portion of the graph is the portion that corresponds to the restrictions *x* and *y* we found in Step 2.

Here is also the formal answers for all the rest of the information that problem asked for.

$$\begin{array}{ll} \text{Range of } x: & 0 \le x < \infty \\ \text{Range of } y: & -8 \le y < \infty \\ \text{Range of } t \text{ for one trace}: & -\infty < t < \infty \\ \text{Total number of traces}: & 1 \end{array}$$

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12. Write down a set of parametric equations for the following equation.

$$y = 3x^2 - \ln(4x+2)$$

Solution
There really isn't a lot to this problem. All we need to do is use the "formulas" right at the end of the notes for this section.

A set of parametric equations for the equation above are,

$$\begin{aligned} x &= t \\ y &= 3t^2 - \ln(4t+2) \end{aligned}$$

---

13. Write down a set of parametric equations for the following equation.

$$x^2 + y^2 = 36$$

The parametric curve resulting from the parametric equations should be at $(6,0)$ when $t = 0$ and the curve should have a counter clockwise rotation.

Solution
If we don't worry about the "starting" point (*i.e.* where the curve is at when $t = 0$) and we don't worry about the direction of motion we know from the notes that the following set of parametric equations will trace out a circle of radius 6 centered at the origin.

$$\begin{aligned} x &= 6\cos(t) \\ y &= 6\sin(t) \end{aligned}$$

All we need to do is verify if the extra requirements are met or not.

First, we can clearly see with a quick evaluation that when $t = 0$ we are at the point $(6,0)$ as we need to be.

Next, we can either use our knowledge from the examples worked in the notes for this section or an analysis similar to some of the earlier problems in this section to verify that circles in this form will always trace out in a counter clockwise rotation.

In other words, the set of parametric equations give above is a set of parametric equations which will trace out the given circle with the given restrictions. So, formally the answer for this problem is,

$$\boxed{\begin{aligned} x &= 6\cos(t) \\ y &= 6\sin(t) \end{aligned}}$$

We'll leave this problem with a final note about the answer here. This is possibly the "simplest" answer we could give but it is completely possible that you may have come up with a different answer to this problem. There are almost always lots of different possible sets of parametric equations that will trace out a particular parametric curve according to some particular set of restrictions.

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14. Write down a set of parametric equations for the following equation.

$$\frac{x^2}{4} + \frac{y^2}{49} = 1$$

The parametric curve resulting from the parametric equations should be at $\left(0,-7\right)$ when $t = 0$ and the curve should have a clockwise rotation.

Solution

If we don't worry about the "starting" point (*i.e.* where the curve is at when $t = 0$) and we don't worry about the direction of motion we know from the notes that the following set of parametric equations will trace out the ellipse given by the equation above.

$$\begin{aligned} x &= 2\cos\left(t\right) \\ y &= 7\sin\left(t\right) \end{aligned}$$

The problem with this set of parametric equations is that when $t = 0$ we are at the point $\left(2,0\right)$ which is not the point we are supposed to be at. Also, from our knowledge of the examples worked in the notes for this section or an analysis similar to some of the earlier problems in this section we can see that the parametric curve traced out by this set of equations will trace out in a counter clockwise rotation – again not what we need.

So, we need to come up with a different set of parametric equations that meets the requirements.

The first thing to acknowledge is that using sine and cosine will always be the easiest way to get a set of parametric equations for an ellipse. However, there is no reason at all to always use cosine for the *x* equation and sine for the *y* equation.

Knowing that we need $x = 0$ and $y = -7$ when $t = 0$ and using the fact that we know that $\sin\left(0\right) = 0$ and $\cos\left(0\right) = 1$ the following set of parametric equations will "start" at the correct point when $t = 0$.

$$\begin{aligned} x &= -2\sin\left(t\right) \\ y &= -7\cos\left(t\right) \end{aligned}$$

All we need to do now is check if this will trace out the ellipse in a clockwise direction.

If we start at $t = 0$ and increase *t* until we reach $t = \frac{\pi}{2}$ we know that sine will increase from 0 to 1. This will in turn mean that *x* must decrease (don't forget the minus sign on the *x* equation) from 0 to -2.

Likewise, increasing *t* from $t = 0$ to $t = \frac{\pi}{2}$ we know that cosine will decrease from 1 to 0. This in turn means that *y* will increase (don't forget the minus sign on the *y* equation!) from -7 to 0.

The only way for both of these things to happen at the same time is for the curve to start at $\left(0,-7\right)$ when $t = 0$ and trace along the ellipse in a clockwise direction until we reach the point $\left(-2,0\right)$ when $t = \frac{\pi}{2}$.

We could continue in this fashion further increasing *t* until it reaches $t = 2\pi$ (which will put us back at the "starting" point) and convince ourselves that the ellipse will continue to trace out in a clockwise direction.

Therefore, one possible set of parametric equations that we could use is,

$$\boxed{\begin{aligned} x &= -2\sin\left(t\right) \\ y &= -7\cos\left(t\right) \end{aligned}}$$

We'll leave this problem with a final note about the answer here. This is possibly the "simplest" answer we could give but it is completely possible that you may have come up with a different answer to this problem. There are almost always lots of different possible sets of parametric equations that will trace out a particular parametric curve according to some particular set of restrictions.

Section 3-2 : Tangents with Parametric Equations

1. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^2 - 2t + 7 \qquad \frac{dy}{dt} = 4t^3$$

Step 2
The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{4t^3}{12t^2 - 2t + 7}}$$

Step 3
For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{4t^3}{12t^2 - 2t + 7}\right) = \frac{\left(12t^2\right)\left(12t^2 - 2t + 7\right) - 4t^3\left(24t - 2\right)}{\left(12t^2 - 2t + 7\right)^2} = \boxed{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^2}}$$

Step 4
The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^2}}{12t^2 - 2t + 7} = \boxed{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^3}}$$

---

2. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x = \mathbf{e}^{-7t} + 2 \qquad y = 6\mathbf{e}^{2t} + \mathbf{e}^{-3t} - 4t$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -7\mathbf{e}^{-7t} \qquad\qquad \frac{dy}{dt} = 12\mathbf{e}^{2t} - 3\mathbf{e}^{-3t} - 4$$

Step 2
The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12\mathbf{e}^{2t} - 3\mathbf{e}^{-3t} - 4}{-7\mathbf{e}^{-7t}} = \boxed{-\tfrac{12}{7}\mathbf{e}^{9t} + \tfrac{3}{7}\mathbf{e}^{4t} + \tfrac{4}{7}\mathbf{e}^{7t}}$$

Step 3
For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-\tfrac{12}{7}\mathbf{e}^{9t} + \tfrac{3}{7}\mathbf{e}^{4t} + \tfrac{4}{7}\mathbf{e}^{7t}\right) = \boxed{-\tfrac{108}{7}\mathbf{e}^{9t} + \tfrac{12}{7}\mathbf{e}^{4t} + 4\mathbf{e}^{7t}}$$

Step 4
The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\tfrac{108}{7}\mathbf{e}^{9t} + \tfrac{12}{7}\mathbf{e}^{4t} + 4\mathbf{e}^{7t}}{-7\mathbf{e}^{-7t}} = \boxed{\tfrac{108}{49}\mathbf{e}^{16t} - \tfrac{12}{49}\mathbf{e}^{11t} - \tfrac{4}{7}\mathbf{e}^{14t}}$$

---

3. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t) \quad \text{at} \quad t = \tfrac{\pi}{2}$$

Step 1
We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt} = -6\sin(3t) - 12\cos(3t) \qquad\qquad \frac{dy}{dt} = 18\sec^2(6t)$$

The first derivative is then,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{18\sec^2(6t)}{-6\sin(3t)-12\cos(3t)}=\frac{3\sec^2(6t)}{-\sin(3t)-2\cos(3t)}$$

Step 2
The slope of the tangent line at $t=\frac{\pi}{2}$ is then,

$$m=\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}}=\frac{3(-1)^2}{-(-1)-2(0)}=3$$

At $t=\frac{\pi}{2}$ the parametric curve is at the point,

$$x_{t=\frac{\pi}{2}}\Big|=2(0)-4(-1)=4 \qquad y_{t=\frac{\pi}{2}}\Big|=3(0)=0 \qquad \Rightarrow \qquad (4,0)$$

Step 3
The (only) tangent line for this problem is then,

$$y=0+3(x-4) \qquad \to \qquad \boxed{y=3x-12}$$

---

4. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x=t^2-2t-11 \qquad y=t(t-4)^3-3t^2(t-4)^2+7 \quad \text{at} \quad (-3,7)$$

Step 1
We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt}=2t-2$$
$$\frac{dy}{dt}=(t-4)^3+3t(t-4)^2-6t(t-4)^2-6t^2(t-4)=(t-4)^3-3t(t-4)^2-6t^2(t-4)$$

The first derivative is then,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{(t-4)^3-3t(t-4)^2-6t^2(t-4)}{2t-2}$$

Hint : Don't forget that because the derivative we found above is in terms of *t* we need to determine the value(s) of *t* that put the parametric curve at the given point.

Step 2

Okay, the derivative we found above is in terms of *t* and we we'll need to next determine the value(s) of *t* that put the parametric curve at $\left(-3,7\right)$.

This is easy enough to do by setting the *x* and *y* coordinates equal to the known parametric equations and determining the value(s) of *t* that satisfy both equations.

Doing that gives,

$$\begin{aligned} -3 &= t^2 - 2t - 11 \\ 0 &= t^2 - 2t - 8 \\ 0 &= \left(t-4\right)\left(t+2\right) \qquad \to \qquad t = -2, t = 4 \end{aligned}$$

$$\begin{aligned} 7 &= t\left(t-4\right)^3 - 3t^2\left(t-4\right)^2 + 7 \\ 0 &= \left(t-4\right)^2\left[t\left(t-4\right) - 3t^2\right] \\ 0 &= \left(t-4\right)^2\left[-4t - 2t^2\right] \\ 0 &= -2t\left(t-4\right)^2\left[2+t\right] \qquad \to \qquad t = -2, t = 0, t = 4 \end{aligned}$$

We can see from this list that the parametric curve will be at $\left(-3,7\right)$ for $t = -2$ and $t = 4$.

Step 3

From the previous step we can see that we will in fact have two tangent lines at the point. Here are the slopes for each tangent line.

The slope of the tangent line at $t = -2$ is,

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -24$$

and the slope of the tangent line at $t = 4$ is,

$$m = \left.\frac{dy}{dx}\right|_{t=4} = 0$$

Step 4

The tangent line for $t = -2$ is then,

$$y = 7 - 24\left(x+3\right) \qquad \to \qquad \boxed{y = -24x - 65}$$

The tangent line for $t = 4$ is then,

$$y = 7 - (0)(x+3) \qquad \to \qquad \boxed{y = 7}$$

Do not get excited about the second tangent line! It is just saying that the second tangent line is a horizontal line.

---

5. Find the values of *t* that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x = t^5 - 7t^4 - 3t^3 \qquad y = 2\cos(3t) + 4t$$

Step 1
We'll need the following derivatives for this problem.

$$\frac{dx}{dt} = 5t^4 - 28t^3 - 9t^2 \qquad \frac{dy}{dt} = -6\sin(3t) + 4$$

Step 2
We know that horizontal tangent lines will occur where $\frac{dy}{dt} = 0$, provided $\frac{dx}{dt} \neq 0$ at the same value of *t*.

So, to find the horizontal tangent lines we'll need to solve,

$$-6\sin(3t) + 4 = 0 \qquad \to \qquad \sin(3t) = \tfrac{2}{3} \qquad \to \qquad 3t = \sin^{-1}\left(\tfrac{2}{3}\right) = 0.7297$$

Also, a quick glance at a unit circle we can see that a second angle is,

$$3t = \pi - 0.7297 = 2.4119$$

All possible values of *t* that will give horizontal tangent lines are then,

$$\begin{array}{l} 3t = 0.7297 + 2\pi n \\ 3t = 2.4119 + 2\pi n \end{array} \qquad \to \qquad \begin{array}{l} t = 0.2432 + \frac{2}{3}\pi n \\ t = 0.8040 + \frac{2}{3}\pi n \end{array}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Note that we don't officially know these do in fact give horizontal tangent lines until we also determine that $\frac{dx}{dt} \neq 0$ at these points. We'll be able to determine that after the next step.

Step 3

We know that vertical tangent lines will occur where $\frac{dx}{dt} = 0$, provided $\frac{dy}{dt} \neq 0$ at the same value of *t*.

So, to find the vertical tangent lines we'll need to solve,

$$5t^4 - 28t^3 - 9t^2 = 0$$

$$t^2\left(5t^2 - 28t - 9\right) = 0 \qquad \to \qquad t = 0,\ t = \frac{28 \pm \sqrt{964}}{10} \qquad \to \qquad t = 0,\ t = -0.3048,\ t = 5.9048$$

Step 4

From a quick inspection of the two lists of *t* values from Step 2 and Step 3 we can see there are no values in common between the two lists. Therefore, any values of *t* that gives $\frac{dy}{dt} = 0$ will not give $\frac{dx}{dt} = 0$ and visa-versa.

Therefore the values of *t* that gives horizontal tangent lines are,

$$\boxed{\begin{array}{l} t = 0.2432 + \frac{2}{3}\pi n \\ t = 0.8040 + \frac{2}{3}\pi n \end{array}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

The values of *t* that gives vertical tangent lines are,

$$\boxed{t = 0,\ t = -0.3048,\ t = 5.9048}$$

Section 3-3 : Area with Parametric Equations

1. Determine the area of the region below the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once from right to left for the given range of *t*. You should only use the given parametric equations to determine the answer.

$$x = 4t^3 - t^2 \qquad y = t^4 + 2t^2 \qquad 1 \le t \le 3$$

Solution

There really isn't too much to this problem. Just recall that the formula from the notes assumes that $x = f\left(t\right)$ and $y = g\left(t\right)$. So, the area under the curve is,

$$\begin{aligned} A &= \int_1^3 \left(t^4 + 2t^2\right)\left(12t^2 - 2t\right) dt \\ &= \int_1^3 12t^6 - 2t^5 + 24t^4 - 4t^3\, dt \\ &= \left(\tfrac{12}{7}t^7 - \tfrac{1}{3}t^6 + \tfrac{24}{5}t^5 - t^4\right)\Big|_1^3 = \boxed{\tfrac{481568}{105} = 4586.3619} \end{aligned}$$

---

2. Determine the area of the region below the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once from right to left for the given range of *t*. You should only use the given parametric equations to determine the answer.

$$x = 3 - \cos^3\left(t\right) \qquad y = 4 + \sin\left(t\right) \qquad 0 \le t \le \pi$$

Solution

There really isn't too much to this problem. Just recall that the formula from the notes assumes that $x = f\left(t\right)$ and $y = g\left(t\right)$. So, the area under the curve is,

$$\begin{aligned} A &= \int_0^\pi \left(4 + \sin\left(t\right)\right)\left(3\cos^2\left(t\right)\sin\left(t\right)\right) dt \\ &= \int_0^\pi 12\cos^2\left(t\right)\sin\left(t\right) + 3\cos^2\left(t\right)\sin^2\left(t\right) dt \\ &= \int_0^\pi 12\cos^2\left(t\right)\sin\left(t\right) + 3\left[\tfrac{1}{2}\sin\left(2t\right)\right]^2 dt \\ &= \int_0^\pi 12\cos^2\left(t\right)\sin\left(t\right) + \tfrac{3}{4}\sin^2\left(2t\right) dt \\ &= \int_0^\pi 12\cos^2\left(t\right)\sin\left(t\right) + \tfrac{3}{8}\left(1 - \cos\left(4t\right)\right) dt \\ &= \left(-4\cos^3\left(t\right) + \tfrac{3}{8}t - \tfrac{3}{32}\sin\left(4t\right)\right)\Big|_0^\pi = \boxed{8 + \tfrac{3}{8}\pi} \end{aligned}$$

You did recall how to do all the trig manipulations and trig integrals to do this integral correct? If not you should go back to the Integrals Involving Trig Functions section to do some review/problems.

Section 3-4 : Arc Length with Parametric Equations

1. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + \left(8 - t\right)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad\qquad \frac{dy}{dt} = -\tfrac{3}{2}\left(8 - t\right)^{\frac{1}{2}}$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\tfrac{3}{2}\left(8-t\right)^{\frac{1}{2}}\right]^2}\,dt = \sqrt{144t + \tfrac{9}{4}\left(8-t\right)}\,dt = \sqrt{\tfrac{567}{4}t + 18}\,dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_0^4 \sqrt{\tfrac{567}{4}t + 18}\,dt$$

Step 4
This is a simple integral to compute with a quick substitution. Here is the integral work,

$$L = \int_0^4 \sqrt{\tfrac{567}{4}t + 18}\,dt = \tfrac{4}{567}\left(\tfrac{2}{3}\right)\left(\tfrac{567}{4}t + 18\right)^{\frac{3}{2}}\Big|_0^4 = \boxed{\tfrac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865}$$

---

2. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 3t + 1 \qquad y = 4 - t^2 \qquad -2 \le t \le 0$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 3 \qquad\qquad \frac{dy}{dt} = -2t$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[3\right]^2 + \left[-2t\right]^2}\,dt = \sqrt{9 + 4t^2}\,dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_{-2}^{0} \sqrt{9 + 4t^2}\,dt$$

Step 4
This integral will require a trig substitution (as will quite a few arc length integrals!).

Here is the trig substitution we'll need for this integral.

$$t = \tfrac{3}{2}\tan\theta \qquad dt = \tfrac{3}{2}\sec^2\theta\,d\theta$$
$$\sqrt{9 + 4t^2} = \sqrt{9 + 9\tan^2\theta} = 3\sqrt{1 + \tan^2\theta} = 3\sqrt{\sec^2\theta} = 3\left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$\begin{array}{llllll} t = -2: & -2 = \tfrac{3}{2}\tan\theta & \to & \tan\theta = -\tfrac{4}{3} & \to & \theta = \tan^{-1}\left(-\tfrac{4}{3}\right) = -0.9273 \\ t = 0: & 0 = \tfrac{3}{2}\tan\theta & \to & \tan\theta = 0 & \to & \theta = 0 \end{array}$$

Okay, the corresponding range of θ for this problem is $-0.9273 \le \theta \le 0$ (fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{9 + 4t^2} = 3\sec\theta$$

The integral is then,

$$\begin{aligned} L &= \int_{-2}^{0} \sqrt{9 + 4t^2}\,dt = \int_{-0.9273}^{0} \left(3\sec\theta\right)\left(\tfrac{3}{2}\sec^2\theta\right)d\theta \\ &= \int_{-0.9273}^{0} \tfrac{9}{2}\sec^3\theta\,d\theta = \tfrac{9}{4}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big]\Big|_{-0.9273}^{0} = \boxed{7.4719} \end{aligned}$$

---

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad\qquad -52\pi \le t \le 34\pi$$

Hint : Be very careful with this problem. Note the two quantities we are being asked to find, how they relate to each other and which of the two that we know how to compute from the material in this section.

Step 1
This is a problem that many students have issues with. First note that we are being asked to find both the total distance traveled by the particle AND the length of the curve. Also, recall that of these two quantities we only discussed how to determine the length of a curve in this section.

Therefore, let's concentrate on finding the length of the curve first, then we'll worry about the total distance traveled.

Step 2
To find the length we'll need the following two derivatives,

$$\frac{dx}{dt} = \cos\left(\tfrac{1}{4}t\right) \qquad\qquad \frac{dy}{dt} = \cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)$$

The *ds* for this problem is then,

$$ds = \sqrt{\left[\cos\left(\tfrac{1}{4}t\right)\right]^2 + \left[\cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)\right]^2}\,dt = \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Now, this is where many students run into issues with this problem. Many students use the following integral to determine the length of the curve.

$$\int_{-52\pi}^{34\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Can you see what is wrong with this integral?

Step 3
Remember from the discussion in this section that in order to use the arc length formula the curve can only trace out exactly once over the range of the limits in the integral.

Therefore, we can't even write down the formula that we did in the previous step until we first determine if the curve traces out exactly once in the given range of *t*'s.

If it turns out that the curve traces out more than once in the given range of *t*'s then the integral we wrote down in the previous step is simply wrong. We will then need to determine a range of *t*'s for one trace so we can write down the proper integral for the length.

Luckily enough for us we actually did this in a practice problem in the Parametric Equations and Curves section. Examining this set of parametric equations was problem #8 from that section. From the solution to that problem we found a couple of pieces of information that will be needed for this problem.

First, we determined that the curve does trace out more than once in the given range of *t*'s for the problem. In fact, we determined that the curve traced out 21.5 times over the given range of *t*'s.

Secondly, and more importantly at this point, we also determined that the curve would trace out exactly once in the range of $-2\pi \le t \le 2\pi$. Note that we actually listed several possible ranges of *t*'s for one trace and we can use any of them. This is simply the first one found and so it's the one we decided to use for this problem.

So, we can now see that the integral we wrote down in the previous step was in fact not correct and will not give the length of the curve.

Using the range of *t*'s we found in the earlier problem we can see that the integral for the length is,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\, dt$$

Before proceeding with the next step we should address the fact that, in this case, we already had the information in hand to write down the proper integral for the length. In most cases this will not be the case and you will need to go back and do a shortened version of the analysis we did in the Parametric Equations and Curves section. We don't need all the information but to get the pertinent information we will need to go through most of the analysis.

If you need a refresher on how to do that analysis you should go back to that section and work through a few of the practice problems.

Step 4

Okay, let's get to work on evaluating the integral. At first glance this looks like a really unpleasant integral (and there's no ignoring the fact that it's not a super easy integral) however if we're careful it isn't as difficult as it might appear at first glance.

First, let's notice that we can do a little simplification as follows,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right)\left(1 + \sin^2\left(\tfrac{1}{4}t\right)\right)}\, dt = \int_{-2\pi}^{2\pi} \left|\cos\left(\tfrac{1}{4}t\right)\right| \sqrt{1 + \sin^2\left(\tfrac{1}{4}t\right)}\, dt$$

As always, be very careful with the absolute value bars! Depending on the range of *t*'s we used for the integral it might not be possible to just drop them. So, the next thing let's do is determine how to deal with them.

We know that, for this integral, we used the following range of *t*'s.

$$-2\pi \le t \le 2\pi$$

Now, notice that we don't just have a *t* in the argument for the trig functions in our integral. In fact what we have is $\frac{1}{4}t$. So, we can see that as *t* ranges over $-2\pi \le t \le 2\pi$ we will get the following range for $\frac{1}{4}t$ (just divide the above inequality by 4!),

$$-\tfrac{\pi}{2} \le \tfrac{1}{4}t \le \tfrac{\pi}{2}$$

We know from our trig knowledge that as long as $\frac{1}{4}t$ stays in this range, which it will for our integral, then $\cos\left(\frac{1}{4}t\right) \ge 0$ and so we can drop the absolute values from the cosine in the integral.

As a final word of warning here keep in mind that for other ranges of *t*'s we might have had negative cosine in the range of *t*'s and so we'd need to add in a minus sign to the integrand when we dropped the absolute value bars. Whether or not we need to do this will depend upon the range of *t*'s we chose to use for one trace and so this quick analysis will need to be done for these kinds of integrals.

Step 5

Okay, at this point the integral for the length of the curve is now,

$$L = \int_{-2\pi}^{2\pi} \cos\left(\tfrac{1}{4}t\right)\sqrt{1+\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

This still looks to be an unpleasant integral. However, in this case note that we can use the following simply substitution to convert it into a relatively easy integral.

$$u = \sin\left(\tfrac{1}{4}t\right) \quad \to \quad \sin^2\left(\tfrac{1}{4}t\right) = u^2 \qquad du = \tfrac{1}{4}\cos\left(\tfrac{1}{4}t\right)dt$$
$$t = -2\pi: \quad u = \sin\left(-\tfrac{1}{2}\pi\right) = -1 \qquad\qquad t = 2\pi: \quad u = \sin\left(\tfrac{1}{2}\pi\right) = 1$$

Under this substitution the integral of the length of the curve is then,

$$L = \int_{-1}^{1} 4\sqrt{1+u^2}\,du$$

Step 6

Now, at this point we can see that we have a fairly simple trig substitution we'll need to evaluate to find the length of the curve. The trig substitution we'll need for this integral is,

$$u = \tan\theta \qquad du = \sec^2\theta\,d\theta \qquad\qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$\begin{array}{llllll} u=-1: & -1 = \tan\theta & \to & \tan\theta = -1 & \to & \theta = -\tfrac{\pi}{4} \\ u = 1: & 1 = \tan\theta & \to & \tan\theta = 1 & \to & \theta = \tfrac{\pi}{4} \end{array}$$

Okay, the corresponding range of θ for this problem is $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (first and fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{1+u^2} = \sec\theta$$

The length of the curve is then,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right)\left(1+\sin^2\left(\tfrac{1}{4}t\right)\right)}\,dt = \int_{-1}^{1} 4\sqrt{1+u^2}\,du$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\sec^3\theta\,d\theta = 2\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \boxed{9.1824}$$

Step 7
Okay, that was quite a bit of work to get the length of the curve. We now need to recall that we were also asked to get the total distance traveled by the particle. This is actually quite easy to get now that we have the length of the curve.

To get the total distance traveled all we need to recall is that we noted in Step 3 above that we determined in problem #8 from the Parametric Equations and Curves section that the curve will trace out 21.5 times. Since we also know the length of a single trace of the curve we know that the total distance traveled by the particle must be,

$$\text{Total Distance Traveled} = (9.1824)(21.5) = \boxed{197.4216}$$

---

4. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$x = 2 + t^2 \qquad y = \mathbf{e}^t \sin(2t) \qquad 0 \le t \le 3$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = \mathbf{e}^t\sin(2t) + 2\mathbf{e}^t\cos(2t)$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{[2t]^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt = \sqrt{4t^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_0^3 \sqrt{4t^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt$$

---

5. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$x = \cos^3\left(2t\right) \qquad y = \sin\left(1 - t^2\right) \qquad -\tfrac{3}{2} \le t \le 0$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -6\cos^2\left(2t\right)\sin\left(2t\right) \qquad\qquad \frac{dy}{dt} = -2t\cos\left(1 - t^2\right)$$

Step 2
We'll need the *ds* for this problem.

$$\begin{aligned} ds &= \sqrt{\left[-6\cos^2\left(2t\right)\sin\left(2t\right)\right]^2 + \left[-2t\cos\left(1 - t^2\right)\right]^2}\,dt \\ &= \sqrt{36\cos^4\left(2t\right)\sin^2\left(2t\right) + 4t^2\cos^2\left(1 - t^2\right)}\,dt \end{aligned}$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_{-\frac{3}{2}}^{0} \sqrt{36\cos^4\left(2t\right)\sin^2\left(2t\right) + 4t^2\cos^2\left(1 - t^2\right)}\,dt$$

Section 3-5 : Surface Area with Parametric Equations

1. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$\text{Rotate } x = 3 + 2t \qquad y = 9 - 3t \qquad 1 \le t \le 4 \quad \text{about the } y\text{-axis.}$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = -3$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[2\right]^2 + \left[-3\right]^2}\, dt = \sqrt{13}\, dt$$

Step 3
The integral for the surface area is then,

$$SA = \int 2\pi x\, ds = \int_1^4 2\pi\left(3 + 2t\right)\sqrt{13}\, dt = 2\pi\sqrt{13}\int_1^4 3 + 2t\, dt$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4
This is a really simple integral…

$$SA = 2\pi\sqrt{13}\int_1^4 3 + 2t\, dt = 2\pi\sqrt{13}\left(3t + t^2\right)\Big|_1^4 = \boxed{48\pi\sqrt{13}}$$

---

2. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$\text{Rotate } x = 9 + 2t^2 \qquad y = 4t \qquad 0 \le t \le 2 \quad \text{about the } x\text{-axis.}$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[4t\right]^2 + \left[4\right]^2}\,dt = \sqrt{16t^2 + 16}\,dt = 4\sqrt{t^2+1}\,dt$$

Note that we factored a 16 out of the root to make the rest of the work a little simpler to deal with.

Step 3
The integral for the surface area is then,

$$SA = \int 2\pi y\,ds = \int_0^2 2\pi\left(4t\right)\left(4\sqrt{t^2+1}\right)dt = 32\pi\int_0^2 t\sqrt{t^2+1}\,dt$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4
This is a simple integral to compute with a quick substitution. Here is the integral work,

$$SA = 32\pi\int_0^2 t\sqrt{t^2+1}\,dt = \tfrac{32}{3}\pi\left(t^2+1\right)^{\frac{3}{2}}\Big|_0^2 = \boxed{\tfrac{32}{3}\pi\left(5^{\frac{3}{2}}-1\right) = 341.1464}$$

---

3. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$\text{Rotate } x = 3\cos\left(\pi t\right) \qquad y = 5t+2 \qquad 0 \le t \le \tfrac{1}{2} \text{ about the } y\text{-axis.}$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -3\pi\sin\left(\pi t\right) \qquad\qquad \frac{dy}{dt} = 5$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[-3\pi\sin\left(\pi t\right)\right]^2 + \left[5\right]^2}\,dt = \sqrt{9\pi^2\sin^2\left(\pi t\right) + 25}\,dt$$

Step 3
The integral for the surface area is then,

$$\begin{aligned} SA = \int 2\pi x\, ds &= \int_0^{\frac{1}{2}} 2\pi\left(3\cos\left(\pi t\right)\right)\sqrt{9\pi^2\sin^2\left(\pi t\right)+25}\,dt \\ &= 6\pi\int_0^{\frac{1}{2}}\cos\left(\pi t\right)\sqrt{9\pi^2\sin^2\left(\pi t\right)+25}\,dt \end{aligned}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4
Okay, this is a particularly unpleasant looking integral but we need to be able to deal with these kinds of integrals on occasion. We'll be able to do quite a bit of simplification if we first use the following substitution.

$$\begin{array}{lll} & u = \sin\left(\pi t\right) \quad \to \quad \sin^2\left(\pi t\right) = u^2 & du = \pi\cos\left(\pi t\right) \\ t = 0: & u = \sin\left(0\right) = 0 & t = \tfrac{1}{2}: \quad u = \sin\left(\tfrac{1}{2}\pi\right) = 1 \end{array}$$

With this substitution the integral becomes,

$$SA = 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du$$

Step 5
This integral can be evaluated with the following (somewhat messy…) trig substitution.

$$t = \tfrac{5}{3\pi}\tan\theta \qquad dt = \tfrac{5}{3\pi}\sec^2\theta\,d\theta$$

$$\sqrt{9\pi^2u^2+25} = \sqrt{25\tan^2\theta+25} = 5\sqrt{\tan^2\theta+1} = 5\sqrt{\sec^2\theta} = 5\left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$\begin{array}{llllll} u = 0: & 0 = \tfrac{5}{3\pi}\tan\theta & \to & \tan\theta = 0 & \to & \theta = 0 \\ u = 1: & 1 = \tfrac{5}{3\pi}\tan\theta & \to & \tan\theta = \tfrac{3\pi}{5} & \to & \theta = \tan^{-1}\left(\tfrac{3\pi}{5}\right) = 1.0830 \end{array}$$

Okay, the corresponding range of θ for this problem is $0 \le \theta \le 1.0830$ (first quadrant) and in this range we know that secant is positive. Therefore, the root becomes,

$$\sqrt{9\pi^2u^2+25} = 5\sec\theta$$

The surface area is then,

$$\begin{aligned} SA &= \int_0^{\frac{1}{2}} 2\pi\left(3\cos\left(\pi t\right)\right)\sqrt{9\pi^2\sin^2\left(\pi t\right)+25}\,dt \\ &= 6\int_0^1 \sqrt{9\pi^2 u^2+25}\,du \\ &= 6\int_0^{1.0830}\left(5\sec\theta\right)\left(\tfrac{5}{3\pi}\sec^2\theta\right)d\theta \\ &= 6\int_0^{1.0830}\tfrac{25}{3\pi}\sec^3\theta\,d\theta \\ &= \tfrac{25}{\pi}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^{1.0830} = \boxed{43.0705} \end{aligned}$$

This problem was a little messy but don't let that make you decide that you can't do these types of problems! They can be done and often can be simplified with some relatively simple substitutions.

---

4. Set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$\text{Rotate } x = 1+\ln\left(5+t^2\right) \qquad y = 2t-2t^2 \qquad 0 \le t \le 2 \quad \text{about the } x\text{-axis.}$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = \frac{2t}{5+t^2} \qquad\qquad \frac{dy}{dt} = 2-4t$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[\frac{2t}{5+t^2}\right]^2+\left[2-4t\right]^2}\,dt = \sqrt{\frac{4t^2}{\left(5+t^2\right)^2}+\left(2-4t\right)^2}\,dt$$

Step 3
The integral for the surface area is then,

$$SA = \int 2\pi y\,ds = \boxed{\int_0^2 2\pi\left(2t-2t^2\right)\sqrt{\frac{4t^2}{\left(5+t^2\right)^2}+\left(2-4t\right)^2}\,dt}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

---

5. Set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$\text{Rotate } x = 1 + 3t^2 \qquad y = \sin\left(2t\right)\cos\left(\tfrac{1}{4}t\right) \qquad 0 \le t \le \tfrac{1}{2} \text{ about the } y\text{-axis.}$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 2\cos\left(2t\right)\cos\left(\tfrac{1}{4}t\right) - \tfrac{1}{4}\sin\left(2t\right)\sin\left(\tfrac{1}{4}t\right)$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[6t\right]^2 + \left[2\cos\left(2t\right)\cos\left(\tfrac{1}{4}t\right) - \tfrac{1}{4}\sin\left(2t\right)\sin\left(\tfrac{1}{4}t\right)\right]^2}\,dt$$

Step 3
The integral for the surface area is then,

$$SA = \int 2\pi x\,ds = \boxed{\int_0^{\frac{1}{2}} 2\pi\left(1 + 3t^2\right)\sqrt{36t^2 + \left(2\cos\left(2t\right)\cos\left(\tfrac{1}{4}t\right) - \tfrac{1}{4}\sin\left(2t\right)\sin\left(\tfrac{1}{4}t\right)\right)^2}\,dt}$$

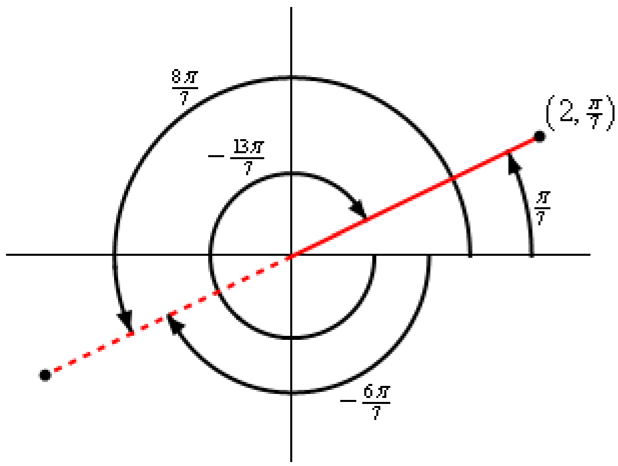
Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Section 3-6 : Polar Coordinates

1. For the point with polar coordinates $\left(2,\frac{\pi}{7}\right)$ determine three different sets of coordinates for the same point all of which have angles different from $\frac{\pi}{7}$ and are in the range $-2\pi \le \theta \le 2\pi$.

Step 1

This problem is really as exercise in how well we understand the unit circle. Here is a quick sketch of the point and some angles.



We can see that the negative angle ending at the solid red line that is in the range specified in the problem statement is simply $\frac{\pi}{7}-2\pi=-\frac{13\pi}{7}$.

If we extend the solid line into the third quadrant (*i.e.* the dashed red line) then the positive angle ending at the dashed red line is $\frac{\pi}{7}+\pi=\frac{8\pi}{7}$. Likewise, the negative angle ending at the dashed red line is $\frac{\pi}{7}-\pi=-\frac{6\pi}{7}$.

With these angles getting the other three points should be pretty simple.

Step 2

For the first "new" point we can use the negative angle that ends on the solid red line to get the point.

$$\boxed{\left(2,-\frac{13\pi}{7}\right)}$$

Step 3

For the remaining points recall that if we use a negative *r* then we go "backwards" from where the angle ends to get the point. So, if we use $r=-2$, any angle that ends on the dashed red line will go "backwards" into the first quadrant 2 units to get to the point.

This gives the remaining two points using both the positive and negative angle ending on the dashed red line,

$$\boxed{\left(-2,-\tfrac{6\pi}{7}\right)} \qquad \boxed{\left(-2,\tfrac{8\pi}{7}\right)}$$

---

2. The polar coordinates of a point are $\left(-5, 0.23\right)$. Determine the Cartesian coordinates for the point.

Solution
There really isn't too much to this problem. From the point we can see that we have $r = -5$ and $\theta = 0.23$ (in radians of course!). Once we have these all we need to is plug into the formulas from this section to get,

$$x = r\cos\theta = \left(-5\right)\cos\left(0.23\right) = -4.8683 \qquad y = r\sin\theta = \left(-5\right)\sin\left(0.23\right) = -1.1399$$

So, the Cartesian coordinates for the point are then,

$$\boxed{\left(-4.8683, -1.1399\right)}$$

---

3. The Cartesian coordinate of a point are $\left(2, -6\right)$. Determine a set of polar coordinates for the point.

Step 1
Let's first determine *r*. That's always simple.

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(2\right)^2 + \left(-6\right)^2} = \sqrt{40} = 2\sqrt{10}$$

Step 2
Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the *r* we found in the first step.

Here are the two possible values of θ.

$$\theta_1 = \tan^{-1}\left(\tfrac{y}{x}\right) = \tan^{-1}\left(\tfrac{-6}{2}\right) = -1.2490 \qquad \theta_2 = \theta_1 + \pi = 1.8926$$

So, we can see that $-\frac{\pi}{2} = -1.57 < \theta_1 = -1.2490 < 0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2} = 1.57 < \theta_2 = 1.8926 < \pi = 3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the fourth quadrant and so, for this problem, θ_1 is the correct value.

The polar coordinates of the point using the *r* from the first step and θ from this step is,

$$\boxed{\left(2\sqrt{10},-1.2490\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

---

4. The Cartesian coordinate of a point are $\left(-8,1\right)$. Determine a set of polar coordinates for the point.

Step 1

Let's first determine *r*. That's always simple.

$$r=\sqrt{x^2+y^2}=\sqrt{\left(-8\right)^2+\left(1\right)^2}=\sqrt{65}$$

Step 2

Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the *r* we found in the first step.

Here are the two possible values of θ.

$$\theta_1=\tan^{-1}\left(\tfrac{y}{x}\right)=\tan^{-1}\left(\tfrac{1}{-8}\right)=-0.1244 \qquad\qquad \theta_2=\theta_1+\pi=3.0172$$

So, we can see that $-\frac{\pi}{2}=-1.57<\theta_1=-0.1244<0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2}=1.57<\theta_2=3.0172<\pi=3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the second quadrant and so, for this problem, θ_2 is the correct value.

The polar coordinates of the point using the *r* from the first step and θ from this step is,

$$\boxed{\left(\sqrt{65},\ 3.0172\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

---

5. Convert the following equation into an equation in terms of polar coordinates.

$$\frac{4x}{3x^2+3y^2}=6-xy$$

Solution
Basically, what we need to do here is to convert all the *x*'s and *y*'s into *r*'s and θ's using the following formulas.

$$x=r\cos\theta \qquad y=r\sin\theta \qquad r^2=x^2+y^2$$

Don't forget about the last one! If it is possible to use this formula (and you can see where we'll use it in the case can't you?) it will save a lot of work!

First let's substitute in the equations as needed.

$$\frac{4\left(r\cos\theta\right)}{3r^2}=6-\left(r\cos\theta\right)\left(r\sin\theta\right)$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{\frac{4\cos\theta}{3r}=6-r^2\cos\theta\sin\theta}$$

---

6. Convert the following equation into an equation in terms of polar coordinates.

$$x^2=\frac{4x}{y}-3y^2+2$$

Solution
Basically, what we need to do here is to convert all the *x*'s and *y*'s into *r*'s and θ's using the following formulas.

$$x=r\cos\theta \qquad y=r\sin\theta \qquad r^2=x^2+y^2$$

Don't forget about the last one! If it is possible to use this formula (which won't do us a lot of good in this problem) it will save a lot of work!

First let's substitute in the equations as needed.

$$\left(r\cos\theta\right)^2=\frac{4\left(r\cos\theta\right)}{r\sin\theta}-3\left(r\sin\theta\right)^2+2$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{r^2\cos^2\theta = 4\cot\theta - 3r^2\sin^2\theta + 2}$$

---

7. Convert the following equation into an equation in terms of Cartesian coordinates.

$$6r^3\sin\theta = 4 - \cos\theta$$

Solution

There is a variety of ways to work this problem. One way is to first multiply everything by *r* and then doing a little rearranging as follows,

$$6r^4\sin\theta = 4r - r\cos\theta \qquad \Rightarrow \qquad 6r^3\left(r\sin\theta\right) = 4r - r\cos\theta$$

We can now use the following formulas to finish this problem.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2}$$

Here is the answer for this problem,

$$\boxed{6y\left[\sqrt{x^2+y^2}\,\right]^3 = 4\sqrt{x^2+y^2} - x}$$

---

8. Convert the following equation into an equation in terms of Cartesian coordinates.

$$\frac{2}{r} = \sin\theta - \sec\theta$$

Solution

There is a variety of ways to work this problem. One way is to first do the following rearranging/rewriting of the equation.

$$\frac{2}{r} = \sin\theta - \frac{1}{\cos\theta} \qquad \to \qquad \frac{2\cos\theta}{r} = \sin\theta\cos\theta - 1$$

At this point we can multiply everything by r^2 and do a little rearranging as follows,

$$2r\cos\theta = r^2\sin\theta\cos\theta - r^2 \qquad \to \qquad 2r\cos\theta = \left(r\sin\theta\right)\left(r\cos\theta\right) - r^2$$

We can now use the following formulas to finish this problem.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Here is the answer for this problem,

$$\boxed{2x = yx - \left(x^2 + y^2\right)}$$

---

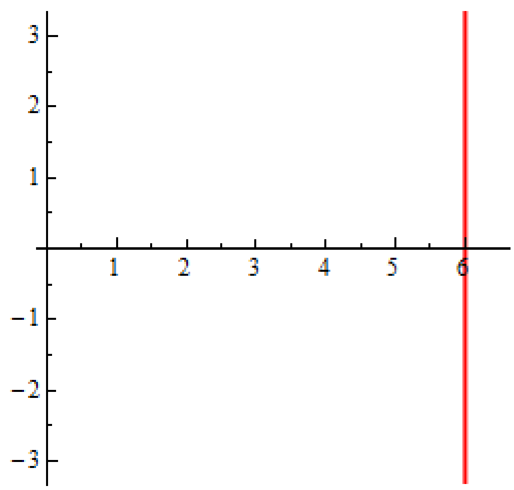
9. Sketch the graph of the following polar equation.

$$\cos\theta = \frac{6}{r}$$

Solution
Multiplying both sides by *r* gives,

$$r\cos\theta = 6$$

and we know from the notes on this section that this is simply the vertical line $x = 6$. So here is the graph of this function.



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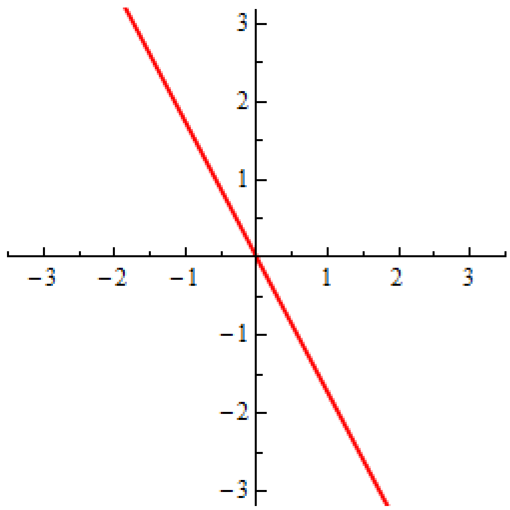
10. Sketch the graph of the following polar equation.

$$\theta = -\frac{\pi}{3}$$

Solution
We know from the notes on this section that this is simply the line that goes through the origin and has slope of $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.

So here is the graph of this function.



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11. Sketch the graph of the following polar equation.

$$r = -14\cos\theta$$

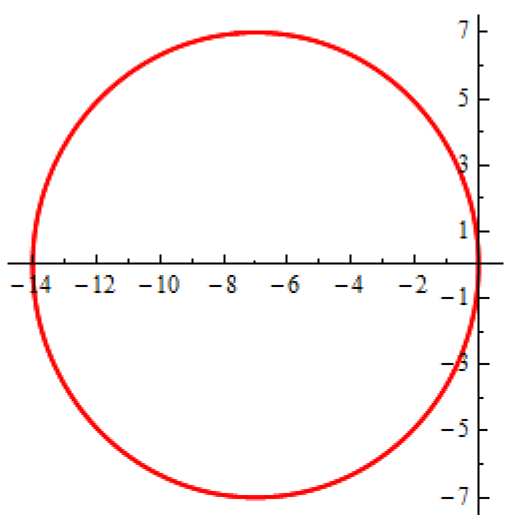
Solution

We can rewrite this as,

$$r = 2\left(-7\right)\cos\theta$$

and so we know from the notes on this section that this is simply the circle with radius 7 and center $\left(-7,0\right)$.

So here is the graph of this function.



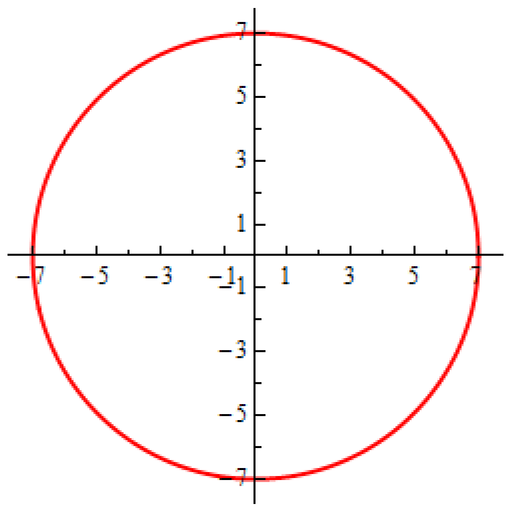
12. Sketch the graph of the following polar equation.

$$r = 7$$

Solution

We know from the notes on this section that this is simply the circle with radius 7 and centered at the origin.

So here is the graph of this function.



13. Sketch the graph of the following polar equation.

$$r = 9\sin\theta$$

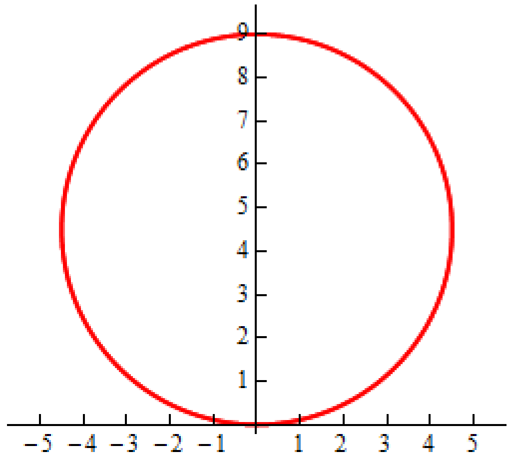
Solution

We can rewrite this as,

$$r = 2\left(\tfrac{9}{2}\right)\sin\theta$$

and so we know from the notes on this section that this is simply the circle with radius $\frac{9}{2}$ and center $\left(0, \frac{9}{2}\right)$.

So here is the graph of this function.



14. Sketch the graph of the following polar equation.

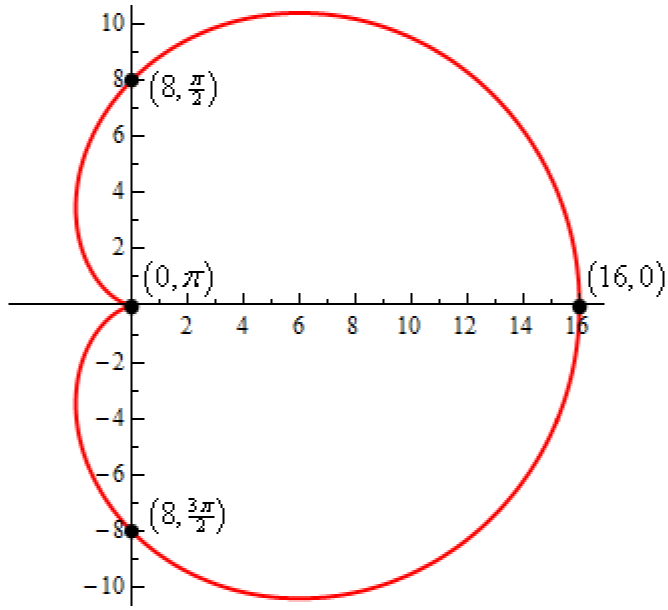
$$r = 8 + 8\cos\theta$$

Solution

We know from the notes on this section that this is a cardioid and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	16	$(16,0)$
$\frac{\pi}{2}$	8	$\left(8,\frac{\pi}{2}\right)$
π	0	$(0,\pi)$
$\frac{3\pi}{2}$	8	$\left(8,\frac{3\pi}{2}\right)$
2π	16	$(16,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The “tick marks” on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

---

15. Sketch the graph of the following polar equation.

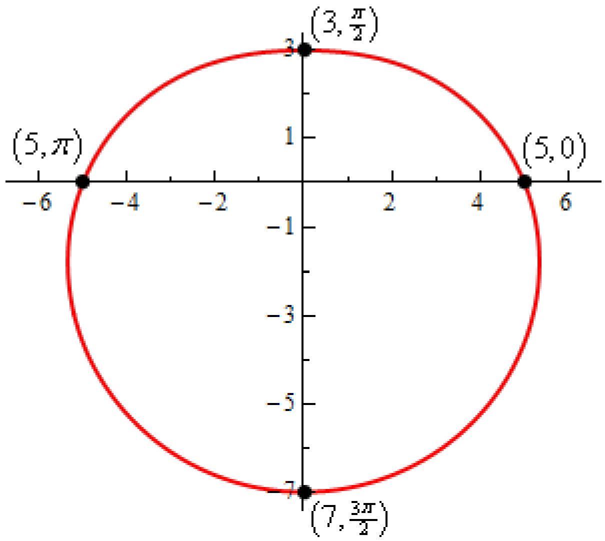
$$r = 5 - 2\sin\theta$$

Solution

We know from the notes on this section that this is a limacon without an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	5	$(5,0)$
$\frac{\pi}{2}$	3	$\left(3,\frac{\pi}{2}\right)$
π	5	$(5,\pi)$
$\frac{3\pi}{2}$	7	$\left(7,\frac{3\pi}{2}\right)$
2π	5	$(5,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

Also, many of these graphs are vaguely heart shaped although as this sketch has shown many do and this one is more circular than heart shaped.

---

16. Sketch the graph of the following polar equation.

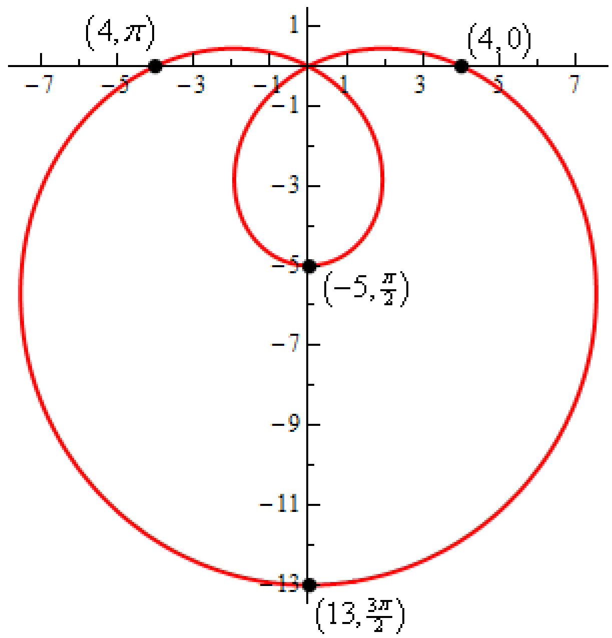
$$r = 4 - 9\sin\theta$$

Solution

We know from the notes on this section that this is a limacon with an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	4	$(4,0)$
$\frac{\pi}{2}$	-5	$(-5,\frac{\pi}{2})$
π	4	$(4,\pi)$
$\frac{3\pi}{2}$	13	$(13,\frac{3\pi}{2})$
2π	4	$(4,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The “tick marks” on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

Section 3-7 : Tangents with Polar Coordinates

1. Find the tangent line to $r = \sin\left(4\theta\right)\cos\left(\theta\right)$ at $\theta = \dfrac{\pi}{6}$.

Step 1

First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 4\cos\left(4\theta\right)\cos\left(\theta\right) - \sin\left(4\theta\right)\sin\left(\theta\right)$$

Step 2

Next using the formula from the notes on this section we have,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{\left(4\cos\left(4\theta\right)\cos\left(\theta\right) - \sin\left(4\theta\right)\sin\left(\theta\right)\right)\sin\theta + \left(\sin\left(4\theta\right)\cos\left(\theta\right)\right)\cos\theta}{\left(4\cos\left(4\theta\right)\cos\left(\theta\right) - \sin\left(4\theta\right)\sin\left(\theta\right)\right)\cos\theta - \left(\sin\left(4\theta\right)\cos\left(\theta\right)\right)\sin\theta}\end{aligned}$$

This is a very messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do…

Step 3

Next, we'll need to evaluate both the derivative from the previous step as well as *r* at $\theta = \dfrac{\pi}{6}$.

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{6}} = \frac{1}{3\sqrt{3}} \qquad\qquad \left. r\right|_{\theta = \frac{\pi}{6}} = \frac{3}{4}$$

You can see why we need both of these right?

Step 4

Last, we need the *x* and *y* coordinate that we'll be at when $\theta = \dfrac{\pi}{6}$. These values are easy enough to find given that we know what *r* is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos\left(\theta\right) = \tfrac{3}{4}\cos\left(\tfrac{\pi}{6}\right) = \tfrac{3\sqrt{3}}{8} \qquad\qquad y = r\sin\left(\theta\right) = \tfrac{3}{4}\sin\left(\tfrac{\pi}{6}\right) = \tfrac{3}{8}$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5
The tangent line is then,

$$\boxed{y = \tfrac{3}{8} + \tfrac{1}{3\sqrt{3}}\left(x - \tfrac{3\sqrt{3}}{8}\right) = \tfrac{1}{3\sqrt{3}}x + \tfrac{1}{4}}$$

---

2. Find the tangent line to $r = \theta - \cos\left(\theta\right)$ at $\theta = \dfrac{3\pi}{4}$.

Step 1
First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 1 + \sin\left(\theta\right)$$

Step 2
Next using the formula from the notes on this section we have,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\\ &= \frac{\left(1+\sin\left(\theta\right)\right)\sin\theta + \left(\theta - \cos\left(\theta\right)\right)\cos\theta}{\left(1+\sin\left(\theta\right)\right)\cos\theta - \left(\theta - \cos\left(\theta\right)\right)\sin\theta}\end{aligned}$$

This is a somewhat messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do…

Step 3
Next, we'll need to evaluate both the derivative from the previous step as well as *r* at $\theta = \dfrac{3\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{3\pi}{4}} = 0.2843 \qquad\qquad \left.r\right|_{\theta=\frac{3\pi}{4}} = 3.0633$$

You can see why we need both of these right?

Step 4

Last, we need the *x* and *y* coordinate that we'll be at when $\theta = \dfrac{3\pi}{4}$. These values are easy enough to find given that we know what *r* is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos\left(\theta\right) = 3.0633\cos\left(\tfrac{3\pi}{4}\right) = -2.1661 \qquad y = r\sin\left(\theta\right) = 3.0633\sin\left(\tfrac{3\pi}{4}\right) = 2.1661$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5

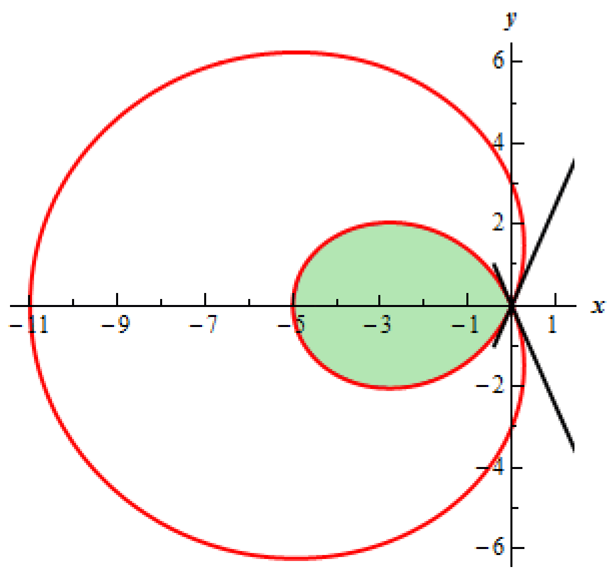
The tangent line is then,

$$\boxed{y = 2.1661 + 0.2843\left(x + 2.1661\right) = 0.2843x + 2.7819}$$

Section 3-8 : Area with Polar Coordinates

1. Find the area inside the inner loop of $r = 3 - 8\cos\theta$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ that the graph goes through the origin (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graph goes through the origin we know that we must have $r = 0$. So,

$$\begin{aligned} 3 - 8\cos\theta &= 0 \\ \cos\theta &= \tfrac{3}{8} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(\tfrac{3}{8}\right) = 1.1864 \end{aligned}$$

This is the angle in the first quadrant where the graph goes through the origin.

We next need the angle in the fourth quadrant. We need to be a little careful with this second angle. We need to always remember that the limits on the integral we'll eventually be computing must go from smaller to larger value. Also, as the angle moves from the smaller to larger value they must trace out the boundary curve of the region we are interested in.

From a quick sketch of a unit circle we can quickly get two possible values for the angle in the fourth quadrant.

$$\theta = 2\pi - 1.1864 = 5.0968 \qquad\qquad \theta = -1.1864$$

Depending upon the problem we are being asked to do either of these could be the one we need. However, in this case we can see that if we use the first angle (*i.e.* the positive angle) we actually end up tracing out the outer portion of the curve and that isn't what we want here. However, if we use the second angle (*i.e.* the negative angle) we will trace out the inner loop as we move from this angle to the angle in the first quadrant.

So, for this particular problem, we need to use $\theta = -1.1864$.

The ranges of θ for this problem is then $-1.1864 \le \theta \le 1.1864$.

Step 3
The area of the inner loop is then,

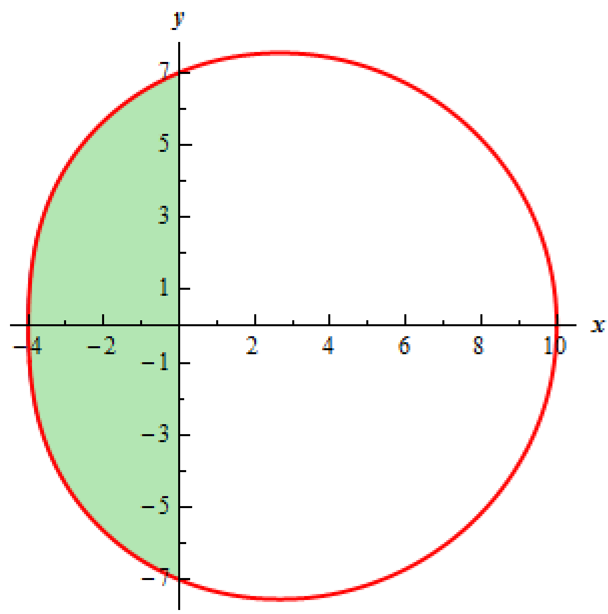
$$\begin{aligned} A &= \int_{-1.1864}^{1.1864} \tfrac{1}{2}\left(3-8\cos\theta\right)^2 \, d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 9-48\cos\theta+64\cos^2\left(\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 9-48\cos\theta+32\left(1+\cos\left(2\theta\right)\right) d\theta \\ &= \tfrac{1}{2}\int_{-1.1864}^{1.1864} 41-48\cos\theta+32\cos\left(2\theta\right) d\theta \\ &= \tfrac{1}{2}\left(41\theta-48\sin\left(\theta\right)+16\sin\left(2\theta\right)\right)\Big|_{-1.1864}^{1.1864} = \boxed{15.2695} \end{aligned}$$

Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the [Integrals Involving Trig Functions](#) section.

---

2. Find the area inside the graph of $r = 7+3\cos\theta$ and to the left of the *y*-axis.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
For this problem there isn't too much difficulty in getting the limits for the problem. We will need to use the limits $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ to trace out the portion of the graph to the left of the *y*-axis.

Remember that it is important to trace out the portion of the curve defining the area we are interested in as the θ's increase from the smaller to larger value.

Step 3
The area is then,

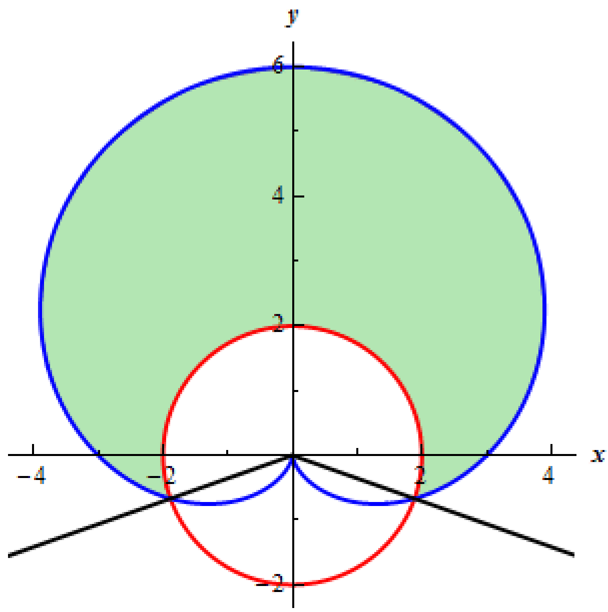
$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{1}{2}\left(7+3\cos\theta\right)^2 d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49+42\cos\theta+9\cos^2\left(\theta\right)d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49+42\cos\theta+\tfrac{9}{2}\left(1+\cos\left(2\theta\right)\right)d\theta \\ &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{107}{2}+42\cos\theta+\tfrac{9}{2}\cos\left(2\theta\right)d\theta \\ &= \tfrac{1}{2}\left(\tfrac{107}{2}\theta+42\sin\left(\theta\right)+\tfrac{9}{4}\sin\left(2\theta\right)\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{42.0376} \end{aligned}$$

Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

---

3. Find the area that is inside $r = 3+3\sin\theta$ and outside $r = 2$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$3 + 3\sin\theta = 2$$
$$\sin\theta = -\tfrac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{3}\right) = -0.3398$$

This is the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

The ranges of θ for this problem is then $-0.3398 \le \theta \le 3.4814$.

Step 3
From the graph we can see that $r = 3 + 3\sin\theta$ is the "outer" graph for this region and $r = 2$ is the "inner" graph.

The area is then,

$$\begin{aligned}
A &= \int_{-0.3398}^{3.4814} \tfrac{1}{2}\left[\left(3+3\sin\theta\right)^2 - \left(2\right)^2\right] d\theta \\
&= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin\theta + 9\sin^2\left(\theta\right) d\theta \\
&= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin\theta + \tfrac{9}{2}\left(1-\cos\left(2\theta\right)\right) d\theta \\
&= \tfrac{1}{2}\int_{-0.3398}^{3.4814} \tfrac{19}{2} + 18\sin\theta - \tfrac{9}{2}\cos\left(2\theta\right) d\theta \\
&= \tfrac{1}{2}\left(\tfrac{19}{2}\theta - 18\cos\left(\theta\right) - \tfrac{9}{4}\sin\left(2\theta\right)\right)\Big|_{-0.3398}^{3.4814} = \boxed{33.7074}
\end{aligned}$$

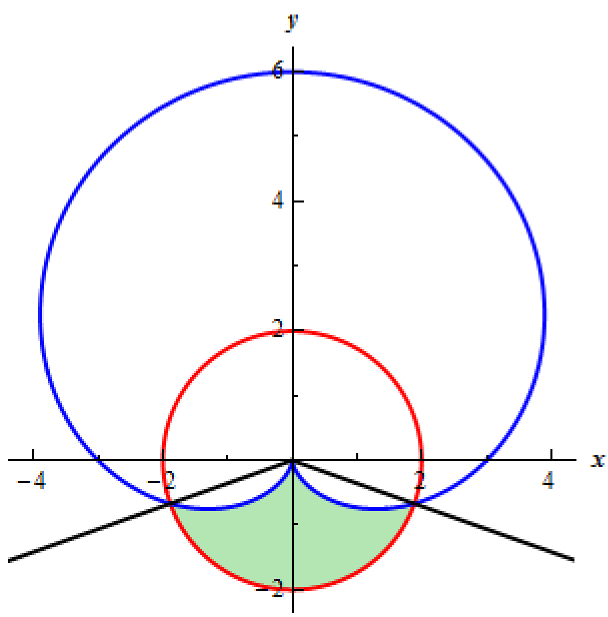
Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

---

4. Find the area that is inside $r = 2$ and outside $r = 3 + 3\sin\theta$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$\begin{aligned} 3+3\sin\theta &= 2 \\ \sin\theta &= -\tfrac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{3}\right) = -0.3398 \end{aligned}$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

Now, we'll have a problem if we use these two angles for our area integral. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. These two clearly will not do that. In fact, they trace out the area from the previous problem.

To fix this problem it is probably easiest to use a quick sketch of a unit circle to see that another value for the angle in the fourth quadrant is,

$$\theta = 2\pi - 0.3398 = 5.9434$$

Using this angle along with the angle we already have in the third quadrant will trace out the area we are interested in.

Therefore, the ranges of θ for this problem is then $3.4814 \le \theta \le 5.9434$.

Step 3

From the graph we can see that $r = 2$ is the "outer" graph for this region and $r = 3+3\sin\theta$ is the "inner" graph.

The area is then,

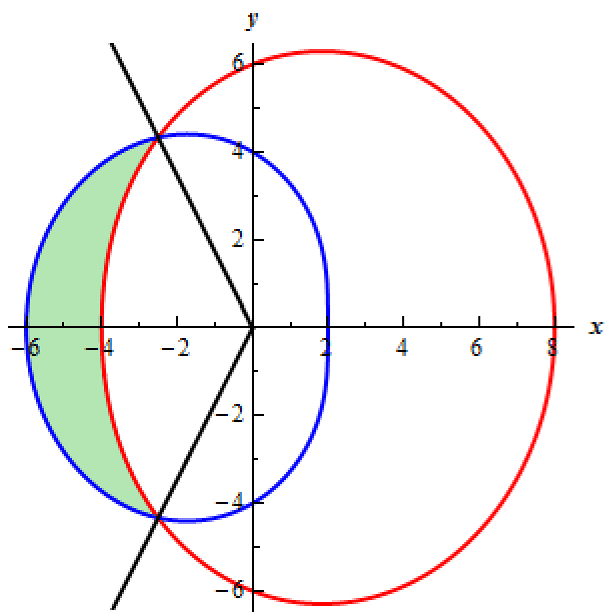
$$\begin{aligned} A &= \int_{3.4814}^{5.9434} \tfrac{1}{2}\left[(2)^2 - (3+3\sin\theta)^2\right]d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -5-18\sin\theta-9\sin^2(\theta)\,d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -5-18\sin\theta-\tfrac{9}{2}\left(1-\cos(2\theta)\right)d\theta \\ &= \tfrac{1}{2}\int_{3.4814}^{5.9434} -\tfrac{19}{2}-18\sin\theta+\tfrac{9}{2}\cos(2\theta)\,d\theta \\ &= \tfrac{1}{2}\left(-\tfrac{19}{2}\theta+18\cos(\theta)+\tfrac{9}{4}\sin(2\theta)\right)\Big|_{3.4814}^{5.9434} = \boxed{3.8622} \end{aligned}$$

Do not get too excited about all the minus signs in the second step above. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from the integral!

---

5. Find the area that is inside $r = 4 - 2\cos\theta$ and outside $r = 6 + 2\cos\theta$.

Step 1
First, here is a quick sketch of the graph of the region we are interested in.



Step 2
Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$6 + 2\cos\theta = 4 - 2\cos\theta$$
$$\cos\theta = -\tfrac{1}{2} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(-\tfrac{1}{2}\right) = \tfrac{2\pi}{3}$$

This is the value for the angle in the second quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly see two possible values for the angle in the third quadrant where the two graphs intersect.

$$\theta = 2\pi - \tfrac{2\pi}{3} = \tfrac{4\pi}{3} \qquad\qquad \theta = -\tfrac{2\pi}{3}$$

Now, we need to recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. Keeping this in mind and we can see that we'll need to use the positive angle for this problem. If we used the negative angle we'd be tracing out the "right" portions of our curves and we need to trace out the "left" portions of our curves.

Therefore, the ranges of θ for this problem is then $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.

Step 3

From the graph we can see that $r = 4 - 2\cos\theta$ is the "outer" graph for this region and $r = 6 + 2\cos\theta$ is the "inner" graph.

The area is then,

$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \tfrac{1}{2}\left[\left(4 - 2\cos\theta\right)^2 - \left(6 + 2\cos\theta\right)^2\right] d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} -10 - 20\cos\theta \, d\theta \\ &= \left(-10\theta - 20\sin\left(\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{13.6971} \end{aligned}$$

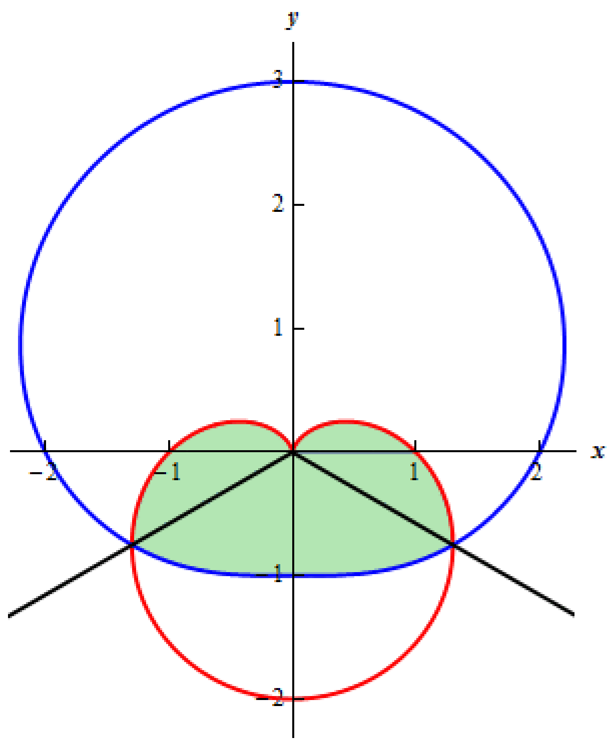
Do not get too excited about all the minus signs in the integral. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from out integral!

---

6. Find the area that is inside both $r = 1 - \sin\theta$ and $r = 2 + \sin\theta$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of *r*. So,

$$\begin{aligned} 2+\sin\theta &= 1-\sin\theta \\ \sin\theta &= -\tfrac{1}{2} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{2}\right) = -\tfrac{\pi}{6} \end{aligned}$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect. From a quick sketch of a unit circle we can see that a second possible value for this angle is,

$$\theta = 2\pi - \tfrac{\pi}{6} = \tfrac{11\pi}{6}$$

From a quick sketch of a unit circle we can quickly get a value for the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + \tfrac{\pi}{6} = \tfrac{7\pi}{6}$$

Okay. We now have a real problem. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area.

If we use $-\tfrac{\pi}{6} \le \theta \le \tfrac{7\pi}{6}$ we actually end up tracing out the large "open" or unshaded region that lies above the shaded region.

Likewise, if we use $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$ we actually end up tracing out the smaller "open" or unshaded region that lies below the shaded region.

In other words, we can't find the shaded area simply by using the formula from this section.

Step 3
As we saw in the previous step we can't just compute an integral in order to get the area of the shaded region. However, that doesn't mean that we can't find the area of the shaded region. We just need to work a little harder at it for this problem.

To find the area of the shaded area we can notice that the shaded area is really nothing more than the remainder of the area inside $r = 2 + \sin\theta$ once we take out the portion that is also outside $r = 1 - \sin\theta$.

Another way to look at is that the shaded area is simply the remainder of the area inside $r = 1 - \sin\theta$ once we take out the portion that is also outside $r = 2 + \sin\theta$.

We can use either of these ideas to find the area of the shaded region. We'll use the first one for no other reason that it was the first one listed.

If we knew the total area that is inside $r = 2 + \sin\theta$ (which we can find with a simple integral) and if we also knew the area that is inside $r = 2 + \sin\theta$ and outside $r = 1 - \sin\theta$ then the shaded area is nothing more than the difference between these two areas.

Step 4
Okay, let's start this off by getting the total area that is inside $r = 2 + \sin\theta$. This can be found by evaluating the following integral.

$$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}\left(2 + \sin\theta\right)^2 d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \sin^2\left(\theta\right) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \tfrac{1}{2}\left(1 - \cos\left(2\theta\right)\right) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos\left(2\theta\right) d\theta \\ &= \tfrac{1}{2}\left(\tfrac{9}{2}\theta - 4\cos\left(\theta\right) - \tfrac{1}{4}\sin\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{\tfrac{9\pi}{2}} \end{aligned}$$

Note that we need to do a full "revolution" to get all the area inside $r = 2 + \sin\theta$ and so we used the range $0 \le \theta \le 2\pi$ for this integral.

Step 5
Now, the area that is inside $r = 2 + \sin\theta$ and outside $r = 1 - \sin\theta$ is,

$$
\begin{aligned}
A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \tfrac{1}{2}\left[\left(2+\sin\theta\right)^2 - \left(1-\sin\theta\right)^2\right] d\theta \\
&= \tfrac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 3+6\sin\theta \, d\theta \\
&= \tfrac{1}{2}\left(3\theta - 6\cos\left(\theta\right)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} = \boxed{2\pi + 3\sqrt{3}}
\end{aligned}
$$

Note that we found the limits for this region in Step 2.

Step 6

Finally, the shaded area is simply,

$$A = \tfrac{9\pi}{2} - \left(2\pi + 3\sqrt{3}\right) = \boxed{\tfrac{5}{2}\pi - 3\sqrt{3} = 2.6578}$$

Section 3-9 : Arc Length with Polar Coordinates

1. Determine the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = -4\sin\theta\,,\ 0 \le \theta \le \pi$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -4\cos\theta$$

Step 2

We'll need the *ds* for this problem.

$$\begin{aligned} ds &= \sqrt{\left[-4\sin\theta\right]^2 + \left[-4\cos\theta\right]^2}\, d\theta \\ &= \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta = 4\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = 4d\theta \end{aligned}$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_0^\pi 4\, d\theta$$

Step 4

This is a really simple integral to compute. Here is the integral work,

$$L = \int_0^\pi 4\, d\theta = 4\theta\Big|_0^\pi = \boxed{4\pi}$$

---

2. Set up, but do not evaluate, and integral that gives the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \theta\cos\theta\,,\ 0 \le \theta \le \pi$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = \cos\theta - \theta\sin\theta$$

Step 2

We'll need the *ds* for this problem.

$$ds = \sqrt{\left[\theta\cos\theta\right]^2 + \left[\cos\theta - \theta\sin\theta\right]^2}\, d\theta$$

Step 3
The integral for the arc length is then,

$$\boxed{L = \int_0^{\pi} \sqrt{\left[\theta\cos\theta\right]^2 + \left[\cos\theta - \theta\sin\theta\right]^2}\, d\theta}$$

---

3. Set up, but do not evaluate, and integral that gives the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \cos\left(2\theta\right) + \sin\left(3\theta\right),\ 0 \le \theta \le 2\pi$$

Step 1
The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -2\sin\left(2\theta\right) + 3\cos\left(3\theta\right)$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[\cos\left(2\theta\right) + \sin\left(3\theta\right)\right]^2 + \left[-2\sin\left(2\theta\right) + 3\cos\left(3\theta\right)\right]^2}\, d\theta$$

Step 3
The integral for the arc length is then,

$$\boxed{L = \int_0^{2\pi} \sqrt{\left[\cos\left(2\theta\right) + \sin\left(3\theta\right)\right]^2 + \left[-2\sin\left(2\theta\right) + 3\cos\left(3\theta\right)\right]^2}\, d\theta}$$

Section 3-10 : Surface Area with Polar Coordinates

1. Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. You may assume that the curve traces out exactly once for the given range of θ.

$$r = 5 - 4\sin\theta, \; 0 \le \theta \le \pi \text{ rotated about the } x\text{-axis.}$$

Step 1
The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -4\cos\theta$$

Step 2
We'll need the *ds* for this problem.

$$\begin{aligned} ds &= \sqrt{\left[5 - 4\sin\theta\right]^2 + \left[-4\cos\theta\right]^2}\, d\theta \\ &= \sqrt{25 - 40\sin\theta + 16\sin^2\theta + 16\cos^2\theta}\, d\theta = \sqrt{41 - 40\sin\theta}\, d\theta \end{aligned}$$

Step 3
The integral for the surface area is then,

$$SA = \int 2\pi y\, ds = \boxed{\int_0^{\pi} 2\pi\left(5 - 4\sin\theta\right)\sin\theta\sqrt{41 - 40\sin\theta}\, d\theta}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about. Also, do not forget to substitute the polar conversion formula for *y*!

---

2. Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \cos^2\theta, \; -\tfrac{\pi}{6} \le \theta \le \tfrac{\pi}{6} \text{ rotated about the } y\text{-axis.}$$

Step 1
The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -2\cos\theta\sin\theta$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{\left[\cos^2\theta\right]^2 + \left[-2\cos\theta\sin\theta\right]^2}\,d\theta = \sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta}\,d\theta$$

Step 3
The integral for the surface area is then,

$$\begin{aligned} SA &= \int 2\pi x\,ds = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\pi\left(\cos^2\theta\right)\cos\theta\sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta}\,d\theta \\ &= \boxed{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\pi\cos^3\theta\sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta}\,d\theta} \end{aligned}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about. Also, do not forget to substitute the polar conversion formula for *x*!

Section 3-11 : Arc Length and Surface Area Revisited

Problems have not yet been written for this section and probably won't be to be honest since this is just a summary section.

Chapter 4 : Series & Sequences

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Sequences – In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.

More on Sequences – In this section we will continue examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.

Series – The Basics – In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).

Convergence/Divergence of Series – In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.

Special Series – In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.

Integral Test – In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

Comparison Test/Limit Comparison Test – In this section we will discuss using the Comparison Test and Limit Comparison Tests to determine if an infinite series converges or diverges. In order to use either test the terms of the infinite series must be positive. Proofs for both tests are also given.

Alternating Series Test – In this section we will discuss using the Alternating Series Test to determine if an infinite series converges or diverges. The Alternating Series Test can be used only if the terms of the series alternate in sign. A proof of the Alternating Series Test is also given.

Absolute Convergence – In this section we will have a brief discussion on absolute convergence and conditionally convergent and how they relate to convergence of infinite series.

Ratio Test – In this section we will discuss using the Ratio Test to determine if an infinite series converges absolutely or diverges. The Ratio Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Ratio Test is also given.

Root Test – In this section we will discuss using the Root Test to determine if an infinite series converges absolutely or diverges. The Root Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Root Test is also given.

Strategy for Series – In this section we give a general set of guidelines for determining which test to use in determining if an infinite series will converge or diverge. Note as well that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines. A summary of all the various tests, as well as conditions that must be met to use them, we discussed in this chapter are also given in this section.

Estimating the Value of a Series – In this section we will discuss how the Integral Test, Comparison Test, Alternating Series Test and the Ratio Test can, on occasion, be used to estimating the value of an infinite series.

Power Series – In this section we will give the definition of the power series as well as the definition of the radius of convergence and interval of convergence for a power series. We will also illustrate how the Ratio Test and Root Test can be used to determine the radius and interval of convergence for a power series.

Power Series and Functions – In this section we discuss how the formula for a convergent Geometric Series can be used to represent some functions as power series. To use the Geometric Series formula, the function must be able to be put into a specific form, which is often impossible. However, use of this formula does quickly illustrate how functions can be represented as a power series. We also discuss differentiation and integration of power series.

Taylor Series – In this section we will discuss how to find the Taylor/Maclaurin Series for a function. This will work for a much wider variety of function than the method discussed in the previous section at the expense of some often unpleasant work. We also derive some well known formulas for Taylor series of $\mathbf{e}^x$, $\cos(x)$ and $\sin(x)$ around $x = 0$.

Applications of Series – In this section we will take a quick look at a couple of applications of series. We will illustrate how we can find a series representation for indefinite integrals that cannot be evaluated by any other method. We will also see how we can use the first few terms of a power series to approximate a function.

Binomial Series – In this section we will give the Binomial Theorem and illustrate how it can be used to quickly expand terms in the form $(a+b)^n$ when *n* is an integer. In addition, when *n* is not an integer an extension to the Binomial Theorem can be used to give a power series representation of the term.

Section 4-1 : Sequences

1. List the first 5 terms of the following sequence.

$$\left\{ \frac{4n}{n^2-7} \right\}_{n=0}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n=0$, plug in the first five values of *n* into the formula for the sequence terms. Doing that gives,

$$\begin{aligned}
n=0: &\qquad \frac{4(0)}{(0)^2-7}=0 \\
n=1: &\qquad \frac{4(1)}{(1)^2-7}=\frac{4}{-6}=-\frac{2}{3} \\
n=2: &\qquad \frac{4(2)}{(2)^2-7}=\frac{8}{-3}=-\frac{8}{3} \\
n=3: &\qquad \frac{4(3)}{(3)^2-7}=\frac{12}{2}=6 \\
n=4: &\qquad \frac{4(4)}{(4)^2-7}=\frac{16}{9}
\end{aligned}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{0,\ -\frac{2}{3},\ -\frac{8}{3},\ 6,\ \frac{16}{9},\ \ldots\right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "…" at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

---

2. List the first 5 terms of the following sequence.

$$\left\{ \frac{(-1)^{n+1}}{2n+(-3)^n} \right\}_{n=2}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n = 2$, plug in the first five values of *n* into the formula for the sequence terms. Doing that gives,

$$n = 2: \qquad \frac{(-1)^{2+1}}{2(2)+(-3)^2} = \frac{-1}{13} = -\frac{1}{13}$$

$$n = 3: \qquad \frac{(-1)^{3+1}}{2(3)+(-3)^3} = \frac{1}{-21} = -\frac{1}{21}$$

$$n = 4: \qquad \frac{(-1)^{4+1}}{2(4)+(-3)^4} = \frac{-1}{89} = -\frac{1}{89}$$

$$n = 5: \qquad \frac{(-1)^{5+1}}{2(5)+(-3)^5} = \frac{1}{-233} = -\frac{1}{233}$$

$$n = 6: \qquad \frac{(-1)^{6+1}}{2(6)+(-3)^6} = \frac{-1}{741} = -\frac{1}{741}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{-\frac{1}{13}, -\frac{1}{21}, -\frac{1}{89}, -\frac{1}{233}, -\frac{1}{741}, \ldots\right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "…" at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

---

3. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{\frac{n^2 - 7n + 3}{1 + 10n - 4n^2}\right\}_{n=3}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n \to \infty} \frac{n^2 - 7n + 3}{1 + 10n - 4n^2} = -\frac{1}{4}$$

You do recall how to take limits at infinity right? If not you should go back into the Calculus I material do some refreshing on limits at infinity as well at L'Hospital's rule.

Step 2
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is $-\frac{1}{4}$.

---

4. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{ \frac{(-1)^{n-2} n^2}{4+n^3} \right\}_{n=0}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty} \frac{(-1)^{n-2} n^2}{4+n^3}$$

However, because of the $(-1)^{n-2}$ we can't compute this limit using our knowledge of computing limits from Calculus I.

Step 2
Recall however, that we had a nice Fact in the notes from this section that had us computing not the limit above but instead computing the limit of the absolute value of the sequence terms.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-2} n^2}{4+n^3} \right| = \lim_{n\to\infty} \frac{n^2}{4+n^3} = 0$$

This is a limit that we can compute because the absolute value got rid of the alternating sign, *i.e.* the $(-1)^{n+2}$.

Step 3
Now, by the Fact from class we know that because the limit of the absolute value of the sequence terms was zero (and recall that to use that fact the limit MUST be zero!) we also know the following limit.

$$\lim_{n\to\infty} \frac{(-1)^{n-2} n^2}{4+n^3} = 0$$

Step 4
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is zero.

---

5. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{\frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}}\right\}_{n=1}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}}=\lim_{n\to\infty}\frac{5\mathbf{e}^{5n}}{-2\mathbf{e}^{2n}}=\lim_{n\to\infty}\frac{5}{-2}\mathbf{e}^{3n}=-\infty$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not you should go back into the Calculus I material do some refreshing.

Step 2
We can see that the limit of the terms existed and but was infinite and so we know that the sequence **diverges**.

---

6. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{\frac{\ln\left(n+2\right)}{\ln\left(1+4n\right)}\right\}_{n=1}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{\ln\left(n+2\right)}{\ln\left(1+4n\right)}=\lim_{n\to\infty}\frac{{}^{1}\!/_{n+2}}{{}^{4}\!/_{1+4n}}=\lim_{n\to\infty}\frac{1+4n}{4\left(n+2\right)}=1$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not, you should go back into the Calculus I material do some refreshing.

Step 2
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is one.

Section 4-2 : More on Sequences

1. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ \frac{1}{4n} \right\}_{n=1}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1
For this problem let's get the bounded information first as that seems to be pretty simple.

First note that because both the numerator and denominator are positive then the quotient is also positive and so we can see that the sequence must be **bounded below by zero**.

Next let's note that because we are starting with $n = 1$ the denominator will always be $4n \geq 4 > 1$ and so we can also see that the sequence must be **bounded above by one**. Note that, in this case, this not the "best" upper bound for the sequence but the problem didn't ask for that. For this sequence we'll be able to get a better one once we have the increasing/decreasing information.

Because the sequence is bounded above and bounded below the sequence is also **bounded**.

Step 2
For the increasing/decreasing information we can see that, for our range of $n \geq 1$, we have,

$$4n < 4(n+1)$$

and so,

$$\frac{1}{4n} > \frac{1}{4(n+1)}$$

If we define $a_n = \frac{1}{4n}$ this in turn tells us that $a_n > a_{n+1}$ for all $n \geq 1$ and so the sequence is **decreasing** and hence **monotonic**.

Note that because we have now determined that the sequence is decreasing we can see that the "best" upper bound would be the first term of the sequence or, $\frac{1}{4}$.

2. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ n\left(-1\right)^{n+2} \right\}_{n=0}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1
For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

In this case let's just write out the first few terms of the sequence.

$$\left\{ n\left(-1\right)^{n+2} \right\}_{n=0}^{\infty} = \left\{0,\ -1,\ 2,\ -3,\ 4,\ -5,\ 6,\ -7,\ldots\right\}$$

Just from the first three terms we can see that this sequence is **not an increasing sequence** and it is **not a decreasing sequence** and therefore is **not monotonic**.

Step 2
Now let's see what bounded information we can get.

From the first few terms of the sequence we listed out above we can see that each successive term will get larger and change signs. Therefore, there cannot be an upper or a lower bound for the sequence. No matter what value we would try to use for an upper or a lower bound all we would need to do is take *n* large enough and we would eventually get a sequence term that would go past the proposed bound.

Therefore, this sequence is **not bounded**.

---

3. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ 3^{-n} \right\}_{n=0}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence it might be a little easier to find the bounds (if any exist) if you first have the increasing/decreasing information.

Step 1

For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

We' all agree that, for our range of $n \ge 0$, we have,

$$n < n+1$$

This in turn gives,

$$3^{-n} = \frac{1}{3^n} > \frac{1}{3^{n+1}} = 3^{-(n+1)}$$

So, if we define $a_n = 3^{-n}$ we have $a_n > a_{n+1}$ for all $n \ge 0$ and so the sequence is **decreasing** and hence is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive and so the sequence is **bounded below by zero**.

Next, we saw in the first step that the sequence was decreasing and so the first term will be the largest term and so the sequence is **bounded above by** $3^{-(0)} = 1$ (*i.e.* the $n = 0$ sequence term).

Therefore, because this sequence is bounded below and bounded above the sequence is **bounded**.

---

4. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ \frac{2n^2 - 1}{n} \right\}_{n=2}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f(x) = \frac{2x^2 - 1}{x} \qquad \Rightarrow \qquad f'(x) = \frac{2x^2 + 1}{x^2}$$

We can clearly see that the derivative will always be positive for $x \neq 0$ and so the function is increasing for $x \neq 0$. Therefore, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n = 2$, must also be **increasing** and hence it is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive for our range of $n \geq 2$ and so the sequence is **bounded below by zero**. We could also use the fact that the sequence is increasing the first term would have to be the smallest term in the sequence and so a better lower bound would be the first sequence term which is $\frac{7}{2}$. Either would work for this problem.

Now let's see what we can determine about an upper bound (provided it has one of course…).

We know that the function is increasing but that doesn't mean there is no upper bound. Take a look at Problems 1 and 3 above. Each of those were decreasing sequences and yet they had a lower bound. Do not make the mistake of assuming that an increasing sequence will not have an upper bound or a decreasing sequence will not have a lower bound. Sometimes they will and sometimes they won't.

For this sequence we'll need to approach any potential upper bound a little differently than the previous problems. Let's first compute the following limit of the terms,

$$\lim_{n \to \infty} \frac{2n^2 - 1}{n} = \lim_{n \to \infty} \left(2n - \frac{1}{n} \right) = \infty$$

Since the limit of the terms is infinity we can see that the terms will increase without bound. Therefore, in this case, there really is **no upper bound** for this sequence. Please remember the warning above however! Just because this increasing sequence had no upper bound does not mean that every increasing sequence will not have an upper bound.

Finally, because this sequence is bounded below but not bounded above the sequence is **not bounded**.

---

5. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ \frac{4-n}{2n+3} \right\}_{n=1}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{4-x}{2x+3} \qquad\qquad \Rightarrow \qquad\qquad f'\left(x\right)=\frac{-11}{\left(2x+3\right)^{2}}$$

We can clearly see that the derivative will always be negative for $x \ne -\frac{3}{2}$ and so the function is decreasing for $x \ne -\frac{3}{2}$. Therefore, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n=1$, must also be **decreasing** and hence it is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, because the sequence is decreasing we can see that the first term of the sequence will be the largest and hence will also be an upper bound for the sequence. So, the sequence is **bounded above** by $\frac{3}{5}$ (*i.e.* the $n=1$ sequence term).

Next let's look for the lower bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n\to\infty}\frac{4-n}{2n+3}=-\frac{1}{2}$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n\to\infty$ the sequence terms must be getting closer and closer to $-\frac{1}{2}$.

Now, for a second, let's suppose that that $-\frac{1}{2}$ is not a lower bound for the sequence terms and let's also keep in mind that we've already determined that the sequence is decreasing (means that each successive term must be smaller than (*i.e.* below) the previous one…).

So, if $-\frac{1}{2}$ is not a lower bound then we know that somewhere there must be sequence terms below (or smaller than) $-\frac{1}{2}$. However, because we also know that terms must be getting closer and closer to $-\frac{1}{2}$ and we've now assumed there are terms below $-\frac{1}{2}$ the only way for that to happen at this point is for at least a few sequence terms to increase up towards $-\frac{1}{2}$ (remember we've assumed there are terms below this!). That can't happen however because we know the sequence is a decreasing sequence.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the decreasing nature of the sequence by first assuming that $-\frac{1}{2}$ was not a lower bound. Since making this assumption led us to something that can't possibly be true that in turn means that $-\frac{1}{2}$ must in fact be a lower bound since we've shown that sequence terms simply cannot go below this value!

Therefore, the sequence is **bounded below** by $-\frac{1}{2}$.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a quick word of caution. The limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence. It will only be a bound under certain circumstances and so we can't simply compute the limit and assume it will be a bound for every sequence! Can you see a condition that will allow the limit to be a bound?

---

6. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{-n}{n^2+25}\right\}_{n=2}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{-x}{x^2+25} \qquad \Rightarrow \qquad f'\left(x\right)=\frac{x^2-25}{\left(x^2+25\right)^2}$$

Hopefully, it's fairly clear that the critical points of the function are $x=\pm 5$. We'll leave it to you to draw a quick number line or sign chart to verify that the function will be decreasing in the range $2 \le x < 5$ and increasing in the range $x > 5$. Note that we just looked at the ranges of *x* that correspond to the ranges of *n* for our sequence here.

Now, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n=2$, has terms that increase and terms that decrease and hence the sequence is **not an increasing sequence** and the sequence is **not a decreasing sequence**. That also means that the sequence is **not monotonic**.

Step 2
Now let's see what bounded information we can get.

In this case, unlike many of the previous problems in this section, we don't have a monotonic sequence. However, we can still use the increasing/decreasing information above to help us out with the bounds.

First, we know that the sequence is decreasing in the range $2 \le n < 5$ and increasing in the range $n > 5$. From our Calculus I knowledge we know that this means $n=5$ must be a minimum of the sequence terms and hence the sequence is **bounded below** by $\frac{-5}{50}=-\frac{1}{10}$ (*i.e.* the $n=5$ sequence term).

Next let's look for the upper bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n\to\infty}\frac{-n}{n^2+25}=0$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n\to\infty$ the sequence terms must be getting closer and closer to zero.

Now, for a second, let's look at just the portion of the sequence with $n>5$ and let's further suppose that zero is not an upper bound for the sequence terms with $n>5$. Let's also keep in mind that we've already determined that the sequence is increasing for $n>5$ (means that each successive term must be larger than (*i.e.* above) the previous one…).

So, if zero is not an upper bound (for $n > 5$) then we know that somewhere there must be sequence terms with $n > 5$ above (or larger than) zero. So, we know that terms must be getting closer and closer to zero and we've now assumed there are terms above zero. Therefore the only way for the terms to approach the limit of zero is for at least a few sequence terms with $n > 5$ to decrease down towards zero (remember we've assumed there are terms above this!). That can't happen however because we know that for $n > 5$ the sequence is increasing.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the increasing nature of the sequence for $n > 5$ by first assuming that zero was not an upper bound for the portion of the sequence with $n > 5$. Since making this assumption led us to something that can't possibly be true that in turn means that zero must in fact be an upper bound for the portion of the sequence with $n > 5$ since we've shown that sequence terms simply cannot go above this value!

Note that we've not yet actually shown that zero in an upper bound for the sequence and in fact it might not actually be an upper bound. However, what we have shown is that it is an upper bound for the vast majority of the sequence, *i.e.* for the portion of the sequence with $n > 5$.

All we need to do to finish the upper bound portion of this problem off is check what the first few terms of the sequence are doing. There are several ways to do this. One is to just compute the remaining initial terms of the sequence to see if they are above or below zero. For this sequence that isn't too bad as there are only 4 terms ($n = 2,3,4,5$). However, if there'd been several hundred terms that wouldn't be so easy so let's take a look at another approach that will always be easy to do in this case because we have the increasing/decreasing information for this initial portion of the sequence.

Let's simply note that for the first part of this sequence we've already shown that the sequence is decreasing. Therefore, the very first sequence term of $-\frac{2}{29}$ (*i.e.* the $n = 2$ sequence term) will be the largest term for this initial bit of the sequence that is decreasing. This term is clearly less than zero and so zero will also be larger than all the remaining terms in the initial decreasing portion of the sequence and hence the sequence is **bounded above by zero**.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a couple of quick words of caution.

First, the limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence so be careful to not just always assume that the limit is an upper/lower bound for a sequence.

Second, as this problem has shown determining the bounds of a sequence can sometimes be a fairly involved process that involves quite a bit of work and lots of various pieces of knowledge about the other behavior of the sequence.

Section 4-3 : Series - The Basics

1. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=1}^{\infty}\left(n2^n - 3^{1-n}\right)$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by two so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to decrease by the same amount (two in this case…).

Doing this gives the following series.

$$\sum_{n=1}^{\infty}\left(n2^n - 3^{1-n}\right) = \sum_{n=3}^{\infty}\left(\left(n-2\right)2^{n-2} - 3^{1-(n-2)}\right) = \boxed{\sum_{n=3}^{\infty}\left(\left(n-2\right)2^{n-2} - 3^{3-n}\right)}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n-2$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

---

2. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=7}^{\infty}\frac{4-n}{n^2+1}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to decrease the initial value of the index by four so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to increase by the same amount (four in this case…).

Doing this gives the following series.

$$\sum_{n=7}^{\infty}\frac{4-n}{n^2+1} = \sum_{n=3}^{\infty}\frac{4-\left(n+4\right)}{\left(n+4\right)^2+1} = \boxed{\sum_{n=3}^{\infty}\frac{-n}{\left(n+4\right)^2+1}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n+4$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

---

3. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by one so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to decrease by the same amount (one in this case…).

Doing this gives the following series.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1-3}(n-1+2)}{5^{1+2(n-1)}} = \boxed{\sum_{n=3}^{\infty} \frac{(-1)^{n-4}(n+1)}{5^{2n-1}}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n-1$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

---

4. Strip out the first 3 terms from the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1}$.

Solution

Remember that when we say we are going to "strip out" terms from a series we aren't really getting rid of them. All we are doing is writing the first few terms of the series as a summation in front of the series.

So, for this series stripping out the first three terms gives,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1} &= \frac{2^{-1}}{1^2+1} + \frac{2^{-2}}{2^2+1} + \frac{2^{-3}}{3^2+1} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} \\ &= \frac{1}{4} + \frac{1}{20} + \frac{1}{80} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} \\ &= \boxed{\frac{5}{16} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1}}\end{aligned}$$

This first step isn't really all that necessary but was included here to make it clear that we were plugging in $n = 1$, $n = 2$ and $n = 3$ (*i.e.* the first three values of *n*) into the general series term. Also, don't forget to change the starting value of *n* to reflect the fact that we've "stripped out" the first three values of *n* or terms.

---

5. Given that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1.6865$ determine the value of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3+1}$.

Step 1

First notice that if we strip out the first two terms from the series that starts at $n = 0$ the result will involve a series that starts at $n = 2$.

Doing this gives,

$$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = \frac{1}{0^3+1} + \frac{1}{1^3+1} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Step 2

Now, for this situation we are given the value of the series that starts at $n = 0$ and are asked to determine the value of the series that starts at $n = 2$. To do this all we need to do is plug in the known value of the series that starts at $n = 0$ into the "equation" above and "solve" for the value of the series that starts at $n = 2$.

This gives,

$$1.6865 = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} \qquad \Rightarrow \qquad \sum_{n=2}^{\infty} \frac{1}{n^3+1} = 1.6865 - \frac{3}{2} = \boxed{0.1865}$$

Section 4-4 : Convergence/Divergence of Series

1. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=1}^{\infty} n\,2^n$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\boxed{\begin{aligned} s_1 &= (1)2^1 = 2 \\ s_2 &= (1)2^1 + (2)2^2 = 10 \\ s_3 &= (1)2^1 + (2)2^2 + (3)2^3 = 34 \end{aligned}}$$

---

2. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=3}^{\infty} \frac{2n}{n+2}$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\boxed{\begin{aligned} s_3 &= \frac{2(3)}{3+2} = \frac{6}{5} \\ s_4 &= \frac{2(3)}{3+2} + \frac{2(4)}{4+2} = \frac{38}{15} \\ s_5 &= \frac{2(3)}{3+2} + \frac{2(4)}{4+2} + \frac{2(5)}{5+2} = \frac{416}{105} \end{aligned}}$$

---

3. Assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{5+8n^2}{2-7n^2}$$

Solution
There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5+8n^2}{2-7n^2} = -\frac{8}{7}$$

Now, we can see that this limit exists and is finite (*i.e.* is not plus/minus infinity). Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **converges** and its value is,

$$\sum_{n=0}^{\infty} a_n = -\frac{8}{7}$$

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

---

4. Assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{n^2}{5+2n}$$

Solution
There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n^2}{5+2n} = \infty$$

Now, we can see that this limit exists and is infinite. Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **diverges**.

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

---

5. Show that the following series is divergent.

$$\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$$

Solution
First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3n\,\mathbf{e}^n}{n^2+1} = \infty \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergence**.

---

6. Show that the following series is divergent.

$$\sum_{n=5}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}$$

Solution
First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{6+8n+9n^2}{3+2n+n^2} = 9 \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergence**.

Section 4-5 : Special Series

1. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n=0$ and so we can put this into the form,

$$\sum_{n=0}^{\infty} a\,r^{n}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

In this case it's pretty simple to put the series into the form above so here is that work.

$$\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n} = \sum_{n=0}^{\infty} 3^{2}3^{n}\,2^{1}2^{-3n} = \sum_{n=0}^{\infty} (9)(2)\frac{3^{n}}{2^{3n}} = \sum_{n=0}^{\infty} 18\frac{3^{n}}{8^{n}} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^{n}$$

Make sure you properly deal with any negative exponents that might happen to be in the terms!

Also recall that all the exponents must be simply *n* and can't be 3*n* or anything else. So, for this problem, we'll need to use basic exponent rules to write $2^{3n} = \left(2^{3}\right)^{n} = 8^{n}$.

Step 3
With the series in "proper" form we can see that $a=18$ and $r=\frac{3}{8}$. Therefore, because we can clearly see that $\left|r\right| = \frac{3}{8} < 1$, the series will **converge** and its value is,

$$\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^{n} = \frac{18}{1-\frac{3}{8}} = \boxed{\frac{144}{5}}$$

---

2. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{5}{6n}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a harmonic series.

Step 2
So because this is a harmonic series we know that it will **diverge**.

---

3. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n=1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\,r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

So, let's get started on the work to put the series into the form above. First, let's get take care of the fact that both the *n*'s in the exponents are negative and they should be positive. Converting to positive *n*'s gives,

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}} = \sum_{n=1}^{\infty} \frac{8^{n-2}}{(-6)^{n-3}}$$

Note that how you chose to deal with the 3 and the 2 in the respective exponents is up to you. You can either do it the way we did here or strip them out and then move the terms to the numerator or denominator.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$8^{n-2} = 8^{n-1}8^{-1} \qquad\qquad (-6)^{n-3} = (-6)^{n-1}(-6)^{-2}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty}\frac{(-6)^{3-n}}{8^{2-n}}=\sum_{n=1}^{\infty}\frac{8^{n-1}8^{-1}}{(-6)^{n-1}(-6)^{-2}}=\sum_{n=1}^{\infty}\frac{(-6)^{2}}{8^{1}}\left(\frac{8}{-6}\right)^{n-1}=\sum_{n=1}^{\infty}\frac{9}{2}\left(-\frac{4}{3}\right)^{n-1}$$

Step 3

With the series in "proper" form we can see that $a=\frac{9}{2}$ and $r=-\frac{4}{3}$. Therefore, because we can clearly see that $|r|=\left|-\frac{4}{3}\right|=\frac{4}{3}>1$, the series will **diverge.**

---

4. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty}\frac{3}{n^2+7n+12}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2

Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{3}{n^2+7n+12}=\frac{3}{(n+3)(n+4)}=\frac{A}{n+3}+\frac{B}{n+4}\qquad\rightarrow\qquad 3=A(n+4)+B(n+3)$$

$$\begin{array}{ll} n=-3 & 3=A \\ n=-4 & 3=-B \end{array}\qquad\rightarrow\qquad \begin{array}{l} A=3 \\ B=-3 \end{array}$$

The series term in partial fraction form is then,

$$\frac{3}{n^2+7n+12}=\frac{3}{n+3}-\frac{3}{n+4}$$

Step 3

The partial sums for this series are then,

$$s_n=\sum_{i=1}^{n}\left[\frac{3}{i+3}-\frac{3}{i+4}\right]$$

Step 4
Expanding the partial sums from the previous step give,

$$s_n = \sum_{i=1}^{n}\left[\frac{3}{i+3}-\frac{3}{i+4}\right] = \left[\frac{3}{4}-\cancel{\frac{3}{5}}\right]+\left[\cancel{\frac{3}{5}}-\cancel{\frac{3}{6}}\right]+\left[\cancel{\frac{3}{6}}-\cancel{\frac{3}{7}}\right]+\cdots$$
$$+\left[\cancel{\frac{3}{n+1}}-\cancel{\frac{3}{n+2}}\right]+\left[\cancel{\frac{3}{n+2}}-\cancel{\frac{3}{n+3}}\right]+\left[\cancel{\frac{3}{n+3}}-\frac{3}{n+4}\right]$$
$$=\frac{3}{4}-\frac{3}{n+4}$$

It is important when doing this expanding to expand out from both the initial and final values of *i* and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Note that at this point we now know that the series was a telescoping series since we got all the "interior" terms to cancel out.

Step 5
At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{3}{4}-\frac{3}{n+4}\right]=\frac{3}{4}$$

Step 6
The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=1}^{\infty}\frac{3}{n^2+7n+12}=\frac{3}{4}$$

---

5. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty}\frac{5^{n+1}}{7^{n-2}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n=1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\,r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$5^{n+1} = 5^{n-1}5^{2} \qquad\qquad 7^{n-2} = 7^{n-1}7^{-1}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} \frac{5^{n-1}5^{2}}{7^{n-1}7^{-1}} = \sum_{n=1}^{\infty} (25)(7)\frac{5^{n-1}}{7^{n-1}} = \sum_{n=1}^{\infty} 175\left(\frac{5}{7}\right)^{n-1}$$

Step 3
With the series in "proper" form we can see that $a = 175$ and $r = \frac{5}{7}$. Therefore, because we can clearly see that $\left|r\right| = \frac{5}{7} < 1$, the series will **converge** and its value is,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} 175\left(\frac{5}{7}\right)^{n-1} = \frac{175}{1-\frac{5}{7}} = \boxed{\frac{1225}{2}}$$

---

6. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Now, while we have correctly identified this as a geometric series it doesn't start at either of the two standard starting values of *n*, *i.e.* $n = 0$ or $n = 1$.

This won't stop us from determining if the series converges or diverges because that only depends on the value of *r* which we can determine regardless of the starting value of *n* with enough work. However, if the series does converge we won't be able to use the formula for determining the value of the series as that also needs the value of *a* and that does require the series to start at one of the two standard starting values.

We have two options for taking care of this problem. One is to use an index shift to convert this into a series that starts at one of the standard starting values of *n*. In most cases this is probably the only real option.

However, in this case let's notice that this series is almost identical to the series from the previous problem. The only difference is that this series starts at $n = 2$ while the series in the previous problem starts at $n = 1$. This means that we can use the results of the previous problem to greatly reduce the amount of work needed here.

Step 3
We know that the series in the previous problem converged and since we're only changing the starting value of *n* that will not affect the convergence of the series. Therefore, the series in this problem will also **converge**.

Since we also know that the value of the series in the previous series is $\frac{1225}{2}$ we can find the value of the series in this problem. All we need to do is strip out one term from the series in the previous problem to get,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \frac{5^2}{7^{-1}} + \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

Then using the value we found in the previous problem can get the value of the series from this problem as follows,

$$\frac{1225}{2} = 175 + \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} \qquad \Rightarrow \qquad \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \frac{1225}{2} - 175 = \boxed{\frac{875}{2}}$$

On a quick side note if you did chose to do an index shift here are the two series (for each possible starting value of *n*) that you should have gotten.

$$\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} \frac{5^{n+2}}{7^{n-1}} = \sum_{n=0}^{\infty} \frac{5^{n+3}}{7^{n}}$$

Both of the last two are in the "standard" form and can be used to arrive at the same result as above.

---

7. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=4}^{\infty} \frac{10}{n^2 - 4n + 3}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2

Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{10}{n^2-4n+3} = \frac{10}{(n-1)(n-3)} = \frac{A}{n-1} + \frac{B}{n-3} \qquad \to \qquad 10 = A(n-3) + B(n-1)$$

$$\begin{array}{ll} n=1 & 10=-2A \\ n=3 & 10=2B \end{array} \qquad \to \qquad \begin{array}{l} A=-5 \\ B=5 \end{array}$$

The series term in partial fraction form is then,

$$\frac{10}{n^2-4n+3} = \frac{5}{n-3} - \frac{5}{n-1}$$

Step 3

The partial sums for this series are then,

$$s_n = \sum_{i=4}^{n} \left[\frac{5}{i-3} - \frac{5}{i-1} \right]$$

Step 4

Expanding the partial sums from the previous step give,

$$\begin{aligned} s_n &= \sum_{i=4}^{n} \left[\frac{5}{i-3} - \frac{5}{i-1} \right] \\ &= \left[\frac{5}{1} - \cancel{\frac{5}{3}} \right] + \left[\frac{5}{2} - \cancel{\frac{5}{4}} \right] + \left[\cancel{\frac{5}{3}} - \cancel{\frac{5}{5}} \right] + \left[\cancel{\frac{5}{4}} - \cancel{\frac{5}{6}} \right] + \left[\cancel{\frac{5}{5}} - \cancel{\frac{5}{7}} \right] + \cdots \\ &\qquad + \left[\cancel{\frac{5}{n-7}} - \cancel{\frac{5}{n-5}} \right] + \left[\cancel{\frac{5}{n-6}} - \cancel{\frac{5}{n-4}} \right] + \left[\cancel{\frac{5}{n-5}} - \cancel{\frac{5}{n-3}} \right] + \\ &\qquad \left[\cancel{\frac{5}{n-4}} - \frac{5}{n-2} \right] + \left[\cancel{\frac{5}{n-3}} - \frac{5}{n-1} \right] \\ &= 5 + \frac{5}{2} - \frac{5}{n-2} - \frac{5}{n-1} \end{aligned}$$

It is important when doing this expanding to expand out from both the initial and final values of *i* and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Also, as seen above these can be quite messy to expand out until everything starts to cancel out so don't get too excited about it when it does get messy like this. It just happens sometimes and we have to be careful with all the expansion.

Note that at this point we now know that the series was a telescoping series since we got almost all the "interior" terms to cancel out.

Step 5
At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{15}{2} - \frac{5}{n-2} - \frac{5}{n-1}\right] = \frac{15}{2}$$

Step 6
The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=4}^{\infty} \frac{10}{n^2 - 4n + 3} = \frac{15}{2}$$

Section 4-6 : Integral Test

1. Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

Solution

There really isn't all that much to this problem. We could use the Integral Test on this series or we could just use the *p*-series Test we discussed in the notes for this section.

We can clearly see that $p = \pi > 1$ and so by the *p*-series Test this series must **converge.**

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2}{3+5n}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{2}{3+5n}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3

In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\int_0^t \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\left(\frac{2}{5}\ln|3+5x|\right)\Bigg|_0^t = \lim_{t\to\infty}\left(\frac{2}{5}\ln|3+5t| - \frac{2}{5}\ln|3|\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{(2n+7)^3}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{1}{(2n+7)^3}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{1}{(2n+7)^3} > \frac{1}{(2(n+1)+7)^3} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_2^\infty \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\left.\left(-\frac{1}{4}\frac{1}{(2x+7)^2}\right)\right|_2^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{4}\frac{1}{(2t+7)^2}+\frac{1}{4}\frac{1}{(11)^2}\right) = \frac{1}{484}$$

Step 5
Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^2}{n^3+1}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{n^2}{n^3+1}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in $n+1$ into the series term as we've done in the first couple of problems in this section. Doing that would suggest that both the numerator and denominator will increase and so it's not all that clear cut of a case that the terms will be decreasing.

Therefore, we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{x^2}{x^3+1} \qquad\qquad f'(x) = \frac{2x - x^4}{\left(x^3+1\right)^2} = \frac{x\left(2-x^3\right)}{\left(x^3+1\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $0 < x < \sqrt[3]{2} = 1.2599$ and will decrease for $\sqrt[3]{2} = 1.2599 < x < \infty$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are not always decreasing but will be decreasing for $n > \sqrt[3]{2}$ which is sufficient for us to use to say that this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|x^3+1\right|\right)\Bigg|_0^t = \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|t^3+1\right| - \ln(1)\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{3}{n^2 - 3n + 2}$$

We can clearly see that for $n \ge 3$ (which matches our range of *n* for the series) we will have,

$$n^2 \ge 3n \qquad \Rightarrow \qquad n^2 - 3n \ge 0 \qquad \Rightarrow \qquad n^2 - 3n + 2 \ge n^2 - 3n \ge 0$$

Therefore, the series terms are positive and so that condition is met.

Note that on occasion we'll need to do more than just state that the series terms are positive by inspection and do a little work to show that the terms really are positive!

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in *n* + 1 into the series term as we've done in the first couple of problems in this section.

Doing that the first term in the denominator would be getting larger which would suggest the series term is decreasing. However, because the second term in the denominator is subtracted off if we increase *n* that would suggest the denominator is getting smaller and hence the series term is increasing.

Because we have these "competing" interests we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f\left(x\right) = \frac{3}{x^2 - 3x + 2} \qquad\qquad f'\left(x\right) = \frac{9 - 6x}{\left(x^2 - 3x + 2\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $x < \frac{3}{2}$ and will decrease for $x > \frac{3}{2}$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are always decreasing for the range *n* in our series and so this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test. The integral we'll need to compute is,

$$\int_3^\infty \frac{3}{x^2 - 3x + 2}\,dx$$

This integral will however require us to do some quick partial fractions in order to do the evaluation. Here is that quick work.

$$\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \qquad \to \qquad 3 = A(x-2) + B(x-1)$$

$$\begin{aligned} x = 1: \quad 3 &= -A \\ x = 2: \quad 3 &= B \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -3 \\ B &= 3 \end{aligned}$$

The integral is then,

$$\begin{aligned} \int_3^\infty \frac{3}{x-2} - \frac{3}{x-1}\,dx = \lim_{t\to\infty}\int_3^t \frac{3}{x-2} - \frac{3}{x-1}\,dx &= \lim_{t\to\infty}\left(3\ln|x-2| - 3\ln|x-1|\right)\Big|_3^t \\ &= \lim_{t\to\infty}\left[3\ln|t-2| - 3\ln|t-1| - \left(3\ln|1| - 3\ln|2|\right)\right] \\ &= \lim_{t\to\infty}\left[3\ln\left|\frac{t-2}{t-1}\right| + 3\ln|2|\right] = 3\ln\left(\frac{1}{1}\right) + 3\ln(2) = 3\ln(2) \end{aligned}$$

Be careful with the limit of the first two terms! To correctly compute the limit they need to be combined using logarithm properties as shown and we can then do a L'Hospital's Rule on the argument of the log to compute the limit.

Step 5

Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

Section 4-7 : Comparison Test/Limit Comparison Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$$

Step 1
First, the series terms are,

$$a_n=\left(\frac{1}{n^2}+1\right)^2$$

and it should pretty obvious in this case that they are positive and so we know that we can use the Comparison Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
For most of the Comparison Test problems we usually guess the convergence and proceed from there. However, in this case it is hopefully clear that for any *n*,

$$\left(\frac{1}{n^2}+1\right)^2>\left(1\right)^2=1$$

Now, let's take a look at the following series,

$$\sum_{n=1}^{\infty}1$$

Because $\lim\limits_{n\to\infty}1=1\neq0$ we can see from the Divergence Test that this series will be divergent.

So we've found a divergent series with terms that are smaller than the original series terms. Therefore, by the Comparison Test the series in the problem statement must also be **divergent**.

As a final note for this problem notice that we didn't actually need to do a Comparison Test to arrive at this answer. We could have just used the Divergence Test from the beginning since,

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+1\right)^2 = 1 \neq 0$$

This is something that you should always keep in mind with series convergence problems. The Divergence Test is a quick test that can, on occasion, be used to quickly determine that a series diverges and hence avoid a lot of the hassles of some of the other tests.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$$

Step 1
First, the series terms are,

$$a_n = \frac{n^2}{n^3-3}$$

and it should pretty obvious that as long as $n > \sqrt[3]{3}$ (which we'll always have for this series) that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-3" in the denominator won't really affect the size of the denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

We also know that the series,

$$\sum_{n=4}^{\infty}\frac{1}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case it should be pretty clear that,

$$n^3 > n^3 - 3$$

Therefore, we'll have the following relationship.

$$\frac{n^2}{n^3} < \frac{n^2}{n^3 - 3}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is larger than the denominator in the right term. Therefore, the rational expression on the left must be smaller than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3} = \sum_{n=4}^{\infty} \frac{1}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$$

Step 1
First, the series terms are,

$$a_n = \frac{7}{n(n+1)}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+1" in the denominator won't really affect the size of the denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{7}{n(n)} = \frac{7}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{7}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case it should be pretty clear that,

$$n < n+1 \qquad \Rightarrow \qquad n\left(n\right) < n\left(n+1\right)$$

Therefore, we'll have the following relationship.

$$\frac{7}{n\left(n\right)} > \frac{7}{n\left(n+1\right)}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is smaller than the denominator in the right term. Therefore, the rational expression on the left must be larger than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{7}{n\left(n\right)} = \sum_{n=2}^{\infty} \frac{7}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=7}^{\infty} \frac{4}{n^2 - 2n - 3}$$

Step 1
First, the series terms are,

$$a_n = \frac{4}{n^2 - 2n - 3}$$

You can verify that for $n \geq 7$ we have $n^2 > 2n+3$ and so $n^2 - 2n - 3 = n^2 - (2n+3) > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the denominator will increase at a much faster rate than the $-2n-3$ (a linear term) portion of the denominator. Therefore the n^2 portion of the denominator will, in all likelihood, define the behavior of the denominator and so the terms should behave like,

$$b_n = \frac{4}{n^2}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms on the denominator. Doing that however gives the following inequality,

$$n^2 > n^2 - 2n - 3$$

This in turn gives the following relationship.

$$\frac{4}{n^2} < \frac{4}{n^2 - 2n - 3}$$

The denominator on the left is larger and so the rational expression on the left must be smaller. This leads to the problem. While the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will definitely converge (as discussed above) it's terms are smaller than the series terms in the problem statement. Just because a series with smaller terms converges does not, in any way, imply a series with larger terms will also converge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n \to \infty} \left[\frac{4}{n^2 - 2n - 3} \frac{n^2}{4} \right] = \lim_{n \to \infty} \left[\frac{n^2}{n^2 - 2n - 3} \right] = 1$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e. c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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5. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n-1}{\sqrt{n^6+1}}$$

Step 1
First, the series terms are,

$$a_n = \frac{n-1}{\sqrt{n^6+1}}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-1" in the numerator and the "+1" in the denominator won't really affect the size of the numerator and denominator respectively for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{n}{\sqrt{n^6}} = \frac{n}{n^3} = \frac{1}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$n > n - 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}}$$

Now, in the denominator it again is hopefully clear that,

$$n^6 < n^6 + 1$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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6. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2(n)}$$

Step 1
First, the series terms are,

$$a_n = \frac{2n^3+7}{n^4 \sin^2(n)}$$

and it should pretty obvious that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+7" in the numerator and the "$\sin^2(n)$" in the denominator won't really affect the size of the numerator and denominator respectively for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{2n^3}{n^4} = \frac{2}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$2n^3 < 2n^3 + 7$$

Using this we can make the numerator smaller to get the following relationship,

$$\frac{2n^3+7}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4\sin^2\left(n\right)}$$

Now, we know that $0 \le \sin^2\left(n\right) \le 1$ and so in the denominator we can see that if we replace the $\sin^2\left(n\right)$ with its largest possible value we have,

$$n^4\sin^2\left(n\right) < n^4\left(1\right) = n^4$$

Using this we can make the denominator larger (and hence make the rational expression smaller) to get,

$$\frac{2n^3+7}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4} = \frac{2}{n}$$

Step 4
Now, the series,

$$\sum_{n=1}^{\infty}\frac{2}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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7. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^n \sin^2\left(5n\right)}{4^n + \cos^2\left(n\right)}$$

Step 1
First, the series terms are,

$$a_n = \frac{2^n \sin^2\left(5n\right)}{4^n + \cos^2\left(n\right)}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The trig functions in the numerator and in the denominator won't really affect the size of the numerator and denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{2^n}{4^n} = \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n$$

We also know that the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

will converge because it is a geometric series with $r = \frac{1}{2} < 1$.

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We know that $0 \le \sin^2\left(5n\right) \le 1$ and so replacing the $\sin^2\left(5n\right)$ in the numerator with the largest possible value we get,

$$2^n \sin^2\left(5n\right) < 2^n\left(1\right) = 2^n$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{2^n \sin^2\left(5n\right)}{4^n + \cos^2\left(n\right)} < \frac{2^n}{4^n + \cos^2\left(n\right)}$$

Now, in the denominator we know that $0 \le \cos^2\left(n\right) \le 1$ and so replacing the $\cos^2\left(n\right)$ with the smallest possible value we get,

$$4^n + \cos^2\left(n\right) > 4^n + 0 = 4^n$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{2^n \sin^2\left(5n\right)}{4^n + \cos^2\left(n\right)} < \frac{2^n}{4^n + \cos^2\left(n\right)} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

Step 4
Now, the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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8. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

Step 1
First, the series terms are,

$$a_n = \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
In this case let's first notice the exponential in the numerator will go to zero as *n* goes to infinity. Let's also notice that the denominator is just a polynomial. In cases like this the exponential is going to go to zero so fast that behavior of the denominator will not matter at all and in all probability the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We can use some quick Calculus I to prove that $\mathbf{e}^{-n}$ is a decreasing function and so,

$$\mathbf{e}^{-n} < \mathbf{e}^{-3} < 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{\mathbf{e}^{-n}}{n^2+2n} < \frac{1}{n^2+2n}$$

Now, in the denominator it should be fairly clear that,

$$n^2+2n > n^2$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{\mathbf{e}^{-n}}{n^2+2n} < \frac{1}{n^2+2n} < \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

is a convergent series (*p*-series Test with $p = 2 > 1$) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

---

9. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4n^2-n}{n^3+9}$$

Step 1
First, the series terms are,

$$a_n = \frac{4n^2-n}{n^3+9}$$

You can verify that for $n \ge 1$ we have $4n^2 > n$ and so $4n^2 - n > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the numerator will increase at a much faster rate than the $-n$ (a linear term) portion of the numerator. Therefore the n^2 portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n = \frac{4n^2}{n^3} = \frac{4}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{4}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverge we'll need to find a series with smaller terms that we know, or can prove, diverge.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

We now have a problem however. The obvious thing to try is to drop the last term in both the numerator and the denominator. Doing that however gives the following inequalities,

$$4n^2 - n < 4n^2 \qquad\qquad n^3 + 9 > n^3$$

Using these two in the series terms gives the following relationship,

$$\frac{4n^2 - n}{n^3 + 9} < \frac{4n^2}{n^3 + 9} < \frac{4n^2}{n^3} = \frac{4}{n}$$

Now the series,

$$\sum_{n=0}^{\infty} \frac{4}{n}$$

will definitely diverge (as discussed above) it's terms are larger than the series terms in the problem statement. Just because a series with larger terms diverges does not, in any way, imply a series with smaller terms will also diverge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n \to \infty} \left[\frac{4n^2 - n}{n^3 + 9} \; \frac{n}{4} \right] = \lim_{n \to \infty} \left[\frac{4n^3 - n^2}{4n^3 + 36} \right] = 1$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e.* *c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series diverges and so the series given in the problem statement must also **diverge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

---

10. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{\sqrt{2n^2+4n+1}}{n^3+9}$$

Step 1
First, the series terms are,

$$a_n=\frac{\sqrt{2n^2+4n+1}}{n^3+9}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the $2n^2$ (a quadratic term) in the numerator will increase at a much faster rate than the $4n+1$ (a linear term) portion of the numerator. Therefore the $2n^2$ portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n=\frac{\sqrt{2n^2}}{n^3}=\frac{\sqrt{2}}{n^2}$$

We also know that the series,

$$\sum_{n=1}^{\infty}\frac{\sqrt{2}}{n^2}$$

will converge by the *p*-series Test ($p=2>1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms in the numerator and the last term in the denominator. Doing that however gives the following inequalities,

$$2n^2 < 2n^2 + 4n + 1 \qquad\qquad n^3 + 9 > n^3$$

This leads to a real problem! If we use the inequality for the numerator we're going to get a smaller rational expression and if we use the inequality for the denominator we're going to get a larger rational expression. Because these two can't both be used at the same time it will make it fairly difficult to use the Comparison Test since neither one individually give a series we can quickly deal with.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n\to\infty} \left[\frac{\sqrt{2n^2+4n+1}}{n^3+9} \, \frac{n^2}{\sqrt{2}} \right] \\ &= \lim_{n\to\infty} \left[\frac{n^2 \sqrt{n^2\left(2+\frac{4}{n}+\frac{1}{n^2}\right)}}{\sqrt{2}\, n^3 \left(1+\frac{9}{n^3}\right)} \right] = \lim_{n\to\infty} \left[\frac{n^2 \left(n\right) \sqrt{2+\frac{4}{n}+\frac{1}{n^2}}}{\sqrt{2}\, n^3 \left(1+\frac{9}{n^3}\right)} \right] = \frac{\sqrt{2}}{\sqrt{2}} = 1 \end{aligned}$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e. c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

Section 4-8 : Alternating Series Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{7+2n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{7+2n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{7+2n} > \frac{1}{7+2(n+1)}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3 + 4n + 1}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{n^3 + 4n + 1}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n^3 + 4n + 1} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{n^3 + 4n + 1} > \frac{1}{(n+1)^3 + 4(n+1) + 1}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{(-1)^n \left(2^n + 3^n\right)}$$

Step 1

Do not get excited about the $\left(-1\right)^n$ is in the denominator! This is still an alternating series! All the $\left(-1\right)^n$ does is change the sign regardless of whether or not it is in the numerator.

Also note that we could just as easily rewrite the terms as,

$$\frac{1}{\left(-1\right)^n\left(2^n+3^n\right)}=\frac{\left(-1\right)^n}{\left(-1\right)^n}\frac{1}{\left(-1\right)^n\left(2^n+3^n\right)}=\frac{\left(-1\right)^n}{\left(-1\right)^{2n}\left(2^n+3^n\right)}=\frac{\left(-1\right)^n}{\left(2^n+3^n\right)}$$

Note that $\left(-1\right)^{2n}=1$ because the exponent is always even!

So, we now know that this is an alternating series with,

$$b_n=\frac{1}{2^n+3^n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{2^n+3^n}=0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{2^n+3^n}>\frac{1}{2^{n+1}+3^{n+1}}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2+9}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{n}{n^2+9}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+9} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{x}{x^2+9} \qquad\qquad f'(x) = \frac{9-x^2}{(x^2+9)^2}$$

It should be pretty clear that the function will be increasing in $0 \le x < 3$ and decreasing in $x > 3$ (the range of *x* that corresponds to our range of *n*).

So, the b_n do not actually form a decreasing sequence but they are decreasing for $n > 3$ and so we can say that they are eventually decreasing and as discussed in the notes that will be sufficient for us.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1-n}{3n-n^2}$$

and b_n are positive for $n \ge 4$ and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-n}{3n-n^2} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{1-x}{3x-x^2} \qquad\qquad f'(x) = \frac{-x^2+2x-3}{\left(3x-x^2\right)^2}$$

The numerator of the derivative is never zero for any real number (we'll leave that to you to verify) and since it is clearly negative at $x = 0$ we know that the function will always be decreasing for $x \ge 4$.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

Section 4-9 : Absolute Convergence

1. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$$

Step 1
Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n^3+1} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Now, notice that,

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

and we know by the *p*-series test that

$$\sum_{n=2}^{\infty} \frac{1}{n^3}$$

converges.

Therefore, by the Comparison Test we know that the series from the problem statement,

$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

will also converge.

Step 2
So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

---

2. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-3}}{\sqrt{n}}$$

Step 1
Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-3}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$$

Now, by the by the *p*-series test we can see that this series will diverge.

Step 2
So, at this point we know that the series in the problem statement is not absolutely convergent so all we need to do is check to see if it's conditionally convergent or divergent. To do this all we need to do is check the convergence of the series in the problem statement.

The series in the problem statement is an alternating series with,

$$b_n = \frac{1}{\sqrt{n}}$$

Clearly the b_n are positive so we can use the Alternating Series Test on this series. It is hopefully clear that the b_n are a decreasing sequence and $\lim\limits_{n\to\infty} b_n = 0$.

Therefore, by the Alternating Series Test the series from the problem statement is convergent.

Step 3
So, because the series with the absolute value diverges and the series from the problem statement converges we know that the series in the problem statement is **conditionally convergent**.

---

3. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+1}$$

Step 1
Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=3}^{\infty}\left|\frac{(-1)^{n+1}(n+1)}{n^3+1}\right| = \sum_{n=3}^{\infty}\frac{n+1}{n^3+1}$$

We know by the *p*-series test that the following series converges.

$$\sum_{n=3}^{\infty}\frac{1}{n^2}$$

If we now compute the following limit,

$$c = \lim_{n\to\infty}\left[\frac{n+1}{n^3+1}\frac{n^2}{1}\right] = \lim_{n\to\infty}\left[\frac{n^3+n^2}{n^3+1}\right] = 1$$

we know by the Limit Comparison Test that the two series in this problem have the same convergence because *c* is neither zero or infinity and because $\sum_{n=3}^{\infty}\frac{1}{n^2}$ converges we know that the series from the problem statement must also converge.

Step 2
So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

Section 4-10 : Ratio Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{1-2(n+1)}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{-1-2n}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^2+1}\frac{n^2+1}{3^2}\right| = \lim_{n\to\infty}\left|\frac{n^2+1}{9\left[(n+1)^2+1\right]}\right| = \frac{1}{9}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{1}{9} < 1$ and so by the Ratio Test the series **converges**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(2(n+1)\right)!}{5(n+1)+1}\frac{5n+1}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(5n+1)}{5n+6}\right| = \infty$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \infty > 1$ and so by the Ratio Test the series **diverges**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{1+3(n+1)}(n+1+1)}{(n+1)^2 5^{1+n+1}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(-2)^3(n+2)}{(n+1)^2(5)}\frac{n^2}{(n+1)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{-8n^2(n+2)}{5(n+1)^2(n+1)}\right| = \frac{8}{5}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{8}{5} > 1$ and so by the Ratio Test the series **diverges**.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4(n+1)}}{(n+1-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)(n-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4}}{n-1}\right| = 0$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 0 < 1$ and so by the Ratio Test the series **converges**.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+7}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1+1}}{6(n+1)+7}\frac{6n+7}{(-1)^{n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{6n+13}\frac{6n+7}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(6n+7)}{6n+13}\right| = 1$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 1$ and so by the Ratio Test tells us nothing about this series.

Step 3
Just because the Ratio Test doesn't tell us anything doesn't mean we can't determine if this series converges or diverges.

In fact, it's actually quite simple to do in this case. This is an Alternating Series with,

$$b_n = \frac{1}{6n+7}$$

The b_n are clearly positive and it should be pretty obvious (hopefully) that they also form a decreasing sequence. Finally, we also can see that $\lim\limits_{n\to\infty} b_n = 0$ and so by the Alternating Series Test this series will **converge**.

Note, that if this series were not in this section doing this as an Alternating Series from the start would probably have been the best way of approaching this problem.

Section 4-11 : Root Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$$

Step 1
We'll need to compute *L*.

$$L=\lim_{n\to\infty}\sqrt[n]{\left|a_n\right|}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2}\right|=\left(-\frac{3}{2}\right)^{2}=\frac{9}{4}$$

Step 2
Okay, we can see that $L=\frac{9}{4}>1$ and so by the Root Test the series **diverges**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^{1-3n}}{4^{2n}}$$

Step 1
We'll need to compute *L*.

$$L=\lim_{n\to\infty}\sqrt[n]{\left|a_n\right|}=\lim_{n\to\infty}\left|\frac{n^{1-3n}}{4^{2n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{n^{\frac{1}{n}-3}}{4^{2}}\right|=\left|\frac{n^{\frac{1}{n}}\;n^{-3}}{4^{2}}\right|=\frac{(1)(0)}{16}=0$$

Step 2
Okay, we can see that $L=0<1$ and so by the Root Test the series **converges**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty}\frac{(-5)^{1+2n}}{2^{5n-3}}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|a_n\right|} = \lim_{n\to\infty} \left|\frac{\left(-5\right)^{1+2n}}{2^{5n-3}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{\left(-5\right)^{\frac{1}{n}+2}}{2^{5-\frac{3}{n}}}\right| = \left|\frac{\left(-5\right)^{2}}{2^{5}}\right| = \frac{25}{32}$$

Step 2
Okay, we can see that $L = \frac{25}{32} < 1$ and so by the Root Test the series **converges**.

Section 4-12 : Strategy for Series

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

Section 4-13 : Estimating the Value of a Series

1. Use the Integral Test and $n = 10$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$.

Step 1
Since we are being asked to use the Integral Test to estimate the value of the series we should first make sure that the Integral Test can actually be used on this series.

First, the series terms are clearly positive so that condition is met.

Now, let's do a little Calculus I on the following function.

$$f(x) = \frac{x}{\left(x^2+1\right)^2} \qquad\qquad f'(x) = \frac{1-3x^2}{\left(x^2+1\right)^3}$$

The derivative of the function will be negative for $x > \frac{1}{\sqrt{3}} = 0.5774$ and so the function will be decreasing in this range. Because the function and the series terms are the same we can also see that the series terms are decreasing for the range of *n* in our series.

Therefore, the conditions required to use the Integral Test are met! Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n = 10$. This is,

$$s_{10} = \sum_{n=1}^{10} \frac{n}{\left(n^2+1\right)^2} = 0.392632317$$

Step 3
Now, to increase the accuracy of the partial sum from the previous step we know we can use each of the following two integrals.

$$\int_{10}^{\infty} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty}\int_{10}^{t} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2\left(x^2+1\right)}\right]\Bigg|_{10}^{t} = \lim_{t\to\infty}\left[\frac{1}{202} - \frac{1}{2\left(t^2+1\right)}\right] = \frac{1}{202}$$

$$\int_{11}^{\infty} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty}\int_{11}^{t} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2\left(x^2+1\right)}\right]\Bigg|_{11}^{t} = \lim_{t\to\infty}\left[\frac{1}{244} - \frac{1}{2\left(t^2+1\right)}\right] = \frac{1}{244}$$

Step 4
Okay, we know from the notes in this section that if *s* represents that actual value of the series that it must be in the following range.

$$0.392632317 + \frac{1}{244} < s < 0.392632317 + \frac{1}{202}$$
$$0.396730678 < s < 0.397582813$$

Step 5
Finally, if we average the two numbers above we can get a better estimate of,

$$\boxed{s \approx 0.397156745}$$

---

2. Use the Comparison Test and $n = 20$ to estimate the value of $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3 \ln\left(n\right)}$.

Step 1
Since we are being asked to use the Comparison Test to estimate the value of the series we should first make sure that the Comparison Test can actually be used on this series.

In this case that is easy enough because, for our range of *n*, the series terms are clearly positive and so we can use the Comparison Test.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n = 20$. This is,

$$s_{20} = \sum_{n=3}^{20} \frac{1}{n^3 \ln\left(n\right)} = 0.057315878$$

Step 3
Now, let's see if we can get can get an error estimate on this approximation of the series value. To do that we'll first need to do the Comparison Test on this series.

That is easy enough for this series once we notice that $\ln\left(n\right)$ is an increasing function and so $\ln\left(n\right) \ge \ln\left(3\right)$. Therefore, we get,

$$\frac{1}{n^3 \ln(n)} \le \frac{1}{n^3 \ln(3)} = \frac{1}{\ln(3)} \frac{1}{n^3}$$

Step 4

We now know, from the discussion in the notes, that an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) is,

$$\begin{aligned} R_{20} \le T_{20} = \sum_{n=21}^{\infty} \frac{1}{n^3 \ln(3)} &< \int_{20}^{\infty} \frac{1}{x^3 \ln(3)}\,dx \\ &= \lim_{t\to\infty} \int_{20}^{t} \frac{1}{x^3 \ln(3)}\,dx = \lim_{t\to\infty} \left(-\frac{1}{2x^2 \ln(3)} \right)\Bigg|_{20}^{t} \\ &= \lim_{t\to\infty} \left(\frac{1}{800\ln(3)} - \frac{1}{2t^2 \ln(3)} \right) = \frac{1}{800\ln(3)} \end{aligned}$$

Step 5

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.057315878}$$

and the error on this estimate will be no more than $\frac{1}{800\ln(3)} = 0.001137799$.

---

3. Use the Alternating Series Test and $n = 16$ to estimate the value of $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$.

Step 1

Since we are being asked to use the Alternating Series Test to estimate the value of the series we should first make sure that the Alternating Series Test can actually be used on this series.

First, note that the b_n for this series are,

$$b_n = \frac{n}{n^2+1}$$

and they are positive and with a quick derivative we can see they are decreasing and so the Alternating Series Test can be used here.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n = 16$. This is,

$$s_{16} = \sum_{n=2}^{16} \frac{(-1)^n n}{n^2+1} = 0.260554530$$

Step 3
Now, we know, from the discussion in the notes, that an upper bound on the absolute value of the remainder (*i.e.* the error between the approximation and exact value) is nothing more than,

$$b_{17} = \frac{17}{290} = 0.058620690$$

Step 4
So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.260554530}$$

and the error on this estimate will be no more than 0.058620690.

---

4. Use the Ratio Test and $n = 8$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{3^{1+n}}{n\,2^{3+2n}}$.

Step 1
First notice that the terms are positive and so we can use the Ratio Test to do the estimate. Remember that this is a requirement only to use the Ratio Test to get an estimate of the series value and is not an actual requirement to use the Ratio Test to determine if the series converges or diverges.

So, let's start off with the partial sum using $n = 8$. This is,

$$s_8 = \sum_{n=1}^{8} \frac{3^{1+n}}{n\,2^{3+2n}} = 0.509881435$$

Step 2
Now, to get an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) we need the following ratio,

$$r_n = \frac{a_{n+1}}{a_n} = \frac{3^{2+n}}{(n+1)2^{5+2n}} \frac{n\,2^{3+2n}}{3^{1+n}} = \frac{3n}{4(n+1)}$$

We'll also potentially need the limit,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3n}{4(n+1)} = \frac{3}{4}$$

Step 3

Next, we need to know if the r_n form an increasing or decreasing sequence. A quick application of Calculus I will answer this.

$$f(x) = \frac{3x}{4(x+1)} \qquad\qquad f'(x) = \frac{3}{4(x+1)^2} > 0$$

As noted above the derivative is always positive and so the function, and hence the r_n are increasing.

Step 4

The upper bound on the remainder is then,

$$R_8 \le \frac{a_9}{1-L} = \frac{\frac{6561}{2{,}097{,}152}}{1-\frac{3}{4}} = 0.012514114$$

Step 5

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.509881435}$$

and the error on this estimate will be no more than 0.012514114.

Section 4-14 : Power Series

1. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\left(n^2+1\right)}(4x-12)^n$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{(4x-12)^{n+1}}{(-3)^{3+n}\left((n+1)^2+1\right)}\frac{(-3)^{2+n}\left(n^2+1\right)}{(4x-12)^n}\right| = \lim_{n\to\infty}\left|\frac{(4x-12)}{(-3)\left((n+1)^2+1\right)}\frac{\left(n^2+1\right)}{1}\right|$$

$$= |4x-12| \lim_{n\to\infty}\frac{\left(n^2+1\right)}{3\left((n+1)^2+1\right)} = \frac{1}{3}|4x-12|$$

Step 2

So, we know that the series will converge if,

$$\frac{1}{3}|4x-12| < 1 \qquad \rightarrow \qquad \frac{4}{3}|x-3| < 1 \qquad \rightarrow \qquad |x-3| < \frac{3}{4}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{3}{4}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{3}{4} < x-3 < \frac{3}{4} \qquad \rightarrow \qquad \frac{9}{4} < x < \frac{15}{4}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{9}{4} \ : \ \sum_{n=0}^{\infty}\frac{1}{(-3)^{2+n}\left(n^2+1\right)}(-3)^n = \sum_{n=0}^{\infty}\frac{1}{(-3)^2\left(n^2+1\right)} = \sum_{n=0}^{\infty}\frac{1}{9\left(n^2+1\right)}$$

$$x = \frac{15}{4} \ : \ \sum_{n=0}^{\infty}\frac{1}{(-1)^{2+n}(3)^{2+n}\left(n^2+1\right)}(3)^n = \sum_{n=0}^{\infty}\frac{1}{(-1)^{2+n}(3)^2\left(n^2+1\right)} = \sum_{n=0}^{\infty}\frac{(-1)^{2+n}}{9\left(n^2+1\right)}$$

Now, we can do a quick Comparison Test on the first series to see that it converges and we can do a quick Alternating Series Test on the second series to see that is also converges.

We'll leave it to you to verify both of these statements.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \quad : \quad \frac{9}{4} \le x \le \frac{15}{4}} \qquad\qquad \boxed{R = \frac{3}{4}}$$

---

2. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}} \left(2x+17\right)^n$$

Step 1
Okay, let's start off with the Root Test to get our hands on *L*.

$$L = \lim_{n\to\infty} \left| \frac{n^{2n+1}}{4^{3n}} \left(2x+17\right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n^{2+\frac{1}{n}}}{4^3} \left(2x+17\right) \right| = \left|2x+17\right| \lim_{n\to\infty} \frac{n^{2+\frac{1}{n}}}{4^3}$$

Okay, we can see that , in this case, *L* will be infinite provided $x \ne -\frac{17}{2}$ and so the series will diverge for $x \ne -\frac{17}{2}$. We also know that the power series will converge for $x = -\frac{17}{2}$ (this is the value of *a* for this series!).

Step 2
Therefore, we know that the interval of convergence is $\boxed{x = -\frac{17}{2}}$ and the radius of convergence is $\boxed{R = 0}$.

---

3. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n+1}{\left(2n+1\right)!} \left(x-2\right)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)^{n+1}}{(2n+3)!}\frac{(2n+1)!}{(n+1)(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)}{(2n+3)(2n+2)(2n+1)!}\frac{(2n+1)!}{(n+1)}\right|$$
$$= |x-2|\lim_{n\to\infty}\frac{n+2}{(2n+3)(2n+2)(n+1)} = 0$$

Okay, we can see that , in this case, $L = 0$ for every *x*.

Step 2

Therefore, we know that the interval of convergence is $\boxed{-\infty < x < \infty}$ and the radius of convergence is $\boxed{R = \infty}$.

---

4. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty}\frac{4^{1+2n}}{5^{n+1}}(x+3)^n$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{4^{3+2n}(x+3)^{n+1}}{5^{n+2}}\frac{5^{n+1}}{4^{1+2n}(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{4^2(x+3)}{5}\right| = |x+3|\lim_{n\to\infty}\frac{16}{5} = \frac{16}{5}|x+3|$$

Step 2

So, we know that the series will converge if,

$$\frac{16}{5}|x+3| < 1 \qquad \to \qquad |x+3| < \frac{5}{16}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{5}{16}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{5}{16} < x+3 < \frac{5}{16} \qquad \to \qquad -\frac{53}{16} < x < -\frac{43}{16}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = -\frac{53}{16} \; : \; \sum_{n=0}^{\infty} \frac{4^1 4^{2n}}{5^n 5^1}\left(-\frac{5}{16}\right)^n = \sum_{n=0}^{\infty} \frac{4\left(16^n\right)}{5^n\left(5\right)} \frac{\left(-1\right)^n 5^n}{16^n} = \sum_{n=0}^{\infty} \frac{4\left(-1\right)^n}{5}$$

$$x = -\frac{43}{16} \; : \; \sum_{n=0}^{\infty} \frac{4^1 4^{2n}}{5^n 5^1}\left(\frac{5}{16}\right)^n = \sum_{n=0}^{\infty} \frac{4\left(16^n\right)}{5^n\left(5\right)} \frac{5^n}{16^n} = \sum_{n=0}^{\infty} \frac{4}{5}$$

Now,

$$\lim_{n\to\infty} \frac{4\left(-1\right)^n}{5} \quad - \quad \text{Does not exist} \qquad\qquad \lim_{n\to\infty} \frac{4}{5} = \frac{4}{5}$$

Therefore, each of these two series diverge by the Divergence Test.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \;\; : \;\; -\frac{53}{16} < x < -\frac{43}{16}} \qquad\qquad \boxed{R = \frac{5}{16}}$$

---

5. For the following power series determine the interval and radius of convergence.

$$\sum_{n=1}^{\infty} \frac{6^n}{n}\left(4x-1\right)^{n-1}$$

Step 1
Okay, let’s start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{6^{n+1}\left(4x-1\right)^n}{n+1}\frac{n}{6^n\left(4x-1\right)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{6n\left(4x-1\right)}{n+1}\right| = \left|4x-1\right|\lim_{n\to\infty}\frac{6n}{n+1} = 6\left|4x-1\right|$$

Step 2
So, we know that the series will converge if,

$$6\left|4x-1\right| < 1 \qquad \to \qquad 24\left|x-\tfrac{1}{4}\right| < 1 \qquad \to \qquad \left|x-\tfrac{1}{4}\right| < \frac{1}{24}$$

Step 3
So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{1}{24}}$.

Step 4
Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{1}{24} < x - \frac{1}{4} < \frac{1}{24} \qquad \to \qquad \frac{5}{24} < x < \frac{7}{24}$$

Step 5
To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{5}{24} \; : \; \sum_{n=1}^{\infty} \frac{6^n}{n}\left(-\frac{1}{6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{6^n}{n}\frac{\left(-1\right)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{6\left(-1\right)^{n-1}}{n}$$

$$x = \frac{7}{24} \; : \; \sum_{n=1}^{\infty} \frac{6^n}{n}\left(\frac{1}{6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{6^n}{n}\frac{1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{6}{n}$$

Now, the first series is an alternating harmonic series which we know converges (or you could just do a quick Alternating Series Test to verify this) and the second series diverges by the *p*-series test.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \; : \; \frac{5}{24} \le x < \frac{7}{24}} \qquad \boxed{R = \frac{1}{24}}$$

Section 4-15 : Power Series and Functions

1. Write the following function as a power series and give the interval of convergence.

$$f\left(x\right)=\frac{6}{1+7x^4}$$

Step 1

First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f\left(x\right)=6\frac{1}{1+7x^4}$$

Step 2

Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator to get the following form of the function,

$$f\left(x\right)=6\frac{1}{1-\left(-7x^4\right)}$$

Step 3

At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f\left(x\right)=6\frac{1}{1-\left(-7x^4\right)}=6\sum_{n=0}^{\infty}\left(-7x^4\right)^n \qquad \text{provided } \left|-7x^4\right|<1$$

Step 4

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f\left(x\right)=\sum_{n=0}^{\infty}6\left(-7\right)^n\left(x^4\right)^n=\sum_{n=0}^{\infty}6\left(-7\right)^n x^{4n} \qquad \text{provided } \left|-7x^4\right|<1$$

Step 5

To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|-7x^4\right|<1 \quad\to\quad 7\left|x\right|^4<1 \quad\to\quad \left|x\right|^4<\frac{1}{7} \quad\to\quad \left|x\right|<\frac{1}{7^{\frac{1}{4}}} \quad\to\quad -\frac{1}{7^{\frac{1}{4}}}<x<\frac{1}{7^{\frac{1}{4}}}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6
The answers for this problem are then,

$$\boxed{\text{Power Series} \ : \ \frac{6}{1+7x^4} = \sum_{n=0}^{\infty} 6\left(-7\right)^n x^{4n}} \qquad \boxed{\text{Interval} \ : \ -\frac{1}{7^{\frac{1}{4}}} < x < \frac{1}{7^{\frac{1}{4}}}}$$

---

2. Write the following function as a power series and give the interval of convergence.

$$f\left(x\right) = \frac{x^3}{3-x^2}$$

Step 1
First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f\left(x\right) = x^3 \frac{1}{3-x^2}$$

Step 2
Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 3 out of the denominator as follows,

$$f\left(x\right) = \frac{x^3}{3}\frac{1}{1-\frac{1}{3}x^2}$$

Step 3
At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f\left(x\right) = \frac{x^3}{3}\frac{1}{1-\frac{1}{3}x^2} = \frac{x^3}{3}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}x^2\right)^n \qquad \text{provided } \left|\tfrac{1}{3}x^2\right| < 1$$

Step 4
Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f\left(x\right)=\frac{x^3}{3}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}x^2\right)^n=\sum_{n=0}^{\infty}\tfrac{1}{3}x^3\left(\tfrac{1}{3}\right)^n\left(x^2\right)^n=\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^{n+1}x^{2n+3}\qquad\text{provided }\left|\tfrac{1}{3}x^2\right|<1$$

Step 5
To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\tfrac{1}{3}x^2\right|<1\quad\to\quad\tfrac{1}{3}\left|x\right|^2<1\quad\to\quad\left|x\right|^2<3\quad\to\quad\left|x\right|<\sqrt{3}\quad\to\quad-\sqrt{3}<x<\sqrt{3}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6
The answers for this problem are then,

$$\boxed{\text{Power Series}\ :\ \frac{x^3}{3-x^2}=\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^{n+1}x^{2n+3}}\qquad\boxed{\text{Interval}\ :\ -\sqrt{3}<x<\sqrt{3}}$$

---

3. Write the following function as a power series and give the interval of convergence.

$$f\left(x\right)=\frac{3x^2}{5-2\sqrt[3]{x}}$$

Step 1
First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f\left(x\right)=3x^2\frac{1}{5-2\sqrt[3]{x}}$$

Step 2
Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 5 out of the denominator as follows,

$$f\left(x\right)=\frac{3x^2}{5}\frac{1}{1-\frac{2}{5}\sqrt[3]{x}}$$

Step 3
At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f\left(x\right)=\frac{3x^2}{5}\frac{1}{1-\frac{2}{5}\sqrt[3]{x}}=\frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\tfrac{2}{5}\sqrt[3]{x}\right)^n \qquad \text{provided } \left|\tfrac{2}{5}\sqrt[3]{x}\right|<1$$

Step 4

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f\left(x\right)=\frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\tfrac{2}{5}\sqrt[3]{x}\right)^n=\sum_{n=0}^{\infty}\tfrac{3}{5}x^2\left(\tfrac{2}{5}\right)^n\left(x^{\frac{1}{3}}\right)^n=\sum_{n=0}^{\infty}\tfrac{3}{5}\left(\tfrac{2}{5}\right)^n x^{\frac{1}{3}n+2} \qquad \text{provided } \left|\tfrac{2}{5}\sqrt[3]{x}\right|<1$$

Step 5

To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\tfrac{2}{5}\sqrt[3]{x}\right|<1 \quad\rightarrow\quad \tfrac{2}{5}\left|x\right|^{\frac{1}{3}}<1 \quad\rightarrow\quad \left|x\right|^{\frac{1}{3}}<\tfrac{5}{2} \quad\rightarrow\quad \left|x\right|<\tfrac{125}{8} \quad\rightarrow\quad -\tfrac{125}{8}<x<\tfrac{125}{8}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6

The answers for this problem are then,

$$\boxed{\text{Power Series} \ : \ \frac{3x^2}{5-2\sqrt[3]{x}}=\sum_{n=0}^{\infty}\tfrac{3}{5}\left(\tfrac{2}{5}\right)^n x^{\frac{1}{3}n+2}} \qquad \boxed{\text{Interval} \ : \ -\tfrac{125}{8}<x<\tfrac{125}{8}}$$

---

4. Give a power series representation for the derivative of the following function.

$$g\left(x\right)=\frac{5x}{1-3x^5}$$

Hint : While we could differentiate the function and then attempt to find a power series representation that seems like a lot of work. It's a good think that we know how to differentiate power series.

Step 1

First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$g\left(x\right)=5x\frac{1}{1-3x^5}=5x\sum_{n=0}^{\infty}\left(3x^5\right)^n=\sum_{n=0}^{\infty}5x\left(3^n\right)x^{5n}=\sum_{n=0}^{\infty}5\left(3^n\right)x^{5n+1}$$

Step 2
Now, we know how to differentiate power series and we know that the derivative of the power series representation of a function is the power series representation of the derivative of the function.

Therefore,

$$g'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} 5\left(3^n\right)x^{5n+1}\right] = \boxed{\sum_{n=0}^{\infty} 5\left(5n+1\right)\left(3^n\right)x^{5n}}$$

Remember that to differentiate a power series all we need to do is differentiate the term of the power series with respect to *x*.

---

5. Give a power series representation for the integral of the following function.

$$h(x) = \frac{x^4}{9+x^2}$$

Hint : Integrating this function seems like (potentially) a lot of work, not to mention determining a power series representation of the result. It's a good think that we know how to integrate power series.

Step 1
First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$h(x) = \frac{x^4}{9}\frac{1}{1-\left(-\frac{1}{9}x^2\right)} = \frac{x^4}{9}\sum_{n=0}^{\infty}\left(-\tfrac{1}{9}x^2\right)^n = \sum_{n=0}^{\infty}\tfrac{1}{9}x^4\left(-1\right)^n\left(\tfrac{1}{9}\right)^n x^{2n} = \sum_{n=0}^{\infty}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+4}$$

Step 2
Now, we know how to integrate power series and we know that the integral of the power series representation of a function is the power series representation of the integral of the function.

Therefore,

$$\int h(x)\,dx = \int \sum_{n=0}^{\infty}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+4}\,dx = \boxed{c + \sum_{n=0}^{\infty}\tfrac{1}{2n+5}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+5}}$$

Remember that to integrate a power series all we need to do is integrate the term of the power series and we can't forget to add on the "+c" since we're doing an indefinite integral.

Section 4-16 : Taylor Series

1. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f\left(x\right)=\cos\left(4x\right)$ about $x=0$.

Step 1

There really isn't all that much to do here for this problem. We are working with cosine and want the Taylor series about $x=0$ and so we can use the Taylor series for cosine derived in the notes to get,

$$\cos\left(4x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}\left(4x\right)^{2n}}{\left(2n\right)!}$$

Step 2

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

In this case we don't have anything out in front of the series to worry about so all we need to do is use the basic exponent rules on the 2*x* term to get,

$$\cos\left(4x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}4^{2n}x^{2n}}{\left(2n\right)!}=\boxed{\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}16^{n}x^{2n}}{\left(2n\right)!}}$$

---

2. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f\left(x\right)=x^{6}\mathbf{e}^{2x^{3}}$ about $x=0$.

Step 1

There really isn't all that much to do here for this problem. We are working with the exponential function and want the Taylor series about $x=0$ and so we can use the Taylor series for the exponential function derived in the notes to get,

$$x^{6}\mathbf{e}^{2x^{3}}=x^{6}\sum_{n=0}^{\infty}\frac{\left(2x^{3}\right)^{n}}{n!}$$

Note that we only convert the exponential using the Taylor series derived in the notes and, at this point, we just leave the x^{6} alone in front of the series.

Step 2

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$x^6\mathbf{e}^{2x^3} = x^6\sum_{n=0}^{\infty}\frac{\left(2x^3\right)^n}{n!} = \sum_{n=0}^{\infty}x^6\,\frac{2^n\left(x^3\right)^n}{n!} = \boxed{\sum_{n=0}^{\infty}\frac{2^n\,x^{3n+6}}{n!}}$$

---

3. Find the Taylor Series for $f\left(x\right)=\mathbf{e}^{-6x}$ about $x=-4$.

Step 1

Because we are working about $x=-4$ in this problem we are not able to just use the formula derived in class for the exponential function because that requires us to be working about $x=0$.

Step 2

So, we'll need to start over from the beginning and start taking some derivatives of the function.

$$\begin{aligned}
n=0: &\quad f\left(x\right)=\mathbf{e}^{-6x}\\
n=1: &\quad f'\left(x\right)=-6\mathbf{e}^{-6x}\\
n=2: &\quad f''\left(x\right)=\left(-6\right)^2\mathbf{e}^{-6x}\\
n=3: &\quad f^{(3)}\left(x\right)=\left(-6\right)^3\mathbf{e}^{-6x}\\
n=4: &\quad f^{(4)}\left(x\right)=\left(-6\right)^4\mathbf{e}^{-6x}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula.

Step 3

It is now time to see if we can get a formula for the general term in the Taylor Series.

In this case, it is (hopefully) pretty simple to catch the pattern in the derivatives above. The general term is given by,

$$f^{(n)}\left(x\right)=\left(-6\right)^n\mathbf{e}^{-6x} \qquad n=0,1,2,3,\ldots$$

As noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of *n*'s and we need to know that before we start writing down the Taylor Series.

Step 4
Now, recall that we don't really want the general term at any *x*. We want the general term at $x = -4$. This is,

$$f^{(n)}\left(-4\right) = \left(-6\right)^n \mathbf{e}^{24} \qquad n = 0,1,2,3,\ldots$$

Step 5
Okay, at this point we can formally write down the Taylor Series for this problem.

$$\mathbf{e}^{-6x} = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(-4\right)}{n!}\left(x+4\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{\left(-6\right)^n \mathbf{e}^{24}}{n!}\left(x+4\right)^n}$$

---

4. Find the Taylor Series for $f\left(x\right) = \ln\left(3+4x\right)$ about $x = 0$.

Step 1
Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n &= 0: & f\left(x\right) &= \ln\left(3+4x\right)\\
n &= 1: & f'\left(x\right) &= \frac{4}{3+4x} = 4\left(3+4x\right)^{-1}\\
n &= 2: & f''\left(x\right) &= -4^2\left(3+4x\right)^{-2}\\
n &= 3: & f^{(3)}\left(x\right) &= 4^3\left(2\right)\left(3+4x\right)^{-3}\\
n &= 4: & f^{(4)}\left(x\right) &= -4^4\left(2\right)\left(3\right)\left(3+4x\right)^{-4}\\
n &= 5: & f^{(5)}\left(x\right) &= 4^5\left(2\right)\left(3\right)\left(4\right)\left(3+4x\right)^{-5}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n = 4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case we "merged" all the 4's that came from the chain rule into a single term but left it as an exponent rather than get an actual value. This is not uncommon with these kinds of problems. The exponents we dropped down for the derivatives we left alone with the exception of dealing with the signs.

Step 2

It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned} & f^{(0)}(x)=\ln(3+4x) && n=0 \\ & f^{(n)}(x)=(-1)^{n+1}4^n(n-1)!(3+4x)^{-n} && n=1,2,3,\ldots \end{aligned}$$

As noted this formula works all the way back to $n=1$ but clearly does not work for $n=0$. It is problems like this one that make it clear why we always need to check our proposed formula for the general solution to see just how far back it works.

Step 3

Now, recall that we don't really want the general term at any *x*. We want the general term at $x=0$. This is,

$$\begin{aligned} f^{(0)}(0) &= \ln(3) && n=0 \\ f^{(n)}(0) &= (-1)^{n+1}4^n(n-1)!(3)^{-n} \\ &= (-1)^{n+1}4^n(n-1)!\frac{1}{3^n} \\ &= (-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)! && n=1,2,3,\ldots \end{aligned}$$

We did a little simplification for the second one just to make it a little simpler.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem. However, before we actually do that recall that our general term formula did not work for $n=0$ and so we'll need to first strip that out of the series before we put the general formula in.

$$\begin{aligned} \ln(3+4x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\ &= f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\ &= \ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)!}{n!}x^n \\ &= \boxed{\ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n}x^n} \end{aligned}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

5. Find the Taylor Series for $f\left(x\right)=\dfrac{7}{x^4}$ about $x=-3$.

Step 1
Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0: &\quad f\left(x\right)=\frac{7}{x^4}=7x^{-4}\\
n=1: &\quad f'\left(x\right)=-7\left(4\right)x^{-5}\\
n=2: &\quad f''\left(x\right)=7\left(4\right)\left(5\right)x^{-6}\\
n=3: &\quad f^{(3)}\left(x\right)=-7\left(4\right)\left(5\right)\left(6\right)x^{-7}\\
n=4: &\quad f^{(4)}\left(x\right)=7\left(4\right)\left(5\right)\left(6\right)\left(7\right)x^{-8}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case the only "simplification" we did was to multiply out the minus signs that came from the exponents upon doing the derivatives. That is a fairly common thing to do with these kinds of problems.

Step 2
It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(n)}\left(x\right)&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)}{\left(2\right)\left(3\right)}\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)x^{-8}\\
&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)}{6}x^{-8}\\
&=\frac{7}{6}\left(-1\right)^n\left(n+3\right)!x^{-(n+4)} \qquad\qquad n=0,1,2,3,\ldots
\end{aligned}$$

This formula may have been a little trickier to get. We almost had a factorial in the derivatives but each one was missing the $\left(2\right)\left(3\right)$ part that would be needed to get the factorial to show up. Because that was all that was missing and it was missing in each of the derivatives we multiplied each derivative by

$\frac{(2)(3)}{(2)(3)}$ (*i.e.* a really fancy way of writing one…). We could then use the numerator of this to complete the factorial and the denominator was just left alone.

Also, as noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of *n*'s and we need to know that before we start writing down the Taylor Series.

Step 3
Now, recall that we don't really want the general term at any *x*. We want the general term at $x = -3$. This is,

$$\begin{aligned} f^{(n)}(-3) &= \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n+4)} \\ &= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}} \\ &= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(-1)^4 (3)^{n+4}} \\ &= \frac{7(n+3)!}{6(3)^{n+4}} \qquad\qquad n = 1,2,3,\ldots \end{aligned}$$

We did a little simplification here so we could cancel out all the alternating signs that were present in the term.

Step 4
Okay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{7(n+3)!}{6(3)^{n+4}\, n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

---

6. Find the Taylor Series for $f(x) = 7x^2 - 6x + 1$ about $x = 2$.

Step 1

First, let's not get too excited about the fact that we have a polynomial here for this problem. It works exactly the same way with a few small differences.

We'll start off by taking a few derivatives of the function and evaluating them at $x = 2$

$$
\begin{array}{lll}
n=0: & f(x) = 7x^2 - 6x + 1 & f(2) = 17 \\
n=1: & f'(x) = 14x - 6 & f'(2) = 22 \\
n=2: & f''(x) = 14 & f''(2) = 14 \\
n \geq 3: & f^{(n)}(x) = 0 & f^{(n)}(2) = 0
\end{array}
$$

Okay, this is where one of the differences between a polynomial and the other types of functions we typically see with Taylor Series problems. After some point all the derivatives will be zero. That is not something to get excited about. In fact, it actually makes the problem a little easier!

Because all the derivatives are zero after some point we don't need a formula for the general term. All we need are the values of the non-zero derivative terms.

Step 2
Once we have the values from the previous step all we need to do is write down the Taylor Series. To do that all we need to do is strip all the non-zero terms from the series and then acknowledge that the remainder will just be zero (all the remaining terms are zero after all!).

Doing this gives,

$$
\begin{aligned}
7x^2 - 6x + 1 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + f'(2)(x-2) + \tfrac{1}{2} f''(2)(x-2)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \boxed{17 + 22(x-2) + 7(x-2)^2}
\end{aligned}
$$

It looks a little strange but there it is. Do not multiply/simplify this out. This really is the answer we are looking for.

Also, don't think that this is a problem that is just done to make you work another problem. There are applications of series (beyond the scope of this course however…) that really do require this kind of thing to be done as strange as that might sound!

Section 4-17 : Applications of Series

1. Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\mathbf{e}^x - 1}{x}\,dx$$

Step 1
This problem isn't quite as hard as it might first appear. We know how to integrate a series so all we really need to do here is find a Taylor series for the integrand and then integrate that.

Step 2
Okay, let's start out by noting that we are working about $x = 0$ and that means we can use the formula for the Taylor Series of the exponential function. For reference purposes this is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Next, let's strip out the $n = 0$ term from this and then subtract one. Doing this gives,

$$\mathbf{e}^x - 1 = \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right] - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Of course, in doing the above step all we really managed to do was eliminate the $n = 0$ term from the series. In fact, that was not a bad thing to have happened as well see shortly.

Finally, let's divide the whole thing by *x*. This gives,

$$\frac{\mathbf{e}^x - 1}{x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

We moved the *x* that was outside the series into the series. This is required in order to do the integral of the series. We only want a single *x* in the problem and we now have that.

Also note that while the function on the left has a division by zero issue the series on the right does not have this problem. All of the *x*'s in the series have positive or zero exponents! This is a really good thing.

Of course, the other good thing that we have at this point is that we've managed to find a series representation for the integrand!

Step 3
All we need to do now is compute the integral of the series to get a series representation of the integral.

$$\int \frac{\mathbf{e}^x - 1}{x}\,dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\,dx = \boxed{c + \sum_{n=1}^{\infty} \frac{x^n}{(n)(n!)}}$$

---

2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8, -2]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Luckily enough for us we did that in Problem 3 of the previous section. Here is the Taylor Series we derived in that problem.

$$\mathbf{e}^{-6x} = \sum_{n=0}^{\infty} \frac{(-6)^n \mathbf{e}^{24}}{n!}(x+4)^n$$

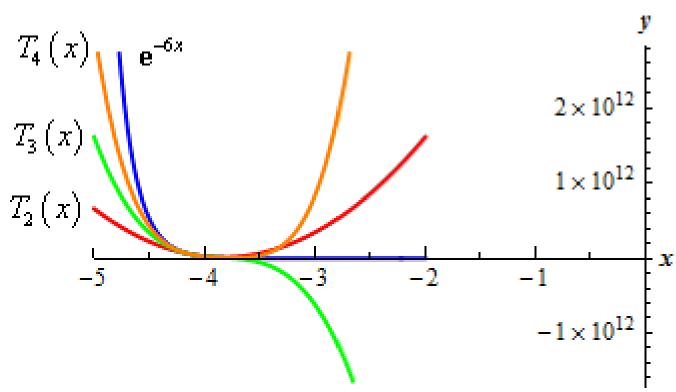
Step 2

Here are the three Taylor polynomials needed for this problem.

$$T_2(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2$$
$$T_3(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3$$
$$T_4(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3 + 54\mathbf{e}^{24}(x+4)^4$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay "near" $x = -4$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

---

3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \ln(3+4x)$ about $x = 0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $\left[-\frac{1}{2}, 2\right]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = \ln(3+4x)$ about $x = 0$. Luckily enough for us we did that in Problem 4 of the previous section. Here is the Taylor Series we derived in that problem.

$$\ln(3+4x) = \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n} x^n$$

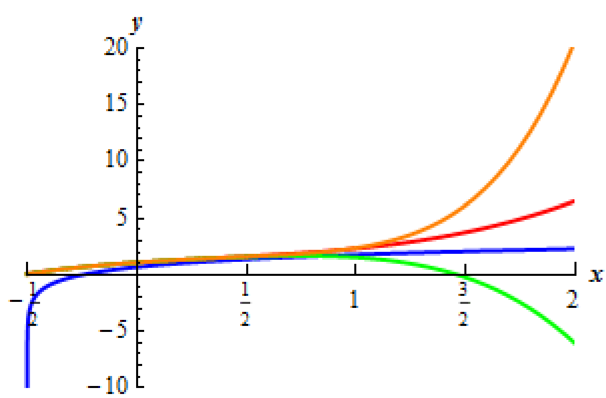
Step 2

Here are the three Taylor polynomials needed for this problem.

$$\begin{aligned}
T_3(x) &= \ln(3) + \tfrac{4}{3}x - \tfrac{8}{9}x^2 + \tfrac{64}{81}x^3 \\
T_4(x) &= \ln(3) + \tfrac{4}{3}x - \tfrac{8}{9}x^2 + \tfrac{64}{81}x^3 - \tfrac{64}{81}x^4 \\
T_5(x) &= \ln(3) + \tfrac{4}{3}x - \tfrac{8}{9}x^2 + \tfrac{64}{81}x^3 - \tfrac{64}{81}x^4 + \tfrac{1024}{1215}x^5
\end{aligned}$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay "near" $x = 0$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

Section 4-18 : Binomial Series

1. Use the Binomial Theorem to expand $\left(4+3x\right)^5$.

Solution

Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}\left(4+3x\right)^5 &= \sum_{i=0}^{5}\binom{5}{i}4^{5-i}\left(3x\right)^i \\ &= \binom{5}{0}\left(4^5\right)+\binom{5}{1}\left(4^4\right)\left(3x\right)^1+\binom{5}{2}\left(4^3\right)\left(3x\right)^2+\binom{5}{3}\left(4^2\right)\left(3x\right)^3+\binom{5}{4}\left(4^1\right)\left(3x\right)^4 \\ &\qquad +\binom{5}{5}\left(3x\right)^5 \\ &= 4^5+\left(5\right)\left(4^4\right)\left(3x\right)+\frac{5\left(4\right)}{2!}\left(4^3\right)\left(3x\right)^2+\frac{5\left(4\right)\left(3\right)}{3!}\left(4^2\right)\left(3x\right)^3+\left(5\right)\left(4\right)\left(3x\right)^4+\left(3x\right)^5 \\ &= \boxed{1024+3840x+5760x^2+4320x^3+1620x^4+243x^5}\end{aligned}$$

---

2. Use the Binomial Theorem to expand $\left(9-x\right)^4$.

Solution

Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}\left(9-x\right)^4 &= \sum_{i=0}^{4}\binom{4}{i}9^{4-i}\left(-x\right)^i \\ &= \binom{4}{0}\left(9^4\right)+\binom{4}{1}\left(9^3\right)\left(-x\right)^1+\binom{4}{2}\left(9^2\right)\left(-x\right)^2+\binom{4}{3}\left(9^1\right)\left(-x\right)^3+\binom{4}{4}\left(-x\right)^4 \\ &= 9^4+\left(4\right)\left(9^3\right)\left(-x\right)+\frac{4\left(3\right)}{2!}\left(9^2\right)\left(-x\right)^2+\left(4\right)\left(9^1\right)\left(-x\right)^3+\left(-x\right)^4 \\ &= \boxed{6561-2916x+486x^2-36x^3+x^4}\end{aligned}$$

---

3. Write down the first four terms in the binomial series for $\left(1+3x\right)^{-6}$.

Step 1
First, we need to make sure it is in the proper form to use the Binomial Series from the notes which in this case we are already in the proper form with $k=-6$.

Step 2
Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}\left(1+3x\right)^{-6} &= \sum_{i=0}^{\infty}\binom{-6}{i}\left(3x\right)^{i}\\ &= 1+\left(-6\right)\left(3x\right)^{1}+\frac{\left(-6\right)\left(-7\right)}{2!}\left(3x\right)^{2}+\frac{\left(-6\right)\left(-7\right)\left(-8\right)}{3!}\left(3x\right)^{3}+\cdots\\ &= \boxed{1-18x+189x^{2}-1512x^{3}+\cdots}\end{aligned}$$

---

4. Write down the first four terms in the binomial series for $\sqrt[3]{8-2x}$.

Step 1
First, we need to make sure it is in the proper form to use the Binomial Series. Here is the proper form for this function,

$$\sqrt[3]{8-2x}=\left(8\left(1-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}}=\left(8\right)^{\frac{1}{3}}\left(1-\tfrac{1}{4}x\right)^{\frac{1}{3}}=2\left(1+\left(-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}}$$

Recall that for proper from we need it to be in the form "1+" and so we needed to factor the 8 out of the root and "move" the minus sign into the second term. Also, as we can see we will have $k=\frac{1}{3}$

Step 2
Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}\sqrt[3]{8-2x} &= 2\left(1+\left(-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}}\\ &= 2\sum_{i=0}^{\infty}\binom{\frac{1}{3}}{i}\left(-\tfrac{1}{4}x\right)^{i}\\ &= 2\left[1+\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{4}x\right)^{1}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\tfrac{1}{4}x\right)^{2}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(-\tfrac{1}{4}x\right)^{3}+\cdots\right]\\ &= \boxed{2-\tfrac{1}{6}x-\tfrac{1}{72}x^{2}-\tfrac{5}{2592}x^{3}+\cdots}\end{aligned}$$

Chapter 5 : Vectors

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Basic Concepts – In this section we will introduce some common notation for vectors as well as some of the basic concepts about vectors such as the magnitude of a vector and unit vectors. We also illustrate how to find a vector from its starting and end points.

Vector Arithmetic – In this section we will discuss the mathematical and geometric interpretation of the sum and difference of two vectors. We also define and give a geometric interpretation for scalar multiplication. We also give some of the basic properties of vector arithmetic and introduce the common *i*, *j*, *k* notation for vectors.

Dot Product – In this section we will define the dot product of two vectors. We give some of the basic properties of dot products and define orthogonal vectors and show how to use the dot product to determine if two vectors are orthogonal. We also discuss finding vector projections and direction cosines in this section.

Cross Product – In this section we define the cross product of two vectors and give some of the basic facts and properties of cross products.

Section 5-1 : Vectors - The Basics

1. Give the vector for the line segment from $\left(-9,2\right)$ to $\left(4,-1\right)$. Find its magnitude and determine if the vector is a unit vector.

Step 1
Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point. Always keep in mind that the starting and ending points are important!

Here is the vector for this line segment.

$$\vec{v} = \left\langle 4-\left(-9\right),-1-2\right\rangle = \boxed{\left\langle 13,-3\right\rangle}$$

Step 2
To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\left\|\vec{v}\right\| = \sqrt{\left(13\right)^2+\left(-3\right)^2} = \boxed{\sqrt{178}}$$

Step 3
Because we can see that $\left\|\vec{v}\right\| = \sqrt{178} \neq 1$ we know that this vector is **not** a unit vector.

---

2. Give the vector for the line segment from $\left(4,5,6\right)$ to $\left(4,6,6\right)$. Find its magnitude and determine if the vector is a unit vector.

Step 1
Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point. Always keep in mind that the starting and ending points are important!

Here is the vector for this line segment.

$$\vec{v} = \left\langle 4-4,6-5,6-6\right\rangle = \boxed{\left\langle 0,1,0\right\rangle}$$

Step 2
To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\left\|\vec{v}\right\| = \sqrt{\left(0\right)^2+\left(1\right)^2+\left(0\right)^2} = \boxed{1}$$

Step 3

Because we can see that $\|\vec{v}\| = 1$ we know that this vector **is** a unit vector.

---

3. Give the vector for the position vector for $\left(-3,2,10\right)$. Find its magnitude and determine if the vector is a unit vector.

Step 1
Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

Just recall that the starting point for any position vector is the origin and the ending point is the point we're working with. In other words, the components of the position vector are simply the coordinates of the point.

Here is the position vector for this point.

$$\boxed{\vec{v} = \left\langle -3,2,10 \right\rangle}$$

Step 2
To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{\left(-3\right)^2 + \left(2\right)^2 + \left(10\right)^2} = \boxed{\sqrt{113}}$$

Step 3
Because we can see that $\|\vec{v}\| = \sqrt{113} \ne 1$ we know that this vector is **not** a unit vector.

---

4. Give the vector for the position vector for $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Find its magnitude and determine if the vector is a unit vector.

Step 1
Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

Just recall that the starting point for any position vector is the origin and the ending point is the point we're working with. In other words, the components of the position vector are simply the coordinates of the point.

Here is the position vector for this point.

$$\boxed{\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle}$$

Step 2
To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{4}{4}} = \boxed{1}$$

Step 3
Because we can see that $\|\vec{v}\| = 1$ we know that this vector **is** a unit vector.

---

5. The vector $\vec{v} = \langle 6, -4, 0 \rangle$ starts at the point $P = (-2, 5, -1)$. At what point does the vector end?

Step 1
To answer this problem we just need to recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

So, if the ending point of the vector is given by $Q = (x_2, y_2, z_2)$ then we know that the vector $\vec{v}$ can be written as,

$$\vec{v} = \overrightarrow{PQ} = \langle x_2 + 2, y_2 - 5, z_2 + 1 \rangle$$

Step 2
But we also know just what the components of $\vec{v}$ are so we can set the vector from Step 1 above equal to what we know $\vec{v}$ is. Doing this gives,

$$\langle x_2 + 2, y_2 - 5, z_2 + 1 \rangle = \langle 6, -4, 0 \rangle$$

Step 3
Now, if two vectors are equal the corresponding components must be equal. Or,

$$\begin{aligned} x_2 + 2 &= 6 & &\Rightarrow & x_2 &= 4 \\ y_2 - 5 &= -4 & &\Rightarrow & y_2 &= 1 \\ z_2 + 1 &= 0 & &\Rightarrow & z_2 &= -1 \end{aligned}$$

As noted above this results in three very simple equations that we can solve to determine the coordinates of the ending point.

The endpoint of the vector is then,

$$\boxed{Q = \left(4, 1, -1\right)}$$

Section 5-2 : Vector Arithmetic

1. Given $\vec{a} = \langle 8,5\rangle$ and $\vec{b} = \langle -3,6\rangle$ compute each of the following.

(a) $6\vec{a}$

(b) $7\vec{b} - 2\vec{a}$

(c) $\left\| 10\vec{a} + 3\vec{b} \right\|$

(a) $6\vec{a}$

This is just a scalar multiplication problem. Just remember to multiply each component by the scalar, 6 in this case.

$$6\vec{a} = 6\langle 8,5\rangle = \boxed{\langle 48,30\rangle}$$

(b) $7\vec{b} - 2\vec{a}$

Here we'll just do each of the scalar multiplications and then do the subtraction. With the subtraction just remember to subtract corresponding components from each vector and recall that order is important here since we are doing subtraction!

$$7\vec{b} - 2\vec{a} = 7\langle -3,6\rangle - 2\langle 8,5\rangle = \langle -21,42\rangle - \langle 16,10\rangle = \boxed{\langle -37,32\rangle}$$

(c) $\left\| 10\vec{a} + 3\vec{b} \right\|$

So, first we compute the vector inside the magnitude bars and the compute the magnitude.

$$10\vec{a} + 3\vec{b} = 10\langle 8,5\rangle + 3\langle -3,6\rangle = \langle 80,50\rangle + \langle -9,18\rangle = \langle 71,68\rangle$$

The magnitude is then,

$$\left\| 10\vec{a} + 3\vec{b} \right\| = \sqrt{\left(71\right)^2 + \left(68\right)^2} = \boxed{\sqrt{9665}}$$

---

2. Given $\vec{u} = 8\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{v} = 7\vec{j} - 4\vec{k}$ compute each of the following.

(a) $-3\vec{v}$

(b) $12\vec{u} + \vec{v}$

(c) $\left\| -9\vec{v} - 2\vec{u} \right\|$

(a) $-3\vec{v}$

This is just a scalar multiplication problem. Just remember to multiply each component by the scalar, -3 in this case.

$$-3\vec{v} = -3\left(7\vec{j} - 4\vec{k}\right) = \boxed{-21\vec{j} + 12\vec{k}}$$

(b) $12\vec{u} + \vec{v}$

Here we'll just do each of the scalar multiplications and then do the subtraction. With the addition just remember to add corresponding components from each vector.

$$12\vec{u} + \vec{v} = 12\left(8\vec{i} - \vec{j} + 3\vec{k}\right) + \left(7\vec{j} - 4\vec{k}\right) = \left(96\vec{i} - 12\vec{j} + 36\vec{k}\right) + \left(7\vec{j} - 4\vec{k}\right) = \boxed{96\vec{i} - 5\vec{j} + 32\vec{k}}$$

(c) $\left\|-9\vec{v} - 2\vec{u}\right\|$

So, first we compute the vector inside the magnitude bars and the compute the magnitude.

$$\begin{aligned} -9\vec{v} - 2\vec{u} &= -9\left(7\vec{j} - 4\vec{k}\right) - 2\left(8\vec{i} - \vec{j} + 3\vec{k}\right) \\ &= \left(-63\vec{j} + 36\vec{k}\right) - \left(16\vec{i} - 2\vec{j} + 6\vec{k}\right) = -16\vec{i} - 61\vec{j} + 30\vec{k} \end{aligned}$$

Be careful with the lack of an $\vec{i}$ component in the first vector here. Just recall that means the coefficient of $\vec{i}$ in the first vector is just zero!

The magnitude is then,

$$\left\|-9\vec{v} - 2\vec{u}\right\| = \sqrt{\left(-16\right)^2 + \left(-61\right)^2 + \left(30\right)^2} = \boxed{\sqrt{4877}}$$

---

3. Find a unit vector that points in the same direction as $\vec{q} = \vec{i} + 3\vec{j} + 9\vec{k}$.

Step 1

Of course, the first step here really should be to check and see if we are lucky enough to actually have a unit vector already. It's unlikely we do have a unit vector but you never know until you check!

$$\left\|\vec{q}\right\| = \sqrt{\left(1\right)^2 + \left(3\right)^2 + \left(9\right)^2} = \sqrt{91}$$

Okay, as we pretty much had already guessed, this isn't a unit vector (its magnitude isn't one!) but we can use this to help find the answer.

Step 2

Recall that all we need to do to turn any vector into a unit vector is divide the vector by its magnitude. Doing that for this vector gives,

$$\vec{u} = \frac{\vec{q}}{\|\vec{q}\|} = \frac{1}{\sqrt{91}}\left(\vec{i} + 3\vec{j} + 9\vec{k}\right) = \boxed{\frac{1}{\sqrt{91}}\vec{i} + \frac{3}{\sqrt{91}}\vec{j} + \frac{9}{\sqrt{91}}\vec{k}}$$

As a quick check, not really required of course, we can compute a quick magnitude to verify that we do in fact have a unit vector.

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{91}}\right)^2 + \left(\frac{3}{\sqrt{91}}\right)^2 + \left(\frac{9}{\sqrt{91}}\right)^2} = \sqrt{\frac{91}{91}} = 1$$

So, we do have a unit vector!

---

4. Find a vector that points in the same direction as $\vec{c} = \langle -1, 4\rangle$ with a magnitude of 10.

Step 1
At first glance this doesn't appear to be all that similar to the previous problem. However, it's actually very similar.

First, let's check to see what the magnitude of this vector is.

$$\|\vec{c}\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{17}$$

Step 2
Okay, oddly enough let's determine a unit vector that points in the same direction. This doesn't seem all that useful but it's actually a very good thing to do in this case.

The unit vector is,

$$\vec{u} = \frac{\vec{c}}{\|\vec{c}\|} = \frac{1}{\sqrt{17}}\langle -1, 4\rangle = \boxed{\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right\rangle}$$

Now, let's think about what we did here. We took the original vector and multiplied it by a number, $\frac{1}{\sqrt{17}}$ in this case, to change its magnitude. The result is a new vector, pointing in the same direction as the original vector, and has a new magnitude of one.

So, how can we use this new vector (and the process by which we found it) to get an answer for this problem?

Step 3

We know that scalar multiplication can change the magnitude of a vector. We've got a vector with magnitude of one that points in the correct direction. To convert this into a vector with magnitude of 10 all we need to do is multiply this new unit vector by 10 to get,

$$\vec{v} = 10\vec{u} = 10\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle = \left\langle -\frac{10}{\sqrt{17}}, \frac{40}{\sqrt{17}} \right\rangle$$

Now, let's verify that this does what we want it to do with a quick magnitude computation.

$$\|\vec{v}\| = \sqrt{\left(-\frac{10}{\sqrt{17}}\right)^2 + \left(\frac{40}{\sqrt{17}}\right)^2} = \sqrt{\frac{1700}{17}} = \sqrt{100} = 10$$

So, we do have a vector with magnitude 10 as predicted!

---

5. Determine if $\vec{a} = \langle 3, -5, 1 \rangle$ and $\vec{b} = \langle 6, -2, 2 \rangle$ are parallel vectors.

Step 1

Recall that two vectors are parallel if they are scalar multiples of each other. In other words, these two vectors will be scalar multiples if we can find a number *k* such that,

$$\vec{a} = k\,\vec{b}$$

Step 2

Let's just take a look at the first component from each vector. It is obvious that $6 = 2(3)$. So, to convert the first components we'd need to multiply $\vec{a}$ by 2. However, if we did that we'd get,

$$2\vec{a} = \langle 6, -10, 2 \rangle \neq \vec{b}$$

This is clearly not $\vec{b}$. The first component is correct and the third component is correct but the second isn't correct. Therefore, there is no single number, *k*, that we can use to convert $\vec{a}$ into $\vec{b}$ through scalar multiplication.

This in turn means that $\vec{a}$ and $\vec{b}$ **cannot possibly be parallel**.

---

6. Determine if $\vec{v} = 9\vec{i} - 6\vec{j} - 24\vec{k}$ and $\vec{w} = \langle -15, 10, 40 \rangle$ are parallel vectors.

Step 1
Recall that two vectors are parallel if they are scalar multiples of each other. In other words, these two vectors will be scalar multiples if we can find a number *k* such that,

$$\vec{v} = k\,\vec{w}$$

Step 2
Let's just take a look at the first component from each vector. It is should be clear that $-15 = \left(-\frac{5}{3}\right)\left(9\right)$.
So, to convert the first components we'd need to multiply $\vec{v}$ by $-\frac{5}{3}$.
if we did that we'd get,

$$-\tfrac{5}{3}\vec{v} = \left\langle -15, 10, 40 \right\rangle = \vec{w}$$

So, we were able to find a number *k* that we could use to convert $\vec{v}$ into $\vec{w}$ through scalar multiplication and so the **two vectors are parallel**.

---

7. Prove the property : $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

Step 1
These types of proofs always seem mysterious to students the first time they run across them. The main reason for the mystery is probably that it just seems obvious that it is true. That tends to make is difficult to prove.

We know that this property is true for numbers. However, we can't assume that just because it's true for numbers that it will be true for all other types of objects, vectors in this case!

So, let's start off with two general vectors.

$$\vec{v} = \left\langle v_1, v_2, \ldots, v_n \right\rangle \qquad\qquad \vec{w} = \left\langle w_1, w_2, \ldots w_n \right\rangle$$

To do this type of proof all we need to do is start with the left side perform the indicated operation, addition in this case, and then use properties about numbers that we already know to be true to try and manipulate it to look like the right side.

Step 2
So, let's start off with the vector addition on the left side. All we want to do here is use the definition of vector addition to write the sum of the two vectors. This is,

$$\vec{v} + \vec{w} = \left\langle v_1, v_2, \ldots, v_n \right\rangle + \left\langle w_1, w_2, \ldots w_n \right\rangle = \left\langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \right\rangle$$

Step 3

Okay, as we noted above we know that $2+3=3+2$. In other words, we know that the order we do addition of numbers doesn't matter.

Why bring this up again?

Well, note that each of the components of the "new" vector on the right side is just a sum of two numbers. Therefore, we can use this property to flip the order of the addition in each of the components.

Doing this gives,

$$\vec{v}+\vec{w}=\left\langle v_1+w_1, v_2+w_2, \ldots, v_n+w_n\right\rangle=\left\langle w_1+v_1, w_2+v_2, \ldots, w_n+v_n\right\rangle$$

Step 4

Now, recall that according to the definition of vector arithmetic the first number in the sum in each component of the vector on the right is the component of the first vector while the second number in the sum is the component of the second vector.

So, all we need to do now is "undo" the sum that gave the vector on the right to get,

$$\vec{v}+\vec{w}=\left\langle w_1+v_1, w_2+v_2, \ldots, w_n+v_n\right\rangle=\vec{w}+\vec{v}$$

This is exactly what we were asked to prove and so we are done!

Section 5-3 : Dot Product

1. Determine the dot product, $\vec{a} \bullet \vec{b}$ if $\vec{a} = \langle 9,5,-4,2 \rangle$ and $\vec{b} = \langle -3,-2,7,-1 \rangle$.

Solution
Not really a whole lot to do here. We just need to run through the definition of the dot product.

$$\vec{a} \bullet \vec{b} = (9)(-3)+(5)(-2)+(-4)(7)+(2)(-1) = \boxed{-67}$$

---

2. Determine the dot product, $\vec{a} \bullet \vec{b}$ if $\vec{a} = \langle 0,4,-2 \rangle$ and $\vec{b} = 2\vec{i} - \vec{j} + 7\vec{k}$.

Solution
Not really a whole lot to do here. We just need to run through the definition of the dot product and do not get excited about the "mixed" notation here. We know that they are equivalent notations!

$$\vec{a} \bullet \vec{b} = (0)(2)+(4)(-1)+(-2)(7) = \boxed{-18}$$

---

3. Determine the dot product, $\vec{a} \bullet \vec{b}$ if $\|\vec{a}\| = 5$, $\|\vec{b}\| = \frac{3}{7}$ and the angle between the two vectors is $\theta = \frac{\pi}{12}$.

Solution
Not really a whole lot to do here. We just need to run through the formula from the geometric interpretation of the dot product.

$$\vec{a} \bullet \vec{b} = (5)\left(\frac{3}{7}\right)\cos\left(\frac{\pi}{12}\right) = \boxed{2.0698}$$

---

4. Determine the angle between $\vec{v} = \langle 1,2,3,4 \rangle$ and $\vec{w} = \langle 0,-1,4,-2 \rangle$.

Solution
Not really a whole lot to do here. All we really need to do is rewrite the formula from the geometric interpretation of the dot product as,

$$\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

This will allow us to quickly determine the angle between the two vectors.

We'll first need the following quantities (we'll leave it to you to verify the arithmetic involved in these computations….).

$$\vec{v} \bullet \vec{w} = 2 \qquad \|\vec{v}\| = \sqrt{30} \qquad \|\vec{w}\| = \sqrt{21}$$

The angle between the vectors is then,

$$\cos\theta = \frac{2}{\sqrt{30}\,\sqrt{21}} = 0.07968 \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(0.07968\right) = \boxed{1.49103 \text{ radians}}$$

---

5. Determine the angle between $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = \left\langle -9, 1, -5 \right\rangle$.

Solution

Not really a whole lot to do here. All we really need to do is rewrite the formula from the geometric interpretation of the dot product as,

$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$$

This will allow us to quickly determine the angle between the two vectors.

We'll first need the following quantities (we'll leave it to you to verify the arithmetic involved in these computations….).

$$\vec{a} \bullet \vec{b} = 4 \qquad \|\vec{a}\| = \sqrt{14} \qquad \|\vec{b}\| = \sqrt{107}$$

The angle between the vectors is then,

$$\cos\theta = \frac{4}{\sqrt{14}\,\sqrt{107}} = 0.1033 \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(0.1034\right) = \boxed{1.4673 \text{ radians}}$$

---

6. Determine if $\vec{q} = \left\langle 4, -2, 7 \right\rangle$ and $\vec{p} = -3\vec{i} + \vec{j} + 2\vec{k}$ are parallel, orthogonal or neither.

Step 1
Based on a quick inspection of the components we can see that the first and second components of the two vectors have opposite signs and the third doesn't. This means there is no possible way for these two vectors to be scalar multiples since there is no number that will change the sign on the first two components and leave the sign of the third component unchanged.

Therefore, we can quickly see that the two vectors are **not parallel**.

Step 2
Let's do a quick dot product on the two vectors next.

$$\vec{q} \bullet \vec{p} = 0$$

Okay, the dot product is zero and we know from the notes that this in turn means that the two vectors must be **orthogonal**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

---

7. Determine if $\vec{a} = \langle 3,10 \rangle$ and $\vec{b} = \langle 4,-1 \rangle$ are parallel, orthogonal or neither.

Step 1
Based on a quick inspection of the components we can see that the first components of the vectors have the same sign and the second have opposite signs. This means there is no possible way for these two vectors to be scalar multiples since there is no number that will change the sign on the second components and leave the sign of the first component unchanged.

Therefore, we can quickly see that the two vectors are **not parallel**.

Step 2
Let's do a quick dot product on the two vectors next.

$$\vec{a} \bullet \vec{b} = 2$$

Okay, the dot product is not zero and we know from the notes that this in turn means that the two vectors are **not orthogonal**.

The answer to the problem is therefore the two vectors are **neither parallel or orthogonal**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

---

8. Determine if $\vec{w} = \vec{i} + 4\vec{j} - 2\vec{k}$ and $\vec{v} = -3\vec{i} - 12\vec{j} + 6\vec{k}$ are parallel, orthogonal or neither.

Solution
Based on a quick inspection is seems (hopefully) fairly clear that we have,

$$\vec{v} = -3\vec{w}$$

Therefore, the two vectors are **parallel**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

---

9. Given $\vec{a} = \langle -8, 2 \rangle$ and $\vec{b} = \langle -1, -7 \rangle$ compute $\text{proj}_{\vec{a}}\, \vec{b}$.

Solution
All we really need to do here is use the formula from the notes. That will need the following quantities.

$$\vec{a} \bullet \vec{b} = -6 \qquad\qquad \left\| \vec{a} \right\|^2 = 68$$

The projection is then,

$$\text{proj}_{\vec{a}}\, \vec{b} = \frac{-6}{68} \langle -8, 2 \rangle = \boxed{\left\langle \frac{12}{17}, -\frac{3}{17} \right\rangle}$$

---

10. Given $\vec{u} = 7\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = -2\vec{i} + 5\vec{j} - 6\vec{k}$ compute $\text{proj}_{\vec{w}}\, \vec{u}$.

Solution
All we really need to do here is use the formula from the notes. That will need the following quantities.

$$\vec{u} \bullet \vec{w} = -25 \qquad\qquad \left\| \vec{w} \right\|^2 = 65$$

The projection is then,

$$\text{proj}_{\vec{w}}\,\vec{u} = \frac{-25}{65}\left(-2\vec{i} + 5\vec{j} - 6\vec{k}\right) = \boxed{\frac{10}{13}\vec{i} - \frac{25}{13}\vec{j} + \frac{30}{13}\vec{k}}$$

---

11. Determine the direction cosines and direction angles for $\vec{r} = \left\langle -3, -\frac{1}{4}, 1 \right\rangle$.

Solution

All we really need to do here is use the formulas from the notes. That will need the following quantity.

$$\left\|\vec{r}\right\| = \sqrt{\frac{161}{16}} = \frac{\sqrt{161}}{4}$$

The direction cosines and angles are then,

$$\cos\alpha = \frac{-3}{\sqrt{161}/4} = -\frac{12}{\sqrt{161}} \qquad \Rightarrow \qquad \alpha = \cos^{-1}\left(-\frac{12}{\sqrt{161}}\right) = 2.8106 \text{ radians}$$

$$\cos\beta = \frac{-1/4}{\sqrt{161}/4} = -\frac{1}{\sqrt{161}} \qquad \Rightarrow \qquad \beta = \cos^{-1}\left(-\frac{1}{\sqrt{161}}\right) = 1.6497 \text{ radians}$$

$$\cos\gamma = \frac{1}{\sqrt{161}/4} = \frac{4}{\sqrt{161}} \qquad \Rightarrow \qquad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{161}}\right) = 1.2501 \text{ radians}$$

Section 5-4 : Cross Product

1. If $\vec{w} = \langle 3, -1, 5 \rangle$ and $\vec{v} = \langle 0, 4, -2 \rangle$ compute $\vec{v} \times \vec{w}$.

Solution
Not really a whole lot to do here. We just need to run through one of the various methods for computing the cross product. We'll be using the "trick" we used in the notes.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -2 \\ 3 & -1 & 5 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 0 & 4 \\ 3 & -1 \end{matrix}$$

$$= 20\vec{i} - 6\vec{j} + 0\vec{k} - 0\vec{j} - 2\vec{i} - 12\vec{k} = \boxed{18\vec{i} - 6\vec{j} - 12\vec{k}}$$

---

2. If $\vec{w} = \langle 1, 6, -8 \rangle$ and $\vec{v} = \langle 4, -2, -1 \rangle$ compute $\vec{w} \times \vec{v}$.

Solution
Not really a whole lot to do here. We just need to run through one of the various methods for computing the cross product. We'll be using the "trick" we used in the notes.

$$\vec{w} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & -8 \\ 4 & -2 & -1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 6 \\ 4 & -2 \end{matrix}$$

$$= -6\vec{i} - 32\vec{j} - 2\vec{k} - \left(-\vec{j}\right) - 16\vec{i} - 24\vec{k} = \boxed{-22\vec{i} - 31\vec{j} - 26\vec{k}}$$

---

3. Find a vector that is orthogonal to the plane containing the points $P = (3, 0, 1)$, $Q = (4, -2, 1)$ and $R = (5, 3, -1)$.

Step 1
We first need two vectors that are both parallel to the plane. Using the points that we are given (all in the plane) we can quickly get quite a few vectors that are parallel to the plane. We'll use the following two vectors.

$$\overrightarrow{PQ} = \langle 1, -2, 0 \rangle \qquad \overrightarrow{PR} = \langle 2, 3, -2 \rangle$$

Step 2
Now we know that the cross product of any two vectors will be orthogonal to the two original vectors. Since the two vectors from Step 1 are parallel to the plane (they actually lie in the plane in this case!) we know that the cross product must then also be orthogonal, or normal, to the plane.

So, using the "trick" we used in the notes the cross product is,

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= \left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ 2 & 3 & -2\end{matrix}\right|\begin{matrix}\vec{i} & \vec{j}\\ 1 & -2\\ 2 & 3\end{matrix}\\ &= 4\vec{i}+0\vec{j}+3\vec{k}-\left(-2\vec{j}\right)-0\vec{i}-\left(-4\vec{k}\right)=\boxed{4\vec{i}+2\vec{j}+7\vec{k}}\end{aligned}$$

---

4. Are the vectors $\vec{u}=\left\langle 1,2,-4\right\rangle$, $\vec{v}=\left\langle -5,3,-7\right\rangle$ and $\vec{w}=\left\langle -1,4,2\right\rangle$ are in the same plane?

Solution
As discussed in the notes to answer this question all we need to do is compute the following quantity,

$$\begin{aligned}\vec{u}\bullet\left(\vec{v}\times\vec{w}\right) &= \left|\begin{matrix}1 & 2 & -4\\ -5 & 3 & -7\\ -1 & 4 & 2\end{matrix}\right|\begin{matrix}1 & 2\\ -5 & 3\\ -1 & 4\end{matrix}\\ &= 6+14+80-\left(-20\right)-\left(-28\right)-12=136\end{aligned}$$

Okay, since this is not zero we know that they **are not in the same plane**.

Chapter 6 : 3-Dimensional Space

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

The 3-D Coordinate System – In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Section 6-1 : The 3-D Coordinate System

1. Give the projection of $P = (3, -4, 6)$ onto the three coordinate planes.

Solution
There really isn't a lot to do with this problem. We know that the *xy*-plane is given by the equation $z = 0$ and so the projection into the *xy*-plane for any point is simply found by setting the *z* coordinate to zero. We can find the projections for the other two coordinate planes in a similar fashion.

So, the projects are then,

$$\begin{aligned} xy-\text{plane} &: \ (3,-4,0) \\ xz-\text{plane} &: \ (3,0,6) \\ yz-\text{plane} &: \ (0,-4,6) \end{aligned}$$

---

2. Which of the points $P = (4, -2, 6)$ and $Q = (-6, -3, 2)$ is closest to the *yz*-plane?

Step 1
The shortest distance between any point and any of the coordinate planes will be the distance between that point and its projection onto that plane.

Let's call the projections of *P* and Q onto the *yz*-plane $\overline{P}$ and $\overline{Q}$ respectively. They are,

$$\overline{P} = (0,-2,6) \qquad\qquad \overline{Q} = (0,-3,2)$$

Step 2
To determine which of these is closest to the *yz*-plane we just need to compute the distance between the point and its projection onto the *yz*-plane.

Note as well that because only the *x*-coordinate of the two points are different the distance between the two points will just be the absolute value of the difference between two *x* coordinates.

Therefore,

$$d\left(P,\overline{P}\right) = 4 \qquad\qquad d\left(Q,\overline{Q}\right) = 6$$

Based on this is should be pretty clear that $P = (4, -2, 6)$ is closest to the *yz*-plane.

---

3. Which of the points $P=(-1,4,-7)$ and $Q=(6,-1,5)$ is closest to the *z*-axis?

Step 1

First, let's note that the coordinates of any point on the *z*-axis will be $(0,0,z)$.

Also, the shortest distance from any point not on the *z*-axis to the *z*-axis will occur if we draw a line straight from the point to the *z*-axis in such a way that it forms a right angle with the *z*-axis.

So, if we start with any point not on the *z*-axis, say (x_1, y_1, z_1), the point on the *z*-axis that will be closest to this point is $(0,0,z_1)$.

Let's call the point closest to *P* and *Q* on the *z*-axis closest to be $\bar{P}$ and $\bar{Q}$ respectively. They are,

$$\bar{P}=(0,0,-7) \qquad\qquad \bar{Q}=(0,0,5)$$

Step 2

To determine which of these is closest to the *z*-axis we just need to compute the distance between the point and its projection onto the *z*-axis.

The distances are,

$$d(P,\bar{P})=\sqrt{(-1-0)^2+(4-0)^2+(-7-(-7))^2}=\sqrt{17}$$
$$d(Q,\bar{Q})=\sqrt{(6-0)^2+(-1-0)^2+(5-5)^2}=\sqrt{37}$$

Based on this is should be pretty clear that $P=(-1,4,-7)$ is closest to the *z*-axis.

---

4. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$7x^2-9y^3=3x+1$$

Solution

First notice that because there are two variables in this equation it cannot have a graph in $\mathbb{R}$ since equations in that coordinate system can only have a single variable.

There are two variables in the equation so we know that it will have a graph in $\mathbb{R}^2$.

Likewise, the fact that the equation has two variables means that it will also have a graph in $\mathbb{R}^3$. Although in this case the third variable, *z*, can have any value.

---

5. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$x^3 + \sqrt{y^2 + 1} - 6z = 2$$

Solution

This equation has three variables and so it will have a graph in $\mathbb{R}^3$.

On other hand because the equation has three variables in it there will be no graph in $\mathbb{R}^2$ (can have at most two variables) and it will not have a graph in $\mathbb{R}$ (can only have a single variable).

Section 6-2 : Equations of Lines

1. Give the equation of the line through the points $\left(2,-4,1\right)$ and $\left(0,4,-10\right)$ in vector form, parametric form and symmetric form.

Step 1
Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We have two points that are on the line. We can use either point and depending on your choice of points you may have different answers that we get here. We will use the first point listed above for our point for no other reason that it is the first one listed.

The parallel vector is really simple to get as well since we can always form the vector from the first point to the second point and this vector will be on the line and so will also be parallel to the line. The vector is,

$$\vec{v} = \left\langle -2,8,-11\right\rangle$$

Step 2
The vector form of the line is,

$$\boxed{\vec{r}\left(t\right) = \left\langle 2,-4,1\right\rangle + t\left\langle -2,8,-11\right\rangle = \left\langle 2-2t,-4+8t,1-11t\right\rangle}$$

Step 3
The parametric form of the line is,

$$\boxed{x = 2-2t \qquad y = -4+8t \qquad z = 1-11t}$$

Step 4
To get the symmetric form all we need to do is solve each of the parametric equations for *t* and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{2-x}{2} = \frac{4+y}{8} = \frac{1-z}{11}}$$

---

2. Give the equation of the line through the point $\left(-7,2,4\right)$ and parallel to the line given by $x = 5-8t$, $y = 6+t$, $z = -12t$ in vector form, parametric form and symmetric form.

Step 1
Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We were given a point on the line so no need to worry about that for this problem.

The parallel vector is really simple to get as well since we were told that the new line must be parallel to the given line. We also know that the coefficients of the *t*'s in the equation of the line forms a vector parallel to the line.

So,

$$\vec{v} = \left\langle -8, 1, -12 \right\rangle$$

is a vector that is parallel to the given line.

Now, if $\vec{v}$ is parallel to the given line and the new line must be parallel to the given line then $\vec{v}$ must also be parallel to the new line.

Step 2
The vector form of the line is,

$$\boxed{\vec{r}\left(t\right) = \left\langle -7, 2, 4 \right\rangle + t\left\langle -8, 1, -12 \right\rangle = \left\langle -7 - 8t, 2 + t, 4 - 12t \right\rangle}$$

Step 3
The parametric form of the line is,

$$\boxed{x = -7 - 8t \qquad y = 2 + t \qquad z = 4 - 12t}$$

Step 4
To get the symmetric form all we need to do is solve each of the parametric equations for *t* and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{-7 - x}{8} = y - 2 = \frac{4 - z}{12}}$$

---

3. Is the line through the points $\left(2, 0, 9\right)$ and $\left(-4, 1, -5\right)$ parallel, orthogonal or neither to the line given by $\vec{r}\left(t\right) = \left\langle 5, 1 - 9t, -8 - 4t \right\rangle$?

Step 1
Let's start this off simply by getting vectors parallel to each of the lines.

For the line through the points $(2,0,9)$ and $(-4,1,-5)$ we know that the vector between these two points will lie on the line and hence be parallel to the line. This vector is,

$$\vec{v}_1 = \langle 6,-1,14\rangle$$

For the second line the coefficients of the *t*'s are the components of the parallel vector so this vector is,

$$\vec{v}_2 = \langle 0,-9,-4\rangle$$

Step 2

Now, from the first components of these vectors it is hopefully clear that they are not scalar multiples. There is no number we can multiply to zero to get 6.

Likewise, we can only multiply 6 by zero to get 0. However, if we multiply the first vector by zero all the components would be zero and that is clearly not the case.

Therefore, they are not scalar multiples and so these two vectors are not parallel. This also in turn means that **the two lines can't possibly be parallel** either (since each vector is parallel to its respective line).

Step 3

Next,

$$\vec{v}_1 \bullet \vec{v}_2 = -47$$

The dot product is not zero and so these vectors aren't orthogonal. Because the two vectors are parallel to their respective lines this also means that **the two lines are not orthogonal**.

---

4. Determine the intersection point of the line given by $x = 8 + t$, $y = 5 + 6t$, $z = 4 - 2t$ and the line given by $\vec{r}(t) = \langle -7 + 12t, 3 - t, 14 + 8t\rangle$ or show that they do not intersect.

Step 1

If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same *x*, *y* and *z* coordinates.

Step 2

This means that we can set up the following system of equations.

$$\begin{aligned} 8+t_1 &= -7+12t_2 \\ 5+6t_1 &= 3-t_2 \\ 4-2t_1 &= 14+8t_2 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 3
Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first two equations. We'll worry about the third equation eventually.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$\begin{aligned} t_1 = -15+12t_2 \qquad \to \qquad 5+6\left(-15+12t_2\right) &= 3-t_2 \\ -85+72t_2 &= 3-t_2 \\ 73t_2 &= 88 \qquad \to \qquad t_2 = \tfrac{88}{73} \\ & \qquad\qquad\qquad t_1 = -15+12\left(\tfrac{88}{73}\right) = -\tfrac{39}{73} \end{aligned}$$

Step 4
Okay, this is a somewhat "messy" solution, but they will often be that way so we shouldn't get too excited about it!

Now, recall that to get this solution we used the first two equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the *x* and *y* coordinates will be the same (remember we used the *x* and *y* equations to find this solution….).

At this point we need to recall that we did have a third equation that also needs to be satisfied at these values of *t*. In other words, we need to plug $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ into the third equation and see what we get. Doing this gives,

$$\tfrac{370}{73} = 4-2\left(-\tfrac{39}{73}\right) \neq 14+8\left(\tfrac{88}{73}\right) = \tfrac{1726}{73}$$

Okay, the two sides are not the same. Just what does this mean? In terms of systems of equations it means that $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ are NOT a solution to the system of equations in Step 2. Had they been a solution then we would have gotten the same number from both sides.

In terms of whether or not the lines intersect we need to only recall that the third equation corresponds to the *z* coordinates of the lines. We know that at $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ the two lines will have the same *x* and *y* coordinates (since they came from solving the first two equations). However, we've just shown that they will not give the same *z* coordinate.

In other words, there are no values of t_1 and t_2 for which the two lines will have the same *x*, *y* and *z* coordinates. Hence, we can now say that the **two lines do not intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 3. Different sets of equations will lead to different values of t_1 and t_2 but they will still not satisfy the remaining equation for this problem and we will get the same result of the lines not intersecting.

---

5. Determine the intersection point of the line through the points $\left(1,-2,13\right)$ and $\left(2,0,-5\right)$ and the line given by $\vec{r}\left(t\right) = \left\langle 2+4t,-1-t,3\right\rangle$ or show that they do not intersect.

Step 1
Because we don't have the equation for the first line that will be the first thing we'll need to do. The vector between the two points (and hence parallel to the line) is,

$$\vec{v} = \left\langle 1,2,-18\right\rangle$$

Using the first point listed the equation of the first line is then,

$$\vec{r}\left(t\right) = \left\langle 1,-2,13\right\rangle + t\left\langle 1,2,-18\right\rangle = \left\langle 1+t,-2+2t,13-18t\right\rangle$$

Step 2
If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same *x*, *y* and *z* coordinates.

Step 3
This means that we can set up the following system of equations.

$$\begin{aligned} 1+t_1 &= 2+4t_2 \\ -2+2t_1 &= -1-t_2 \\ 13-18t_1 &= 3 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 4
Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first and third equation. We'll worry about the second equation eventually.

Note that for this system the third equation should definitely be used here since we can quickly just solve that for t_1.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$t_1 = \tfrac{5}{9} \qquad \to \qquad 1 + \tfrac{5}{9} = 2 + 4t_2 \qquad \to \qquad t_2 = -\tfrac{1}{9}$$

Step 5
Now, recall that to get this solution we used the first and third equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the *x* and *z* coordinates will be the same (remember we used the *x* and *z* equations to find this solution….).

At this point we need to recall that we did have another equation that also needs to be satisfied at these values of *t*. In other words, we need to plug $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ into the second equation and see what we get. Doing this gives,

$$-2 + 2\left(\tfrac{5}{9}\right) = -\tfrac{8}{9} = -1 - \left(-\tfrac{1}{9}\right)$$

Okay, the two sides are the same. Just what does this mean? In terms of systems of equations it means that $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ is a solution to the system of equations in Step 3.

In terms of whether or not the lines intersect we now know that at $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ the two lines will have the same *x*, *y* and *z* coordinates (since they satisfy all three equations).

In other words, these **two lines do intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 4. Different sets of equations will lead to the same values of t_1 and t_2 leading to the two lines intersecting.

---

6. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xy*-plane? If so, give the point.

Step 1
If the line intersects the *xy*-plane there will be a point on the line that is also in the *xy*-plane. Recall as well that any point in the *xy*-plane will have a *z* coordinate of $z = 0$.

Step 2
So, to determine if the line intersects the *xy*-plane all we need to do is set the equation for the *z* coordinate equal to zero and solve it for *t*, if that's possible.

Doing this gives,

$$12 - 11t = 0 \qquad \to \qquad t = \tfrac{12}{11}$$

Step 3
So, we were able to solve for *t* in this case and so we can now say that **the line does intersect the *xy*-plane**.

Step 4
All we need to do to finish this off this problem is find the full point of intersection. We can find this simply by plugging $t = \frac{12}{11}$ into the *x* and *y* portions of the equation of the line.

Doing this gives,

$$x = 9 + 21\left(\tfrac{12}{11}\right) = \tfrac{351}{11} \qquad y = -7$$

The point of intersection is then : $\boxed{\left(\frac{351}{11}, -7, 0\right)}$.

---

7. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xz*-plane? If so, give the point.

Step 1
If the line intersects the *xz*-plane there will be a point on the line that is also in the *xz*-plane. Recall as well that any point in the *xz*-plane will have a *y* coordinate of $y = 0$.

Step 2
So, to determine if the line intersects the *xz*-plane all we need to do is set the equation for the *y* coordinate equal to zero and solve it for *t*, if that's possible.

However, in this case we can see that is clearly not possible since the *y* equation is simply $y = -7$ and this can clearly never be zero.

Step 3
Therefore, **the line does not intersect the *xz*-plane**.

Section 6-3 : Equations of Planes

1. Write down the equation of the plane containing the points $(4,-3,1)$, $(-3,-1,1)$ and $(4,-2,8)$.

Step 1
To make the work on this problem a little easier let's "name" the points as,

$$P = (4,-3,1) \qquad Q = (-3,-1,1) \qquad R = (4,-2,8)$$

Now, we know that in order to write down the equation of a plane we'll need a point (we have three so that's not a problem!) and a vector that is normal to the plane.

Step 2
We'll need to do a little work to get a normal vector.

First, we'll need two vectors that lie in the plane and we can get those from the three points we're given. Note that there are lots of possible vectors that we could use here. Here are the two that we'll use for this problem.

$$\overrightarrow{PQ} = \langle -7,2,0\rangle \qquad \overrightarrow{PR} = \langle 0,1,7\rangle$$

Step 3
Now, these two vectors lie in the plane and we know that the cross product of any two vectors will be orthogonal to both of the vectors. Therefore, the cross product of these two vectors will also be orthogonal (and so normal!) to the plane.

So, let's get the cross product.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -7 & 2 & 0 \\ 0 & 1 & 7 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -7 & 2 \\ 0 & 1 \end{matrix} = 14\vec{i} - 7\vec{k} - \left(-49\vec{j}\right) = 14\vec{i} + 49\vec{j} - 7\vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Step 4
Now all we need to do is write down the equation.

We have three points to choose form here. We'll use the first point simply because it is the first point listed. Any of the others could also be used.

The equation of the plane is,

$$\boxed{14(x-4)+49(y+3)-7(z-1)=0 \qquad \to \qquad 14x+49y-7z=-98}$$

Note that depending on your choice of vectors in Step 2, the order you chose to use them in the cross product computation in Step 3 and the point chosen here will all affect your answer. However, regardless of your choices the equation you get will be an acceptable answer provided you did all the work correctly.

---

2. Write down the equation of the plane containing the point $(3,0,-4)$ and orthogonal to the line given by $\vec{r}(t)=\langle 12-t,1+8t,4+6t\rangle$.

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is orthogonal to the line given in the problem statement. This means that the plane will also be orthogonal to any vector that just happens to be parallel to the line.

From the equation of the line we know that the coefficients of the *t*'s are the components of a vector that is parallel to the line. So, a vector parallel to the line is then,

$$\vec{v}=\langle -1,8,6\rangle$$

Now, as mentioned above because this vector is parallel to the line then it will also need to be orthogonal to the plane and hence be normal to the plane. So, a normal vector for the plane is,

$$\vec{n}=\langle -1,8,6\rangle$$

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{-(x-3)+8(y-0)+6(z+4)=0 \qquad \to \qquad -x+8y+6z=-27}$$

---

3. Write down the equation of the plane containing the point $(-8,3,7)$ and parallel to the plane given by $4x+8y-2z=45$.

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is parallel to the plane given in the problem statement. This means that any vector normal to one plane will be normal to both planes.

From the equation of the plane we were given we know that the coefficients of the *x*, *y* and *z* are the components of a vector that is normal to the plane. So, a vector normal to the given plane is then,

$$\vec{n} = \langle 4, 8, -2 \rangle$$

Now, as mentioned above because this vector is normal to the given plane then it will also need to be normal to the plane we want to find the equation for.

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{4(x+8)+8(y-3)-2(z-7)=0 \qquad \to \qquad 4x+8y-2z=-22}$$

---

4. Determine if the plane given by $4x-9y-z=2$ and the plane given by $x+2y-14z=-6$ are parallel, orthogonal or neither.

Step 1
Let's start off this problem by noticing that the vector $\vec{n}_1 = \langle 4, -9, -1 \rangle$ will be normal to the first plane and the vector $\vec{n}_2 = \langle 1, 2, -14 \rangle$ will be normal to the second plane.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2
If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the first component of each vector have the same sign and the same can be said for the third component. However, the second component of each vector has opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the first and third component the same and simultaneously changing the sign on the second component. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \bullet \vec{n}_2 = 0$$

The dot product is zero and so the two normal vectors are orthogonal. Therefore, **the two planes are orthogonal**.

---

5. Determine if the plane given by $-3x + 2y + 7z = 9$ and the plane containing the points $\left(-2,6,1\right)$, $\left(-2,5,0\right)$ and $\left(-1,4,-3\right)$ are parallel, orthogonal or neither.

Step 1
Let's start off this problem by noticing that the vector $\vec{n}_1 = \left\langle -3,2,7\right\rangle$ will be normal to the first plane and it would be nice to have a normal vector for the second plane.

We know (Problem 1 from this section!) how to determine a normal vector given three points in the plane. Here is that work.

$$P = \left(-2,6,1\right) \qquad Q = \left(-2,5,0\right) \qquad R = \left(-1,4,-3\right)$$
$$\overrightarrow{QP} = \left\langle 0,1,1\right\rangle \qquad \overrightarrow{RQ} = \left\langle -1,1,3\right\rangle$$

$$\vec{n}_2 = \overrightarrow{QP} \times \overrightarrow{RQ} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ 0 & 1 \\ -1 & 1 \end{array} = 3\vec{i} - \vec{j} - \vec{i} - \left(-\vec{k}\right) = 2\vec{i} - \vec{j} + \vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2

If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the third component of each vector have the same sign while the first and second components each have opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the third component the same and simultaneously changing the sign on the first and second components. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \bullet \vec{n}_2 = -1$$

The dot product is not zero and so the two normal vectors are not orthogonal. Therefore, **the two planes are not orthogonal**.

---

6. Determine if the line given by $\vec{r}\left(t\right) = \left\langle -2t, 2+7t, -1-4t \right\rangle$ intersects the plane given by $4x + 9y - 2z = -8$ or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of *t* such that if we plug that *t* into the equation of the line we'd get a point that lies on the plane. We also know that if a point $\left(x, y, z\right)$ is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

$$\left(-2t, 2+7t, -1-4t\right)$$

for all values of *t*.

Step 3
So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$4\left(-2t\right) + 9\left(2+7t\right) - 2\left(-1-4t\right) = -8$$

Step 4

Let's solve this for *t* as follows,

$$63t + 20 = -8 \qquad \rightarrow \qquad t = -\tfrac{4}{9}$$

Step 5
We were able to find a *t* from this equation. What that means is that this is the value of *t* that, once we plug into the equation of the line, gives the point of intersection of the line and plane.

So, **the line and plane do intersect** and they will intersect at the point $\left(\tfrac{8}{9}, -\tfrac{10}{9}, \tfrac{7}{9}\right)$.

Note that all we did to get the point is plug $t = -\tfrac{4}{9}$ into the general form for points on the line we wrote down in Step 2.

---

7. Determine if the line given by $\vec{r}\left(t\right) = \left\langle 4 + t, -1 + 8t, 3 + 2t \right\rangle$ intersects the plane given by $2x - y + 3z = 15$ or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of *t* such that if we plug that *t* into the equation of the line we'd get a point that lies on the plane. We also know that if a point $\left(x, y, z\right)$ is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

$$\left(4 + t, -1 + 8t, 3 + 2t\right)$$

for all values of *t*.

Step 3
So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$2\left(4 + t\right) - \left(-1 + 8t\right) + 3\left(3 + 2t\right) = 15$$

Step 4
Let's solve this for *t* as follows,

$$18 = 15 \text{ ??}$$

Step 5
Hmmm…

So, either we've just managed to prove that 18 and 15 are in fact the same number or there is something else going on here.

Clearly 18 and 15 are not the same number and so something else must be going on. In fact, all this means is that there is no *t* that will satisfy the equation we wrote down in Step 3. This in turn means that **the line and plane do not intersect.**

---

8. Find the line of intersection of the plane given by $3x+6y-5z=-3$ and the plane given by $-2x+7y-z=24$.

Step 1
Okay, we know that we need a point and vector parallel to the line in order to write down the equation of the line. In this case neither has been given to us.

First let's note that any point on the line of intersection must also therefore be in both planes and it's actually pretty simple to find such a point. Whatever our line of intersection is it must intersect at least one of the coordinate planes. It doesn't have to intersect all three of the coordinate planes but it will have to intersect at least one.

So, let's see if it intersects the *xy*-plane. Because the point on the intersection line must also be in both planes let's set $z=0$ (so we'll be in the *xy*-plane!) in both of the equations of our planes.

Doing this gives,

$$\begin{aligned} 3x+6y&=-3 \\ -2x+7y&=24 \end{aligned}$$

Step 2
This is a simple system to solve so we'll leave it to you to verify that the solution is,

$$x=-5 \qquad y=2$$

The fact that we were able to find a solution to the system from Step 1 means that the line of intersection does in fact intersect the *xy*-plane and it does so at the point $(-5,2,0)$. This is also then a point on the line of intersection.

Note that if the system from Step 1 didn't have a solution then the line of intersection would not have intersected the *xy*-plane and we'd need to try one of the remaining coordinate planes.

Step 3
Okay, now we need a vector that is parallel to the line of intersection. This might be a little hard to visualize, but if you think about it the line of intersection would have to be orthogonal to both of the

normal vectors from the two planes. This in turn means that any vector orthogonal to the two normal vectors must then be parallel to the line of intersection.

Nicely enough we know that the cross product of any two vectors will be orthogonal to each of the two vectors. So, here are the two normal vectors for our planes and their cross product.

$$\vec{n}_1 = \langle 3,6,-5\rangle \qquad\qquad \vec{n}_2 = \langle -2,7,-1\rangle$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & -5 \\ -2 & 7 & -1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 3 & 6 \\ -2 & 7 \end{matrix}$$

$$= -6\vec{i} + 10\vec{j} + 21\vec{k} - \left(-3\vec{j}\right) - \left(-35\vec{i}\right) - \left(-12\vec{k}\right) = 29\vec{i} + 13\vec{j} + 33\vec{k}$$

Note that we used the “trick” discussed in the notes to compute the cross product here.

Step 4

So, we now have enough information to write down the equation of the line of intersection of the two planes. The equation is,

$$\boxed{\vec{r}\left(t\right) = \langle -5,2,0\rangle + t\langle 29,13,33\rangle = \langle -5+29t, 2+13t, 33t\rangle}$$

---

9. Determine if the line given by $x = 8-15t$, $y = 9t$, $z = 5+18t$ and the plane given by $10x-6y-12z = 7$ are parallel, orthogonal or neither.

Step 1

Let’s start off this problem by noticing that the vector $\vec{v} = \langle -15,9,18\rangle$ will be parallel to the line and the vector $\vec{n} = \langle 10,-6,-12\rangle$ will be normal to the plane.

Now try to visualize the line and plane and their corresponding vectors. What would the line and plane look like if the two vectors were orthogonal to each other? What would the line and plane look like if the two vectors were parallel to each other?

Step 2

If the two vectors are orthogonal to each other the line would be parallel to the plane. If you think about this it does make sense. If $\vec{v}$ is orthogonal to $\vec{n}$ then it must be parallel to the plane because $\vec{n}$ is orthogonal to the plane. Then because the line is parallel to $\vec{v}$ it must also be parallel to the plane.

So, let’s do a quick dot product here.

$$\vec{v} \bullet \vec{n} = -420$$

The dot product is not zero and so the two vectors aren't orthogonal to each other. Therefore, the **line and plane are not parallel**.

Step 3

If the two vectors are parallel to each other the line would be orthogonal to the plane. If you think about this it does make sense. The line is parallel to $\vec{v}$ which we've just assumed is parallel to $\vec{n}$. We also know that $\vec{n}$ is orthogonal to the plane and so anything that is parallel to $\vec{n}$ (the line for instance) must also be orthogonal to the plane.

In this case it looks like we have the following relationship between the two vectors.

$$\vec{v} = -\tfrac{3}{2}\vec{n}$$

The two vectors are parallel and so the **line and plane are orthogonal**.

Section 6-4 : Quadric Surfaces

1. Sketch the following quadric surface.

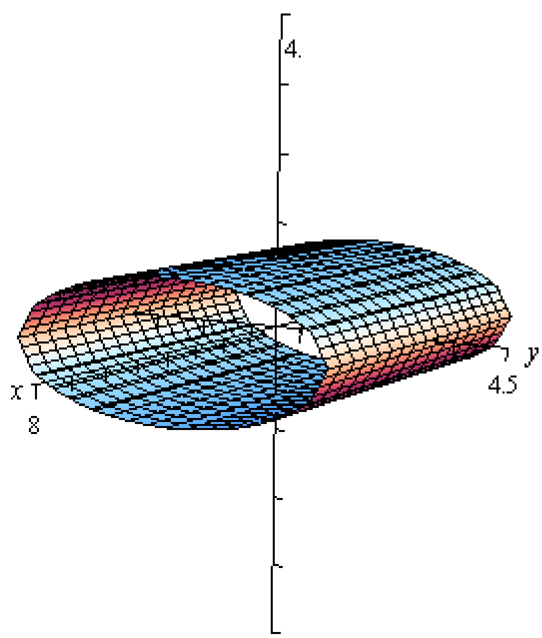
$$\frac{y^2}{9} + z^2 = 1$$

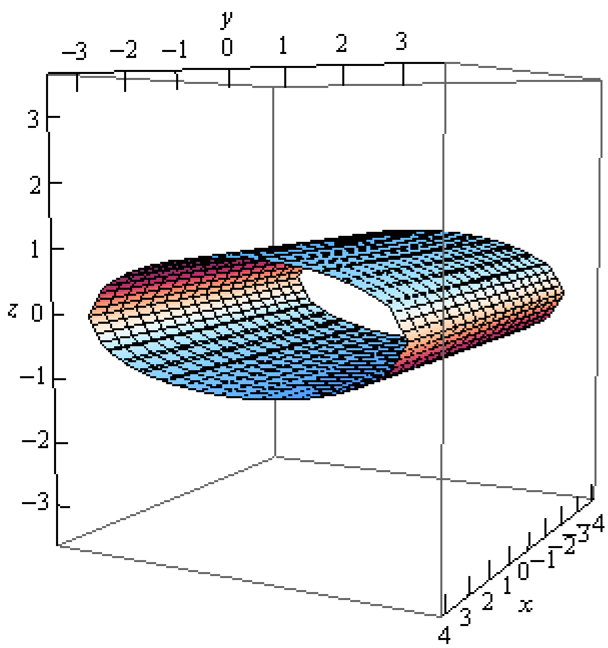
Solution

This is a cylinder that is centered on the *x*-axis. The cross sections of the cylinder will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





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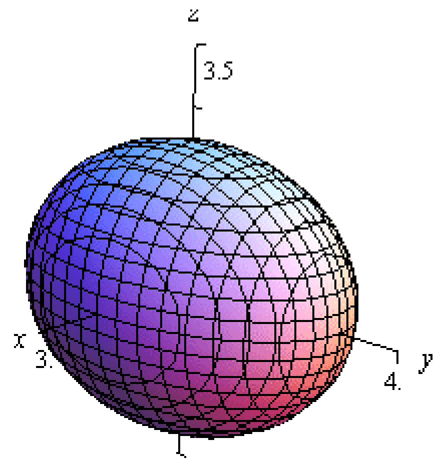
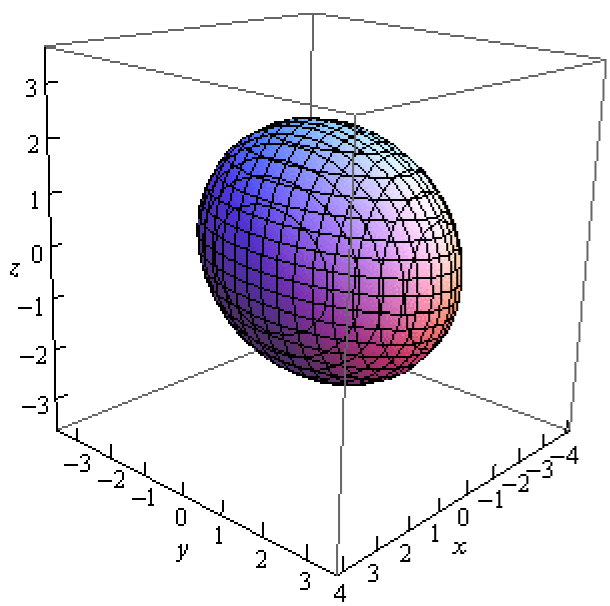
2. Sketch the following quadric surface.

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{6}=1$$

Solution
This is an ellipsoid and because the numbers in the denominators of each of the terms are not the same we know that it won't be a sphere.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





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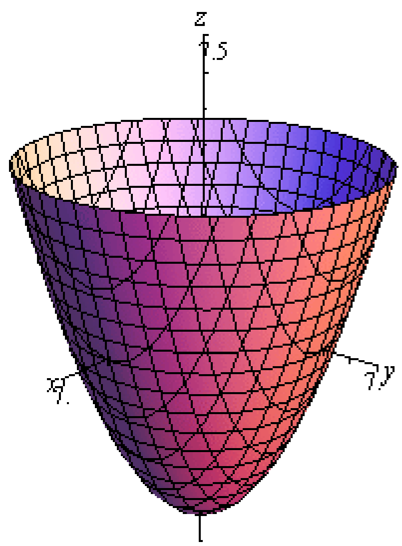
3. Sketch the following quadric surface.

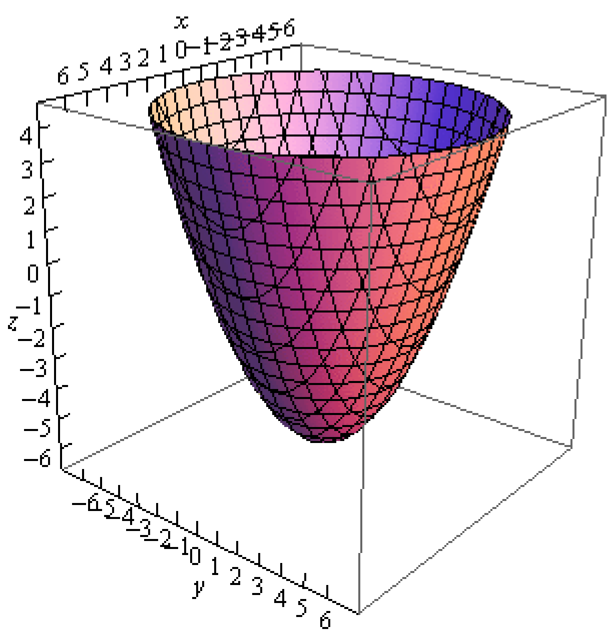
$$z = \frac{x^2}{4} + \frac{y^2}{4} - 6$$

Solution

This is an elliptic paraboloid that is centered on the *z*-axis. Because the *x* and *y* terms are positive we know that it will open upwards. The "-6" tells us that the surface will start at $z = -6$. We can also say that because the coefficients of the *x* and *y* terms are identical the cross sections of the surface will be circles.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





4. Sketch the following quadric surface.

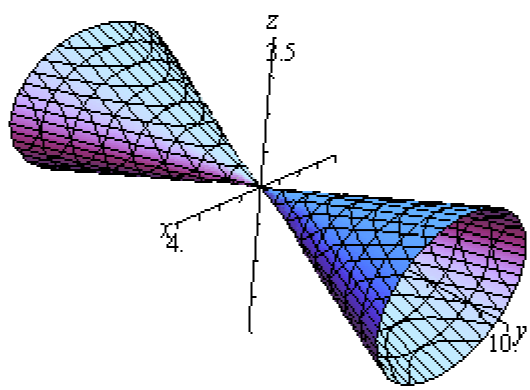
$$y^2 = 4x^2 + 16z^2$$

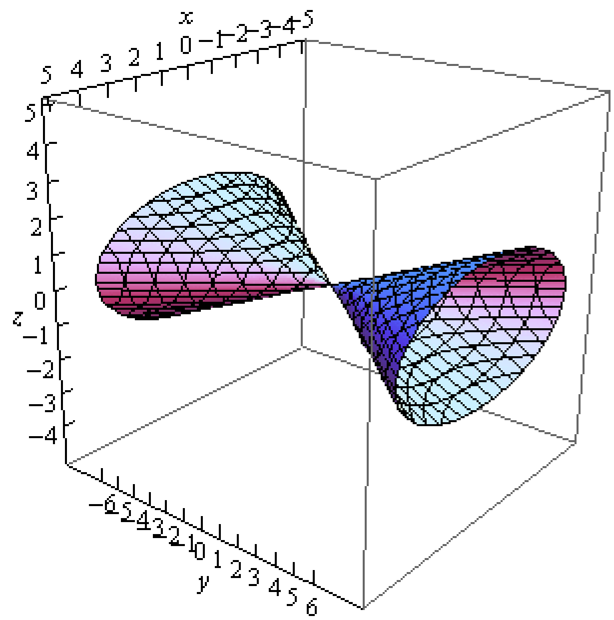
Solution

This is a cone that is centered on the *y*-axis and because the coefficients of the *x* and *z* terms are different the cross sections of the surface will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





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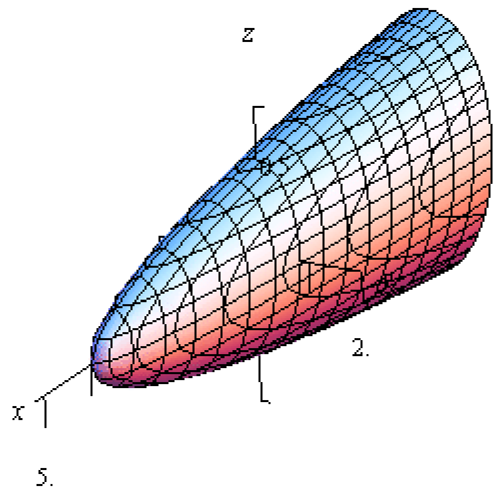
5. Sketch the following quadric surface.

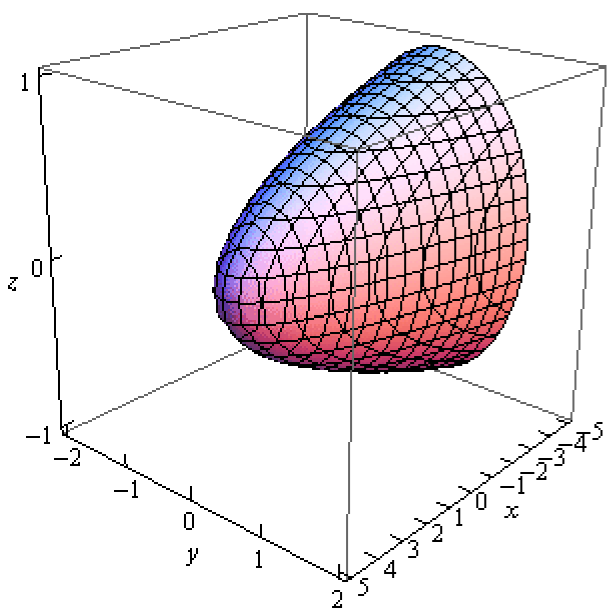
$$x = 4 - 5y^2 - 9z^2$$

Solution
This is an elliptic paraboloid that is centered on the *x*-axis. Because the *y* and *z* terms are negative we know that it will open in the negative *x* direction. The "4" tells us that the surface will start at $x = 4$. We can also say that because the coefficients of the *y* and *z* terms are different the cross sections of the surface will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.


z
2.
x
5.


1
z 0
−1
−2
−1
0
y
1
2
5
4
3
2
1
0
−1
−2
−3
−4
−5
x

Section 6-5 : Functions of Several Variables

1. Find the domain of the following function.

$$f(x, y) = \sqrt{x^2 - 2y}$$

Solution

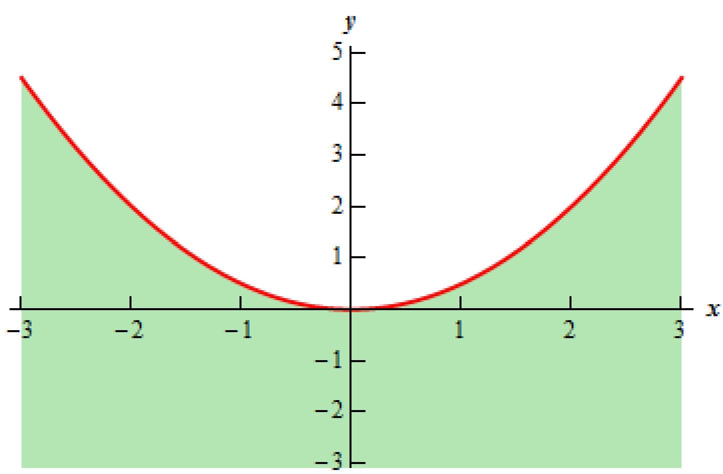
There really isn't all that much to this problem. We know that we can't have negative numbers under the square root and so the we'll need to require that whatever (x, y) is it will need to satisfy,

$$x^2 - 2y \ge 0$$

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$x^2 \ge 2y \qquad \Rightarrow \qquad y \le \frac{1}{2}x^2$$

So, it looks like we need to be on or below the parabola above. The domain is illustrated by the green area and red line in the sketch below.



2. Find the domain of the following function.

$$f(x, y) = \ln(2x - 3y + 1)$$

Solution

There really isn't all that much to this problem. We know that we can't have negative numbers or zero in a logarithm so we'll need to require that whatever (x, y) is it will need to satisfy,

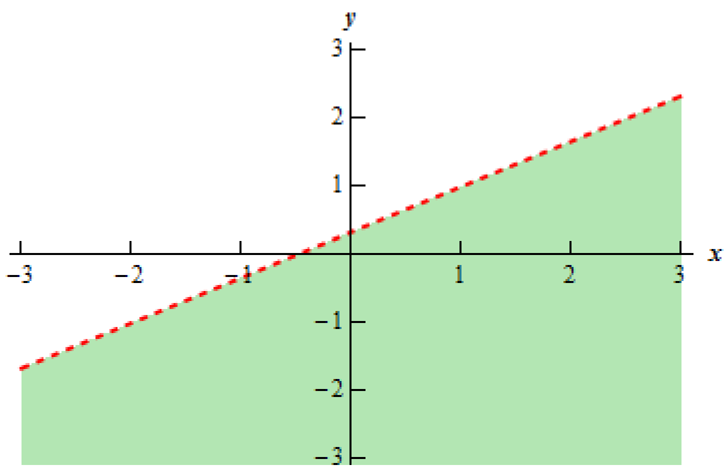
$$2x - 3y + 1 > 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$2x + 1 > 3y \qquad \Rightarrow \qquad y < \frac{2}{3}x + \frac{1}{3}$$

So, it looks like we need to be below the line above. The domain is illustrated by the green area in the sketch below.



Note that we dashed the graph of the "bounding" line to illustrate that we don't take points from the line itself.

---

3. Find the domain of the following function.

$$f(x, y, z) = \frac{1}{x^2 + y^2 + 4z}$$

Solution

There really isn't all that much to this problem. We know that we can't have division by zero so we'll need to require that whatever (x, y, z) is it will need to satisfy,

$$x^2 + y^2 + 4z \neq 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to identify the domain a little better.

$$4z \neq -x^2 - y^2 \qquad \Rightarrow \qquad z \neq -\frac{x^2}{4} - \frac{y^2}{4}$$

So, it looks like we need to avoid points, (x, y, z), that are on the elliptic paraboloid given by the equation above.

---

4. Find the domain of the following function.

$$f(x, y) = \frac{1}{x} + \sqrt{y+4} - \sqrt{x+1}$$

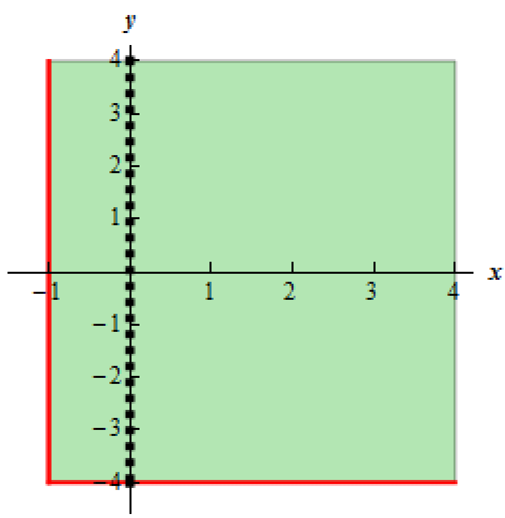
Solution

There really isn't all that much to this problem. We know that we can't have division by zero and we can't take square roots of negative numbers and so we'll need to require that whatever (x, y) is it will need to satisfy the following three conditions.

$$x \geq -1, \quad x \neq 0 \qquad\qquad y \geq -4$$

This is also our domain since these are the only conditions require in order for the function to exist.

A sketch of the domain is shown below. We can take any point in the green area or on the red lines with the exception of the *y*-axis (*i.e.* $x \neq 0$) as indicated by the black dashes on the *y*-axis.



---

5. Identify and sketch the level curves (or contours) for the following function.

$$2x - 3y + z^2 = 1$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$2x - 3y + k^2 = 1$$

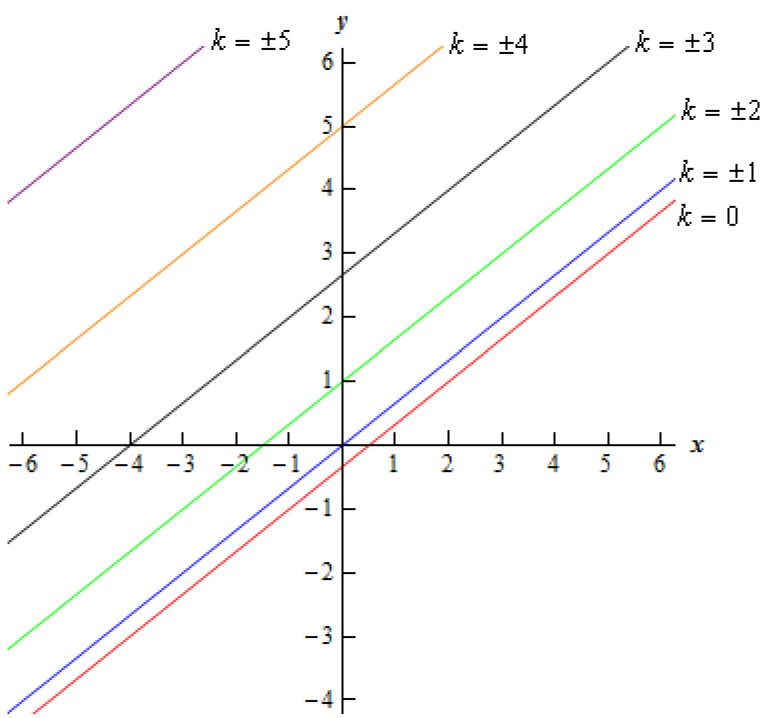
Step 2
A quick rewrite of the equation from the previous step gives us,

$$y = \frac{2}{3}x + \frac{k^2 - 1}{3}$$

So, the level curves for this function will be lines with slope $\frac{2}{3}$ and a *y*-intercept of $\left(0, \frac{k^2-1}{3}\right)$.

Note as well that there will be no restrictions on the values of *k* that we can use, as there sometimes are. Also note that the sign of *k* will not matter so, with the exception of the level curve for $k = 0$, each level curve will in fact arise from two different values of *k*.

Step 3
Below is a sketch of some level curves for some values of *k* for this function.



6. Identify and sketch the level curves (or contours) for the following function.

$$4z + 2y^2 - x = 0$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$4k + 2y^2 - x = 0$$

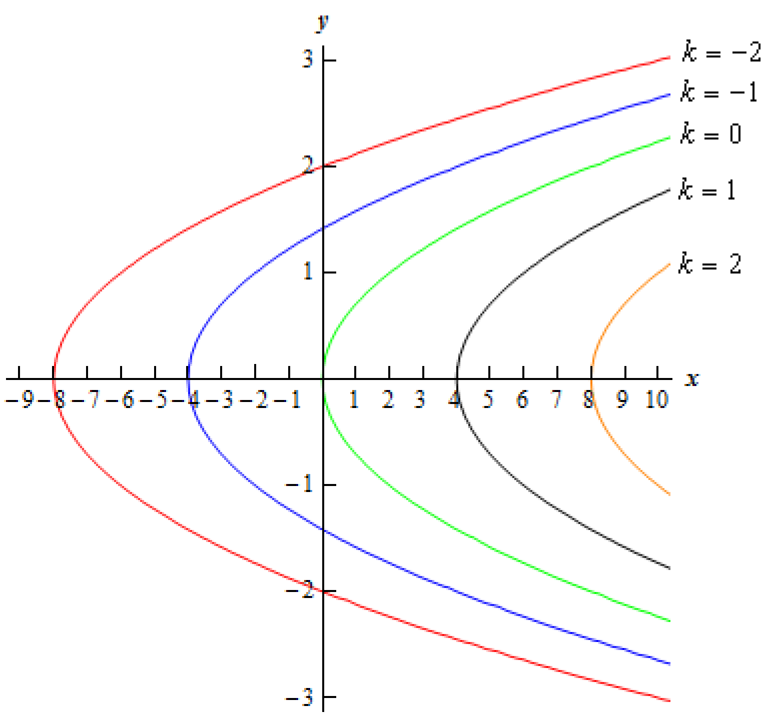
Step 2
A quick rewrite of the equation from the previous step gives us,

$$x = 2y^2 + 4k$$

So, the level curves for this function will be parabolas opening to the right and starting at 4*k*.

Note as well that there will be no restrictions on the values of *k* that we can use, as there sometimes are.

Step 3
Below is a sketch of some level curves for some values of *k* for this function.



---

7. Identify and sketch the level curves (or contours) for the following function.

$$y^2 = 2x^2 + z$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$y^2 = 2x^2 + k$$

Step 2
For this problem the value of *k* will affect the type of graph of the level curve.

First, if $k = 0$ the equation will be,

$$y^2 = 2x^2 \qquad \Rightarrow \qquad y = \pm\sqrt{2}\; x$$

So, in this case the level curve (actually curves if you think about it) will be two lines through the origin. One is increasing and the other is decreasing.

Next, let's take a look at what we get if $k > 0$. In this case a quick rewrite of the equation from Step 2 gives,

$$\frac{y^2}{k} - \frac{2x^2}{k} = 1$$

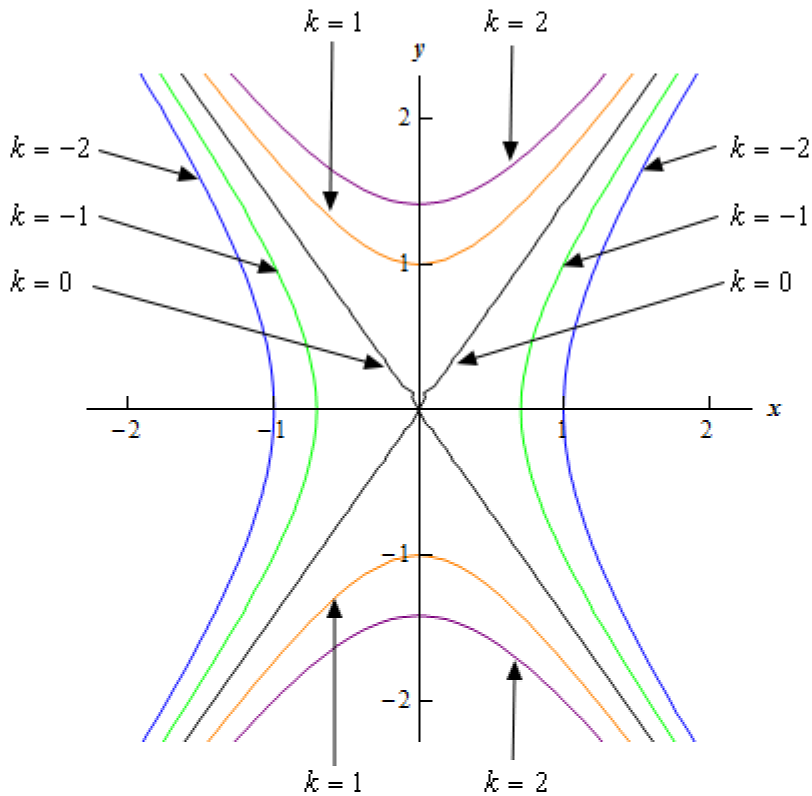
Because we know that *k* is positive we see that we have a hyperbola with the *y* term the positive term and the *x* term the negative term. This means that the hyperbola will be symmetric about the *y*-axis and opens up and down.

Finally, what do we get if $k < 0$. In this case the equation is,

$$-\frac{2x^2}{k} + \frac{y^2}{k} = 1$$

Now, be careful with this equation. In this case we have negative values of *k*. This means that the *x* term is in fact positive (the minus sign will cancel against the minus sign in the *k*). Likewise, the *y* term will be negative (it's got a negative *k* in the denominator). Therefore, we'll have a hyperbola that is symmetric about the *x*-axis and opens right and left.

Step 3
Below is a sketch of some level curves for some values of *k* for this function.



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8. Identify and sketch the traces for the following function.

$$2x-3y+z^2=1$$

Step 1

We have two traces. One we get by plugging $x=a$ into the equation and the other we get by plugging $y=b$ into the equation. Here is what we get for each of these.

$$\begin{aligned} x=a \quad &: \quad 2a-3y+z^2=1 \quad \to \quad y=\tfrac{1}{3}z^2+\tfrac{2a-1}{3} \\ y=b \quad &: \quad 2x-3b+z^2=1 \quad \to \quad x=-\tfrac{1}{2}z^2+\tfrac{3b+1}{2} \end{aligned}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of *x* and *y*. Instead we have one equation in terms of *x* and *z* and another in terms of *y* and *z*.

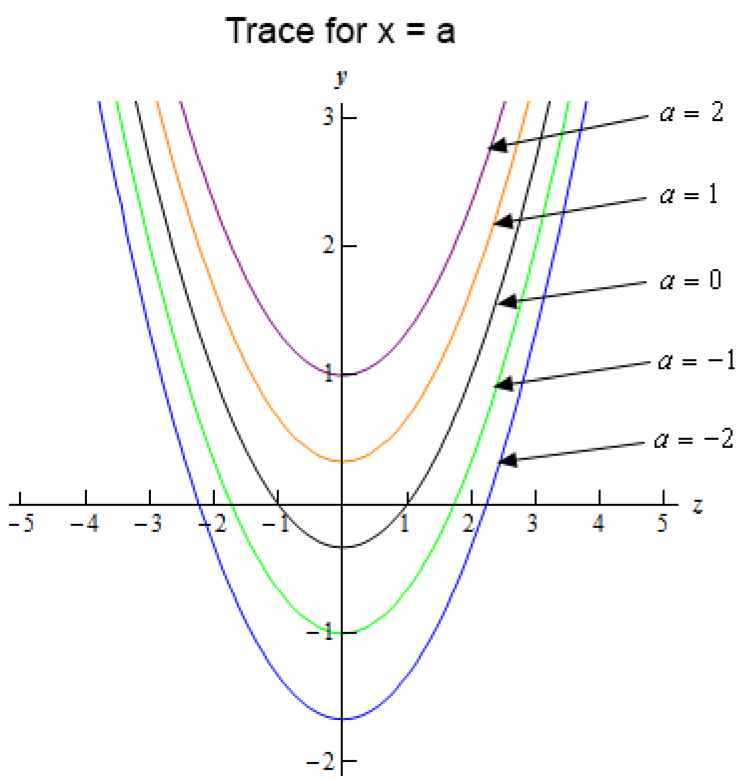
Do not get excited about this! They work the same way that equations in terms of *x* and *y* work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

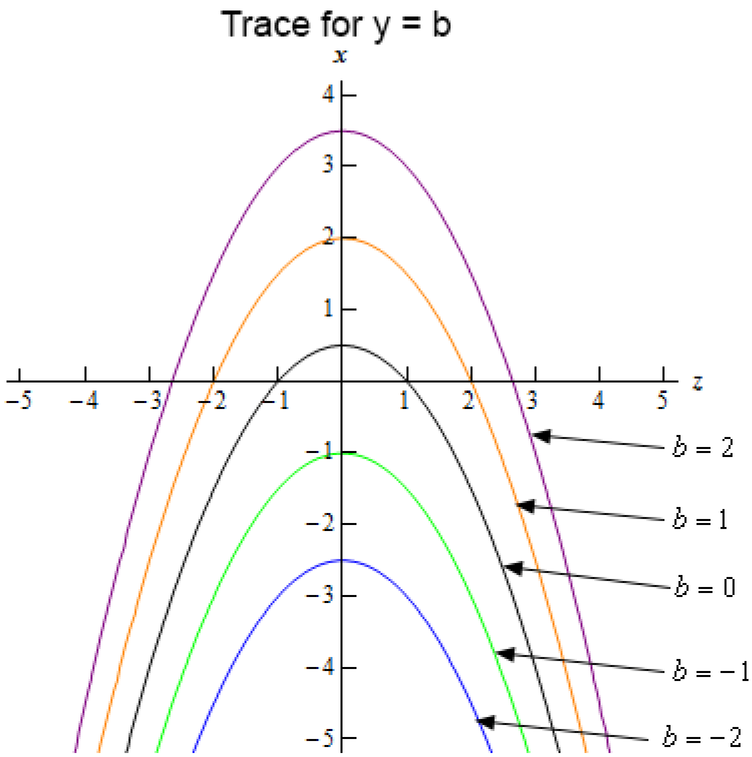
Just because we have an *x* doesn't mean that it must be the horizontal axis and just because we have a *y* doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a *z* in them and it is squared we'll let *z* be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens upwards with vertex at $\left(0, \frac{2a-1}{3}\right)$ and for the $y = b$ trace we'll have a parabola that opens downwards with vertex at $\left(0, \frac{3b+1}{2}\right)$.

Step 3
Below is a sketch for each of the traces.





---

9. Identify and sketch the traces for the following function.

$$4z + 2y^2 - x = 0$$

Step 1

We have two traces. One we get by plugging $x = a$ into the equation and the other we get by plugging $y = b$ into the equation. Here is what we get for each of these.

$$\begin{array}{lclcl} x = a & : & 4z + 2y^2 - a = 0 & \rightarrow & z = -\frac{1}{2}y^2 + \frac{a}{4} \\ y = b & : & 4z + 2b^2 - x = 0 & \rightarrow & x = 4z + 2b^2 \end{array}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of *x* and *y*. Instead we have one equation in terms of *x* and *z* and another in terms of *y* and *z*.

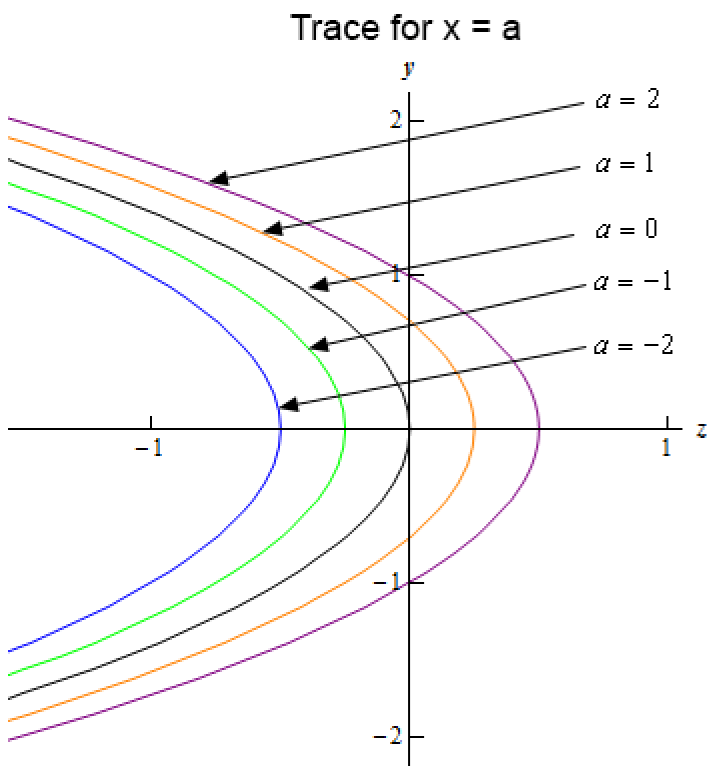
Do not get excited about this! They work the same way that equations in terms of *x* and *y* work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

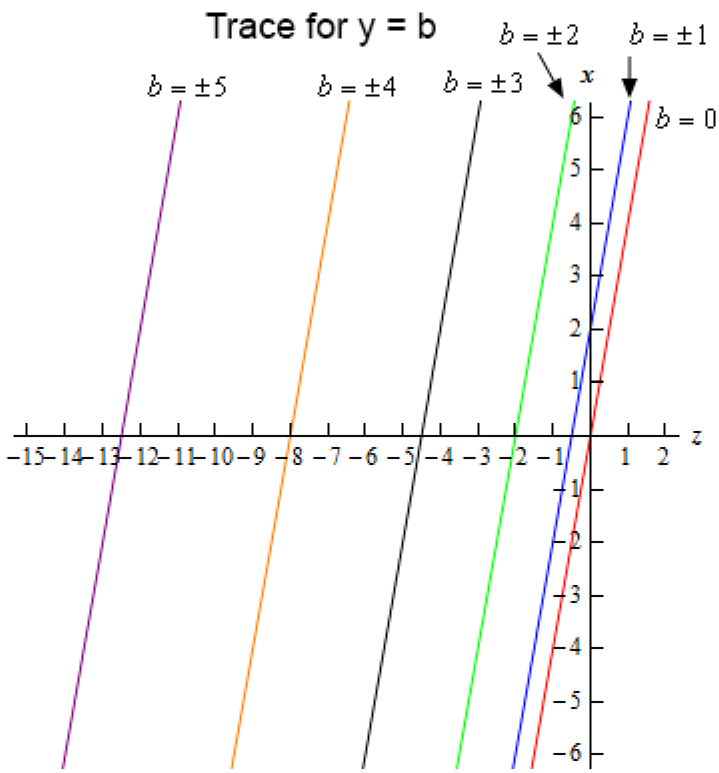
Just because we have an *x* doesn't mean that it must be the horizontal axis and just because we have a *y* doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a *z* in them we'll let *z* be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens to the left with vertex at $\left(\frac{a}{4}, 0\right)$ and for the $y = b$ trace we'll have a line with slope of 4 and an *x*-intercept at $\left(0, 2b^2\right)$.

Step 3
Below is a sketch for each of the traces.




Trace for y = b
b = ±5
b = ±4
b = ±3
b = ±2
b = ±1
b = 0
x
z
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−15
−14
−13
−12
−11
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
1
2

Section 6-6 : Vector Functions

1. Find the domain for the vector function : $\vec{r}\left(t\right)=\left\langle t^2+1,\dfrac{1}{t+2},\sqrt{t+4}\right\rangle$

Step 1
The domain of the vector function is simply the largest possible set of *t*'s that we can use in all the components of the vector function.

The first component will exist for all values of *t* and so we won't exclude any values of *t* from that component.

The second component clearly requires us to avoid $t=-2$ so we don't have division by zero in that component.

We'll also need to require that $t \ge -4$ so avoid taking the square root of negative numbers in the third component.

Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{t \ge -4, \quad t \ne -2}$$

Note that we can't forget to add the $t \ne -2$ onto this since -2 is larger than -4 and would be included otherwise!

---

2. Find the domain for the vector function : $\vec{r}\left(t\right)=\left\langle \ln\left(4-t^2\right),\sqrt{t+1}\right\rangle$

Step 1
The domain of the vector function is simply the largest possible set of *t*'s that we can use in all the components of the vector function.

We know that we can't take logarithms of negative values or zero and so from the first term we know that we'll need to require that,

$$4-t^2>0 \qquad \to \qquad -2<t<2$$

We'll also need to require that $t \ge -1$ so avoid taking the square root of negative numbers in the second component.

Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{-1 \le t < 2}$$

Remember that we want the largest possible set of *t*'s for which all the components will exist. So we can't take values of $-2 < t < -1$ because even though those are okay in the first component but they aren't in the second component. Likewise, even though we can include $t \ge 2$ in the second component we can't plug them into the first component and so we can't include them in the domain of the function.

---

3. Sketch the graph of the vector function : $\vec{r}\left(t\right) = \left\langle 4t, 10 - 2t \right\rangle$

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of *t* that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$\begin{aligned} x &= 4t \\ y &= 10 - 2t \end{aligned}$$

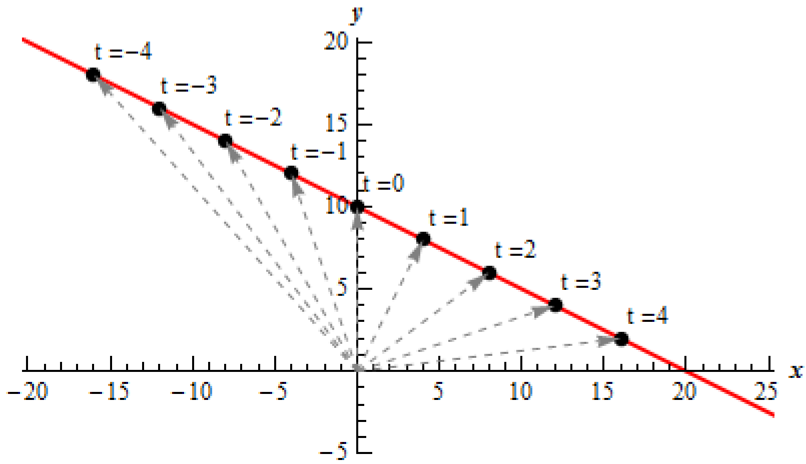
Step 2
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only *x* and *y* that will have the same graph as the vector function.

We can do this as follows,

$$x = 4t \quad \to \quad t = \tfrac{1}{4}x \quad \to \qquad y = 10 - 2\left(\tfrac{1}{4}x\right) = 10 - \tfrac{1}{2}x$$

So, it looks like the graph of the vector function will be a line with slope $-\frac{1}{2}$ and *y*-intercept of $\left(0,10\right)$.

Step 3
A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of *t*'s just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only *x* and *y*).

---

4. Sketch the graph of the vector function : $\vec{r}\left(t\right)=\left\langle t+1,\frac{1}{4}t^2+3\right\rangle$

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of *t* that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$x=t+1$$
$$y=\tfrac{1}{4}t^2+3$$

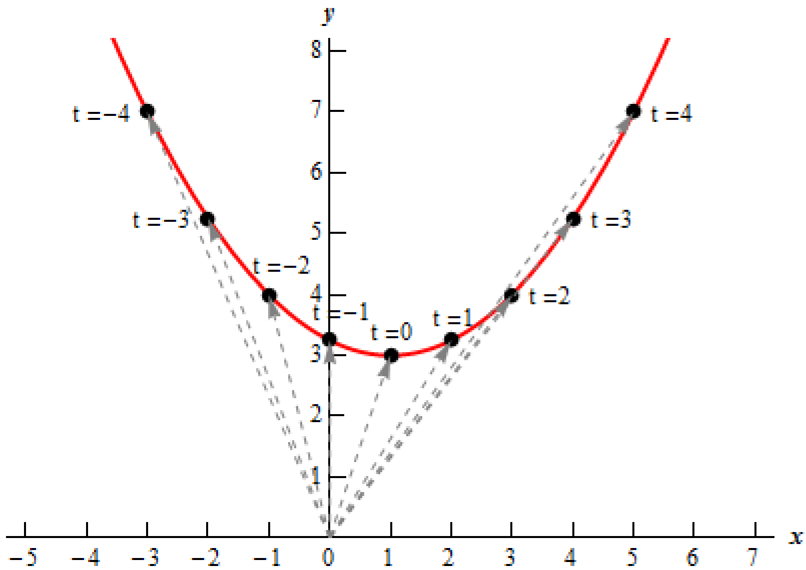
Step 2
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only *x* and *y* that will have the same graph as the vector function.

We can do this as follows,

$$x=t+1 \quad \to \quad t=x-1 \quad \to \quad y=\tfrac{1}{4}\left(x-1\right)^2+3$$

So, it looks like the graph of the vector function will be a parabola with vertex $(1,3)$ and opening upwards.

Step 3
A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of *t*'s just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only *x* and *y*).

---

5. Sketch the graph of the vector function : $\vec{r}(t) = \langle 4\sin(t), 8\cos(t) \rangle$

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of *t* that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

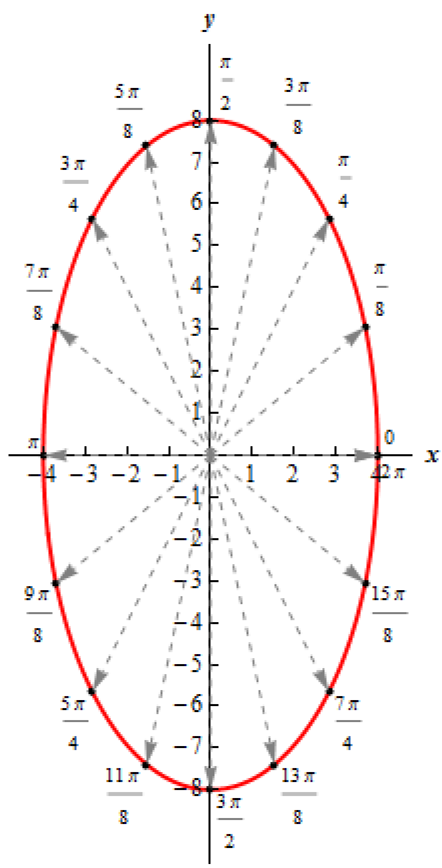
$$x = 4\sin(t)$$
$$y = 8\cos(t)$$

Step 2

Now, recall from our look at parametric equations we now know that this will be the graph of an ellipse centered at the origin with right/left points a distance of 4 from the origin and top/bottom points a distance of 8 from the origin.

Step 3

A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of *t*'s just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only *x* and *y*).

---

6. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 3\cos(6t), -4, \sin(6t) \rangle$$

Step 1
To identify the graph of this vector function (without graphing) let's first write down the set of parametric equations we get from this vector function. They are,

$$\begin{aligned} x &= 3\cos(6t) \\ y &= -4 \\ z &= \sin(6t) \end{aligned}$$

Step 2
Now, from the *x* and *z* equations we can see that we have an ellipse in the *xz*-plane that is given by the following equation.

$$\frac{x^2}{9} + z^2 = 1$$

From the *y* equation we know that this ellipse will not actually be in the *xz*-plane but parallel to the *xz*-plane at $y = -4$.

---

7. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 2 - t, 4 + 7t, -1 - 3t \rangle$$

Solution
There really isn't a lot to do with this problem. The equation should look very familiar to you. We saw quite a few of these types of equations in the Equations of Lines and Equations of Planes sections.

From those sections we know that the graph of this equation is a line in $\mathbb{R}^3$ that goes through the point $(2, 4, -1)$ and parallel to the vector $\vec{v} = \langle -1, 7, -3 \rangle$.

---

8. Write down the equation of the line segment starting at $(1, 3)$ and ending at $(-4, 6)$.

Solution
There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1 - t)\langle 1, 3 \rangle + t\langle -4, 6 \rangle \quad 0 \le t \le 1}$$

Don't forget the limits on *t*! Without that you have the full line that goes through those two points instead of the line segment from $(1,3)$ to $(-4,6)$.

---

9. Write down the equation of the line segment starting at $(0,2,-1)$ and ending at $(7,-9,2)$.

Solution
There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1-t)\langle 0,2,-1\rangle + t\langle 7,-9,2\rangle \quad 0 \le t \le 1}$$

Don't forget the limits on *t*! Without that you have the full line that goes through those two points instead of the line segment from $(0,2,-1)$ to $(7,-9,2)$.

Section 6-7 : Calculus with Vector Functions

1. Evaluate the following limit.

$$\lim_{t \to 1} \left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1} \right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned} \lim_{t \to 1} \left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1} \right\rangle &= \left\langle \lim_{t \to 1} \mathbf{e}^{t-1}, \lim_{t \to 1} 4t, \lim_{t \to 1} \frac{t-1}{t^2-1} \right\rangle \\ &= \left\langle \lim_{t \to 1} \mathbf{e}^{t-1}, \lim_{t \to 1} 4t, \lim_{t \to 1} \frac{1}{2t} \right\rangle = \left\langle \mathbf{e}^{0}, 4, \frac{1}{2} \right\rangle = \boxed{\left\langle 1, 4, \frac{1}{2} \right\rangle} \end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the third term.

---

2. Evaluate the following limit.

$$\lim_{t \to -2} \left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2} \vec{i} + \vec{j} + \left(t^2+6t\right)\vec{k} \right)$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned} \lim_{t \to -2} \left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2} \vec{i} + \vec{j} + \left(t^2+6t\right)\vec{k} \right) &= \lim_{t \to -2} \frac{1-\mathbf{e}^{t+2}}{t^2+t-2} \vec{i} + \lim_{t \to -2} \vec{j} + \lim_{t \to -2} \left(t^2+6t\right)\vec{k} \\ &= \lim_{t \to -2} \frac{-\mathbf{e}^{t+2}}{2t+1} \vec{i} + \lim_{t \to -2} \vec{j} + \lim_{t \to -2} \left(t^2+6t\right)\vec{k} = \boxed{\frac{1}{3}\vec{i} + \vec{j} - 8\vec{k}} \end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the first term.

---

3. Evaluate the following limit.

$$\lim_{t \to \infty} \left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, \mathbf{e}^{-t} \right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\lim_{t\to\infty}\left\langle \frac{1}{t^2},\frac{2t^2}{1-t-t^2},\mathbf{e}^{-t}\right\rangle = \left\langle \lim_{t\to\infty}\frac{1}{t^2},\lim_{t\to\infty}\frac{2t^2}{1-t-t^2},\lim_{t\to\infty}\mathbf{e}^{-t}\right\rangle$$

$$= \left\langle \lim_{t\to\infty}\frac{1}{t^2},\lim_{t\to\infty}\frac{2t^2}{t^2\left(\frac{1}{t^2}-\frac{1}{t}-1\right)},\lim_{t\to\infty}\mathbf{e}^{-t}\right\rangle = \boxed{\left\langle 0,-2,0\right\rangle}$$

Don't forget your basic limit at infinity processes/facts.

---

4. Compute the derivative of the following limit.

$$\vec{r}\left(t\right)=\left(t^3-1\right)\vec{i}+\mathbf{e}^{2t}\,\vec{j}+\cos\left(t\right)\vec{k}$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}'\left(t\right)=3t^2\vec{i}+2\mathbf{e}^{2t}\,\vec{j}-\sin\left(t\right)\vec{k}}$$

---

5. Compute the derivative of the following limit.

$$\vec{r}\left(t\right)=\left\langle \ln\left(t^2+1\right),t\mathbf{e}^{-t},4\right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}'\left(t\right)=\left\langle \frac{2t}{t^2+1},\mathbf{e}^{-t}-t\mathbf{e}^{-t},0\right\rangle}$$

Make sure you haven't forgotten your basic differentiation formulas such as the chain rule (the first term) and the product rule (the second term).

---

6. Compute the derivative of the following limit.

$$\vec{r}\left(t\right)=\left\langle \frac{t+1}{t-1},\tan\left(4t\right),\sin^{2}\left(t\right)\right\rangle$$

Solution
There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\begin{aligned}\vec{r}'\left(t\right)&=\left\langle \frac{\left(1\right)\left(t-1\right)-\left(t+1\right)\left(1\right)}{\left(t-1\right)^{2}},4\sec^{2}\left(4t\right),2\sin\left(t\right)\cos\left(t\right)\right\rangle \\ &=\boxed{\left\langle \frac{-2}{\left(t-1\right)^{2}},4\sec^{2}\left(4t\right),2\sin\left(t\right)\cos\left(t\right)\right\rangle}\end{aligned}$$

Make sure you haven't forgotten your basic differentiation formulas such as the quotient rule (the first term) and the chain rule (the third term).

---

7. Evaluate $\int\vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=t^{3}\,\vec{i}-\frac{2t}{t^{2}+1}\vec{j}+\cos^{2}\left(3t\right)\vec{k}$.

Solution
There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int\vec{r}\left(t\right)dt&=\int t^{3}\,dt\,\vec{i}-\int\frac{2t}{t^{2}+1}dt\,\vec{j}+\int\cos^{2}\left(3t\right)dt\,\vec{k} \\ &=\int t^{3}\,dt\,\vec{i}-\int\frac{2t}{t^{2}+1}dt\,\vec{j}+\int\tfrac{1}{2}\left(1+\cos\left(6t\right)\right)dt\,\vec{k} \\ &=\boxed{\tfrac{1}{4}t^{4}\,\vec{i}-\ln\left|t^{2}+1\right|\vec{j}+\tfrac{1}{2}\left(t+\tfrac{1}{6}\sin\left(6t\right)\right)\vec{k}+\vec{c}}\end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (second term) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (third term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

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8. Evaluate $\int_{-1}^{2} \vec{r}\left(t\right)dt$ where $\vec{r}\left(t\right) = \left\langle 6, 6t^2 - 4t, t\mathbf{e}^{2t} \right\rangle$

Solution

There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec{r}\left(t\right)dt &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \int t\mathbf{e}^{2t}\,dt \right\rangle \\ &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{2}\int \mathbf{e}^{2t}\,dt \right\rangle = \left\langle 6t, 2t^3 - 2t^2, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{4}\mathbf{e}^{2t} \right\rangle\end{aligned}$$

Don't forget the basic integration rules such integration by parts (third term). Also recall that one way to do definite integral is to first do the indefinite integral and then do the evaluation.

The answer for this problem is then,

$$\begin{aligned}\int_{-1}^{2} \vec{r}\left(t\right)dt &= \left\langle 6t, 2t^3 - 2t^2, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{4}\mathbf{e}^{2t} \right\rangle\Big|_{-1}^{2} \\ &= \left\langle 12, 8, \tfrac{3}{4}\mathbf{e}^{4} \right\rangle - \left\langle -6, -4, -\tfrac{3}{4}\mathbf{e}^{-2} \right\rangle = \boxed{\left\langle 18, 12, \tfrac{3}{4}\left(\mathbf{e}^{4} + \mathbf{e}^{-2}\right) \right\rangle}\end{aligned}$$

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

---

9. Evaluate $\int \vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right) = \left\langle \left(1-t\right)\cos\left(t^2 - 2t\right), \cos\left(t\right)\sin\left(t\right), \sec^2\left(4t\right) \right\rangle$.

Solution

There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec{r}\left(t\right)dt &= \left\langle \int \left(1-t\right)\cos\left(t^2 - 2t\right)dt, \int \cos\left(t\right)\sin\left(t\right)dt, \int \sec^2\left(4t\right)dt \right\rangle \\ &= \left\langle \int \left(1-t\right)\cos\left(t^2 - 2t\right)dt, \int \tfrac{1}{2}\sin\left(2t\right)dt, \int \sec^2\left(4t\right)dt \right\rangle \\ &= \boxed{\left\langle -\tfrac{1}{2}\sin\left(t^2 - 2t\right), -\tfrac{1}{4}\cos\left(2t\right), \tfrac{1}{4}\tan\left(4t\right) \right\rangle + \vec{c}}\end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (all terms) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (second term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

Section 6-8 : Tangent, Normal and Binormal Vectors

1. Find the unit tangent vector for the vector function : $\vec{r}\left(t\right)=\left\langle t^2+1,3-t,t^3\right\rangle$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec{r}'\left(t\right)=\left\langle 2t,-1,3t^2\right\rangle$$

$$\left\|\vec{r}'\left(t\right)\right\|=\sqrt{\left(2t\right)^2+\left(-1\right)^2+\left(3t^2\right)^2}=\sqrt{1+4t^2+9t^4}$$

Step 2

The unit tangent vector for this vector function is then,

$$\vec{T}\left(t\right)=\frac{1}{\sqrt{1+4t^2+9t^4}}\left\langle 2t,-1,3t^2\right\rangle=\boxed{\left\langle\frac{2t}{\sqrt{1+4t^2+9t^4}},-\frac{1}{\sqrt{1+4t^2+9t^4}},\frac{3t^2}{\sqrt{1+4t^2+9t^4}}\right\rangle}$$

---

2. Find the unit tangent vector for the vector function : $\vec{r}\left(t\right)=t\mathbf{e}^{2t}\,\vec{i}+\left(2-t^2\right)\vec{j}-\mathbf{e}^{2t}\vec{k}$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec{r}'\left(t\right)=\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)\vec{i}-2t\,\vec{j}-2\mathbf{e}^{2t}\vec{k}$$

$$\left\|\vec{r}'\left(t\right)\right\|=\sqrt{\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)^2+\left(-2t\right)^2+\left(-2\mathbf{e}^{2t}\right)^2}=\sqrt{\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)^2+4t^2+4\mathbf{e}^{4t}}$$

Step 2

The unit tangent vector for this vector function is then,

$$\boxed{\vec{T}\left(t\right)=\frac{1}{\sqrt{\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)^2+4t^2+4\mathbf{e}^{4t}}}\left(\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)\vec{i}-2t\,\vec{j}-2\mathbf{e}^{2t}\vec{k}\right)}$$

Note that because of the "mess" with this one we didn't distribute the magnitude through to each term as we usually do with these. This problem is a good example of just how "messy" these can get.

---

3. Find the tangent line to $\vec{r}\left(t\right)=\cos\left(4t\right)\vec{i}+3\sin\left(4t\right)\vec{j}+t^3\vec{k}$ at $t=\pi$.

Step 1
First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec{r}'\left(t\right)=-4\sin\left(4t\right)\vec{i}+12\cos\left(4t\right)\vec{j}+3t^2\vec{k}$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2
Now we actually need the tangent vector at the value of *t* given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of *t* from the problem statement. These are,

$$\vec{r}\left(\pi\right)=\cos\left(4\pi\right)\vec{i}+3\sin\left(4\pi\right)\vec{j}+\pi^3\vec{k}=\vec{i}+\pi^3\vec{k}$$
$$\vec{r}'\left(\pi\right)=-4\sin\left(4\pi\right)\vec{i}+12\cos\left(4\pi\right)\vec{j}+3\pi^2\vec{k}=12\vec{j}+3\pi^2\vec{k}$$

Step 3
To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec{r}\left(t\right)=\vec{i}+\pi^3\vec{k}+t\left(12\vec{j}+3\pi^2\vec{k}\right)=\vec{i}+12t\,\vec{j}+\left(\pi^3+3\pi^2t\right)\vec{k}}$$

---

4. Find the tangent line to $\vec{r}\left(t\right)=\left\langle 7\mathbf{e}^{2-t},\dfrac{16}{t^3},5-t\right\rangle$ at $t=2$.

Step 1
First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec{r}'\left(t\right)=\left\langle -7\mathbf{e}^{2-t},-\frac{48}{t^4},-1\right\rangle$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2
Now we actually need the tangent vector at the value of *t* given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of *t* from the problem statement. These are,

$$\vec{r}(2) = \langle 7,2,3 \rangle$$
$$\vec{r}\,'(2) = \langle -7,-3,-1 \rangle$$

Step 3
To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec{r}(t) = \langle 7,2,3 \rangle + t\langle -7,-3,-1 \rangle = \langle 7-7t, 2-3t, 3-t \rangle}$$

---

5. Find the unit normal and the binormal vectors for the following vector function.

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), 3 \rangle$$

Step 1
We first need the unit tangent vector so let's get that.

$$\vec{r}\,'(t) = \langle -2\sin(2t), 2\cos(2t), 0 \rangle \qquad \left\| \vec{r}\,'(t) \right\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$$
$$\vec{T}(t) = \frac{1}{2}\langle -2\sin(2t), 2\cos(2t), 0 \rangle = \langle -\sin(2t), \cos(2t), 0 \rangle$$

Step 2
The unit normal vector is then,

$$\vec{T}\,'(t) = \langle -2\cos(2t), -2\sin(2t), 0 \rangle \qquad \left\| \vec{T}\,'(t) \right\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2$$
$$\boxed{\vec{N}(t) = \frac{1}{2}\langle -2\cos(2t), -2\sin(2t), 0 \rangle = \langle -\cos(2t), -\sin(2t), 0 \rangle}$$

Step 3

Finally, the binormal vector is,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(2t) & \cos(2t) & 0 \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix}$$

$$= \sin^2(2t)\vec{k} - \left(-\cos^2(2t)\vec{k}\right) = \left(\sin^2(2t) + \cos^2(2t)\right)\vec{k} = \boxed{\vec{k} = \langle 0,0,1 \rangle = \vec{B}(t)}$$

Section 6-9 : Arc Length with Vector Functions

1. Determine the length of $\vec r\left(t\right)=\left(3-4t\right)\vec i+6t\,\vec j-\left(9+2t\right)\vec k$ from $-6\le t\le 8$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right)=-4\vec i+6\,\vec j-2\vec k$$

$$\left\|\vec r'\left(t\right)\right\|=\sqrt{16+36+4}=\sqrt{56}=2\sqrt{14}$$

Step 2

The length of the curve is then,

$$L=\int_{-6}^{8}2\sqrt{14}\,dt=2\sqrt{14}t\Big|_{-6}^{8}=\boxed{28\sqrt{14}}$$

---

2. Determine the length of $\vec r\left(t\right)=\left\langle \frac{1}{3}t^3,4t,\sqrt{2}t^2\right\rangle$ from $0\le t\le 2$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right)=\left\langle t^2,4,2\sqrt{2}t\right\rangle$$

$$\left\|\vec r'\left(t\right)\right\|=\sqrt{t^4+16+8t^2}=\sqrt{t^4+8t^2+16}=\sqrt{\left(t^2+4\right)^2}=t^2+4$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Step 2

The length of the curve is then,

$$L=\int_{0}^{2}t^2+4\,dt=\left(\frac{1}{3}t^3+4t\right)\Big|_{0}^{2}=\boxed{\frac{32}{3}}$$

Note that if we'd not simplified the magnitude this would have been a very difficult integral to compute!

---

3. Find the arc length function for $\vec r\left(t\right)=\left\langle t^2,2t^3,1-t^3\right\rangle$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right)=\left\langle 2t,6t^2,-3t^2\right\rangle$$

$$\left\|\vec r'\left(t\right)\right\|=\sqrt{4t^2+36t^4+9t^4}=\sqrt{t^2\left(4+45t^2\right)}=\sqrt{t^2}\sqrt{4+45t^2}=\left|t\right|\sqrt{4+45t^2}=t\sqrt{4+45t^2}$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Note as well that because we are assuming that we are starting at $t=0$ for this kind of problem it is safe to assume that $t\ge 0$ and so $\sqrt{t^2}=\left|t\right|=t$.

Step 2
The arc length function is then,

$$s\left(t\right)=\int_0^t u\sqrt{4+45u^2}\,du=\tfrac{1}{135}\left(4+45u^2\right)^{\frac{3}{2}}\Big|_0^t=\boxed{\tfrac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}}-8\right]}$$

---

4. Find the arc length function for $\vec r\left(t\right)=\left\langle 4t,-2t,\sqrt{5}\,t^2\right\rangle$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right)=\left\langle 4,-2,2\sqrt{5}\,t\right\rangle$$

$$\left\|\vec r'\left(t\right)\right\|=\sqrt{16+4+20t^2}=\sqrt{20+20t^2}=\sqrt{20}\sqrt{1+t^2}=2\sqrt{5}\sqrt{1+t^2}$$

Step 2
The arc length function is then,

$$s\left(t\right)=\int_0^t 2\sqrt{5}\sqrt{1+u^2}\,du$$

Do not always expect these integrals to be "simple" integrals. They will often require techniques more involved than just a standard Calculus I substitution. In this case we need the following trig substitution.

$$u = \tan\theta \qquad du = \sec^2\theta\, d\theta \qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right|$$

The limits of the integral become,

$$u = 0 : 0 = \tan\theta \quad \to \quad \theta = 0 \qquad u = t > 0 : \; t = \tan\theta \quad \to \quad \theta = \tan^{-1}(t)$$

Now, as noted we know that $t > 0$ and so we can safely assume that from the $u = t$ limit we will get $0 < \theta < \frac{\pi}{2}$. This in turn means that we will always be in the first quadrant and we know that secant is positive in the first quadrant. Therefore, we can remove the absolute values bars on the secant above.

The arc length function is now,

$$\begin{aligned} s(t) &= \int_0^{\tan^{-1}(t)} 2\sqrt{5}\sec^3\theta\, d\theta = \sqrt{5}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\tan^{-1}(t)} \\ &= \sqrt{5}\left[\sec\left(\tan^{-1}(t)\right)\tan\left(\tan^{-1}(t)\right) + \ln\left|\sec\left(\tan^{-1}(t)\right) + \tan\left(\tan^{-1}(t)\right)\right|\right] \end{aligned}$$

Now we know that $\tan\left(\tan^{-1}(t)\right) = t$ so that will simplify our answer a little. Let's take a look at the secant term to see if we can simplify that as well. First, from our limit work recall that $\theta = \tan^{-1}(t)$. Or with a little rewrite we have,

$$\tan\theta = t = \frac{t}{1} = \frac{\text{opposite}}{\text{adjacent}}$$

Construct a right triangle with opposite side being *t* and the adjacent side being 1. The hypotenuse is then $\sqrt{t^2+1}$. This in turn means that $\sec\theta = \sqrt{t^2+1}$. So,

$$\sec\left(\tan^{-1}(t)\right) = \sec(\theta) = \sqrt{t^2+1}$$

With this simplification our arc length function is then,

$$s(t) = \sqrt{5}\left[t\sqrt{t^2+1} + \ln\left|\sqrt{t^2+1} + t\right|\right]$$

There was some slightly unpleasant simplification here but once we did that we got a much nicer arc length function.

---

5. Determine where on the curve given by $\vec{r}(t) = \left\langle t^2, 2t^3, 1-t^3\right\rangle$ we are after traveling a distance of 20.

Step 1
From Problem 3 above we know that the arc length function for this vector function is,

$$s(t) = \tfrac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}} - 8\right]$$

We need to solve this for *t*. Doing this gives,

$$\begin{aligned}
\left(4+45t^2\right)^{\frac{3}{2}} - 8 &= 135s \\
\left(4+45t^2\right)^{\frac{3}{2}} &= 135s + 8 \\
4+45t^2 &= \left(135s+8\right)^{\frac{2}{3}} \\
t^2 &= \tfrac{1}{45}\left[\left(135s+8\right)^{\frac{2}{3}} - 4\right] \qquad \to \qquad t = \sqrt{\tfrac{1}{45}\left[\left(135s+8\right)^{\frac{2}{3}} - 4\right]}
\end{aligned}$$

Note that we only used the positive *t* after taking the root because the implicit assumption from the arc length function is that *t* is positive.

Step 2
We could use this to reparametrize the vector function however that would lead to a particularly unpleasant function in this case.

The key here is to simply realize that what we are being asked to compute is the value of the reparametrized vector function, $\vec{r}\left(t(s)\right)$ when $s = 20$. Or, in other words, we want to compute $\vec{r}\left(t(20)\right)$.

So, first,

$$t(20) = \sqrt{\tfrac{1}{45}\left[\left(135(20)+8\right)^{\frac{2}{3}} - 4\right]} = 2.05633$$

Our position after traveling a distance of 20 is then,

$$\vec{r}\left(t(20)\right) = \vec{r}\left(2.05633\right) = \boxed{\left\langle 4.22849, 17.39035, -7.69518 \right\rangle}$$

Section 6-10 : Curvature

1. Find the curvature of $\vec r\left(t\right)=\left\langle \cos\left(2t\right),-\sin\left(2t\right),4t\right\rangle$.

Step 1

We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector won't be that bad to work with so let's go with that formula. Here's the unit tangent vector work.

$$\vec r'\left(t\right)=\left\langle -2\sin\left(2t\right),-2\cos\left(2t\right),4\right\rangle \qquad \left\|\vec r'\left(t\right)\right\|=\sqrt{4\sin^2\left(2t\right)+4\cos^2\left(2t\right)+16}=\sqrt{20}=2\sqrt5$$

$$\vec T\left(t\right)=\tfrac{1}{2\sqrt5}\left\langle -2\sin\left(2t\right),-2\cos\left(2t\right),4\right\rangle=\left\langle -\tfrac{\sin(2t)}{\sqrt5},-\tfrac{\cos(2t)}{\sqrt5},\tfrac{2}{\sqrt5}\right\rangle$$

Step 2

Now, what we really need is the magnitude of the derivative of the unit tangent vector so here is that work,

$$\vec T'\left(t\right)=\left\langle -\tfrac{2}{\sqrt5}\cos\left(2t\right),\tfrac{2}{\sqrt5}\sin\left(2t\right),0\right\rangle \qquad \left\|\vec T'\left(t\right)\right\|=\sqrt{\tfrac45\cos^2\left(2t\right)+\tfrac45\sin^2\left(2t\right)}=\tfrac{2}{\sqrt5}$$

Step 3

The curvature is then,

$$\kappa=\frac{\left\|\vec T'\left(t\right)\right\|}{\left\|\vec r'\left(t\right)\right\|}=\frac{2/\sqrt5}{2\sqrt5}=\boxed{\frac15}$$

So, in this case, the curvature will be independent of *t*. That won't always be the case so don't expect this to happen every time.

---

2. Find the curvature of $\vec r\left(t\right)=\left\langle 4t,-t^2,2t^3\right\rangle$.

Step 1

We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector will involve lots of quotients that would probably be unpleasant to take the derivative of. So, let's go with the cross product formula this time.

We'll need the first and second derivative of the vector function. Here are those.

$$\vec r\,'\left(t\right)=\left\langle 4,-2t,6t^2\right\rangle \qquad\qquad \vec r\,''\left(t\right)=\left\langle 0,-2,12t\right\rangle$$

Step 2
Next, we need the cross product of these two derivatives. Here is that work.

$$\vec r\,'\left(t\right)\times\vec r\,''\left(t\right)=\begin{vmatrix}\vec i & \vec j & \vec k\\ 4 & -2t & 6t^2\\ 0 & -2 & 12t\end{vmatrix}=-24t^2\,\vec i-8\vec k-48t\,\vec j+12t^2\,\vec i=-12t^2\,\vec i-48t\,\vec j-8\vec k$$

Step 3
We now need a couple of magnitudes.

$$\left\|\vec r\,'\left(t\right)\times\vec r\,''\left(t\right)\right\|=\sqrt{144t^4+2304t^2+64} \qquad\qquad \left\|\vec r\,'\left(t\right)\right\|=\sqrt{16+4t^2+36t^4}$$

The curvature is then,

$$\kappa=\frac{\left\|\vec r\,'\left(t\right)\times\vec r\,''\left(t\right)\right\|}{\left\|\vec r\,'\left(t\right)\right\|^3}=\boxed{\frac{\sqrt{144t^4+2304t^2+64}}{\left(16+4t^2+36t^4\right)^{\frac{3}{2}}}}$$

A fairly messy formula here, but these will often be that way.

Section 6-11 : Velocity and Acceleration

1. An objects acceleration is given by $\vec{a} = 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}$. The objects initial velocity is $\vec{v}\left(0\right) = \vec{j} - 3\vec{k}$ and the objects initial position is $\vec{r}\left(0\right) = -5\vec{i} + 2\vec{j} - 3\vec{k}$. Determine the objects velocity and position functions.

Step 1

To determine the velocity function all we need to do is integrate the acceleration function.

$$\vec{v}\left(t\right) = \int 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}\,dt = \tfrac{3}{2}t^2\,\vec{i} + 4\mathbf{e}^{-t}\,\vec{j} + 4t^3\vec{k} + \vec{c}$$

Don't forget the "constant" of integration, which in this case is actually the vector $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$. To determine the constant of integration all we need is to use the value $\vec{v}\left(0\right)$ that we were given in the problem statement.

$$\vec{j} - 3\vec{k} = \vec{v}\left(0\right) = 4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

To determine the values of c_1, c_2, and c_3 all we need to do is set the various components equal.

$$\begin{aligned} \vec{i} &: 0 = c_1 \\ \vec{j} &: 1 = 4 + c_2 \\ \vec{k} &: -3 = c_3 \end{aligned} \qquad \Rightarrow \qquad c_1 = 0, \quad c_2 = -3, \quad c_3 = -3$$

The velocity is then,

$$\boxed{\vec{v}\left(t\right) = \tfrac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\vec{j} + \left(4t^3 - 3\right)\vec{k}}$$

Step 2

The position function is simply the integral of the velocity function we found in the previous step.

$$\vec{r}\left(t\right) = \int \tfrac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\vec{j} + \left(4t^3 - 3\right)\vec{k}\,dt = \tfrac{1}{2}t^3\,\vec{i} + \left(-4\mathbf{e}^{-t} - 3t\right)\vec{j} + \left(t^4 - 3t\right)\vec{k} + \vec{c}$$

We'll use the value of $\vec{r}\left(0\right)$ from the problem statement to determine the value of the constant of integration.

$$-5\vec{i} + 2\vec{j} - 3\vec{k} = \vec{r}\left(0\right) = -4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

$$\begin{aligned}\vec{i} &: -5 = c_1 \\ \vec{j} &: 2 = -4 + c_2 \qquad \Rightarrow \qquad c_1 = -5, \quad c_2 = 6, \quad c_3 = -3 \\ \vec{k} &: -3 = c_3\end{aligned}$$

The position function is then,

$$\boxed{\vec{r}\left(t\right) = \left(\tfrac{1}{2}t^3 - 5\right)\vec{i} + \left(-4\mathbf{e}^{-t} - 3t + 6\right)\vec{j} + \left(t^4 - 3t - 3\right)\vec{k}}$$

---

2. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}\left(t\right) = \left\langle \cos\left(2t\right), -\sin\left(2t\right), 4t \right\rangle$.

Step 1

First, we need the first and second derivatives of the position function.

$$\vec{r}'\left(t\right) = \left\langle -2\sin\left(2t\right), -2\cos\left(2t\right), 4 \right\rangle \qquad \vec{r}''\left(t\right) = \left\langle -4\cos\left(2t\right), 4\sin\left(2t\right), 0 \right\rangle$$

Step 2

Next, we'll need the following quantities.

$$\left\| \vec{r}'\left(t\right) \right\| = \sqrt{4\sin^2\left(2t\right) + 4\cos^2\left(2t\right) + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{r}'\left(t\right) \bullet \vec{r}''\left(t\right) = 8\sin\left(2t\right)\cos\left(2t\right) - 8\sin\left(2t\right)\cos\left(2t\right) + 0 = 0$$

$$\begin{aligned}\vec{r}'\left(t\right) \times \vec{r}''\left(t\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\left(2t\right) & -2\cos\left(2t\right) & 4 \\ -4\cos\left(2t\right) & 4\sin\left(2t\right) & 0 \end{vmatrix} \\ &= -16\cos\left(2t\right)\vec{j} - 8\sin^2\left(2t\right)\vec{k} - 8\cos^2\left(2t\right)\vec{k} - 16\sin\left(2t\right)\vec{i} \\ &= -16\sin\left(2t\right)\vec{i} - 16\cos\left(2t\right)\vec{j} - 8\vec{k}\end{aligned}$$

$$\left\| \vec{r}'\left(t\right) \times \vec{r}''\left(t\right) \right\| = \sqrt{256\sin^2\left(2t\right) + 256\cos^2\left(2t\right) + 64} = \sqrt{320} = 8\sqrt{5}$$

Step 3

The tangential component of the acceleration is,

$$a_T = \frac{\vec{r}'(t) \bullet \vec{r}''(t)}{\left\| \vec{r}'(t) \right\|} = \boxed{0}$$

The normal component of the acceleration is,

$$a_N = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{8\sqrt{5}}{2\sqrt{5}} = \boxed{4}$$

Section 6-12 : Cylindrical Coordinates

1. Convert the Cartesian coordinates for $\left(4,-5,2\right)$ into Cylindrical coordinates.

Step 1

From the point we're given we have,

$$x = 4 \qquad\qquad y = -5 \qquad\qquad z = 2$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2

Remember as well that for *r* and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting *r* is easy.

$$r = \sqrt{\left(4\right)^2 + \left(-5\right)^2} = \sqrt{41}$$

Step 3

Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-5}{4}\right) = -0.8961 \qquad\qquad \theta_2 = -0.8961 + \pi = 2.2455$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the fourth quadrant and θ_2 is in the second quadrant. Likewise, from our *x* and *y* coordinates the point is in the fourth quadrant (as we look at the point from above).

This in turn means that we need to use θ_1 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{41},-0.8961,2\right)}$$

---

2. Convert the Cartesian coordinates for $\left(-4,-1,8\right)$ into Cylindrical coordinates.

Step 1

From the point we're given we have,

$$x = -4 \qquad y = -1 \qquad z = 8$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2
Remember as well that for r and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting r is easy.

$$r = \sqrt{\left(-4\right)^2 + \left(-1\right)^2} = \sqrt{17}$$

Step 3
Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-1}{-4}\right) = 0.2450 \qquad \theta_2 = 0.2450 + \pi = 3.3866$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the first quadrant and θ_2 is in the third quadrant. Likewise, from our x and y coordinates the point is in the third quadrant (as we look at the point from above).

This in turn means that we need to use θ_2 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{17}, 3.3866, 8\right)}$$

---

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

$$x^3 + 2x^2 - 6z = 4 - 2y^2$$

Step 1
There really isn't a whole lot to do here. All we need to do is plug in the following x and y polar conversion formulas into the equation where (and if) possible.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Step 2
However, first we'll do a little rewrite.

$$x^3+2x^2+2y^2-6z=4 \qquad \to \qquad x^3+2\left(x^2+y^2\right)-6z=4$$

Step 3
Now let's use the formulas from Step 1 to convert the equation into Cylindrical coordinates.

$$\left(r\cos\theta\right)^3+2\left(r^2\right)-6z=4 \qquad \to \qquad \boxed{r^3\cos^3\theta+2r^2-6z=4}$$

---

4. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$z\,r=2-r^2$$

Solution
There is not really a lot to do here other than plug in $r=\sqrt{x^2+y^2}$ into the equation. Doing this is,

$$\boxed{z\sqrt{x^2+y^2}=2-\left(x^2+y^2\right)}$$

---

5. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$4\sin\left(\theta\right)-2\cos\left(\theta\right)=\frac{r}{z}$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following *x* and *y* polar conversion formulas in the equation where (and if) possible.

$$x=r\cos\theta \qquad y=r\sin\theta \qquad r^2=x^2+y^2$$

Step 2
To make the conversion a little easier let's multiply the equation by *r* to get,

$$4r\sin\left(\theta\right)-2r\cos\left(\theta\right)=\frac{r^2}{z}$$

Step 3
Now let's use the formulas from Step 1 to convert the equation into Cartesian coordinates.

$$\boxed{4y - 2x = \frac{x^2 + y^2}{z}}$$

---

6. Identify the surface generated by the equation : $r^2 - 4r\cos\left(\theta\right) = 14$

Step 1
To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$x^2 + y^2 - 4x = 14$$

Step 2
To identify this equation (and you do know what it is!) let's complete the square on the *x* part of the equation.

$$\begin{aligned} x^2 - 4x + y^2 &= 14 \\ x^2 - 4x + 4 + y^2 &= 14 + 4 \\ \left(x - 2\right)^2 + y^2 &= 18 \end{aligned}$$

So, we can see that this is a cylinder whose central axis is a vertical line parallel to the *z*-axis and goes through the point (2, 0) in the *xy*-plane and the radius of the cylinder is $\sqrt{18}$.

---

7. Identify the surface generated by the equation : $z = 7 - 4r^2$

Step 1
To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$z = 7 - 4\left(x^2 + y^2\right) = 7 - 4x^2 - 4y^2$$

Step 2
From the Cartesian equation in Step 1 we can see that the surface generated by the equation is an elliptic paraboloid that starts at $z = 7$ and opens down.

Section 6-13 : Spherical Coordinates

1. Convert the Cartesian coordinates for $\left(3,-4,1\right)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$x = 3 \qquad\qquad y = -4 \qquad\qquad z = 1$$

Step 2
Let's first determine ρ.

$$\rho = \sqrt{\left(3\right)^2 + \left(-4\right)^2 + \left(1\right)^2} = \sqrt{26}$$

Step 3
We can now determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{1}{\sqrt{26}} \qquad\qquad \varphi = \cos^{-1}\left(\frac{1}{\sqrt{26}}\right) = 1.3734$$

Step 4
Let's use the *x* conversion formula to determine θ.

$$\cos\theta = \frac{x}{\rho\sin\varphi} = \frac{3}{\sqrt{26}\sin\left(1.3734\right)} = 0.6 \qquad \to \qquad \theta_1 = \cos^{-1}\left(0.6\right) = 0.9273$$

This angle is in the first quadrant and if we sketch a quick unit circle we see that a second angle in the fourth quadrant is $\theta_2 = 2\pi - 0.9273 = 5.3559$.

If we look at the three dimensional coordinate system from above we can see that from our *x* and *y* coordinates the point is in the fourth quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{26}, 5.3559, 1.3734\right)}$$

---

2. Convert the Cartesian coordinates for $\left(-2,-1,-7\right)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$x = -2 \qquad y = -1 \qquad z = -7$$

Step 2
Let's first determine ρ.

$$\rho = \sqrt{(-2)^2 + (-1)^2 + (-7)^2} = \sqrt{54}$$

Step 3
We can now determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{-7}{\sqrt{54}} \qquad \varphi = \cos^{-1}\left(\frac{-7}{\sqrt{54}}\right) = 2.8324$$

Step 4
Let's use the *y* conversion formula to determine θ.

$$\sin\theta = \frac{-1}{\rho\sin\varphi} = \frac{-1}{\sqrt{54}\sin(2.8324)} = -0.4472 \qquad \to \qquad \theta_1 = \sin^{-1}(-0.4472) = -0.4636$$

This angle is in the fourth quadrant and if we sketch a quick unit circle we see that a second angle in the third quadrant is $\theta_2 = \pi + 0.4636 = 3.6052$.

If we look at the three dimensional coordinate system from above we can see that from our *x* and *y* coordinates the point is in the third quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{54}, 3.6052, 2.8324\right)}$$

---

3. Convert the Cylindrical coordinates for $(2, 0.345, -3)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$r = 2 \qquad \theta = 0.345 \qquad z = -3$$

So, we already have the value of θ for the Spherical coordinates.

Step 2
Next, we can determine ρ.

$$\rho = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

Step 3
Finally, we can determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{-3}{\sqrt{13}} \qquad \varphi = \cos^{-1}\left(\frac{-3}{\sqrt{13}}\right) = 2.5536$$

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{13}, 0.345, 2.5536\right)}$$

---

4. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$x^2 + y^2 = 4x + z - 2$$

Step 1
All we need to do here is plug in the following conversion formulas into the equation and do a little simplification.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

Step 2
Plugging the conversion formula in gives,

$$\left(\rho\sin\varphi\cos\theta\right)^2 + \left(\rho\sin\varphi\sin\theta\right)^2 = 4\left(\rho\sin\varphi\cos\theta\right) + \rho\cos\varphi - 2$$

The first two terms can be simplified as follows.

$$\begin{aligned}\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2\\ \rho^2\sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right) &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2\\ \rho^2\sin^2\varphi &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2\end{aligned}$$

---

5. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\rho^2 = 3 - \cos\varphi$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Step 2
To make this problem a little easier let's first multiply the equation by ρ. Doing this gives,

$$\rho^3 = 3\rho - \rho\cos\varphi$$

Doing this makes recognizing the right most term a little easier.

Step 3
Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} = 3\sqrt{x^2 + y^2 + z^2} - z}$$

---

6. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\csc\varphi = 2\cos\theta + 4\sin\theta$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Step 2
To make this problem a little easier let's first do some rewrite on the equation.

First, let's deal with the cosecant.

$$\frac{1}{\sin\varphi} = 2\cos\theta + 4\sin\theta \qquad \to \qquad 1 = 2\sin\varphi\cos\theta + 4\sin\varphi\sin\theta$$

Next, let's multiply everything by ρ to get,

$$\rho = 2\rho\sin\varphi\cos\theta + 4\rho\sin\varphi\sin\theta$$

Doing this makes recognizing the terms on the right a little easier.

Step 3
Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\sqrt{x^2 + y^2 + z^2} = 2x + 4y}$$

---

7. Identify the surface generated by the given equation : $\varphi = \frac{4\pi}{5}$

Solution
Okay, as we discussed this type of equation in the notes for this section. We know that all points on the surface generated must be of the form $\left(\rho, \theta, \frac{4\pi}{5}\right)$.

So, we can rotate around the *z*-axis as much as want them to (*i.e.* θ can be anything) and we can move as far as we want from the origin (*i.e.* ρ can be anything). All we need to do is make sure that the point will always make an angle of $\frac{4\pi}{5}$ with the positive z-axis.

In other words, we have a cone. It will open downwards and the "wall" of the cone will form an angle of $\frac{4\pi}{5}$ with the positive *z*-axis and it will form an angle of $\frac{\pi}{5}$ with the negative *z*-axis.

---

8. Identify the surface generated by the given equation : $\rho = -2\sin\varphi\cos\theta$

Step 1
Let's first multiply each side of the equation by ρ to get,

$$\rho^2 = -2\rho\sin\varphi\cos\theta$$

Step 2
We can now easily convert this to Cartesian coordinates to get,

$$\begin{aligned} x^2 + y^2 + z^2 &= -2x \\ x^2 + 2x + y^2 + z^2 &= 0 \end{aligned}$$

Let’s complete the square on the *x* portion to get,

$$\begin{aligned} x^2 + 2x + 1 + y^2 + z^2 &= 0 + 1 \\ (x+1)^2 + y^2 + z^2 &= 1 \end{aligned}$$

Step 3

So, it looks like we have a sphere with radius 1 that is centered at $(-1, 0, 0)$.

