CALCULUS III

Solutions to Practice Problems

Paul Dawkins

Table of Contents

Preface

Here are the solutions to the practice problems for the Calculus III notes.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Outline

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.
Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system.

Partial Derivatives – In this chapter we'll take a brief look at limits of functions of more than one variable and then move into derivatives of functions of more than one variable. As we'll see if we can do derivatives of functions with one variable it isn't much more difficult to do derivatives of functions of more than one variable (with a very important subtlety). We will also discuss interpretations of partial derivatives, higher order partial derivatives and the chain rule as applied to functions of more than one variable. We will also define and discuss directional derivatives.
Limits – In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.
Partial Derivatives – In this section we will look at the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (*i.e.* without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.
Interpretations of Partial Derivatives – In the section we will take a look at a couple of important interpretations of partial derivatives. First, the always important, rate of change of the function. Although we now have multiple 'directions' in which the function can change (unlike in Calculus I). We will also see that partial derivatives give the slope of tangent lines to the traces of the function.
Higher Order Partial Derivatives – In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, *etc*. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.
Differentials – In this section we extend the idea of differentials we first saw in Calculus I to functions of several variables.
Chain Rule – In the section we extend the idea of the chain rule to functions of several variables. In particular, we will see that there are multiple variants to the chain rule here all depending on how many variables our function is dependent on and how each of those variables can, in turn, be written in terms of different variables. We will also give a nice method for writing down the chain rule for pretty much an ysituation you might run into when dealing with functions of multiple variables. In addition, we will derive a very quick way of doing implicit differentiation so we no longer need to go through the process we first did back in Calculus I.
Directional Derivatives – In the section we introduce the concept of directional derivatives, including how to compute them and see a couple of nice facts pertaining to directional derivatives.

Applications of Partial Derivatives – In this chapter we will take a look at several applications of partial derivatives. We will find the equation of tangent planes to surfaces and we will revisit on of the more important applications of derivatives from earlier Calculus classes. We will spend a significant amount of time finding relative and absolute extrema of functions of multiple variables. We will also introduce Lagrange multipliers to find the absolute extrema of a function subject to one or more constraints.

Change of Variables – In previous sections we've converted Cartesian coordinates in Polar, Cylindrical and Spherical coordinates. In this section we will generalize this idea and discuss how we convert integrals in Cartesian coordinates into alternate coordinate systems. Included will be a derivation of the *dV* conversion formula when converting to Spherical coordinates.
Surface Area – In this section we will show how a double integral can be used to determine the surface area of the portion of a surface that is over a region in two dimensional space.
Area and Volume Revisited – In this section we summarize the various area and volume formulas from this chapter.

Line Integrals – In this chapter we will introduce a new kind of integral : Line Integrals. With Line Integrals we will be integrating functions of two or more variables where the independent variables now are defined by curves rather than regions as with double and triple integrals. We will also investigate conservative vector fields and discuss Green's Theorem in this chapter.
Vector Fields – In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length.
Line Integrals – Part I – In this section we will start looking at line integrals. In particular we will look at line integrals with respect to arc length.
Line Integrals – Part II – In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to *x*, *y*, and/or *z*. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.
Line Integrals of Vector Fields – In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to *x*, *y* and *z*.
Fundamental Theorem for Line Integrals – In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.
Conservative Vector Fields – In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.
Green's Theorem – In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Surface Integrals – In this chapter we look at yet another kind on integral : Surface Integrals. With Surface Integrals we will be integrating functions of two or more variables where the independent variables are now on the surface of three dimensional solids. We will also look at Stokes' Theorem and the Divergence Theorem.
Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.
Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Chapter 1 : 3-Dimensional Space

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

The 3-D Coordinate System – In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Section 1-1 : The 3-D Coordinate System

1. Give the projection of $P = (3,-4,6)$ onto the three coordinate planes.

Solution
There really isn't a lot to do with this problem. We know that the *xy*-plane is given by the equation $z = 0$ and so the projection into the *xy*-plane for any point is simply found by setting the *z* coordinate to zero. We can find the projections for the other two coordinate planes in a similar fashion.

So, the projects are then,

$$xy - \text{plane} : \ (3,-4,0)$$
$$xz - \text{plane} : \ (3,0,6)$$
$$yz - \text{plane} : \ (0,-4,6)$$

---

2. Which of the points $P = (4,-2,6)$ and $Q = (-6,-3,2)$ is closest to the *yz*-plane?

Step 1
The shortest distance between any point and any of the coordinate planes will be the distance between that point and its projection onto that plane.

Let's call the projections of *P* and Q onto the *yz*-plane $\overline{P}$ and $\overline{Q}$ respectively. They are,

$$\overline{P} = (0,-2,6) \qquad\qquad \overline{Q} = (0,-3,2)$$

Step 2
To determine which of these is closest to the *yz*-plane we just need to compute the distance between the point and its projection onto the *yz*-plane.

Note as well that because only the *x*-coordinate of the two points are different the distance between the two points will just be the absolute value of the difference between two *x* coordinates.

Therefore,

$$d\left(P,\overline{P}\right) = 4 \qquad\qquad d\left(Q,\overline{Q}\right) = 6$$

Based on this is should be pretty clear that $P = (4,-2,6)$ is closest to the *yz*-plane.

---

3. Which of the points $P = (-1,4,-7)$ and $Q = (6,-1,5)$ is closest to the *z*-axis?

Step 1
First, let's note that the coordinates of any point on the *z*-axis will be $(0,0,z)$.

Also, the shortest distance from any point not on the *z*-axis to the *z*-axis will occur if we draw a line straight from the point to the *z*-axis in such a way that it forms a right angle with the *z*-axis.

So, if we start with any point not on the *z*-axis, say (x_1, y_1, z_1), the point on the *z*-axis that will be closest to this point is $(0,0,z_1)$.

Let's call the point closest to *P* and *Q* on the *z*-axis closest to be $\overline{P}$ and $\overline{Q}$ respectively. They are,

$$\overline{P} = (0,0,-7) \qquad\qquad \overline{Q} = (0,0,5)$$

Step 2
To determine which of these is closest to the *z*-axis we just need to compute the distance between the point and its projection onto the *z*-axis.

The distances are,

$$d\left(P,\overline{P}\right) = \sqrt{(-1-0)^2 + (4-0)^2 + (-7-(-7))^2} = \sqrt{17}$$
$$d\left(Q,\overline{Q}\right) = \sqrt{(6-0)^2 + (-1-0)^2 + (5-5)^2} = \sqrt{37}$$

Based on this is should be pretty clear that $P = (-1,4,-7)$ is closest to the *z*-axis.

---

4. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$7x^2 - 9y^3 = 3x + 1$$

Solution
First notice that because there are two variables in this equation it cannot have a graph in $\mathbb{R}$ since equations in that coordinate system can only have a single variable.

There are two variables in the equation so we know that it will have a graph in $\mathbb{R}^2$.

Likewise, the fact that the equation has two variables means that it will also have a graph in $\mathbb{R}^3$. Although in this case the third variable, *z*, can have any value.

---

5. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$x^3 + \sqrt{y^2 + 1} - 6z = 2$$

Solution

This equation has three variables and so it will have a graph in $\mathbb{R}^3$.

On other hand because the equation has three variables in it there will be no graph in $\mathbb{R}^2$ (can have at most two variables) and it will not have a graph in $\mathbb{R}$ (can only have a single variable).

Section 1-2 : Equations of Lines

1. Give the equation of the line through the points $\left(2,-4,1\right)$ and $\left(0,4,-10\right)$ in vector form, parametric form and symmetric form.

Step 1

Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We have two points that are on the line. We can use either point and depending on your choice of points you may have different answers that we get here. We will use the first point listed above for our point for no other reason that it is the first one listed.

The parallel vector is really simple to get as well since we can always form the vector from the first point to the second point and this vector will be on the line and so will also be parallel to the line. The vector is,

$$\vec{v} = \left\langle -2,8,-11\right\rangle$$

Step 2

The vector form of the line is,

$$\boxed{\vec{r}\left(t\right) = \left\langle 2,-4,1\right\rangle + t\left\langle -2,8,-11\right\rangle = \left\langle 2-2t,-4+8t,1-11t\right\rangle}$$

Step 3

The parametric form of the line is,

$$\boxed{x = 2-2t \qquad y = -4+8t \qquad z = 1-11t}$$

Step 4

To get the symmetric form all we need to do is solve each of the parametric equations for *t* and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{2-x}{2} = \frac{4+y}{8} = \frac{1-z}{11}}$$

---

2. Give the equation of the line through the point $\left(-7,2,4\right)$ and parallel to the line given by $x = 5-8t$, $y = 6+t$, $z = -12t$ in vector form, parametric form and symmetric form.

Step 1
Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We were given a point on the line so no need to worry about that for this problem.

The parallel vector is really simple to get as well since we were told that the new line must be parallel to the given line. We also know that the coefficients of the *t*'s in the equation of the line forms a vector parallel to the line.

So,

$$\vec{v} = \langle -8, 1, -12 \rangle$$

is a vector that is parallel to the given line.

Now, if $\vec{v}$ is parallel to the given line and the new line must be parallel to the given line then $\vec{v}$ must also be parallel to the new line.

Step 2
The vector form of the line is,

$$\boxed{\vec{r}(t) = \langle -7, 2, 4 \rangle + t\langle -8, 1, -12 \rangle = \langle -7 - 8t, 2 + t, 4 - 12t \rangle}$$

Step 3
The parametric form of the line is,

$$\boxed{x = -7 - 8t \qquad y = 2 + t \qquad z = 4 - 12t}$$

Step 4
To get the symmetric form all we need to do is solve each of the parametric equations for *t* and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{-7 - x}{8} = y - 2 = \frac{4 - z}{12}}$$

---

3. Is the line through the points $(2, 0, 9)$ and $(-4, 1, -5)$ parallel, orthogonal or neither to the line given by $\vec{r}(t) = \langle 5, 1 - 9t, -8 - 4t \rangle$?

Step 1
Let's start this off simply by getting vectors parallel to each of the lines.

For the line through the points $(2,0,9)$ and $(-4,1,-5)$ we know that the vector between these two points will lie on the line and hence be parallel to the line. This vector is,

$$\vec{v}_1 = \langle 6,-1,14 \rangle$$

For the second line the coefficients of the *t*'s are the components of the parallel vector so this vector is,

$$\vec{v}_2 = \langle 0,-9,-4 \rangle$$

Step 2
Now, from the first components of these vectors it is hopefully clear that they are not scalar multiples. There is no number we can multiply to zero to get 6.

Likewise, we can only multiply 6 by zero to get 0. However, if we multiply the first vector by zero all the components would be zero and that is clearly not the case.

Therefore, they are not scalar multiples and so these two vectors are not parallel. This also in turn means that **the two lines can't possibly be parallel** either (since each vector is parallel to its respective line).

Step 3
Next,

$$\vec{v}_1 \bullet \vec{v}_2 = -47$$

The dot product is not zero and so these vectors aren't orthogonal. Because the two vectors are parallel to their respective lines this also means that **the two lines are not orthogonal**.

---

4. Determine the intersection point of the line given by $x = 8+t$, $y = 5+6t$, $z = 4-2t$ and the line given by $\vec{r}(t) = \langle -7+12t, 3-t, 14+8t \rangle$ or show that they do not intersect.

Step 1
If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same x, y and z coordinates.

Step 2
This means that we can set up the following system of equations.

$$\begin{aligned} 8+t_1 &= -7+12t_2 \\ 5+6t_1 &= 3-t_2 \\ 4-2t_1 &= 14+8t_2 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 3

Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first two equations. We'll worry about the third equation eventually.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$t_1 = -15+12t_2 \quad \to \quad \begin{aligned} 5+6\left(-15+12t_2\right) &= 3-t_2 \\ -85+72t_2 &= 3-t_2 \\ 73t_2 &= 88 \end{aligned} \quad \to \quad \begin{aligned} t_2 &= \tfrac{88}{73} \\ t_1 &= -15+12\left(\tfrac{88}{73}\right) = -\tfrac{39}{73} \end{aligned}$$

Step 4

Okay, this is a somewhat "messy" solution, but they will often be that way so we shouldn't get too excited about it!

Now, recall that to get this solution we used the first two equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the *x* and *y* coordinates will be the same (remember we used the *x* and *y* equations to find this solution….).

At this point we need to recall that we did have a third equation that also needs to be satisfied at these values of *t*. In other words, we need to plug $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ into the third equation and see what we get. Doing this gives,

$$\tfrac{370}{73} = 4-2\left(-\tfrac{39}{73}\right) \neq 14+8\left(\tfrac{88}{73}\right) = \tfrac{1726}{73}$$

Okay, the two sides are not the same. Just what does this mean? In terms of systems of equations it means that $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ are NOT a solution to the system of equations in Step 2. Had they been a solution then we would have gotten the same number from both sides.

In terms of whether or not the lines intersect we need to only recall that the third equation corresponds to the *z* coordinates of the lines. We know that at $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ the two lines will have the same *x* and *y* coordinates (since they came from solving the first two equations). However, we've just shown that they will not give the same *z* coordinate.

In other words, there are no values of t_1 and t_2 for which the two lines will have the same *x*, *y* and *z* coordinates. Hence, we can now say that the **two lines do not intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 3. Different sets of equations will lead to different values of t_1 and t_2 but they will still not satisfy the remaining equation for this problem and we will get the same result of the lines not intersecting.

---

5. Determine the intersection point of the line through the points $(1,-2,13)$ and $(2,0,-5)$ and the line given by $\vec{r}(t) = \langle 2+4t, -1-t, 3\rangle$ or show that they do not intersect.

Step 1

Because we don't have the equation for the first line that will be the first thing we'll need to do. The vector between the two points (and hence parallel to the line) is,

$$\vec{v} = \langle 1, 2, -18\rangle$$

Using the first point listed the equation of the first line is then,

$$\vec{r}(t) = \langle 1,-2,13\rangle + t\langle 1,2,-18\rangle = \langle 1+t, -2+2t, 13-18t\rangle$$

Step 2

If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same *x*, *y* and *z* coordinates.

Step 3

This means that we can set up the following system of equations.

$$\begin{aligned} 1 + t_1 &= 2 + 4t_2 \\ -2 + 2t_1 &= -1 - t_2 \\ 13 - 18t_1 &= 3 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 4
Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first and third equation. We'll worry about the second equation eventually.

Note that for this system the third equation should definitely be used here since we can quickly just solve that for t_1.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$t_1 = \tfrac{5}{9} \qquad \to \qquad 1 + \tfrac{5}{9} = 2 + 4t_2 \qquad \to \qquad t_2 = -\tfrac{1}{9}$$

Step 5
Now, recall that to get this solution we used the first and third equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the *x* and *z* coordinates will be the same (remember we used the *x* and *z* equations to find this solution….).

At this point we need to recall that we did have another equation that also needs to be satisfied at these values of *t*. In other words, we need to plug $t_1 = \tfrac{5}{9}$ and $t_2 = -\tfrac{1}{9}$ into the second equation and see what we get. Doing this gives,

$$-2 + 2\left(\tfrac{5}{9}\right) = -\tfrac{8}{9} = -1 - \left(-\tfrac{1}{9}\right)$$

Okay, the two sides are the same. Just what does this mean? In terms of systems of equations it means that $t_1 = \tfrac{5}{9}$ and $t_2 = -\tfrac{1}{9}$ is a solution to the system of equations in Step 3.

In terms of whether or not the lines intersect we now know that at $t_1 = \tfrac{5}{9}$ and $t_2 = -\tfrac{1}{9}$ the two lines will have the same *x*, *y* and *z* coordinates (since they satisfy all three equations).

In other words, these **two lines do intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 4. Different sets of equations will lead to the same values of t_1 and t_2 leading to the two lines intersecting.

---

6. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xy*-plane? If so, give the point.

Step 1
If the line intersects the xy-plane there will be a point on the line that is also in the xy-plane. Recall as well that any point in the xy-plane will have a z coordinate of $z = 0$.

Step 2
So, to determine if the line intersects the *xy*-plane all we need to do is set the equation for the *z* coordinate equal to zero and solve it for *t*, if that's possible.

Doing this gives,

$$12 - 11t = 0 \qquad \rightarrow \qquad t = \tfrac{12}{11}$$

Step 3
So, we were able to solve for *t* in this case and so we can now say that **the line does intersect the *xy*-plane**.

Step 4
All we need to do to finish this off this problem is find the full point of intersection. We can find this simply by plugging $t = \tfrac{12}{11}$ into the *x* and *y* portions of the equation of the line.

Doing this gives,

$$x = 9 + 21\left(\tfrac{12}{11}\right) = \tfrac{351}{11} \qquad y = -7$$

The point of intersection is then : $\boxed{\left(\tfrac{351}{11}, -7, 0\right)}$.

---

7. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xz*-plane? If so, give the point.

Step 1
If the line intersects the xz-plane there will be a point on the line that is also in the xz-plane. Recall as well that any point in the xz-plane will have a y coordinate of $y = 0$.

Step 2
So, to determine if the line intersects the *xz*-plane all we need to do is set the equation for the *y* coordinate equal to zero and solve it for *t*, if that's possible.

However, in this case we can see that is clearly not possible since the *y* equation is simply $y = -7$ and this can clearly never be zero.

Step 3
Therefore, **the line does not intersect the *xz*-plane**.

Section 1-3 : Equations of Planes

1. Write down the equation of the plane containing the points $\left(4,-3,1\right)$, $\left(-3,-1,1\right)$ and $\left(4,-2,8\right)$.

Step 1
To make the work on this problem a little easier let's "name" the points as,

$$P=\left(4,-3,1\right) \qquad Q=\left(-3,-1,1\right) \qquad R=\left(4,-2,8\right)$$

Now, we know that in order to write down the equation of a plane we'll need a point (we have three so that's not a problem!) and a vector that is normal to the plane.

Step 2
We'll need to do a little work to get a normal vector.

First, we'll need two vectors that lie in the plane and we can get those from the three points we're given. Note that there are lots of possible vectors that we could use here. Here are the two that we'll use for this problem.

$$\overrightarrow{PQ}=\left\langle -7,2,0\right\rangle \qquad \overrightarrow{PR}=\left\langle 0,1,7\right\rangle$$

Step 3
Now, these two vectors lie in the plane and we know that the cross product of any two vectors will be orthogonal to both of the vectors. Therefore, the cross product of these two vectors will also be orthogonal (and so normal!) to the plane.

So, let's get the cross product.

$$\vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ -7 & 2 & 0 \\ 0 & 1 & 7 \end{array}\right|\begin{array}{cc} \vec{i} & \vec{j} \\ -7 & 2 \\ 0 & 1 \end{array}=14\vec{i}-7\vec{k}-\left(-49\vec{j}\right)=14\vec{i}+49\vec{j}-7\vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Step 4
Now all we need to do is write down the equation.

We have three points to choose form here. We'll use the first point simply because it is the first point listed. Any of the others could also be used.

The equation of the plane is,

$$\boxed{14(x-4)+49(y+3)-7(z-1)=0 \qquad \to \qquad 14x+49y-7z=-98}$$

Note that depending on your choice of vectors in Step 2, the order you chose to use them in the cross product computation in Step 3 and the point chosen here will all affect your answer. However, regardless of your choices the equation you get will be an acceptable answer provided you did all the work correctly.

---

2. Write down the equation of the plane containing the point $(3,0,-4)$ and orthogonal to the line given by $\vec{r}(t)=\langle 12-t,1+8t,4+6t\rangle$.

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is orthogonal to the line given in the problem statement. This means that the plane will also be orthogonal to any vector that just happens to be parallel to the line.

From the equation of the line we know that the coefficients of the *t*'s are the components of a vector that is parallel to the line. So, a vector parallel to the line is then,

$$\vec{v}=\langle -1,8,6\rangle$$

Now, as mentioned above because this vector is parallel to the line then it will also need to be orthogonal to the plane and hence be normal to the plane. So, a normal vector for the plane is,

$$\vec{n}=\langle -1,8,6\rangle$$

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{-(x-3)+8(y-0)+6(z+4)=0 \qquad \to \qquad -x+8y+6z=-27}$$

---

3. Write down the equation of the plane containing the point $(-8,3,7)$ and parallel to the plane given by $4x+8y-2z=45$.

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is parallel to the plane given in the problem statement. This means that any vector normal to one plane will be normal to both planes.

From the equation of the plane we were given we know that the coefficients of the *x*, *y* and *z* are the components of a vector that is normal to the plane. So, a vector normal to the given plane is then,

$$\vec{n} = \langle 4, 8, -2 \rangle$$

Now, as mentioned above because this vector is normal to the given plane then it will also need to be normal to the plane we want to find the equation for.

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{4(x+8)+8(y-3)-2(z-7)=0 \qquad \to \qquad 4x+8y-2z=-22}$$

---

4. Determine if the plane given by $4x-9y-z=2$ and the plane given by $x+2y-14z=-6$ are parallel, orthogonal or neither.

Step 1
Let's start off this problem by noticing that the vector $\vec{n}_1 = \langle 4, -9, -1 \rangle$ will be normal to the first plane and the vector $\vec{n}_2 = \langle 1, 2, -14 \rangle$ will be normal to the second plane.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2
If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the first component of each vector have the same sign and the same can be said for the third component. However, the second component of each vector has opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the first and third component the same and simultaneously changing the sign on the second component. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \bullet \vec{n}_2 = 0$$

The dot product is zero and so the two normal vectors are orthogonal. Therefore, **the two planes are orthogonal**.

---

5. Determine if the plane given by $-3x + 2y + 7z = 9$ and the plane containing the points $\left(-2,6,1\right)$, $\left(-2,5,0\right)$ and $\left(-1,4,-3\right)$ are parallel, orthogonal or neither.

Step 1
Let's start off this problem by noticing that the vector $\vec{n}_1 = \left\langle -3,2,7\right\rangle$ will be normal to the first plane and it would be nice to have a normal vector for the second plane.

We know (Problem 1 from this section!) how to determine a normal vector given three points in the plane. Here is that work.

$$P = \left(-2,6,1\right) \qquad Q = \left(-2,5,0\right) \qquad R = \left(-1,4,-3\right)$$

$$\overrightarrow{QP} = \left\langle 0,1,1\right\rangle \qquad \overrightarrow{RQ} = \left\langle -1,1,3\right\rangle$$

$$\vec{n}_2 = \overrightarrow{QP} \times \overrightarrow{RQ} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ 0 & 1 \\ -1 & 1 \end{array} = 3\vec{i} - \vec{j} - \vec{i} - \left(-\vec{k}\right) = 2\vec{i} - \vec{j} + \vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2
If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the third component of each vector have the same sign while the first and second components each have opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the third component the same and simultaneously changing the sign on the first and second components. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \bullet \vec{n}_2 = -1$$

The dot product is not zero and so the two normal vectors are not orthogonal. Therefore, **the two planes are not orthogonal**.

---

6. Determine if the line given by $\vec{r}\left(t\right) = \left\langle -2t, 2+7t, -1-4t \right\rangle$ intersects the plane given by $4x + 9y - 2z = -8$ or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of *t* such that if we plug that *t* into the equation of the line we'd get a point that lies on the plane. We also know that if a point $\left(x, y, z\right)$ is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

$$\left(-2t, 2+7t, -1-4t\right)$$

for all values of *t*.

Step 3
So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$4\left(-2t\right)+9\left(2+7t\right)-2\left(-1-4t\right)=-8$$

Step 4
Let's solve this for *t* as follows,

$$63t+20=-8 \qquad \rightarrow \qquad t=-\tfrac{4}{9}$$

Step 5
We were able to find a *t* from this equation. What that means is that this is the value of *t* that, once we plug into the equation of the line, gives the point of intersection of the line and plane.

So, **the line and plane do intersect** and they will intersect at the point $\left(\frac{8}{9},-\frac{10}{9},\frac{7}{9}\right)$.

Note that all we did to get the point is plug $t=-\frac{4}{9}$ into the general form for points on the line we wrote down in Step 2.

---

7. Determine if the line given by $\vec{r}\left(t\right)=\left\langle 4+t,-1+8t,3+2t\right\rangle$ intersects the plane given by $2x-y+3z=15$ or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of *t* such that if we plug that *t* into the equation of the line we'd get a point that lies on the plane. We also know that if a point $\left(x,y,z\right)$ is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

$$\left(4+t,-1+8t,3+2t\right)$$

for all values of *t*.

Step 3
So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$2\left(4+t\right)-\left(-1+8t\right)+3\left(3+2t\right)=15$$

Step 4
Let's solve this for *t* as follows,

$$18 = 15 \ ??$$

Step 5
Hmmm…

So, either we've just managed to prove that 18 and 15 are in fact the same number or there is something else going on here.

Clearly 18 and 15 are not the same number and so something else must be going on. In fact, all this means is that there is no *t* that will satisfy the equation we wrote down in Step 3. This in turn means that **the line and plane do not intersect.**

---

8. Find the line of intersection of the plane given by $3x + 6y - 5z = -3$ and the plane given by $-2x + 7y - z = 24$.

Step 1
Okay, we know that we need a point and vector parallel to the line in order to write down the equation of the line. In this case neither has been given to us.

First let's note that any point on the line of intersection must also therefore be in both planes and it's actually pretty simple to find such a point. Whatever our line of intersection is it must intersect at least one of the coordinate planes. It doesn't have to intersect all three of the coordinate planes but it will have to intersect at least one.

So, let's see if it intersects the *xy*-plane. Because the point on the intersection line must also be in both planes let's set $z = 0$ (so we'll be in the *xy*-plane!) in both of the equations of our planes.

Doing this gives,

$$\begin{aligned} 3x + 6y &= -3 \\ -2x + 7y &= 24 \end{aligned}$$

Step 2
This is a simple system to solve so we'll leave it to you to verify that the solution is,

$$x = -5 \qquad y = 2$$

The fact that we were able to find a solution to the system from Step 1 means that the line of intersection does in fact intersect the *xy*-plane and it does so at the point $\left(-5, 2, 0\right)$. This is also then a point on the line of intersection.

Note that if the system from Step 1 didn't have a solution then the line of intersection would not have intersected the *xy*-plane and we'd need to try one of the remaining coordinate planes.

Step 3
Okay, now we need a vector that is parallel to the line of intersection. This might be a little hard to visualize, but if you think about it the line of intersection would have to be orthogonal to both of the normal vectors from the two planes. This in turn means that any vector orthogonal to the two normal vectors must then be parallel to the line of intersection.

Nicely enough we know that the cross product of any two vectors will be orthogonal to each of the two vectors. So, here are the two normal vectors for our planes and their cross product.

$$\vec{n}_1 = \left\langle 3,6,-5 \right\rangle \qquad\qquad \vec{n}_2 = \left\langle -2,7,-1 \right\rangle$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & -5 \\ -2 & 7 & -1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 3 & 6 \\ -2 & 7 \end{matrix}$$

$$= -6\vec{i} + 10\vec{j} + 21\vec{k} - \left(-3\vec{j}\right) - \left(-35\vec{i}\right) - \left(-12\vec{k}\right) = 29\vec{i} + 13\vec{j} + 33\vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Step 4
So, we now have enough information to write down the equation of the line of intersection of the two planes. The equation is,

$$\boxed{\vec{r}\left(t\right) = \left\langle -5,2,0 \right\rangle + t\left\langle 29,13,33 \right\rangle = \left\langle -5+29t, 2+13t, 33t \right\rangle}$$

---

9. Determine if the line given by $x = 8 - 15t$, $y = 9t$, $z = 5 + 18t$ and the plane given by $10x - 6y - 12z = 7$ are parallel, orthogonal or neither.

Step 1
Let's start off this problem by noticing that the vector $\vec{v} = \left\langle -15,9,18 \right\rangle$ will be parallel to the line and the vector $\vec{n} = \left\langle 10,-6,-12 \right\rangle$ will be normal to the plane.

Now try to visualize the line and plane and their corresponding vectors. What would the line and plane look like if the two vectors were orthogonal to each other? What would the line and plane look like if the two vectors were parallel to each other?

Step 2

If the two vectors are orthogonal to each other the line would be parallel to the plane. If you think about this it does make sense. If $\vec{v}$ is orthogonal to $\vec{n}$ then it must be parallel to the plane because $\vec{n}$ is orthogonal to the plane. Then because the line is parallel to $\vec{v}$ it must also be parallel to the plane.

So, let's do a quick dot product here.

$$\vec{v} \bullet \vec{n} = -420$$

The dot product is not zero and so the two vectors aren't orthogonal to each other. Therefore, the **line and plane are not parallel**.

Step 3
If the two vectors are parallel to each other the line would be orthogonal to the plane. If you think about this it does make sense. The line is parallel to $\vec{v}$ which we've just assumed is parallel to $\vec{n}$. We also know that $\vec{n}$ is orthogonal to the plane and so anything that is parallel to $\vec{n}$ (the line for instance) must also be orthogonal to the plane.

In this case it looks like we have the following relationship between the two vectors.

$$\vec{v} = -\tfrac{3}{2}\vec{n}$$

The two vectors are parallel and so the **line and plane are orthogonal**.

Section 1-4 : Quadric Surfaces

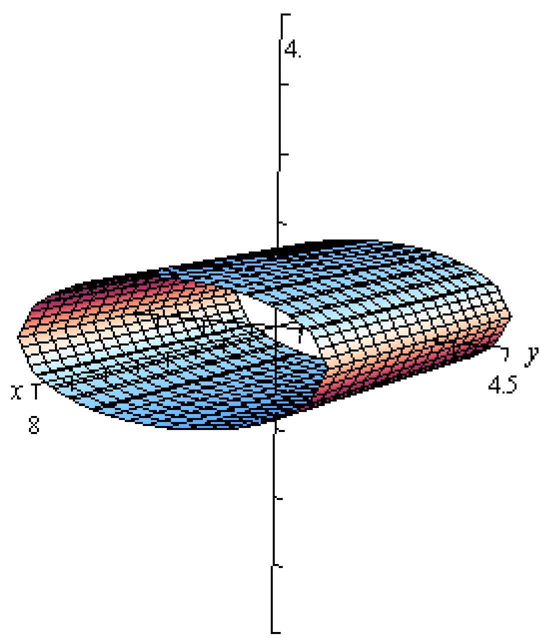
1. Sketch the following quadric surface.

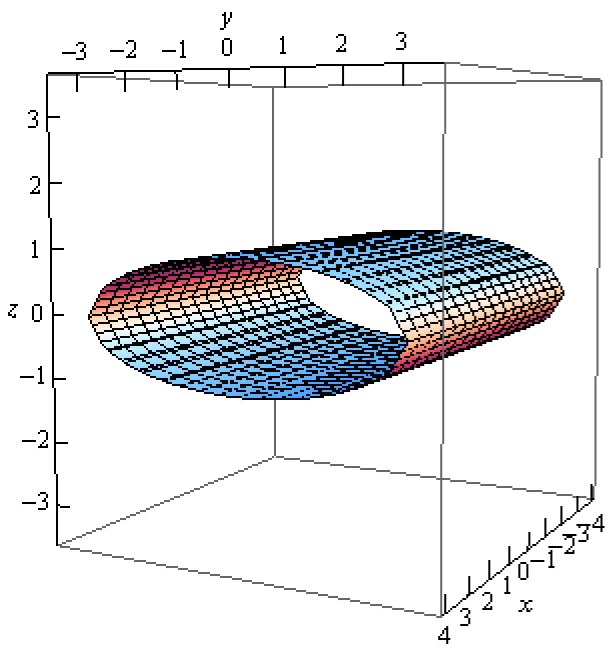
$$\frac{y^2}{9} + z^2 = 1$$

Solution
This is a cylinder that is centered on the *x*-axis. The cross sections of the cylinder will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.



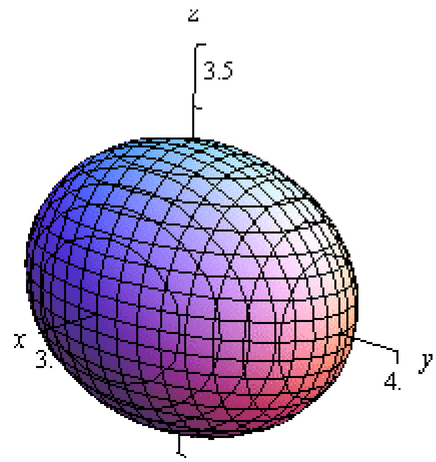


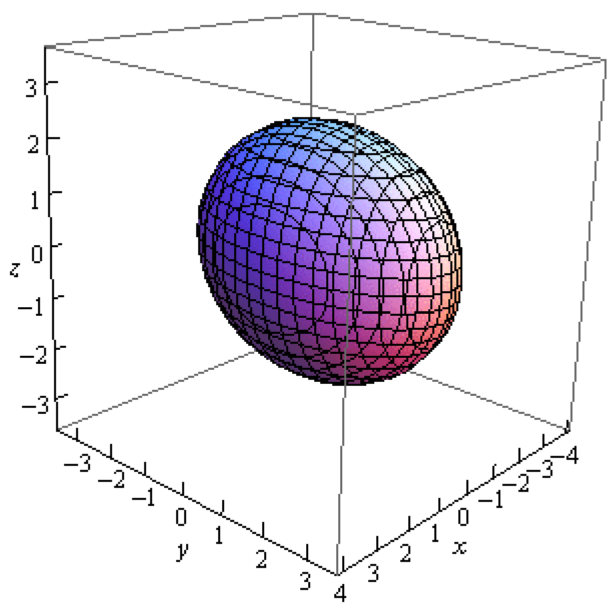
2. Sketch the following quadric surface.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{6} = 1$$

Solution
This is an ellipsoid and because the numbers in the denominators of each of the terms are not the same we know that it won't be a sphere.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





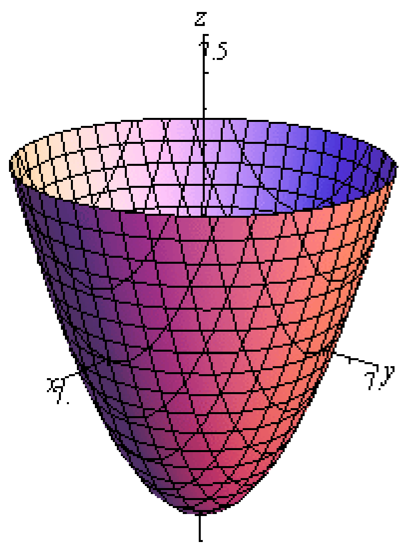
3. Sketch the following quadric surface.

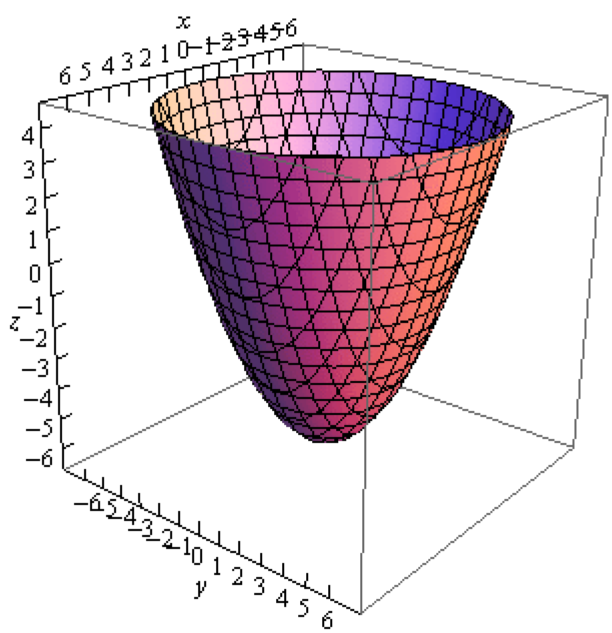
$$z = \frac{x^2}{4} + \frac{y^2}{4} - 6$$

Solution

This is an elliptic paraboloid that is centered on the *z*-axis. Because the *x* and *y* terms are positive we know that it will open upwards. The "-6" tells us that the surface will start at $z = -6$. We can also say that because the coefficients of the *x* and *y* terms are identical the cross sections of the surface will be circles.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





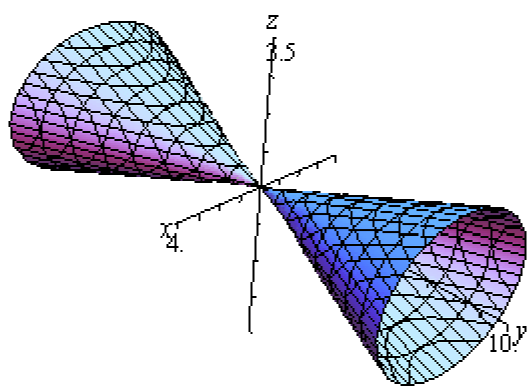
4. Sketch the following quadric surface.

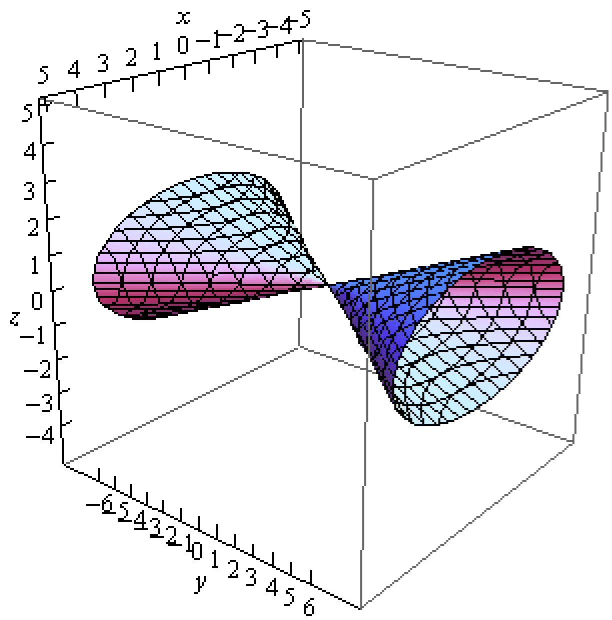
$$y^2 = 4x^2 + 16z^2$$

Solution
This is a cone that is centered on the *y*-axis and because the coefficients of the *x* and *z* terms are different the cross sections of the surface will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.





---

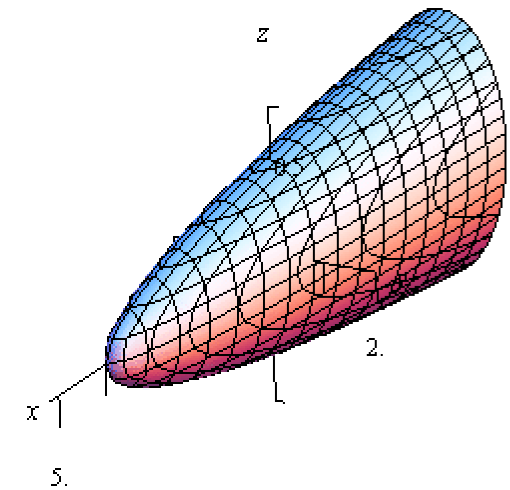
5. Sketch the following quadric surface.

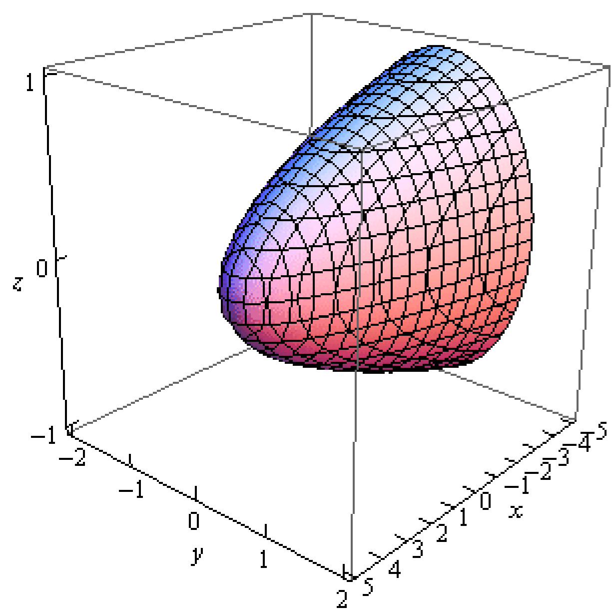
$$x = 4 - 5y^2 - 9z^2$$

Solution
This is an elliptic paraboloid that is centered on the *x*-axis. Because the *y* and *z* terms are negative we know that it will open in the negative *x* direction. The "4" tells us that the surface will start at $x = 4$. We can also say that because the coefficients of the *y* and *z* terms are different the cross sections of the surface will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.


z
2.
x
5.


1
z 0
−1
−2
−1
0
y
1
2
5
4
3
2
1
0
−1
−2
−3
−4
−5
x

Section 1-5 : Functions of Several Variables

1. Find the domain of the following function.

$$f(x,y) = \sqrt{x^2 - 2y}$$

Solution

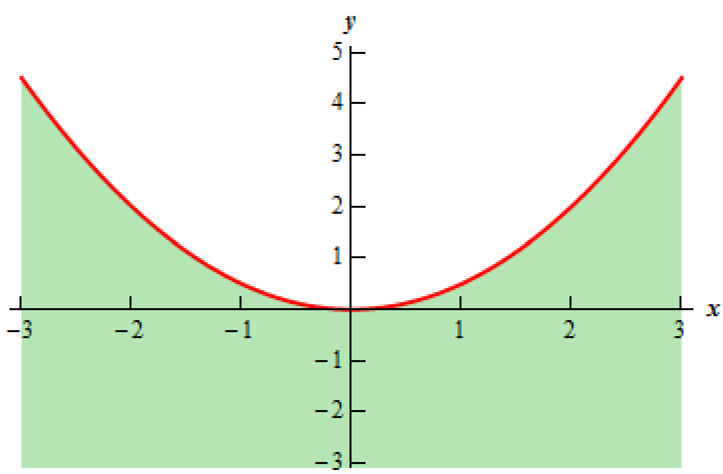
There really isn't all that much to this problem. We know that we can't have negative numbers under the square root and so the we'll need to require that whatever (x,y) is it will need to satisfy,

$$x^2 - 2y \ge 0$$

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$x^2 \ge 2y \qquad \Rightarrow \qquad y \le \frac{1}{2}x^2$$

So, it looks like we need to be on or below the parabola above. The domain is illustrated by the green area and red line in the sketch below.



---

2. Find the domain of the following function.

$$f(x,y) = \ln(2x - 3y + 1)$$

Solution

There really isn't all that much to this problem. We know that we can't have negative numbers or zero in a logarithm so we'll need to require that whatever (x,y) is it will need to satisfy,

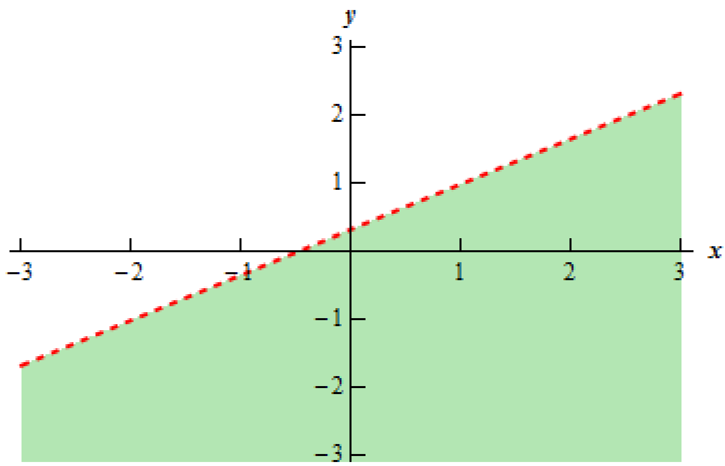
$$2x - 3y + 1 > 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$2x + 1 > 3y \qquad \Rightarrow \qquad y < \frac{2}{3}x + \frac{1}{3}$$

So, it looks like we need to be below the line above. The domain is illustrated by the green area in the sketch below.



Note that we dashed the graph of the "bounding" line to illustrate that we don't take points from the line itself.

3. Find the domain of the following function.

$$f(x, y, z) = \frac{1}{x^2 + y^2 + 4z}$$

Solution

There really isn't all that much to this problem. We know that we can't have division by zero so we'll need to require that whatever (x, y, z) is it will need to satisfy,

$$x^2 + y^2 + 4z \neq 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to identify the domain a little better.

$$4z \neq -x^2 - y^2 \qquad \Rightarrow \qquad z \neq -\frac{x^2}{4} - \frac{y^2}{4}$$

So, it looks like we need to avoid points, (x, y, z), that are on the elliptic paraboloid given by the equation above.

---

4. Find the domain of the following function.

$$f(x,y) = \frac{1}{x} + \sqrt{y+4} - \sqrt{x+1}$$

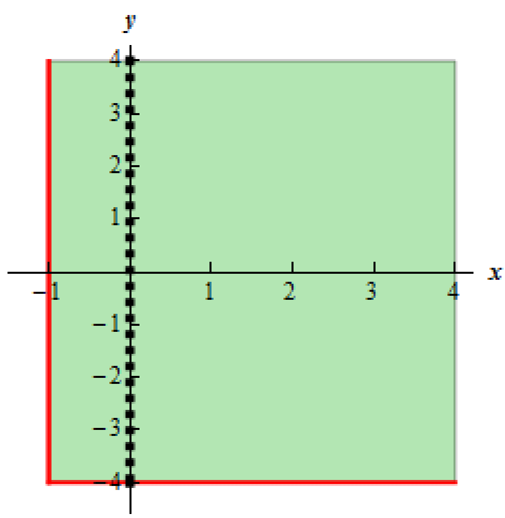
Solution

There really isn't all that much to this problem. We know that we can't have division by zero and we can't take square roots of negative numbers and so we'll need to require that whatever (x, y) is it will need to satisfy the following three conditions.

$$x \geq -1, \quad x \neq 0 \qquad\qquad y \geq -4$$

This is also our domain since these are the only conditions require in order for the function to exist.

A sketch of the domain is shown below. We can take any point in the green area or on the red lines with the exception of the *y*-axis (*i.e.* $x \neq 0$) as indicated by the black dashes on the *y*-axis.



---

5. Identify and sketch the level curves (or contours) for the following function.

$$2x - 3y + z^2 = 1$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$2x - 3y + k^2 = 1$$

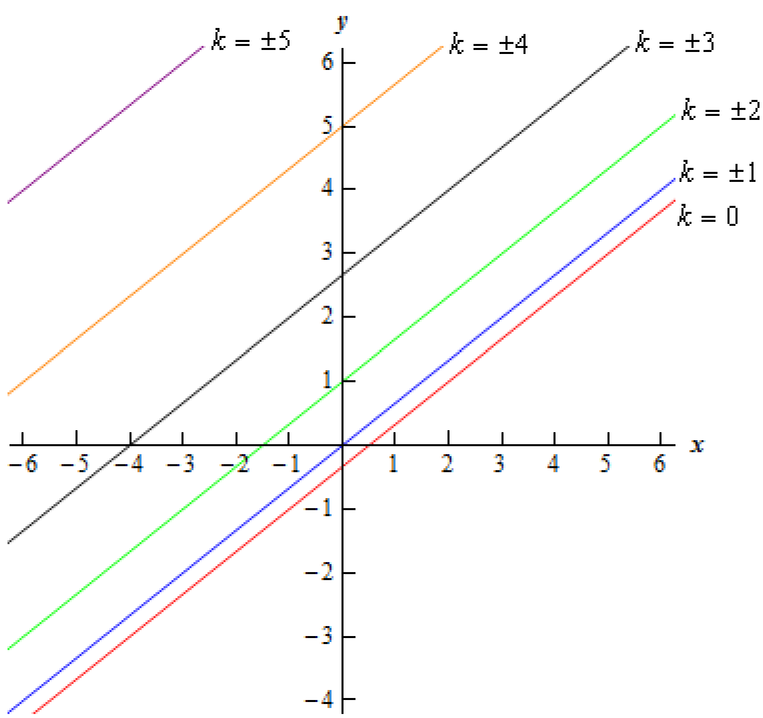
Step 2
A quick rewrite of the equation from the previous step gives us,

$$y = \frac{2}{3}x + \frac{k^2 - 1}{3}$$

So, the level curves for this function will be lines with slope $\frac{2}{3}$ and a *y*-intercept of $\left(0, \frac{k^2-1}{3}\right)$.

Note as well that there will be no restrictions on the values of *k* that we can use, as there sometimes are. Also note that the sign of *k* will not matter so, with the exception of the level curve for $k = 0$, each level curve will in fact arise from two different values of *k*.

Step 3
Below is a sketch of some level curves for some values of *k* for this function.



6. Identify and sketch the level curves (or contours) for the following function.

$$4z + 2y^2 - x = 0$$

Step 1

We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$4k + 2y^2 - x = 0$$

Step 2

A quick rewrite of the equation from the previous step gives us,

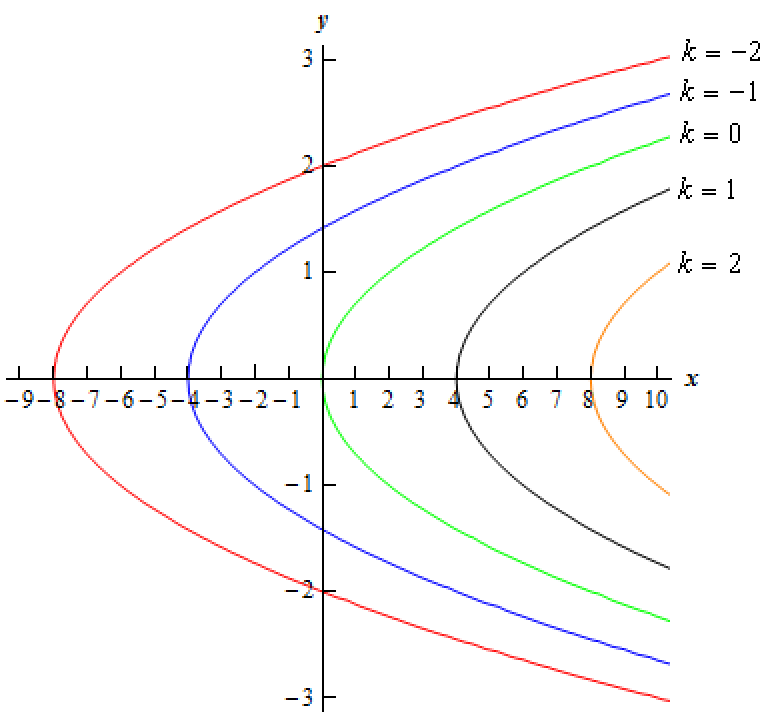
$$x = 2y^2 + 4k$$

So, the level curves for this function will be parabolas opening to the right and starting at 4*k*.

Note as well that there will be no restrictions on the values of *k* that we can use, as there sometimes are.

Step 3

Below is a sketch of some level curves for some values of *k* for this function.



---

7. Identify and sketch the level curves (or contours) for the following function.

$$y^2 = 2x^2 + z$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$y^2 = 2x^2 + k$$

Step 2
For this problem the value of *k* will affect the type of graph of the level curve.

First, if $k = 0$ the equation will be,

$$y^2 = 2x^2 \qquad \Rightarrow \qquad y = \pm\sqrt{2}\ x$$

So, in this case the level curve (actually curves if you think about it) will be two lines through the origin. One is increasing and the other is decreasing.

Next, let's take a look at what we get if $k > 0$. In this case a quick rewrite of the equation from Step 2 gives,

$$\frac{y^2}{k} - \frac{2x^2}{k} = 1$$

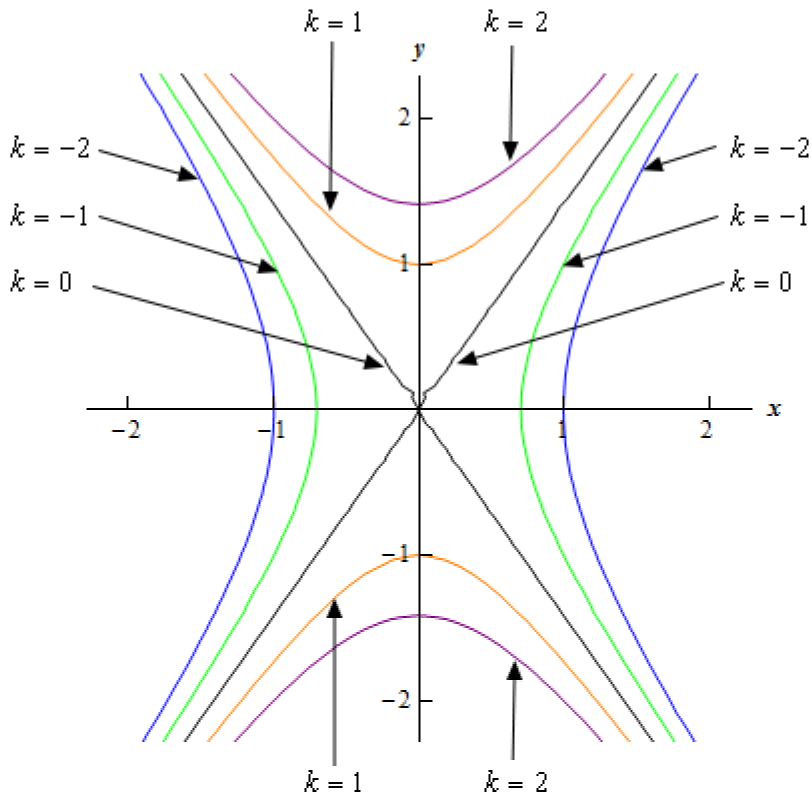
Because we know that *k* is positive we see that we have a hyperbola with the *y* term the positive term and the *x* term the negative term. This means that the hyperbola will be symmetric about the *y*-axis and opens up and down.

Finally, what do we get if $k < 0$. In this case the equation is,

$$-\frac{2x^2}{k} + \frac{y^2}{k} = 1$$

Now, be careful with this equation. In this case we have negative values of *k*. This means that the *x* term is in fact positive (the minus sign will cancel against the minus sign in the *k*). Likewise, the *y* term will be negative (it's got a negative *k* in the denominator). Therefore, we'll have a hyperbola that is symmetric about the *x*-axis and opens right and left.

Step 3
Below is a sketch of some level curves for some values of *k* for this function.



---

8. Identify and sketch the traces for the following function.

$$2x-3y+z^2=1$$

Step 1

We have two traces. One we get by plugging $x=a$ into the equation and the other we get by plugging $y=b$ into the equation. Here is what we get for each of these.

$$\begin{aligned} x=a &\quad : \quad 2a-3y+z^2=1 \quad \rightarrow \quad y=\tfrac{1}{3}z^2+\tfrac{2a-1}{3} \\ y=b &\quad : \quad 2x-3b+z^2=1 \quad \rightarrow \quad x=-\tfrac{1}{2}z^2+\tfrac{3b+1}{2} \end{aligned}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of *x* and *y*. Instead we have one equation in terms of *x* and *z* and another in terms of *y* and *z*.

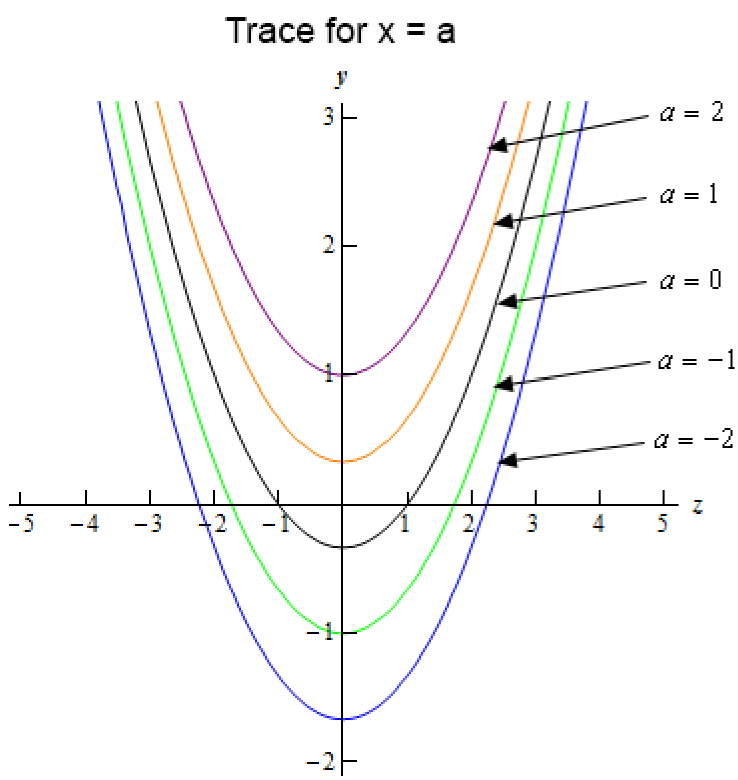
Do not get excited about this! They work the same way that equations in terms of *x* and *y* work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

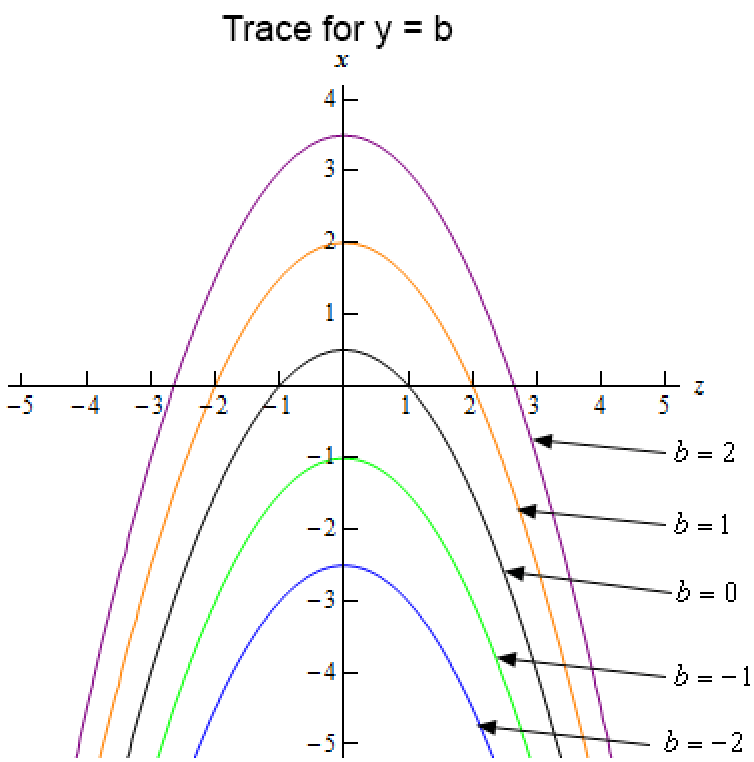
Just because we have an *x* doesn't mean that it must be the horizontal axis and just because we have a *y* doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a *z* in them and it is squared we'll let *z* be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens upwards with vertex at $\left(0, \frac{2a-1}{3}\right)$ and for the $y = b$ trace we'll have a parabola that opens downwards with vertex at $\left(0, \frac{3b+1}{2}\right)$.

Step 3
Below is a sketch for each of the traces.





---

9. Identify and sketch the traces for the following function.

$$4z + 2y^2 - x = 0$$

Step 1

We have two traces. One we get by plugging $x = a$ into the equation and the other we get by plugging $y = b$ into the equation. Here is what we get for each of these.

$$\begin{aligned} x = a \quad &: \quad 4z + 2y^2 - a = 0 \quad \rightarrow \quad z = -\tfrac{1}{2}y^2 + \tfrac{a}{4} \\ y = b \quad &: \quad 4z + 2b^2 - x = 0 \quad \rightarrow \quad x = 4z + 2b^2 \end{aligned}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of *x* and *y*. Instead we have one equation in terms of *x* and *z* and another in terms of *y* and *z*.

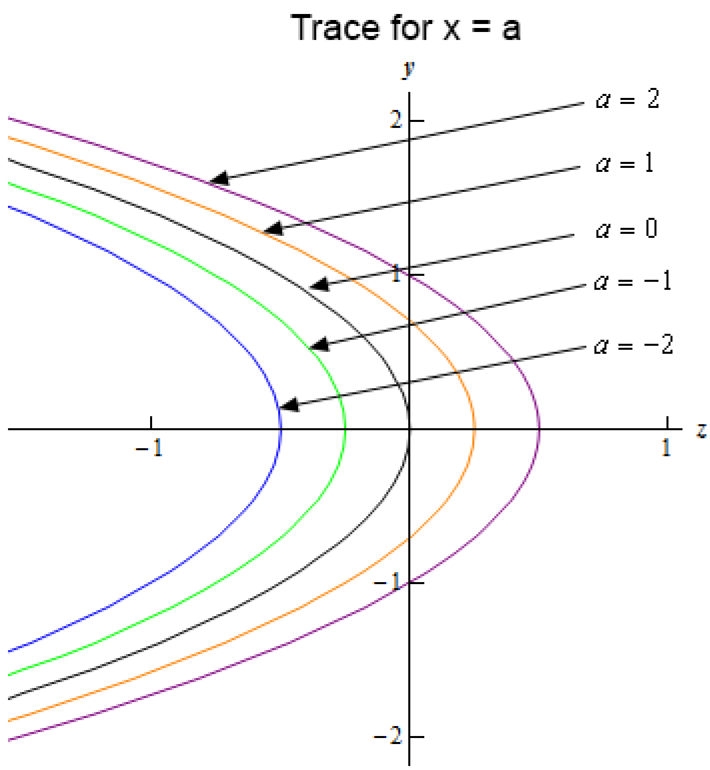
Do not get excited about this! They work the same way that equations in terms of *x* and *y* work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

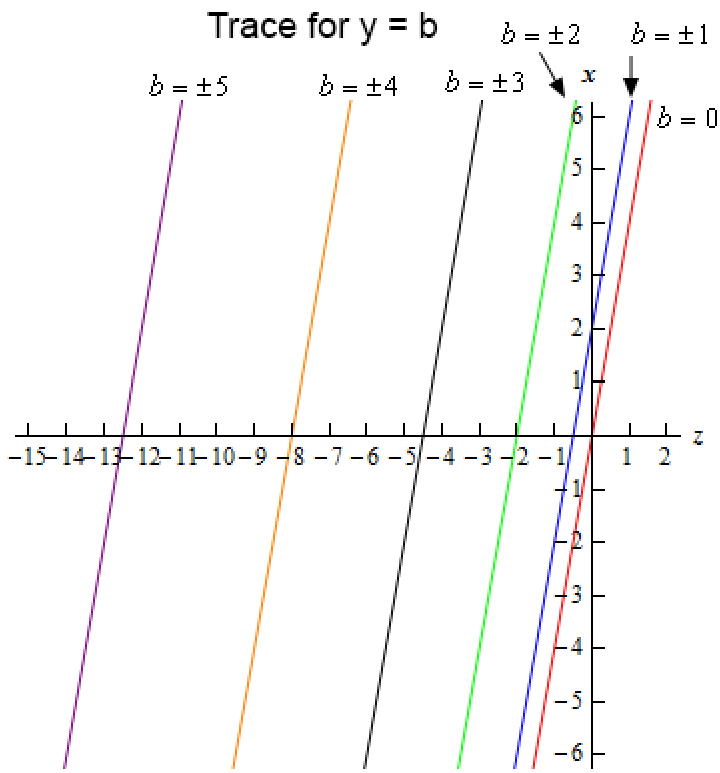
Just because we have an *x* doesn't mean that it must be the horizontal axis and just because we have a *y* doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a *z* in them we'll let *z* be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens to the left with vertex at $\left(\frac{a}{4}, 0\right)$ and for the $y = b$ trace we'll have a line with slope of 4 and an *x*-intercept at $\left(0, 2b^2\right)$.

Step 3
Below is a sketch for each of the traces.





Trace for y = b
b = ±2
b = ±1
b = ±5
b = ±4
b = ±3
b = 0
x
z

Section 1-6 : Vector Functions

1. Find the domain for the vector function : $\vec r\left(t \right) = \left\langle {t^2} + 1,\frac{1}{t + 2},\sqrt {t + 4} \right\rangle$

Step 1
The domain of the vector function is simply the largest possible set of *t*'s that we can use in all the components of the vector function.

The first component will exist for all values of *t* and so we won't exclude any values of *t* from that component.

The second component clearly requires us to avoid $t = - 2$ so we don't have division by zero in that component.

We'll also need to require that $t \ge - 4$ so avoid taking the square root of negative numbers in the third component.

Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{t \ge - 4,\quad t \ne - 2}$$

Note that we can't forget to add the $t \ne - 2$ onto this since -2 is larger than -4 and would be included otherwise!

---

2. Find the domain for the vector function : $\vec r\left(t \right) = \left\langle \ln \left(4 - {t^2} \right),\sqrt {t + 1} \right\rangle$

Step 1
The domain of the vector function is simply the largest possible set of *t*'s that we can use in all the components of the vector function.

We know that we can't take logarithms of negative values or zero and so from the first term we know that we'll need to require that,

$$4 - {t^2} > 0\qquad \to \qquad - 2 < t < 2$$

We'll also need to require that $t \ge - 1$ so avoid taking the square root of negative numbers in the second component.

Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{-1 \le t < 2}$$

Remember that we want the largest possible set of *t*'s for which all the components will exist. So we can't take values of $-2 < t < -1$ because even though those are okay in the first component but they aren't in the second component. Likewise, even though we can include $t \ge 2$ in the second component we can't plug them into the first component and so we can't include them in the domain of the function.

---

3. Sketch the graph of the vector function : $\vec{r}\left(t\right) = \left\langle 4t, 10 - 2t \right\rangle$

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of *t* that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$\begin{aligned} x &= 4t \\ y &= 10 - 2t \end{aligned}$$

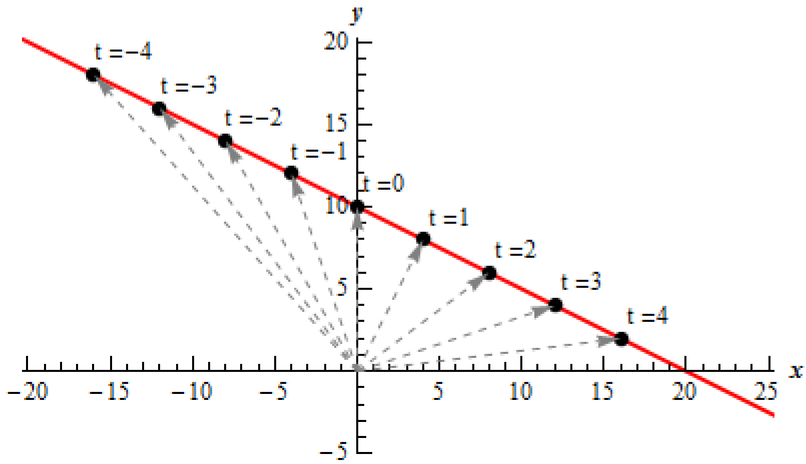
Step 2
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only *x* and *y* that will have the same graph as the vector function.

We can do this as follows,

$$x = 4t \quad \to \quad t = \tfrac{1}{4}x \quad \to \quad y = 10 - 2\left(\tfrac{1}{4}x\right) = 10 - \tfrac{1}{2}x$$

So, it looks like the graph of the vector function will be a line with slope $-\frac{1}{2}$ and *y*-intercept of $\left(0,10\right)$.

Step 3
A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of t's just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only x and y).

---

4. Sketch the graph of the vector function : $\vec{r}\left(t\right)=\left\langle t+1,\frac{1}{4}t^{2}+3\right\rangle$

Step 1

One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of t that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$x=t+1$$
$$y=\frac{1}{4}t^{2}+3$$

Step 2

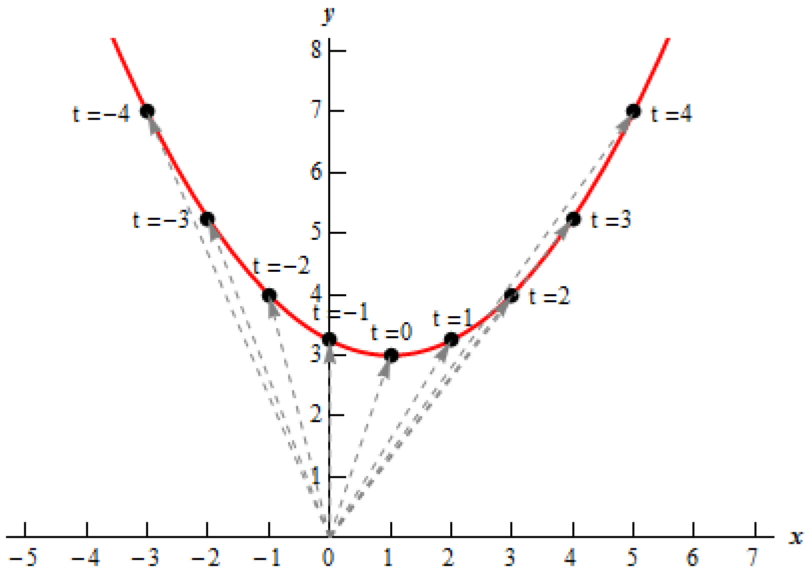
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only x and y that will have the same graph as the vector function.

We can do this as follows,

$$x=t+1 \quad\to\quad t=x-1 \quad\to\quad y=\frac{1}{4}\left(x-1\right)^{2}+3$$

So, it looks like the graph of the vector function will be a parabola with vertex $(1,3)$ and opening upwards.

Step 3
A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of *t*'s just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only *x* and *y*).

---

5. Sketch the graph of the vector function : $\vec{r}(t) = \langle 4\sin(t), 8\cos(t)\rangle$

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

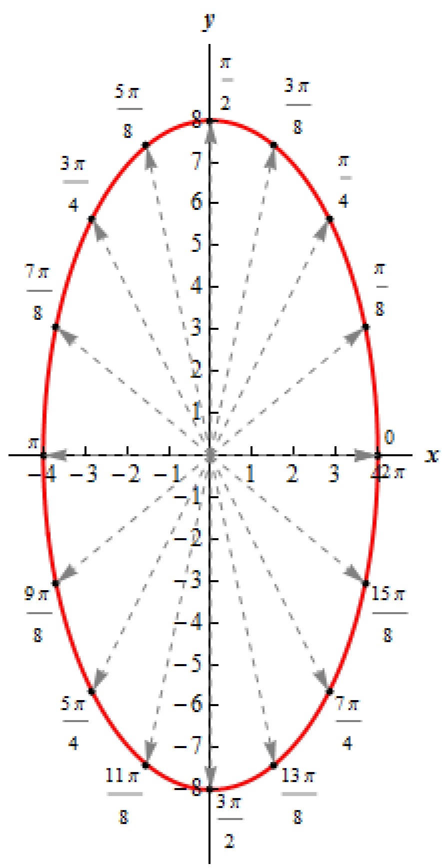
This will work provided we pick the "correct" values of *t* that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$x = 4\sin(t)$$
$$y = 8\cos(t)$$

Step 2
Now, recall from our look at parametric equations we now know that this will be the graph of an ellipse centered at the origin with right/left points a distance of 4 from the origin and top/bottom points a distance of 8 from the origin.

Step 3
A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of t's just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only *x* and *y*).

---

6. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 3\cos(6t), -4, \sin(6t) \rangle$$

Step 1
To identify the graph of this vector function (without graphing) let's first write down the set of parametric equations we get from this vector function. They are,

$$\begin{aligned} x &= 3\cos(6t) \\ y &= -4 \\ z &= \sin(6t) \end{aligned}$$

Step 2
Now, from the *x* and *z* equations we can see that we have an ellipse in the *xz*-plane that is given by the following equation.

$$\frac{x^2}{9} + z^2 = 1$$

From the *y* equation we know that this ellipse will not actually be in the *xz*-plane but parallel to the *xz*-plane at $y = -4$.

---

7. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 2 - t, 4 + 7t, -1 - 3t \rangle$$

Solution
There really isn't a lot to do with this problem. The equation should look very familiar to you. We saw quite a few of these types of equations in the Equations of Lines and Equations of Planes sections.

From those sections we know that the graph of this equation is a line in $\mathbb{R}^3$ that goes through the point $(2, 4, -1)$ and parallel to the vector $\vec{v} = \langle -1, 7, -3 \rangle$.

---

8. Write down the equation of the line segment starting at $(1, 3)$ and ending at $(-4, 6)$.

Solution
There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1 - t)\langle 1, 3 \rangle + t\langle -4, 6 \rangle \quad 0 \le t \le 1}$$

Don't forget the limits on *t*! Without that you have the full line that goes through those two points instead of the line segment from $(1,3)$ to $(-4,6)$.

---

9. Write down the equation of the line segment starting at $(0,2,-1)$ and ending at $(7,-9,2)$.

Solution
There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1-t)\langle 0,2,-1\rangle + t\langle 7,-9,2\rangle \quad 0 \le t \le 1}$$

Don't forget the limits on *t*! Without that you have the full line that goes through those two points instead of the line segment from $(0,2,-1)$ to $(7,-9,2)$.

Section 1-7 : Calculus with Vector Functions

1. Evaluate the following limit.

$$\lim_{t\to 1}\left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1}\right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t\to 1}\left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1}\right\rangle &= \left\langle \lim_{t\to 1}\mathbf{e}^{t-1}, \lim_{t\to 1}4t, \lim_{t\to 1}\frac{t-1}{t^2-1}\right\rangle \\ &= \left\langle \lim_{t\to 1}\mathbf{e}^{t-1}, \lim_{t\to 1}4t, \lim_{t\to 1}\frac{1}{2t}\right\rangle = \left\langle \mathbf{e}^{0}, 4, \frac{1}{2}\right\rangle = \boxed{\left\langle 1, 4, \frac{1}{2}\right\rangle}\end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the third term.

---

2. Evaluate the following limit.

$$\lim_{t\to -2}\left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\vec{j}+\left(t^2+6t\right)\vec{k}\right)$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t\to -2}\left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\vec{j}+\left(t^2+6t\right)\vec{k}\right) &= \lim_{t\to -2}\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\lim_{t\to -2}\vec{j}+\lim_{t\to -2}\left(t^2+6t\right)\vec{k} \\ &= \lim_{t\to -2}\frac{-\mathbf{e}^{t+2}}{2t+1}\vec{i}+\lim_{t\to -2}\vec{j}+\lim_{t\to -2}\left(t^2+6t\right)\vec{k} = \boxed{\frac{1}{3}\vec{i}+\vec{j}-8\vec{k}}\end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the first term.

---

3. Evaluate the following limit.

$$\lim_{t \to \infty} \left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, \mathbf{e}^{-t} \right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t \to \infty} \left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, \mathbf{e}^{-t} \right\rangle &= \left\langle \lim_{t \to \infty} \frac{1}{t^2}, \lim_{t \to \infty} \frac{2t^2}{1-t-t^2}, \lim_{t \to \infty} \mathbf{e}^{-t} \right\rangle \\ &= \left\langle \lim_{t \to \infty} \frac{1}{t^2}, \lim_{t \to \infty} \frac{2t^2}{t^2\left(\frac{1}{t^2}-\frac{1}{t}-1\right)}, \lim_{t \to \infty} \mathbf{e}^{-t} \right\rangle = \boxed{\left\langle 0, -2, 0 \right\rangle}\end{aligned}$$

Don't forget your basic limit at infinity processes/facts.

---

4. Compute the derivative of the following limit.

$$\vec{r}\left(t\right) = \left(t^3 - 1\right)\vec{i} + \mathbf{e}^{2t}\,\vec{j} + \cos\left(t\right)\vec{k}$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}'\left(t\right) = 3t^2\vec{i} + 2\mathbf{e}^{2t}\,\vec{j} - \sin\left(t\right)\vec{k}}$$

---

5. Compute the derivative of the following limit.

$$\vec{r}\left(t\right) = \left\langle \ln\left(t^2+1\right), t\mathbf{e}^{-t}, 4 \right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}'\left(t\right) = \left\langle \frac{2t}{t^2+1}, \mathbf{e}^{-t} - t\mathbf{e}^{-t}, 0 \right\rangle}$$

Make sure you haven't forgotten your basic differentiation formulas such as the chain rule (the first term) and the product rule (the second term).

---

6. Compute the derivative of the following limit.

$$\vec{r}\left(t\right)=\left\langle \frac{t+1}{t-1},\tan\left(4t\right),\sin^2\left(t\right)\right\rangle$$

Solution
There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\begin{aligned}\vec{r}'\left(t\right)&=\left\langle \frac{\left(1\right)\left(t-1\right)-\left(t+1\right)\left(1\right)}{\left(t-1\right)^2},4\sec^2\left(4t\right),2\sin\left(t\right)\cos\left(t\right)\right\rangle\\&=\boxed{\left\langle \frac{-2}{\left(t-1\right)^2},4\sec^2\left(4t\right),2\sin\left(t\right)\cos\left(t\right)\right\rangle}\end{aligned}$$

Make sure you haven't forgotten your basic differentiation formulas such as the quotient rule (the first term) and the chain rule (the third term).

---

7. Evaluate $\int\vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=t^3\,\vec{i}-\frac{2t}{t^2+1}\vec{j}+\cos^2\left(3t\right)\vec{k}$.

Solution
There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int\vec{r}\left(t\right)dt&=\int t^3\,dt\,\vec{i}-\int\frac{2t}{t^2+1}dt\,\vec{j}+\int\cos^2\left(3t\right)dt\,\vec{k}\\&=\int t^3\,dt\,\vec{i}-\int\frac{2t}{t^2+1}dt\,\vec{j}+\int\tfrac{1}{2}\left(1+\cos\left(6t\right)\right)dt\,\vec{k}\\&=\boxed{\tfrac{1}{4}t^4\,\vec{i}-\ln\left|t^2+1\right|\vec{j}+\tfrac{1}{2}\left(t+\tfrac{1}{6}\sin\left(6t\right)\right)\vec{k}+\vec{c}}\end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (second term) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (third term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

8. Evaluate $\int_{-1}^{2} \vec r\left(t\right)dt$ where $\vec r\left(t\right) = \left\langle 6, 6t^2 - 4t, t\mathbf{e}^{2t}\right\rangle$

Solution
There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec r\left(t\right)dt &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \int t\mathbf{e}^{2t}\,dt\right\rangle\\ &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{2}\int \mathbf{e}^{2t}\,dt\right\rangle = \left\langle 6t, 2t^3 - 2t^2, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{4}\mathbf{e}^{2t}\right\rangle\end{aligned}$$

Don't forget the basic integration rules such integration by parts (third term). Also recall that one way to do definite integral is to first do the indefinite integral and then do the evaluation.

The answer for this problem is then,

$$\begin{aligned}\int_{-1}^{2} \vec r\left(t\right)dt &= \left.\left\langle 6t, 2t^3 - 2t^2, \tfrac{1}{2}t\mathbf{e}^{2t} - \tfrac{1}{4}\mathbf{e}^{2t}\right\rangle\right|_{-1}^{2}\\ &= \left\langle 12, 8, \tfrac{3}{4}\mathbf{e}^{4}\right\rangle - \left\langle -6, -4, -\tfrac{3}{4}\mathbf{e}^{-2}\right\rangle = \boxed{\left\langle 18, 12, \tfrac{3}{4}\left(\mathbf{e}^{4} + \mathbf{e}^{-2}\right)\right\rangle}\end{aligned}$$

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

---

9. Evaluate $\int \vec r\left(t\right)dt$, where $\vec r\left(t\right) = \left\langle \left(1-t\right)\cos\left(t^2 - 2t\right), \cos\left(t\right)\sin\left(t\right), \sec^2\left(4t\right)\right\rangle$.

Solution
There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec r\left(t\right)dt &= \left\langle \int \left(1-t\right)\cos\left(t^2 - 2t\right)dt, \int \cos\left(t\right)\sin\left(t\right)dt, \int \sec^2\left(4t\right)dt\right\rangle\\ &= \left\langle \int \left(1-t\right)\cos\left(t^2 - 2t\right)dt, \int \tfrac{1}{2}\sin\left(2t\right)dt, \int \sec^2\left(4t\right)dt\right\rangle\\ &= \boxed{\left\langle -\tfrac{1}{2}\sin\left(t^2 - 2t\right), -\tfrac{1}{4}\cos\left(2t\right), \tfrac{1}{4}\tan\left(4t\right)\right\rangle + \vec c}\end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (all terms) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (second term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

Section 1-8 : Tangent, Normal and Binormal Vectors

1. Find the unit tangent vector for the vector function : $\vec r\left(t\right) = \left\langle t^2 + 1, 3 - t, t^3 \right\rangle$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec r'\left(t\right) = \left\langle 2t, -1, 3t^2 \right\rangle$$

$$\left\| \vec r'\left(t\right) \right\| = \sqrt{\left(2t\right)^2 + \left(-1\right)^2 + \left(3t^2\right)^2} = \sqrt{1 + 4t^2 + 9t^4}$$

Step 2

The unit tangent vector for this vector function is then,

$$\vec T\left(t\right) = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}}\left\langle 2t, -1, 3t^2 \right\rangle = \boxed{\left\langle \frac{2t}{\sqrt{1 + 4t^2 + 9t^4}}, -\frac{1}{\sqrt{1 + 4t^2 + 9t^4}}, \frac{3t^2}{\sqrt{1 + 4t^2 + 9t^4}} \right\rangle}$$

---

2. Find the unit tangent vector for the vector function : $\vec r\left(t\right) = t\mathbf{e}^{2t}\,\vec i + \left(2 - t^2\right)\vec j - \mathbf{e}^{2t}\vec k$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec r'\left(t\right) = \left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)\vec i - 2t\,\vec j - 2\mathbf{e}^{2t}\vec k$$

$$\left\| \vec r'\left(t\right) \right\| = \sqrt{\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)^2 + \left(-2t\right)^2 + \left(-2\mathbf{e}^{2t}\right)^2} = \sqrt{\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)^2 + 4t^2 + 4\mathbf{e}^{4t}}$$

Step 2

The unit tangent vector for this vector function is then,

$$\boxed{\vec T\left(t\right) = \frac{1}{\sqrt{\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)^2 + 4t^2 + 4\mathbf{e}^{4t}}}\left(\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)\vec i - 2t\,\vec j - 2\mathbf{e}^{2t}\vec k\right)}$$

Note that because of the "mess" with this one we didn't distribute the magnitude through to each term as we usually do with these. This problem is a good example of just how "messy" these can get.

---

3. Find the tangent line to $\vec{r}\left(t\right)=\cos\left(4t\right)\vec{i}+3\sin\left(4t\right)\vec{j}+t^3\vec{k}$ at $t=\pi$.

Step 1

First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec{r}'\left(t\right)=-4\sin\left(4t\right)\vec{i}+12\cos\left(4t\right)\vec{j}+3t^2\vec{k}$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2

Now we actually need the tangent vector at the value of *t* given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of *t* from the problem statement. These are,

$$\begin{aligned}\vec{r}\left(\pi\right)&=\cos\left(4\pi\right)\vec{i}+3\sin\left(4\pi\right)\vec{j}+\pi^3\vec{k}=\vec{i}+\pi^3\vec{k}\\ \vec{r}'\left(\pi\right)&=-4\sin\left(4\pi\right)\vec{i}+12\cos\left(4\pi\right)\vec{j}+3\pi^2\vec{k}=12\vec{j}+3\pi^2\vec{k}\end{aligned}$$

Step 3

To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec{r}\left(t\right)=\vec{i}+\pi^3\vec{k}+t\left(12\vec{j}+3\pi^2\vec{k}\right)=\vec{i}+12t\,\vec{j}+\left(\pi^3+3\pi^2 t\right)\vec{k}}$$

---

4. Find the tangent line to $\vec{r}\left(t\right)=\left\langle 7\mathbf{e}^{2-t},\dfrac{16}{t^3},5-t\right\rangle$ at $t=2$.

Step 1

First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec{r}'\left(t\right)=\left\langle -7\mathbf{e}^{2-t},-\frac{48}{t^4},-1\right\rangle$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2
Now we actually need the tangent vector at the value of *t* given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of *t* from the problem statement. These are,

$$\vec{r}(2) = \langle 7, 2, 3 \rangle$$
$$\vec{r}\,'(2) = \langle -7, -3, -1 \rangle$$

Step 3
To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec{r}(t) = \langle 7, 2, 3 \rangle + t\langle -7, -3, -1 \rangle = \langle 7 - 7t, 2 - 3t, 3 - t \rangle}$$

---

5. Find the unit normal and the binormal vectors for the following vector function.

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), 3 \rangle$$

Step 1
We first need the unit tangent vector so let's get that.

$$\vec{r}\,'(t) = \langle -2\sin(2t), 2\cos(2t), 0 \rangle \qquad \left\| \vec{r}\,'(t) \right\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$$
$$\vec{T}(t) = \frac{1}{2}\langle -2\sin(2t), 2\cos(2t), 0 \rangle = \langle -\sin(2t), \cos(2t), 0 \rangle$$

Step 2
The unit normal vector is then,

$$\vec{T}\,'(t) = \langle -2\cos(2t), -2\sin(2t), 0 \rangle \qquad \left\| \vec{T}\,'(t) \right\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2$$
$$\boxed{\vec{N}(t) = \frac{1}{2}\langle -2\cos(2t), -2\sin(2t), 0 \rangle = \langle -\cos(2t), -\sin(2t), 0 \rangle}$$

Step 3

Finally, the binormal vector is,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(2t) & \cos(2t) & 0 \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix}$$

$$= \sin^2(2t)\vec{k} - \left(-\cos^2(2t)\vec{k}\right) = \left(\sin^2(2t) + \cos^2(2t)\right)\vec{k} = \boxed{\vec{k} = \langle 0,0,1\rangle = \vec{B}(t)}$$

Section 1-9 : Arc Length with Vector Functions

1. Determine the length of $\vec r\left(t\right) = \left(3-4t\right)\vec i + 6t\,\vec j - \left(9+2t\right)\vec k$ from $-6 \le t \le 8$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right) = -4\vec i + 6\,\vec j - 2\vec k$$

$$\left\|\vec r'\left(t\right)\right\| = \sqrt{16+36+4} = \sqrt{56} = 2\sqrt{14}$$

Step 2
The length of the curve is then,

$$L = \int_{-6}^{8} 2\sqrt{14}\,dt = \left.2\sqrt{14}t\right|_{-6}^{8} = \boxed{28\sqrt{14}}$$

---

2. Determine the length of $\vec r\left(t\right) = \left\langle \frac{1}{3}t^3, 4t, \sqrt{2}t^2\right\rangle$ from $0 \le t \le 2$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec r'\left(t\right) = \left\langle t^2, 4, 2\sqrt{2}t\right\rangle$$

$$\left\|\vec r'\left(t\right)\right\| = \sqrt{t^4+16+8t^2} = \sqrt{t^4+8t^2+16} = \sqrt{\left(t^2+4\right)^2} = t^2+4$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Step 2
The length of the curve is then,

$$L = \int_0^2 t^2+4\,dt = \left.\left(\tfrac{1}{3}t^3+4t\right)\right|_0^2 = \boxed{\tfrac{32}{3}}$$

Note that if we'd not simplified the magnitude this would have been a very difficult integral to compute!

---

3. Find the arc length function for $\vec{r}\left(t\right)=\left\langle t^2,2t^3,1-t^3\right\rangle$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}'\left(t\right)=\left\langle 2t,6t^2,-3t^2\right\rangle$$

$$\left\|\vec{r}'\left(t\right)\right\|=\sqrt{4t^2+36t^4+9t^4}=\sqrt{t^2\left(4+45t^2\right)}=\sqrt{t^2}\sqrt{4+45t^2}=\left|t\right|\sqrt{4+45t^2}=t\sqrt{4+45t^2}$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Note as well that because we are assuming that we are starting at $t=0$ for this kind of problem it is safe to assume that $t\geq 0$ and so $\sqrt{t^2}=\left|t\right|=t$.

Step 2
The arc length function is then,

$$s\left(t\right)=\int_0^t u\sqrt{4+45u^2}\,du=\tfrac{1}{135}\left(4+45u^2\right)^{\frac{3}{2}}\Big|_0^t=\boxed{\tfrac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}}-8\right]}$$

---

4. Find the arc length function for $\vec{r}\left(t\right)=\left\langle 4t,-2t,\sqrt{5}\,t^2\right\rangle$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}'\left(t\right)=\left\langle 4,-2,2\sqrt{5}\,t\right\rangle$$

$$\left\|\vec{r}'\left(t\right)\right\|=\sqrt{16+4+20t^2}=\sqrt{20+20t^2}=\sqrt{20}\sqrt{1+t^2}=2\sqrt{5}\sqrt{1+t^2}$$

Step 2
The arc length function is then,

$$s\left(t\right)=\int_0^t 2\sqrt{5}\sqrt{1+u^2}\,du$$

Do not always expect these integrals to be "simple" integrals. They will often require techniques more involved than just a standard Calculus I substitution. In this case we need the following trig substitution.

$$u = \tan\theta \qquad du = \sec^2\theta\,d\theta \qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right|$$

The limits of the integral become,

$$u = 0 : 0 = \tan\theta \quad \to \quad \theta = 0 \qquad u = t > 0 : \; t = \tan\theta \quad \to \quad \theta = \tan^{-1}\left(t\right)$$

Now, as noted we know that $t > 0$ and so we can safely assume that from the $u = t$ limit we will get $0 < \theta < \frac{\pi}{2}$. This in turn means that we will always be in the first quadrant and we know that secant is positive in the first quadrant. Therefore, we can remove the absolute values bars on the secant above.

The arc length function is now,

$$\begin{aligned} s\left(t\right) &= \int_0^{\tan^{-1}(t)} 2\sqrt{5}\sec^3\theta\,d\theta = \sqrt{5}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^{\tan^{-1}(t)} \\ &= \sqrt{5}\left[\sec\left(\tan^{-1}\left(t\right)\right)\tan\left(\tan^{-1}\left(t\right)\right) + \ln\left|\sec\left(\tan^{-1}\left(t\right)\right)+\tan\left(\tan^{-1}\left(t\right)\right)\right|\right] \end{aligned}$$

Now we know that $\tan\left(\tan^{-1}\left(t\right)\right) = t$ so that will simplify our answer a little. Let's take a look at the secant term to see if we can simplify that as well. First, from our limit work recall that $\theta = \tan^{-1}\left(t\right)$. Or with a little rewrite we have,

$$\tan\theta = t = \frac{t}{1} = \frac{\text{opposite}}{\text{adjacent}}$$

Construct a right triangle with opposite side being *t* and the adjacent side being 1. The hypotenuse is then $\sqrt{t^2+1}$. This in turn means that $\sec\theta = \sqrt{t^2+1}$. So,

$$\sec\left(\tan^{-1}\left(t\right)\right) = \sec\left(\theta\right) = \sqrt{t^2+1}$$

With this simplification our arc length function is then,

$$s\left(t\right) = \sqrt{5}\left[t\sqrt{t^2+1} + \ln\left|\sqrt{t^2+1}+t\right|\right]$$

There was some slightly unpleasant simplification here but once we did that we got a much nicer arc length function.

---

5. Determine where on the curve given by $\vec{r}\left(t\right) = \left\langle t^2, 2t^3, 1-t^3\right\rangle$ we are after traveling a distance of 20.

Step 1
From Problem 3 above we know that the arc length function for this vector function is,

$$s\left(t\right)=\tfrac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}}-8\right]$$

We need to solve this for *t*. Doing this gives,

$$\begin{aligned}\left(4+45t^2\right)^{\frac{3}{2}}-8&=135s\\ \left(4+45t^2\right)^{\frac{3}{2}}&=135s+8\\ 4+45t^2&=\left(135s+8\right)^{\frac{2}{3}}\\ t^2&=\tfrac{1}{45}\left[\left(135s+8\right)^{\frac{2}{3}}-4\right]\qquad\to\qquad t=\sqrt{\tfrac{1}{45}\left[\left(135s+8\right)^{\frac{2}{3}}-4\right]}\end{aligned}$$

Note that we only used the positive *t* after taking the root because the implicit assumption from the arc length function is that *t* is positive.

Step 2
We could use this to reparametrize the vector function however that would lead to a particularly unpleasant function in this case.

The key here is to simply realize that what we are being asked to compute is the value of the reparametrized vector function, $\vec{r}\left(t\left(s\right)\right)$ when $s=20$. Or, in other words, we want to compute $\vec{r}\left(t\left(20\right)\right)$.

So, first,

$$t\left(20\right)=\sqrt{\tfrac{1}{45}\left[\left(135\left(20\right)+8\right)^{\frac{2}{3}}-4\right]}=2.05633$$

Our position after traveling a distance of 20 is then,

$$\vec{r}\left(t\left(20\right)\right)=\vec{r}\left(2.05633\right)=\boxed{\left\langle 4.22849,17.39035,-7.69518\right\rangle}$$

Section 1-10 : Curvature

1. Find the curvature of $\vec r\left(t\right)=\left\langle \cos\left(2t\right),-\sin\left(2t\right),4t\right\rangle$.

Step 1
We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector won't be that bad to work with so let's go with that formula. Here's the unit tangent vector work.

$$\vec r'\left(t\right)=\left\langle -2\sin\left(2t\right),-2\cos\left(2t\right),4\right\rangle \qquad \left\|\vec r'\left(t\right)\right\|=\sqrt{4\sin^2\left(2t\right)+4\cos^2\left(2t\right)+16}=\sqrt{20}=2\sqrt{5}$$
$$\vec T\left(t\right)=\tfrac{1}{2\sqrt{5}}\left\langle -2\sin\left(2t\right),-2\cos\left(2t\right),4\right\rangle=\left\langle -\tfrac{\sin(2t)}{\sqrt{5}},-\tfrac{\cos(2t)}{\sqrt{5}},\tfrac{2}{\sqrt{5}}\right\rangle$$

Step 2
Now, what we really need is the magnitude of the derivative of the unit tangent vector so here is that work,

$$\vec T'\left(t\right)=\left\langle -\tfrac{2}{\sqrt{5}}\cos\left(2t\right),\tfrac{2}{\sqrt{5}}\sin\left(2t\right),0\right\rangle \qquad \left\|\vec T'\left(t\right)\right\|=\sqrt{\tfrac{4}{5}\cos^2\left(2t\right)+\tfrac{4}{5}\sin^2\left(2t\right)}=\tfrac{2}{\sqrt{5}}$$

Step 3
The curvature is then,

$$\kappa=\frac{\left\|\vec T'\left(t\right)\right\|}{\left\|\vec r'\left(t\right)\right\|}=\frac{2/\sqrt{5}}{2\sqrt{5}}=\boxed{\frac{1}{5}}$$

So, in this case, the curvature will be independent of *t*. That won't always be the case so don't expect this to happen every time.

---

2. Find the curvature of $\vec r\left(t\right)=\left\langle 4t,-t^2,2t^3\right\rangle$.

Step 1
We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector will involve lots of quotients that would probably be unpleasant to take the derivative of. So, let's go with the cross product formula this time.

We'll need the first and second derivative of the vector function. Here are those.

$$\vec{r}'(t) = \left\langle 4, -2t, 6t^2 \right\rangle \qquad \vec{r}''(t) = \left\langle 0, -2, 12t \right\rangle$$

Step 2

Next, we need the cross product of these two derivatives. Here is that work.

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2t & 6t^2 \\ 0 & -2 & 12t \end{vmatrix} = -24t^2\,\vec{i} - 8\vec{k} - 48t\,\vec{j} + 12t^2\,\vec{i} = -12t^2\,\vec{i} - 48t\,\vec{j} - 8\vec{k}$$

Step 3

We now need a couple of magnitudes.

$$\left\| \vec{r}'(t) \times \vec{r}''(t) \right\| = \sqrt{144t^4 + 2304t^2 + 64} \qquad \left\| \vec{r}'(t) \right\| = \sqrt{16 + 4t^2 + 36t^4}$$

The curvature is then,

$$\kappa = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3} = \boxed{\frac{\sqrt{144t^4 + 2304t^2 + 64}}{\left(16 + 4t^2 + 36t^4\right)^{\frac{3}{2}}}}$$

A fairly messy formula here, but these will often be that way.

Section 1-11 : Velocity and Acceleration

1. An objects acceleration is given by $\vec{a} = 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}$. The objects initial velocity is $\vec{v}\left(0\right) = \vec{j} - 3\vec{k}$ and the objects initial position is $\vec{r}\left(0\right) = -5\vec{i} + 2\vec{j} - 3\vec{k}$. Determine the objects velocity and position functions.

Step 1
To determine the velocity function all we need to do is integrate the acceleration function.

$$\vec{v}\left(t\right) = \int 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}\,dt = \tfrac{3}{2}t^2\,\vec{i} + 4\mathbf{e}^{-t}\,\vec{j} + 4t^3\vec{k} + \vec{c}$$

Don't forget the "constant" of integration, which in this case is actually the vector $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$. To determine the constant of integration all we need is to use the value $\vec{v}\left(0\right)$ that we were given in the problem statement.

$$\vec{j} - 3\vec{k} = \vec{v}\left(0\right) = 4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

To determine the values of c_1, c_2, and c_3 all we need to do is set the various components equal.

$$\begin{aligned} \vec{i} &: 0 = c_1 \\ \vec{j} &: 1 = 4 + c_2 \\ \vec{k} &: -3 = c_3 \end{aligned} \qquad \Rightarrow \qquad c_1 = 0, \;\; c_2 = -3, \;\; c_3 = -3$$

The velocity is then,

$$\boxed{\vec{v}\left(t\right) = \tfrac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\vec{j} + \left(4t^3 - 3\right)\vec{k}}$$

Step 2
The position function is simply the integral of the velocity function we found in the previous step.

$$\vec{r}\left(t\right) = \int \tfrac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\vec{j} + \left(4t^3 - 3\right)\vec{k}\,dt = \tfrac{1}{2}t^3\,\vec{i} + \left(-4\mathbf{e}^{-t} - 3t\right)\vec{j} + \left(t^4 - 3t\right)\vec{k} + \vec{c}$$

We'll use the value of $\vec{r}\left(0\right)$ from the problem statement to determine the value of the constant of integration.

$$-5\vec{i} + 2\vec{j} - 3\vec{k} = \vec{r}\left(0\right) = -4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

$$\begin{aligned} \vec{i} &: -5 = c_1 \\ \vec{j} &: 2 = -4 + c_2 \qquad \Rightarrow \qquad c_1 = -5, \quad c_2 = 6, \quad c_3 = -3 \\ \vec{k} &: -3 = c_3 \end{aligned}$$

The position function is then,

$$\boxed{\vec{r}(t) = \left(\tfrac{1}{2}t^3 - 5\right)\vec{i} + \left(-4\mathbf{e}^{-t} - 3t + 6\right)\vec{j} + \left(t^4 - 3t - 3\right)\vec{k}}$$

---

2. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}(t) = \left\langle \cos(2t), -\sin(2t), 4t \right\rangle$.

Step 1

First, we need the first and second derivatives of the position function.

$$\vec{r}'(t) = \left\langle -2\sin(2t), -2\cos(2t), 4 \right\rangle \qquad \vec{r}''(t) = \left\langle -4\cos(2t), 4\sin(2t), 0 \right\rangle$$

Step 2

Next, we'll need the following quantities.

$$\left\| \vec{r}'(t) \right\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{r}'(t) \bullet \vec{r}''(t) = 8\sin(2t)\cos(2t) - 8\sin(2t)\cos(2t) + 0 = 0$$

$$\begin{aligned} \vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(2t) & -2\cos(2t) & 4 \\ -4\cos(2t) & 4\sin(2t) & 0 \end{vmatrix} \\ &= -16\cos(2t)\vec{j} - 8\sin^2(2t)\vec{k} - 8\cos^2(2t)\vec{k} - 16\sin(2t)\vec{i} \\ &= -16\sin(2t)\vec{i} - 16\cos(2t)\vec{j} - 8\vec{k} \end{aligned}$$

$$\left\| \vec{r}'(t) \times \vec{r}''(t) \right\| = \sqrt{256\sin^2(2t) + 256\cos^2(2t) + 64} = \sqrt{320} = 8\sqrt{5}$$

Step 3

The tangential component of the acceleration is,

$$a_T = \frac{\vec{r}'(t) \bullet \vec{r}''(t)}{\left\|\vec{r}'(t)\right\|} = \boxed{0}$$

The normal component of the acceleration is,

$$a_N = \frac{\left\|\vec{r}'(t) \times \vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|} = \frac{8\sqrt{5}}{2\sqrt{5}} = \boxed{4}$$

Section 1-12 : Cylindrical Coordinates

1. Convert the Cartesian coordinates for $\left(4,-5,2\right)$ into Cylindrical coordinates.

Step 1
From the point we're given we have,

$$x = 4 \qquad y = -5 \qquad z = 2$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2
Remember as well that for *r* and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting *r* is easy.

$$r = \sqrt{\left(4\right)^2 + \left(-5\right)^2} = \sqrt{41}$$

Step 3
Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-5}{4}\right) = -0.8961 \qquad \theta_2 = -0.8961 + \pi = 2.2455$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the fourth quadrant and θ_2 is in the second quadrant. Likewise, from our *x* and *y* coordinates the point is in the fourth quadrant (as we look at the point from above).

This in turn means that we need to use θ_1 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{41}, -0.8961, 2\right)}$$

---

2. Convert the Cartesian coordinates for $\left(-4,-1,8\right)$ into Cylindrical coordinates.

Step 1

From the point we're given we have,

$$x = -4 \qquad y = -1 \qquad z = 8$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2
Remember as well that for r and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting r is easy.

$$r = \sqrt{\left(-4\right)^2 + \left(-1\right)^2} = \sqrt{17}$$

Step 3
Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-1}{-4}\right) = 0.2450 \qquad \theta_2 = 0.2450 + \pi = 3.3866$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the first quadrant and θ_2 is in the third quadrant. Likewise, from our x and y coordinates the point is in the third quadrant (as we look at the point from above).

This in turn means that we need to use θ_2 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{17}, 3.3866, 8\right)}$$

---

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

$$x^3 + 2x^2 - 6z = 4 - 2y^2$$

Step 1
There really isn't a whole lot to do here. All we need to do is plug in the following x and y polar conversion formulas into the equation where (and if) possible.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Step 2
However, first we'll do a little rewrite.

$$x^3 + 2x^2 + 2y^2 - 6z = 4 \qquad \to \qquad x^3 + 2\left(x^2 + y^2\right) - 6z = 4$$

Step 3
Now let's use the formulas from Step 1 to convert the equation into Cylindrical coordinates.

$$\left(r\cos\theta\right)^3 + 2\left(r^2\right) - 6z = 4 \qquad \to \qquad \boxed{r^3\cos^3\theta + 2r^2 - 6z = 4}$$

---

4. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$z\,r = 2 - r^2$$

Solution
There is not really a lot to do here other than plug in $r = \sqrt{x^2 + y^2}$ into the equation. Doing this is,

$$\boxed{z\sqrt{x^2 + y^2} = 2 - \left(x^2 + y^2\right)}$$

---

5. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$4\sin\left(\theta\right) - 2\cos\left(\theta\right) = \frac{r}{z}$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following *x* and *y* polar conversion formulas in the equation where (and if) possible.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2$$

Step 2
To make the conversion a little easier let's multiply the equation by *r* to get,

$$4r\sin\left(\theta\right) - 2r\cos\left(\theta\right) = \frac{r^2}{z}$$

Step 3
Now let's use the formulas from Step 1 to convert the equation into Cartesian coordinates.

$$\boxed{4y - 2x = \frac{x^2 + y^2}{z}}$$

---

6. Identify the surface generated by the equation : $r^2 - 4r\cos\left(\theta\right) = 14$

Step 1
To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$x^2 + y^2 - 4x = 14$$

Step 2
To identify this equation (and you do know what it is!) let's complete the square on the x part of the equation.

$$\begin{aligned} x^2 - 4x + y^2 &= 14 \\ x^2 - 4x + 4 + y^2 &= 14 + 4 \\ \left(x-2\right)^2 + y^2 &= 18 \end{aligned}$$

So, we can see that this is a cylinder whose central axis is a vertical line parallel to the *z*-axis and goes through the point (2, 0) in the *xy*-plane and the radius of the cylinder is $\sqrt{18}$.

---

7. Identify the surface generated by the equation : $z = 7 - 4r^2$

Step 1
To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$z = 7 - 4\left(x^2 + y^2\right) = 7 - 4x^2 - 4y^2$$

Step 2
From the Cartesian equation in Step 1 we can see that the surface generated by the equation is an elliptic paraboloid that starts at $z = 7$ and opens down.

Section 1-13 : Spherical Coordinates

1. Convert the Cartesian coordinates for $\left(3,-4,1\right)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$x = 3 \qquad y = -4 \qquad z = 1$$

Step 2
Let's first determine ρ.

$$\rho = \sqrt{\left(3\right)^2 + \left(-4\right)^2 + \left(1\right)^2} = \sqrt{26}$$

Step 3
We can now determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{1}{\sqrt{26}} \qquad \varphi = \cos^{-1}\left(\frac{1}{\sqrt{26}}\right) = 1.3734$$

Step 4
Let's use the *x* conversion formula to determine θ.

$$\cos\theta = \frac{x}{\rho\sin\varphi} = \frac{3}{\sqrt{26}\sin\left(1.3734\right)} = 0.6 \quad \to \quad \theta_1 = \cos^{-1}\left(0.6\right) = 0.9273$$

This angle is in the first quadrant and if we sketch a quick unit circle we see that a second angle in the fourth quadrant is $\theta_2 = 2\pi - 0.9273 = 5.3559$.

If we look at the three dimensional coordinate system from above we can see that from our *x* and *y* coordinates the point is in the fourth quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{26}, 5.3559, 1.3734\right)}$$

---

2. Convert the Cartesian coordinates for $\left(-2,-1,-7\right)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$x = -2 \qquad y = -1 \qquad z = -7$$

Step 2
Let's first determine ρ.

$$\rho = \sqrt{(-2)^2 + (-1)^2 + (-7)^2} = \sqrt{54}$$

Step 3
We can now determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{-7}{\sqrt{54}} \qquad \varphi = \cos^{-1}\left(\frac{-7}{\sqrt{54}}\right) = 2.8324$$

Step 4
Let's use the *y* conversion formula to determine θ.

$$\sin\theta = \frac{-1}{\rho\sin\varphi} = \frac{-1}{\sqrt{54}\sin(2.8324)} = -0.4472 \quad \to \quad \theta_1 = \sin^{-1}(-0.4472) = -0.4636$$

This angle is in the fourth quadrant and if we sketch a quick unit circle we see that a second angle in the third quadrant is $\theta_2 = \pi + 0.4636 = 3.6052$.

If we look at the three dimensional coordinate system from above we can see that from our *x* and *y* coordinates the point is in the third quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{54}, 3.6052, 2.8324\right)}$$

---

3. Convert the Cylindrical coordinates for $(2, 0.345, -3)$ into Spherical coordinates.

Step 1
From the point we're given we have,

$$r = 2 \qquad \theta = 0.345 \qquad z = -3$$

So, we already have the value of θ for the Spherical coordinates.

Step 2
Next, we can determine ρ.

$$\rho = \sqrt{\left(2\right)^2 + \left(-3\right)^2} = \sqrt{13}$$

Step 3
Finally, we can determine φ.

$$\cos\varphi = \frac{z}{\rho} = \frac{-3}{\sqrt{13}} \qquad \varphi = \cos^{-1}\left(\frac{-3}{\sqrt{13}}\right) = 2.5536$$

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{13}, 0.345, 2.5536\right)}$$

---

4. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$x^2 + y^2 = 4x + z - 2$$

Step 1
All we need to do here is plug in the following conversion formulas into the equation and do a little simplification.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

Step 2
Plugging the conversion formula in gives,

$$\left(\rho\sin\varphi\cos\theta\right)^2 + \left(\rho\sin\varphi\sin\theta\right)^2 = 4\left(\rho\sin\varphi\cos\theta\right) + \rho\cos\varphi - 2$$

The first two terms can be simplified as follows.

$$\begin{aligned} \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2 \\ \rho^2\sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right) &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2 \\ \rho^2\sin^2\varphi &= 4\rho\sin\varphi\cos\theta + \rho\cos\varphi - 2 \end{aligned}$$

---

5. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\rho^2 = 3 - \cos\varphi$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Step 2
To make this problem a little easier let's first multiply the equation by ρ. Doing this gives,

$$\rho^3 = 3\rho - \rho\cos\varphi$$

Doing this makes recognizing the right most term a little easier.

Step 3
Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} = 3\sqrt{x^2 + y^2 + z^2} - z}$$

---

6. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\csc\varphi = 2\cos\theta + 4\sin\theta$$

Step 1
There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Step 2
To make this problem a little easier let's first do some rewrite on the equation.

First, let's deal with the cosecant.

$$\frac{1}{\sin\varphi} = 2\cos\theta + 4\sin\theta \qquad \to \qquad 1 = 2\sin\varphi\cos\theta + 4\sin\varphi\sin\theta$$

Next, let's multiply everything by ρ to get,

$$\rho = 2\rho\sin\varphi\cos\theta + 4\rho\sin\varphi\sin\theta$$

Doing this makes recognizing the terms on the right a little easier.

Step 3
Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\sqrt{x^2+y^2+z^2} = 2x+4y}$$

---

7. Identify the surface generated by the given equation : $\varphi = \dfrac{4\pi}{5}$

Solution
Okay, as we discussed this type of equation in the notes for this section. We know that all points on the surface generated must be of the form $\left(\rho, \theta, \frac{4\pi}{5}\right)$.

So, we can rotate around the *z*-axis as much as want them to (*i.e.* θ can be anything) and we can move as far as we want from the origin (*i.e.* ρ can be anything). All we need to do is make sure that the point will always make an angle of $\frac{4\pi}{5}$ with the positive z-axis.

In other words, we have a cone. It will open downwards and the "wall" of the cone will form an angle of $\frac{4\pi}{5}$ with the positive *z*-axis and it will form an angle of $\frac{\pi}{5}$ with the negative *z*-axis.

---

8. Identify the surface generated by the given equation : $\rho = -2\sin\varphi\cos\theta$

Step 1
Let's first multiply each side of the equation by ρ to get,

$$\rho^2 = -2\rho\sin\varphi\cos\theta$$

Step 2
We can now easily convert this to Cartesian coordinates to get,

$$\begin{aligned} x^2 + y^2 + z^2 &= -2x \\ x^2 + 2x + y^2 + z^2 &= 0 \end{aligned}$$

Let’s complete the square on the *x* portion to get,

$$\begin{aligned} x^2 + 2x + 1 + y^2 + z^2 &= 0 + 1 \\ \left(x+1\right)^2 + y^2 + z^2 &= 1 \end{aligned}$$

Step 3

So, it looks like we have a sphere with radius 1 that is centered at $\left(-1,0,0\right)$.

Chapter 2 : Partial Derivatives

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Limits – In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.

Partial Derivatives – In this section we will look at the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (*i.e.* without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.

Interpretations of Partial Derivatives – In the section we will take a look at a couple of important interpretations of partial derivatives. First, the always important, rate of change of the function. Although we now have multiple 'directions' in which the function can change (unlike in Calculus I). We will also see that partial derivatives give the slope of tangent lines to the traces of the function.

Higher Order Partial Derivatives – In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, *etc*. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.

Differentials – In this section we extend the idea of differentials we first saw in Calculus I to functions of several variables.

Chain Rule – In the section we extend the idea of the chain rule to functions of several variables. In particular, we will see that there are multiple variants to the chain rule here all depending on how many variables our function is dependent on and how each of those variables can, in turn, be written in terms of different variables. We will also give a nice method for writing down the chain rule for pretty much any situation you might run into when dealing with functions of multiple variables. In addition, we will derive a very quick way of doing implicit differentiation so we no longer need to go through the process we first did back in Calculus I.

Directional Derivatives – In the section we introduce the concept of directional derivatives. With directional derivatives we can now ask how a function is changing if we allow all the independent variables to change rather than holding all but one constant as we had to do with partial derivatives. In addition, we will define the gradient vector to help with some of the notation and work here. The gradient vector will be very useful in some later sections as well. We will also give a nice fact that will allow us to determine the direction in which a given function is changing the fastest.

Section 2-1 : Limits

1. Evaluate the following limit.

$$\lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - z\mathbf{e}^{2y}}{6x+2y-3z}$$

Solution
In this case there really isn't all that much to do. We can see that the denominator exists and will not be zero at the point in question and the numerator also exists at the point in question. Therefore, all we need to do is plug in the point to evaluate the limit.

$$\lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - z\mathbf{e}^{2y}}{6x+2y-3z} = \boxed{\frac{5}{18}}$$

---

2. Evaluate the following limit.

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

Step 1
Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike most of the examples of this type however, that doesn't just mean that the limit won't exist.

In this case notice that we can factor and simplify the function as follows,

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y)\to(2,1)} \frac{x(x-2y)}{(x-2y)(x+2y)} = \lim_{(x,y)\to(2,1)} \frac{x}{x+2y}$$

We may not be used to factoring these kinds of polynomials but we can't forget that factoring is still a possibility that we need to address for these limits.

Step 2
Now, that we've factored and simplified the function we can see that we've lost the division by zero issue and so we can now evaluate the limit. Doing this gives,

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y)\to(2,1)} \frac{x}{x+2y} = \boxed{\frac{1}{2}}$$

---

3. Evaluate the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x}$$

Step 1
Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike the second problem above however there is no factoring that can be done to make this into a "doable" limit.

Therefore, we will proceed with the problem as if the limit doesn't exist.

Step 2
So, since we've made the assumption that the limit probably doesn't exist that means we need to find two different paths upon which the limit has different values.

There are many different paths to try but let's start this off with the *x*-axis (*i.e.* $y=0$).

Along this path we get,

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(x,0)\to(0,0)} \frac{x}{7x} = \lim_{(x,0)\to(0,0)} \frac{1}{7} = \frac{1}{7}$$

Step 3
Now let's try the *y*-axis (*i.e.* $x=0$) and see what we get.

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(0,y)\to(0,0)} \frac{-4y}{6y} = \lim_{(0,y)\to(0,0)} \frac{-2}{3} = -\frac{2}{3}$$

Step 4
So, we have two different paths that give different values of the limit and so we now know that this limit **does not exist**.

---

4. Evaluate the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^6}{xy^3}$$

Step 1
Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike the second problem above however there is no factoring that can be done to make this into a "doable" limit.

Therefore, we will proceed with the problem as if the limit doesn't exist.

Step 2

So, since we've made the assumption that the limit probably doesn't exist that means we need to find two different paths upon which the limit has different values.

In this case note that using the *x*-axis or *y*-axis will not work as either one will result in a division by zero issue. So, let's start off using the path $x = y^3$.

Along this path we get,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} = \lim_{(y^3,y)\to(0,0)} \frac{y^6 - y^6}{y^3 y^3} = \lim_{(y^3,y)\to(0,0)} 0 = 0$$

Step 3
Now let's try $x = y$ for the second path.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} = \lim_{(y,y)\to(0,0)} \frac{y^2 - y^6}{yy^3} = \lim_{(y,y)\to(0,0)} \frac{1 - y^4}{y^2} = \lim_{(y,y)\to(0,0)} \left(\frac{1}{y^2} - y^2 \right) = \infty$$

Step 4
So, we have two different paths that give different values of the limit and so we now know that this limit **does not exist**.

Section 2-2 : Partial Derivatives

1. Find all the 1st order partial derivatives of the following function.

$$f\left(x,y,z\right)=4x^3y^2-\mathbf{e}^zy^4+\frac{z^3}{x^2}+4y-x^{16}$$

Solution

So, this is clearly a function of *x*, *y* and *z* and so we'll have three 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the *x* partial derivative all *y*'s and *z*'s are treated as constants. This in turn means that, for the *x* partial derivative, the second and fourth terms are considered to be constants (they don't contain any *x*'s) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Here are the three 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x}&=f_x=12x^2y^2-\frac{2z^3}{x^3}-16x^{15}\\\frac{\partial f}{\partial y}&=f_y=8x^3y-4\mathbf{e}^zy^3+4\\\frac{\partial f}{\partial z}&=f_z=-\mathbf{e}^zy^4+\frac{3z^2}{x^2}\end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

2. Find all the 1st order partial derivatives of the following function.

$$w=\cos\left(x^2+2y\right)-\mathbf{e}^{4x-z^4y}+y^3$$

Solution

This function isn't written explicitly with the $\left(x,y,z\right)$ part but it is (hopefully) clearly a function of *x*, *y* and *z* and so we'll have three 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the *x* partial derivative all *y*'s and *z*'s are treated as constants. This in turn means that, for the *x* partial derivative, the third term is considered to be a constant (it doesn't contain any x's) and so differentiates to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Be careful with chain rule. Again, one of the biggest issues with partial derivatives is students forgetting the "rules" of partial derivatives when it comes to differentiating the inside function. Just remember that you're just doing the partial derivative of a function and remember which variable we are differentiating with respect to.

Here are the three 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= w_x = -2x\sin\left(x^2+2y\right) - 4\mathbf{e}^{4x-z^4y} \\ \frac{\partial w}{\partial y} &= w_y = -2\sin\left(x^2+2y\right) + z^4\mathbf{e}^{4x-z^4y} + 3y^2 \\ \frac{\partial w}{\partial z} &= w_z = 4z^3y\mathbf{e}^{4x-z^4y} \end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

3. Find all the 1st order partial derivatives of the following function.

$$f\left(u,v,p,t\right) = 8u^2t^3p - \sqrt{v}\,p^2t^{-5} + 2u^2t + 3p^4 - v$$

Solution

So, this is clearly a function of u, v, p, and t and so we'll have four 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the u partial derivative all v's, p's and t's are treated as constants. This in turn means that, for the u partial derivative, the second, fourth and fifth terms are considered to be constants (they don't contain any u's) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Here are the four 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}&\frac{\partial f}{\partial u}=f_u=16ut^3p+4ut\\&\frac{\partial f}{\partial v}=f_v=-\tfrac{1}{2}v^{-\frac{1}{2}}p^2t^{-5}-1\\&\frac{\partial f}{\partial p}=f_p=8u^2t^3-2\sqrt{v}\,pt^{-5}+12p^3\\&\frac{\partial f}{\partial t}=f_t=24u^2t^2p+5\sqrt{v}\,p^2t^{-6}+2u^2\end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

4. Find all the 1st order partial derivatives of the following function.

$$f\left(u,v\right)=u^2\sin\left(u+v^3\right)-\sec\left(4u\right)\tan^{-1}\left(2v\right)$$

Solution
For this problem it looks like we'll have two 1st order partial derivatives to compute.

Be careful with product rules with partial derivatives. For example, the second term, while definitely a product, will not need the product rule since each "factor" of the product only contains *u*'s or *v*'s. On the other hand, the first term will need a product rule when doing the *u* partial derivative since there are *u*'s in both of the "factors" of the product. However, just because we had to product rule with first term for the *u* partial derivative doesn't mean that we'll need to product rule for the *v* partial derivative as only the second "factor" in the product has a *v* in it.

Basically, be careful to not "overthink" product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the "factors" of the product. Do both "factors" have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the two 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}&\frac{\partial f}{\partial u}=f_u=2u\sin\left(u+v^3\right)+u^2\cos\left(u+v^3\right)-4\sec\left(4u\right)\tan\left(4u\right)\tan^{-1}\left(2v\right)\\&\frac{\partial f}{\partial v}=f_v=3v^2u^2\cos\left(u+v^3\right)-\frac{2\sec\left(4u\right)}{1+4v^2}\end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

5. Find all the 1st order partial derivatives of the following function.

$$f\left(x,z\right)=\mathbf{e}^{-x}\sqrt{z^4+x^2}-\frac{2x+3z}{4z-7x}$$

Solution

For this problem it looks like we'll have two 1st order partial derivatives to compute.

Be careful with product rules and quotient rules with partial derivatives. For example, the first term, while clearly a product, will only need the product rule for the *x* derivative since both "factors" in the product have *x*'s in them. On the other hand, the first "factor" in the first term does not contain a *z* and so we won't need to do the product rule for the *z* derivative. In this case the second term will require a quotient rule for both derivatives.

Basically, be careful to not "overthink" product/quotient rules with partial derivatives. Do them when required but make sure to not do them just because you see a product/quotient. When you see a product/quotient look at the "factors" of the product/quotient. Do both "factors" have the variable you are differentiating with respect to or not and use the product/quotient rule only if they both do.

Here are the two 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x}&=f_x=-\mathbf{e}^{-x}\left(z^4+x^2\right)^{\frac{1}{2}}+x\,\mathbf{e}^{-x}\left(z^4+x^2\right)^{-\frac{1}{2}}-\frac{29z}{\left(4z-7x\right)^2}\\ \frac{\partial f}{\partial z}&=f_z=2z^3\mathbf{e}^{-x}\left(z^4+x^2\right)^{-\frac{1}{2}}+\frac{29x}{\left(4z-7x\right)^2}\end{aligned}}$$

Note that we did a little bit of simplification in the derivative work here and didn't actually show the "first" step of the problem under the assumption that by this point of your mathematical career you can do the product and quotient rule and don't really need us to show that step to you.

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

6. Find all the 1st order partial derivatives of the following function.

$$g\left(s,t,v\right)=t^2\ln\left(s+2t\right)-\ln\left(3v\right)\left(s^3+t^2-4v\right)$$

Solution

For this problem it looks like we'll have three 1st order partial derivatives to compute.

Be careful with product rules with partial derivatives. The first term will only need a product rule for the *t* derivative and the second term will only need the product rule for the *v* derivative. Do not "overthink" product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the "factors" of the product. Do both "factors" have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the three 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial g}{\partial s} &= g_s = \frac{t^2}{s+2t} - 3s^2\ln\left(3v\right)\\ \frac{\partial g}{\partial t} &= g_t = 2t\ln\left(s+2t\right) + \frac{2t^2}{s+2t} - 2t\ln\left(3v\right)\\ \frac{\partial g}{\partial v} &= g_v = 4\ln\left(3v\right) - \frac{s^3+t^2-4v}{v}\end{aligned}}$$

Make sure you can differentiate natural logarithms as they will come up fairly often. Recall that, with the chain rule, we have,

$$\frac{d}{dx}\left[\ln\left(f\left(x\right)\right)\right] = \frac{f'\left(x\right)}{f\left(x\right)}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

7. Find all the 1st order partial derivatives of the following function.

$$R\left(x,y\right) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}$$

Solution

For this problem it looks like we'll have two 1st order partial derivatives to compute.

Be careful with quotient rules with partial derivatives. For example the first term, while clearly a quotient, will not require the quotient rule for the *x* derivative and will only require the quotient rule for the *y* derivative if we chose to leave the y^2+1 in the denominator (recall we could just bring it up to the numerator as $\left(y^2+1\right)^{-1}$ if we wanted to). The second term on the other hand clearly has *y*'s in both the numerator and the denominator and so will require a quotient rule for the *y* derivative.

Here are the two 1st order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial R}{\partial x} &= R_x = \frac{2x}{y^2+1} + \frac{2xy^2}{\left(x^2+y\right)^2}\\ \frac{\partial R}{\partial y} &= R_y = -\frac{2yx^2}{\left(y^2+1\right)^2} - \frac{2yx^2+y^2}{\left(x^2+y\right)^2}\end{aligned}}$$

Note that we did a little bit of simplification in the derivative work here and didn't actually show the "first" step of the problem under the assumption that by this point of your mathematical career you can do the quotient rule and don't really need us to show that step to you.

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

8. Find all the 1st order partial derivatives of the following function.

$$z = \frac{p^2\left(r+1\right)}{t^3} + pr\,\mathbf{e}^{2p+3r+4t}$$

Solution

For this problem it looks like we'll have three 1st order partial derivatives to compute. Here they are,

$$\boxed{\begin{aligned}\frac{\partial z}{\partial p} &= z_p = \frac{2p\left(r+1\right)}{t^3} + r\,\mathbf{e}^{2p+3r+4t} + 2pr\,\mathbf{e}^{2p+3r+4t}\\ \frac{\partial z}{\partial r} &= z_r = \frac{p^2}{t^3} + p\,\mathbf{e}^{2p+3r+4t} + 3pr\,\mathbf{e}^{2p+3r+4t}\\ \frac{\partial z}{\partial t} &= z_t = -\frac{3p^2\left(r+1\right)}{t^4} + 4pr\,\mathbf{e}^{2p+3r+4t}\end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

9. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the following function.

$$x^2\sin\left(y^3\right) + x\mathbf{e}^{3z} - \cos\left(z^2\right) = 3y - 6z + 8$$

Step 1

Okay, we are basically being asked to do implicit differentiation here and recall that we are assuming that *z* is in fact $z\left(x,y\right)$ when we do our derivative work.

Let's get $\frac{\partial z}{\partial x}$ first and that requires us to differentiate with respect to *x*. Just recall that any product involving *x* and *z* will require the product rule because we're assuming that *z* is a function of *x*. Also recall to properly chain rule any derivative of *z* to pick up the $\frac{\partial z}{\partial x}$ when differentiating the "inside" function.

Differentiating the equation with respect to *x* gives,

$$2x\sin\left(y^3\right)+\mathbf{e}^{3z}+3\frac{\partial z}{\partial x}x\mathbf{e}^{3z}+2z\frac{\partial z}{\partial x}\sin\left(z^2\right)=-6\frac{\partial z}{\partial x}$$

Solving for $\frac{\partial z}{\partial x}$ gives,

$$2x\sin\left(y^3\right)+\mathbf{e}^{3z}=\left(-6-3x\mathbf{e}^{3z}-2z\sin\left(z^2\right)\right)\frac{\partial z}{\partial x}\qquad\to\qquad\boxed{\frac{\partial z}{\partial x}=\frac{2x\sin\left(y^3\right)+\mathbf{e}^{3z}}{-6-3x\mathbf{e}^{3z}-2z\sin\left(z^2\right)}}$$

Step 2

Now we get to do it all over again except this time we'll differentiate with respect to *y* in order to find $\frac{\partial z}{\partial y}$. So, differentiating gives,

$$3y^2x^2\cos\left(y^3\right)+3\frac{\partial z}{\partial y}x\mathbf{e}^{3z}+2z\frac{\partial z}{\partial y}\sin\left(z^2\right)=3-6\frac{\partial z}{\partial y}$$

Solving for $\frac{\partial z}{\partial y}$ gives,

$$3y^2x^2\cos\left(y^3\right)-3=\left(-6-3x\mathbf{e}^{3z}-2z\sin\left(z^2\right)\right)\frac{\partial z}{\partial y}\qquad\to\qquad\boxed{\frac{\partial z}{\partial y}=\frac{3y^2x^2\cos\left(y^3\right)-3}{-6-3x\mathbf{e}^{3z}-2z\sin\left(z^2\right)}}$$

Section 2-3 : Interpretations of Partial Derivatives

1. Determine if $f\left(x,y\right)=x\ln\left(4y\right)+\sqrt{x+y}$ is increasing or decreasing at $\left(-3,6\right)$ if

(a) we allow *x* to vary and hold *y* fixed.
(b) we allow *y* to vary and hold *x* fixed.

(a) we allow *x* to vary and hold *y* fixed.
So, we want to determine the increasing/decreasing nature of a function at a point. We know that this means a derivative from our basic Calculus knowledge. Also, from the problem statement we know we want to allow *x* to vary while *y* is held fixed. This means that we will need the *x* partial derivative.

The *x* partial derivative and its value at the point is,

$$f_x\left(x,y\right)=\ln\left(4y\right)+\tfrac{1}{2}\left(x+y\right)^{-\frac{1}{2}}\qquad\rightarrow\qquad f_x\left(-3,6\right)=\ln\left(24\right)+\tfrac{1}{2\sqrt{3}}=3.4667$$

So, we can see that $f_x\left(-3,6\right)>0$ and so at $\left(-3,6\right)$ if we allow *x* to vary and hold *y* fixed the function will be increasing.

(b) we allow *y* to vary and hold *x* fixed.
This part is pretty much the same as the previous part. The only difference is that here we are allowing *y* to vary and we'll hold *x* fixed. This means we'll need the *y* partial derivative.

The *y* partial derivative and its value at the point is,

$$f_y\left(x,y\right)=\tfrac{x}{y}+\tfrac{1}{2}\left(x+y\right)^{-\frac{1}{2}}\qquad\rightarrow\qquad f_y\left(-3,6\right)=-\tfrac{1}{2}+\tfrac{1}{2\sqrt{3}}=-0.2113$$

So, we can see that $f_y\left(-3,6\right)<0$ and so at $\left(-3,6\right)$ if we allow *y* to vary and hold *x* fixed the function will be decreasing.

Note that, because of the three dimensional nature of the graph of this function we can't expect the increasing/decreasing nature of the function in one direction to be the same as in any other direction!

---

2. Determine if $f\left(x,y\right)=x^2\sin\left(\frac{\pi}{y}\right)$ is increasing or decreasing at $\left(-2,\frac{3}{4}\right)$ if

(a) we allow *x* to vary and hold *y* fixed.
(b) we allow *y* to vary and hold *x* fixed.

(a) we allow *x* to vary and hold *y* fixed.

So, we want to determine the increasing/decreasing nature of a function at a point. We know that this means a derivative from our basic Calculus knowledge. Also, from the problem statement we know we want to allow *x* to vary while *y* is held fixed. This means that we will need the *x* partial derivative.

The *x* partial derivative and its value at the point is,

$$f_x(x,y) = 2x\sin\left(\tfrac{\pi}{y}\right) \qquad \to \qquad f_x\left(-2,\tfrac{3}{4}\right) = 2\sqrt{3}$$

So, we can see that $f_x\left(-2,\tfrac{3}{4}\right) > 0$ and so at $\left(-2,\tfrac{3}{4}\right)$ if we allow *x* to vary and hold *y* fixed the function will be increasing.

(b) we allow *y* to vary and hold *x* fixed.
This part is pretty much the same as the previous part. The only difference is that here we are allowing *y* to vary and we'll hold *x* fixed. This means we'll need the *y* partial derivative.

The *y* partial derivative and its value at the point is,

$$f_y(x,y) = -\tfrac{\pi x^2}{y^2}\cos\left(\tfrac{\pi}{y}\right) \qquad \to \qquad f_y\left(-2,\tfrac{3}{4}\right) = \tfrac{32\pi}{9}$$

So, we can see that $f_y\left(-2,\tfrac{3}{4}\right) > 0$ and so at $\left(-2,\tfrac{3}{4}\right)$ if we allow *y* to vary and hold *x* fixed the function will be increasing.

Note that, because of the three dimensional nature of the graph of this function we can't expect the increasing/decreasing nature of the function in one direction to be the same as in any other direction! In this case it did happen to be the same behavior but there is no reason to expect that in general.

---

3. Write down the vector equations of the tangent lines to the traces for $f(x,y) = x\,\mathbf{e}^{2x-y^2}$ at $(2,0)$.

Step 1
We know that there are two traces. One for $x = 2$ (*i.e. x* is fixed and *y* is allowed to vary) and one for $y = 0$ (*i.e. y* is fixed and *x* is allowed to vary). We also know that $f_y(2,0)$ will be the slope for the first trace (*y* varies and *x* is fixed!) and $f_x(2,0)$ will be the slope for the second trace (*x* varies and *y* is fixed!).

So, we'll need the value of both of these partial derivatives. Here is that work,

$$f_x(x,y) = \mathbf{e}^{2x-y^2} + 2x\,\mathbf{e}^{2x-y^2} \qquad \to \qquad f_x(2,0) = 5\mathbf{e}^4 = 272.9908$$
$$f_y(x,y) = -2yx\,\mathbf{e}^{2x-y^2} \qquad \to \qquad f_y(2,0) = 0$$

Step 2
Now, we need to write down the vector equation of the line and so we don't (at some level) need the "slopes" as listed in the previous step. What we need are tangent vectors that give these slopes.

Recall from the notes that the tangent vector for the first trace is,

$$\left\langle 0,1,f_y\left(2,0\right)\right\rangle = \left\langle 0,1,0\right\rangle$$

Likewise, the tangent vector for the second trace is,

$$\left\langle 1,0,f_x\left(2,0\right)\right\rangle = \left\langle 1,0,5\mathbf{e}^4\right\rangle$$

Step 3
Next, we'll also need the position vector for the point on the surface that we are looking at. This is,

$$\left\langle 2,0,f\left(2,0\right)\right\rangle = \left\langle 2,0,2\mathbf{e}^4\right\rangle$$

Step 4
Finally, the vector equation of the tangent line for the first trace is,

$$\boxed{\vec r\left(t\right) = \left\langle 2,0,2\mathbf{e}^4\right\rangle + t\left\langle 0,1,0\right\rangle = \left\langle 2,t,2\mathbf{e}^4\right\rangle}$$

and the trace for the second trace is,

$$\boxed{\vec r\left(t\right) = \left\langle 2,0,2\mathbf{e}^4\right\rangle + t\left\langle 1,0,5\mathbf{e}^4\right\rangle = \left\langle 2+t,0,2\mathbf{e}^4+5\mathbf{e}^4 t\right\rangle}$$

Section 2-4 : Higher Order Partial Derivatives

1. Verify Clairaut's Theorem for the following function.

$$f\left(x,y\right)=x^3y^2-\frac{4y^6}{x^3}$$

Step 1

First, we know we'll need the two 1st order partial derivatives. Here they are,

$$f_x=3x^2y^2+12x^{-4}y^6 \qquad f_y=2x^3y-24x^{-3}y^5$$

Step 2

Now let's compute each of the mixed second order partial derivatives.

$$f_{x\,y}=\left(f_x\right)_y=6x^2y+72x^{-4}y^5 \qquad f_{y\,x}=\left(f_y\right)_x=6x^2y+72x^{-4}y^5$$

Okay, we can see that $f_{x\,y}=f_{y\,x}$ and so Clairaut's theorem has been verified for this function.

---

2. Verify Clairaut's Theorem for the following function.

$$A\left(x,y\right)=\cos\left(\frac{x}{y}\right)-x^7y^4+y^{10}$$

Step 1

First, we know we'll need the two 1st order partial derivatives. Here they are,

$$A_x=-\frac{1}{y}\sin\left(\frac{x}{y}\right)-7x^6y^4 \qquad A_y=\frac{x}{y^2}\sin\left(\frac{x}{y}\right)-4x^7y^3+10y^9$$

Step 2

Now let's compute each of the mixed second order partial derivatives.

$$A_{x\,y}=\left(A_x\right)_y=\frac{1}{y^2}\sin\left(\frac{x}{y}\right)+\frac{x}{y^3}\cos\left(\frac{x}{y}\right)-28x^6y^3$$
$$A_{y\,x}=\left(A_y\right)_x=\frac{1}{y^2}\sin\left(\frac{x}{y}\right)+\frac{x}{y^3}\cos\left(\frac{x}{y}\right)-28x^6y^3$$

Okay, we can see that $A_{x\,y}=A_{y\,x}$ and so Clairaut's theorem has been verified for this function.

3. Find all 2nd order derivatives for the following function.

$$g\left(u,v\right)=u^3v^4-2u\sqrt{v^3}+u^6-\sin\left(3v\right)$$

Step 1
First, we know we'll need the two 1st order partial derivatives. Here they are,

$$g_u=3u^2v^4-2v^{\frac{3}{2}}+6u^5 \qquad\qquad g_v=4u^3v^3-3uv^{\frac{1}{2}}-3\cos\left(3v\right)$$

Step 2
Now let's compute each of the second order partial derivatives.

$$\begin{aligned} g_{uu}&=\left(g_u\right)_u=6uv^4+30u^4\\ g_{uv}&=\left(g_u\right)_v=12u^2v^3-3v^{\frac{1}{2}}\\ g_{vu}&=g_{uv}=12u^2v^3-3v^{\frac{1}{2}} \qquad\qquad \text{by Clairaut's Theorem}\\ g_{vv}&=\left(g_v\right)_v=12u^3v^2-\tfrac{3}{2}uv^{-\frac{1}{2}}+9\sin\left(3v\right)\end{aligned}$$

4. Find all 2nd order derivatives for the following function.

$$f\left(s,t\right)=s^2t+\ln\left(t^2-s\right)$$

Step 1
First, we know we'll need the two 1st order partial derivatives. Here they are,

$$f_s=2st-\frac{1}{t^2-s} \qquad\qquad f_t=s^2+\frac{2t}{t^2-s}$$

Step 2
Now let's compute each of the second order partial derivatives.

$$f_{ss} = \left(f_s\right)_s = 2t - \frac{1}{\left(t^2 - s\right)^2}$$

$$f_{st} = \left(f_s\right)_t = 2s + \frac{2t}{\left(t^2 - s\right)^2}$$

$$f_{ts} = f_{st} = 2s + \frac{2t}{\left(t^2 - s\right)^2} \qquad \text{by Clairaut's Theorem}$$

$$f_{tt} = \left(f_t\right)_t = \frac{-2t^2 - 2s}{\left(t^2 - s\right)^2}$$

---

5. Find all 2nd order derivatives for the following function.

$$h(x,y) = \mathbf{e}^{x^4 y^6} - \frac{y^3}{x}$$

Step 1
First, we know we'll need the two 1st order partial derivatives. Here they are,

$$h_x = 4x^3 y^6 \mathbf{e}^{x^4 y^6} + \frac{y^3}{x^2} \qquad\qquad h_y = 6y^5 x^4 \mathbf{e}^{x^4 y^6} - \frac{3y^2}{x}$$

Step 2
Now let's compute each of the second order partial derivatives.

$$h_{xx} = \left(h_x\right)_x = 12x^2 y^6 \mathbf{e}^{x^4 y^6} + 16x^6 y^{12} \mathbf{e}^{x^4 y^6} - \frac{2y^3}{x^3}$$

$$h_{xy} = \left(h_x\right)_y = 24x^3 y^5 \mathbf{e}^{x^4 y^6} + 24x^7 y^{11} \mathbf{e}^{x^4 y^6} + \frac{3y^2}{x^2}$$

$$h_{yx} = h_{xy} = 24x^3 y^5 \mathbf{e}^{x^4 y^6} + 24x^7 y^{11} \mathbf{e}^{x^4 y^6} + \frac{3y^2}{x^2} \qquad \text{by Clairaut's Theorem}$$

$$h_{yy} = \left(h_y\right)_y = 30y^4 x^4 \mathbf{e}^{x^4 y^6} + 36y^{10} x^8 \mathbf{e}^{x^4 y^6} - \frac{6y}{x}$$

---

6. Find all 2nd order derivatives for the following function.

$$f(x,y,z) = \frac{x^2 y^6}{z^3} - 2x^6 z + 8y^{-3} x^4 + 4z^2$$

Step 1
First, we know we'll need the three 1st order partial derivatives. Here they are,

$$\begin{aligned}
f_x &= 2xy^6z^{-3} - 12x^5z + 32y^{-3}x^3 \\
f_y &= 6x^2y^5z^{-3} - 24y^{-4}x^4 \\
f_z &= -3x^2y^6z^{-4} - 2x^6 + 8z
\end{aligned}$$

Step 2
Now let's compute each of the second order partial derivatives (and there will be a few of them….).

$$\begin{aligned}
f_{xx} &= \left(f_x\right)_x = 2y^6z^{-3} - 60x^4z + 96y^{-3}x^2 \\
f_{xy} &= \left(f_x\right)_y = 12xy^5z^{-3} - 96y^{-4}x^3 \\
f_{xz} &= \left(f_x\right)_z = -6xy^6z^{-4} - 12x^5 \\
f_{yx} &= f_{xy} = 12xy^5z^{-3} - 96y^{-4}x^3 && \text{by Clairaut's Theorem} \\
f_{yy} &= \left(f_y\right)_y = 30x^2y^4z^{-3} + 96y^{-5}x^4 \\
f_{yz} &= \left(f_y\right)_z = -18x^2y^5z^{-4} \\
f_{zx} &= f_{xz} = -6xy^6z^{-4} - 12x^5 && \text{by Clairaut's Theorem} \\
f_{zy} &= f_{yz} = -18x^2y^5z^{-4} && \text{by Clairaut's Theorem} \\
f_{zz} &= \left(f_z\right)_z = 12x^2y^6z^{-5} + 8
\end{aligned}$$

Note that when we used Clairaut's Theorem here we used the natural extension to the Theorem we gave in the notes.

---

7. Given $f\left(x,y,z\right) = x^4y^3z^6$ find $\dfrac{\partial^6 f}{\partial y\partial z^2\partial y\partial x^2}$.

Step 1
Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to. However, in this case there doesn't seem to be any reason to do anything other than the order shown in the problem statement.

Here is the first derivative we need to take.

$$\frac{\partial f}{\partial x} = 4x^3y^3z^6$$

Step 2
The second derivative is,

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 y^3 z^6$$

Step 3
The third derivative is,

$$\frac{\partial^3 f}{\partial y \partial x^2} = 36x^2 y^2 z^6$$

Step 4
The fourth derivative is,

$$\frac{\partial^4 f}{\partial z \partial y \partial x^2} = 216x^2 y^2 z^5$$

Step 5
The fifth derivative is,

$$\frac{\partial^5 f}{\partial z^2 \partial y \partial x^2} = 1080x^2 y^2 z^4$$

Step 6
The sixth and final derivative we need for this problem is,

$$\boxed{\frac{\partial^6 f}{\partial y \partial z^2 \partial y \partial x^2} = 2160x^2 y z^4}$$

---

8. Given $w = u^2 \mathbf{e}^{-6v} + \cos\left(u^6 - 4u + 1\right)$ find $w_{v\,u\,u\,v\,v}$.

Step 1
Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to. However, in this case there doesn't seem to be any reason to do anything other than the order shown in the problem statement.

Here is the first derivative we need to take.

$$w_v = -6u^2 \mathbf{e}^{-6v}$$

Note that if we'd done a couple of *u* derivatives first the second derivative would have been a product rule. Because we did the *v* derivative first we won't need to worry about the "messy" *u* derivatives of the second term as that term differentiates to zero when differentiating with respect to *v*.

Step 2
The second derivative is,

$$w_{vu} = -12u\mathbf{e}^{-6v}$$

Step 3
The third derivative is,

$$w_{vuu} = -12\mathbf{e}^{-6v}$$

Step 4
The fourth derivative is,

$$w_{vuuv} = 72\mathbf{e}^{-6v}$$

Step 5
The fifth and final derivative we need for this problem is,

$$\boxed{w_{vuuvv} = -432\mathbf{e}^{-6v}}$$

---

9. Given $G(x,y) = y^4 \sin(2x) + x^2\left(y^{10} - \cos\left(y^2\right)\right)^7$ find $G_{yyyxxxy}$.

Step 1
Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to.

In this case the *y* derivatives of the second term will become unpleasant at some point given that we have four of them. However, the second term has an x^2 and there are three *x* derivatives we'll need to do eventually. Therefore, the second term will differentiate to zero with the third *x* derivative. So, let's make heavy use of Clairaut's to do the three *x* derivatives first prior to any of the *y* derivatives so we won't need to deal with the "messy" *y* derivatives with the second term.

Here is the first derivative we need to take.

$$G_x = 2y^4\cos(2x) + 2x\left(y^{10} - \cos\left(y^2\right)\right)^7$$

Note that if we'd done a couple of *y* derivatives first the second would have been a product rule and because we did the *x* derivative first we won't need to every work about the "messy" *y* derivatives of the second term.

Step 2
The second derivative is,

$$G_{xx} = -4y^4 \sin\left(2x\right) + 2\left(y^{10} - \cos\left(y^2\right)\right)^7$$

Step 3
The third derivative is,

$$G_{xxx} = -8y^4 \cos\left(2x\right)$$

Step 4
The fourth derivative is,

$$G_{xxxy} = -32y^3 \cos\left(2x\right)$$

Step 5
The fifth derivative is,

$$G_{xxxyy} = -96y^2 \cos\left(2x\right)$$

Step 6
The sixth derivative is,

$$G_{xxxyyy} = -192y \cos\left(2x\right)$$

Step 7
The seventh and final derivative we need for this problem is,

$$\boxed{G_{yyyxxxy} = G_{xxxyyyy} = -192\cos\left(2x\right)}$$

Section 2-5 : Differentials

1. Compute the differential of the following function.

$$z = x^2 \sin(6y)$$

Solution
Not much to do here. Just recall that the differential in this case is,

$$dz = z_x \, dx + z_y \, dy$$

The differential is then,

$$\boxed{dz = 2x\sin(6y)\,dx + 6x^2\cos(6y)\,dy}$$

---

2. Compute the differential of the following function.

$$f(x,y,z) = \ln\left(\frac{xy^2}{z^3}\right)$$

Solution
Not much to do here. Just recall that the differential in this case is,

$$df = f_x \, dx + f_y \, dy + f_z \, dz$$

The differential is then,

$$\boxed{df = \frac{1}{x}dx + \frac{2}{y}dy - \frac{3}{z}dz}$$

Section 2-6 : Chain Rule

1. Given the following information use the Chain Rule to determine $\dfrac{dz}{dt}$.

$$z = \cos\left(y\,x^2\right) \qquad x = t^4 - 2t, \quad y = 1 - t^6$$

Solution

Okay, we can just use the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= \left[-2xy\sin\left(yx^2\right)\right]\left[4t^3 - 2\right] + \left[-x^2\sin\left(yx^2\right)\right]\left[-6t^5\right]\\ &= \boxed{-2\left(t^4 - 2t\right)\left(1 - t^6\right)\left(4t^3 - 2\right)\sin\left(\left(1 - t^6\right)\left(t^4 - 2t\right)^2\right) + 6t^5\left(t^4 - 2t\right)^2\sin\left(\left(1 - t^6\right)\left(t^4 - 2t\right)^2\right)}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for *x* and *y* in the third step just to get an equation in *t*. For some of these, due to the mess of the final formula, it might have been easier to just leave the *x*'s and *y*'s alone and acknowledge their definition in terms of *t* to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

---

2. Given the following information use the Chain Rule to determine $\dfrac{dw}{dt}$.

$$w = \frac{x^2 - z}{y^4} \qquad x = t^3 + 7, \quad y = \cos\left(2t\right), \quad z = 4t$$

Solution

Okay, we can just use a (hopefully) pretty obvious extension of the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dw}{dt}&=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}\\&=\left[\frac{2x}{y^4}\right]\left[3t^2\right]+\left[\frac{-4\left(x^2-z\right)}{y^5}\right]\left[-2\sin\left(2t\right)\right]+\left[-\frac{1}{y^4}\right]\left[4\right]\\&=\boxed{\frac{6t^2\left(t^3+7\right)}{\cos^4\left(2t\right)}+\frac{8\sin\left(2t\right)\left(\left(t^3+7\right)^2-4t\right)}{\cos^5\left(2t\right)}-\frac{4}{\cos^4\left(2t\right)}}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for *x* and *y* in the third step just to get an equation in *t*. For some of these, due to the mess of the final formula, it might have been easier to just leave the *x*'s and *y*'s alone and acknowledge their definition in terms of *t* to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

---

3. Given the following information use the Chain Rule to determine $\dfrac{dz}{dx}$.

$$z=x^2y^4-2y \qquad\qquad y=\sin\left(x^2\right)$$

Solution

Okay, we can just use the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dz}{dx}&=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}\\&=\left[2xy^4\right]+\left[4x^2y^3-2\right]\left[2x\cos\left(x^2\right)\right]\\&=\boxed{2x\sin^4\left(x^2\right)+2x\left(4x^2\sin^3\left(x^2\right)-2\right)\cos\left(x^2\right)}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for *y* in the third step just to get an equation in *x*. For some of these, due to the mess of the final formula, it might have been easier to just leave the *y*'s alone and acknowledge their definition in terms of *x* to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

---

4. Given the following information use the Chain Rule to determine $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$z = x^{-2} y^{6} - 4x \qquad x = u^{2} v, \quad y = v - 3u$$

Solution

Okay, we can just use the "formulas" from the notes (with a small change to the letters) to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= \left[-2x^{-3}y^{6} - 4\right]\left[2uv\right] + \left[6x^{-2}y^{5}\right]\left[-3\right] \\ &= \boxed{2uv\left(-2u^{-6}v^{-3}\left(v-3u\right)^{6} - 4\right) - 18u^{-4}v^{-2}\left(v-3u\right)^{5}}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= \left[-2x^{-3}y^{6} - 4\right]\left[u^{2}\right] + \left[6x^{-2}y^{5}\right]\left[1\right] \\ &= \boxed{u^{2}\left(-2u^{-6}v^{-3}\left(v-3u\right)^{6} - 4\right) + 6u^{-4}v^{-2}\left(v-3u\right)^{5}}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for *x* and *y* in the third step just to get an equation in *u* and *v*. For some of these, due to the mess of the final formula, it might have been easier to just leave the *x*'s and *y*'s alone and acknowledge their definition in terms of *u* and *v* to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

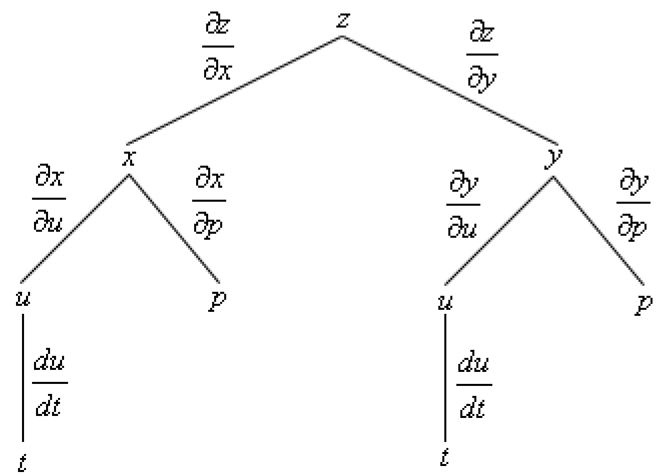
---

5. Given the following information use the Chain Rule to determine z_t and z_p .

$$z = 4y\sin\left(2x\right) \qquad x = 3u - p, \quad y = p^{2}u, \quad u = t^{2} + 1$$

Step 1

Okay, we don't have a formula from the notes for this one so we'll need to derive on up first. To do this we'll need the following tree diagram.



Step 2
Here are the formulas for the two derivatives we're being asked to find.

$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \qquad\qquad z_p = \frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p}$$

Step 3
Here is the work for this problem.

$$\begin{aligned} z_t &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \\ &= \left[8y\cos\left(2x\right)\right]\left[3\right]\left[2t\right] + \left[4\sin\left(2x\right)\right]\left[p^2\right]\left[2t\right] \\ &= \boxed{48ty\cos\left(2x\right) + 8tp^2\sin\left(2x\right)} \end{aligned}$$

$$\begin{aligned} z_p &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} \\ &= \left[8y\cos\left(2x\right)\right]\left[-1\right] + \left[4\sin\left(2x\right)\right]\left[2pu\right] \\ &= \boxed{-8y\cos\left(2x\right) + 8pu\sin\left(2x\right)} \end{aligned}$$

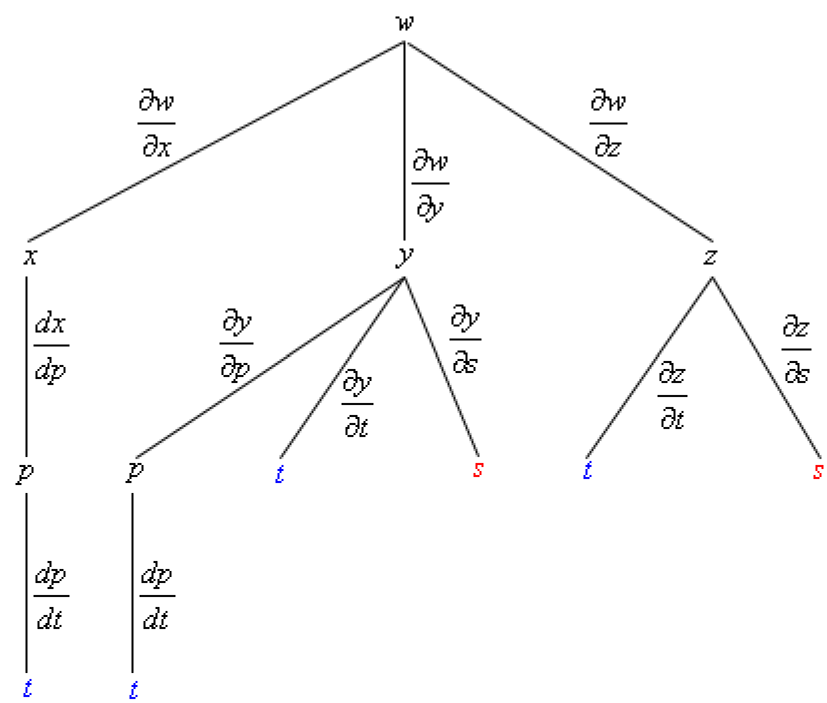
In the second step of each of the derivatives we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we didn't do any "back substitution" in these derivatives due to the mess that we'd get from each of the derivatives after we got done with all the back substitution.

6. Given the following information use the Chain Rule to determine $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

$$w = \sqrt{x^2 + y^2} + \frac{6z}{y} \qquad x = \sin(p), \quad y = p + 3t - 4s, \quad z = \frac{t^3}{s^2}, \quad p = 1 - 2t$$

Step 1
Okay, we don't have a formula from the notes for this one so we'll need to derive on up first. To do this we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Step 2
Here are the formulas for the two derivatives we're being asked to find.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Step 3
Here is the work for this problem.

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\ &= \left[\frac{x}{\sqrt{x^2+y^2}}\right]\left[\cos(p)\right][-2] + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][1][-2] + \\ &\qquad + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][3] + \left[\frac{6}{y}\right]\left[\frac{3t^2}{s^2}\right] \\ &= \boxed{\frac{-2x\cos(p)}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2} + \frac{18t^2}{ys^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][-4] + \left[\frac{6}{y}\right]\left[-\frac{2t^3}{s^3}\right] \\ &= \boxed{\frac{-4y}{\sqrt{x^2+y^2}} + \frac{24z}{y^2} - \frac{12t^3}{ys^3}}\end{aligned}$$

In the second step of each of the derivatives we added brackets just to make it clear which term came from which derivative in the "formula" and in general probably aren't needed. We also did a little simplification as needed to get to the third step.

Also, we didn't do any "back substitution" in these derivatives due to the mess that we'd get from each of the derivatives after we got done with all the back substitution.

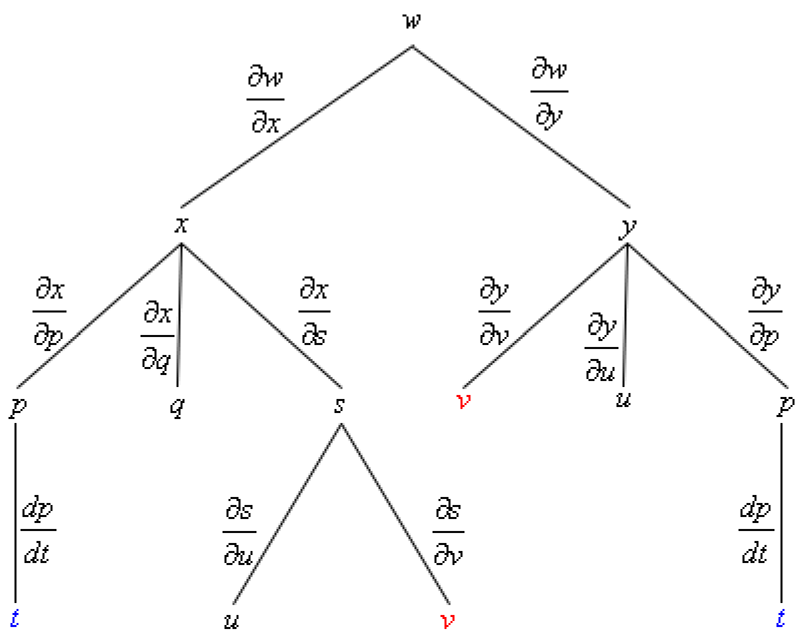
---

7. Determine formulas for $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial v}$ for the following situation.

$$w = w(x,y) \qquad\qquad x = x(p,q,s), \quad y = y(p,u,v), \quad s = s(u,v), \quad p = p(t)$$

Step 1

To determine the formula for these derivatives we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Step 2

Here are the formulas we're being asked to find.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

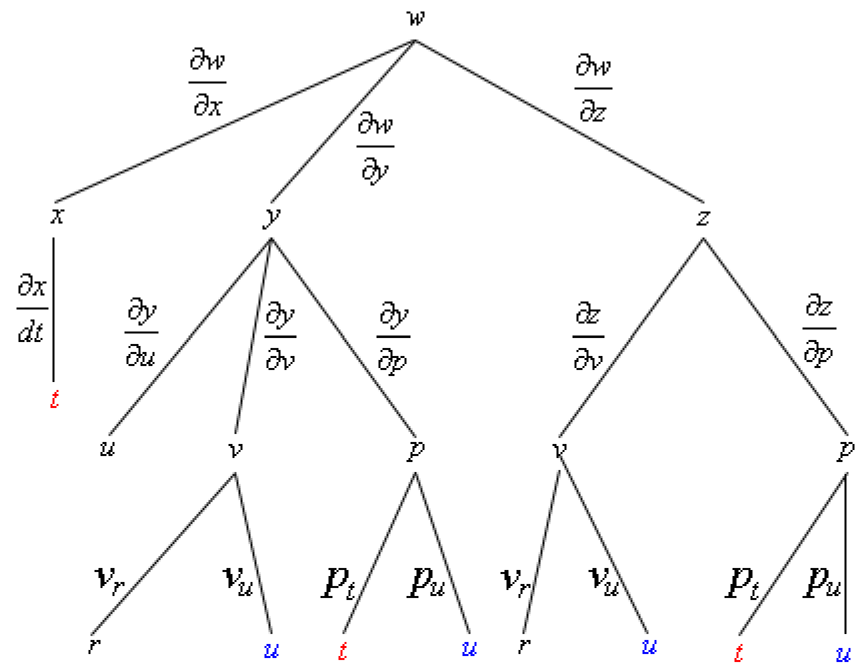
---

8. Determine formulas for $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial u}$ for the following situation.

$$w = w\left(x, y, z\right) \qquad x = x\left(t\right), \quad y = y\left(u, v, p\right), \quad z = z\left(v, p\right), \quad v = v\left(r, u\right), \quad p = p\left(t, u\right)$$

Step 1

To determine the formula for these derivatives we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Also, because the last "row" of branches was getting a little close together we switched to the subscript derivative notation to make it easier to see which derivative was associated with each branch.

Step 2
Here are the formulas we're being asked to find.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p}\frac{\partial p}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p}\frac{\partial p}{\partial u}$$

---

9. Compute $\dfrac{dy}{dx}$ for the following equation.

$$x^2 y^4 - 3 = \sin(xy)$$

Step 1
First a quick rewrite of the equation.

$$x^2 y^4 - 3 - \sin(xy) = 0$$

Step 2
From the rewrite in the previous step we can see that,

$$F(x,y) = x^2 y^4 - 3 - \sin(xy)$$

We can now simply use the formula we derived in the notes to get the derivative.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{2xy^4 - y\cos(xy)}{4x^2y^3 - x\cos(xy)} = \frac{y\cos(xy) - 2xy^4}{4x^2y^3 - x\cos(xy)}}$$

Note that in for the second form of the answer we simply moved the "-" in front of the fraction up to the numerator and multiplied it through. We could just have easily done this with the denominator instead if we'd wanted to.

---

10. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following equation.

$$\mathbf{e}^{zy} + xz^2 = 6xy^4z^3$$

Step 1
First a quick rewrite of the equation.

$$\mathbf{e}^{zy} + xz^2 - 6xy^4z^3 = 0$$

Step 2
From the rewrite in the previous step we can see that,

$$F(x,y) = \mathbf{e}^{zy} + xz^2 - 6xy^4z^3$$

We can now simply use the formulas we derived in the notes to get the derivatives.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \boxed{-\frac{z^2 - 6y^4z^3}{y\mathbf{e}^{zy} + 2xz - 18xy^4z^2} = \frac{6y^4z^3 - z^2}{y\mathbf{e}^{zy} + 2xz - 18xy^4z^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \boxed{-\frac{z\mathbf{e}^{zy} - 24xy^3z^3}{y\mathbf{e}^{zy} + 2xz - 18xy^4z^2} = \frac{24xy^3z^3 - z\mathbf{e}^{zy}}{y\mathbf{e}^{zy} + 2xz - 18xy^4z^2}}$$

Note that in for the second form of each of the answers we simply moved the "-" in front of the fraction up to the numerator and multiplied it through. We could just have easily done this with the denominator instead if we'd wanted to.

---

11. Determine f_{uu} for the following situation.

$$f = f\left(x, y\right) \qquad\qquad x = u^2 + 3v, \quad y = uv$$

Step 1
These kinds of problems always seem mysterious at first. That is probably because we don't actually know what the function itself is. This isn't really a problem. It simply means that the answers can get a little messy as we'll rarely be able to do much in the way of simplification.

So, the first step here is to get the first derivative and this will require the following chain rule formula.

$$f_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Here is the first derivative,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\left[2u\right] + \frac{\partial f}{\partial y}\left[v\right] = 2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

Do not get excited about the "unknown" derivatives in our answer here. They will always be present in these kinds of problems.

Step 2
Now, much as we did in the notes, let's do a little rewrite of the answer above as follows,

$$\frac{\partial}{\partial u}\left(f\right) = 2u\frac{\partial}{\partial x}\left(f\right) + v\frac{\partial}{\partial y}\left(f\right)$$

With this rewrite we now have a "formula" for differentiating any function of *x* and *y* with respect to *u* whenever $x = u^2 + 3v$ and $y = uv$. In other words, whenever we have such a function all we need to do is replace the *f* in the parenthesis with whatever our function is. We'll need this eventually.

Step 3
Now, let's get the second derivative. We know that we find the second derivative as follows,

$$f_{uu} = \frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right)$$

Step 4

Now, recall that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are functions of *x* and *y* which are in turn defined in terms of *u* and *v* as defined in the problem statement. This means that we'll need to do the product rule on the first term since it is a product of two functions that both involve *u*. We won't need to product rule the second term, in this case, because the first function in that term involves only *v*'s.

Here is that work,

$$f_{uu} = 2\frac{\partial f}{\partial x} + 2u\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$$

Because the function is defined only in terms of *x* and *y* we cannot "merge" the *u* and *x* derivatives in the second term into a "mixed order" second derivative. For the same reason we cannot "merge" the *u* and *y* derivatives in the third term.

In each of these cases we are being asked to differentiate functions of *x* and *y* with respect to *u* where *x* and *y* are defined in terms of *u* and *v*.

Step 5

Now, recall the "formula" from Step 2,

$$\frac{\partial}{\partial u}\left(f\right) = 2u\frac{\partial}{\partial x}\left(f\right) + v\frac{\partial}{\partial y}\left(f\right)$$

Recall that this tells us how to differentiate any function of *x* and *y* with respect to *u* as long as *x* and *y* are defined in terms of *u* and *v* as they are in this problem.

Well luckily enough for us both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are functions of *x* and *y* which in turn are defined in terms of *u* and *v* as we need them to be. This means we can use this "formula" for each of the derivatives in the result from Step 4 as follows,

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = 2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y\partial x}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = 2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2u\frac{\partial^2 f}{\partial x\partial y} + v\frac{\partial^2 f}{\partial y^2}$$

Step 6

Okay, all we need to do now is put the results from Step 5 into the result from Step 4 and we'll be done.

$$\begin{aligned} f_{uu} &= 2\frac{\partial f}{\partial x} + 2u\left[2u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y \partial x}\right] + v\left[2u\frac{\partial^2 f}{\partial x \partial y} + v\frac{\partial^2 f}{\partial y^2}\right] \\ &= 2\frac{\partial f}{\partial x} + 4u^2\frac{\partial^2 f}{\partial x^2} + 2uv\frac{\partial^2 f}{\partial y \partial x} + 2uv\frac{\partial^2 f}{\partial x \partial y} + v^2\frac{\partial^2 f}{\partial y^2} \\ &= \boxed{2\frac{\partial f}{\partial x} + 4u^2\frac{\partial^2 f}{\partial x^2} + 4uv\frac{\partial^2 f}{\partial x \partial y} + v^2\frac{\partial^2 f}{\partial y^2}} \end{aligned}$$

Note that we assumed that the two mixed order partial derivative are equal for this problem and so combined those terms. If you can’t assume this or don’t want to assume this then the second line would be the answer.

Section 2-7 : Directional Derivatives

1. Determine the gradient of the following function.

$$f\left(x,y\right)=x^2\sec\left(3x\right)-\frac{x^2}{y^3}$$

Solution
Not really a lot to do for this problem. Here is the gradient.

$$\nabla f=\left\langle f_x,f_y\right\rangle=\boxed{\left\langle 2x\sec\left(3x\right)+3x^2\sec\left(3x\right)\tan\left(3x\right)-\frac{2x}{y^3},\frac{3x^2}{y^4}\right\rangle}$$

---

2. Determine the gradient of the following function.

$$f\left(x,y,z\right)=x\cos\left(xy\right)+z^2y^4-7xz$$

Solution
Not really a lot to do for this problem. Here is the gradient.

$$\nabla f=\left\langle f_x,f_y,f_z\right\rangle=\boxed{\left\langle\cos\left(xy\right)-xy\sin\left(xy\right)-7z,-x^2\sin\left(xy\right)+4z^2y^3,2zy^4-7x\right\rangle}$$

---

3. Determine $D_{\vec{u}}f$ for $f\left(x,y\right)=\cos\left(\dfrac{x}{y}\right)$ in the direction of $\vec{v}=\left\langle 3,-4\right\rangle$.

Step 1
Okay, we know we need the gradient so let's get that first.

$$\nabla f=\left\langle-\frac{1}{y}\sin\left(\frac{x}{y}\right),\frac{x}{y^2}\sin\left(\frac{x}{y}\right)\right\rangle$$

Step 2
Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\|\vec{v}\| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5 \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}\langle 3,-4\rangle = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$$

Step 3
The directional derivative is then,

$$D_{\vec{u}} f = \left\langle -\frac{1}{y}\sin\left(\frac{x}{y}\right), \frac{x}{y^2}\sin\left(\frac{x}{y}\right)\right\rangle \bullet \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$$
$$= \boxed{-\frac{3}{5y}\sin\left(\frac{x}{y}\right) - \frac{4x}{5y^2}\sin\left(\frac{x}{y}\right) = -\frac{1}{5}\left(\frac{3}{y} + \frac{4x}{y^2}\right)\sin\left(\frac{x}{y}\right)}$$

---

4. Determine $D_{\vec{u}} f$ for $f(x,y,z) = x^2y^3 - 4xz$ in the direction of $\vec{v} = \langle -1, 2, 0\rangle$.

Step 1
Okay, we know we need the gradient so let's get that first.

$$\nabla f = \langle 2xy^3 - 4z, 3x^2y^2, -4x\rangle$$

Step 2
Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\|\vec{v}\| = \sqrt{(-1)^2 + (2)^2 + (0)^2} = \sqrt{5} \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}}\langle -1,2,0\rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle$$

Step 3
The directional derivative is then,

$$D_{\vec{u}} f = \langle 2xy^3 - 4z, 3x^2y^2, -4x\rangle \bullet \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle = \boxed{\frac{1}{\sqrt{5}}\left(4z - 2xy^3 + 6x^2y^2\right)}$$

---

5. Determine $D_{\vec{u}} f(3,-1,0)$ for $f(x,y,z) = 4x - y^2\mathbf{e}^{3xz}$ direction of $\vec{v} = \langle -1, 4, 2\rangle$.

Step 1
Okay, we know we need the gradient so let's get that first.

$$\nabla f = \left\langle 4 - 3zy^2\mathbf{e}^{3xz}, -2y\mathbf{e}^{3xz}, -3xy^2\mathbf{e}^{3xz} \right\rangle$$

Because we also know that we'll eventually need this evaluated at the point we may as well get that as well.

$$\nabla f\left(3,-1,0\right) = \left\langle 4,2,-9 \right\rangle$$

Step 2
Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\left\|\vec{v}\right\| = \sqrt{\left(-1\right)^2 + \left(4\right)^2 + \left(2\right)^2} = \sqrt{21} \qquad \vec{u} = \frac{\vec{v}}{\left\|\vec{v}\right\|} = \frac{1}{\sqrt{21}}\left\langle -1,4,2 \right\rangle = \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$$

Step 3
The directional derivative is then,

$$D_{\vec{u}}f\left(3,-1,0\right) = \left\langle 4,2,-9 \right\rangle \bullet \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle = \boxed{-\frac{14}{\sqrt{21}}}$$

---

6. Find the maximum rate of change of $f\left(x,y\right) = \sqrt{x^2 + y^4}$ at $\left(-2,3\right)$ and the direction in which this maximum rate of change occurs.

Step 1
First, we'll need the gradient and its value at $\left(-2,3\right)$.

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2 + y^4}}, \frac{2y^3}{\sqrt{x^2 + y^4}} \right\rangle \qquad \nabla f\left(-2,3\right) = \left\langle \frac{-2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle$$

Step 2
Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at $\left(-2,3\right)$, which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

$$\nabla f\left(-2,3\right) = \boxed{\left\langle \frac{-2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle}$$

Step 3
The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,

$$\left\|\nabla f\left(-2,3\right)\right\| = \sqrt{\frac{4}{85}+\frac{2916}{85}} = \boxed{\sqrt{\frac{584}{17}} = 5.8611}$$

---

7. Find the maximum rate of change of $f\left(x,y,z\right) = \mathbf{e}^{2x}\cos\left(y-2z\right)$ at $\left(4,-2,0\right)$ and the direction in which this maximum rate of change occurs.

Step 1
First, we'll need the gradient and its value at $\left(4,-2,0\right)$.

$$\nabla f = \left\langle 2\mathbf{e}^{2x}\cos\left(y-2z\right), -\mathbf{e}^{2x}\sin\left(y-2z\right), 2\mathbf{e}^{2x}\sin\left(y-2z\right)\right\rangle$$
$$\nabla f\left(4,-2,0\right) = \left\langle 2\mathbf{e}^{8}\cos\left(-2\right), -\mathbf{e}^{8}\sin\left(-2\right), 2\mathbf{e}^{8}\sin\left(-2\right)\right\rangle = \left\langle -2481.03, 2710.58, -5421.15\right\rangle$$

Step 2
Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at $\left(4,-2,0\right)$, which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

$$\nabla f\left(4,-2,0\right) = \boxed{\left\langle -2481.03, 2710.58, -5421.15\right\rangle}$$

Step 3
The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,

$$\left\|\nabla f\left(4,-2,0\right)\right\| = \sqrt{\left(-2481.03\right)^2+\left(2710.58\right)^2+\left(-5421.15\right)^2} = \boxed{6549.17}$$

Chapter 3 : Applications of Partial Derivatives

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f(x,y)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.

Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.

Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).

Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).

Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Section 3-1 : Tangent Planes and Linear Approximations

1. Find the equation of the tangent plane to $z = x^2 \cos(\pi y) - \dfrac{6}{xy^2}$ at $(2,-1)$.

Step 1
First, we know we'll need the two 1st order partial derivatives. Here they are,

$$f_x = 2x\cos(\pi y) + \frac{6}{x^2 y^2} \qquad\qquad f_y = -\pi x^2 \sin(\pi y) + \frac{12}{xy^3}$$

Step 2
Now we also need the two derivatives from the first step and the function evaluated at $(2,-1)$. Here are those evaluations,

$$f(2,-1) = -7 \qquad f_x(2,-1) = -\frac{5}{2} \qquad f_y(2,-1) = -6$$

Step 3
The tangent plane is then,

$$\boxed{z = -7 - \tfrac{5}{2}(x-2) - 6(y+1) = -\tfrac{5}{2}x - 6y - 8}$$

---

2. Find the equation of the tangent plane to $z = x\sqrt{x^2 + y^2} + y^3$ at $(-4,3)$.

Step 1
First, we know we'll need the two 1st order partial derivatives. Here they are,

$$f_x = \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} \qquad\qquad f_y = \frac{xy}{\sqrt{x^2 + y^2}} + 3y^2$$

Step 2
Now we also need the two derivatives from the first step and the function evaluated at $(-4,3)$. Here are those evaluations,

$$f(-4,3) = 7 \qquad f_x(-4,3) = \frac{41}{5} \qquad f_y(-4,3) = \frac{123}{5}$$

Step 3
The tangent plane is then,

$$\boxed{z = 7 + \tfrac{41}{5}\left(x+4\right) + \tfrac{123}{5}\left(y-3\right) = \tfrac{41}{5}x + \tfrac{123}{5}y - 34}$$

---

3. Find the linear approximation to $z = 4x^2 - y\mathbf{e}^{2x+y}$ at $\left(-2,4\right)$.

Step 1
Recall that the linear approximation to a function at a point is really nothing more than the tangent plane to that function at the point.

So, we know that we'll first need the two 1st order partial derivatives. Here they are,

$$f_x = 8x - 2y\mathbf{e}^{2x+y} \qquad\qquad f_y = -\mathbf{e}^{2x+y} - y\mathbf{e}^{2x+y}$$

Step 2
Now we also need the two derivatives from the first step and the function evaluated at $\left(-2,4\right)$. Here are those evaluations,

$$f\left(-2,4\right) = 12 \qquad\qquad f_x\left(-2,4\right) = -24 \qquad\qquad f_y\left(-2,4\right) = -5$$

Step 3
The linear approximation is then,

$$\boxed{L\left(x,y\right) = 12 - 24\left(x+2\right) - 5\left(y-4\right) = -24x - 5y - 16}$$

Section 3-2 : Gradient Vector, Tangent Planes and Normal Lines

1. Find the tangent plane and normal line to $x^2 y = 4z\mathbf{e}^{x+y} - 35$ at $\left(3,-3,2\right)$.

Step 1
First, we need to do a quick rewrite of the equation as,

$$x^2 y - 4z\mathbf{e}^{x+y} = -35$$

Step 2
Now we need the gradient of the function on the left side of the equation from Step 1 and its value at $\left(3,-3,2\right)$. Here are those quantities.

$$\nabla f = \left\langle 2xy - 4z\mathbf{e}^{x+y}, x^2 - 4z\mathbf{e}^{x+y}, -4\mathbf{e}^{x+y}\right\rangle \qquad\qquad \nabla f\left(3,-3,2\right) = \left\langle -26,1,-4\right\rangle$$

Step 3
The tangent plane is then,

$$\boxed{-26\left(x-3\right)+\left(1\right)\left(y+3\right)-4\left(z-2\right)=0} \qquad \Rightarrow \qquad \boxed{-26x+y-4z=-89}$$

The normal line is,

$$\boxed{\vec r\left(t\right)=\left\langle 3,-3,2\right\rangle + t\left\langle -26,1,-4\right\rangle = \left\langle 3-26t,-3+t,2-4t\right\rangle}$$

---

2. Find the tangent plane and normal line to $\ln\left(\dfrac{x}{2y}\right) = z^2\left(x-2y\right)+3z+3$ at $\left(4,2,-1\right)$.

Step 1
First, we need to do a quick rewrite of the equation as,

$$\ln\left(\frac{x}{2y}\right) - z^2\left(x-2y\right) - 3z = 3$$

Step 2
Now we need the gradient of the function on the left side of the equation from Step 1 and its value at $\left(4,2,-1\right)$. Here are those quantities.

$$\nabla f = \left\langle \frac{1}{x} - z^2, -\frac{1}{y} + 2z^2, -2z\left(x - 2y\right) - 3 \right\rangle \qquad \nabla f\left(4,2,-1\right) = \left\langle -\frac{3}{4}, \frac{3}{2}, -3 \right\rangle$$

Step 3
The tangent plane is then,

$$\boxed{-\tfrac{3}{4}\left(x-4\right) + \tfrac{3}{2}\left(y-2\right) - 3\left(z+1\right) = 0} \qquad \Rightarrow \qquad \boxed{-\tfrac{3}{4}x + \tfrac{3}{2}y - 3z = 3}$$

The normal line is,

$$\boxed{\vec r\left(t\right) = \left\langle 4,2,-1\right\rangle + t\left\langle -\tfrac{3}{4}, \tfrac{3}{2}, -3\right\rangle = \left\langle 4 - \tfrac{3}{4}t, 2 + \tfrac{3}{2}t, -1 - 3t\right\rangle}$$

Section 3-3 : Relative Minimums and Maximums

1. Find and classify all the critical points of the following function.

$$f(x,y) = (y-2)x^2 - y^2$$

Step 1

We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$f_x = 2(y-2)x \qquad\qquad f_y = x^2 - 2y$$
$$f_{xx} = 2(y-2) \qquad f_{xy} = 2x \qquad f_{yy} = -2$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 &: \quad 2(y-2)x = 0 \qquad \to \qquad y = 2 \text{ or } x = 0 \\ f_y = 0 &: \quad x^2 - 2y = 0 \end{aligned}$$

As shown above we have two possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

$$y = 2 : \; x^2 - 4 = 0 \quad \to \quad x = \pm 2 \quad \Rightarrow \quad (2,2) \text{ and } (-2,2)$$

$$x = 0 : \; -2y = 0 \quad \to \quad y = 0 \quad \Rightarrow \quad (0,0)$$

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of *x* and *y* values they get. That is not how these types of systems are solved!

We got $x = \pm 2$ above only because we assumed first that $y = 2$ and so that leads to the two solutions listed in that first line above. Likewise, we only got $y = 0$ because we first assumed that $x = 0$ which leads to the third solution listed above in the second line. The points $(0,2)$, $(-2,0)$ and $(2,0)$ are NOT solutions to this system as can be easily checked by plugging them into the second equation in the system.

So, do not just "mix and match" all possible values of *x* and *y* into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in mind what assumptions you had to make in order to get certain *x* or *y* values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has three critical points : $(0,0)$, $(-2,2)$, $(2,2)$.

Also, before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3
Next, we'll need the following,

$$D(x,y) = f_{xx}f_{yy} - \left[f_{xy}\right]^2 = \left[2(y-2)\right]\left[-2\right] - \left[2x\right]^2 = -4(y-2) - 4x^2$$

Step 4
With $D(x,y)$ we can now classify each of the critical points as follows.

$$\begin{array}{llll} (0,0) & : \; D(0,0) = 8 > 0 & f_{xx}(0,0) = -4 < 0 & \text{Relative Maximum} \\ (-2,2) & : \; D(-2,2) = -16 < 0 & & \text{Saddle Point} \\ (2,2) & : \; D(2,2) = -16 < 0 & & \text{Saddle Point} \end{array}$$

Don't forget to check the value of f_{xx} when *D* is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative *D* we don't need the second check as we know the critical point will be a saddle point.

---

2. Find and classify all the critical points of the following function.

$$f(x,y) = 7x - 8y + 2xy - x^2 + y^3$$

Step 1
We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$\begin{array}{lll} f_x = 7 + 2y - 2x & & f_y = -8 + 2x + 3y^2 \\ \quad f_{xx} = -2 & f_{xy} = 2 & f_{yy} = 6y \end{array}$$

Step 2
Now, let's find the critical points for this problem. That means solving the following system.

$$\begin{array}{ll} f_x = 0 \; : & 7 + 2y - 2x = 0 \\ f_y = 0 \; : & -8 + 2x + 3y^2 = 0 \quad \to \quad x = 4 - \tfrac{3}{2}y^2 \end{array}$$

As shown above we solved the second equation for *x* and we can now plug this into the first equation as follows,

$$0 = 7 + 2y - 2\left(4 - \tfrac{3}{2}y^2\right) = 3y^2 + 2y - 1 = (3y-1)(y+1) \quad \to \quad y = -1,\ y = \tfrac{1}{3}$$

This gives two values of *y* which we can now plug back into either of our equations to find corresponding *x* values. Here is that work.

$$y = -1:\ x = 4 - \tfrac{3}{2}(-1)^2 = \tfrac{5}{2} \quad \Rightarrow \quad \left(\tfrac{5}{2}, -1\right)$$

$$y = \tfrac{1}{3}:\ x = 4 - \tfrac{3}{2}\left(\tfrac{1}{3}\right)^2 = \tfrac{23}{6} \quad \Rightarrow \quad \left(\tfrac{23}{6}, \tfrac{1}{3}\right)$$

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of *x* and *y* values they get. That is not how these types of systems are solved!

We got $x = \frac{5}{2}$ above only because we assumed first that $y = -1$ and so that leads to the solution listed in first line above. Likewise, we only got $x = \frac{23}{6}$ because we first assumed that $y = \frac{1}{3}$ which leads to the second solution listed in the second line above. The points $\left(\frac{5}{2}, \frac{1}{3}\right)$ and $\left(\frac{23}{6}, -1\right)$ are NOT a solutions to this system as can be easily checked by plugging these points into the either of the equations in the system.

So, do not just "mix and match" all possible values of *x* and *y* into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in mind what assumptions you had to make in order to get certain *x* or *y* values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has two critical points : $\left(\frac{5}{2}, -1\right), \left(\frac{23}{6}, \frac{1}{3}\right)$.

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3
Next, we'll need the following,

$$D(x,y) = f_{xx}f_{yy} - \left[f_{xy}\right]^2 = [-2][6y] - [2]^2 = -12y - 4$$

Step 4
With $D(x,y)$ we can now classify each of the critical points as follows.

$$\begin{array}{llll} \left(\frac{5}{2}, -1\right) & : \ D\left(\frac{5}{2}, -1\right) = 8 > 0 & f_{xx}\left(\frac{5}{2}, -1\right) = -2 < 0 & \text{Relative Maximum} \\ \left(\frac{23}{6}, \frac{1}{3}\right) & : \ D\left(\frac{23}{6}, \frac{1}{3}\right) = -8 < 0 & & \text{Saddle Point} \end{array}$$

Don't forget to check the value of f_{xx} when *D* is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative *D* we don't need the second check as we know the critical point will be a saddle point.

---

3. Find and classify all the critical points of the following function.

$$f(x,y) = \left(3x + 4x^3\right)\left(y^2 + 2y\right)$$

Step 1
We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

Do not make these derivatives harder than really are! Do not multiply the function out! We just have a function of *x*'s times a function of *y*'s. Take advantage of that when doing the derivatives.

$$f_x = \left(3 + 12x^2\right)\left(y^2 + 2y\right) \qquad\qquad f_y = \left(3x + 4x^3\right)\left(2y + 2\right)$$
$$f_{xx} = 24x\left(y^2 + 2y\right) \qquad f_{xy} = \left(3 + 12x^2\right)\left(2y + 2\right) \qquad f_{yy} = 2\left(3x + 4x^3\right)$$

Step 2
Now, let's find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 &: \quad \left(3 + 12x^2\right)\left(y^2 + 2y\right) = 0 \\ f_y = 0 &: \quad \left(3x + 4x^3\right)\left(2y + 2\right) = 0 \end{aligned}$$

We could start the solution process with either of these equations as both are pretty simple to solve. Let's start with the first equation.

$$\left(3 + 12x^2\right)\left(y^2 + 2y\right) = \left(3 + 12x^2\right)\left(y\right)\left(y + 2\right) = 0 \quad \to \quad y = 0, y = -2, \; x = \pm\tfrac{1}{2}i$$

Okay, we've got something to deal with at this point. We clearly get four different values to work with here. Two of them, however, are complex. One of the unspoken rules here is that we are only going to work with real values and so we will ignore any complex answers and work with only the real values.

So, we now have two possible values of *y* so let's plug each of them into the second equation as follows,

$$y = 0: \; 2\left(3x + 4x^3\right) = 2x\left(3 + 4x^2\right) = 0 \quad \to \quad x = 0, x = \pm\tfrac{\sqrt{3}}{2}i \quad \Rightarrow \quad (0,0)$$

$$y = -2: \; -2\left(3x + 4x^3\right) = 2x\left(3 + 4x^2\right) = 0 \quad \to \quad x = 0, x = \pm\tfrac{\sqrt{3}}{2}i \quad \Rightarrow \quad (0,-2)$$

As with the first part of the solution process we only take the real values and so ignore the complex portions from this part as well.

In the previous two problems we made mention at this point to be careful and not just from up points for all possible combinations of the *x* and *y* values we have at this point.

One of the reasons that students often do that is because of problems like this one where it appears that we are doing just that. However, we haven't just randomly formed all combinations here. It just so happened that when we assumed $y = 0$ and $y = -2$ that we just happened to get the same value of *x*, $x = 0$. In general, this won't happen and so do not read into this problem that we always just form all possible combinations of the *x* and *y* values to get the critical points for a function. We must always pay attention to the assumptions made at the start of each step.

So, in summary, this function has two critical points : $\left(0,-2\right)$, $\left(0,0\right)$.

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3
Next, we'll need the following,

$$\begin{aligned} D\left(x,y\right) &= f_{xx}f_{yy} - \left[f_{xy}\right]^2 \\ &= \left[24x\left(y^2+2y\right)\right]\left[2\left(3x+4x^3\right)\right] - \left[\left(3+12x^2\right)\left(2y+2\right)\right]^2 \\ &= 48x\left(3x+4x^3\right)\left(y^2+2y\right) - \left(3+12x^2\right)^2\left(2y+2\right)^2 \end{aligned}$$

Step 4
With $D\left(x,y\right)$ we can now classify each of the critical points as follows.

$$\begin{array}{lll} \left(0,-2\right) \quad : & D\left(0,-2\right) = -36 < 0 & \qquad \text{Saddle Point} \\ \left(0,0\right) \quad : & D\left(0,0\right) = -36 < 0 & \qquad \text{Saddle Point} \end{array}$$

Don't always expect every problem to have at least one relative extrema. As this example has shown it is completely possible to have only saddle points.

---

4. Find and classify all the critical points of the following function.

$$f\left(x,y\right) = 3y^3 - x^2y^2 + 8y^2 + 4x^2 - 20y$$

Step 1
We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$f_x = -2xy^2 + 8x \qquad\qquad f_y = 9y^2 - 2x^2y + 16y - 20$$

$$f_{xx} = -2y^2 + 8 \qquad f_{xy} = -4xy \qquad f_{yy} = 18y - 2x^2 + 16$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 &: \quad -2xy^2 + 8x = 2x\left(4 - y^2\right) = 0 \qquad \rightarrow \qquad y = \pm 2 \;\text{ or }\; x = 0 \\ f_y = 0 &: \quad 9y^2 - 2x^2y + 16y - 20 = 0 \end{aligned}$$

As shown above we have three possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

$$y = -2 : \; 4x^2 - 16 = 0 \quad \rightarrow \quad x = \pm 2 \quad \Rightarrow \quad (2,-2) \;\text{ and }\; (-2,-2)$$

$$y = 2 : \; -4x^2 + 48 = 0 \quad \rightarrow \quad x = \pm 2\sqrt{3} \quad \Rightarrow \quad \left(2\sqrt{3},2\right) \;\text{ and }\; \left(-2\sqrt{3},2\right)$$

$$x = 0 : \; 9y^2 + 16y - 20 = 0 \quad \rightarrow \quad y = \tfrac{-16 \pm \sqrt{976}}{18} \quad \Rightarrow \quad \left(0, \tfrac{-16-\sqrt{976}}{18}\right) \;\text{ and }\; \left(0, \tfrac{-16+\sqrt{976}}{18}\right)$$

As we noted in the first two problems in this section be careful to only write down the actual solutions as found in the above work. Do not just write down all possible combinations of *x* and *y* from each of the three lines above. If you do that for this problem you will end up with a large number of points that are not critical points.

Also, do not get excited about the "mess" (*i.e.* roots) involved in some of the critical points. They will be a fact of life with these problems on occasion.

So, in summary, this function has the following six critical points.

$$(-2,-2),\; (2,-2),\; \left(2\sqrt{3},2\right),\; \left(-2\sqrt{3},2\right),\; \left(0, \tfrac{-16-\sqrt{976}}{18}\right),\; \left(0, \tfrac{-16+\sqrt{976}}{18}\right).$$

Step 3

Next, we'll need the following,

$$\begin{aligned} D(x,y) &= f_{xx} f_{yy} - \left[f_{xy}\right]^2 \\ &= \left[-2y^2 + 8\right]\left[18y - 2x^2 + 16\right] - \left[-4xy\right]^2 \\ &= \left[-2y^2 + 8\right]\left[18y - 2x^2 + 16\right] - 16x^2y^2 \end{aligned}$$

Step 4

With $D(x,y)$ we can now classify each of the critical points as follows.

$$\begin{array}{llll}
\left(-2,-2\right) & : \; D\left(-2,-2\right) = -256 < 0 & & \text{Saddle Point} \\
\left(2,-2\right) & : \; D\left(2,-2\right) = -256 < 0 & & \text{Saddle Point} \\
\left(-2\sqrt{3},2\right) & : \; D\left(-2\sqrt{3},2\right) = -768 < 0 & & \text{Saddle Point} \\
\left(2\sqrt{3},2\right) & : \; D\left(2\sqrt{3},2\right) = -768 < 0 & & \text{Saddle Point} \\
\left(0,\frac{-16-\sqrt{976}}{18}\right) & : \; D\left(0,\frac{-16-\sqrt{976}}{18}\right) = 180.4 > 0 & f_{xx}\left(0,\frac{-16-\sqrt{976}}{18}\right) = -5.8 < 0 & \text{Relative Maximum} \\
\left(0,\frac{-16+\sqrt{976}}{18}\right) & : \; D\left(0,\frac{-16+\sqrt{976}}{18}\right) = 205.1 > 0 & f_{xx}\left(0,\frac{-16+\sqrt{976}}{18}\right) = 6.6 > 0 & \text{Relative Minimum}
\end{array}$$

Don't forget to check the value of f_{xx} when *D* is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative *D* we don't need the second check as we know the critical point will be a saddle point.

Section 3-4 : Absolute Minimums and Maximums

1. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=192x^3+y^2-4xy^2$ on the triangle with vertices $\left(0,0\right)$, $\left(4,2\right)$ and $\left(-2,2\right)$.

Step 1
We'll need the first order derivatives to start the problem off. Here they are,

$$f_x=576x^2-4y^2 \qquad\qquad f_y=2y-8xy$$

Step 2
We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x=0 &: \quad 576x^2-4y^2=0 \\ f_y=0 &: \quad 2y\left(1-4x\right)=0 \quad \rightarrow \quad y=0 \text{ or } x=\tfrac{1}{4} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

$$y=0:\ 576x^2=0 \ \rightarrow\ x=0 \ \Rightarrow\ \left(0,0\right)$$

$$x=\tfrac{1}{4}:\ 36-4y^2=0 \ \rightarrow\ y=\pm 3 \ \Rightarrow\ \left(\tfrac{1}{4},3\right) \text{ and } \left(\tfrac{1}{4},-3\right)$$

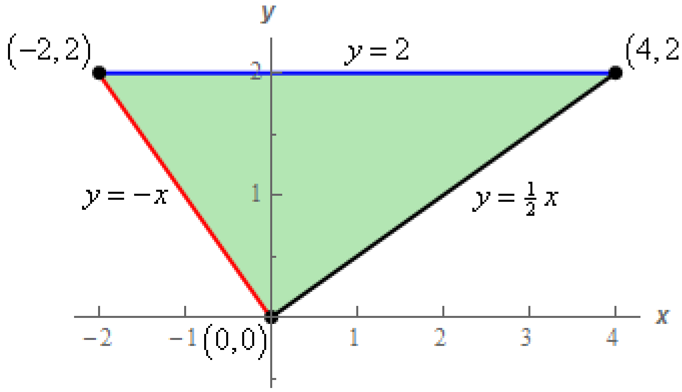
Okay, we have the three critical points listed above. Also recall that we only use critical points that are actually in the region we are working with. In this case, the last two have *y* values that clearly are out of the region (we've sketched the region in the next step if you aren't sure you believe this!) and so we can ignore them.

Therefore, the only critical point from this list that we need to use is the first. Note as well that, in this case, this also happens to be one of the points that define the boundary of the region. This will happen on occasion but won't always.

So, we'll need the function value for the only critical point that is actually in our region. Here is that value,

$$f\left(0,0\right)=0$$

Step 3
Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the triangle can then be defined as follows.

Top : $y = 2, \; -2 \le x \le 4$

Right : $y = \frac{1}{2}x, \; 0 \le x \le 4$

Left : $y = -x, \; -2 \le x \le 0$

Now we need to analyze each of these sides to get potential absolute extrema for $f(x,y)$ that might occur on the boundary.

Step 4
Let's first check out the top : $y = 2, \; -2 \le x \le 4$.

We'll need to identify the points along the top that could be potential absolute extrema for $f(x,y)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 4$.

$$g(x) = f(x,2) = 192x^3 - 16x + 4$$

This is really nothing more than a Calculus I absolute extrema problem so we'll be doing the work here without a lot of explanation. If you don't recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for $g(x)$ are,

$$g'(x) = 576x^2 - 16 = 0 \qquad \rightarrow \qquad x = \pm\tfrac{1}{6}$$

So, these two points as well as the *x* limits for the top give the following four points that are potential absolute extrema for $f(x,y)$.

$$\left(\tfrac{1}{6},2\right) \qquad \left(-\tfrac{1}{6},2\right) \qquad \left(-2,2\right) \qquad \left(4,2\right)$$

Recall that, in this step, we are assuming that $y = 2$! So, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(\tfrac{1}{6},2\right) = \tfrac{20}{9} \qquad f\left(-\tfrac{1}{6},2\right) = \tfrac{52}{9} \qquad f\left(-2,2\right) = -1,500 \qquad f\left(4,2\right) = 12,228$$

Step 5

Next let's check out the right side : $y = \frac{1}{2}x, \ 0 \le x \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $0 \le x \le 4$.

$$g\left(x\right) = f\left(x,\tfrac{1}{2}x\right) = \tfrac{1}{4}x^2 + 191x^3$$

The critical point(s) for the $g\left(x\right)$ from this step are,

$$g'\left(x\right) = \tfrac{1}{2}x + 573x^2 = x\left(\tfrac{1}{2} + 573x\right) = 0 \qquad \to \qquad x = 0, \quad x = -\tfrac{1}{1146}$$

Now, recall what we are restricted to the interval $0 \le x \le 4$ for this portion of the problem and so the second critical point above will not be used as it lies outside this interval.

So, the single point from above that is in the interval $0 \le x \le 4$ as well as the *x* limits for the right give the following two points that are potential absolute extrema for $f\left(x,y\right)$.

$$\left(0,0\right) \qquad \left(4,2\right)$$

Recall that, in this step, we are assuming that $y = \frac{1}{2}x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(0,0\right) = 0 \qquad f\left(4,2\right) = 12,228$$

Before proceeding to the next step note that both of these have already appeared in previous steps. This will happen on occasion but we can't, in many cases, expect this to happen so we do need to go through and do the work for each boundary.

The main exception to this is usually the endpoints of our intervals as they will always be shared in two of the boundary checks and so, once done, don't really need to be checked again. We just included the endpoints here for completeness.

Step 6

Finally, let's check out the left side : $y = -x,\ -2 \le x \le 0$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2 \le x \le 0$.

$$g(x) = f(x,-x) = x^2 + 188x^3$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = 2x + 564x^2 = 2x(1+282x) = 0 \qquad \to \qquad x = 0,\ \ x = -\tfrac{1}{282}$$

Both of these are in the interval $-2 \le x \le 0$ that we are restricted to for this portion of the problem.

So, the two points from above as well as the *x* limits for the right give the following three points that are potential absolute extrema for $f(x,y)$.

$$\left(-\tfrac{1}{282},\tfrac{1}{282}\right) \qquad (0,0) \qquad (-2,2)$$

Recall that, in this step we are assuming that $y = -x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f\left(-\tfrac{1}{282},\tfrac{1}{282}\right) = \tfrac{1}{238{,}572} \qquad f(0,0) = 0 \qquad f(-2,2) = -1{,}500$$

As with the previous step we can note that both of the end points above have already occurred previously in the problem and didn't really need to be checked here. They were just included for completeness.

Step 7
Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$f\left(\tfrac{1}{6},2\right) = \tfrac{20}{9} \qquad f\left(-\tfrac{1}{6},2\right) = \tfrac{52}{9} \qquad f(-2,2) = -1{,}500 \qquad f(4,2) = 12{,}228$$
$$f(0,0) = 0 \qquad f\left(-\tfrac{1}{282},\tfrac{1}{282}\right) = \tfrac{1}{238{,}572}$$

From this list we can see that the absolute maximum of the function will be 12,228 which occurs at $(4,2)$ and the absolute minimum of the function will be -1,500 which occurs at $(-2,2)$.

---

2. Find the absolute minimum and absolute maximum of $f(x,y) = (9x^2 - 1)(1+4y)$ on the rectangle given by $-2 \le x \le 3$, $-1 \le y \le 4$.

Step 1
We'll need the first order derivatives to start the problem off. Here they are,

$$f_x = 18x(1+4y) \qquad\qquad f_y = 4(9x^2 - 1)$$

Step 2
We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x &= 0 \;:\; 18x(1+4y) = 0 \\ f_y &= 0 \;:\; 4(9x^2-1) = 0 \quad \rightarrow \quad x = \pm\tfrac{1}{3} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

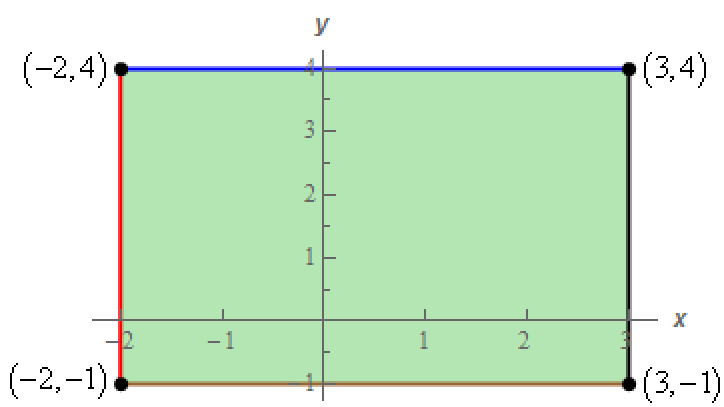
$$x = \tfrac{1}{3}:\; 6(1+4y) = 0 \;\rightarrow\; y = -\tfrac{1}{4} \;\Rightarrow\; \left(\tfrac{1}{3}, -\tfrac{1}{4}\right)$$

$$x = -\tfrac{1}{3}:\; -6(1+4y) = 0 \;\rightarrow\; y = -\tfrac{1}{4} \;\Rightarrow\; \left(-\tfrac{1}{3}, -\tfrac{1}{4}\right)$$

Both of these critical points are in the region we are interested in and so we'll need the function evaluated at both of them. Here are those values,

$$f\left(\tfrac{1}{3}, -\tfrac{1}{4}\right) = 0 \qquad\qquad f\left(-\tfrac{1}{3}, -\tfrac{1}{4}\right) = 0$$

Step 3
Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the rectangle can then be defined as follows.

Top : $y = 4, \quad -2 \le x \le 3$

Bottom : $y = -1, \quad -2 \le x \le 3$

Right : $x = 3, \quad -1 \le y \le 4$

Left : $x = -2, \quad -1 \le y \le 4$

Now we need to analyze each of these sides to get potential absolute extrema for $f\left(x, y\right)$ that might occur on the boundary.

Step 4
Let's first check out the top : $y = 4, \quad -2 \le x \le 3$.

We'll need to identify the points along the top that could be potential absolute extrema for $f\left(x, y\right)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 3$.

$$g\left(x\right) = f\left(x, 4\right) = 17\left(-1 + 9x^2\right)$$

This is really nothing more than a Calculus I absolute extrema problem so we'll be doing the work here without a lot of explanation. If you don't recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for $g\left(x\right)$ are,

$$g'\left(x\right) = 306x = 0 \qquad \rightarrow \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the *x* limits for the top give the following three points that are potential absolute extrema for $f\left(x, y\right)$.

$$\left(0, 4\right) \qquad \left(-2, 4\right) \qquad \left(3, 4\right)$$

Recall that, in this step, we are assuming that $y = 4$! So, the next set of potential absolute extrema for $f\left(x, y\right)$ are then,

$$f\left(0, 4\right) = -17 \qquad f\left(-2, 4\right) = 595 \qquad f\left(3, 4\right) = 1360$$

Step 5
Next, let's check out the bottom : $y = -1, \quad -2 \le x \le 3$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2 \le x \le 3$.

$$g(x) = f(x,-1) = -3\left(-1+9x^2\right)$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = -54x = 0 \qquad \rightarrow \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the *x* limits for the bottom give the following three points that are potential absolute extrema for $f(x,y)$.

$$(0,-1) \qquad (-2,-1) \qquad (3,-1)$$

Recall that, in this step, we are assuming that $y = -1$! So, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(0,-1) = 3 \qquad f(-2,-1) = -105 \qquad f(3,-1) = -240$$

Step 6

Let's now check out the right side : $x = 3, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h(y) = f(3,y) = 80(1+4y)$$

The derivative of the $h(y)$ from this step is,

$$h'(y) = 320$$

In this case there are no critical points of the function along this boundary. So, only the limits for the right side are potential absolute extrema for $f(x,y)$.

$$(3,-1) \qquad (3,4)$$

Recall that, in this step, we are assuming that $x = 3$! Therefore, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(3,-1) = -240 \qquad f(3,4) = 1360$$

Before proceeding to the next step let's note that both of these points have already been listed in previous steps and so did not really need to be written down here. This will always happen with boundary points (as these are here). Boundary points will always show up in multiple boundary steps.

Step 7

Finally, let's check out the left side : $x = -2, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h(y) = f(-2, y) = 35(1 + 4y)$$

The derivative of the $h(y)$ from this step is,

$$h'(y) = 140$$

In this case there are no critical points of the function along this boundary. So, we only the limits for the right side are potential absolute extrema for $f(x, y)$.

$$(-2,-1) \qquad\qquad (-2,4)$$

Recall that, in this step, we are assuming that $x = -2$! Therefore, the next set of potential absolute extrema for $f(x, y)$ are then,

$$f(-2,-1) = -105 \qquad\qquad f(-2,4) = 595$$

As with the previous step both of these are boundary points and have appeared in previous steps. They were simply listed here for completeness.

Step 8
Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$\begin{array}{lll} f(0,4) = -17 & f(-2,4) = 595 & f(3,4) = 1360 \\ f(0,-1) = 3 & f(-2,-1) = -105 & f(3,-1) = -240 \end{array}$$

From this list we can see that the absolute maximum of the function will be 1360 which occurs at $(3,4)$ and the absolute minimum of the function will be -240 which occurs at $(3,-1)$.

Section 3-5 : Lagrange Multipliers

1. Find the maximum and minimum values of $f\left(x,y\right)=81x^2+y^2$ subject to the constraint $4x^2+y^2=9$.

Step 1

Before proceeding with the problem let's note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of *x* is $-\frac{3}{2}\le x\le\frac{3}{2}$ (the largest values would occur if $y=0$). Likewise, the largest possible range of *y* is $-3\le y\le 3$ (with the largest values occurring if $x=0$).

Note that, at this point, we don't know if *x* and/or *y* will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2

The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned}162x&=8x\lambda\\2y&=2y\lambda\\4x^2+y^2&=9\end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

In this case, simply because the numbers are a little smaller, let's start with the second equation. A little rewrite of the equation gives us the following,

$$2y\lambda-2y=2y\left(\lambda-1\right)=0\qquad\to\qquad y=0\quad\text{or}\quad\lambda=1$$

Be careful here to not just divide both sides by *y* to "simplify" the equation. Remember that you can't divide by anything unless you know for a fact that it won't ever be zero. In this case we can see that *y* clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Step 4
We now have two possibilities from Step 2. Either $y = 0$ or $\lambda = 1$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $y = 0$. In this case we can go directly to the constraint to get,

$$4x^2 = 9 \qquad \to \qquad x = \pm\tfrac{3}{2}$$

Therefore, from this part we get two points that are potential absolute extrema,

$$\left(-\tfrac{3}{2}, 0\right) \qquad\qquad \left(\tfrac{3}{2}, 0\right)$$

Step 5
Next, let's assume that $\lambda = 1$. In this case, we can plug this into the first equation to get,

$$162x = 8x \qquad \to \qquad 154x = 0 \qquad \to \qquad x = 0$$

So, under this assumption we must have $x = 0$. We can now plug this into the constraint to get,

$$y^2 = 9 \qquad \to \qquad y = \pm 3$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(0, -3\right) \qquad\qquad \left(0, 3\right)$$

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-\tfrac{3}{2}, 0\right) = \tfrac{729}{4} \qquad f\left(\tfrac{3}{2}, 0\right) = \tfrac{729}{4} \qquad f\left(0, -3\right) = 9 \qquad f\left(0, 3\right) = 9$$

The absolute maximum is then $\frac{729}{4} = 182.25$ which occurs at $\left(-\tfrac{3}{2}, 0\right)$ and $\left(\tfrac{3}{2}, 0\right)$. The absolute minimum is 9 which occurs at $\left(0, -3\right)$ and $\left(0, 3\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

2. Find the maximum and minimum values of $f(x,y) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$.

Step 1
Before proceeding with the problem let's note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of *x* is $-1 \le x \le 1$ (the largest values would occur if $y = 0$). Likewise, the largest possible range of *y* is $-1 \le y \le 1$ (with the largest values occurring if $x = 0$).

Note that, at this point, we don't know if *x* and/or *y* will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 16x &= 2x\lambda \\ -2 &= 2y\lambda \\ x^2 + y^2 &= 1 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system it looks like maybe the first equation will give us some information to start off with so let's start with that equation. A quick rewrite of the equation gives us the following,

$$16x - 2x\lambda = 2x(8 - \lambda) = 0 \qquad \to \qquad x = 0 \quad \text{or} \quad \lambda = 8$$

Be careful here to not just divide both sides by *x* to "simplify" the equation. Remember that you can't divide by anything unless you know for a fact that it won't ever be zero. In this case we can see that *x* clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Step 4
We now have two possibilities from Step 2. Either $x = 0$ or $\lambda = 8$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $x = 0$. In this case we can go directly to the constraint to get,

$$y^2 = 1 \qquad \to \qquad y = \pm 1$$

Therefore, from this part we get two points that are potential absolute extrema,

$$\left(0,-1\right) \qquad\qquad \left(0,1\right)$$

Step 5
Next, let's assume that $\lambda = 8$. In this case, we can plug this into the second equation to get,

$$-2 = 16y \qquad \to \qquad y = -\tfrac{1}{8}$$

So, under this assumption we must have $y = -\frac{1}{8}$. We can now plug this into the constraint to get,

$$x^2 + \tfrac{1}{64} = 1 \qquad \to \qquad x^2 = \tfrac{63}{64} \qquad \to \qquad x = \pm\sqrt{\tfrac{63}{64}} = \pm\tfrac{3\sqrt{7}}{8}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(-\tfrac{3\sqrt{7}}{8},-\tfrac{1}{8}\right) \qquad\qquad \left(\tfrac{3\sqrt{7}}{8},-\tfrac{1}{8}\right)$$

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-\tfrac{3\sqrt{7}}{8},-\tfrac{1}{8}\right) = \tfrac{65}{8} \qquad f\left(\tfrac{3\sqrt{7}}{8},-\tfrac{1}{8}\right) = \tfrac{65}{8} \qquad f\left(0,-1\right) = 2 \qquad f\left(0,1\right) = -2$$

The absolute maximum is then $\frac{65}{8} = 8.125$ which occurs at $\left(-\frac{3\sqrt{7}}{8},-\frac{1}{8}\right)$ and $\left(\frac{3\sqrt{7}}{8},-\frac{1}{8}\right)$. The absolute minimum is -2 which occurs at $\left(0,1\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

---

3. Find the maximum and minimum values of $f\left(x,y,z\right) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

Step 1
Before proceeding with the problem let's note because our constraint is the sum of three terms that are squared (and hence positive) the largest possible range of *x* is $-6 \le x \le 6$ (the largest values would occur if $y = 0$ and $z = 0$). Likewise, we'd get the same ranges for both *y* and *z*.

Note that, at this point, we don't know if *x*, *y* or *z* will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 0 &= 2x\lambda \\ 2y &= 2y\lambda \\ -10 &= 2z\lambda \\ x^2 + y^2 + z^2 &= 36 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system let's start with the third equation and note that because the left side is -10, or more importantly can never by zero, we can see that we must therefore have $z \ne 0$ and $\lambda \ne 0$. The fact that λ can't be zero is really important for this problem.

Step 4
Okay, because we now know that $\lambda \ne 0$ we can see that the only way for the first equation to be true is to have $x = 0$.

Therefore, no matter what else is going on with *y* and *z* in this problem we must always have $x = 0$ and we'll need to keep that in mind.

Step 5

Next, let's take a look at the second equation. A quick rewrite of this equation gives,

$$2y - 2y\lambda = 2y\left(1-\lambda\right) = 0 \qquad \to \qquad y = 0 \quad \text{or} \quad \lambda = 1$$

Step 6
We now have two possibilities from Step 4. Either $y = 0$ or $\lambda = 1$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $y = 0$ and recall from Step 3 that we also know that $x = 0$. In this case we can plug these values into the constraint to get,

$$z^2 = 36 \qquad \to \qquad z = \pm 6$$

Therefore, from this part we get two points that are potential absolute extrema,

$$\left(0,0,-6\right) \qquad\qquad \left(0,0,6\right)$$

Step 7
Next, let's assume that $\lambda = 1$. If we head back to the third equation we can see that we now have,

$$-10 = 2z \qquad \to \qquad z = -5$$

So, under this assumption we must have $z = -5$ and recalling once more from Step 3 that we have $x = 0$ we can now plug these into the constraint to get,

$$y^2 + 25 = 36 \qquad \to \qquad y^2 = 11 \qquad \to \qquad y = \pm\sqrt{11}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(0,-\sqrt{11},-5\right) \qquad\qquad \left(0,\sqrt{11},-5\right)$$

Step 8
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(0,-\sqrt{11},-5\right) = 61 \qquad f\left(0,\sqrt{11},-5\right) = 61 \qquad f\left(0,0,-6\right) = 60 \qquad f\left(0,0,6\right) = -60$$

The absolute maximum is then 61 which occurs at $\left(0,-\sqrt{11},-5\right)$ and $\left(0,\sqrt{11},-5\right)$. The absolute minimum is -60 which occurs at $\left(0,0,6\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

4. Find the maximum and minimum values of $f\left(x,y,z\right)=xyz$ subject to the constraint $x+9y^2+z^2=4$. Assume that $x \ge 0$ for this problem. Why is this assumption needed?

Step 1
Before proceeding with the solution to this problem let's address why the assumption that $x \ge 0$ is needed for this problem.

The answer is simple. Without that assumption this function will not have absolute extrema.

If there are no restrictions on *x* then we could make *x* as large and negative as we wanted to and we could still meet the constraint simply by chose a very large *y* and/or *z*. Note as well that because *y* and *z* are both squared we could chose them to be either negative or positive.

If we took our choices for *x*, *y* and *z* and plugged them into the function then the function would be similarly large. Also, the larger we chose *x* the larger we'd need to choose appropriate *y* and/or *z* and hence the larger our function would become. Finally, as noted above because we could chose *y* and *z* to be either positive or negative we could force the function to be either positive or negative with appropriate choices of signs for *y* and *z*.

In other words, if we have no restriction on *x*, we can make the function arbitrarily large in a positive and negative sense and so this function would not have absolute extrema.

On the other hand, if we put on the restriction on *x* that we have we now have the sum of three positive terms that must equal four. This in turn leads to the following largest possible values of the three variables in the problem.

$$0 \le x \le 4 \qquad\qquad -\tfrac{2}{3} \le y \le \tfrac{2}{3} \qquad\qquad -2 \le z \le 2$$

The largest value of *x* and the extreme values of *y* and *z* would occur when the other two variables are zero and in general there is no way to know ahead of time if any of the variables will in fact take on their largest possible values. However, what we can say now is that because all of our variables are bounded then by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

Note as well that all we really need here is a lower limit for *x*. It doesn't have to be zero that just makes the above analysis a little bit easier. We could have used the restriction that $x \ge -8$ if we'd wanted to. With this restriction we'd still have a bounded set of ranges for *x*, *y* and *z* and so the function would still have absolute extrema.

This problem shows just why this step is so important for these problems. If this problem did not have a restriction on *x* and we neglected to do this step we'd get the (very) wrong answer! We could still go through the process below and we'd get values that would appear to be absolute extrema. However, as we've shown above without any restriction on *x* the function would not have absolute extrema.

The issue here is that the Lagrange multiplier process itself is not set up to detect if absolute extrema exist or not. Before we even start the process we need to first make sure that the values we get out of the process will in fact be absolute extrema (*i.e.* we need to verify that absolute extrema exist).

Step 2
The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} yz &= \lambda \\ xz &= 18y\lambda \\ xy &= 2z\lambda \\ x + 9y^2 + z^2 &= 4 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system let's start out by multiplying the first equation by *x*, multiplying the second equation by *y* and multiplying the third equation by *z*. Doing this gives the following "new" system of equations.

$$\begin{aligned} xyz &= x\lambda \\ xyz &= 18y^2\lambda \\ xyz &= 2z^2\lambda \\ x + 9y^2 + z^2 &= 4 \end{aligned}$$

Let's also note that the constraint won't be true if all three variables are zero simultaneously. One or two of the variables can be zero but we can't have all three be zero.

Step 4
Now, let's set the first and second equations from Step 3 equal. Doing this gives,

$$x\lambda = 18y^2\lambda \quad \to \quad \left(x - 18y^2\right)\lambda = 0 \quad \to \quad x = 18y^2 \;\text{ or }\; \lambda = 0$$

Let's also set the second and third equation from Step 3 equal. Doing this gives,

$$18y^2\lambda = 2z^2\lambda \quad \to \quad \left(18y^2 - 2z^2\right)\lambda = 0 \quad \to \quad z^2 = 9y^2 \;\text{ or }\; \lambda = 0$$

Step 5

Okay, from Step 4 we have two possibilities. Either $\lambda = 0$ or we have $x = 18y^2$ and $z^2 = 9y^2$.

Let's take care of the first possibility, $\lambda = 0$. If we go back to the original system this assumption gives us the following system.

$$\begin{aligned} yz &= 0 & \rightarrow \quad & y = 0 \text{ or } z = 0 \\ xz &= 0 & \rightarrow \quad & x = 0 \text{ or } z = 0 \\ xy &= 0 & \rightarrow \quad & x = 0 \text{ or } y = 0 \\ x + 9y^2 + z^2 &= 4 & & \end{aligned}$$

Step 6

We have all sorts of possibilities from Step 5. From the first equation we have two possibilities. Let's start with $y = 0$. Since the third equation from Step 5 won't really tell us anything (after all it is now 0 = 0) let's move to the second equation. In this case we get either $x = 0$ or $z = 0$.

Recall that at the end of the third step we noticed that we can't have all three of the variables be zero but we could have two of them be zero. So, this leads to the following two cases that we can plug into the constraint to find the value of the third variable.

$$\begin{aligned} y = 0, x = 0 \quad &: \quad z^2 = 4 \quad \rightarrow \quad z = \pm 2 \\ y = 0, z = 0 \quad &: \quad x = 4 \end{aligned}$$

So, this gives us the following three potential absolute extrema.

$$(0,0,-2) \qquad (0,0,2) \qquad (4,0,0)$$

Next, let's take a look at the second possibility from the first equation in Step 5, $z = 0$. In this case the second equation will be 0 = 0 and so will not be of any use. The third however, has the possibilities of $x = 0$ or $y = 0$. The second of these was already addressed above so all we need to look at is,

$$z = 0, x = 0 \quad : \quad 9y^2 = 4 \quad \rightarrow \quad y = \pm \tfrac{2}{3}$$

This leads to two more potential absolute extrema.

$$\left(0,-\tfrac{2}{3},0\right) \qquad \left(0,\tfrac{2}{3},0\right)$$

We could now go back and start with the second or third equation but if we did that you'd just end up with the above possibilities (you might want to verify that for yourself…). Therefore, we get a total of five possible absolute extrema from this Step. They are,

$$(0,0,\pm 2) \qquad \left(0,\pm\tfrac{2}{3},0\right) \qquad (4,0,0)$$

We made heavy use of the "$\pm$" notation here to simplify things a little bit. It's not required but will make the rest of the work with these points a little easier as we'll see eventually.

Step 7
Now, way back in Step 5 we had another possibility : $x = 18y^2$ and $z^2 = 9y^2$. We have to now take a look at this case. In this case we can plug each of these directly into the constraint to get the following,

$$18y^2 + 9y^2 + 9y^2 = 36y^2 = 4 \qquad \rightarrow \qquad y = \pm\tfrac{1}{3}$$

Now we can go back to the two assumptions we started this step off with to get,

$$x = 18\left(\tfrac{1}{9}\right) = 2 \qquad\qquad z^2 = 9\left(\tfrac{1}{9}\right) = 1 \quad \rightarrow \quad z = \pm 1$$

Now, in most cases, we can't just "mix and match" all the values of *x*, *y* and *z* to from points. In this case however, we can do exactly that. The $x = 2$ will arise regardless of the sign on *y* because of the y^2 in the *x* assumption. Likewise, because of the y^2 in the *z* assumption each of the *z*'s can arise for either *y* and so we get all combinations of *x*, *y* and *z* for points in this case.

Therefore, we get the following four possible absolute extrema from this step.

$$\left(2,-\tfrac{1}{3},-1\right) \qquad \left(2,-\tfrac{1}{3},1\right) \qquad \left(2,\tfrac{1}{3},-1\right) \qquad \left(2,\tfrac{1}{3},1\right)$$

Step 8
In total, it looks like we have nine points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points and with nine points that seems like a lot of work.

However, in this case, it's actually quite simple. Recall that the function we're evaluating is $f\left(x,y,z\right) = xyz$. First, this means that if even one of the variables is zero the whole function will be zero. Therefore, the function evaluations for the five points from Step 6 all give,

$$f\left(0,0,\pm 2\right) = f\left(0,\pm\tfrac{2}{3},0\right) = f\left(4,0,0\right) = 0$$

Note the usage of the "$\pm$" notation to "simplify" the work here as well.

Now, the potential points from Step 7 are all the same values, with the exception of signs changing occasionally on the *y* and *z*. That means that the function value here will be either $-\tfrac{2}{3}$ or $\tfrac{2}{3}$ depending on the number of minus signs in the point. So again, not a lot of effort to compute these function values. Here are the evaluations for the points from Step 7.

$$f\left(2,-\tfrac{1}{3},1\right) = f\left(2,\tfrac{1}{3},-1\right) = -\tfrac{2}{3} \qquad\qquad f\left(2,-\tfrac{1}{3},-1\right) = f\left(2,\tfrac{1}{3},1\right) = \tfrac{2}{3}$$

The absolute maximum is then $\frac{2}{3}$ which occurs at $\left(2,-\frac{1}{3},-1\right)$ and $\left(2,\frac{1}{3},1\right)$. The absolute minimum is $-\frac{2}{3}$ which occurs at $\left(2,-\frac{1}{3},1\right)$ and $\left(2,\frac{1}{3},-1\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that some of the solution processes for the systems that arise with Lagrange multipliers can be quite involved. It can be easy to get lost in the details of the solution process and forget to go back and take care of one or more possibilities. You need to always be very careful and before finishing a problem go back and make sure that you've dealt with all the possible solution paths in the problem.

---

5. Find the maximum and minimum values of $f\left(x,y,z\right)=3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$.

Step 1
Before proceeding with the problem let's note that the second constraint is the sum of two terms that are squared (and hence positive). Therefore, the largest possible range of *x* is $-3\le x\le 3$ (the largest values would occur if $z=0$). We'll get a similar range for *z*.

Now, the first constraint is not the sum of two (or more) positive numbers. However, we've already established that *x* is restricted to $-3\le x\le 3$ and this will give $-7\le y\le 1$ as the largest possible range of *y*'s. Note that we can easily get this range by acknowledging that the first constraint is just a line and so the extreme values of *y* will correspond to the extreme values of *x*.

So, because we now know that our answers must occur in these bounded ranges by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 6x &= 4\lambda+2x\mu \\ 1 &= -3\lambda \\ 0 &= 2z\mu \\ 4x-3y &= 9 \\ x^2+z^2 &= 9 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system we get a "freebie" to start off with. Notice that from the second equation we quickly can see that $\lambda = -\frac{1}{3}$ regardless of any of the values of the other variables in the system.

Step 4

Next, from the third equation we can see that we have either $z = 0$ or $\mu = 0$, So, we have 2 possibilities to look at. Let's take a look at $z = 0$ first.

In this case we can go straight to the second constraint to get,

$$x^2 = 9 \qquad \rightarrow \qquad x = \pm 3$$

We can in turn plug each of these possibilities into the first constraint to get values for *y*.

$$\begin{aligned} x = -3 \quad &: \quad -12 - 3y = 9 \qquad \rightarrow \qquad y = -7 \\ x = 3 \quad &: \quad 12 - 3y = 9 \qquad \rightarrow \qquad y = 1 \end{aligned}$$

Okay, from this step we have two possible absolute extrema.

$$\left(-3, -7, 0\right) \qquad \left(3, 1, 0\right)$$

Step 5

Now let's go back and take a look at what happens if $\mu = 0$. If we plug this into the first equation in our system (and recalling that we also know that $\lambda = -\frac{1}{3}$) we get,

$$6x = -\tfrac{4}{3} \qquad \rightarrow \qquad x = -\tfrac{2}{9}$$

We can plug this into each of our constraints to get values of *y* (from the first constraint) and *z* (form the second constraint). Here is that work,

$$\begin{aligned} 4\left(-\tfrac{2}{9}\right) - 3y = 9 \qquad &\rightarrow \qquad y = -\tfrac{89}{27} \\ \left(-\tfrac{2}{9}\right)^2 + z^2 = 9 \qquad &\rightarrow \qquad z = \pm\tfrac{5\sqrt{29}}{9} \end{aligned}$$

This leads to two more potential absolute extrema.

$$\left(-\tfrac{2}{9}, -\tfrac{89}{27}, -\tfrac{5\sqrt{29}}{9}\right) \qquad \left(-\tfrac{2}{9}, -\tfrac{89}{27}, \tfrac{5\sqrt{29}}{9}\right)$$

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-3,-7,0\right)=20 \qquad f\left(3,1,0\right)=28 \qquad f\left(-\tfrac{2}{9},-\tfrac{89}{27},-\tfrac{5\sqrt{29}}{9}\right)=-\tfrac{85}{27} \qquad f\left(-\tfrac{2}{9},-\tfrac{89}{27},\tfrac{5\sqrt{29}}{9}\right)=-\tfrac{85}{27}$$

The absolute maximum is then 28 which occurs at $\left(3,1,0\right)$. The absolute minimum is $-\frac{85}{27}$ which occurs at $\left(-\frac{2}{9},-\frac{89}{27},-\frac{5\sqrt{29}}{9}\right)$ and $\left(-\frac{2}{9},-\frac{89}{27},\frac{5\sqrt{29}}{9}\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that, in this case, the value of the absolute extrema (as opposed to the location) did not actually depend on the value of *z* in any way as the function we were optimizing in this problem did not depend on *z*. This will happen sometimes and we shouldn't get too worried about it when it does.

Note however that we still need the values of *z* for the location of the absolute extrema. We need the values of *z* for the location because the points that give the absolute extrema are also required to satisfy the constraint and the second constraint in our problem does involve *z*'s!

Chapter 4 : Multiple Integrals

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Double Integrals – In this section we will define the double integral.

Iterated Integrals – In this section we will show how Fubini's Theorem can be used to evaluate double integrals where the region of integration is a rectangle.

Double Integrals over General Regions – In this section we will start evaluating double integrals over general regions, *i.e.* regions that aren't rectangles. We will illustrate how a double integral of a function can be interpreted as the net volume of the solid between the surface given by the function and the *xy*-plane.

Double Integrals in Polar Coordinates – In this section we will look at converting integrals (including *dA*) in Cartesian coordinates into Polar coordinates. The regions of integration in these cases will be all or portions of disks or rings and so we will also need to convert the original Cartesian limits for these regions into Polar coordinates.

Triple Integrals – In this section we will define the triple integral. We will also illustrate quite a few examples of setting up the limits of integration from the three dimensional region of integration. Getting the limits of integration is often the difficult part of these problems.

Triple Integrals in Cylindrical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Cylindrical coordinates. We will also be converting the original Cartesian limits for these regions into Cylindrical coordinates.

Triple Integrals in Spherical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Spherical coordinates. We will also be converting the original Cartesian limits for these regions into Spherical coordinates.

Change of Variables – In previous sections we've converted Cartesian coordinates in Polar, Cylindrical and Spherical coordinates. In this section we will generalize this idea and discuss how we convert integrals in Cartesian coordinates into alternate coordinate systems. Included will be a derivation of the *dV* conversion formula when converting to Spherical coordinates.

Surface Area – In this section we will show how a double integral can be used to determine the surface area of the portion of a surface that is over a region in two dimensional space.

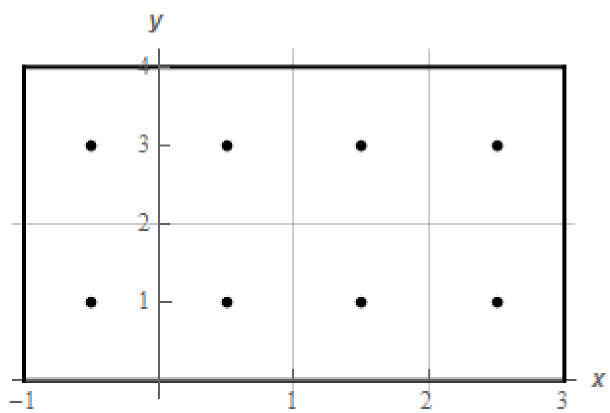
Area and Volume Revisited – In this section we summarize the various area and volume formulas from this chapter.

Section 4-1 : Double Integrals

1. Use the Midpoint Rule to estimate the volume under $f\left(x,y\right)=x^2+y$ and above the rectangle given by $-1\le x\le 3$, $0\le y\le 4$ in the *xy*-plane. Use 4 subdivisions in the *x* direction and 2 subdivisions in the *y* direction.

Step 1

Okay, first let's get a quick sketch of the rectangle we're dealing with here.



The light gray lines show the subdivisions for each direction. The dots in the center of each "block" are the midpoints of each of the blocks.

The coordinates of each of the dots in the lower row are,

$$\left(-\tfrac{1}{2},1\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(\tfrac{3}{2},1\right) \qquad \left(\tfrac{5}{2},1\right)$$

and the coordinates of each of the dots in the upper row are,

$$\left(-\tfrac{1}{2},3\right) \qquad \left(\tfrac{1}{2},3\right) \qquad \left(\tfrac{3}{2},3\right) \qquad \left(\tfrac{5}{2},3\right)$$

Step 2

We know that the volume we are after is simply,

$$V=\iint\limits_R f\left(x,y\right)dA$$

and we also know that the Midpoint Rule for this particular case is the following double summation.

$$\iint\limits_R f\left(x,y\right)dA\approx\sum_{i=1}^{4}\sum_{j=1}^{2}f\left(\overline{x}_i,\overline{y}_j\right)\Delta A \qquad\qquad f\left(x,y\right)=x^2+y$$

Remember that there are four *x* subdivisions and so the *i* summation will go to 4. There are two *y* subdivisions and the *j* summation will go to 2.

The $\left(\overline{x}_i, \overline{y}_j\right)$ in the formula are simply the midpoints of each of the blocks (*i.e.* the points listed in Step 1 above) and ΔA is the area of each of blocks and so is $\Delta A = (1)(2) = 2$.

Therefore, the volume for this problem is approximately,

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2 f\left(\overline{x}_i, \overline{y}_j\right) \qquad\qquad f\left(x,y\right) = x^2 + y$$

Note that we plugged ΔA into the formula.

Step 3
The best way to compute this double summation is probably to first compute the "inner" summation for each value of *i*.

Each of the inner summations are done for a fixed value of *i* as *i* runs from $i = 1$ to $i = 4$. Therefore, the four inner summations are computed using the values of the function at the midpoints for the blocks in each of the columns in the sketch above in Step 1.

This means we need the following summations.

$$\begin{aligned}
i = 1 \;:\; & \sum_{j=1}^{2} 2 f\left(\overline{x}_1, \overline{y}_j\right) = \sum_{j=1}^{2} 2 f\left(-\tfrac{1}{2}, \overline{y}_j\right) = 2\left[f\left(-\tfrac{1}{2}, 1\right) + f\left(-\tfrac{1}{2}, 3\right)\right] = 9 \\
i = 2 \;:\; & \sum_{j=1}^{2} 2 f\left(\overline{x}_2, \overline{y}_j\right) = \sum_{j=1}^{2} 2 f\left(\tfrac{1}{2}, \overline{y}_j\right) = 2\left[f\left(\tfrac{1}{2}, 1\right) + f\left(\tfrac{1}{2}, 3\right)\right] = 9 \\
i = 3 \;:\; & \sum_{j=1}^{2} 2 f\left(\overline{x}_3, \overline{y}_j\right) = \sum_{j=1}^{2} 2 f\left(\tfrac{3}{2}, \overline{y}_j\right) = 2\left[f\left(\tfrac{3}{2}, 1\right) + f\left(\tfrac{3}{2}, 3\right)\right] = 17 \\
i = 4 \;:\; & \sum_{j=1}^{2} 2 f\left(\overline{x}_4, \overline{y}_j\right) = \sum_{j=1}^{2} 2 f\left(\tfrac{5}{2}, \overline{y}_j\right) = 2\left[f\left(\tfrac{5}{2}, 1\right) + f\left(\tfrac{5}{2}, 3\right)\right] = 33
\end{aligned}$$

Step 4
We can now compute the "outer" summation. This is just the sum of all the inner summations we computed in Step 3.

The volume is then approximately,

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2 f\left(\overline{x}_i, \overline{y}_j\right) = 9 + 9 + 17 + 33 = \boxed{68}$$

For reference purposes we will eventually be able to verify that the exact volume is $\frac{208}{3} = 69.333\overline{3}$ and so the approximation in this case is actually fairly close.

Section 4-2 : Iterated Integrals

1. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to *x* first and **(b)** by integrating with respect to *y* first.

$$\iint\limits_R 12x - 18y\,dA \qquad\qquad R = [-1,4]\times[2,3]$$

(a) Evaluate by integrating with respect to *x* first.

Step 1

Not too much to do with this problem other than to do the integral in the order given in the problem statement.

Let's first get the integral set up with the proper order of integration.

$$\iint\limits_R 12x - 18y\,dA = \int_2^3 \int_{-1}^4 12x - 18y\,dx\,dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the *x* integration is first and so the *x* limits need to go on the inner integral and the *y* limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the x integration now.

$$\begin{aligned}\iint\limits_R 12x - 18y\,dA &= \int_2^3 \int_{-1}^4 12x - 18y\,dx\,dy \\ &= \int_2^3 \left(6x^2 - 18xy\right)\Big|_{-1}^4\,dy = \int_2^3 90 - 90y\,dy\end{aligned}$$

Just remember that because we are integrating with respect to *x* in this step we treat all *y*'s as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3
Now all we need to take care of in the *y* integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 12x - 18y\,dA = \int_2^3 90 - 90y\,dy = \left(90y - 45y^2\right)\Big|_2^3 = \boxed{-135}$$

(b) Evaluate by integrating with respect to *y* first.

Step 1
Not too much to do with this problem other than to do the integral in the order given in the problem statement.

Let's first get the integral set up with the proper order of integration.

$$\iint\limits_R 12x - 18y\,dA = \int_{-1}^4 \int_2^3 12x - 18y\,dy\,dx$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to fist and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the *y* integration is first and so the *y* limits need to go on the inner integral and the *x* limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2
Okay, let's do the *y* integration now.

$$\begin{aligned} \iint\limits_R 12x - 18y\,dA &= \int_{-1}^4 \int_2^3 12x - 18y\,dy\,dx \\ &= \int_{-1}^4 \left(12xy - 9y^2\right)\Big|_2^3 dx = \int_{-1}^4 12x - 45\,dx \end{aligned}$$

Just remember that because we are integrating with respect to *y* in this step we treat all *x*'s as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3

Now all we need to take care of in the *x* integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 12x - 18y\,dA = \int_{-1}^{4} 12x - 45\,dx = \left(6x^2 - 45x\right)\Big|_{-1}^{4} = \boxed{-135}$$

The same answer as from **(a)** which is what we should expect of course. The order of integration will not change the answer. One order may be easier/simpler than the other but the final answer will always be the same regardless of the order.

---

2. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R 6y\sqrt{x} - 2y^3\,dA \qquad\qquad R = [1,4]\times[0,3]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to *x* first for no other reason that it will allow us to get rid of the root right away.

So, here is the integral set up to do the *x* integration first.

$$\iint\limits_R 6y\sqrt{x} - 2y^3 \, dA = \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3 \, dx \, dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the *x* integration is first and so the *x* limits need to go on the inner integral and the *y* limits need to go on the out integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the *x* integration now.

$$\begin{aligned}\iint\limits_R 6y\sqrt{x} - 2y^3 \, dA &= \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3 \, dx \, dy \\ &= \int_0^3 \left(4yx^{\frac{3}{2}} - 2xy^3 \right)\Big|_1^4 \, dy = \int_0^3 28y - 6y^3 \, dy\end{aligned}$$

Just remember that because we are integrating with respect to *x* in this step we treat all *y*'s as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3

Now all we need to take care of is the *y* integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 6y\sqrt{x} - 2y^3 \, dA = \int_0^3 28y - 6y^3 \, dy = \left(14y^2 - \tfrac{3}{2}y^4 \right)\Big|_0^3 = \boxed{\tfrac{9}{2}}$$

3. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA \qquad\qquad R = [-1,0]\times[1,2]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to y first for no other reason that it will allow us to get rid of the rational expressions in the integrand after the first integration.

So, here is the integral set up to do the y integration first.

$$\iint_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA = \int_{-1}^{0}\int_{1}^{2} \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dy\,dx$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the *y* integration is first and so the *y* limits need to go on the inner integral and the *x* limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2
Okay, let's do the y integration now.

$$\begin{aligned}\iint_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA &= \int_{-1}^{0}\int_{1}^{2} \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dy\,dx \\ &= \int_{-1}^{0} \left.\left(\tfrac{1}{2}\ln|y|\,\mathbf{e}^x + \frac{4x-1}{y}\right)\right|_{1}^{2} dx = \int_{-1}^{0} \tfrac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 4x + 1\right)dx\end{aligned}$$

Just remember that because we are integrating with respect to *y* in this step we treat all *x*'s as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made. Don't forget to do any simplification after the evaluation as that can sometimes greatly simplify the next integration.

Step 3
Now all we need to take care of in the *x* integration and that is just a simple Calculus I integral. Here is that work.

$$\begin{aligned}\iint_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA &= \int_{-1}^{0} \tfrac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 4x + 1\right)dx \\ &= \left.\left(\tfrac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 2x^2 + x\right)\right)\right|_{-1}^{0} = \boxed{\tfrac{1}{2}\left(\ln(2) - \ln(2)\,\mathbf{e}^{-1} + 3\right) = 1.71908}\end{aligned}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

4. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R \sin\left(2x\right) - \frac{1}{1+6y}\,dA \qquad\qquad R = \left[\tfrac{\pi}{4},\tfrac{\pi}{2}\right]\times\left[0,1\right]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to x first for no other reason that the x limits seem a little messier and so this will get rid of them with the first integration.

So, here is the integral set up to do the x integration first.

$$\iint_R \sin(2x) - \frac{1}{1+6y}\, dA = \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \frac{1}{1+6y}\, dx\, dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the *x* integration is first and so the *x* limits need to go on the inner integral and the *y* limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2
Okay, let's do the *x* integration now.

$$\begin{aligned}\iint_R \sin(2x) - \frac{1}{1+6y}\, dA &= \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \frac{1}{1+6y}\, dx\, dy \\ &= \int_0^1 \left(-\tfrac{1}{2}\cos(2x) - \frac{x}{1+6y}\, dx \right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} dy = \int_0^1 \frac{1}{2} - \frac{\frac{\pi}{4}}{1+6y}\, dy\end{aligned}$$

Just remember that because we are integrating with respect to *x* in this step we treat all *y*'s as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made. Don't forget to do any simplification after the evaluation as that can sometimes greatly simplify the next integration.

Step 3
Now all we need to take care of is the *y* integration and that is just a simple Calculus I integral. Here is that work.

$$\begin{aligned}\iint_R \sin(2x) - \frac{1}{1+6y}\, dA &= \int_0^1 \frac{1}{2} - \frac{\frac{\pi}{4}}{1+6y}\, dy \\ &= \left(\tfrac{1}{2}y - \tfrac{\pi}{24}\ln|1+6y| \right)\Big|_0^1 = \boxed{\tfrac{1}{2} - \tfrac{\pi}{24}\ln(7) = 0.24528}\end{aligned}$$

5. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA \qquad\qquad R=\left[0,2\right]\times\left[0,\sqrt{8}\,\right]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem the order still really doesn't matter all that much as both will require a substitution. However, we will simplify the integrand a little if we first do the *y* substitution since that will eliminate the *y* in front of the exponential. So, in this case, it looks like the integration process might be a little simpler if we integrate with respect to *y* first so let's do that.

Here is the integral set up to do the *y* integration first.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA=\int_0^2\int_0^{\sqrt{8}} y\mathbf{e}^{y^2-4x}\,dy\,dx$$

Step 2

Okay, let's do the *y* integration now.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA=\int_0^2\int_0^{\sqrt{8}} y\mathbf{e}^{y^2-4x}\,dy\,dx \qquad\qquad u=y^2-4x \quad\rightarrow\quad du=2y\,dy$$

$$=\int_0^2\left(\left.\tfrac{1}{2}\mathbf{e}^{y^2-4x}\right)\right|_0^{\sqrt{8}}dx=\int_0^2\tfrac{1}{2}\left(\mathbf{e}^{8-4x}-\mathbf{e}^{-4x}\right)dx$$

Just remember that because we are integrating with respect to *y* in this step we treat all *x*'s as if they were a constant and we know how to deal with constants in integrals.

Be careful with substitutions in the first integration. We showed the substitution we used above as well as the differential. When computing the differential we need to differentiate the right side of the substitution with respect to *y* (since we are doing a *y* integration). In other words, we need to do a partial derivative of the right side and so the "-4*x*" will differentiate to zero when differentiating with respect to *y*!

One of the bigger mistakes that students make here is to leave the "-4*x*" in the differential or to "differentiate" it to "-4" which in turn causes the substitution to not work because there is then no way to get rid of the *y* in front of the exponential.

Mistakes with Calculus I substitutions at this stage is one of the biggest issues that students have when first doing these kinds of integrals so you need to be very careful and always pay attention to which variable you are integrating with respect to and then differentiate your substitution with respect to the same variable.

Step 3
Now all we need to take care of is the *x* integration and that is just Calculus I integral with a couple of simple substitutions (we'll leave the substitution work to you to verify). Here is that work.

$$\begin{aligned}\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA &= \int_0^2 \tfrac{1}{2}\left(\mathbf{e}^{8-4x}-\mathbf{e}^{-4x}\right)dx \\ &= \left(\tfrac{1}{2}\left(-\tfrac{1}{4}\mathbf{e}^{8-4x}+\tfrac{1}{4}\mathbf{e}^{-4x}\right)\right)\Bigg|_0^2 = \boxed{\tfrac{1}{8}\left(\mathbf{e}^{8}+\mathbf{e}^{-8}-2\right)=372.3698}\end{aligned}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

---

6. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA \qquad\qquad R=[0,3]\times[0,2]$$

Step 1
The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem the order still really doesn't matter all that much as both will require a substitution. However, we might get a little more simplification of the integrand if we first do the *y* substitution since that will eliminate the y^2 in front of the root. Of course, the *x* substitution would have eliminated the *x* in front of the root so either way we'll get some simplification, but the *y* integration seems to offer a little bit more simplification.

So, here is the integral set up to do the *y* integration first.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3\int_0^2 xy^2\sqrt{x^2+y^3}\,dy\,dx$$

Step 2
Okay, let's do the *y* integration now.

$$\iint_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3\int_0^2 xy^2\left(x^2+y^3\right)^{\frac{1}{2}}\,dy\,dx \qquad u = x^2+y^3 \;\to\; du = 3y^2\,dy$$
$$= \int_0^3 \left(\tfrac{2}{9}x\left(x^2+y^3\right)^{\frac{3}{2}}\right)\Bigg|_0^2 dx = \int_0^3 \tfrac{2}{9}x\left(x^2+8\right)^{\frac{3}{2}} - \tfrac{2}{9}x^4\,dx$$

Just remember that because we are integrating with respect to *y* in this step we treat all *x*'s as if they were a constant and we know how to deal with constants in integrals.

Be careful with substitutions in the first integration. We showed the substitution we used above as well as the differential. When computing the differential we need to differentiate the right side of the substitution with respect to *y* (since we are doing a *y* integration). In other words, we need to do a partial derivative of the right side and so the x^2 will differentiate to zero when differentiating with respect to *y*!

One of the bigger mistakes that students make here is to leave the x^2 in the differential or to "differentiate" it to "2*x*" which in turn causes the substitution to not work because there is then no way to get rid of the y^2 in front of the root.

Mistakes with Calculus I substitutions at this stage is one of the biggest issues that students have when first doing these kinds of integrals so you need to be very careful and always pay attention to which variable you are integrating with respect to and then differentiate your substitution with respect to the same variable.

Step 3
Now all we need to take care of is the *x* integration and that is just Calculus I integral with a simple substitution (we'll leave the substitution work to you to verify). Here is that work.

$$\iint_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3 \tfrac{2}{9}x\left(x^2+8\right)^{\frac{3}{2}} - \tfrac{2}{9}x^4\,dx$$
$$= \left(\tfrac{2}{45}\left(x^2+8\right)^{\frac{5}{2}} - \tfrac{2}{45}x^5\right)\Bigg|_0^3 = \boxed{\tfrac{2}{45}\left(17^{\frac{5}{2}} - 243 - 128\sqrt{2}\right) = 34.1137}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

---

7. Compute the following double integral over the indicated rectangle.

$$\iint_R xy\cos\left(yx^2\right)dA \qquad R = [-2,3]\times[-1,1]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem however, we have a real difference in the orders. If we do the *y* integration first we will have to do integration by parts. On the other hand, if we do *x* first we only need to do a Calculus I substitution, which is almost always easier/quicker than integration by parts. On top of all that, if you think about how the substitution will go it looks like we'll also lose the integration by parts for the *y* after the substitution is done (if you don't see that don't worry you will eventually with enough practice).

So, it looks like integrating with respect to *x* first is the way to go. Here is the integral set up to do the *x* integration first.

$$\iint\limits_R xy\cos\left(y\,x^2\right)dA = \int_{-1}^{1}\int_{-2}^{3} xy\cos\left(y\,x^2\right)dx\,dy$$

Step 2

Okay, let's do the *x* integration now.

$$\begin{aligned}\iint\limits_R xy\cos\left(y\,x^2\right)dA &= \int_{-1}^{1}\int_{-2}^{3} xy\cos\left(y\,x^2\right)dx\,dy \qquad\qquad u = yx^2 \quad\to\quad du = 2yx\,dx\\ &= \int_{-1}^{1}\left(\tfrac{1}{2}\sin\left(y\,x^2\right)\right)\Big|_{-2}^{3}\,dy = \int_{-1}^{1}\tfrac{1}{2}\left(\sin\left(9y\right)-\sin\left(4y\right)\right)dy\end{aligned}$$

So, as noted above, upon doing the substitution we not only lost the *x* in front of the cosine but we also lost the *y* and that in turn means we won't have to do integration by parts for the *y* integral.

So, in this case doing the *x* integration first completely eliminated the integration by parts from the *y* integration from the problem. This won't always happen but when it does we'll take advantage of it and when it doesn't we'll be doing integration by parts whether we want to or not.

Step 3

Now all we need to take care of is the *y* integration and that is just Calculus I integral with a couple of really simple substitutions (we'll leave the substitution work to you to verify). Here is that work.

$$\begin{aligned}\iint\limits_R xy\cos\left(y\,x^2\right)dA &= \int_{-1}^{1}\tfrac{1}{2}\left(\sin\left(9y\right)-\sin\left(4y\right)\right)dy\\ &= \tfrac{1}{2}\left(\tfrac{1}{4}\cos\left(4y\right)-\tfrac{1}{9}\cos\left(9y\right)\right)\Big|_{-1}^{1}\\ &= \boxed{\tfrac{1}{2}\left[\tfrac{1}{4}\left(\cos\left(4\right)-\cos\left(-4\right)\right)-\tfrac{1}{9}\left(\cos\left(9\right)-\cos\left(-9\right)\right)\right]=0}\end{aligned}$$

Recall that cosine is an even function and so $\cos\left(-\theta\right)=\cos\left(\theta\right)$!

---

8. Compute the following double integral over the indicated rectangle.

$$\iint\limits_{R} xy\cos\left(y\right)-x^2\,dA \qquad\qquad R=\left[1,2\right]\times\left[\tfrac{\pi}{2},\pi\right]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem however, we have a real difference in the orders. If we do the *y* integration first we will have to do integration by parts. On the other hand, the *x* integration is a very simple Calculus I integration.

So, it looks like integrating with respect to *x* first is the way to go. Here is the integral set up to do the *x* integration first.

$$\iint\limits_{R} xy\cos\left(y\right)-x^2\,dA=\int_{\frac{\pi}{2}}^{\pi}\int_{1}^{2} xy\cos\left(y\right)-x^2\,dx\,dy$$

Step 2

Okay, let's do the *x* integration now.

$$\begin{aligned}\iint\limits_{R} xy\cos\left(y\right)-x^2\,dA&=\int_{\frac{\pi}{2}}^{\pi}\int_{1}^{2} xy\cos\left(y\right)-x^2\,dx\,dy\\&=\int_{\frac{\pi}{2}}^{\pi}\left(\tfrac{1}{2}x^2y\cos\left(y\right)-\tfrac{1}{3}x^3\right)\Big|_{1}^{2}\,dy=\int_{\frac{\pi}{2}}^{\pi}\tfrac{3}{2}y\cos\left(y\right)-\tfrac{7}{3}\,dy\end{aligned}$$

Note that for this example, unlike the previous one, the integration by parts did not go away after doing the first integration. Be careful to not just expect things like integration by parts to just disappear after doing the first integration. They often won't, and, in fact, it is possible that they might actually show up after doing the first integration!

Step 3

Now all we need to take care of is the *y* integration. As noted this is integration by parts for the first term and so we should probably split up the integral before doing the integration by parts.

Here is the work for this problem.

$$\iint\limits_R xy\cos(y) - x^2\, dA = \int_{\frac{\pi}{2}}^{\pi} \tfrac{3}{2}y\cos(y)\,dy - \int_{\frac{\pi}{2}}^{\pi} \tfrac{7}{3}\,dy \qquad u = \tfrac{3}{2}y \qquad dv = \cos(y)\,dy$$

$$= \left(\tfrac{3}{2}y\sin(y) + \tfrac{3}{2}\cos(y) - \tfrac{7}{3}y\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \boxed{-\tfrac{3}{2} - \tfrac{23}{12}\pi = -7.5214}$$

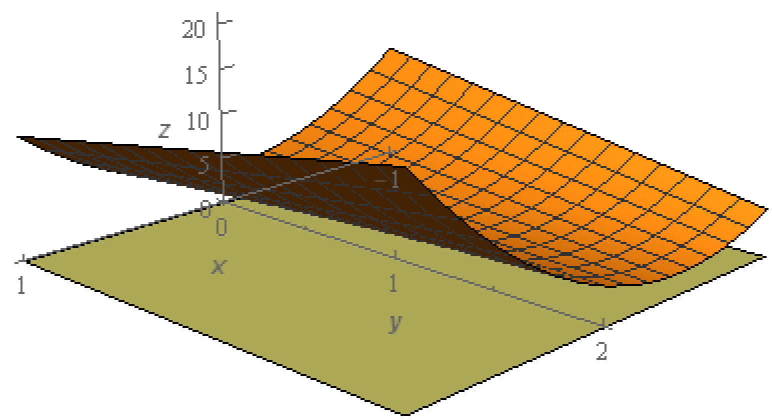
Note that we gave the *u* and *dv* for the integration by parts work but are leaving the details to you to verify the result.

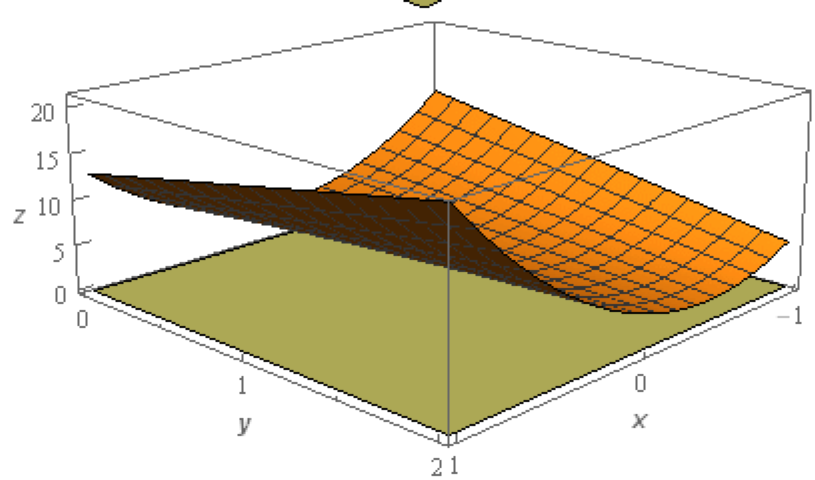
---

9. Determine the volume that lies under $f(x,y) = 9x^2 + 4xy + 4$ and above the rectangle given by $[-1,1]\times[0,2]$ in the *xy*-plane.

Step 1

First, let's start off with a quick sketch of the function.





The greenish rectangle under the surface is *R* from the problem statement and is given to help visualize the relationship between the surface and *R*.

Note that the sketch isn't really needed for this problem. It is just here so we can get a feel for what the problem is asking of us. Also, the sketch is presented in two ways to help "see" the surface. Sometimes it is easier to see what is going on with both a "standard" set of axes and a "box frame" set of axes.

Now, we know from the notes that the volume is given by,

$$V = \iint\limits_R 9x^2 + 4xy + 4\,dA$$

where *R* is the rectangle given in the problem statement.

Step 2

Now, as with the other problems in this section we need to determine the order of integration. In this case there doesn't seem to be much difference between the two so we'll integrate with respect to x first simply because there are more of them and so maybe we'll "simplify" the integral somewhat after doing that integration.

Here is the x integration work.

$$\begin{aligned} V &= \iint\limits_R 9x^2 + 4xy + 4\,dA \\ &= \int_0^2 \int_{-1}^1 9x^2 + 4xy + 4\,dx\,dy = \int_0^2 \left(3x^3 + 2x^2 y + 4x\right)\Big|_{-1}^{1} dy = \int_0^2 14\,dy \end{aligned}$$

In this case the second term involving the *y* actually canceled out when doing the *x* limit evaluation. This will happen on occasion and isn't something we should get excited about when it happens.

Step 3

Finally, all we need to do is the very simply *y* integration. Here is that work.

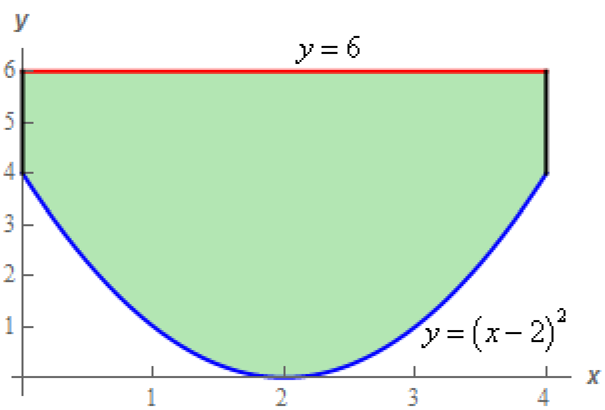
$$V = \int_0^2 14\,dy = \left(14y\right)\Big|_0^2 = \boxed{28}$$

Section 4-3 : Double Integrals over General Regions

1. Evaluate $\iint\limits_D 42y^2 - 12x\,dA$ where $D = \left\{ (x,y) \mid 0 \le x \le 4, (x-2)^2 \le y \le 6 \right\}$

Step 1

Below is a quick sketch of the region *D*.



In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2

With this problem we were pretty much given the order of integration by how the region *D* was specified in the problem statement. Note however, that the sketch shows that this was pretty much the only easy order of integration. The same function is always on the top of the region and the same function is always on the bottom of the region and so it makes sense to integrate *y* first.

If we wanted to integrate *x* first we'd have a messier integration to deal with. First the right/left functions change and so we couldn't do the *x* integration with a single integral. The *x* integration would require two integrals in this case. There is also the fact that the lower portion of the region has the same function for both the right and left sides. The equation could be solved for *x*, as we'd need to in order to *x* integration first, and often that is either very difficult or would give unpleasant limits. It wouldn't be too difficult in this case but it would put roots into the limits and that often makes for messier integration.

So, let's go with the order of integration specified in the problem statement and because we were given *D* in the set builder notation we also were given the limits for both *x* and *y* which is nice as we usually will need to figure those out on our own.

Here is the integral set up for *y* integration first.

$$\iint\limits_D 42y^2 - 12x\,dA = \int_0^4 \int_{(x-2)^2}^6 42y^2 - 12x\,dy\,dx$$

Step 3

Here is the *y* integration.

$$\begin{aligned}\iint\limits_D 42y^2 - 12x\,dA &= \int_0^4 \int_{(x-2)^2}^6 42y^2 - 12x\,dy\,dx \\ &= \int_0^4 \left(14y^3 - 12xy\right)\Big|_{(x-2)^2}^6\,dx \\ &= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2\,dx\end{aligned}$$

Step 4

Now, we did not do any real simplification of the integrand in the last step. There was a reason for that.

After doing the first integration students will often just launch into a "simplification" mode and multiply everything out and "simplify" everything. Sometimes that does need to be done and we don't want to give the impression it is never a good thing or never needs to be done.

However, take a look at the third term above. It could be multiplied out if we wanted to but it would take a little bit of time and there is a chance we'd mess up a sign or coefficient somewhere. We are going to be integrating and the third term can be integrated very quickly with a simple Calculus I substitution. In other words, why bother with the messy multiplication with that term when it does not need to be done.

The fourth term, on the other hand, does need to be multiplied out because of the extra *x* that is in the front of the term.

So, before just launching into "simplification" mode take a quick look at the integrand and see if there are any terms that can be done with a simple substitution as we won't need to mess with those terms. Only multiply out terms that actually need to be multiplied out.

Here is the *x* integration work. We will leave the Algebra details to you to verify and we'll also be leaving the Calculus I substitution work to you to verify.

$$\begin{aligned}\iint\limits_D 42y^2 - 12x\,dA &= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2\,dx \\ &= \int_0^4 3024 - 24x - 48x^2 + 12x^3 - 14(x-2)^6\,dx \\ &= \left(3024x - 12x^2 - 16x^3 + 3x^4 - 2(x-2)^7\right)\Big|_0^4 = \boxed{11136}\end{aligned}$$

Before leaving this problem let's again note how much easier dealing with the third term was because we did not multiply it out and just used a substitution. Made this problem a lot easier.

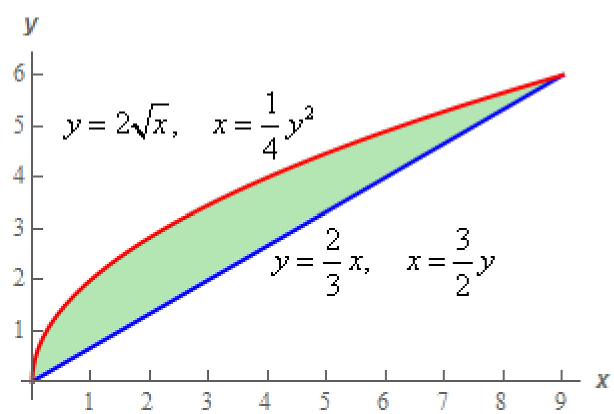
Also note that this problem illustrated an important point that needs to be made with many of these integrals. These integrals will often get very messy after the first integration. You need to be ready for that and expect it to happen on occasion. Just because they start off looking "easy" doesn't mean that they will remain easy throughout the whole problem. Just because it becomes a mess doesn't mean you've made a mistake, although that is unfortunately always a possible reason for a messy integral. It may just mean this is one of those integrals that get somewhat messy before they are done.

---

2. Evaluate $\iint\limits_D 2yx^2 + 9y^3\,dA$ where *D* is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for each curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} 0 \le x \le 9 \\ \frac{2}{3}x \le y \le 2\sqrt{x} \end{array} \qquad \text{OR} \qquad \begin{array}{c} 0 \le y \le 6 \\ \frac{1}{4}y^2 \le x \le \frac{3}{2}y \end{array}$$

Step 3
As noted above either order could be done without much real change in difficulty. So, for this problem let's integrate with respect to *y* first.

Often roots in limits can lead to messier integrands for the second integration. However, in this case notice that both the *y* terms will integrate to terms with even exponents and that will eliminate the root upon evaluation. This order will also keep the exponents slightly smaller which *may* help a little with the second integration.

Here is the integral set up for *y* integration first.

$$\iint\limits_D 2yx^2 + 9y^3\,dA = \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3\,dy\,dx$$

Step 4
Here is the *y* integration.

$$\begin{aligned} \iint\limits_D 2yx^2 + 9y^3\,dA &= \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3\,dy\,dx \\ &= \int_0^9 \left(y^2x^2 + \tfrac{9}{4}y^4 \right)\Big|_{\frac{2}{3}x}^{2\sqrt{x}}\,dx \\ &= \int_0^9 4x\left(x^2\right) + \tfrac{9}{4}\left(16x^2\right) - \left[\tfrac{4}{9}x^2\left(x^2\right) + \tfrac{9}{4}\left(\tfrac{16}{81}x^4\right) \right] dx \\ &= \int_0^9 36x^2 + 4x^3 - \tfrac{8}{9}x^4\,dx \end{aligned}$$

Step 5
In this case there is a small amount of pretty simple simplification that we could do to reduce the complexity of the integrand and so we did that.

All that is left is to do the *x* integration. Here is that work.

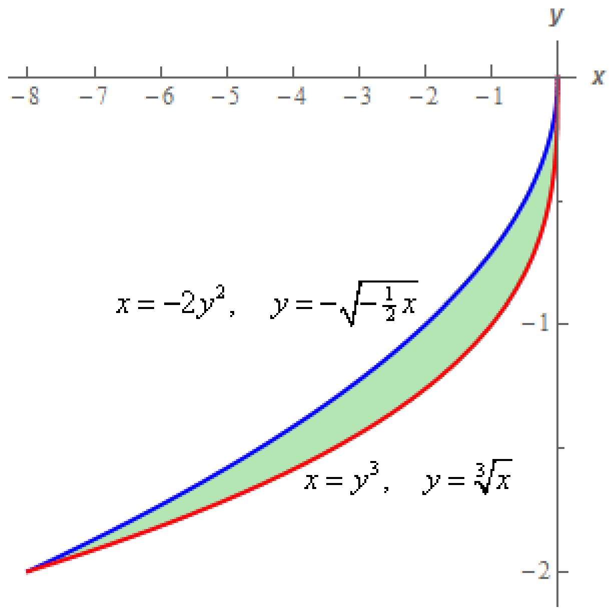
$$\begin{aligned} \iint\limits_D 2yx^2 + 9y^3\,dA &= \int_0^9 36x^2 + 4x^3 - \tfrac{8}{9}x^4\,dx \\ &= \left(12x^3 + x^4 - \tfrac{8}{45}x^5 \right)\Big|_0^9 = \boxed{\tfrac{24057}{5} = 4811.4} \end{aligned}$$

---

3. Evaluate $\iint\limits_D 10x^2y^3 - 6\,dA$ where *D* is the region bounded by $x = -2y^2$ and $x = y^3$.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for each curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} -8 \le x \le 0 \\ \sqrt[3]{x} \le y \le -\sqrt{-\frac{1}{2}x} \end{array} \qquad \text{OR} \qquad \begin{array}{c} -2 \le y \le 0 \\ -2y^2 \le x \le y^3 \end{array}$$

Step 3

As noted above either order could be done without much real change in difficulty. However, for the first set of limits both of the *y* limits are roots and that might make the second integration a little messier. The upper *y* limit is also a little messy with all the minus signs.

So, for this problem let's integrate with respect to *x* first. Here is the integral set up for *x* integration first.

$$\iint\limits_D 10x^2y^3 - 6\,dA = \int_{-2}^{0}\int_{-2y^2}^{y^3} 10x^2y^3 - 6\,dx\,dy$$

Step 4

Here is the x integration.

$$\begin{aligned}\iint\limits_D 10x^2y^3 - 6\,dA &= \int_{-2}^{0}\int_{-2y^2}^{y^3} 10x^2y^3 - 6\,dx\,dy \\ &= \int_{-2}^{0}\left(\tfrac{10}{3}x^3y^3 - 6x\right)\Big|_{-2y^2}^{y^3}\,dy \\ &= \int_{-2}^{0}\tfrac{10}{3}y^9\left(y^3\right) - 6y^3 - \left[\tfrac{10}{3}\left(-8y^6\right)\left(y^3\right) - 6\left(-2y^2\right)\right]dy \\ &= \int_{-2}^{0}\tfrac{10}{3}y^{12} + \tfrac{80}{3}y^9 - 6y^3 - 12y^2\,dy\end{aligned}$$

Step 5

In this case there is a small amount of pretty simple simplification that we could do to reduce the complexity of the integrand and so we did that.

All that is left is to do the *y* integration. Here is that work.

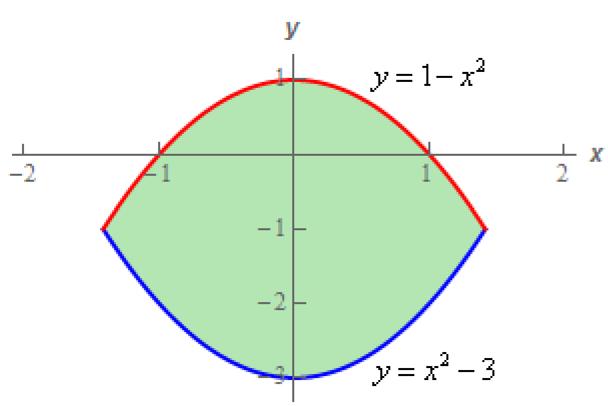
$$\begin{aligned}\iint\limits_D 10x^2y^3 - 6\,dA &= \int_{-2}^{0}\tfrac{10}{3}y^{12} + \tfrac{80}{3}y^9 - 6y^3 - 12y^2\,dy \\ &= \left(\tfrac{10}{39}y^{13} + \tfrac{8}{3}y^{10} - \tfrac{3}{2}y^4 - 4y^3\right)\Big|_{-2}^{0} = \boxed{-\tfrac{8296}{13} = -638.1538}\end{aligned}$$

---

4. Evaluate $\iint\limits_D x\left(y-1\right)dA$ where *D* is the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

Step 1

Below is a quick sketch of the region D.



In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2
With this problem the region is really only set up to integrate *y* first. Integrating *x* first would require two integrals (the right/left functions change) and the limits for the *x*'s would be a little messy to deal with.

So, here are the limits for this integral.

$$-\sqrt{2} \le x \le \sqrt{2}$$
$$x^2 - 3 \le y \le 1 - x^2$$

The *x* limits can easily be found by setting the two equations equal and solving for *x*.

Step 3
Here is the integral set up for *y* integration first.

$$\iint\limits_D x\left(y-1\right)dA = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{1-x^2} x\left(y-1\right)dy\,dx$$

Step 4
Here is the *y* integration.

$$\begin{aligned}\iint\limits_D x\left(y-1\right)dA &= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{1-x^2} x\left(y-1\right)dy\,dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}}\left(x\left(\tfrac{1}{2}y^2 - y\right)\right)\Big|_{x^2-3}^{1-x^2}dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}}\tfrac{1}{2}x\left(1-x^2\right)^2 - \tfrac{1}{2}x\left(x^2-3\right)^2 - x\left(1-x^2\right) + x\left(x^2-3\right)dx\end{aligned}$$

Step 5
With this problem we have two options in dealing with the simplification of the integrand. The 3rd and 4th terms will need to be simplified. The 1st and 2nd terms however can be simplified, and they aren't that hard to simplify or we could do a fairly quick Calculus I substitutions for each of them.

If you multiply everything out you will get the following integral.

$$\iint\limits_D x\left(y-1\right)dA = \int_{-\sqrt{2}}^{\sqrt{2}} 4x^3 - 8x\,dx = \left(x^4 - 4x^2\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{0}$$

There is a lot of cancelation going on with this integrand. It isn't obvious however that there would be that much cancelling at first glance and the multiplication required to do the cancelling is the type where it is easy to miss a minus sign and get the wrong integrand and then a wrong answer.

So, let's also do the substitution path to see the difference. Doing that gives,

$$\begin{aligned}\iint\limits_D x\left(y-1\right)dA &= \int_{-\sqrt{2}}^{\sqrt{2}} \tfrac{1}{2}x\left(1-x^2\right)^2 - \tfrac{1}{2}x\left(x^2-3\right)^2 + 2x^3 - 4x\,dx \\ &= \left(-\tfrac{1}{12}\left(1-x^2\right)^3 - \tfrac{1}{12}\left(x^2-3\right)^3 + \tfrac{1}{2}x^4 - 2x^2\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{0}\end{aligned}$$

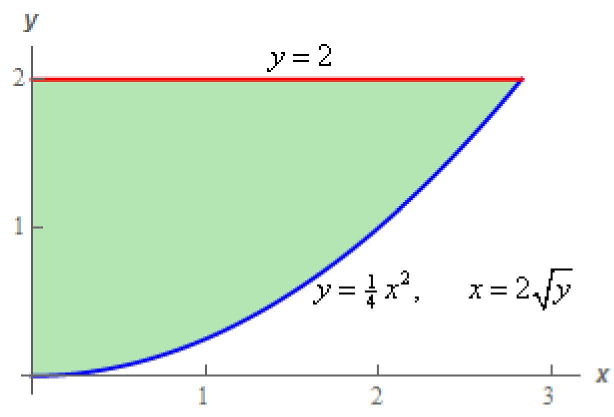
So, the same answer which shouldn't be very surprising, but a slightly messier integration and evaluation process. Which path you chose to take and which path you feel is the easier of the two is probably very dependent on the person. As shown however you will get the same answer so you won't need to worry about that.

---

5. Evaluate $\iint\limits_D 5x^3\cos\left(y^3\right)dA$ where *D* is the region bounded by $y=2$, $y=\frac{1}{4}x^2$ and the *y*-axis.

Step 1

Below is a quick sketch of the region *D*.



Note that we gave both forms of the equation for the lower curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2
With this problem the region can be dealt with either order of integration. The integral on the other hand can't. We simply cannot integrate *y* first as there is no y^2 in front of the cosine we'd need to do the substitution. So, we'll have no choice but to do the *x* integration first.

So, here are the limits for this integral.

$$\begin{gathered} 0 \le y \le 2 \\ 0 \le x \le 2\sqrt{y} \end{gathered}$$

Step 3
Here is the integral set up for *x* integration first.

$$\iint\limits_D 5x^3 \cos\left(y^3\right) dA = \int_0^2 \int_0^{2\sqrt{y}} 5x^3 \cos\left(y^3\right) dx\,dy$$

Step 4
Here is the *x* integration.

$$\begin{aligned} \iint\limits_D 5x^3 \cos\left(y^3\right) dA &= \int_0^2 \int_0^{2\sqrt{y}} 5x^3 \cos\left(y^3\right) dx\,dy \\ &= \int_0^2 \left(\tfrac{5}{4}x^4 \cos\left(y^3\right)\right)\Big|_0^{2\sqrt{y}} dy \\ &= \int_0^2 20y^2 \cos\left(y^3\right) dy \end{aligned}$$

Step 5
Note that while we couldn't do the *y* integration first we can do it now. The y^2 we need for the substitution is now there after doing the *x* integration.

Here is the *y* integration.

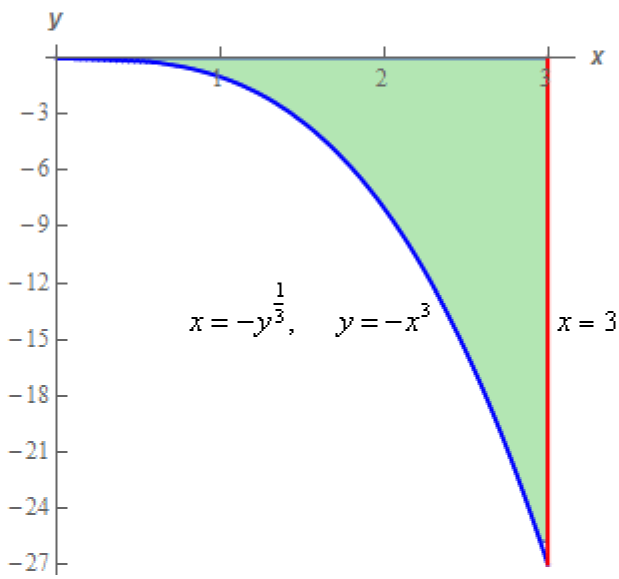
$$\begin{aligned} \iint\limits_D 5x^3 \cos\left(y^3\right) dA &= \int_0^2 20y^2 \cos\left(y^3\right) dy \\ &= \left(\tfrac{20}{3}\sin\left(y^3\right)\right)\Big|_0^2 = \boxed{\tfrac{20}{3}\sin\left(8\right) = 6.5957} \end{aligned}$$

Remember to have your calculator set to radians when computing to decimals.

6. Evaluate $\displaystyle\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA$ where *D* is the region bounded by $x=-y^{\frac{1}{3}}$, $x=3$ and the *x*-axis.

Step 1

Below is a quick sketch of the region *D*.



Note that we gave both forms of the equation for the left curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of *D*.

Even if you can do the integral in either order the sketch of *D* will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} 0\le x\le 3 \\ -x^3\le y\le 0 \end{array} \qquad \text{OR} \qquad \begin{array}{c} -27\le y\le 0 \\ -y^{\frac{1}{3}}\le x\le 3 \end{array}$$

Step 3

As noted above either order could be done without much real change in difficulty. However, it looks like the limits are a little nicer if we integrate with respect to *y* first. One of the *y* limits is zero and the *x* limits are smaller so it looks like this might be a little easier to deal with.

So, for this problem let's integrate with respect to *y* first. Here is the integral set up for *y* integration first.

$$\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA = \int_0^3\int_{-x^3}^0 \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dy\,dx$$

Step 4

Here is the *y* integration.

$$\begin{aligned}\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA &= \int_0^3\int_{-x^3}^0 \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dy\,dx \\ &= \int_0^3 \left(\left.\frac{\frac{3}{2}y^{\frac{2}{3}}}{\left(x^3+1\right)}\right)\right|_{-x^3}^{0} dx \\ &= \int_0^3 -\frac{\frac{3}{2}\left(-x^3\right)^{\frac{2}{3}}}{\left(x^3+1\right)}\,dx = \int_0^3 -\frac{\frac{3}{2}x^2}{\left(x^3+1\right)}\,dx\end{aligned}$$

Be careful with the minus signs in this integration. The minus sign in front of the integrand comes from the evaluation process (recall we first evaluate at zero!). The minus sign in the numerator will be eliminated when we deal with the exponent (the numerator of the exponent is a two so we need to square things!).

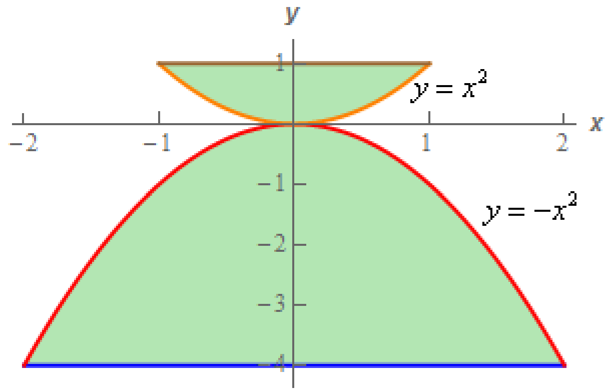
Step 5

Now the *x* integration is a simple Calculus I substitution. We'll leave the substitution work to you to verify. Here is the *x* integration.

$$\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA = \int_0^3 -\frac{\frac{3}{2}x^2}{\left(x^3+1\right)}\,dx = \left.\left(-\frac{1}{2}\ln\left(x^3+1\right)\right)\right|_0^3 = \boxed{-\tfrac{1}{2}\ln\left(28\right) = -1.6661}$$

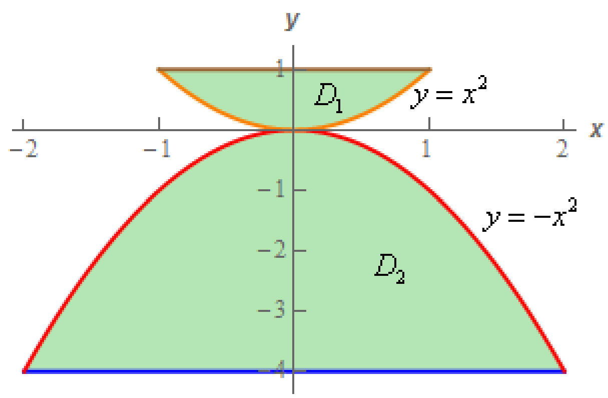
Remember that $\ln(1) = 0$!

7. Evaluate $\displaystyle\iint\limits_D 3-6xy\,dA$ where *D* is the region shown below.



Step 1

First, let's label the two sub regions in D as shown below.



Step 2

Hopefully it is clear that we cannot get a single set of limits that will completely describe *D* so we'll need to split the integral as follows.

$$\iint\limits_D 3-6xy\,dA = \iint\limits_{D_1} 3-6xy\,dA + \iint\limits_{D_2} 3-6xy\,dA$$

Step 3

The region should also (hopefully) make it clear that we'll need to integrate *y* first with both of the integrals. So, here are the limits for each integral.

$$\begin{array}{cc} D_1 & D_2 \\ -1 \le x \le 1 & -2 \le x \le 2 \\ x^2 \le y \le 1 & -4 \le y \le -x^2 \end{array}$$

The integrals are then,

$$\iint\limits_D 3-6xy\,dA = \int_{-1}^{1}\int_{x^2}^{1} 3-6xy\,dy\,dx + \int_{-2}^{2}\int_{-4}^{-x^2} 3-6xy\,dy\,dx$$

Step 4

Not much to do now other than do the integrals. Here is the *y* integration for both of them.

$$\begin{aligned}\iint\limits_D 3-6xy\,dA &= \int_{-1}^{1}\left(3y-3xy^2\right)\Big|_{x^2}^{1}\,dx + \int_{-2}^{2}\left(3y-3xy^2\right)\Big|_{-4}^{-x^2}\,dx \\ &= \int_{-1}^{1} 3x^5-3x^2-3x+3\,dx + \int_{-2}^{2} 12+48x-3x^2-3x^5\,dx\end{aligned}$$

Step 5

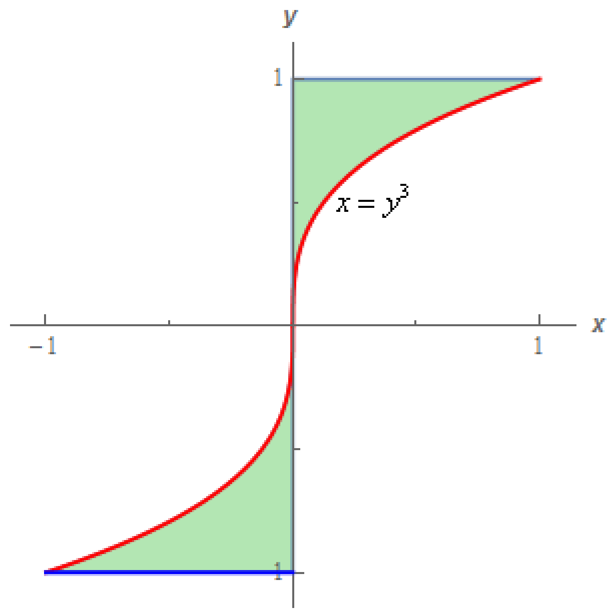
Finally, here is the *x* integration for both of the integrals.

$$\begin{aligned}\iint\limits_D 3-6xy\,dA &= \int_{-1}^{1} 3x^5-3x^2-3x+3\,dx + \int_{-2}^{2} 12+48x-3x^2-3x^5\,dx \\ &= \left(\tfrac{1}{2}x^6-x^3-\tfrac{3}{2}x^2+3x\,dx\right)\Big|_{-1}^{1} + \left(12x+24x^2-x^3-\tfrac{1}{2}x^6\,dx\right)\Big|_{-2}^{2} \\ &= \qquad 4 \qquad + \qquad 32 \\ &= \boxed{36}\end{aligned}$$

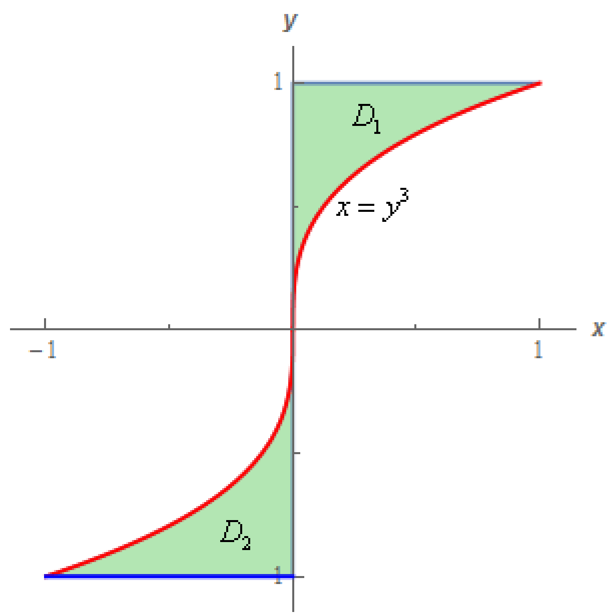
Don't always expect every integral over a region to be done with a single integral. On occasion you will need to split the integral up and do the actual integration over separate sub regions. In this case that was obvious but sometimes it might not be so clear until you get into the problem and realize it would be easier to do over sub regions.

---

8. Evaluate $\iint\limits_D \mathbf{e}^{y^4}\,dA$ where *D* is the region shown below.



Step 1
First, let's label the two sub regions in D as shown below.



Step 2
Despite the fact that each of the regions is bounded by the same curve we cannot get a single set of limits that will completely describe D. In the upper region $x = y^3$ is the right boundary and in the lower region $x = y^3$ is the left boundary.

Therefore, each region will need a separate set of limits and so we'll need to split the integral as follows.

$$\iint\limits_D \mathbf{e}^{y^4}\,dA = \iint\limits_{D_1} \mathbf{e}^{y^4}\,dA + \iint\limits_{D_2} \mathbf{e}^{y^4}\,dA$$

Step 3
Hopefully it is clear that we'll need to integrate *x* first with both of the integrals. So, here are the limits for each integral.

$$\begin{array}{cc} D_1 & D_2 \\ 0 \le y \le 1 & -1 \le y \le 0 \\ 0 \le x \le y^3 & y^3 \le x \le 0 \end{array}$$

The integrals are then,

$$\iint\limits_D \mathbf{e}^{y^4}\,dA = \int_0^1 \int_0^{y^3} \mathbf{e}^{y^4}\,dx\,dy + \int_{-1}^0 \int_{y^3}^0 \mathbf{e}^{y^4}\,dx\,dy$$

Step 4
Not much to do now other than do the integrals. Here is the *x* integration for both of them.

$$\begin{aligned} \iint\limits_D \mathbf{e}^{y^4}\,dA &= \int_0^1 \left(x\mathbf{e}^{y^4} \right)\Big|_0^{y^3} dx + \int_{-1}^0 \left(x\mathbf{e}^{y^4} \right)\Big|_{y^3}^0 dy \\ &= \int_0^1 y^3\mathbf{e}^{y^4}\,dy + \int_{-1}^0 -y^3\mathbf{e}^{y^4}\,dy \end{aligned}$$

Step 5
Finally, here is the *y* integration for both of the integrals.

$$\begin{aligned} \iint\limits_D \mathbf{e}^{y^4}\,dA &= \int_0^1 y^3\mathbf{e}^{y^4}\,dy + \int_{-1}^0 -y^3\mathbf{e}^{y^4}\,dy \\ &= \left(\tfrac{1}{4}\mathbf{e}^{y^4} \right)\Big|_0^1 + \left(-\tfrac{1}{4}\mathbf{e}^{y^4} \right)\Big|_{-1}^0 \\ &= \tfrac{1}{4}(\mathbf{e}-1) + \tfrac{1}{4}(-1+\mathbf{e}) = \boxed{\tfrac{1}{2}(\mathbf{e}-1) = 0.8591} \end{aligned}$$

Don't always expect every integral over a region to be done with a single integral. On occasion you will need to split the integral up and do the actual integration over separate sub regions. In this case that was fairly obvious but sometimes it might not be so clear until you get into the problem and realize it would be easier to do over sub regions.

---

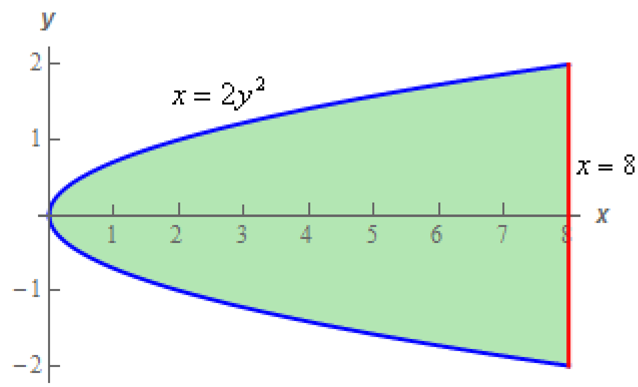
9. Evaluate $\iint\limits_D 7x^2 + 14y\,dA$ where *D* is the region bounded by $x = 2y^2$ and $x = 8$ in the order given below.

(a) Integrate with respect to ***x*** first and then ***y***.

(b) Integrate with respect to ***y*** first and then ***x***.

(a) Integrate with respect to ***x*** first and then ***y***.

Step 1
Here's a quick sketch of the region with the curves labeled for integration with respect to x first.



The limits for the integral for integration with respect to *x* first are then,

$$-2 \le y \le 2$$
$$2y^2 \le x \le 8$$

Plugging these limits into the integral is then,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_{-2}^{2}\int_{2y^2}^{8} 7x^2 + 14y\,dx\,dy$$

Step 2
The *x* integration for this integral is,

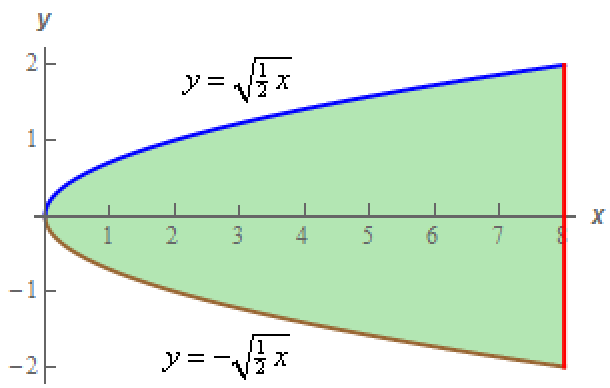
$$\begin{aligned}\iint\limits_D 7x^2 + 14y\,dA &= \int_{-2}^{2}\int_{2y^2}^{8} 7x^2 + 14y\,dx\,dy \\ &= \int_{-2}^{2}\left(\tfrac{7}{3}x^3 + 14xy\right)\Big|_{2y^2}^{8}\,dy \\ &= \int_{-2}^{2}\tfrac{3584}{3} + 112y - 28y^3 - \tfrac{56}{3}y^6\,dy\end{aligned}$$

Step 3
Finally, the *y* integration is,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_{-2}^{2} \tfrac{3584}{3} + 112y - 28y^3 - \tfrac{56}{3}y^6\,dy$$
$$= \left(\tfrac{3584}{3}y + 56y^2 - 7y^4 - \tfrac{8}{3}y^7\right)\Big|_{-2}^{2} = \boxed{4096}$$

(b) Integrate with respect to ***y*** first and then ***x***.

Step 1
Here's a quick sketch of the region with the curves labeled for integration with respect to y first.



Note that in order to do *y* integration first we needed to solve the equation of the parabola for *y* so the top and bottom curve will have distinct equations in terms of *x*, which we need to integrate with respect to *y* first.

The limits for the integral for integration with respect to *y* first are then,

$$0 \le x \le 8$$
$$-\sqrt{\tfrac{1}{2}x} \le y \le \sqrt{\tfrac{1}{2}x}$$

Plugging these limits into the integral is then,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_0^8 \int_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} 7x^2 + 14y\,dy\,dx$$

Step 2
The *y* integration for this integral is,

$$\iint_D 7x^2+14y\,dA = \int_0^8 \int_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} 7x^2+14y\,dy\,dx$$
$$= \int_0^8 \left(7x^2y+7y^2\right)\Big|_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} dx$$
$$= \int_0^8 \tfrac{14}{\sqrt{2}} x^{\frac{5}{2}}\,dx$$

Step 3
Finally, the *x* integration is,

$$\iint_D 7x^2+14y\,dA = \int_0^8 \tfrac{14}{\sqrt{2}} x^{\frac{5}{2}}\,dx = \left(\tfrac{4}{\sqrt{2}} x^{\frac{7}{2}}\right)\Big|_0^8 = \boxed{4096}$$

We got the same result as the first order of integration as we knew we would.

---

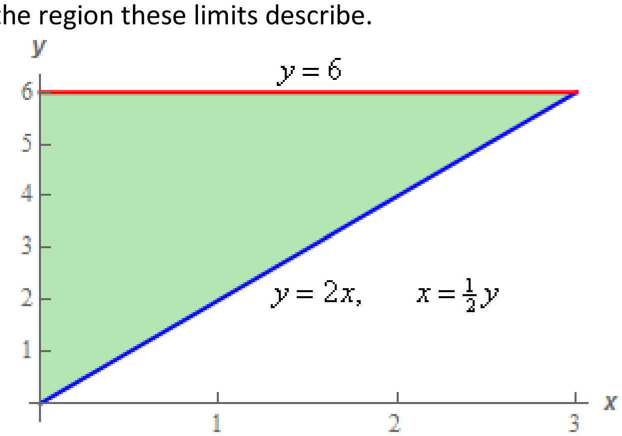
10. Evaluate $\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx$ by first reversing the order of integration.

Step 1
Let's start off by noticing that if we were to integrate with respect to y first we'd need to do a trig substitution (which we'll all be thankful if we don't need to do it!) so interchanging the order in this case might well save us a messy integral.

So, here are the limits we get from the integral.

$$0 \le x \le 3$$
$$2x \le y \le 6$$

Here is a quick sketch of the region these limits describe.



When reversing the order of integration it is often very helpful to have a sketch of the region to make sure we get the correct limits for the reversed order.

Step 2
Okay, if to reverse the order of integration we need to integrate with respect to *x* first. The limits for the reversed order are then,

$$\begin{gathered} 0 \le y \le 6 \\ 0 \le x \le \tfrac{1}{2} y \end{gathered}$$

The integral with reversed order is,

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \int_0^6 \int_0^{\frac{1}{2}y} \sqrt{y^2+2}\,dx\,dy$$

Step 3
Now all we need to do is evaluate the integrals. Here is the *x* integration.

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \int_0^6 \left(x\sqrt{y^2+2} \right) \Big|_0^{\frac{1}{2}y} dy = \int_0^6 \tfrac{1}{2} y \sqrt{y^2+2}\,dy$$

Step 4
Note that because of the *x* integration we'll not need to do a trig substitution. All we need is a simple Calculus I integral. Here is the *y* integration (we'll leave it to you to verify the substitution details).

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \left(\tfrac{1}{6} \left(y^2+2 \right)^{\frac{3}{2}} \right) \Bigg|_0^6 = \boxed{\tfrac{1}{6} \left(38^{\frac{3}{2}} - 2^{\frac{3}{2}} \right) = 38.5699}$$

---

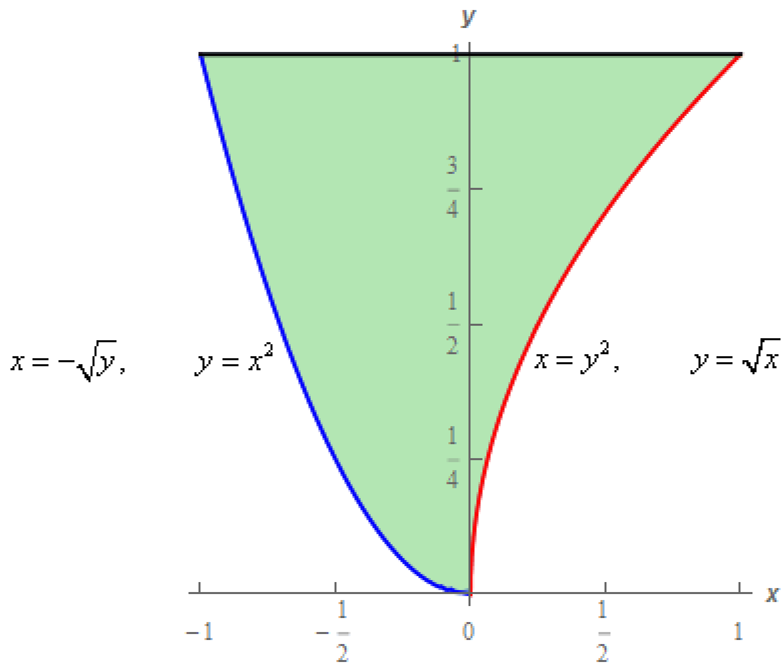
11. Evaluate $\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy$ by first reversing the order of integration.

Step 1
Note that with this problem, unlike the previous problem, there is no issues with this order of integration. We could easily do this integral in the given order. We are only reversing the order in the problem because we were told to do so in the problem statement.

Here are the limits we get from the integral.

$$\begin{gathered} 0 \le y \le 1 \\ -\sqrt{y} \le x \le y^2 \end{gathered}$$

Here is a quick sketch of the region these limits describe.



When reversing the order of integration it is often very helpful to have a sketch of the region to make sure we get the correct limits for the reversed order. That is especially the case with this problem.

Step 2
Okay, if we reverse the order of integration we need to integrate with respect to *y* first. That leads to a small issue however. The lower function changes and so we'll need to split this up into two regions which in turn will mean two integrals when we reverse the order integration.

The limits for each of the regions with the reversed order are then,

$$\begin{array}{ll} -1 \le x \le 0 & \quad 0 \le x \le 1 \\ x^2 \le y \le 1 & \quad \sqrt{x} \le y \le 1 \end{array}$$

The integrals with reversed order are then,

$$\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy = \int_{-1}^0 \int_{x^2}^1 6x - y\,dy\,dx + \int_0^1 \int_{\sqrt{x}}^1 6x - y\,dy\,dx$$

Step 3
Now all we need to do is evaluate the integrals. Here is the *y* integration for each.

$$\begin{aligned} \int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy &= \int_{-1}^0 \left(6xy - \tfrac{1}{2}y^2\right)\Big|_{x^2}^1 dx + \int_0^1 \left(6xy - \tfrac{1}{2}y^2\right)\Big|_{\sqrt{x}}^1 dx \\ &= \int_{-1}^0 \tfrac{1}{2}x^4 - 6x^3 + 6x - \tfrac{1}{2}\,dx + \int_0^1 -6x^{\frac{3}{2}} + \tfrac{13}{2}x - \tfrac{1}{2}\,dx \end{aligned}$$

Step 4
Finally, the x integration.

$$\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y \, dx \, dy = \left(\tfrac{1}{10}x^5 - \tfrac{3}{2}x^4 + 3x^2 - \tfrac{1}{2}x \right)\Big|_{-1}^{0} + \left(-\tfrac{12}{5}x^{\frac{5}{2}} + \tfrac{13}{4}x^2 - \tfrac{1}{2}x \right)\Big|_{0}^{1}$$
$$= -\tfrac{19}{10} + \tfrac{7}{20}$$
$$= \boxed{-\tfrac{31}{20}}$$

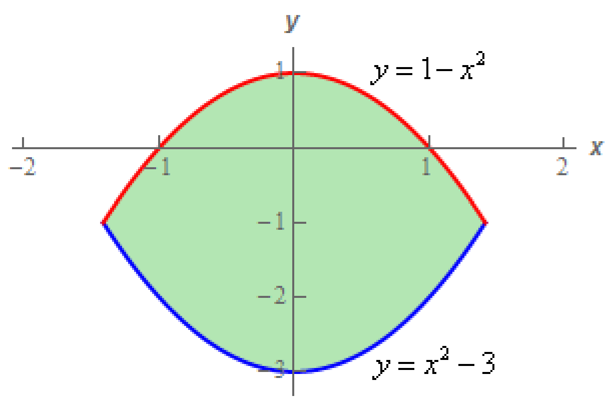
---

12. Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

Step 1
Okay, we know that the area of any region D can be found by evaluating the following double integral.

$$A = \iint\limits_D dA$$

For this problem D is the region sketched below.



Step 2
We've done enough double integrals by this point that it should be pretty obvious the best order of integration is to integrate with respect to y first.

Here are the limits for the integral with this order.

$$-\sqrt{2} \le x \le \sqrt{2}$$
$$x^2 - 3 \le y \le 1 - x^2$$

The x limits can easily be found by setting the two equations equal and solving for x.

Step 3
The integral for the area is then,

$$A = \iint\limits_{D} dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{1-x^2} dy\,dx$$

Step 4

Now all we need to do is evaluate the integral. Here is the y integration.

$$A = \iint\limits_{D} dA = \int_{-\sqrt{2}}^{\sqrt{2}} y\Big|_{x^3-3}^{1-x^2} dx = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2\,dx$$

Step 5

Finally, the x integration and hence the area of D is,

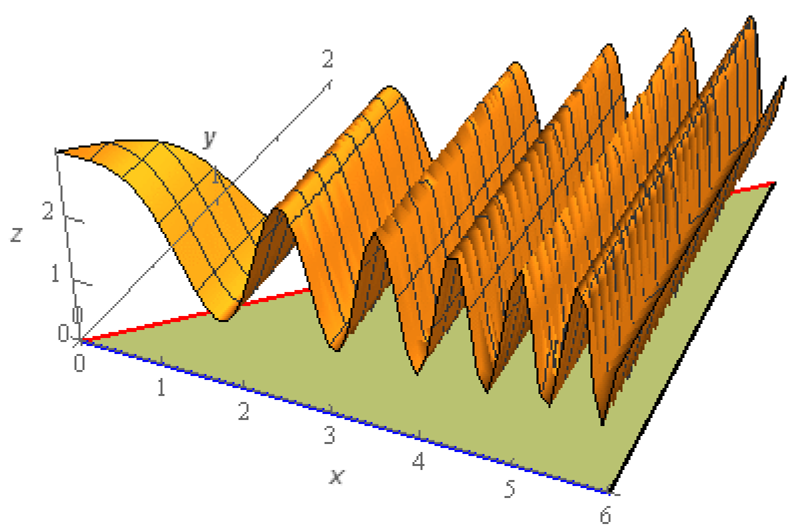
$$A = \left(4x - \tfrac{2}{3}x^3\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{\frac{16\sqrt{2}}{3}}$$

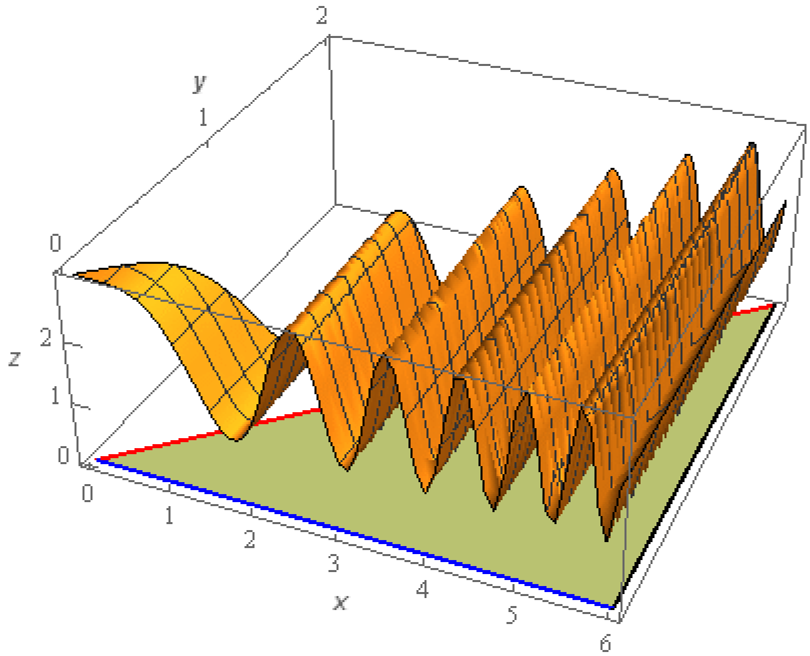
---

13. Use a double integral to determine the volume of the region that is between the *xy*-plane and $f(x,y) = 2 + \cos\left(x^2\right)$ and is above the triangle with vertices $(0,0)$, $(6,0)$ and $(6,2)$.

Step 1

Let's first get a sketch of the function and the triangle that lies under it.





The surface is sketched with a traditional set of axes and well as a "box frame" set of axes. Sometimes it is easier to see what is going on with the surface when both sketches are present.

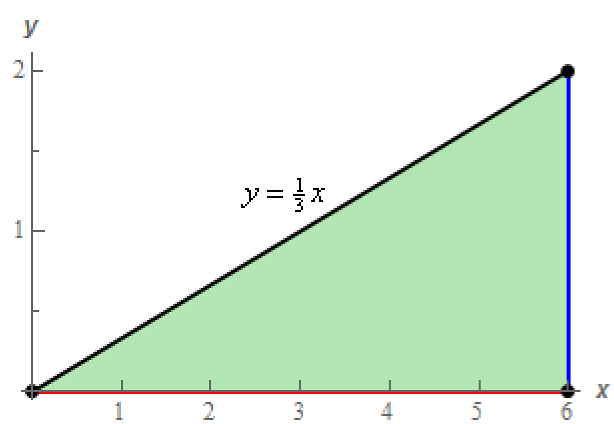
The greenish triangle underneath the surface is the triangle referenced in the problem statement.

Step 2
Now, the volume we are after is given by the following integral,

$$V = \iint\limits_D 2 + \cos\left(x^2\right) dA$$

where *D* is the triangle referenced in the problem statement.

So, in order to evaluate the integral we'll need a sketch of *D* so we can determine an order of integration as well as limits for the integrals.



Step 3
The region *D* can easily be described for either order of integration. However, it should be pretty clear that the integral can't be integrated with respect to *x* first and so we'll need to integrate with respect to *y* first.

Here are the limits for the integral with this order.

$$0 \le x \le 6$$
$$0 \le y \le \tfrac{1}{3}x$$

The integral for the volume is then,

$$V = \iint\limits_D 2 + \cos\left(x^2\right) dA = \int_0^6 \int_0^{\frac{1}{3}x} 2 + \cos\left(x^2\right) dy\, dx$$

Step 4
Now all we need to do is evaluate the integral. Here is the *y* integration.

$$V = \int_0^6 \left(2y + y\cos\left(x^2\right)\right)\Big|_0^{\frac{1}{3}x} dx = \int_0^6 \tfrac{2}{3}x + \tfrac{1}{3}x\cos\left(x^2\right) dx$$

Step 5
Finally, the *x* integration and hence the volume is,

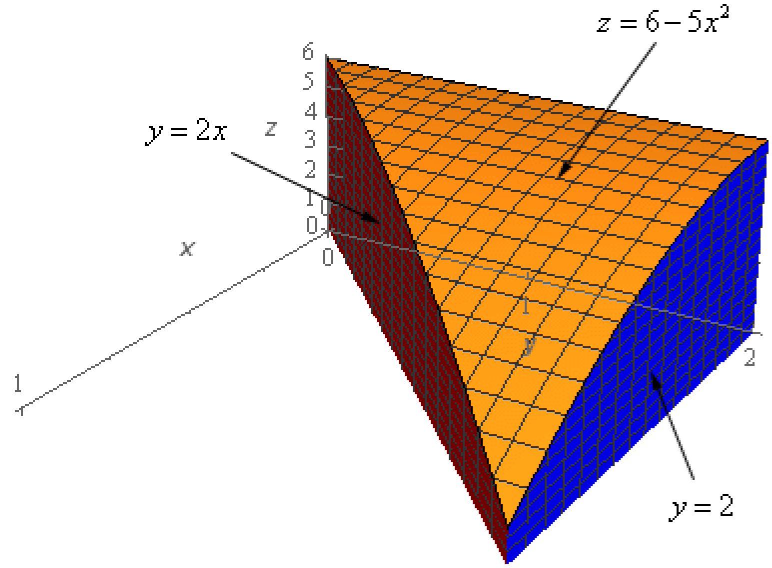
$$V = \left(\tfrac{1}{3}x^2 + \tfrac{1}{6}\sin\left(x^2\right)\right)\Big|_0^6 = \boxed{12 + \tfrac{1}{6}\sin\left(36\right) = 11.8347}$$

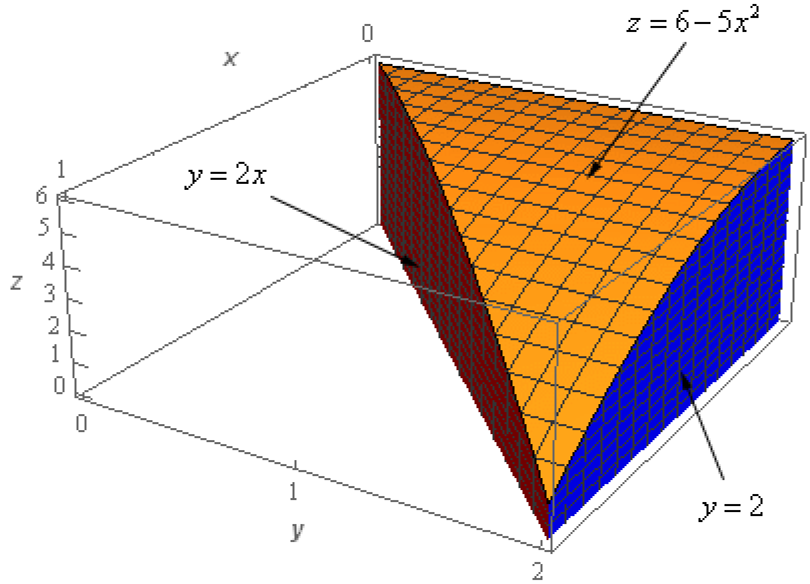
Don't forget to have your calculator set to radians if you are converting to decimals!

---

14. Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x$, $y = 2$, $x = 0$ and the *xy*-plane.

Step 1
Let's first get a sketch of the solid that we're working with. If you are not good at visualizing these types of solids in your head these graphs can be invaluable in helping to get the integral set up.





The surface is sketched with a traditional set of axes and well as a "box frame" set of axes. Sometimes it is easier to see what is going on with the surface when both sketches are present.

The upper surface (the orange surface) is the graph of $z = 6 - 5x^2$. The blue plane is the graph of $y = 2$ which is nothing more than the plane parallel to the *xz*-plane at $y = 2$. The red plane is the graph of $y = 2x$ and this is simply the plane that is perpendicular to the *xy*-plane and goes through the line $y = 2x$ in the *xy*-plane. The surface given by $x = 0$ is simply the *yz*-plane (*i.e* the back of the solid) and is not shown and the *xy*-plane is the bottom of the surface and again is not shown in the sketch.

Step 2

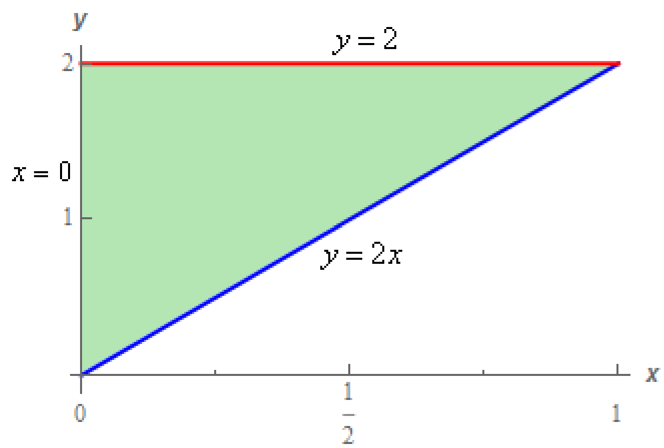
In this section the only method that we have for determining the volume of a solid is to find the volume under a surface. In this case it is hopefully clear that we are looking for the surface that is under

$z = 6 - 5x^2$ and is above the region *D* in the *xy*-plane defined by where the other three planes intersect it. In other words, the region *D* is the regoin in the *xy*-plane that is bounded by $y = 2$, $y = 2x$ and $x = 0$.

The integral for the volume is then,

$$V = \iint\limits_D 6 - 5x^2 \, dA$$

where *D* is sketched below.



Step 3
This integral can be integrated in any order so let's integrate with respect to *y* first to avoid fractions in the limits (which we'd get with one if we integrated with respect to *x* first). Here are the limits for our integral.

$$\begin{gathered} 0 \le x \le 1 \\ 2x \le y \le 2 \end{gathered}$$

The integral for the volume is then,

$$V = \iint\limits_D 6 - 5x^2 \, dA = \int_0^1 \int_{2x}^2 6 - 5x^2 \, dy \, dx$$

Step 4
Now all we need to do is evaluate the integral. Here is the *y* integration.

$$V = \int_0^1 \left(6 - 5x^2\right) y \Big|_{2x}^2 \, dx = \int_0^1 \left(6 - 5x^2\right)\left(2 - 2x\right) dx = \int_0^1 10x^3 - 10x^2 - 12x + 12 \, dx$$

Note that in doing the *y* integration we acknowledged that the whole integrand contained no *y*'s and so could be considered a constant and so would just be multiplied by *y*. We could also have done each term individually but sometimes it is just as easy or even easier to do what we've done here. Of course, we then had to multiply out the integrand for the next step but it wasn't too bad.

Step 5
Finally, the x integration and hence the volume is,

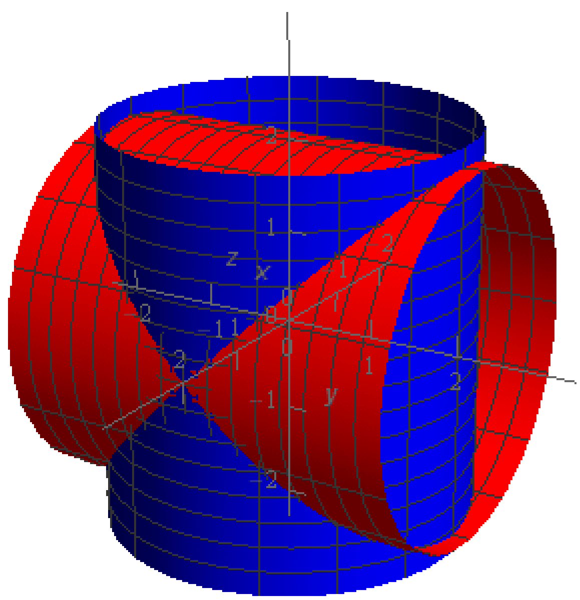
$$V = \left(\tfrac{5}{2}x^4 - \tfrac{10}{3}x^3 - 6x^2 + 12x\right)\Big|_0^1 = \boxed{\tfrac{31}{6}}$$

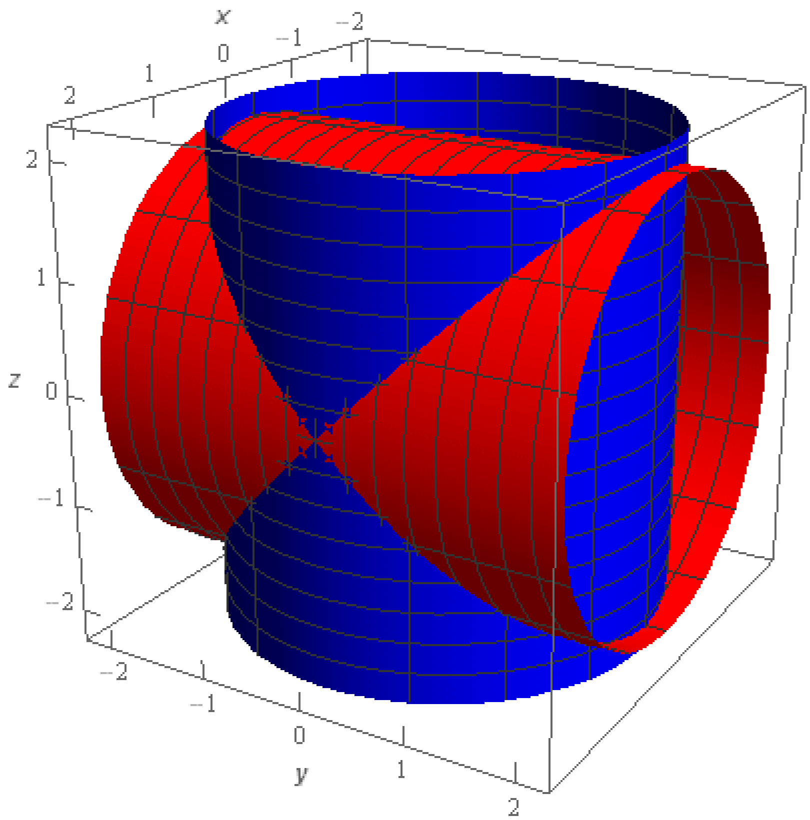
---

15. Use a double integral to determine the volume of the region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Step 1
Okay, probably one of the hardest parts of this problem is the visualization of just what this surface looks like so let's start with that.

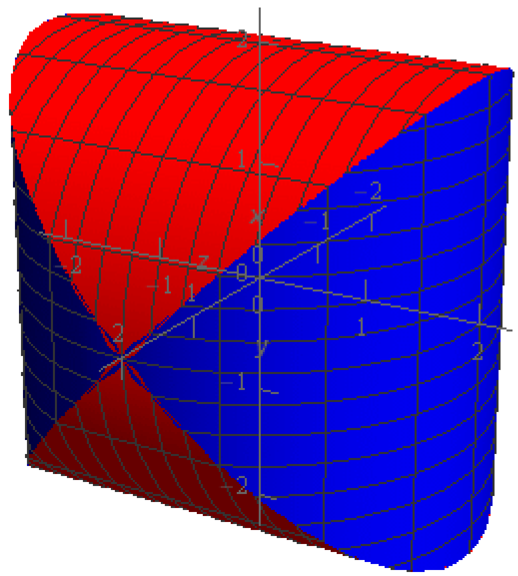
First, $x^2 + y^2 = 4$ is a cylinder of radius 2 centered on the *z*-axis and $x^2 + z^2 = 4$ is a cylinder of radius 2 centered on the *y*-axis and so we will have an intersection of the two around the origin. Here is a sketch of the two cylinders.



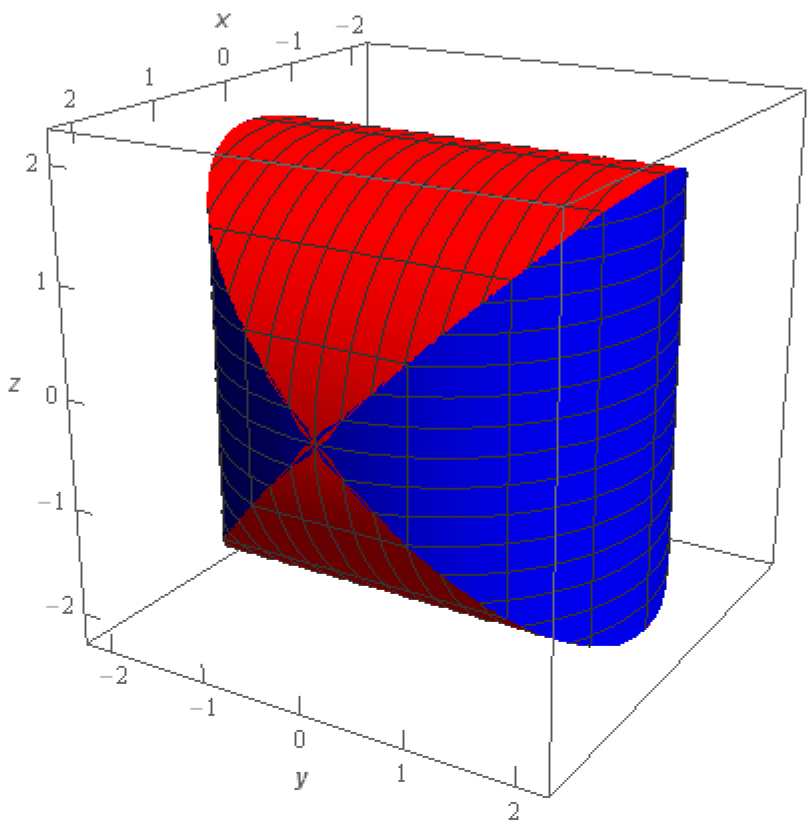


As usual we gave two sketches with different axes systems to help visualize the two cylinders.

Now, the solid we want the volume of is the portion of the red cylinder that lies inside the blue cylinder, or equivalently the portion of the blue cylinder that lies inside the red cylinder. Here are a couple of sketches of only the solid that we are after.



x
y
z



x
y
z

The red portion of this solid is the portion of $x^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 4$ and the blue portion of this solid is the portion of $x^2 + y^2 = 4$ that lies inside $x^2 + z^2 = 4$.

Step 2
Now we need to determine an integral that will give the volume of this solid. In this section we really only talked about finding the solid that was under a surface and above a region *D* in the *xy*-plane.

This solid also exists below the *xy*-plane. For this solid, however, this an easy situation to deal with. Recall that the solid is a portion of the cylinder $x^2 + y^2 = 4$ that is inside the cylinder $x^2 + z^2 = 4$, which is centered on the *z*-axis. This means that the portion of the solid that is below the *xy*-plane is a simply a mirror image of the portion of the solid that is above the *xy*-plane. In other words, the volume of the portion of the solid that lies below the *xy*-plane is the same as the volume of the portion of the solid that lies above the *xy*-plane.

We can use the ideas from this section to easily find the volume of the solid above the *xy*-plane and then the volume of the full solid will simply be double this.

Step 3
So, now we need to actually set up the integral for the volume. First, we can solve $x^2 + z^2 = 4$ for *z* to get,

$$z = \pm\sqrt{4 - x^2}$$

The positive portion of this is the equation for the top half of the cylinder centered on the *y*-axis and the negative portion is the equation for the bottom half of this cylinder.

This means that the integral for the volume of the top half of the solid is,

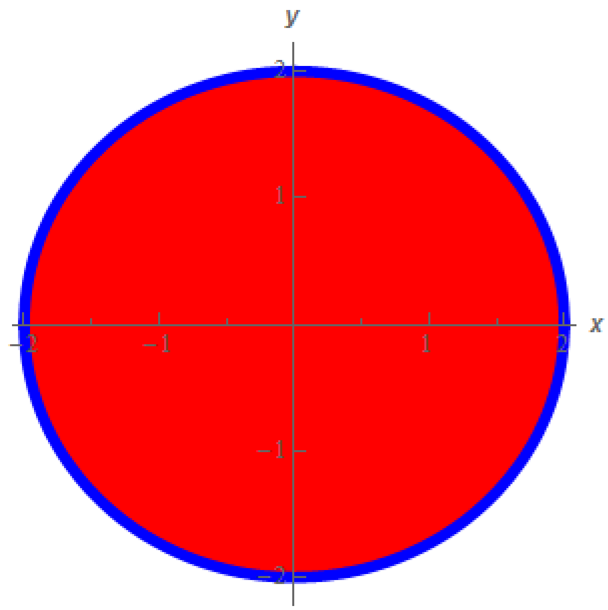
$$V = \iint\limits_D \sqrt{4 - x^2}\, dA$$

The volume of the whole solid is then,

$$V = 2\iint\limits_D \sqrt{4 - x^2}\, dA$$

Step 4
Now, we need to figure out just what the region *D* is. We know that, in general, *D* is the region of the *xy*-plane that we use to get the graph of the surface we are working with, $z = \sqrt{4 - x^2}$, in this case.

But just what is that for this problem? Look at the solid from directly above it. What you see is the following,



The blue circle is the cylinder $x^2 + y^2 = 4$ that is centered on the *z*-axis. Since we are looking at the solid from directly above all we see is the walls of the cylinder which of course is just a circle with equation $x^2 + y^2 = 4$. The red portion of this is the walls of the cylinder $x^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 4$.

This is exactly what we need to determine *D*. Recall once again that *D* is the region in the *xy*-plane we use to graph the upper portion of the surface defining the solid. But from our graph above we see that the upper portion of the surface appears to be a circle of radius 2 centered at the origin. This means that is exactly what *D* is. *D* is just shape that we see when we look at the region from above and so *D* is the circle shown in the graph above.

Step 5
Next let's get the limits for our integral. The region *D* itself doesn't really seem to affect the order of integration and each set of limits are pretty much the same. The only real difference is the *x*'s and *y*'s will be switched around.

Here are the limits for each order of integration.

$$\begin{array}{c} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array} \qquad \text{OR} \qquad \begin{array}{c} -2 \le y \le 2 \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \end{array}$$

So, as noted above there doesn't appear to be much difference in the limits. However, go back and check the integral we need to compute. We can see that if we integrate *x* first we'll need to do a trig substitution. So, let's integrate *y* first.

The integral for the volume of the full solid is then,

$$V = 2\iint\limits_D \sqrt{4-x^2}\,dA = 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx$$

Step 6

Now all we need to do is evaluate the integral. Here is the *y* integration.

$$\begin{aligned} V = 2\iint\limits_D \sqrt{4-x^2}\,dA &= 2\int_{-2}^{2}\left(y\sqrt{4-x^2}\right)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx \\ &= 2\int_{-2}^{2}\left(\sqrt{4-x^2} - \left(-\sqrt{4-x^2}\right)\right)\sqrt{4-x^2}\,dx \\ &= 2\int_{-2}^{2} 2\sqrt{4-x^2}\,\sqrt{4-x^2}\,dx \\ &= 2\int_{-2}^{2} 2\left(4-x^2\right)dx = 4\int_{-2}^{2} 4-x^2\,dx \end{aligned}$$

We put a few more steps into the work that absolutely necessary to make the simplifications clear.

Step 7

Finally, the x integration and hence the volume of the full solid is,

$$V = 4\int_{-2}^{2} 4-x^2\,dx = 4\left(4x - \tfrac{1}{3}x^3\right)\Big|_{-2}^{2} = \boxed{\tfrac{128}{3}}$$

So, as we can see the integration portion of the problem was surprisingly simple. The biggest issue here really is just getting this problem set up.

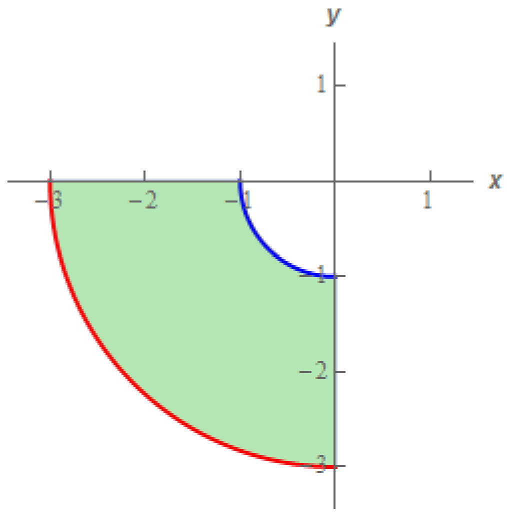
This problem illustrates one of the biggest issues that many students have with some of these problems. You really need to be able to visualize the solids/regions that are being dealt with. Or at the very least have the ability to get the graph of the solid/region. Unfortunately, that is not always something that can be quickly taught. Many folks just seem to naturally be able to visualize these kinds of things but many also are just not able to easily visualize them. If you are in the second set of folks it will make some of these problems a little harder unfortunately but if you persevere you will get through this stuff and hopefully start to be able to do some of the visualization that will be needed on occasion.

Section 4-4 : Double Integrals in Polar Coordinates

1. Evaluate $\iint\limits_D y^2 + 3x\,dA$ where *D* is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Step 1

Below is a quick sketch of the region *D*.



For double integrals in which polar coordinates are going to be used the sketch of *D* is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but it would take two integrals to do the problem and the most of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2

Okay, *D* is just a portion of a ring and so setting up the limits shouldn't be too difficult. Here they are,

$$\pi \le \theta \le \tfrac{3}{2}\pi$$
$$1 \le r \le 3$$

Step 3

The integral in terms of polar coordinates is then,

$$\iint\limits_D y^2 + 3x\,dA = \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 \left(\left(r\sin\theta\right)^2 + 3r\cos\theta\right) r\,dr\,d\theta$$
$$= \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 r^3\sin^2\theta + 3r^2\cos\theta\,dr\,d\theta$$

When converting the integral don't forget to convert the *x* and *y* into polar coordinates. Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra *r* in the integrand. Forgetting the extra *r* is one of the most common mistakes with these kinds of problems.

Step 4
Here is the *r* integration.

$$\iint\limits_D y^2 + 3x\,dA = \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 r^3\sin^2\theta + 3r^2\cos\theta\,dr\,d\theta$$
$$= \int_{\pi}^{\frac{3}{2}\pi} \left(\tfrac{1}{4}r^4\sin^2\theta + r^3\cos\theta\,dr\right)\Big|_1^3\,d\theta$$
$$= \int_{\pi}^{\frac{3}{2}\pi} 20\sin^2\theta + 26\cos\theta\,d\theta$$

Step 5
Finally, here is the θ integration.

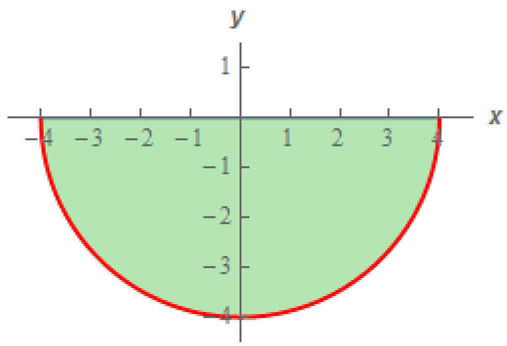
$$\iint\limits_D y^2 + 3x\,dA = \int_{\pi}^{\frac{3}{2}\pi} 20\sin^2\theta + 26\cos\theta\,d\theta$$
$$= \int_{\pi}^{\frac{3}{2}\pi} 10\left(1 - \cos\left(2\theta\right)\right) + 26\cos\theta\,d\theta$$
$$= \left(10\theta - 5\sin\left(2\theta\right) + 26\sin\theta\right)\Big|_{\pi}^{\frac{3}{2}\pi} = \boxed{5\pi - 26}$$

You'll be seeing a fair amount of $\cos^2\theta$, $\sin^2\theta$ and $\sin\theta\cos\theta$ terms in polar integrals so make sure that you know how to integrate these terms! In this case we used a half angle formula to reduce the $\sin^2\theta$ into something we could integrate.

---

2. Evaluate $\iint\limits_D \sqrt{1 + 4x^2 + 4y^2}\,dA$ where *D* is the bottom half of $x^2 + y^2 = 16$.

Step 1
Below is a quick sketch of the region *D*.



For double integrals in which polar coordinates are going to be used the sketch of *D* is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but one of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral. Also note that this integral would be unpleasant in terms of Cartesian coordinates. Hopefully the polar form will be easier to do.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2
Okay, *D* is just a portion of a disk and so setting up the limits shouldn't be too difficult. Here they are,

$$\begin{aligned} \pi \le \theta \le 2\pi \\ 0 \le r \le 4 \end{aligned}$$

Step 3
The integral in terms of polar coordinates is then,

$$\begin{aligned} \iint\limits_D \sqrt{1+4x^2+4y^2}\, dA &= \int_{\pi}^{2\pi}\int_0^4 \left(\sqrt{1+4\left(x^2+y^2\right)}\right) r\, dr\, d\theta \\ &= \int_{\pi}^{2\pi}\int_0^4 r\sqrt{1+4r^2}\, dr\, d\theta \end{aligned}$$

When converting the integral don't forget to convert the *x* and *y* into polar coordinates. In this case don't just substitute the polar conversion formulas in for *x* and *y*! Recall that $x^2+y^2=r^2$ and the integral will be significantly easier to deal with.

Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra *r* in the integrand. Forgetting the extra *r* is one of the most common mistakes with these kinds of problems and in this case without the extra *r* we'd have a much more unpleasant integral to deal with.

Step 4
Here is the r integration.

$$\begin{aligned}\iint\limits_D \sqrt{1+4x^2+4y^2}\,dA &= \int_\pi^{2\pi}\int_0^4 r\sqrt{1+4r^2}\,dr\,d\theta\\ &= \int_\pi^{2\pi}\left.\left(\tfrac{1}{12}\left(1+4r^2\right)^{\frac{3}{2}}\right)\right|_0^4 d\theta\\ &= \int_\pi^{2\pi}\tfrac{1}{12}\left(65^{\frac{3}{2}}-1\right)d\theta\end{aligned}$$

Step 5
Finally, here is the θ integration.

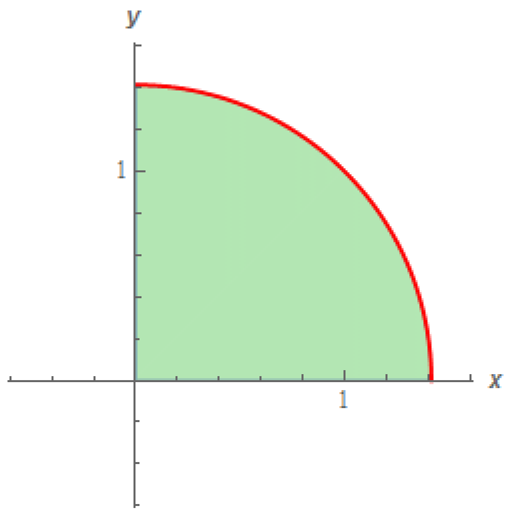
$$\begin{aligned}\iint\limits_D \sqrt{1+4x^2+4y^2}\,dA &= \int_\pi^{2\pi}\tfrac{1}{12}\left(65^{\frac{3}{2}}-1\right)d\theta\\ &= \left.\left(\tfrac{1}{12}\left(65^{\frac{3}{2}}-1\right)\theta\right)\right|_\pi^{2\pi} = \boxed{\tfrac{1}{12}\pi\left(65^{\frac{3}{2}}-1\right)=136.9333}\end{aligned}$$

Note that while this was a really simple integral to evaluate you'll be seeing a fair amount of $\cos^2\theta$, $\sin^2\theta$ and $\sin\theta\cos\theta$ terms in polar integrals so make sure that you know how to integrate these terms!

---

3. Evaluate $\iint\limits_D 4xy-7\,dA$ where *D* is the portion of $x^2+y^2=2$ in the 1st quadrant.

Step 1
Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of *D* is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but one of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2
Okay, *D* is just a portion of a disk and so setting up the limits shouldn't be too difficult. Here they are,

$$\begin{gathered} 0 \le \theta \le \tfrac{1}{2}\pi \\ 0 \le r \le \sqrt{2} \end{gathered}$$

Step 3
The integral in terms of polar coordinates is then,

$$\begin{aligned} \iint\limits_D 4xy - 7\,dA &= \int_0^{\frac{1}{2}\pi} \int_0^{\sqrt{2}} \left(4\left(r\cos\theta\right)\left(r\sin\theta\right) - 7\right) r\,dr\,d\theta \\ &= \int_0^{\frac{1}{2}\pi} \int_0^{\sqrt{2}} 4r^3\cos\theta\sin\theta - 7r\,dr\,d\theta \end{aligned}$$

When converting the integral don't forget to convert the *x* and *y* into polar coordinates.
Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra *r* in the integrand. Forgetting the extra *r* is one of the most common mistakes with these kinds of problems.

Step 4

Here is the r integration.

$$\iint\limits_D 4xy - 7\,dA = \int_0^{\frac{1}{2}\pi}\int_0^{\sqrt{2}} 4r^3\cos\theta\sin\theta - 7r\,dr\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi}\left(r^4\cos\theta\sin\theta - \tfrac{7}{2}r^2\right)\Big|_0^{\sqrt{2}}\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 4\cos\theta\sin\theta - 7\,d\theta$$

Step 5

Finally, here is the θ integration.

$$\iint\limits_D 4xy - 7\,dA = \int_0^{\frac{1}{2}\pi} 4\cos\theta\sin\theta - 7\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 2\sin(2\theta) - 7\,d\theta$$
$$= \left(-\cos(2\theta) - 7\theta\right)\Big|_0^{\frac{1}{2}\pi} = \boxed{2 - \tfrac{7}{2}\pi}$$

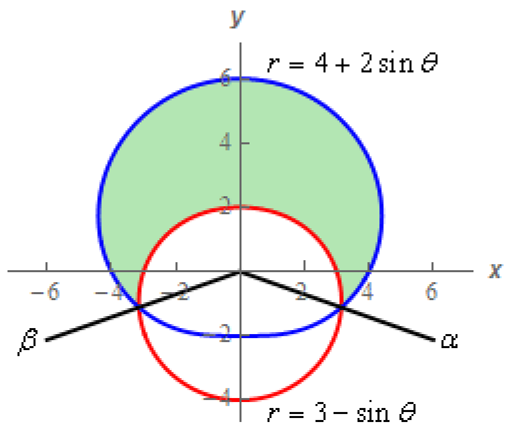
Note that while this was a really simple integral to evaluate you'll be seeing a fair amount of $\cos^2\theta$, $\sin^2\theta$ and $\sin\theta\cos\theta$ terms in polar integrals so make sure that you know how to integrate these terms! In this case we used the double angle formula for sine to quickly reduce the $\cos\theta\sin\theta$ into an easy to integrate term. Of course, we could also have just done a substitution to deal with it but this, in our opinion, is the easiest way to deal with the term.

---

4. Use a double integral to determine the area of the region that is inside $r = 4 + 2\sin\theta$ and outside $r = 3 - \sin\theta$.

Step 1

Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of *D* is often not as useful as for a general region.

In this case however, the sketch is probably more useful than for most of these problems. First, we can readily identify the correct order for the *r* limits (*i.e.* which is the "smaller" and which is the "larger" curve"). Secondly, it makes it clear that we are going to need to determine the limits of $\alpha \le \theta \le \beta$ that we'll need to do the integral.

Step 2
So, as noted above the limits of *r* are (hopefully) pretty clear. To find the limits for θ all we need to do is set the two equations equal. Doing that gives,

$$\begin{aligned} 4 + 2\sin\theta &= 3 - \sin\theta \\ 3\sin\theta &= -1 \\ \sin\theta &= -\tfrac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\tfrac{1}{3}\right) = -0.3398 \end{aligned}$$

This is the angle in the fourth quadrant, *i.e.* $\alpha = -0.3398$. To find the second angle we can note that the angles that the two black lines make with the *x* axis are the same (ignoring the sign of course). Therefore, the second angle will simply be $\beta = \pi + 0.3398 = 3.4814$.

The limits are then,

$$\begin{gathered} -0.3398 \le \theta \le 3.4814 \\ 3 - \sin\theta \le r \le 4 + 2\sin\theta \end{gathered}$$

Step 3
We know that the following double integral will give the area of any region *D*.

$$A = \iint\limits_D dA$$

For our problem the region *D* is the region given in the problem statement and so the integral in terms of polar coordinates is then,

$$A = \iint\limits_D dA = \int_{-0.3398}^{3.4814} \int_{3-\sin\theta}^{4+2\sin\theta} r\,dr\,d\theta$$

When converting the integral don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra *r* in the integrand. Forgetting the extra *r* is one of the most common mistakes with these kinds of problems and that seems to be even truer for area integrals such as this one.

Step 4

Here is the r integration.

$$\begin{aligned} A &= \int_{-0.3398}^{3.4814} \int_{3-\sin\theta}^{4+2\sin\theta} r\,dr\,d\theta \\ &= \int_{-0.3398}^{3.4814} \tfrac{1}{2} r^2 \Big|_{3-\sin\theta}^{4+2\sin\theta} d\theta \\ &= \int_{-0.3398}^{3.4814} \tfrac{1}{2}\left[\left(4+2\sin\theta\right)^2 - \left(3-\sin\theta\right)^2\right] d\theta \end{aligned}$$

Note that if you remember doing these kinds of area problems back in the Calculus II material that integral should look pretty familiar as it follows the formula we used back in that material.

Step 5

Finally, here is the θ integration.

$$\begin{aligned} A &= \int_{-0.3398}^{3.4814} \tfrac{1}{2}\left[\left(4+2\sin\theta\right)^2 - \left(3-\sin\theta\right)^2\right] d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 16 + 16\sin\theta + 4\sin^2\theta - \left(9 - 6\sin\theta + \sin^2\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 7 + 22\sin\theta + 3\sin^2\theta\,d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} 7 + 22\sin\theta + \tfrac{3}{2}\left(1-\cos\left(2\theta\right)\right) d\theta \\ &= \tfrac{1}{2}\int_{-0.3398}^{3.4814} \tfrac{17}{2} + 22\sin\theta - \tfrac{3}{2}\cos\left(2\theta\right) d\theta \\ &= \tfrac{1}{2}\left(\tfrac{17}{2}\theta - 22\cos\theta - \tfrac{3}{4}\sin\left(2\theta\right)\right)\Big|_{-0.3398}^{3.4814} = \boxed{36.5108} \end{aligned}$$

There was a fair amount of simplification that needed to be done for this problem. That will often be the case. None of it was particularly difficult just tedious and easy to make a mistake if you aren't paying attention. The "hardest" part of the simplification here was using a half angle formula to reduce the $\sin^2\theta$ into something we could integrate.

---

5. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\, dy\, dx$$

Step 1
The first thing we should do here is to get a feel for the region that we are integrating over. So, here are the *x* and *y* limits of this integral.

$$0 \le x \le 3$$
$$-\sqrt{9-x^2} \le y \le 0$$

Now, the lower *y* limit tells us that the "lower" curve of the region is given by,

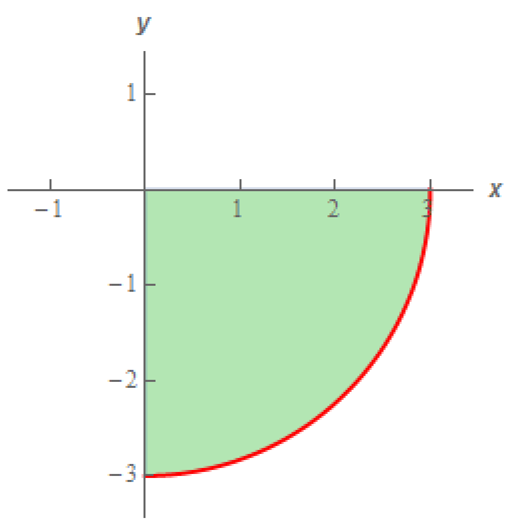
$$y = -\sqrt{9-x^2} \qquad \Rightarrow \qquad x^2 + y^2 = 9$$

As noted above, if we square both sides and do a little rewrite we can see that this is a portion of a circle of radius 3 centered at the origin. In fact, it is the lower portion of the circle (because of the "-" in front of the root).

The upper *y* limit tells us that the region won't go above the *x* axis and so the from the *y* limits we can see that the region is at most the lower half of the circle of radius 3 centered at the origin.

From the *x* limits we can see that *x* will never be negative and so we now know the region can't contain any portion of the left part of the circle. The upper *x* limit tells us that *x* can go all the way out to 3.

Therefore, from the *x* and *y* limits we can see that the region we are integrating over is in fact the portion of the circle of radius 3 centered at the origin that is in the 4th quadrant. Here is a quick sketch for the sake of completeness.



Step 2
Now, we will need the polar limits for our integral and they should be pretty easy to determine from the sketch above. Here they are.

$$\begin{gathered}\tfrac{3}{2}\pi \le \theta \le 2\pi \\ 0 \le r \le 3\end{gathered}$$

Step 3
Okay, we can now convert the limit to an integral involving polar coordinates. To do this just recall the various relationships between the Cartesian and polar coordinates and make use of them to convert any *x*'s and/or *y*'s into *r*'s and/or θ's.

Also, do not forget that the "*dy dx*" in the integral came from the *dA* that would have been in the original double integral (which isn't written down but we know that it could be if we wanted to). We also know how that $dA = r\,dr\,d\theta$ and so for this integral we will have $dy\,dx = r\,dr\,d\theta$.

The integral is then,

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx = \int_{\frac{3}{2}\pi}^{2\pi} \int_0^3 r\mathbf{e}^{r^2}\,dr\,d\theta$$

Note that the extra *r* we picked up from the *dA* is actually needed here to make this a doable integral! It is important to not forget to properly convert the *dA* when converting integrals from Cartesian to polar coordinates.

Step 4
Here is the *r* integration. It needs a simple substitution that we'll leave it to you to verify the results.

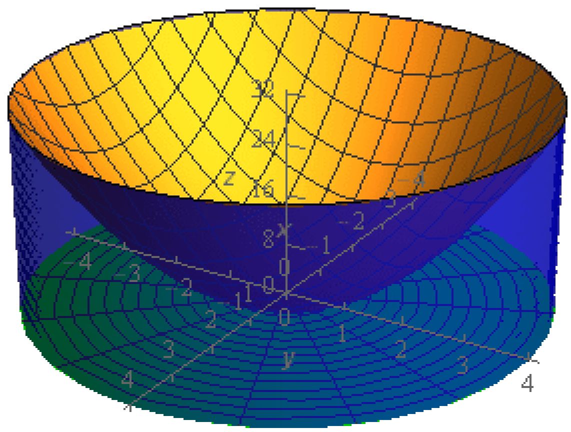
$$\begin{aligned}\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx &= \int_{\frac{3}{2}\pi}^{2\pi} \int_0^3 r\mathbf{e}^{r^2}\,dr\,d\theta \\ &= \int_{\frac{3}{2}\pi}^{2\pi} \left.\tfrac{1}{2}\mathbf{e}^{r^2}\right|_0^3 d\theta \\ &= \int_{\frac{3}{2}\pi}^{2\pi} \tfrac{1}{2}\left(\mathbf{e}^9 - 1\right)d\theta\end{aligned}$$

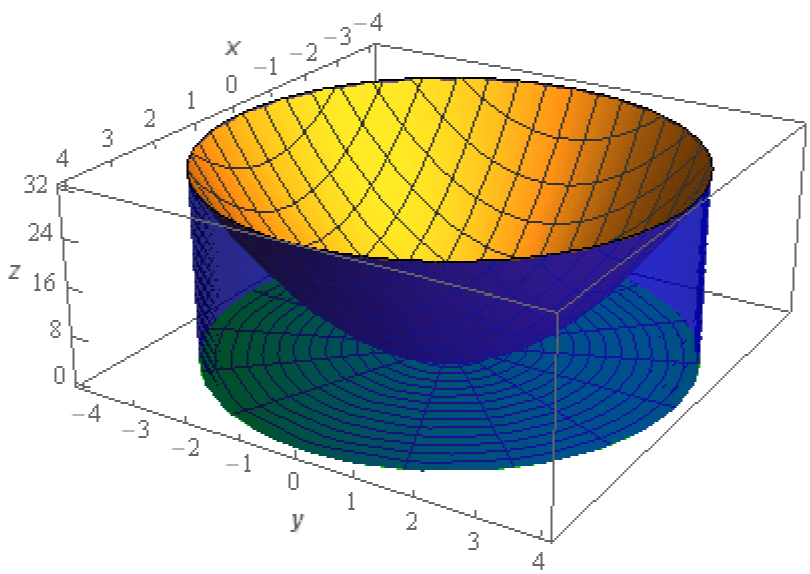
Step 5
Finally, here is the really simple θ integration.

$$\begin{aligned}\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx &= \int_{\frac{3}{2}\pi}^{2\pi} \tfrac{1}{2}\left(\mathbf{e}^9 - 1\right)d\theta \\ &= \left.\tfrac{1}{2}\left(\mathbf{e}^9 - 1\right)\theta\right|_{\frac{3}{2}\pi}^{2\pi} = \boxed{\tfrac{1}{4}\pi\left(\mathbf{e}^9 - 1\right) = 6363.3618}\end{aligned}$$

6. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$ and above the *xy*-plane.

Step 1
Let's start off this problem with a quick sketch of the solid we're looking at here in this problem.





The top of the solid is the paraboloid (the gold colored surface in the sketches). The walls of the solid are the cylinder $x^2 + y^2 = 16$ and is shown with the semi translucent surface in the sketch. The bottom of the solid is the *xy*-plane.

Step 2
So, the volume of the solid that is under the paraboloid and above the region *D* in the *xy*-plane is given by,

$$V = \iint_D 2x^2 + 2y^2 \, dA$$

Step 3
The region D is the region from the xy-plane that we use to sketch the surface we are finding the volume under (*i.e.* the paraboloid).

Determining D in this case is pretty simple if you think about it. The solid is defined by the portion of the paraboloid that is inside the cylinder $x^2 + y^2 = 16$. But this is exactly what defines the portion of the xy-plane that we use to graph the surface. We only use the points from the xy-plane that are inside the cylinder and so the region D is then just the disk defined by $x^2 + y^2 \le 16$.

Another way to think of determining D for this case is to look at the solid from directly above it. The 2D region that you see will be the region D. In this case the region we see is the inside of the cylinder or the disk $x^2 + y^2 \le 16$.

Now, the region D is a disk and so this strongly suggests that we use polar coordinates for this problem. The polar limits for this region D is,

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 4 \end{gathered}$$

Step 4
Okay, let's step up the integral in terms of polar coordinates.

$$V = \iint\limits_D 2x^2 + 2y^2 \, dA = \int_0^{2\pi} \int_0^4 \left(2r^2\right)\left(r\right) dr \, d\theta = \int_0^{2\pi} \int_0^4 2r^3 \, dr \, d\theta$$

Don't forget to convert all the x's and y's into r's and θ's and make sure that you simplify the integrand as much as possible. Also, don't forget to add in the r we get from the dA.

Step 5
Here is the simple r integration.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^4 2r^3 \, dr \, d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2} r^4 \Big|_0^4 \, d\theta = \int_0^{2\pi} 128 \, d\theta \end{aligned}$$

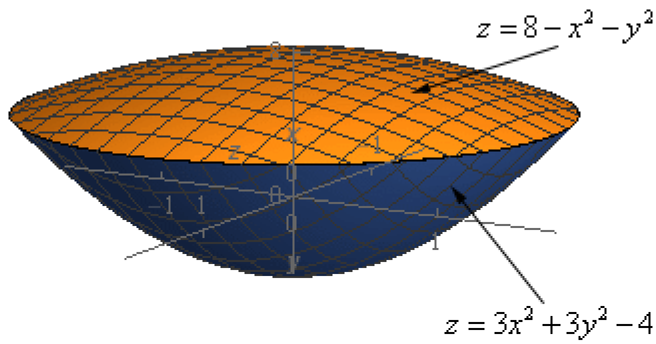
Step 6
Finally, here is the really simple θ integration.

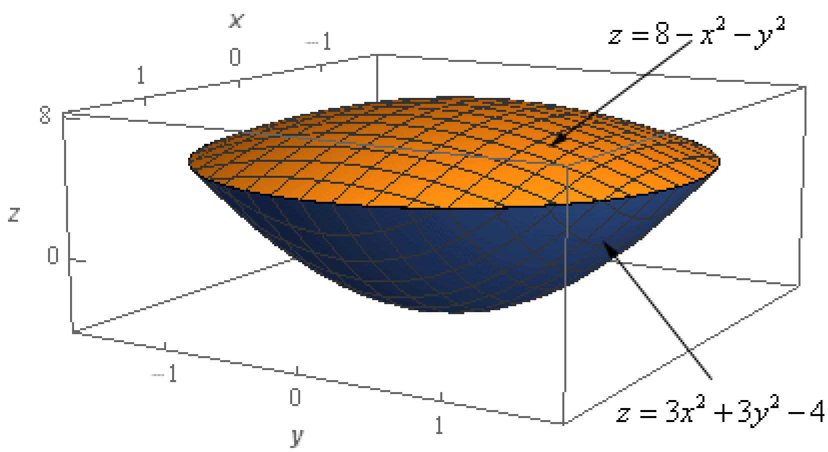
$$V = \int_0^{2\pi} 128 \, d\theta = \boxed{256\pi}$$

Note that this was a very simple integration and so we didn't actually do any of the work and left it to you to verify the details.

7. Use a double integral to determine the volume of the solid that is bounded by $z = 8 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 4$.

Step 1
Let's start off this problem with a quick sketch of the solid we're looking at here in this problem.





The top of the solid is the paraboloid given by $z = 8 - x^2 - y^2$ (the gold colored surface in the sketches) and the bottom of the solid is the paraboloid given by $z = 3x^3 + 3y^3 - 4$ (the blue colored surface in the sketches).

Step 2
To get the volume of this solid we're going to need to know that value of *z* where these two surfaces intersect. To let's solve the equation of the upper paraboloid as follow,

$$z = 8 - x^2 - y^2 \qquad \Rightarrow \qquad x^2 + y^2 = 8 - z$$

Now plug this into the equation of the second paraboloid to get,

$$z = 3\left(x^2 + y^2\right) - 4 = 3\left(8 - z\right) - 4 = 20 - 3z \quad \to \quad 4z = 20 \quad \to \quad z = 5$$

So, we know they will intersect at $z = 5$. Plugging this into the either paraboloid equation and doing a little simplification will give the following equation.

$$x^2 + y^2 = 3$$

Now, what does this tell us? This is the circle where the two paraboloids intersect at $z = 5$. This is also tells us that the region *D* in the *xy*-plane that we are going to use in this problem is the disk defined by $x^2 + y^2 \le 3$. This makes sense if you think about it. It is the region in the *xy*-plane that we'd use to graph each of the paraboloids in the sketches.

Step 3
We don't have a formula to find the volume of this solid at this point so let's see if we can figure out what it is.

Let's start with the volume of just the lower portion of the solid. In other words, what is the volume of the portion of the solid that is below the plane $z = 5$ and above the paraboloid given by $z = 3x^2 + 3y^2 - 4$.

We looked at a solid like this in the notes for this section. Following the same logic in that problem the volume of the lower portion of the solid is given by,

$$V_{\text{lower}} = \iint\limits_D 5 - \left(3x^2 + 3y^2 - 4\right) dA = \iint\limits_D 9 - 3x^2 - 3y^2 \, dA$$

where *D* is the disk $x^2 + y^2 \le 3$ as we discussed above.

Note that even though this paraboloid does slip under the *xy*-plane the formula is still valid.

Step 4
The volume of the upper portion of the solid, *i.e*. the portion under $z = 8 - x^2 - y^2$ and above the plane $z = 5$ can be found with a similar argument to the one we used for the lower region. The volume of the upper region is then,

$$V_{\text{upper}} = \iint\limits_D 8 - x^2 - y^2 - \left(5\right) dA = \iint\limits_D 3 - x^2 - y^2 \, dA$$

where *D* is the disk $x^2 + y^2 \le 3$ as we discussed above.

The volume of the whole solid is then,

$$\begin{aligned} V &= V_{\text{lower}} + V_{\text{upper}} \\ &= \iint\limits_D 9 - 3x^2 - 3y^2 \, dA + \iint\limits_D 3 - x^2 - y^2 \, dA \\ &= \iint\limits_D 9 - 3x^2 - 3y^2 + \left(3 - x^2 - y^2\right) dA \\ &= \iint\limits_D 12 - 4x^2 - 4y^2 \, dA \end{aligned}$$

Note that we could combine the two integrals because they were both over the same region *D*.

Step 5

Okay, as we've already determined *D* is the disk given by $x^2 + y^2 \le 3$ and because this is a disk it makes sense to do this integral in polar coordinates. Here are the polar limits for this integral/region.

$$0 \le \theta \le 2\pi$$
$$0 \le r \le \sqrt{3}$$

Step 6

The volume integral (in terms of polar coordinates) is then,

$$\begin{aligned} V &= \iint\limits_D 12 - 4x^2 - 4y^2 \, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(12 - 4r^2\right)\left(r\right) dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 \, dr \, d\theta \end{aligned}$$

Don't forget to convert all the *x*'s and *y*'s into *r*'s and θ's and make sure that you simplify the integrand as much as possible. Also, don't forget to add in the *r* we get from the *dA*.

Step 7

Here is the r integration.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 \, dr \, d\theta \\ &= \int_0^{2\pi} \left(6r^2 - r^4\right)\Big|_0^{\sqrt{3}} \, d\theta = \int_0^{2\pi} 9 \, d\theta \end{aligned}$$

Step 8

Finally, here is the really simple θ integration.

$$V = \int_0^{2\pi} 9 \, d\theta = \boxed{18\pi}$$

Note that this was a very simple integration and so we didn't actually do any of the work and left it to you to verify the details.

Section 4-5 : Triple Integrals

1. Evaluate $\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2 y - z^3 \, dz\, dy\, dx$

Step 1

There really isn't all that much to this problem. All we need to do is integrate following the given order and recall that just like with double integrals we start with the "inside" integral and work our way out.

So, here is the *z* integration.

$$\begin{aligned}\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2 y - z^3 \, dz\, dy\, dx &= \int_{2}^{3}\int_{-1}^{4}\left(4x^2 yz - \tfrac{1}{4}z^4\right)\Big|_{1}^{0} dy\, dx\\ &= \int_{2}^{3}\int_{-1}^{4} \tfrac{1}{4} - 4x^2 y\, dy\, dx\end{aligned}$$

Remember that triple integration is just like double integration and all the variables other than the one we are integrating with respect to are considered to be constants. So, for the *z* integration the *x*'s and *y*'s are all considered to be constants.

Step 2

Next, we'll do the *y* integration.

$$\begin{aligned}\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2 y - z^3 \, dz\, dy\, dx &= \int_{2}^{3}\left(\tfrac{1}{4}y - 2x^2 y^2\right)\Big|_{-1}^{4} dx\\ &= \int_{2}^{3} \tfrac{5}{4} - 30x^2\, dx\end{aligned}$$

Step 3

Finally, we'll do the *x* integration.

$$\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2 y - z^3 \, dz\, dy\, dx = \left(\tfrac{5}{4}x - 10x^3\right)\Big|_{2}^{3} = \boxed{-\tfrac{755}{4}}$$

So, not too much to do with this problem since the limits were already set up for us.

---

2. Evaluate $\int_{0}^{1}\int_{0}^{z^2}\int_{0}^{3} y\cos\left(z^5\right) dx\, dy\, dz$

Step 1

There really isn't all that much to this problem. All we need to do is integrate following the given order and recall that just like with double integrals we start with the "inside" integral and work our way out.

Also note that the fact that one of the limits is not a constant is not a problem. There is nothing that says that triple integrals set up as this is must only have constants as limits!

So, here is the *x* integration.

$$\begin{aligned}\int_0^1\int_0^{z^2}\int_0^3 y\cos\left(z^5\right)dx\,dy\,dz &= \int_0^1\int_0^{z^2}\left(y\cos\left(z^5\right)x\right)\Big|_0^3\,dy\,dz\\ &= \int_0^1\int_0^{z^2} 3y\cos\left(z^5\right)\,dy\,dz\end{aligned}$$

Remember that triple integration is just like double integration and all the variables other than the one we are integrating with respect to are considered to be constants. So, for the *x* integration the *y*'s and *z*'s are all considered to be constants.

Step 2
Next, we'll do the *y* integration.

$$\begin{aligned}\int_0^1\int_0^{z^2}\int_0^3 y\cos\left(z^5\right)dx\,dy\,dz &= \int_0^1\left(\tfrac{3}{2}y^2\cos\left(z^5\right)\right)\Big|_0^{z^2}\,dz\\ &= \int_0^1 \tfrac{3}{2}z^4\cos\left(z^5\right)dz\end{aligned}$$

Step 3
Finally, we'll do the *z* integration and note that the only way we are able to do this integration is because of the z^4 that is now in the integrand. Without that present we would not be able to do this integral.

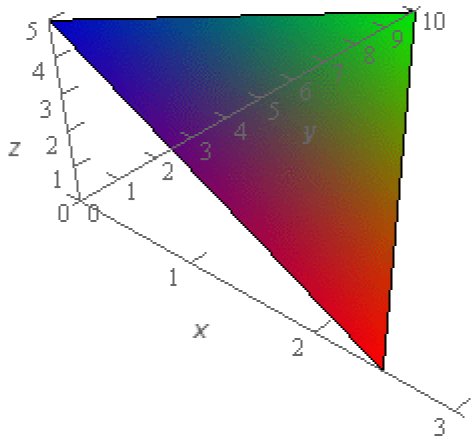
$$\int_0^1\int_0^{z^2}\int_0^3 y\cos\left(z^5\right)dx\,dy\,dz = \left(\tfrac{3}{10}\sin\left(z^5\right)\right)\Big|_0^1 = \boxed{\tfrac{3}{10}\sin\left(1\right) = 0.2524}$$

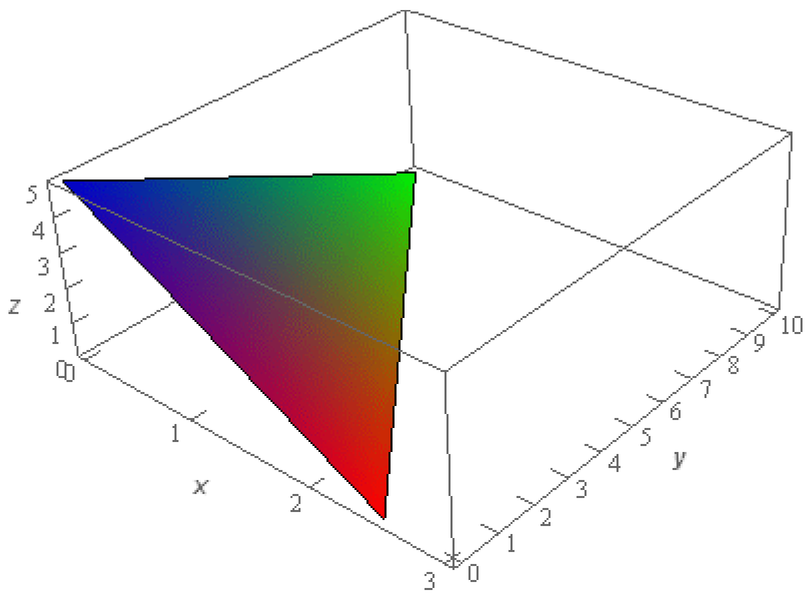
So, not too much to do with this problem since the limits were already set up for us.

---

3. Evaluate $\iiint\limits_E 6z^2\,dV$ where *E* is the region below $4x + y + 2z = 10$ in the first octant.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

For this problem the region *E* is just the region that is under the plane shown above and in the first octant. In other words, the sketch of the plane above is exactly the top of the region *E*. The bottom of the region is the *xy*-plane while the sides are simply the *yz* and *xz*-planes.

Step 2
So, from the sketch above we know that we'll have the following limits for *z*.

$$0 \le z \le 5 - 2x - \tfrac{1}{2}y$$

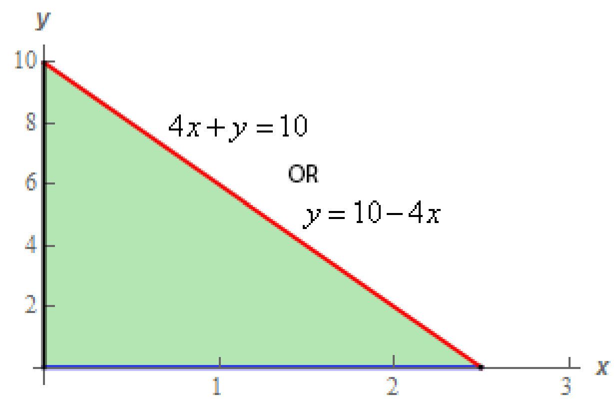
where we got the upper *z* limit simply by solving the equation of the plane for *z*.

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint_E 6z^2\,dV = \iint_D \left[\int_0^{5-2x-\frac{1}{2}y} 6z^2\,dz\right]dA$$

Step 3
Next, we'll need limits for *D* so we can finish setting up the integral. For this problem *D* is simply the region in the *xy*-plane (since we are integrating with respect to *z* first) that we used to graph the plane in Step 1. That also, in this case, makes *D* the bottom of the region.

Here is a sketch of *D*.



The hypotenuse of *D* is simply the intersection of the plane from Step 1 and the *xy*-plane and so we can quickly get its equation by plugging $z = 0$ into the equation of the plane.

Given the nature of this region as well as the function we'll be integrating it looks like we can use either order of integration for *D*. So, to keep the limits at least a little nicer we'll integrate *y*'s and then *x*'s.

Here are the limits for the double integral over *D*.

$$0 \le x \le \tfrac{5}{2}$$
$$0 \le y \le 10 - 4x$$

The upper *x* limit was found simply by plugging $y = 0$ into the equation of the hypotenuse and solving for *x* to determine where the hypotenuse intersected the *x*-axis.

With these limits plugged into the integral we now have,

$$\iiint_E 6z^2\,dV = \int_0^{\frac{5}{2}}\int_0^{10-4x}\int_0^{5-2x-\frac{1}{2}y} 6z^2\,dz\,dy\,dx$$

Step 4
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\iiint\limits_E 6z^2\,dV = \int_0^{\frac{5}{2}}\int_0^{10-4x} \left(2z^3\right)\Big|_0^{5-2x-\frac{1}{2}y}\,dy\,dx$$
$$= \int_0^{\frac{5}{2}}\int_0^{10-4x} 2\left(5-2x-\tfrac{1}{2}y\right)^3\,dy\,dx$$

Do not multiply out the integrand of this integral.

Step 5
As noted in the last step we do not want to multiply out the integrand of this integral. One of the bigger mistakes students make with multiple integrals is to just launch into a simplification mode after the integral and multiply everything out.

Sometimes of course that must be done but, in this case, note that we can easily do the *y* integration with a simple Calculus I substitution. Here is that work.

$$\iiint\limits_E 6z^2\,dV = \int_0^{\frac{5}{2}} -\left(5-2x-\tfrac{1}{2}y\right)^4\Big|_0^{10-4x}\,dx \qquad\qquad u = 5-2x-\tfrac{1}{2}y$$
$$= \int_0^{\frac{5}{2}} \left(5-2x\right)^4\,dx$$

We gave the substitution used in this step but are leaving it to you to verify the details of the substitution.

Step 6
Again, notice that we can either do some "simplification" or we can just do another substitution to finish this integral out. Here is the final integration step for this problem. We'll leave it to you to verify the substitution details.

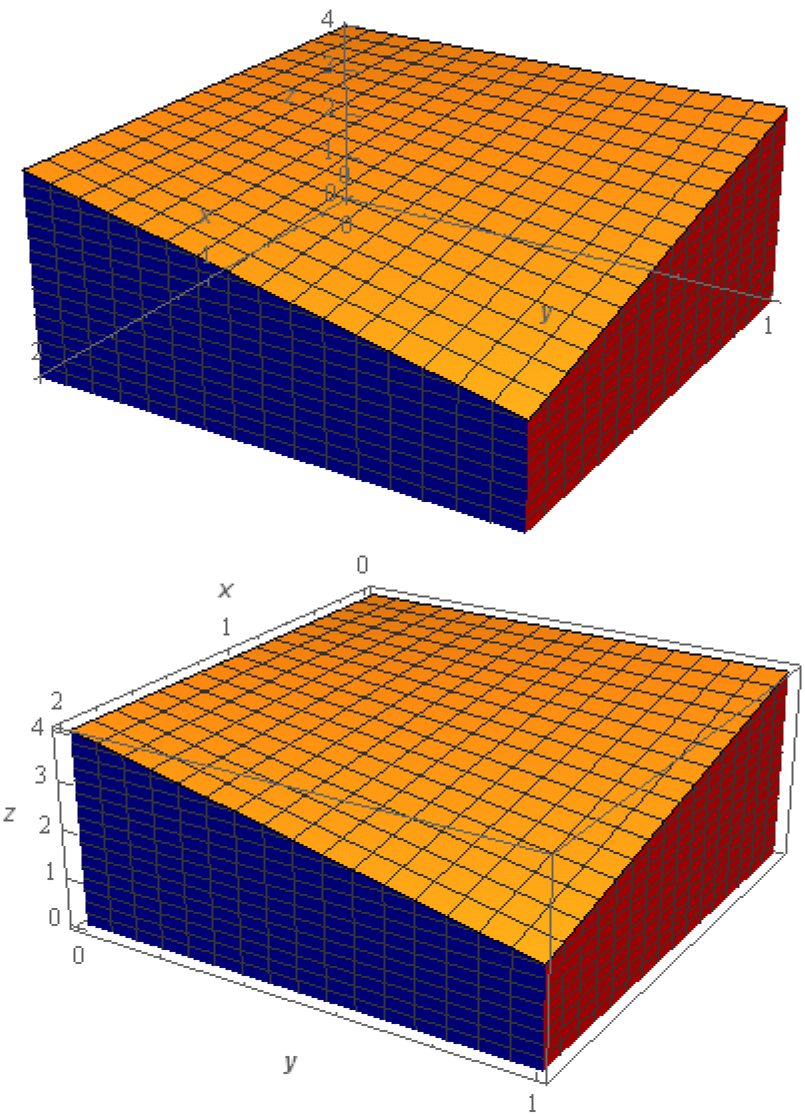
$$\iiint\limits_E 6z^2\,dV = \int_0^{\frac{5}{2}} \left(5-2x\right)^4\,dx \qquad\qquad u = 5-2x$$
$$= -\tfrac{1}{10}\left(5-2x\right)^5\Big|_0^{\frac{5}{2}}$$
$$= \boxed{\tfrac{625}{2}}$$

So, once we got the limits all set up, the integration for this problem wasn't too bad provided we took advantage of the substitutions of course. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

Also, as noted above do not forget about your basic Calculus I substitutions. Using them will often allow us to avoid some messy algebra that will be easy to make a mistake with.

4. Evaluate $\iiint\limits_E 3 - 4x\,dV$ where *E* is the region below $z = 4 - xy$ and above the region in the *xy*-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the orange colored surface) is the graph of $z = 4 - xy$. The two sides shown (the blue and red surfaces) show the two sides of the region that we can see given the orientation of the region. The bottom of the region is the *xy*-plane.

Step 2
So, from the sketch above we know that we'll have the following limits for *z*.

$$0 \le z \le 4 - xy$$

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 3 - 4x\,dV = \iint\limits_D \left[\int_0^{4-xy} 3 - 4x\,dz\right] dA$$

Step 3
Next, we'll need limits for *D* so we can finish setting up the integral. In this case that is really simple as we can see from the problem statement that *D* is just a rectangle in the *xy*-plane and in fact the limits are given in the problem statement as,

$$0 \le x \le 2$$
$$0 \le y \le 1$$

There really isn't any advantage to doing one order vs. the other so, in this case, we'll integrate *y* and then *x*.

With these limits plugged into the integral we now have,

$$\iiint\limits_E 3 - 4x\,dV = \int_0^2 \int_0^1 \int_0^{4-xy} 3 - 4x\,dz\,dy\,dx$$

Step 4
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\begin{aligned}\iiint\limits_E 3 - 4x\,dV &= \int_0^2 \int_0^1 \left(3 - 4x\right) z\Big|_0^{4-xy}\,dy\,dx\\ &= \int_0^2 \int_0^1 \left(3 - 4x\right)\left(4 - xy\right) dy\,dx\\ &= \int_0^2 \int_0^1 4x^2 y - 3xy - 16x + 12\,dy\,dx\end{aligned}$$

Note that because the integrand had no *z*'s in it we treated the whole integrand as a constant and just added a single *z* as shown above. We could just as easily integrated each term individually but it seemed easier to just deal with the integrand as a single term.

Step 5
Now let's do the *y* integration.

$$\iiint\limits_E 3-4x\,dV = \int_0^2 \left(2x^2y^2 - \tfrac{3}{2}xy^2 - 16xy + 12y\right)\Big|_0^1\,dx$$
$$= \int_0^2 12 - \tfrac{35}{2}x + 2x^2\,dx$$

Step 6
Finally, let's do the *x* integration.

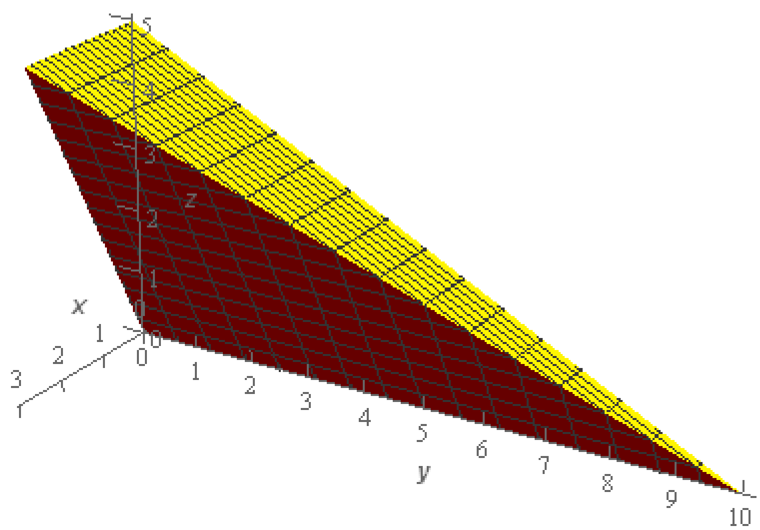
$$\iiint\limits_E 3-4x\,dV = \left(12x - \tfrac{35}{4}x^2 + \tfrac{2}{3}x^3\right)\Big|_0^2 = \boxed{-\tfrac{17}{3}}$$

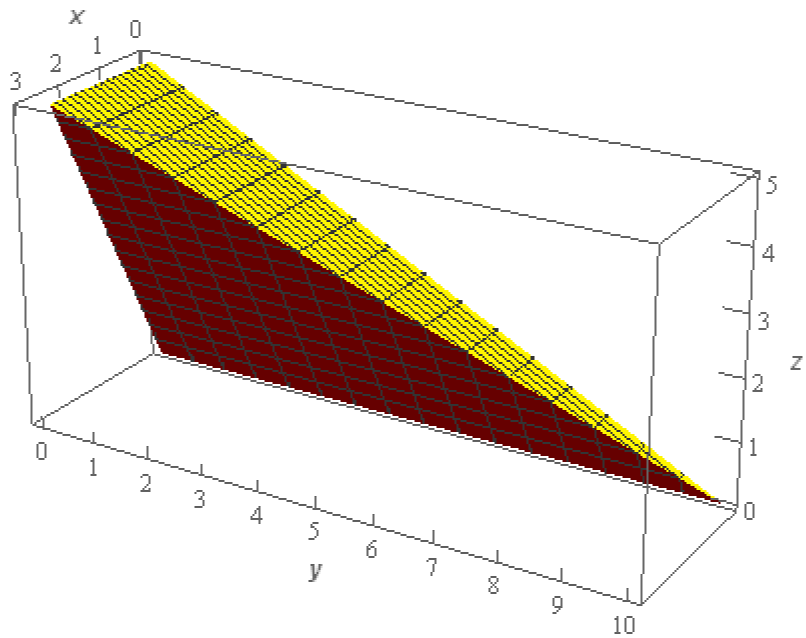
So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

---

5. Evaluate $\iiint\limits_E 12y - 8x\,dV$ where *E* is the region behind $y = 10 - 2z$ and in front of the region in the *xz*-plane bounded by $z = 2x$, $z = 5$ and $x = 0$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the yellow colored surface) is the graph of the portion $y = 10 - 2z$ that lies in front of the region *xz*-plane given in the problem statement.

The red surface is the plane defined by $z = 2x$. This is the plane that intersects the *xz*-plane at the line $z = 2x$ and is bounded above by the surface $y = 10 - 2z$.

Step 2

The region in the *xz*-plane bounded by $z = 2x$, $z = 5$ and $x = 0$ that is referenced in the problem statement is the region *D* and so we know that, for this problem, we'll need to be integrating *y* first (since *D* is in the *xz*-plane).

Therefore, we have the following limits for *y*.

$$0 \le y \le 10 - 2z$$

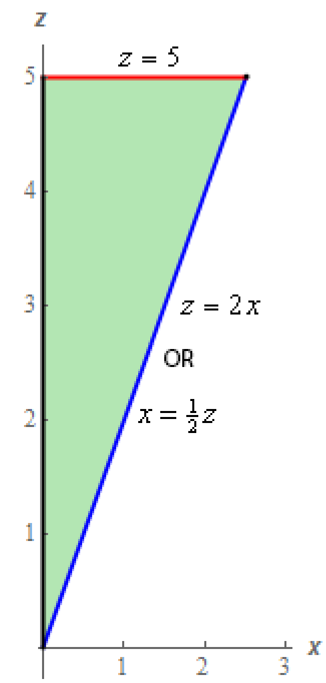
With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 12y - 8x\,dV = \iint\limits_D \left[\int_0^{10-2z} 12y - 8x\,dy\right] dA$$

Step 3

Next, we'll need limits for *D* so we can finish setting up the integral. For this problem *D* is the region in the *xz*-plane from the problem statement as we noted above.

Here is a quick sketch of *D*.



Another way to think this region is that it is the “back” of the solid sketched in Step 1 and the solid we sketched in Step 1 is all behind this region. In other words the positive y-axis is goes directly back into the page while the negative *y*-axis comes directly out of the page.

As noted in the sketch of *D* we can easily defined by either of the following sets of limits.

$$\begin{array}{c} 0 \le x \le \frac{5}{2} \\ 2x \le z \le 5 \end{array} \qquad \text{OR} \qquad \begin{array}{c} 0 \le z \le 5 \\ 0 \le x \le \frac{1}{2}z \end{array}$$

The integrand doesn't really suggest one of these would be easier than the other so we'll use the 2nd set of limits for no other reason than the lower limits for both are zero which might make things a little nicer in the integration process.

With these limits plugged into the integral we now have,

$$\iiint\limits_E 12y - 8x\,dV = \int_0^5 \int_0^{\frac{1}{2}z} \int_0^{10-2z} 12y - 8x\,dy\,dx\,dz$$

Step 4
Okay, now all we need to do is evaluate the integral. Here is the *y* integration.

$$\begin{aligned} \iiint\limits_E 12y - 8x\,dV &= \int_0^5 \int_0^{\frac{1}{2}z} \left(6y^2 - 8xy\right)\Big|_0^{10-2z}\,dx\,dz \\ &= \int_0^5 \int_0^{\frac{1}{2}z} 6\left(10-2z\right)^2 - 8x\left(10-2z\right)dx\,dz \end{aligned}$$

Note that we did no simplification here because it is not yet clear that we need to do any simplification. The next integration is with respect to *x* and the first term is a constant as far as that integration is concerned and the second term is multiplied by an *x* and so it will might well be easier to leave it in that form for the *x* integration.

Step 5
Now let's do the *x* integration.

$$\begin{aligned}\iiint\limits_E 12y - 8x\,dV &= \int_0^5 \left[6\left(10-2z\right)^2 x - 4x^2\left(10-2z\right)\right]\Big|_0^{\frac{1}{2}z}\,dz\\ &= \int_0^5 3z\left(10-2z\right)^2 - z^2\left(10-2z\right)dz\\ &= \int_0^5 14z^3 - 130z^2 + 300z\,dz\end{aligned}$$

Step 6
Finally, let's do the *z* integration.

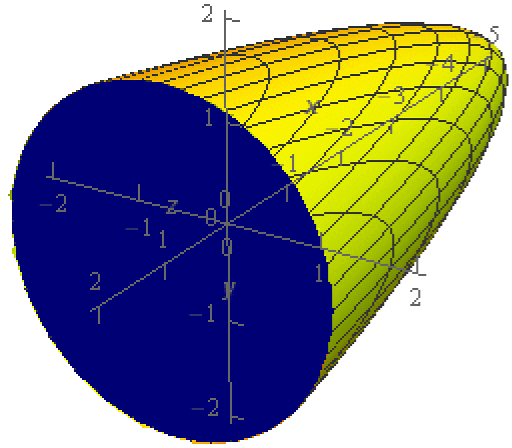
$$\iiint\limits_E 12y - 8x\,dV = \left(\tfrac{7}{2}z^4 - \tfrac{130}{3}z^3 + 150z^2\right)\Big|_0^5 = \boxed{\tfrac{3125}{6}}$$

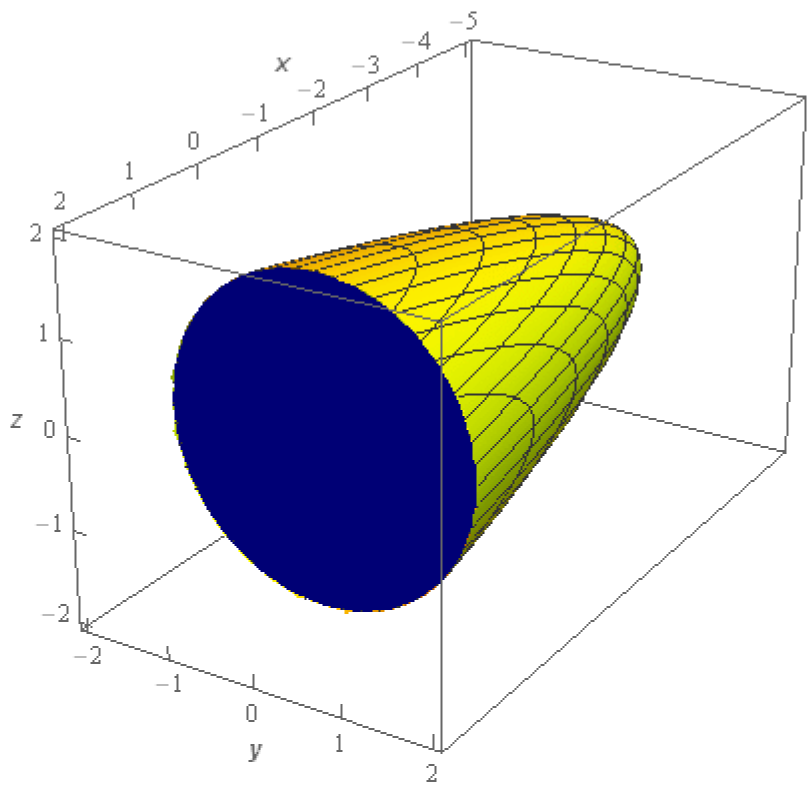
So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

---

6. Evaluate $\iiint\limits_E yz\,dV$ where *E* is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

We know that $x = 2y^2 + 2z^2 - 5$ is an elliptic paraboloid that is centered on the *x*-axis and opens in the positive *x* direction as shown in the sketch above. The blue "cap" on the surface is the portion of the plane $x = 1$ that fits just inside the paraboloid.

Step 2
Okay, for this problem it is hopefully clear that we'll need to integrate with respect to *x* first and that the region *D* will be in the y*z*-plane.

Therefore, we have the following limits for *x*.

$$2y^2 + 2z^2 - 5 \le x \le 1$$

Remember that when setting these kinds of limits up we go from back to front and so the paraboloid will be the lower limit since it is the surface that is in the "back" of the solid while $x = 1$ is the "front" of the solid and so is the upper limit.

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E yz\,dV = \iint\limits_D \left[\int_{2y^2+2z^2-5}^{1} yz\,dx\right] dA$$

Step 3
Next, we'll need limits for *D* so we can finish setting up the integral. To determine *D* we'll need the intersection of the two surfaces. The intersection is,

$$2y^2 + 2z^2 - 5 = 1 \qquad \to \qquad 2y^2 + 2z^2 = 6 \qquad \to \qquad y^2 + z^2 = 3$$

Now, if we looked at the solid from the "front", *i.e.* from along the positive *x*-axis we'd see the disk $y^2 + z^2 \le 3$ and so this is the region *D*.

The region *D* is a disk which clearly suggests polar coordinates, however, it won't be the "standard" *xy* polar coordinates. Since *D* is in the *yz*-plane let's use the following "modified" polar coordinates.

$$y = r\sin\theta \qquad z = r\cos\theta \qquad\qquad y^2 + z^2 = r^2$$

This "definition" of polar coordinates for our problem isn't needed quite yet but will be eventually.

At this point all we need are the polar limits for this circle and those should fairly clearly be given by,

$$\begin{aligned} &0 \le \theta \le 2\pi \\ &0 \le r \le \sqrt{3} \end{aligned}$$

With all the previous problems we'd write the integral down with these limits at this point as well. However, since we are going to have to convert to polar coordinates we'll hold off writing down the integral in polar coordinates until we do the first integration.

Step 4
So, here is the *x* integration.

$$\begin{aligned}\iiint\limits_E yz\,dV &= \iint\limits_D \left(xyz\right)\Big|_{2y^2+2z^2-5}^{1}\,dA\\ &= \iint\limits_D \left[1-\left(2y^2+2z^2-5\right)\right]yz\,dA\\ &= \iint\limits_D \left[6-2\left(y^2+z^2\right)\right]yz\,dA\end{aligned}$$

We did a small amount of simplification here in preparation for the next step.

Step 5
Now we need to convert the integral over to polar coordinates using the modified version we defined in Step 3.

Here is the integral when converted to polar coordinates.

$$\begin{aligned}\iiint\limits_E yz\,dV &= \int_0^{2\pi}\int_0^{\sqrt{3}}\left[6-2r^2\right]\left(r\sin\theta\right)\left(r\cos\theta\right)r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}}\left[6r^3-2r^5\right]\sin\theta\cos\theta\,dr\,d\theta\end{aligned}$$

Step 6
Now let's do the *r* integration.

$$\begin{aligned}\iiint\limits_E yz\,dV &= \int_0^{2\pi}\left[\tfrac{3}{2}r^4-\tfrac{1}{3}r^6\right]\sin\theta\cos\theta\Big|_0^{\sqrt{3}}\,d\theta\\ &= \int_0^{2\pi}\tfrac{9}{2}\sin\theta\cos\theta\,d\theta\end{aligned}$$

Step 7
Finally, let's do the θ integration and notice that we're going to use the double angle formula for sine to "simplify" the integral slightly prior to the integration.

$$\iiint\limits_E yz\,dV = \int_0^{2\pi}\tfrac{9}{4}\sin\left(2\theta\right)d\theta = \left(-\tfrac{9}{8}\cos\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{0}$$

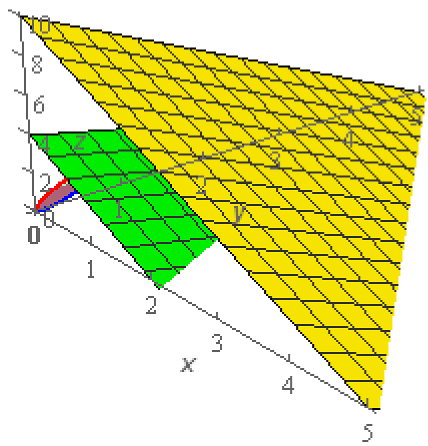
So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

Also, as we saw in this example it is not unusual for polar coordinates to show in the "outer" double integral and there is no reason to expect they will always be the "standard" *xy* definition of polar coordinates and so you will need to be ready to use them in any of the three orientations (*xy*, *xz* or *yz*) in which they may show up.

---

7. Evaluate $\iiint\limits_E 15z\,dV$ where *E* is the region between $2x + y + z = 4$ and $4x + 4y + 2z = 20$ that is in front of the region in the *yz*-plane bounded by $z = 2y^2$ and $z = \sqrt{4y}$.

Step 1
This region always seems pretty difficult to visualize. So, let's start off with the following sketch.

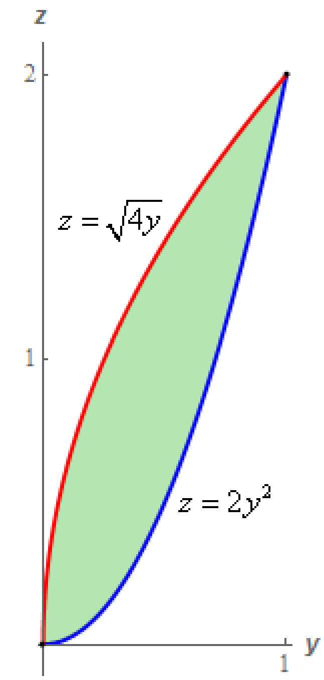


Note that the "orientation" of the *x* and *y* axes are different in this sketch form all the other 3D sketches we've done to this point. This is done to help with the visualization. Without the reorientation it would be very difficult to visualize the bounded region.

Do not always expect that the orientation of the axes to always remain fixed and never changing. Sometimes the orientation will need to change so we can visualize a particular surface or region.

Okay, onto the trying to visualize the region we are working with here.

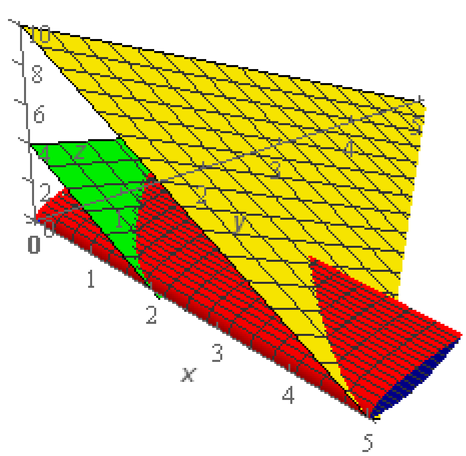
The yellow plane is the graph of $4x + 4y + 2z = 20$ in the 1st octant (the only portion we'll need here) and the green plane is the graph of $2x + y + z = 4$. At the origin (kind of hard to see) is the sketch of the bounded region in the *yz*-plane referenced in the problem statement. Because this bounded region is hard to see on the graph above here is separate graph of just the bounded region.



Now, to try and visualize the region *E*, imagine that there is a lump of cookie dough between the two planes in the first sketch above and the bounded region in the *yz*-plane is a cookie cutter. We move the bounded region away from the *yz*-plane making sure to always keep it parallel to the *yz*-plane.

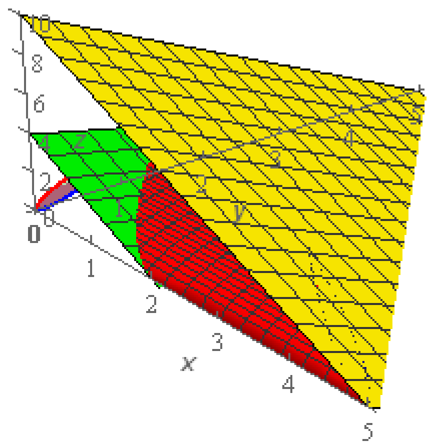
As the bounded region moves out the top of it will first cut into the "cookie dough" when it hits the green plane and will be fully in the cookie dough when the bottom of the bounded region hits the green plane. The bounded region will continue to cut through the cookie dough until it reaches the yellow plane with the top of the bounded region exiting the cookie dough first when it hits the yellow plane. The bounded region will be fully out of the cookie dough when the bottom of the bounded region hits the yellow plane.

Here is a sketch of the path that the bounded region takes as it moves away from the *yz*-plane.

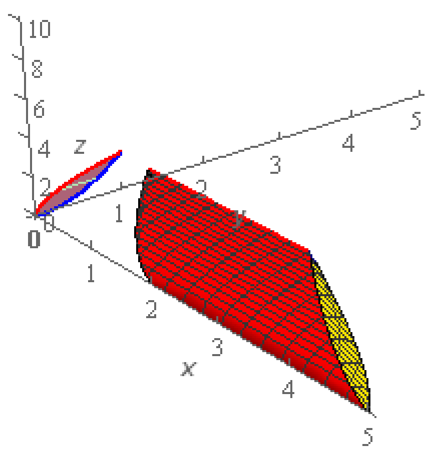


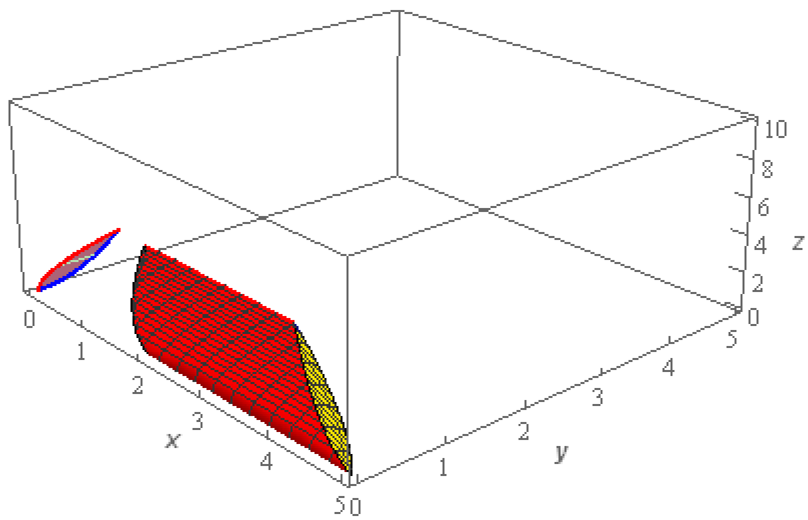
Note that this sketch shows the full path of the bounded region regardless of whether or not it is between the two planes. Or using the analogy from above regardless of whether or not it is cutting out a region from the cookie dough between the two planes. The region *E* that we are after is just the portion of this path that lies between the two planes.

Here is a sketch of the path of the bounded region that is only between the two planes.



The top of the region *E* is the red portion shown above. It is kind of hard however to visualize the full region with the two planes still in the sketch. So, here is another sketch with two planes removed from the sketch.





The region E is then the solid shown above. The "front" of E is sloped and follows the yellow plane (and hence is also shaded yellow to make this clearer) and the "rear" of E is sloped and follows the green plane.

With this final sketch of the region E we included both the "traditional" axis system as well as the "boxed" axes system to help visualize the object. We also kept the same x, y, z scale as the previous images to helpful help with the visualization even though there is a lot of "wasted" space in the right side of the axes system.

Step 2

Okay, for this problem the problem statement tells us that the bounded region, *i.e.* D, is in the yz-plane and we we'll need to integrate with respect to x first.

This means we'll need to solve each of the equations of the planes for x and we'll integrate from the back (or green) plane up to the front (or yellow) plane.

Therefore, we have the following limits for x.

$$2-\tfrac{1}{2}y-\tfrac{1}{2}z \le x \le 5-y-\tfrac{1}{2}z$$

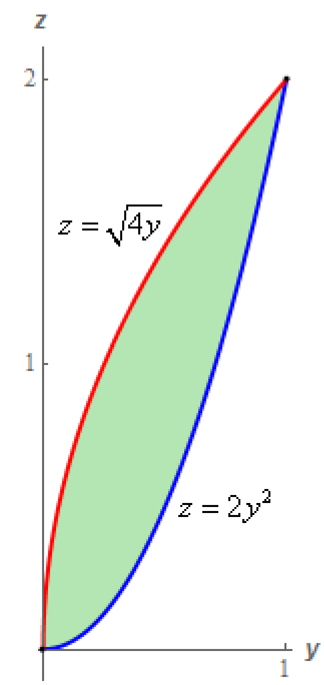
Remember that when setting these kinds of limits up we go from back to front and so the paraboloid will be the lower limit since it is the surface that is in the "back" of the solid while $x=1$ is the "front" of the solid and so is the upper limit.

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 15z\,dV = \iint\limits_D \left[\int_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} 15z\,dx\right]dA$$

Step 3

Next, we'll need limits for *D* so we can finish setting up the integral. As noted above the region *D* is just the bounded region given in the problem statement. For reference purposes here is a copy of the sketch of *D* we gave in the Step 1.



As we can see this region could be integrated in either order but regardless of the order one of the limits will be a quadratic term and one will be a square root. Therefore, we might as well take use the limits as they were given in the problem statement.

So, here are the limits for *D*.

$$0 \le y \le 1$$
$$2y^2 \le z \le \sqrt{4y}$$

With these limits plugged into the integral we now have,

$$\iiint\limits_E 15z\,dV = \int_0^1 \int_{2y^2}^{\sqrt{4y}} \int_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} 15z\,dx\,dz\,dy$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the *x* integration.

$$\begin{aligned}\iiint\limits_E 15z\,dV &= \int_0^1 \int_{2y^2}^{\sqrt{4y}} \left(15zx\right)\Big|_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} dz\,dy \\ &= \int_0^1 \int_{2y^2}^{\sqrt{4y}} 15z\left(3-\tfrac{1}{2}y\right) dz\,dy\end{aligned}$$

Step 5

Now let's do the *z* integration.

$$\iiint\limits_E 15z\,dV = \int_0^1 \tfrac{15}{2}z^2\left(3-\tfrac{1}{2}y\right)\Big|_{2y^2}^{\sqrt{4y}}\,dy$$
$$= \int_0^1 15y^5 - 90y^4 - 15y^2 + 90y\,dy$$

Step 6
Finally, let's do the *y* integration.

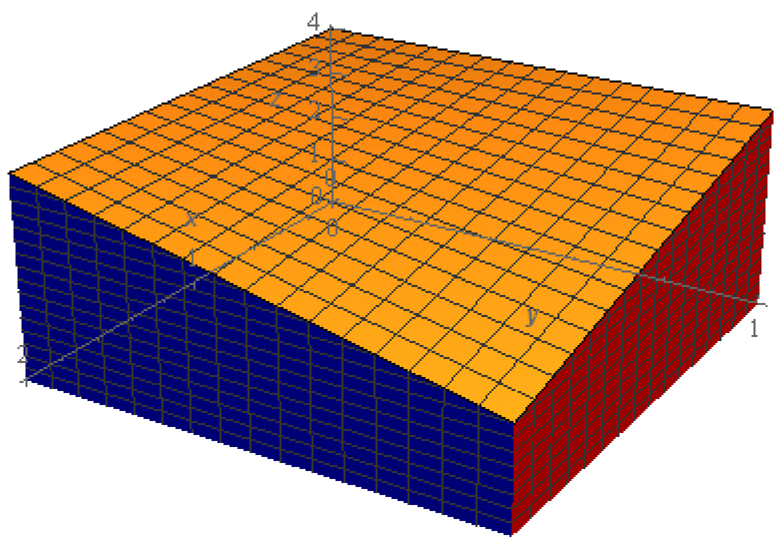
$$\iiint\limits_E 15z\,dV = \left(\tfrac{5}{2}y^6 - 18y^5 - 5y^3 + 45y^2\right)\Big|_0^1 = \boxed{\tfrac{49}{2}}$$

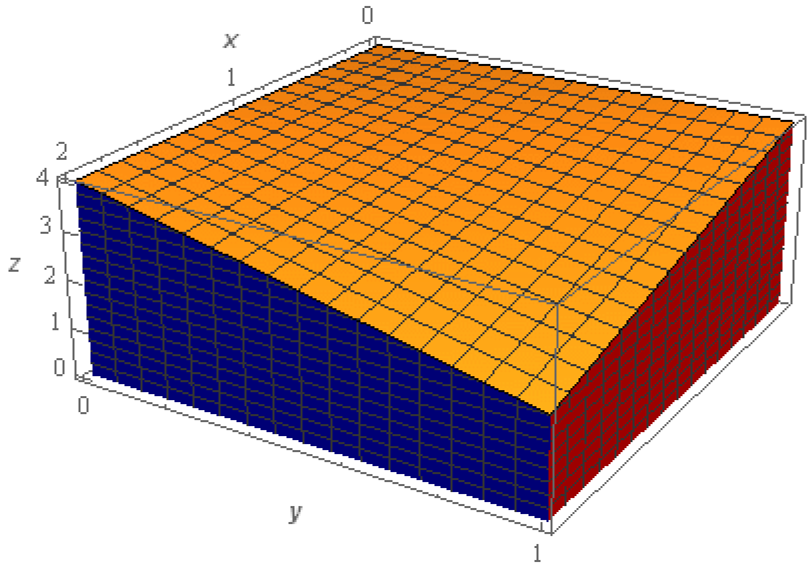
So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once the limits do get set up the integration is often pretty simple and there is no doubt that visualizing the region and getting the limits set up for this problem was probably more difficult that with many of the other problems.

---

8. Use a triple integral to determine the volume of the region below $z = 4 - xy$ and above the region in the *xy*-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

Step 1
Okay, let's start off with a quick sketch of the region we want the volume of so we can get a feel for what we're dealing with. We'll call this region *E*.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the orange colored surface) is the graph of $z = 4 - xy$. The two sides shown (the blue and red surfaces) show the two sides of the region that we can see given the orientation of the region. The bottom of the region is the *xy*-plane.

Step 2
The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

Step 3
So, we now need to get the limits set up for the integral. From the sketch above we know that we'll have the following limits for *z*.

$$0 \le z \le 4 - xy$$

We'll also need limits for *D*. In this case that is really simple as we can see from the problem statement that *D* is just a rectangle in the *xy*-plane and in fact the limits are given in the problem statement as,

$$\begin{gathered} 0 \le x \le 2 \\ 0 \le y \le 1 \end{gathered}$$

There really isn't any advantage to doing one order vs. the other so, in this case, we'll integrate *y* and then *x*.

Now, plugging all these limits into the integral the volume is,

$$V = \iiint\limits_E dV = \int_0^2 \int_0^1 \int_0^{4-xy} dz\,dy\,dx$$

Step 4
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\begin{aligned} V &= \int_0^2 \int_0^1 z\Big|_0^{4-xy} dy\,dx \\ &= \int_0^2 \int_0^1 4 - xy\,dy\,dx \end{aligned}$$

Step 5
Now let's do the *y* integration.

$$\begin{aligned} V &= \int_0^2 \left(4y - \tfrac{1}{2}xy^2\right)\Big|_0^1 dx \\ &= \int_0^2 4 - \tfrac{1}{2}x\,dx \end{aligned}$$

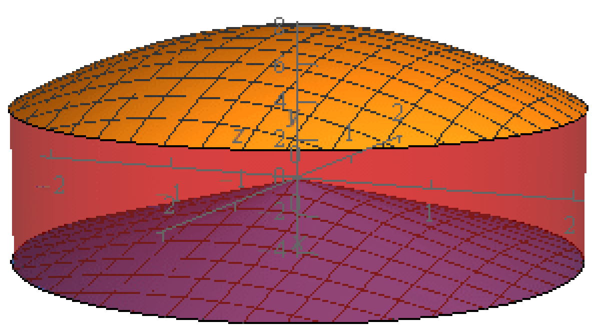
Step 6
Finally, let's do the *x* integration to get the volume of the region.

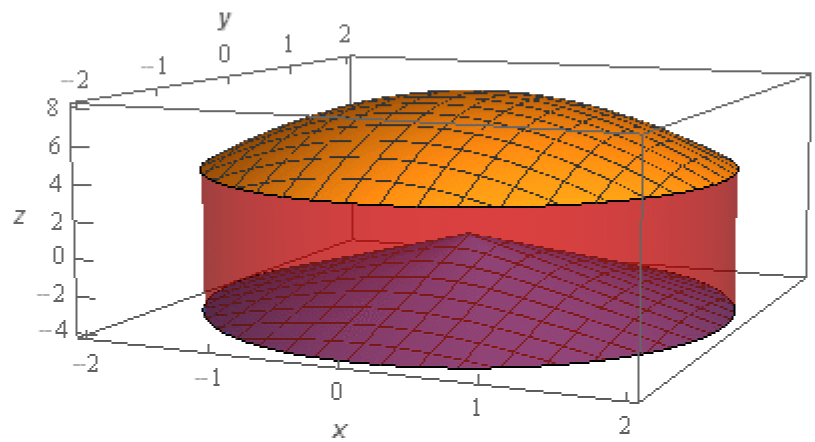
$$V = \iiint\limits_E dV = \left(4x - \tfrac{1}{4}x^2\right)\Big|_0^2 = \boxed{7}$$

---

9. Use a triple integral to determine the volume of the region that is below $z = 8 - x^2 - y^2$ above $z = -\sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$.

Step 1
Okay, let's start off with a quick sketch of the region we want the volume of so we can get a feel for what we're dealing with. We'll call this region *E*.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top of the region (the orange colored surface) is the portion of the graph of the elliptic paraboloid $z = 8 - x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 4$. The bottom of the region is the portion of the graph of the cone $z = -\sqrt{4x^2 + 4y^2}$ that is inside the cylinder $x^2 + y^2 = 4$. The walls of the region (which are translucent to show the bottom portion) is the cylinder $x^2 + y^2 = 4$.

Step 2
The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

Step 3
So, we now need to get the limits set up for the integral. From the sketch above we can see that we'll need to integrate with respect to z first so here are those limits.

$$-\sqrt{4x^2 + 4y^2} \le z \le 8 - x^2 - y^2$$

We'll also need limits for *D*. In this case *D* is just the disk given by $x^2 + y^2 \le 4$ (*i.e.* the portion of the *xy*-plane that is inside the cylinder. This is the region in the *xy*-plane that we need to graph the paraboloid and cone and so is *D*.

Because *D* is a disk it makes sense to use polar coordinates for integrating over *D*. Here are the limits for *D*.

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

Now, let's set up the volume integral as follows.

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} dz \right] dA$$

Because we know that we'll need to do the outer double integral in polar coordinates we'll hold off putting those limits in until we have the *z* integration done.

Step 4
Okay, let's do the *z* integration.

$$\begin{aligned} V &= \iint\limits_D z\Big|_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} dA \\ &= \iint\limits_D 8 - x^2 - y^2 + \sqrt{4x^2+4y^2}\, dA \end{aligned}$$

Step 5
Now let's convert the integral over to polar coordinates. Don't forget that $x^2 + y^2 = r^2$ and that $dA = r\,dr\,d\theta$.

The volume integral in terms of polar coordinates is then,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 \left[8 - r^2 + 2r \right] r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^2 8r - r^3 + 2r^2\,dr\,d\theta \end{aligned}$$

Step 6
The *r* integration is then,

$$V = \int_0^{2\pi} \left(4r^2 - \tfrac{1}{4}r^4 + \tfrac{2}{3}r^3 \right)\Big|_0^2 d\theta = \int_0^{2\pi} \tfrac{52}{3}\,d\theta$$

Step 7
Finally, we can compute the very simple θ integral to get the volume of the region.

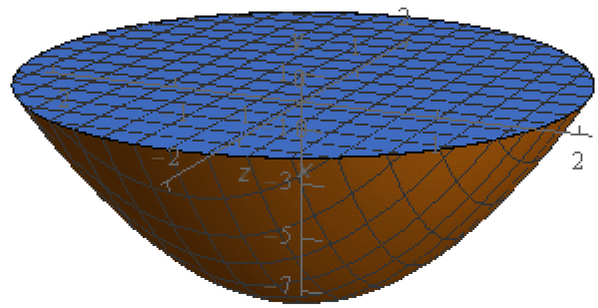
$$V = \int_0^{2\pi} \tfrac{52}{3}\,d\theta = \boxed{\tfrac{104}{3}\pi}$$

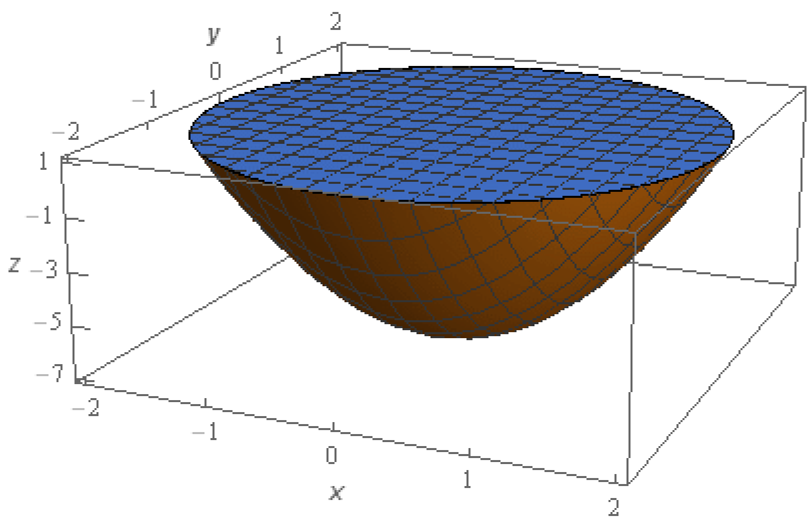
Section 4-6 : Triple Integrals in Cylindrical Coordinates

1. Evaluate $\iiint\limits_E 4xy\,dV$ where *E* is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$.

Step 1

Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

Step 2

So, from the sketch above it should be pretty clear that we'll need to integrate *z* first and so we'll have the following limits for *z*.

$$2x^2 + 2y^2 - 7 \le z \le 1$$

Step 3

For this problem *D* is the disk that "caps" the region sketched in Step 1. We can determine the equation of the disk by setting the two equations from the problem statement equal and doing a little rewriting.

$$2x^2 + 2y^2 - 7 = 1 \qquad \to \qquad 2x^2 + 2y^2 = 8 \qquad \to \qquad x^2 + y^2 = 4$$

So, *D* is the disk $x^2 + y^2 \le 4$ and it should be pretty clear that we'll need to use cylindrical coordinates for this integral.

Here are the cylindrical coordinates for this problem.

$$\begin{aligned} 0 \le \theta &\le 2\pi \\ 0 \le r &\le 2 \\ 2r^2 - 7 \le z &\le 1 \end{aligned}$$

Don't forget to convert the *z* limits from Step 2 into cylindrical coordinates as well.

Step 4
Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\begin{aligned} \iiint\limits_E 4xy\,dV &= \int_0^{2\pi} \int_0^2 \int_{2r^2-7}^1 4\left(r\cos\theta\right)\left(r\sin\theta\right) r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^2 \int_{2r^2-7}^1 4r^3 \cos\theta \sin\theta\,dz\,dr\,d\theta \end{aligned}$$

Don't forget to convert the *x* and *y*'s into cylindrical coordinates and also don't forget that $dV = r\,dz\,dr\,d\theta$ and so we pick up another *r* when converting the *dV* to cylindrical coordinates.

Step 5
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\begin{aligned} \iiint\limits_E 4xy\,dV &= \int_0^{2\pi} \int_0^2 \left(4r^3 \cos\theta \sin\theta z\right)\Big|_{2r^2-7}^1 dr\,d\theta \\ &= \int_0^{2\pi} \int_0^2 4r^3\left(8 - 2r^2\right)\cos\theta \sin\theta\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^2 \left(32r^3 - 8r^5\right)\cos\theta \sin\theta\,dr\,d\theta \end{aligned}$$

Step 6
Next let's do the *r* integration.

$$\iiint\limits_E 4xy\,dV = \int_0^{2\pi} \left(8r^4 - \tfrac{4}{3}r^6\right)\cos\theta\sin\theta\Big|_0^2\,d\theta$$
$$= \int_0^{2\pi} \tfrac{128}{3}\cos\theta\sin\theta\,d\theta$$

Step 7
Finally, we'll do the θ integration.

$$\iiint\limits_E 4xy\,dV = \int_0^{2\pi} \tfrac{64}{3}\sin\left(2\theta\right)d\theta = -\tfrac{32}{3}\cos\left(2\theta\right)\Big|_0^{2\pi} = \boxed{0}$$

Note that we used the double angle formula for sine to simplify the integrand a little prior to the integration. We could also have done one of two substitutions for this step if we'd wanted to (and we'd get the same answer of course).

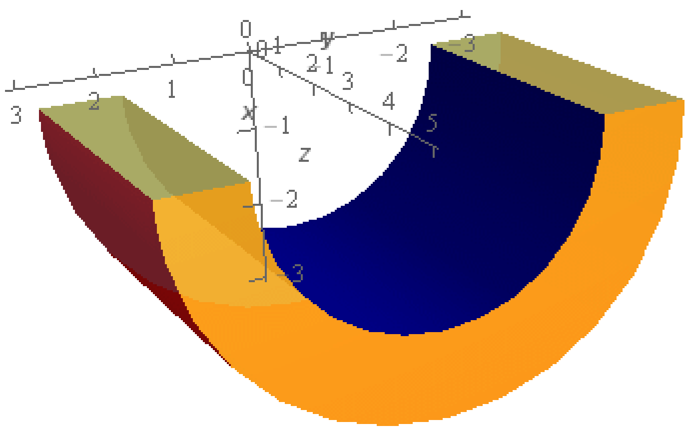
---

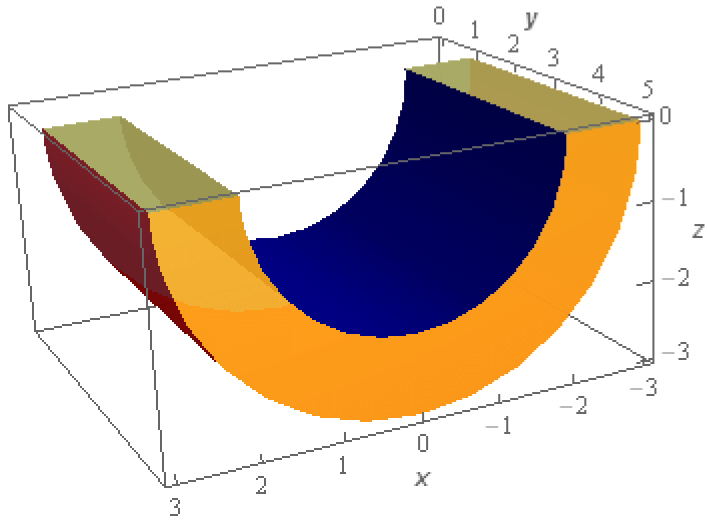
2. Evaluate $\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV$ where *E* is the region between the two cylinders $x^2+z^2=4$ and $x^2+z^2=9$ with $1 \le y \le 5$ and $z \le 0$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.

We know that $x^2+z^2=4$ and $x^2+z^2=9$ are cylinders of radius 2 and 3 respectively that are centered on the *y*-axis. The range $1 \le y \le 5$ tells us that we will only have the cylinders in this range of *y*'s. Finally, the $z \le 0$ tells us that we will only have the lower half of each of the cylinders.

Here then is the sketch of *E*.





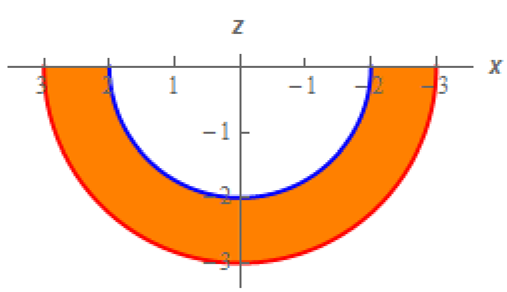
We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The "front" of *E* is just the portion of the plane $y = 5$ that "caps" the front and is the orange ring in the sketch. The "back" of *E* is the portion of the plane $y = 1$ that caps the back of the region and is not shown in the sketch due to the orientation of axis system.

Step 2
So, from the sketch above it looks like the region *D* will be back in the *xz*-plane and so we'll need to integrate with respect to *y* first. In this case this is even easier because both the front and back portions of the surfaces are just the planes $y = 5$ and $y = 1$ respectively. That means that the *y* limits are,

$$1 \le y \le 5$$

Step 3
Now, let's think about the *D* for this problem. If we look at the object from along the *y* axis we see the lower half of the ring with radii 2 and 3 as shown below.



Note that the *x*-axis orientation is switched from the standard orientation to more accurately match what you'd see if you did look at *E* from along the *y*-axis. In other words, the positive *x* values are on the left side and the negative *x* values are on the right side. The orientation of the *x*-axis doesn't change that we still see a portion of a ring that lies in the *xz*-plane and this is in fact the *D* for this problem.

Because *D* is in the *xz*-plane and it is a portion of a ring that means that we'll need to use the following "modified" version of cylindrical coordinates.

$$\begin{aligned} x &= r\cos\theta \\ y &= y \\ z &= r\sin\theta \end{aligned}$$

This also matches up with the fact that we need to integrate *y* first (as we determined in Step 2) and the first variable of integration with cylindrical coordinates is always the "free" variable (*i.e.* not the one involving the trig functions).

So, we can easily describe the ring in terms of *r* and θ so here are the cylindrical coordinates for this problem.

$$\begin{aligned} &\pi \le \theta \le 2\pi \\ &2 \le r \le 3 \\ &1 \le y \le 5 \end{aligned}$$

Step 4
Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = \int_\pi^{2\pi}\int_2^3\int_1^5 r\,\mathbf{e}^{-r^2}\,dy\,dr\,d\theta$$

Don't forget $x^2+z^2=r^2$ under our modified cylindrical coordinates and also don't forget that $dV = r\,dy\,dr\,d\theta$ and so we pick up another *r* when converting the *dV* to cylindrical coordinates (that will be very helpful with the *r* integration).

Step 5
Okay, now all we need to do is evaluate the integral. Here is the *y* integration.

$$\begin{aligned} \iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV &= \int_\pi^{2\pi}\int_2^3 r\,\mathbf{e}^{-r^2}\,y\Big|_1^5\,dr\,d\theta \\ &= \int_\pi^{2\pi}\int_2^3 4r\,\mathbf{e}^{-r^2}\,dr\,d\theta \end{aligned}$$

Step 6
Next let's do the *r* integration.

$$\begin{aligned} \iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV &= \int_\pi^{2\pi}\left(-2\,\mathbf{e}^{-r^2}\right)\Big|_2^3\,d\theta \\ &= \int_\pi^{2\pi} 2\left(\mathbf{e}^{-4}-\mathbf{e}^{-9}\right)d\theta \end{aligned}$$

Step 7
Finally, we'll do the θ integration.

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = 2\left(\mathbf{e}^{-4}-\mathbf{e}^{-9}\right)\Big|_{\pi}^{2\pi} = \boxed{2\pi\left(\mathbf{e}^{-4}-\mathbf{e}^{-9}\right) = 0.1143}$$

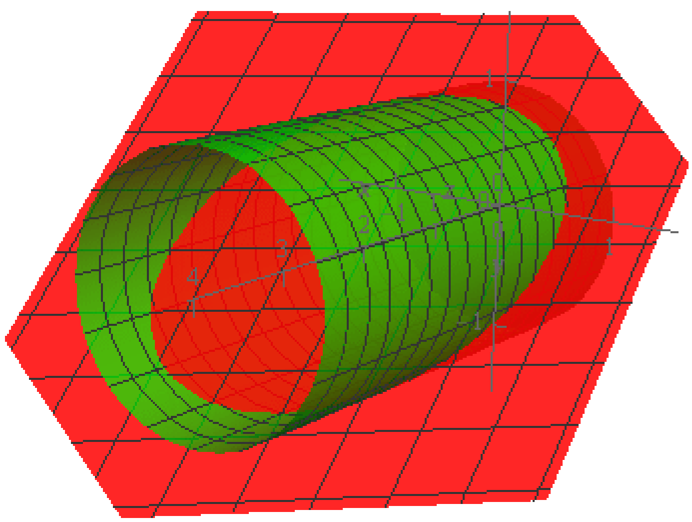
The trickiest part of this one was probably the sketch of *E*. Once you see that and how to get the *D* for the integral the rest of the problem was pretty simple for the most part.

---

3. Evaluate $\iiint\limits_E z\,dV$ where *E* is the region between the two planes $x+y+z=2$ and $x=0$ and inside the cylinder $y^2+z^2=1$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.

Let's start off with a quick sketch of the cylinder and the plane $x+y+z=2$.

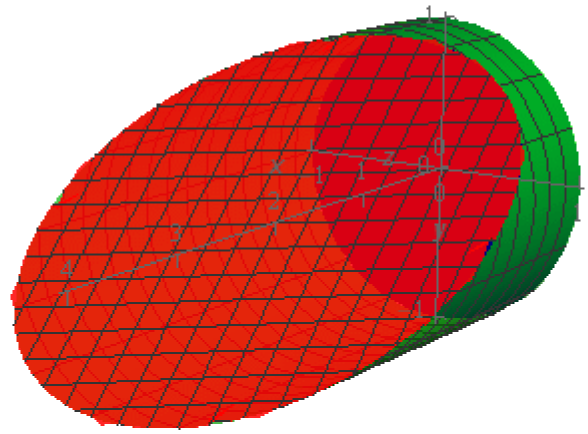


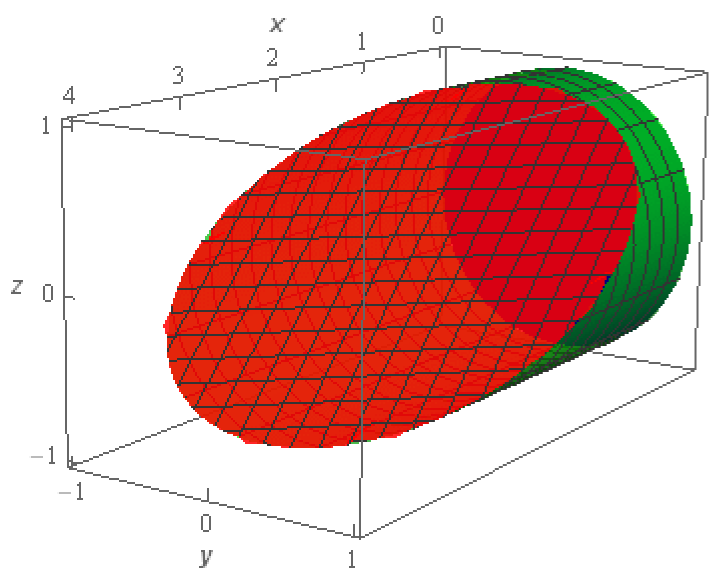
The region *E* we're working with here is inside the cylinder and between the two planes given in the problem statement. What this basically means is that the two planes will cap the cylinder.

In the graph above we put in more of the cylinder than needed and more of the plane $x+y+z=2$ than needed just to help illustrate the relation between the two surfaces. The plane $x=0$ is just the *yz*-plane and "caps" the back of the cylinder and so isn't included in the sketch.

Now, let's get rid of the portion of the cylinder that is in front of $x+y+z=2$ since it's not part of the region and let's get rid of the portion of $x+y+z=2$ that is not inside the cylinder. The resulting sketch is the region *E*.

Here then is the sketch of *E*.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

Step 2
So, from the sketch it is hopefully clear that the region *D* will be in the *yz*-plane and so we'll need to integrate with respect to *x* first. That means that the *x* limits are,

$$0 \le x \le 2 - y - z$$

For the upper *x* limit all we need to do is solve the equation of the plane for *x*.

Step 3

For this problem *D* is simply the disk $y^2 + z^2 \le 1$. Because *D* is in the *yz*-plane and is a disk we'll need to use the following "modified" version of cylindrical coordinates.

$$x = x$$
$$y = r\sin\theta$$
$$z = r\cos\theta$$

This also matches up with the fact that we need to integrate *x* first (as we determined in Step 2) and the first variable of integration with cylindrical coordinates is always the "free" variable (*i.e.* not the one involving the trig functions).

So, we can easily describe the disk in terms of *r* and θ so here are the cylindrical coordinates for this problem.

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$
$$0 \le x \le 2 - r\sin\theta - r\cos\theta$$

Don't forget to convert the *y* and *z* in the *x* upper limit into cylindrical coordinate as well.

Step 4

Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\begin{aligned}\iiint\limits_E z\,dV &= \int_0^{2\pi}\int_0^1\int_0^{2-r\sin\theta-r\cos\theta}\left(r\cos\theta\right)r\,dx\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1\int_0^{2-r\sin\theta-r\cos\theta} r^2\cos\theta\,dx\,dr\,d\theta\end{aligned}$$

Don't forget to convert the integrand to our modified cylindrical coordinates and also don't forget that $dV = r\,dx\,dr\,d\theta$ and so we pick up another *r* when converting the *dV* to cylindrical coordinates.

Step 5

Okay, now all we need to do is evaluate the integral. Here is the *x* integration.

$$\begin{aligned}\iiint\limits_E z\,dV &= \int_0^{2\pi}\int_0^1\left.\left(r^2\cos\theta\,x\right)\right|_0^{2-r\sin\theta-r\cos\theta}dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 r^2\cos\theta\left(2-r\sin\theta-r\cos\theta\right)dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 2r^2\cos\theta - r^3\cos\theta\sin\theta - r^3\cos^2\theta\,dr\,d\theta\end{aligned}$$

Step 6
Next let's do the *r* integration.

$$\iiint\limits_E z\,dV = \int_0^{2\pi} \left(\tfrac{2}{3} r^3 \cos\theta - \tfrac{1}{4} r^4 \cos\theta \sin\theta - \tfrac{1}{4} r^4 \cos^2\theta \right) \Big|_0^1 \, d\theta$$
$$= \int_0^{2\pi} \tfrac{2}{3}\cos\theta - \tfrac{1}{4}\cos\theta\sin\theta - \tfrac{1}{4}\cos^2\theta \, d\theta$$

Step 7
Finally, we'll do the θ integration.

$$\iiint\limits_E z\,dV = \int_0^{2\pi} \tfrac{2}{3}\cos\theta - \tfrac{1}{8}\sin\left(2\theta\right) - \tfrac{1}{8}\left(1 + \cos\left(2\theta\right)\right) d\theta$$
$$= \left(\tfrac{2}{3}\sin\theta + \tfrac{1}{16}\cos\left(2\theta\right) - \tfrac{1}{8}\left(\theta + \tfrac{1}{2}\sin\left(2\theta\right)\right)\right)\Big|_0^{2\pi} = \boxed{-\tfrac{\pi}{4}}$$

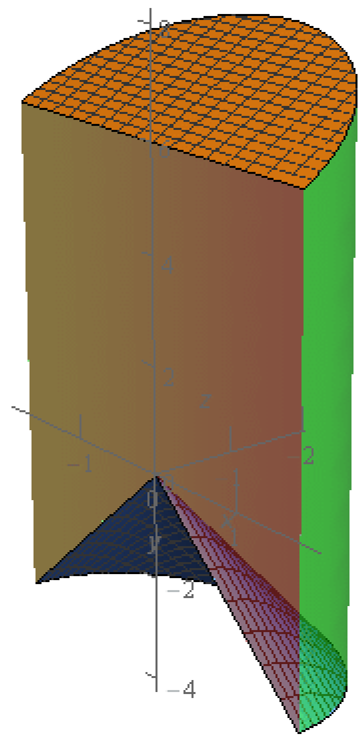
Don't forget to simplify the integrand before doing the final integration. In this case we used the sine double angle on the second term and the cosine half angle formula on the third term to simplify the integrand to allow us to quickly do this integration.

---

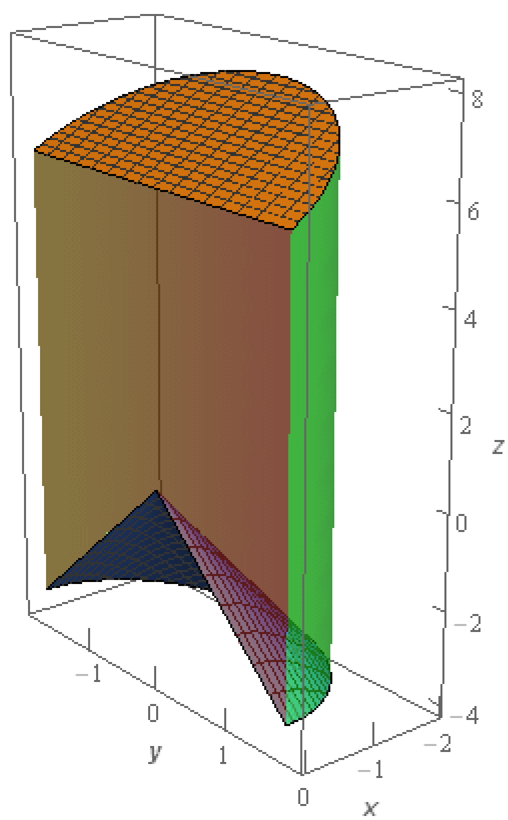
4. Use a triple integral to determine the volume of the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.

Here then is the sketch of *E*.



-1
2
4
2
-2
-4
0
y
x
z



8
6
4
2
z
0
-2
-4
-1
0
y
1
0
-1
-2
x

We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The plane $z = 6 - x$ is the top "cap" on the cylinder and the cone $z = -\sqrt{4x^2 + 4y^2}$ is the bottom "cap" on the cylinder. We only have half of the cylinder because of the $x \le 0$ portion of the problem statement.

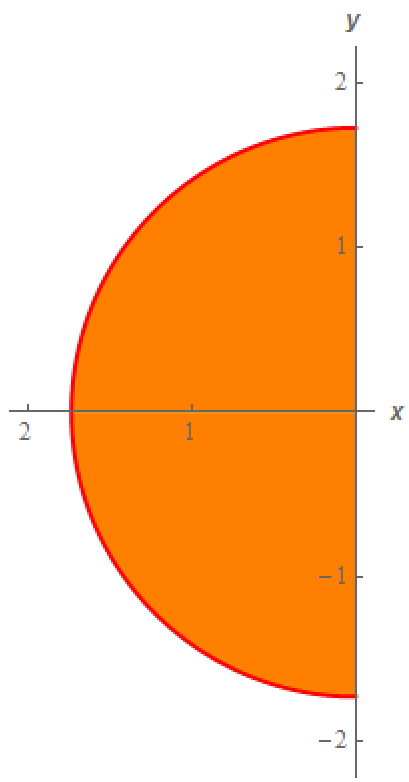
Step 2
The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

Step 3
So, from the sketch it is hopefully clear that the region *D* will be in the *xy*-plane and so we'll need to integrate with respect to *z* first. That means that the *z* limits are,

$$-\sqrt{4x^2 + 4y^2} \le z \le 6 - x$$

Step 4
For this problem *D* is simply the portion of the disk $x^2 + y^2 \le \sqrt{3}$ with $x \le 0$. Here is a quick sketch of *D* to maybe help with the limits.



Since *D* is clearly a portion of a disk it makes sense that we'll be using cylindrical coordinates. So, here are the cylindrical coordinates for this problem.

$$\begin{gathered} \tfrac{\pi}{2} \le \theta \le \tfrac{3\pi}{2} \\ 0 \le r \le \sqrt{3} \\ -2r \le z \le 6 - r\cos\theta \end{gathered}$$

Don't forget to convert the *z* limits into cylindrical coordinates.

Step 5
Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$V = \iiint\limits_E dV = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} \int_{-2r}^{6-r\cos\theta} r\,dz\,dr\,d\theta$$

Don't forget that $dV = r\,dx\,dr\,d\theta$ and so we pick up an *r* when converting the *dV* to cylindrical coordinates.

Step 6
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\begin{aligned} V &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} \left(rz\right)\Big|_{-2r}^{6-r\cos\theta} dr\,d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} 6r - r^2\cos\theta + 2r^2\,dr\,d\theta \end{aligned}$$

Step 7
Next let's do the *r* integration.

$$\begin{aligned} V &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(3r^2 - \tfrac{1}{3}r^3\cos\theta + \tfrac{2}{3}r^3\right)\Big|_0^{\sqrt{3}} d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 9 - \sqrt{3}\cos\theta + 2\sqrt{3}\,d\theta \end{aligned}$$

Step 8
Finally, we'll do the θ integration.

$$V = \left(9\theta + 2\sqrt{3}\theta - \sqrt{3}\sin\theta\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{2\sqrt{3} + \left(9 + 2\sqrt{3}\right)\pi = 42.6212}$$

---

5. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{0}^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y \;\; dz\,dy\,dx$$

Step 1
First let's just get the Cartesian limits from the integral.

$$\begin{gathered} 0 \le x \le \sqrt{5} \\ -\sqrt{5-x^2} \le y \le 0 \\ x^2+y^2-11 \le z \le 9-3x^2-3y^2 \end{gathered}$$

Step 2
Now we need to convert the integral into cylindrical coordinates. Let's first deal with the limits.

We are integrating *z* first in the integral set up to use Cartesian coordinates and so we'll integrate that first in the integral set up to use cylindrical coordinates as well. It is easy to convert the *z* limits to cylindrical coordinates as follows.

$$r^2-11 \le z \le 9-3r^2$$

Step 3
Now, the *x* and *y* limits. These are the two "outer" integrals in the original integral and so they also define *D*. So, let's see if we can determine what *D* is first. Once we have that we should be able to determine the *r* and θ limits for our integral in cylindrical coordinates.

The lower *y* limit is $y=-\sqrt{5-x^2}$ and we can see that *D* will be at most the lower portion of the disk of radius $\sqrt{5}$ centered at the origin.

From the *x* limits we see that *x* must be positive and so *D* is the portion of the disk of radius $\sqrt{5}$ that is in the 4th quadrant.

We now know what *D* is here so the full set of limits for the integral is,

$$\begin{gathered} \tfrac{3\pi}{2} \le \theta \le 2\pi \\ 0 \le r \le \sqrt{5} \\ r^2-11 \le z \le 9-3r^2 \end{gathered}$$

Step 4
Okay, let's convert the integral into cylindrical coordinates.

$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y \;\; dz\,dy\,dx = \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} \left(2r\cos\theta - 3r\sin\theta\right) r\,dz\,dr\,d\theta$$
$$= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} r^2\left(2\cos\theta - 3\sin\theta\right) dz\,dr\,d\theta$$

Don't forget that the $dz\,dy\,dz$ in the Cartesian form of the integral comes from the *dV* in the original triple integral. We also know that, in terms of cylindrical coordinates we have $dV = r\,dz\,dr\,d\theta$ and so we know that $dz\,dy\,dx = r\,dz\,dr\,d\theta$ and we'll pick up an extra *r* in the integrand.

Step 5
Okay, now all we need to do is evaluate the integral. Here is the *z* integration.

$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y \;\; dz\,dy\,dx = \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} \left(r^2\left(2\cos\theta - 3\sin\theta\right) z\right)\Big|_{r^2-11}^{9-3r^2} dr\,d\theta$$
$$= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} r^2\left(2\cos\theta - 3\sin\theta\right)\left(20-4r^2\right) dr\,d\theta$$
$$= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} \left(2\cos\theta - 3\sin\theta\right)\left(20r^2-4r^4\right) dr\,d\theta$$

Step 6
Next let's do the *r* integration.

$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y \;\; dz\,dy\,dx = \int_{\frac{3\pi}{2}}^{2\pi} \left(\left(2\cos\theta - 3\sin\theta\right)\left(\tfrac{20}{3}r^3 - \tfrac{4}{5}r^5\right)\right)\Big|_0^{\sqrt{5}} d\theta$$
$$= \int_{\frac{3\pi}{2}}^{2\pi} \tfrac{40}{3}\sqrt{5}\left(2\cos\theta - 3\sin\theta\right) d\theta$$

Step 7
Finally, we'll do the θ integration.

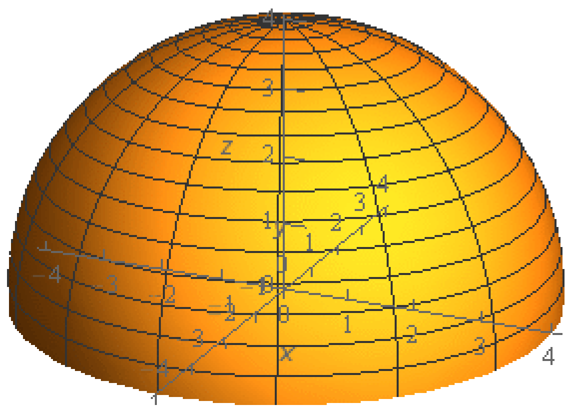
$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y \;\; dz\,dy\,dx = \left(\tfrac{40}{3}\sqrt{5}\left(2\sin\theta + 3\cos\theta\right)\right)\Big|_{\frac{3\pi}{2}}^{2\pi} = \boxed{\tfrac{200}{3}\sqrt{5}}$$

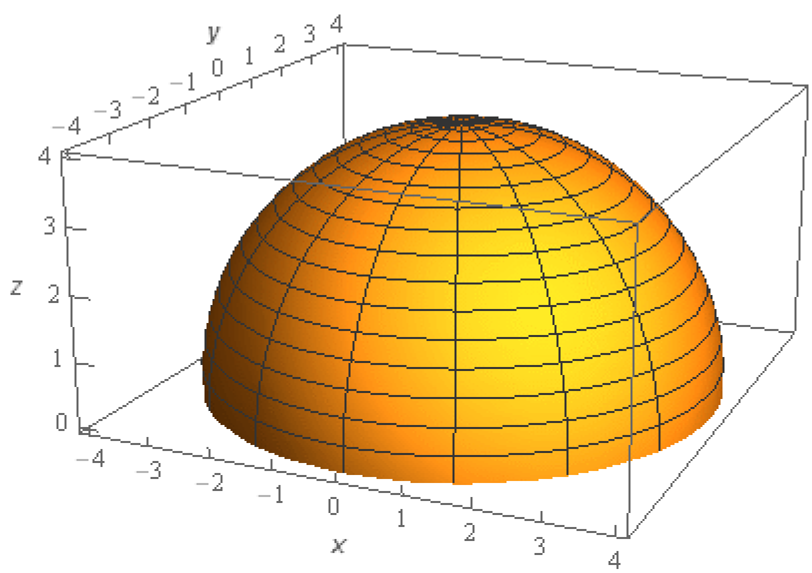
Section 4-7 : Triple Integrals in Spherical Coordinates

1. Evaluate $\iiint\limits_E 10xz + 3\,dV$ where *E* is the region portion of $x^2 + y^2 + z^2 = 16$ with $z \ge 0$.

Step 1

Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the upper half of a sphere of radius 4.

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral and the limits are,

$$0 \le \varphi \le \frac{\pi}{2}$$
$$0 \le \theta \le 2\pi$$
$$0 \le \rho \le 4$$

Remember that φ is the angle from the positive *z*-axis that we rotate through as we cover the region and θ is the angle we rotate around the *z*-axis as we cover the region.

In this case we have the full upper half of the sphere and so θ will range from 0 to 2π while φ will range from 0 to $\frac{\pi}{2}$.

Step 3
Plugging these limits into the integral and converting to spherical coordinates gives,

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 \left[10\left(\rho\sin\varphi\cos\theta\right)\left(\rho\cos\varphi\right)+3\right]\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 10\rho^4\sin^2\varphi\cos\varphi\cos\theta + 3\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Don't forget to convert the *x* and *z* into spherical coordinates and also don't forget that $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the *dV* to spherical coordinates.

Step 4
Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(2\rho^5\sin^2\varphi\cos\varphi\cos\theta + \rho^3\sin\varphi\right)\Big|_0^4\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2048\sin^2\varphi\cos\varphi\cos\theta + 64\sin\varphi\;d\theta\,d\varphi$$

Step 5
Next let's do the θ integration.

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\left(2048\sin^2\varphi\cos\varphi\sin\theta + 64\theta\sin\varphi\right)\Big|_0^{2\pi}\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}} 128\pi\sin\varphi\,d\varphi$$

Step 6

Finally, we'll do the φ integration.

$$\iiint\limits_E 10xz + 3\,dV = \left(-128\pi\cos\varphi\right)\Big|_0^{\frac{\pi}{2}} = \boxed{128\pi}$$

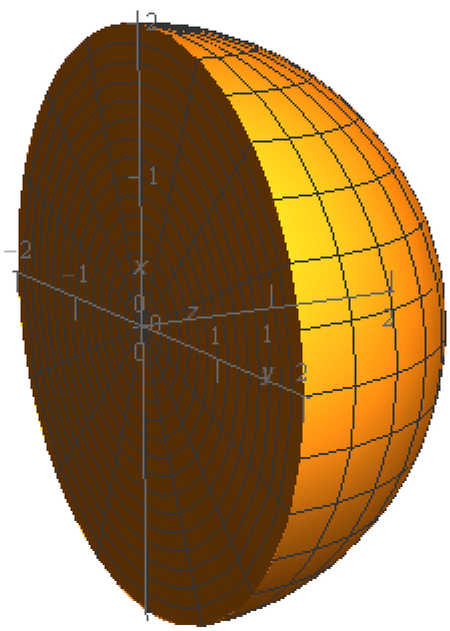
Note that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to. In this case, it might have been "simpler" to do the φ first or second as that would have greatly reduced the integrand for the remaining integral(s).

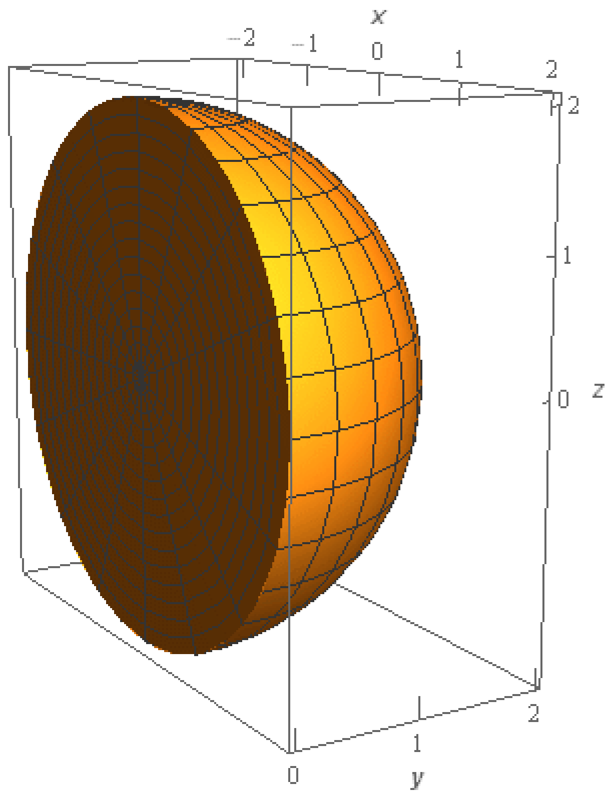
---

2. Evaluate $\iiint\limits_E x^2 + y^2\,dV$ where *E* is the region portion of $x^2 + y^2 + z^2 = 4$ with $y \geq 0$.

Step 1

Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the portion of the sphere of radius 2 with $y \ge 0$.

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral and the limits are,

$$0 \le \varphi \le \pi$$
$$0 \le \theta \le \pi$$
$$0 \le \rho \le 2$$

Remember that φ is the angle from the positive *z*-axis that we rotate through as we cover the region and θ is the angle we rotate around the *z*-axis as we cover the region.

In this case we have only the portion of the sphere with $y \ge 0$ and so θ will range from 0 to π (remember that we measure θ from the positive *x*-axis). Because we want the full half of the sphere with $y \le 0$ we know that φ will range from 0 to π.

Step 3

Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\int_0^{\pi}\int_0^2\left[\left(\rho\sin\varphi\cos\theta\right)^2+\left(\rho\sin\varphi\sin\theta\right)^2\right]\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\pi}\int_0^{\pi}\int_0^2\left[\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta\right]\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\pi}\int_0^{\pi}\int_0^2\left[\rho^2\sin^2\varphi\left(\cos^2\theta+\sin^2\theta\right)\right]\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\pi}\int_0^{\pi}\int_0^2\rho^4\sin^3\varphi\,d\rho\,d\theta\,d\varphi\end{aligned}$$

Don't forget to convert the *x* and *y* into spherical coordinates and also don't forget that $dV=\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the *dV* to spherical coordinates.

In this case we also did a fair amount of simplification that will definitely make the integration easier to deal with. Don't forget to do this kind of simplification when possible!

Step 4

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\int_0^{\pi}\left.\left(\tfrac{1}{5}\rho^5\sin^3\varphi\right)\right|_0^2 d\theta\,d\varphi\\ &= \int_0^{\pi}\int_0^{\pi}\tfrac{32}{5}\sin^3\varphi\,d\theta\,d\varphi\end{aligned}$$

Step 5

Next let's do the θ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\left.\left(\tfrac{32}{5}\theta\sin^3\varphi\right)\right|_0^{\pi} d\varphi\\ &= \int_0^{\pi}\tfrac{32}{5}\pi\sin^3\varphi\,d\varphi\end{aligned}$$

Step 6

Finally, we'll do the φ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\tfrac{32}{5}\pi\sin^2\varphi\sin\varphi\,d\varphi\\ &= \int_0^{\pi}\tfrac{32}{5}\pi\left(1-\cos^2\varphi\right)\sin\varphi\,d\varphi\\ &= \left.\left(-\tfrac{32}{5}\pi\left(\cos\varphi-\tfrac{1}{3}\cos^3\varphi\right)\right)\right|_0^{\pi} = \boxed{\tfrac{128}{15}\pi}\end{aligned}$$

You do recall how to do the kinds of trig integrals we did in this step don't you? If not you should head back and review some of the Calculus II material as these will be showing up on occasion.

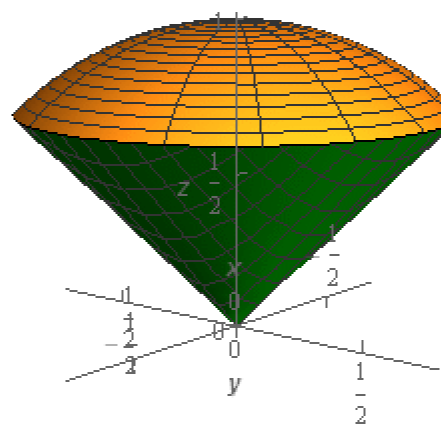
Note that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

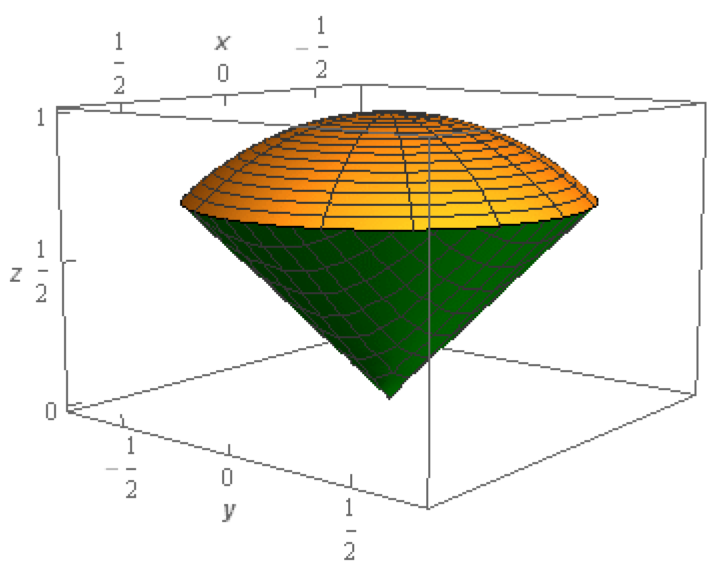
---

3. Evaluate $\iiint\limits_E 3z\,dV$ where E is the region below $x^2+y^2+z^2=1$ and inside $z=\sqrt{x^2+y^2}$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the region below the sphere of radius 1 (the orange surface in the sketches) that is inside (*i.e.* above) the cone (the green surface in the sketches).

Step 2
Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral.

The limits for ρ and θ should be pretty clear as those just correspond to the radius of the sphere and to how much of the sphere we get by rotating about the *z*-axis.

The limits for φ however aren't explicitly given in any way but we can get them from the equation of the cone. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2+y^2}=r$ and we know that, in terms of spherical coordinates, $r=\rho\sin\varphi$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos\varphi=\rho\sin\varphi \qquad \to \qquad \tan\varphi=1 \qquad \to \qquad \varphi=\frac{\pi}{4}$$

So, the equation of the cone is given by $\varphi=\frac{\pi}{4}$ in terms of spherical coordinates. Because the region we are working on is above the cone we know that φ must therefore range from 0 to $\frac{\pi}{4}$.

The limits are then,

$$\begin{aligned} &0\le\varphi\le\tfrac{\pi}{4}\\ &0\le\theta\le 2\pi\\ &0\le\rho\le 1\end{aligned}$$

Step 3
Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned}\iiint\limits_E 3z\,dV &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \left(3\rho\cos\varphi\right)\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 3\rho^3\cos\varphi\sin\varphi\,d\rho\,d\theta\,d\varphi\end{aligned}$$

Don't forget to convert the *z* into spherical coordinates and also don't forget that $dV=\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the *dV* to spherical coordinates.

Step 4
Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E 3z\,dV &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\left(\tfrac{3}{4}\rho^4\cos\varphi\sin\varphi\right)\Big|_0^1\,d\theta\,d\varphi\\ &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\tfrac{3}{4}\cos\varphi\sin\varphi\,d\theta\,d\varphi\end{aligned}$$

Step 5
Next let's do the θ integration.

$$\iiint\limits_E 3z\,dV = \int_0^{\frac{\pi}{4}} \left(\tfrac{3}{4}\theta\cos\varphi\sin\varphi\right)\Big|_0^{2\pi}\,d\varphi$$
$$= \int_0^{\frac{\pi}{4}} \tfrac{3}{2}\pi\cos\varphi\sin\varphi\,d\varphi$$

Step 6
Finally, we'll do the φ integration.

$$\iiint\limits_E 3z\,dV = \int_0^{\frac{\pi}{4}} \tfrac{3}{4}\pi\sin\left(2\varphi\right)d\varphi$$
$$= \left(-\tfrac{3}{8}\pi\cos\left(2\varphi\right)\right)\Big|_0^{\frac{\pi}{4}} = \boxed{\tfrac{3}{8}\pi}$$

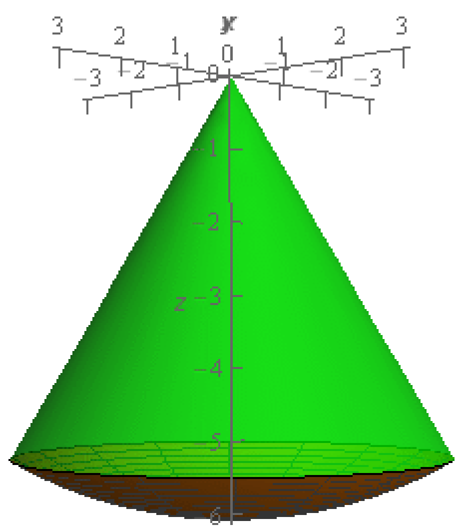
We used the double angle formula for sine to reduce the integral to something that we could quickly do.

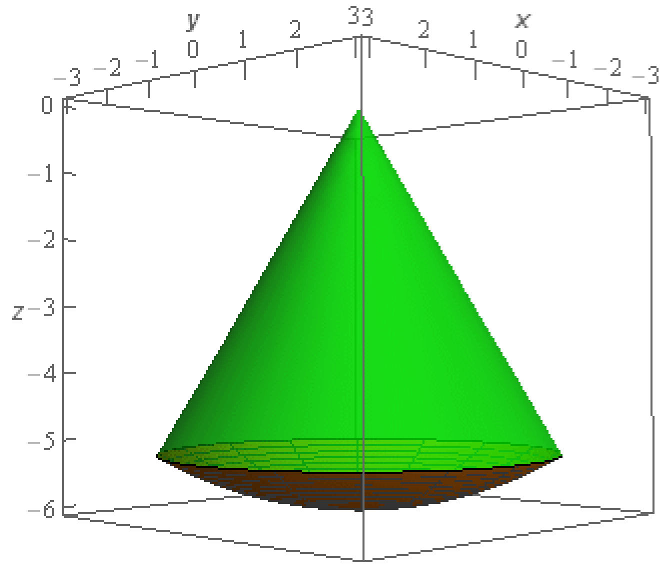
Not that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

---

4. Evaluate $\iiint\limits_E x^2\,dV$ where *E* is the region above $x^2+y^2+z^2=36$ and inside $z=-\sqrt{3x^2+3y^2}$.

Step 1
Okay, let's start off with a quick sketch of the region *E* so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the region above the sphere of radius 6 (the dark orange surface at the bottom in the sketches) that is inside (*i.e.* below) the cone (the green surface in the sketches).

Step 2
Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral.

The limits for ρ and θ should be pretty clear as those just correspond to the radius of the sphere and to how much of the sphere we get by rotating about the *z*-axis.

The limits for φ however aren't explicitly given in any way but we can get them from the equation of the cone. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2 + y^2} = r$ and we know that, in terms of spherical coordinates, $r = \rho\sin\varphi$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos\varphi = -\sqrt{3}\rho\sin\varphi \qquad \to \qquad \tan\varphi = -\frac{1}{\sqrt{3}} \qquad \to \qquad \varphi = \frac{5\pi}{6}$$

So, the equation of the cone is given by $\varphi = \frac{5\pi}{6}$ in terms of spherical coordinates. Because the region we are working on is below the cone we know that φ must therefore range from $\frac{5\pi}{6}$ to π.

The limits are then,

$$\begin{aligned} &\tfrac{5\pi}{6} \le \varphi \le \pi \\ &0 \le \theta \le 2\pi \\ &0 \le \rho \le 6 \end{aligned}$$

Step 3
Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^6 \left(\rho\sin\varphi\cos\theta\right)^2\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^6 \rho^4\sin^3\varphi\cos^2\theta\,d\rho\,d\theta\,d\varphi\end{aligned}$$

Don't forget to convert the *z* into spherical coordinates and also don't forget that $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the *dV* to spherical coordinates.

Step 4
Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\left.\left(\tfrac{1}{5}\rho^5\sin^3\varphi\cos^2\theta\right)\right|_0^6 d\theta\,d\varphi\\ &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\tfrac{7776}{5}\sin^3\varphi\cos^2\theta\,d\theta\,d\varphi\end{aligned}$$

Step 5
Next let's do the θ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\tfrac{7776}{5}\sin^3\varphi\left(\tfrac{1}{2}\right)\left(1+\cos\left(2\theta\right)\right)d\theta\,d\varphi\\ &= \int_{\frac{5\pi}{6}}^{\pi}\left.\left(\tfrac{3888}{5}\sin^3\varphi\left(\theta+\tfrac{1}{2}\sin\left(2\theta\right)\right)\right)\right|_0^{2\pi}d\varphi\\ &= \int_{\frac{5\pi}{6}}^{\pi}\tfrac{7776}{5}\pi\sin^3\varphi\,d\varphi\end{aligned}$$

Step 6
Finally, we'll do the φ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\tfrac{7776}{5}\pi\sin^2\varphi\sin\varphi\,d\varphi\\ &= \int_{\frac{5\pi}{6}}^{\pi}\tfrac{7776}{5}\pi\left(1-\cos^2\varphi\right)\sin\varphi\,d\varphi\\ &= \left.\left(-\tfrac{7776}{5}\pi\left(\cos\varphi-\tfrac{1}{3}\cos^3\varphi\right)\right)\right|_{\frac{5\pi}{6}}^{\pi} = \boxed{\tfrac{7776}{5}\pi\left(\tfrac{2}{3}-\tfrac{3\sqrt{3}}{8}\right)=83.7799}\end{aligned}$$

You do recall how to do the kinds of trig integrals we did in the last two steps don't you? If not you should head back and review some of the Calculus II material as these will be showing up on occasion.

Not that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

---

5. Evaluate the following integral by first converting to an integral in spherical coordinates.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \;\; dz\,dy\,dx$$

Step 1
First let's just get the Cartesian limits from the integral.

$$-1 \le x \le 0$$
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$
$$\sqrt{6x^2+6y^2} \le z \le \sqrt{7-x^2-y^2}$$

Step 2
Now we need to convert the integral into spherical coordinates. Let's first take care of the limits first.

From the upper *z* limit we see that we are under $z = \sqrt{7-x^2-y^2}$ (which is just the equation for the upper portion of a sphere of radius $\sqrt{7}$).

From the lower *z* limit we see that we are above $z = \sqrt{6x^2+6y^2}$ (which is just the equation of a cone).

So, we appear to be inside an "ice cream cone" shaped region as we usually are when dealing with spherical coordinates.

This leads us to the following ρ limits.

$$0 \le \rho \le \sqrt{7}$$

Step 3
The limits for φ we can get them from the equation of the cone that is the lower *z* limit referenced in the previous step. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2+y^2} = r$ and we know that, in terms of spherical coordinates, $r = \rho\sin\varphi$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos\varphi = \sqrt{6}\rho\sin\varphi \quad \to \quad \tan\varphi = \frac{1}{\sqrt{6}} \quad \to \quad \varphi = \tan^{-1}\left(\frac{1}{\sqrt{6}}\right) = 0.3876$$

Because the region we are working on is above the cone we know that φ must therefore range from 0 to 0.3876.

Step 4
Finally, let's get the θ limits. For reference purposes here are the *x* and *y* limits we found in Step 1.

$$\begin{gathered} -1 \le x \le 0 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{gathered}$$

All the *y* limits tell us is that the region *D* from the original Cartesian coordinates integral is a portion of the circle of radius 1. Note that this should make sense as this is also the intersection of the sphere and cone we get from the *z* limits (we'll leave it to you to verify this statement).

Now, from the *x* limits we see that we must have the left side of the circle of radius 1 and so the limits for θ are then,

$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

The full set of spherical coordinate limits for the integral are then,

$$\begin{aligned} &0 \le \varphi \le \tan^{-1}\left(\tfrac{1}{\sqrt{6}}\right) = 0.3876 \\ &\tfrac{\pi}{2} \le \theta \le \tfrac{3\pi}{2} \\ &0 \le \rho \le \sqrt{7} \end{aligned}$$

Step 5
Okay, let's convert the integral in to spherical coordinates.

$$\begin{aligned} \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \; dz\,dy\,dx &= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{\sqrt{7}} \left(18\rho\sin\varphi\sin\theta\right)\left(\rho^2\sin\varphi\right)d\rho\,d\theta\,d\varphi \\ &= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{\sqrt{7}} 18\rho^3\sin^2\varphi\sin\theta\,d\rho\,d\theta\,d\varphi \end{aligned}$$

Don't forget to convert the *y* into spherical coordinates. Also, don't forget that the $dz\,dy\,dx$ come from the *dV* in the original triple integral. We also know that, in terms of spherical coordinates, $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$ and so we in turn know that,

$$dz\,dy\,dx = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Step 6
Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \;\; dz\,dy\,dx = \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\tfrac{9}{2}\rho^4 \sin^2\varphi \sin\theta\right)\Big|_{0}^{\sqrt{7}} d\theta\,d\varphi$$

$$= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{441}{2}\sin^2\varphi \sin\theta\,d\theta\,d\varphi$$

Step 7

Next let’s do the θ integration.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \;\; dz\,dy\,dx = \int_{0}^{0.3876} \left(-\tfrac{441}{2}\sin^2\varphi \cos\theta\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\varphi$$

$$= \int_{0}^{0.3876} 0\,d\varphi$$

$$= \boxed{0}$$

So, as noted above once we got the integrand down to zero there was no reason to continue integrating as the answer will continue to be zero for the rest of the problem.

Don’t get excited about it when these kinds of things happen. They will on occasion and all it means is that we get to stop integrating a little sooner that we would have otherwise.

Section 4-8 : Change of Variables

1. Compute the Jacobian of the following transformation.

$$x = 4u - 3v^2 \qquad y = u^2 - 6v$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & -6v \\ 2u & -6 \end{vmatrix} = -24 - (-12uv) = \boxed{-24 + 12uv}$$

---

2. Compute the Jacobian of the following transformation.

$$x = u^2 v^3 \qquad y = 4 - 2\sqrt{u}$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2uv^3 & 3u^2v^2 \\ -u^{-\frac{1}{2}} & 0 \end{vmatrix} = 0 - \left(-3u^{\frac{3}{2}}v^2\right) = \boxed{3u^{\frac{3}{2}}v^2}$$

---

3. Compute the Jacobian of the following transformation.

$$x = \frac{v}{u} \qquad y = u^2 - 4v^2$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 2u & -8v \end{vmatrix} = \frac{8v^2}{u^2} - (2) = \boxed{\frac{8v^2}{u^2} - 2}$$

---

4. If *R* is the region inside $\frac{x^2}{4}+\frac{y^2}{36}=1$ determine the region we would get applying the transformation $x=2u$, $y=6v$ to *R*.

Solution
There really isn't a lot to this problem.

It should be pretty clear that the outer boundary of *R* is an ellipse. That isn't really important to this problem but this problem will lead to seeing how to set up a nice transformation for elliptical regions.

To determine the transformation of this region all we need to do is plug the transformation boundary equation for *R*. Doing this gives,

$$\frac{\left(2u\right)^2}{4}+\frac{\left(6v\right)^2}{36}=1 \qquad \to \qquad \frac{4u^2}{4}+\frac{36v^2}{36}=1 \qquad \to \qquad u^2+v^2=1$$

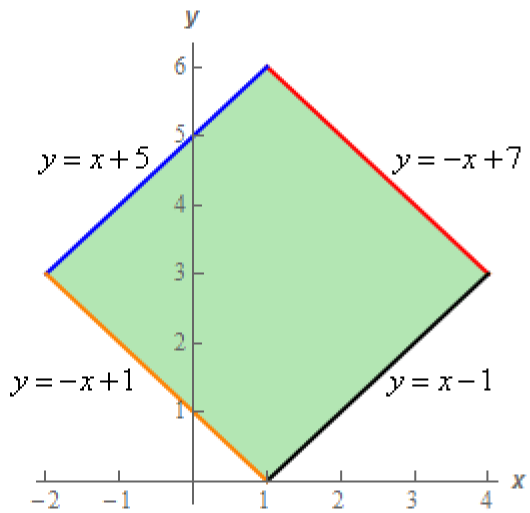
So, the boundary equation for *R* transforms into the equation for the unit circle and so, under this transformation, we can transform an ellipse into a circle (a unit circle in fact…).

You can see how to determine a transformation that will transform an elliptical region into a circular region can't you? Integrating over an elliptical region would probably be pretty unpleasant but integrating over a unit disk will probably be much nicer so this is a nice transformation to understand how to get!

---

5. If *R* is the parallelogram with vertices $\left(1,0\right)$, $\left(4,3\right)$, $\left(1,6\right)$ and $\left(-2,3\right)$ determine the region we would get applying the transformation $x=\frac{1}{2}\left(v-u\right)$, $y=\frac{1}{2}\left(v+u\right)$ to *R*.

Step 1
Let's start off with a sketch of *R*.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

Step 2
Okay, now all we need to do is apply the transformation to each of the boundary equations. To do this we simply plug the transformation into each of the boundary equations.

Let's start with $y = x + 5$. Applying the transformation to this equation gives,

$$\begin{aligned} \tfrac{1}{2}(v+u) &= \tfrac{1}{2}(v-u) + 5 \\ v + u &= v - u + 10 \\ 2u &= 10 \qquad \Rightarrow \qquad \underline{u = 5} \end{aligned}$$

So, this boundary will transform into the equation $u = 5$.

Step 3
Let's now apply the transformation to $y = -x + 7$.

$$\begin{aligned} \tfrac{1}{2}(v+u) &= -\tfrac{1}{2}(v-u) + 7 \\ v + u &= -v + u + 14 \\ 2v &= 14 \qquad \Rightarrow \qquad \underline{v = 7} \end{aligned}$$

This boundary transforms in the equation $v = 7$.

Step 4
Next, let's apply the transformation to $y = x - 1$.

$$\begin{aligned} \tfrac{1}{2}(v+u) &= \tfrac{1}{2}(v-u)-1 \\ v+u &= v-u-2 \\ 2u &= -2 \qquad \Rightarrow \qquad \underline{u=-1} \end{aligned}$$

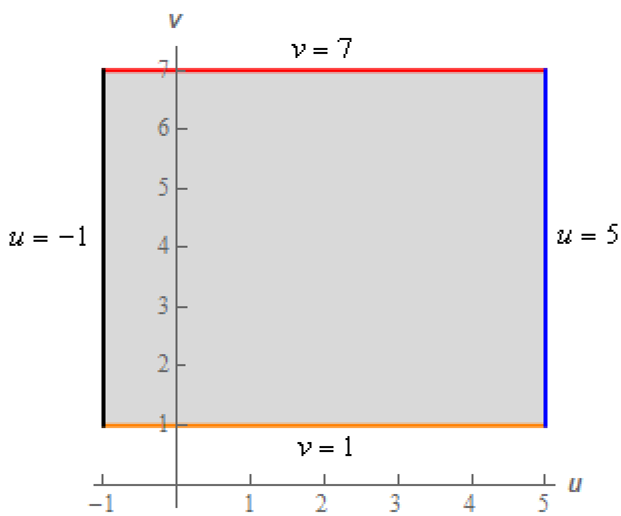
So, this boundary will transform into the equation $u=-1$.

Step 5
Finally let's apply the transformation to $y=-x+1$.

$$\begin{aligned} \tfrac{1}{2}(v+u) &= -\tfrac{1}{2}(v-u)+1 \\ v+u &= -v+u+2 \\ 2v &= 2 \qquad \Rightarrow \qquad \underline{v=1} \end{aligned}$$

This boundary transforms in the equation $v=1$.

Step 6
Sketching the transformed equations gives the following region.

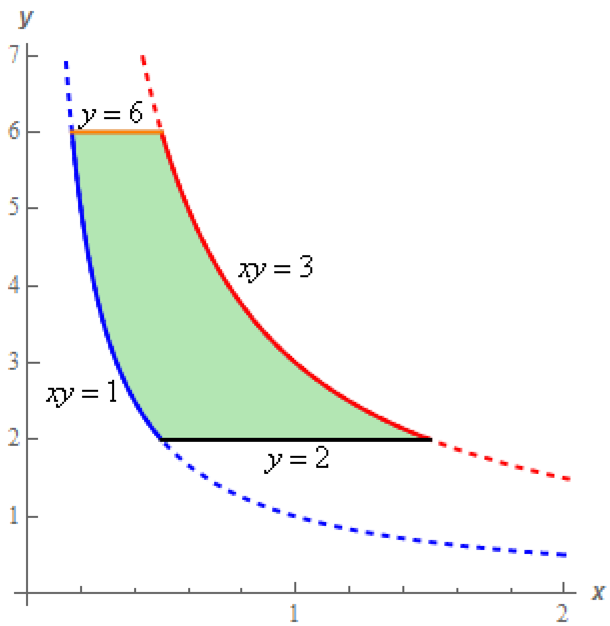


So, we transform the diamond shaped region into a rectangle under the transformation.

Note that we chose *u* to be the horizontal axis and *v* to be vertical axis in the transformed region for this problem. There is no real reason for doing that other than it is just what we've always done. Regardless of your choices here make sure to label the axes to make it clear!

6. If *R* is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ determine the region we would get applying the transformation $x = \dfrac{v}{6u}$, $y = 2u$ to *R*.

Step 1
Let's start off with a sketch of *R*.



The equations of each of the boundaries of the region are given in the sketch. We also included "extensions" of the two curves (the dotted portions) just to show a fuller sketch of the two curves.

Step 2
Okay, now all we need to do is apply the transformation to each of the boundary equations. To do this we simply plug the transformation into each of the boundary equations.

Let's start with $y = 6$. Applying the transformation to this equation gives,

$$2u = 6 \qquad \Rightarrow \qquad \underline{u = 3}$$

So, this boundary will transform into the equation $u = 3$.

Step 3
Let's now apply the transformation to $xy = 3$.

$$\left(\frac{v}{6u}\right)(2u) = 3$$
$$\frac{v}{3} = 3 \qquad \Rightarrow \qquad \underline{v = 9}$$

This boundary transforms in the equation $v = 9$.

Step 4

Next, let's apply the transformation to $y = 2$.

$$2u = 2 \qquad \Rightarrow \qquad \underline{u = 1}$$

So, this boundary will transform into the equation $u = 1$.

Step 5

Finally let's apply the transformation to $xy = 1$.

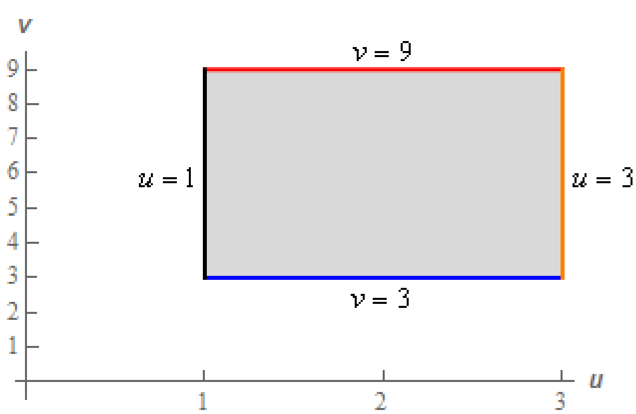
$$\left(\frac{v}{6u}\right)(2u) = 1$$

$$\frac{v}{3} = 1 \qquad \Rightarrow \qquad \underline{v = 3}$$

This boundary transforms in the equation $v = 3$.

Step 6

Sketching the transformed equations gives the following region.



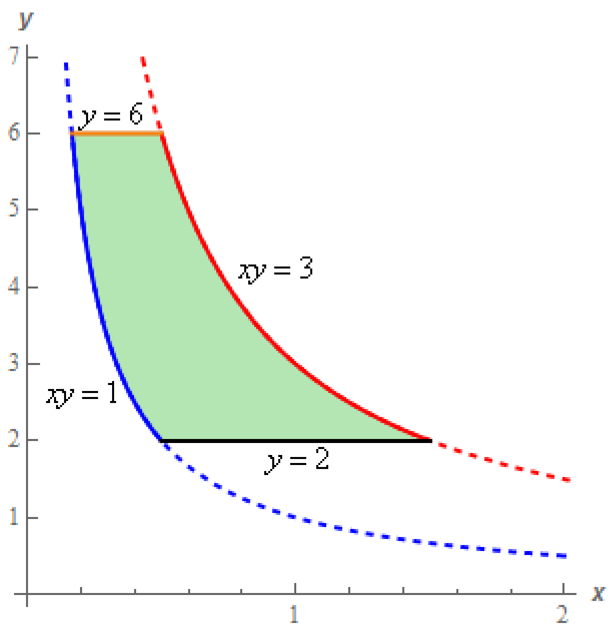
So, we transform the odd shaped region into a rectangle under the transformation.

Note that we chose *u* to be the horizontal axis and *v* to be vertical axis in the transformed region for this problem. There is no real reason for doing that other than it is just what we've always done. Regardless of your choices here make sure to label the axes to make it clear!

7. Evaluate $\iint\limits_R xy^3\,dA$ where *R* is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}$, $y = 2u$.

Step 1
The first thing we need to do is determine the transformation of *R*. We actually determined the transformation of *R* in the previous example. However, let's go through the process again (with a few details omitted) just to have it here in this problem.

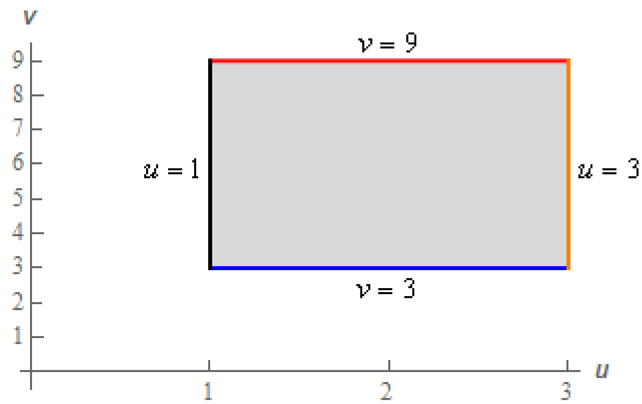
First, a sketch of *R*.



Now, let's transform each of the boundary curves.

$$\begin{array}{llcl}
y = 6: & 2u = 6 & \to & u = 3 \\
xy = 3: & \left(\dfrac{v}{6u}\right)(2u) = 3 & \to & v = 9 \\
y = 2: & 2u = 2 & \to & u = 1 \\
xy = 1: & \left(\dfrac{v}{6u}\right)(2u) = 1 & \to & v = 3
\end{array}$$

Here is a sketch of the transformed region.



So, the limits for the transformed region are,

$$
\begin{gathered}
1 \le u \le 3 \\
3 \le v \le 9
\end{gathered}
$$

Step 2
We'll need the Jacobian of this transformation next.

$$
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{vmatrix} = 0 - \frac{2}{6u} = -\frac{1}{3u}
$$

Step 3
We can now write the integral in terms of the "new" *uv* coordinates system.

$$
\begin{aligned}
\iint\limits_R xy^3\,dA &= \int_1^3 \int_3^9 \left(\frac{v}{6u}\right)(2u)^3 \left|-\frac{1}{3u}\right| dv\,du \\
&= \int_1^3 \int_3^9 \tfrac{4}{9} v\,u\,dv\,du
\end{aligned}
$$

Don't forget to add in the Jacobian and don't forget that we need absolute value bars on it. In this case we know that the range of *u* we're working on (given in Step 1) is positive we know that the quantity in the absolute value bars is negative and so we can drop the absolute value bars by also dropping the minus sign.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4
Finally, let's evaluate the integral.

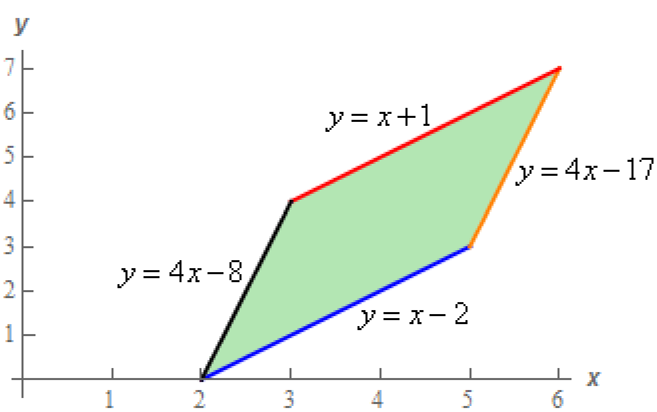
$$\iint\limits_R xy^3\,dA = \int_1^3\int_3^9 \tfrac{4}{9}vu\,dv\,du$$
$$= \int_1^3 \left(\tfrac{2}{9}v^2u\right)\Big|_3^9\,du$$
$$= \int_1^3 16u\,du$$
$$= \left(8u^2\right)\Big|_1^3$$
$$= \boxed{64}$$

---

8. Evaluate $\iint\limits_R 6x - 3y\,dA$ where *R* is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$ and $(3,4)$ using the transformation $x = \dfrac{1}{3}(v-u)$, $y = \dfrac{1}{3}(4v-u)$ to *R*.

Step 1
The first thing we need to do is determine the transformation of *R*.

First, a sketch of *R*.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

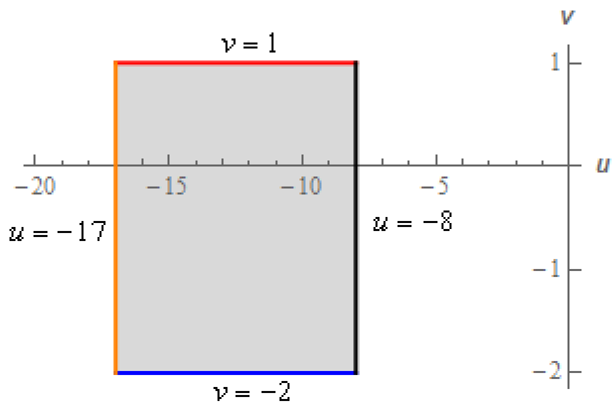
Note that integrating over this region would require three integrals regardless of the order. You can see how each order of integration would require three integrals correct?

Now, let's transform each of the boundary curves.

$$y = x+1: \qquad \tfrac{1}{3}(4v-u) = \tfrac{1}{3}(v-u)+1 \quad \to \quad 3v = 3 \quad \to \quad v = 1$$

$$y = 4x - 17: \quad \tfrac{1}{3}(4v - u) = \tfrac{4}{3}(v - u) - 17 \quad \to \quad 3u = -51 \quad \to \quad u = -17$$
$$y = x - 2: \quad \tfrac{1}{3}(4v - u) = \tfrac{1}{3}(v - u) - 2 \quad \to \quad 3v = -6 \quad \to \quad v = -2$$
$$y = 4x - 8: \quad \tfrac{1}{3}(4v - u) = \tfrac{4}{3}(v - u) - 8 \quad \to \quad 3u = -24 \quad \to \quad u = -8$$

Here is a sketch of the transformed region and note that the transformed region will be much easier to integrate over than the original region.



So, the limits for the transformed region are,

$$-17 \le u \le -8$$
$$-2 \le v \le 1$$

Note as well that this is going to be a much nice region to integrate over than the original region.

Step 2
We'll need the Jacobian of this transformation next.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{vmatrix} = -\frac{4}{9} - \left(-\frac{1}{9}\right) = -\frac{1}{3}$$

Step 3
We can now write the integral in terms of the "new" *uv* coordinates system.

$$\iint\limits_R 6x - 3y\,dA = \int_{-17}^{-8}\int_{-2}^{1}\left[6\left(\tfrac{1}{3}\right)(v-u) - 3\left(\tfrac{1}{3}\right)(4v-u)\right]\left|-\tfrac{1}{3}\right|dv\,du$$
$$= \int_{-17}^{-8}\int_{-2}^{1} -\tfrac{1}{3}(2v+u)\,dv\,du$$

Don't forget to add in the Jacobian and don't forget that we need absolute value bars on it. In this case we can just drop the absolute value bars by also dropping the minus sign since we just have a number in the absolute value.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4
Finally, let's evaluate the integral.

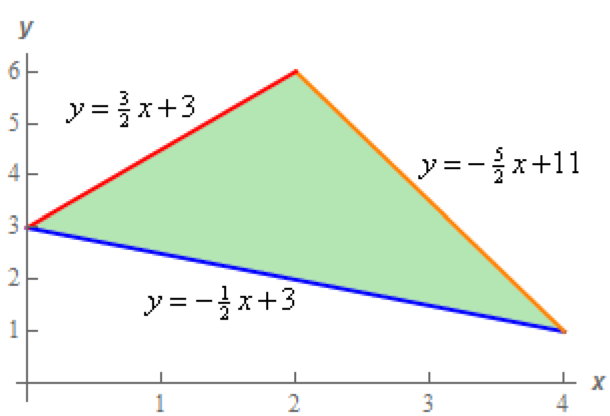
$$\begin{aligned}\iint_R 6x-3y\,dA &= \int_{-17}^{-8}\int_{-2}^{1} -\tfrac{1}{3}\left(2v+u\right)dv\,du \\ &= \int_{-17}^{-8}\left(-\tfrac{1}{3}\left(v^2+uv\right)\right)\Big|_{-2}^{1}\,du \\ &= \int_{-17}^{-8} 1-u\,du \\ &= \left(u-\tfrac{1}{2}u^2\right)\Big|_{-17}^{-8} \\ &= \boxed{\tfrac{243}{2}}\end{aligned}$$

---

9. Evaluate $\iint\limits_R x+2y\,dA$ where *R* is the triangle with vertices $(0,3)$, $(4,1)$ and $(2,6)$ using the transformation $x=\dfrac{1}{2}(u-v)$, $y=\dfrac{1}{4}(3u+v+12)$ to *R*.

Step 1
The first thing we need to do is determine the transformation of *R*.

First, a sketch of *R*.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

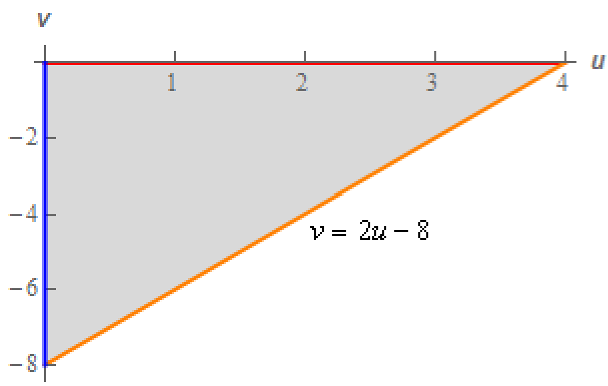
Note that integrating over this region would require two integrals regardless of the order. You can see how each order of integration would require two integrals correct?

Now, let's transform each of the boundary curves.

$$\begin{array}{lll} y=\frac{3}{2}x+3: & \frac{1}{4}(3u+v+12)=\frac{3}{2}\left(\frac{1}{2}\right)(u-v)+3 & \to \quad v=0 \\ y=-\frac{5}{2}x+11: & \frac{1}{4}(3u+v+12)=-\frac{5}{2}\left(\frac{1}{2}\right)(u-v)+11 & \to \quad v=2u-8 \\ y=-\frac{1}{2}x+3: & \frac{1}{4}(3u+v+12)=-\frac{1}{2}\left(\frac{1}{2}\right)(u-v)+3 & \to \quad u=0 \end{array}$$

Note that, in this case, the first and last boundary equation we looked at above just ended up being transformed into the *u*-axis and *v*-axis respectively. That will happen on occasion and might well end up making our life a little easier when it comes to evaluating the integral.

Here is a sketch of the transformed region and note that the transformed region will be much easier to integrate over than the original region.



So, the limits for the transformed region are,

$$\begin{gathered} 0 \le u \le 4 \\ 2u-8 \le v \le 0 \end{gathered}$$

Note as well that this is going to be a much nice region to integrate over than the original region.

Step 2
We'll need the Jacobian of this transformation next.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{8} - \left(-\frac{3}{8}\right) = \frac{1}{2}$$

Step 3
We can now write the integral in terms of the "new" *uv* coordinates system.

$$\begin{aligned}\iint\limits_R x+2y\,dA &= \int_0^4 \int_{2u-8}^0 \left[\tfrac{1}{2}(u-v) + 2\left(\tfrac{1}{4}\right)(3u+v+12)\right]\left|\tfrac{1}{2}\right| dv\,du \\ &= \int_0^4 \int_{2u-8}^0 u+3\,dv\,du\end{aligned}$$

Don't forget to add in the Jacobian and don't forget that the absolute value bars on it. In this case we can just drop the absolute value bars since we just have a positive number in the absolute value.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4
Finally, let's evaluate the integral.

$$\begin{aligned}\iint\limits_R x+2y\,dA &= \int_0^4 \int_{2u-8}^0 u+3\,dv\,du \\ &= \int_0^4 (u+3)v\Big|_{2u-8}^0 du \\ &= \int_0^4 (u+3)(-2u+8)\,du \\ &= \int_0^4 24+2u-2u^2\,du \\ &= \left(24u+u^2-\tfrac{2}{3}u^3\right)\Big|_0^4 \\ &= \boxed{\tfrac{208}{3}}\end{aligned}$$

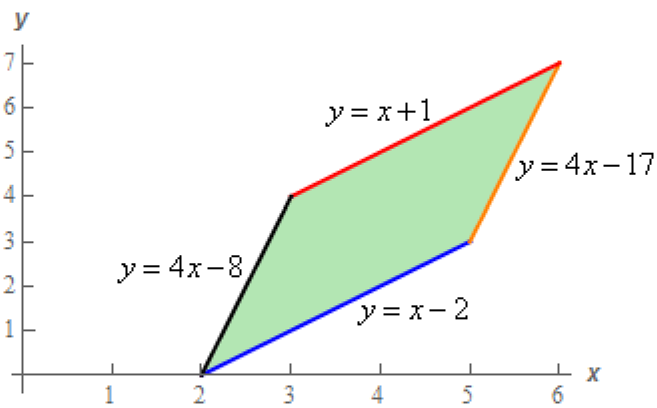
---

10. Derive the transformation used in problem 8.

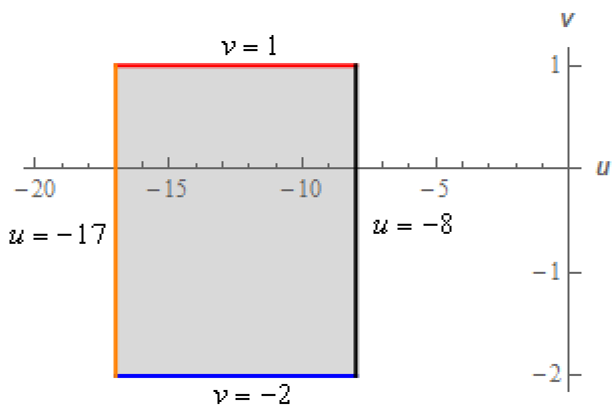
Step 1
Okay, for reference purposes we want to derive the following transformation,

$$x = \frac{1}{3}(v-u) \qquad\qquad y = \frac{1}{3}(4v-u)$$

We used this transformation on the following region.



The result of the applying the transformation to the region above got us the following "new" region.



We need to derive the transformation. This seems to be a nearly impossible task at first glance. However, it isn't as difficult as it might appear to be. Or maybe we should say it won't be as difficult as it seems to be once you see how to do it.

Before we start with the derivation process we should point out that there are actually quite a few possible transformations that we could use on the original region and get a rectangle for the new region. They won't however all yield the same rectangle. We are going to get a specific transformation but you should be able how to modify what we're doing here to get a different transformation.

Step 2
The first thing that we're going to do is rewrite all the boundary equations for the original region as follows.

$$\begin{array}{lcl} y = x + 1 & \to & y - x = 1 \\ y = 4x - 17 & \to & y - 4x = -17 \\ y = x - 2 & \to & y - x = -2 \\ y = 4x - 8 & \to & y - 4x = -8 \end{array}$$

Now, let's notice that because our original region had two sets of parallel sides our new equations can also be organized into two pairs of equations as follows,

$$\begin{aligned} y - x &= 1 \qquad \text{AND} \qquad y - x = -2 \\ y - 4x &= -17 \qquad \text{AND} \qquad y - 4x = -8 \end{aligned}$$

The left side of each pair of equations is identical (*i.e.* they came from parallel sides of our original region). This is going to make life easier for us in order to derive a derivation that will give a "nice" region. Of course, in this case, we already know what the new region is going to be but in general we wouldn't.

Step 3
Now, we want to define new variables *u* and *v* in terms of the old variables *x* and *y* in such a way that we will get the rectangle we are looking for.

Let's define *u* and *v* as follows (we'll explain why in a bit),

$$u = y - 4x \qquad\qquad v = y - x$$

With this definition note that we can already see what our "new" region will be. Putting these definitions into the equations we wrote down in Step 2 gives us,

$$\begin{aligned} v &= 1 \qquad \text{AND} \qquad v = -2 \\ u &= -17 \qquad \text{AND} \qquad u = -8 \end{aligned}$$

This is exactly the new region that we got in Problem 8. This tells us that we'll get the transformation in Problem 8 from this definition (we'll do that work in the next step).

It should also make it a little clearer why we chose the definitions of *u* and *v* as we did above. We know that we wanted an easier region to integration over and let's be honest a rectangle is really easy to integrate over so trying to get a rectangle for the new region seemed like a good idea.

This is where the two pairs of equations comes into play. Because we were able to pair up the equations so the left side was identical in each pair all we had to do was define one of the left sides as *u* and the other as *v* and we would know at that point that we'd get a rectangle as our new region.

Note however, that it should also now be clear that there would be another "simple" definition for *u* and *v* that we could have used (*i.e.* $u = y - x$, $v = y - 4x$). It would have resulted in a similar, yet different, rectangle and a different transformation. There was no real reason for choosing our definition of *u* and *v* as we did over the other possibility other than we knew the region we wanted to get. Back in Problem 8 we basically just chose one at random.

Now, there are probably quite a few questions in your mind at this point and we'll address as many of them as we can in the last step. Let's finish the actual problem out first before we do that however just so we don't forget what we're doing here.

Step 4

Okay, let's finish the actual problem before we answer some questions in the final step. We need to use our definition of *u* and *v* from the last step to get the transformation we used in Problem 8.

First, let's just start off with our definition of *u* and *v* again.

$$u = y - 4x$$
$$v = y - x$$

Now, what we have here is a system of two equations. All we need to do to finish this problem out is solve them for *x* and *y*.

To do this let's first solve the *v* definition for *y*.

$$v = y - x \qquad \to \qquad y = v + x$$

Plug this into the *u* definition and solve for *x*.

$$u = y - 4x \qquad \to \qquad u = v + x - 4x = v - 3x \qquad \to \qquad x = \tfrac{1}{3}\left(v - u\right)$$

That should look familiar of course. Now, all we need to do is find the *y* transformation. To do this we'll just plug our *x* transformation into $y = v + x$ and do a little simplification.

$$y = v + x = v + \tfrac{1}{3}\left(v - u\right) = \tfrac{4}{3}v - \tfrac{1}{3}u = \tfrac{1}{3}\left(4v - u\right)$$

So, there we go. The transformation has been derived and now that you see how to do it we can see that it wasn't really all that bad despite first appearances.

Step 5

Okay, now let's see if we can address at least some of the questions you might have about deriving transformations.

Can we always transform any region into a rectangle?

The short answer is definitely NO. In order to quickly get a transformation that yielded a rectangle here we relied on the fact that we had an original region that had four linear sides and we were able to form them into two pairs of parallel sides. Once that was done it was easy to get a transformation that would yield a rectangle.

Many regions can't be transformed (or at least not easily transformed) into a rectangle.

So, what if we'd had a region with four linear sides but we couldn't form them into two sets of parallel sides?

In this case the problem would still work pretty much the same way but we'd have lots more options for definition of *u* and *v* and the result wouldn't be a rectangle.

In a case like this, because all the sides are linear, we can still write each of them in as follows,

$$\begin{aligned}\text{Side 1}: &\ a_1y+b_1x=c_1\\ \text{Side 2}: &\ a_2y+b_2x=c_2\\ \text{Side 3}: &\ a_3y+b_3x=c_3\\ \text{Side 4}: &\ a_4y+b_4x=c_4\end{aligned}$$

At this point we could define *u* to be any of the left sides and *v* to be any other remaining left sides. This would guarantee one side will be vertical and one side will be horizontal in the new region. The remaining two sides of the original region would then be transformed into new lines and hopefully the resulting region would be easy to work with.

As you can see there would be lots of possible definitions of *u* and *v* here which in turn would lead to lots of different transformations.

What if you had more or less than four linear sides?
The process described in this problem should still work to derive a transformation. Whether or not the resulting region is easy to integrate over may be a completely different questions however and there won't necessarily be any way to know until you've derived the transformation and done the work to get the new region.

What if you had at least one side that was not linear?
With these types of regions then the process described in this problem may still work but it also might not. It will depend on just how "messy" the equation of the non-linear boundary is and if we can even solve the definitions of *u* and *v* for *x* and *y* as we did in Step 4 of this problem.

Unfortunately, there really isn't any good answer to this question. There are many ways of getting transformations. We've simply described one of the easier methods of deriving a transformation for the type of regions that you're most likely to run into in this class. The method will work on occasion for regions that have one or more non-linear boundary but there is no reason to expect it to work with every possible region.

Are all transformations derived in this fashion?
As alluded to in the answer to the previous question, absolutely not. There are many types of regions where this process simply won't work, or at least won't work easily.

A good example is the ellipse we looked at in Problem 4. If you've done that problem then you'll probably already know that you can pretty much get a transformation that will transform the ellipse into a circle by inspection. The process described in this problem simply won't work on that region.

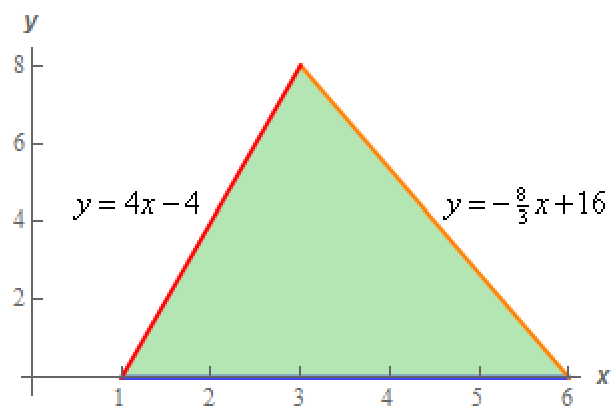
The point of this problem was not to teach you now to derive transformations in general. It was simply to show you how many of the transformations in these problems were derived and to show you one method that, when it can be used, is fairly simple to do.

Are there any variations to this derivation method?
Absolutely. See the next problem for at least one.

11. Derive a transformation that will convert the triangle with vertices $(1,0)$, $(6,0)$ and $(3,8)$ into a right triangle with the right angle occurring at the origin of the *uv* system.

Step 1
We're going to be using slight modification of the process used in the previous problem here. If you haven't read through that problem you should before proceeding with this problem as we're not going to be going over the explanation in quite as much detail here.

So, let's just start off with a sketch of the triangle.



We'll leave it to you to verify the algebra details of deriving the equations of the two sloped sides. The bottom side is of course just defined by $y = 0$.

Step 2
This region is clearly a little different from the region in the previous problem but the next step is still the same. We start with writing the equations of each of the sides as follows.

$$\begin{aligned} y - 4x &= -4 \\ y + \tfrac{8}{3}x &= 16 \\ y &= 0 \end{aligned}$$

Again, what we simply put all the variables on one side and the constant on the other. We also aren't going to worry about the fact that the third equation doesn't have an *x*. Not only will that not be a problem it actually is going to make this problem a little easier.

Step 3
Now, if we followed the process from the last problem we'd define *u* to be one of the equations and *v* to be the other. Let's do that real quick and see what we get.

Which equation we use to define *u* and *v* technically doesn't matter. However, keep in mind that we're eventually going to have to solve the resulting set of equations for *x* and *y* so let's pick definitions with that in mind.

Let's use the following definitions of *u* and *v* to get our transformation.

$$u = y - 4x \qquad\qquad v = y$$

The second definition is really just renaming *y* as *v*. This isn't a problem and in fact will make the rest of the problem a little easier. Note as well that all were really saying here is that the bottom side of the triangle won't change under this transformation.

Okay, let's think a little bit about what we have here. This will transform $y - 4x = -4$ into $u = -4$ and $y = 0$ into $v = 0$. These two side will form the right angle of the resulting triangle (which will occur where they intersect) and $y + \frac{8}{3}x = 16$ will transform into the hypotenuse of this new triangle. If you aren't sure you believe that then you should actually go through the work to verify it.

So, assuming you either believe us or have done the work to verify that what we've said is true we now have an issue that needs to be addressed.

The problem statement said that the right angle should occur at the origin. Under this transformation however the right angle will occur at $\left(-4, 0\right)$ which means this isn't the transformation we were asked to find unfortunately.

Note that we're not saying that this isn't a valid transformation. It absolutely is a valid, and not a particularly bad, transformation. It simply isn't the transformation we were asked to find and so we can't use it for this problem.

Fixing it however isn't hard and if you really followed how we set this up you might see how to quickly "fix" things up.

Step 4

So, we've decided that the definition of *u* and *v* we got in the last step isn't the one we want for this problem so we need to fix things up a little bit.

The problem with our definition in the previous step was really that our definition of *u* transformed $y - 4x = -4$ into $u = -4$ and in order for the right angle to be at the origin we really need to have a side transform into $u = 0$. Note our definition of *v* is already giving us a side with equation $v = 0$ so we don't need to worry about that one.

So, to fix this let's write the first equation as follows,

$$y - 4x + 4 = 0$$

Seems like a simple enough thing to do here. We simply added 4 to both sides to turn the right side into a zero. With this rewrite we can now see that in order to get a side to transform into the equation $u = 0$ all we need to do is define *u* as follows,

$$u = y - 4x + 4$$

This will transform the first equation into $u = 0$ as we need and if we keep our definition of *v* as $v = y$ the third equation will still transform into $v = 0$ and so the right angle of the new triangle will now be at the origin and the hypotenuse of the new triangle will come from the applying the transformation to the third equation.

Step 5
We now have our definitions of *u* and *v*, given below for reference purposes, so all we need to do is solve the system of equation for *x* and *y*.

$$\begin{aligned} u &= y - 4x + 4 \\ v &= y \end{aligned}$$

In this case, solving for *x* and *y* is really simple. We already know that $y = v$ and so from the first equation we get,

$$u = v - 4x + 4 \qquad \to \qquad x = \tfrac{1}{4}\left(v - u + 4\right)$$

So, it looks like our transformation will be,

$$\begin{aligned} x &= \tfrac{1}{4}\left(v - u + 4\right) \\ y &= v \end{aligned}$$

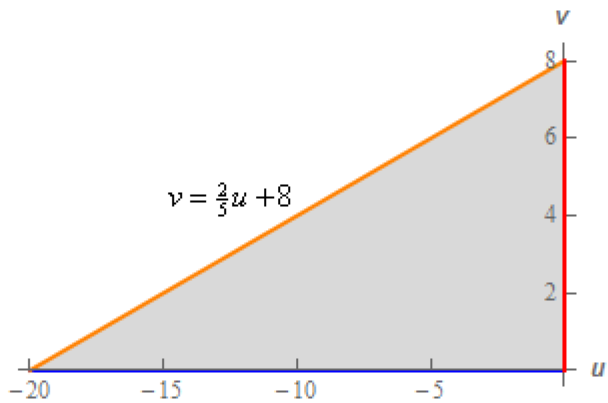
Step 6
Now, technically we've done what the problem asked us to do. However, it probably wouldn't be a bad thing to verify all the claims we made above by actually applying this transformation to the three equations and making sure we do get a right triangle with the right angle at the origin as we claimed.

Theoretically we wouldn't need to apply the transformation to the first and third equation since we used those to define *u* and *v* that in turn gave us our transformation. However, it wouldn't be a bad idea to apply the transformation to them to verify that we get what we expect to get. If we don't then that would mean we probably made a mistake in the previous step.

So, here is the transformation applied to the three equations.

$$\begin{array}{lllll} y = 4x - 4: & v = 4\left(\tfrac{1}{4}\right)\left(v - u + 4\right) - 4 & \to & u = 0 \\ y = -\tfrac{8}{3}x + 16: & v = -\tfrac{8}{3}\left(\tfrac{1}{4}\right)\left(v - u + 4\right) + 16 & \to & v = \tfrac{2}{5}u + 8 \\ y = 0: & v = 0 & & \end{array}$$

Okay, so the first and third equations transformed as expected. Adding in the third equation and we get the following region.



So, the resulting region is what we expected it to be. A right triangle with the right angle occurring at the origin.

Note however, that we did get something (maybe) unexpected here. The original triangle was completely in the 1st quadrant and the "new" triangle is completely in the 2nd quadrant. This kind of thing can happen with these types of problems so don't worry about it when it does.

We got a triangle in the 2nd quadrant there because all the *u*'s ended up being negative. This suggests that maybe if we'd defined *u* to be $u = -\left(y - 4x + 4\right) = -y + 4x - 4$ we might get a triangle in the 1st quadrant instead.

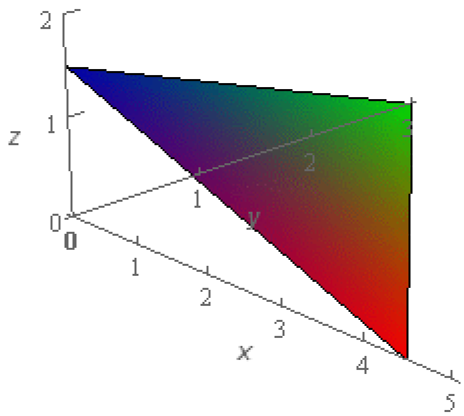
This isn't guaranteed to work in general, but it does work this time. You might want to go through and verify this if you want the practice.

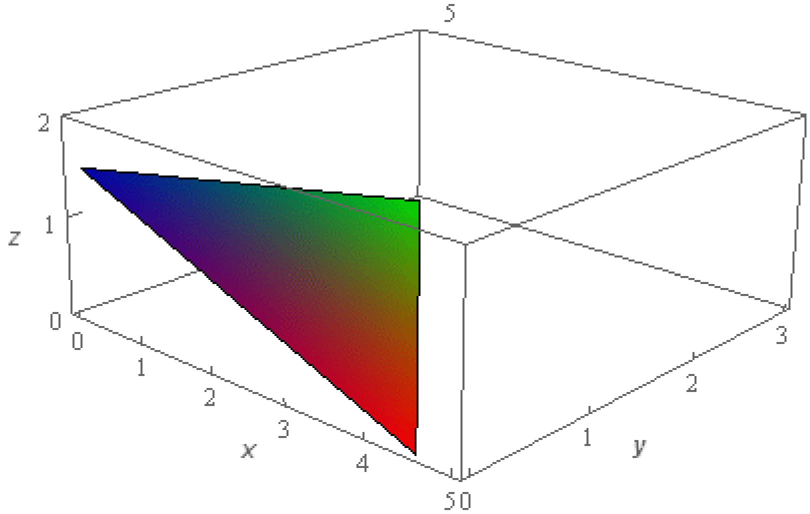
Section 4-9 : Surface Area

1. Determine the surface area of the portion of $2x+3y+6z=9$ that is in the 1st octant.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

Step 2

Let's first set up the integral for the surface area of this surface. This surface doesn't force a region *D* in any of the coordinates planes so we can work with any of them that we want to.

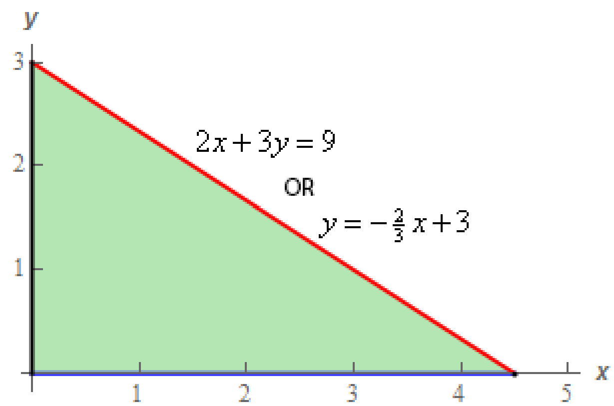
So, let's work with *D* in the *xy*-plane. This, in turn, means we'll first need to solve the equation of the plane for *z* to get,

$$z=\tfrac{3}{2}-\tfrac{1}{3}x-\tfrac{1}{2}y$$

Now, the integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[-\tfrac{1}{3}\right]^2 + \left[-\tfrac{1}{2}\right]^2 + 1}\, dA = \iint\limits_D \sqrt{\tfrac{49}{36}}\, dA = \iint\limits_D \tfrac{7}{6}\, dA$$

Step 3
Now, we'll need to figure out the region *D* we'll need to use in our integral. In this case *D* is simply the triangle in the *xy*-plane that is directly below the plane. Here is a quick sketch of *D*.



The equation of the hypotenuse can be simply found by plugging $z = 0$ into the equation for the plane since it is simply nothing more than where the plane intersects with the *xy*-plane.

Here are a set of limits for this *D*.

$$0 \le x \le \tfrac{9}{2}$$
$$0 \le y \le -\tfrac{2}{3}x + 3$$

Step 4
Now, normally we'd proceed to evaluate the integral at this point and we could do that if we wanted to. However, we don't need to do that in this case.

Let's do a quick rewrite the surface area integral as follows.

$$S = \tfrac{7}{6}\iint\limits_D dA$$

At this point we can see that the integrand of the integral is only 1.

Step 5
Because the integrand is only 1 we can use the fact that the value of this integral is nothing more than the area of *D* and since *D* is just a right triangle we can quickly compute the area of *D*.

The surface area is then,

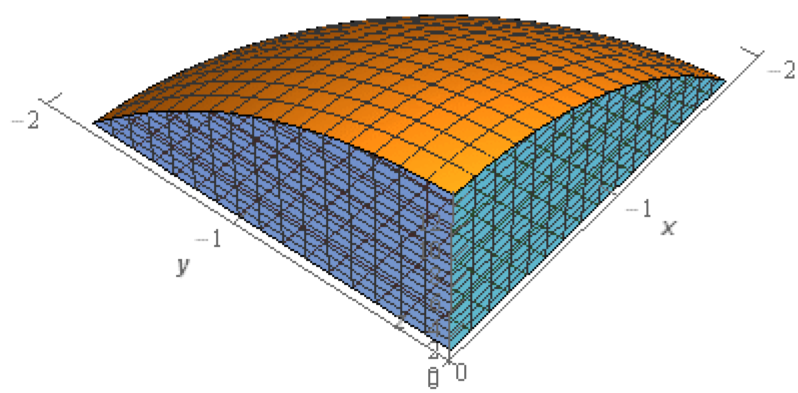
$$S = \tfrac{7}{6}\iint\limits_D dA = \tfrac{7}{6}\left(\text{Area of } D\right) = \tfrac{7}{6}\left(\tfrac{1}{2}\right)\left(\tfrac{9}{2}\right)\left(3\right) = \boxed{\tfrac{63}{8}}$$

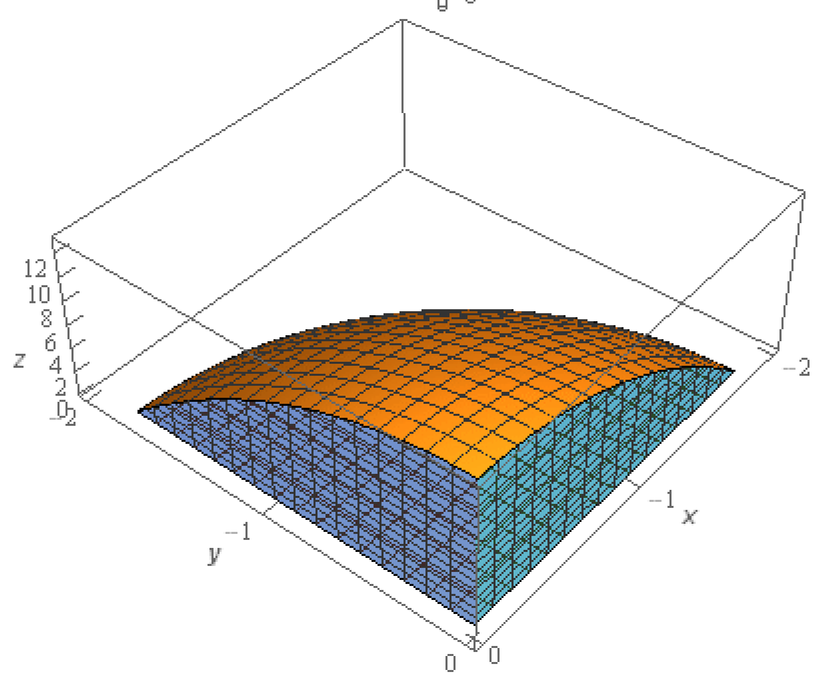
Don't forget this nice little fact about the value of $\iint\limits_D dA$. It doesn't come up often but when it does and *D* is an easy to compute area it can greatly reduce the amount of work for the integral evaluation.

---

2. Determine the surface area of the portion of $z = 13 - 4x^2 - 4y^2$ that is above $z = 1$ with $x \le 0$ and $y \le 0$.

Step 1
Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.





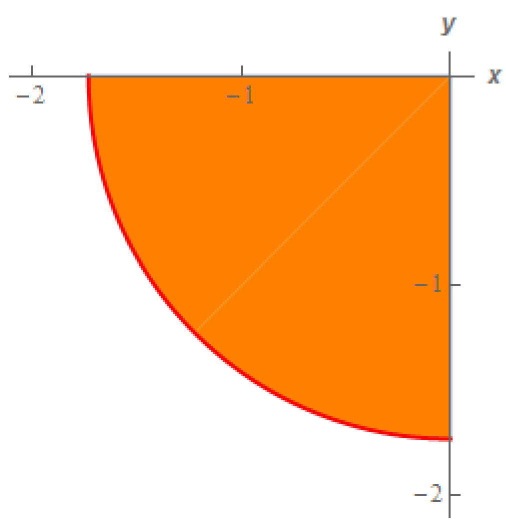
We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

The surface we are after is the portion of the elliptic paraboloid (the orange surface in the sketch) that is above $z = 1$ and in the 3rd quadrant of the *xy*-plane. The bluish "walls" are simply there to provide a frame of reference to help visualize the surface and are not actually part of the surface we are interested in.

Step 2
The integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[-8x\right]^2 + \left[-8y\right]^2 + 1}\,dA = \iint\limits_D \sqrt{64x^2 + 64y^2 + 1}\,dA$$

Step 3
Now, as implied in the last step *D* must be in the *xy*-plane and it should be (hopefully) pretty obvious that it will be a circular region. If you look at the surface from directly above you will see the following region.



To determine equation of the circle all we need to do is set the equation of the elliptic paraboloid equal to $z = 1$ to get,

$$1 = 13 - 4x^2 - 4y^2 \quad \to \quad 4x^2 + 4y^2 = 12 \quad \to \quad x^2 + y^2 = 3$$

So, it is the portion of the circle of radius $\sqrt{3}$ in the 3rd quadrant.

At this point it should also be clear that we'll need to evaluate the integral in terms of polar coordinates. In terms of polar coordinates the limits for *D* are,

$$\pi \le \theta \le \tfrac{3\pi}{2}$$
$$0 \le r \le \sqrt{3}$$

Step 4
Now, let's convert the integral into polar coordinates.

$$S = \iint\limits_D \sqrt{64x^2 + 64y^2 + 1}\, dA = \int_{\pi}^{\frac{3}{2}\pi} \int_0^{\sqrt{3}} r\sqrt{64r^2 + 1}\, dr\, d\theta$$

Don't forget to pick up the extra *r* from converting the *dA* into polar coordinates. If you need a refresher on converting integrals to polar coordinates then you should go back and work some problems from that section.

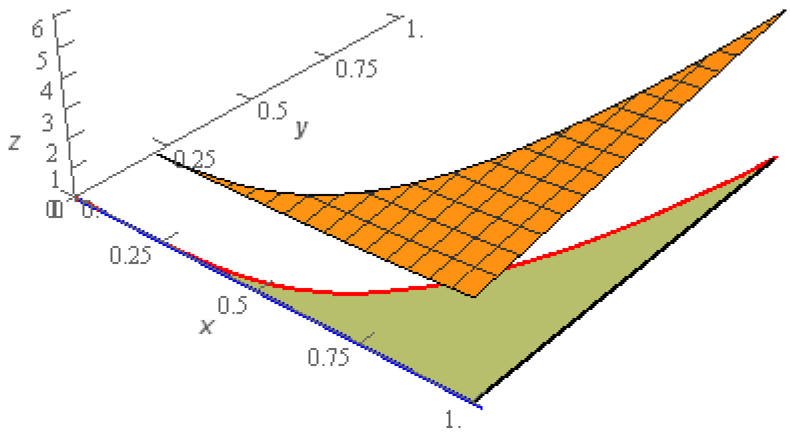
Step 5
Okay, all we need to do then is evaluate the integral.

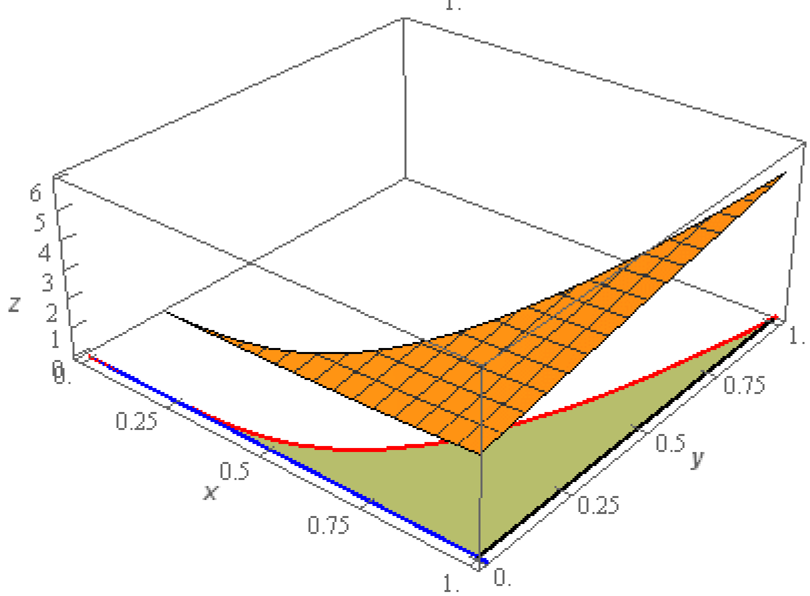
$$\begin{aligned} S &= \int_{\pi}^{\frac{3}{2}\pi} \int_0^{\sqrt{3}} r\sqrt{64r^2 + 1}\, dr\, d\theta \\ &= \int_{\pi}^{\frac{3}{2}\pi} \tfrac{1}{192}\left(64r^2 + 1\right)^{\frac{3}{2}}\Big|_0^{\sqrt{3}}\, d\theta \\ &= \int_{\pi}^{\frac{3}{2}\pi} \tfrac{1}{192}\left(193^{\frac{3}{2}} - 1\right) d\theta \\ &= \tfrac{1}{192}\left(193^{\frac{3}{2}} - 1\right)\theta\Big|_{\pi}^{\frac{3}{2}\pi} = \boxed{\tfrac{\pi}{384}\left(193^{\frac{3}{2}} - 1\right) = 21.9277} \end{aligned}$$

---

3. Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the *xy*-plane bounded by $y = x^5$, $x = 1$ and the *x*-axis.

Step 1
Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

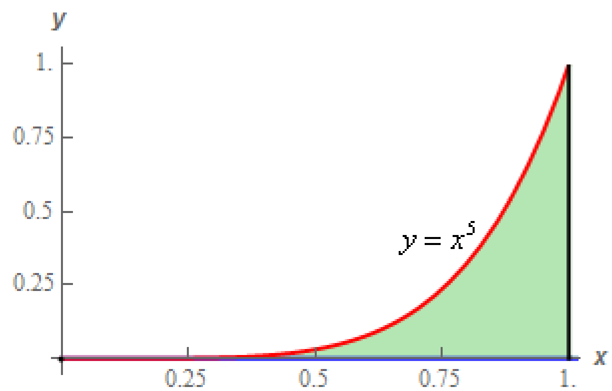
The surface we are after is the orange portion that is above the *xy*-plane and the greenish region in the *xy*-plane is the region over which we are graphing the surface, *i.e.* it is the region *D* we'll use in the integral.

Step 2

The integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[x^3\right]^2 + \left[2\right]^2 + 1}\, dA = \iint\limits_D \sqrt{x^6 + 5}\, dA$$

Step 3
Now, as mentioned in Step 1 the region D is shown in the sketches of the surface. Here is a 2D sketch of D for the sake of completeness.



The limits for this region are,

$$\begin{aligned} &0 \le x \le 1 \\ &0 \le y \le x^5 \end{aligned}$$

Note as well that the integrand pretty much requires us to do the integration in this order.

Step 4
With the limits from Step 3 the integral becomes,

$$S = \iint\limits_D \sqrt{x^6 + 5}\, dA = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx$$

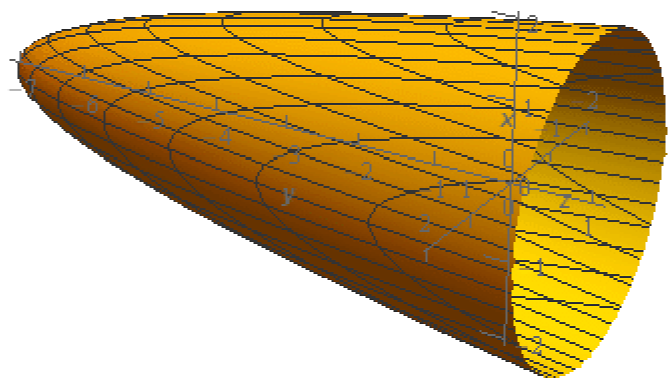
Step 5
Okay, all we need to do then is evaluate the integral.

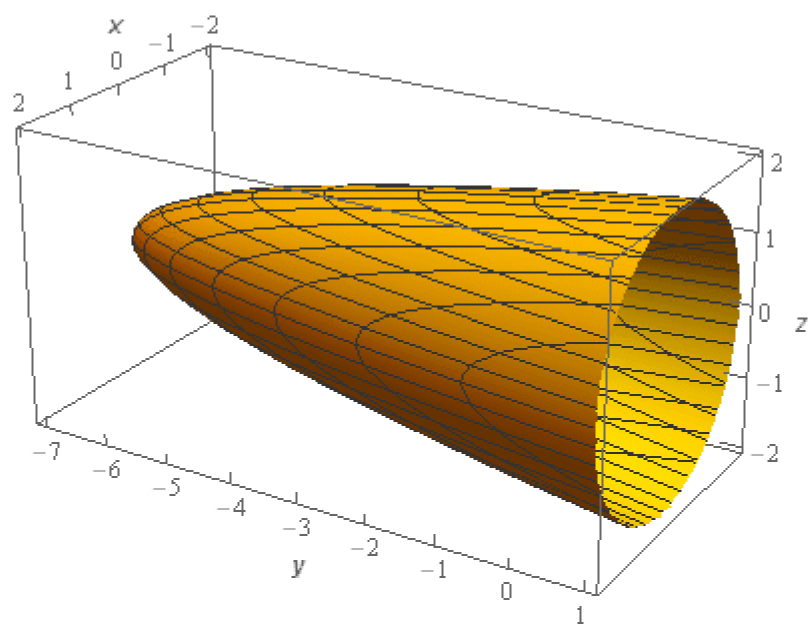
$$\begin{aligned} S &= \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx \\ &= \int_0^1 \left(y\sqrt{x^6 + 5} \right)\Big|_0^{x^5} dx \\ &= \int_0^1 x^5 \sqrt{x^6 + 5}\, dx \\ &= \tfrac{1}{9}\left(x^6 + 5\right)^{\frac{3}{2}}\Big|_0^1 = \boxed{\tfrac{1}{9}\left(6^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = 0.3907} \end{aligned}$$

4. Determine the surface area of the portion of $y = 2x^2 + 2z^2 - 7$ that is inside the cylinder $x^2 + z^2 = 4$.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.





We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

So, we have an elliptic paraboloid centered on the *y*-axis. This also means that the region *D* for our integral will be in the *xz*-plane and we'll be needing polar coordinates for the integral.

Step 2

As noted above the region *D* is going to be in the *xz*-plane and the surface is given in the form $y = f\left(x,z\right)$. The formula for surface area we gave in the notes is only for a region *D* that is in the *xy*-plane with the surface given by $z = f\left(x,y\right)$.

However, it shouldn't be too difficult to see that all we need to do is modify the formula in the following manner to get one for this setup.

$$S = \iint\limits_D \sqrt{\left[f_x\right]^2 + \left[f_z\right]^2 + 1}\,dA \qquad y = f\left(x,z\right) \qquad D \text{ is in the } xz\text{-plane}$$

So, the integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[4x\right]^2 + \left[4z\right]^2 + 1}\,dA = \iint\limits_D \sqrt{16x^2 + 16z^2 + 1}\,dA$$

Step 3
Now, as we noted in Step 1 the region *D* is in the *xz*-plane and because we are after the portion of the elliptical paraboloid that is inside the cylinder given by $x^2 + z^2 = 4$ we can see that the region *D* must therefore be the disk $x^2 + z^2 \le 4$.

Also as noted in Step 1 we'll be needing polar coordinates for this integral so here are the limits for the integral in terms of polar coordinates.

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{gathered}$$

Step 4
Now, let's convert the integral into polar coordinates. However, they won't be the "standard" polar coordinates. Because *D* is in the *xz*-plane we'll need to use the following "modified" polar coordinates.

$$\begin{aligned} x &= r\cos\theta \\ z &= r\sin\theta \\ x^2 + z^2 &= r^2 \end{aligned}$$

Converting the integral to polar coordinates then gives,

$$S = \iint\limits_D \sqrt{16x^2 + 16z^2 + 1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{16r^2 + 1}\,dr\,d\theta$$

Don't forget to pick up the extra *r* from converting the *dA* into polar coordinates. It is the same *dA* as we use for the "standard" polar coordinates.

Step 5
Okay, all we need to do then is evaluate the integral.

$$
\begin{aligned}
S &= \int_0^{2\pi}\int_0^2 r\sqrt{16r^2+1}\,dr\,d\theta \\
&= \int_0^{2\pi} \tfrac{1}{48}\left(16r^2+1\right)^{\frac{3}{2}}\Big|_0^2\,d\theta \\
&= \int_0^{2\pi} \tfrac{1}{48}\left(65^{\frac{3}{2}}-1\right)d\theta \\
&= \tfrac{1}{48}\left(65^{\frac{3}{2}}-1\right)\theta\Big|_0^{2\pi} = \boxed{\tfrac{\pi}{24}\left(65^{\frac{3}{2}}-1\right) = 68.4667}
\end{aligned}
$$

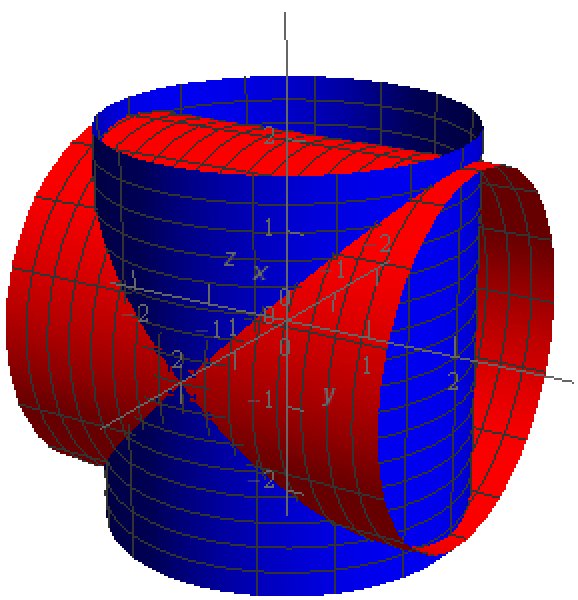
---

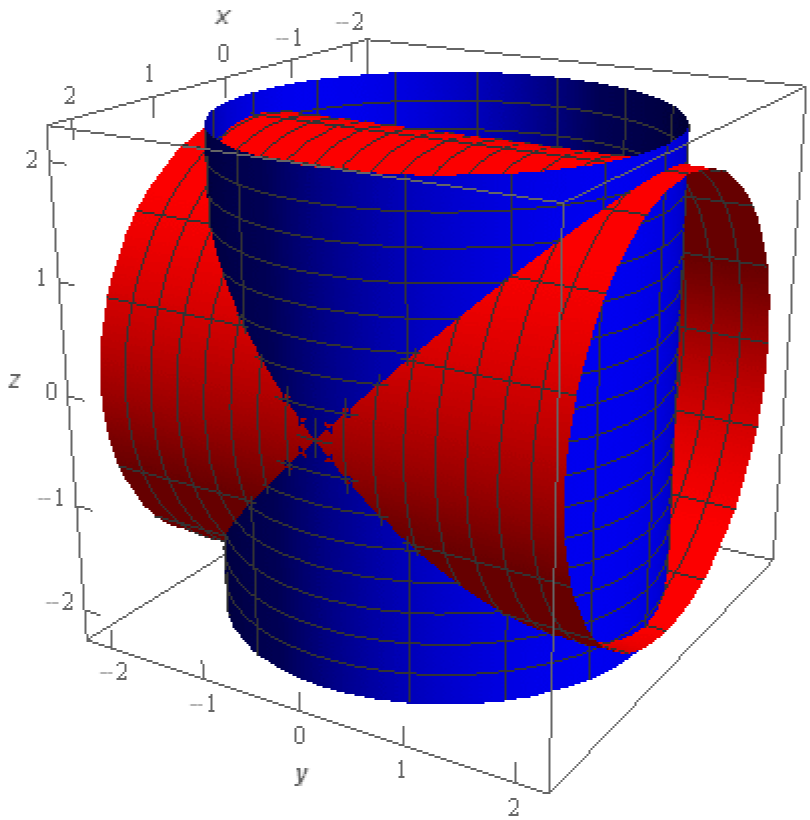
5. Determine the surface area region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Step 1

We looked at the volume of this region back in the Double Integral practice problem section. Now we are going to look at the surface area of this region.

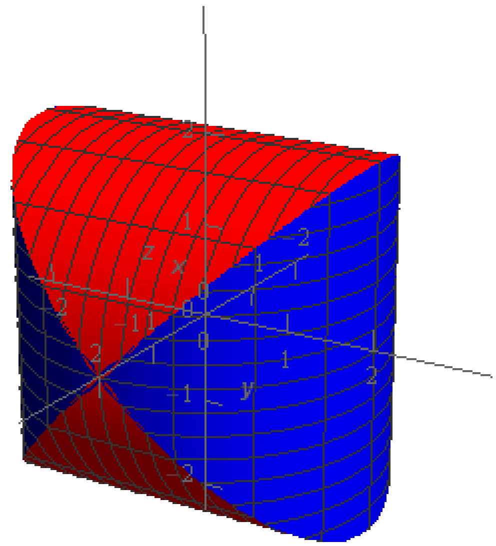
Visualizing the surface/region we're looking at is probably one of the harder parts of this problem. So, let's start off with a sketch of the two cylinders.

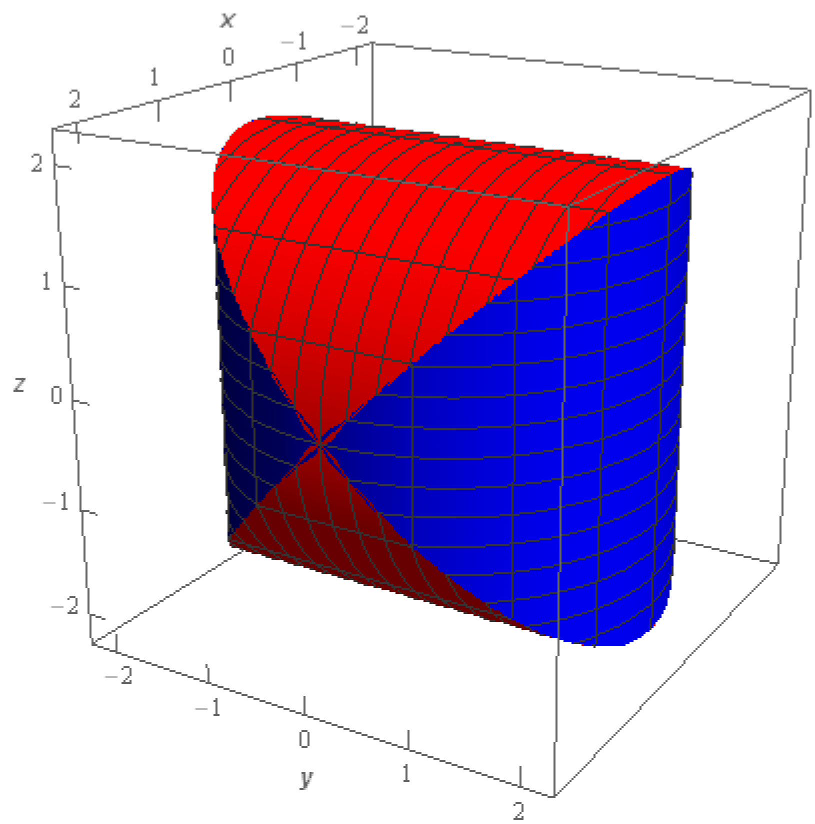




The blue cylinder is the cylinder centered on the *z*-axis (*i.e.* $x^2 + y^2 = 4$) and the red cylinder is the cylinder centered on the *y*-axis (*i.e.* $x^2 + z^2 = 4$).

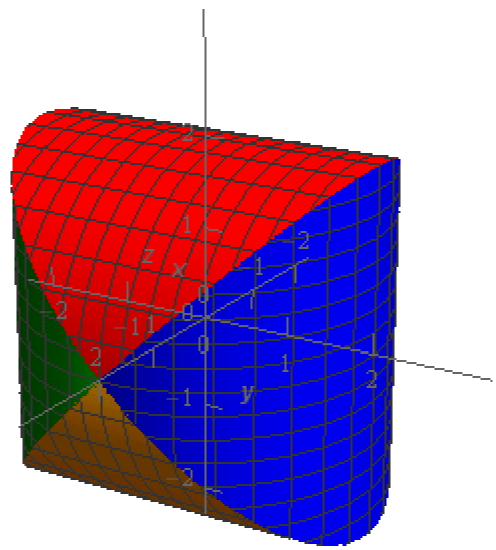
Now, the surface/region we need is the intersection of these two cylinders. So, taking away the "extra" bits of the cylinders that are outside of the intersection we get the following sketches.

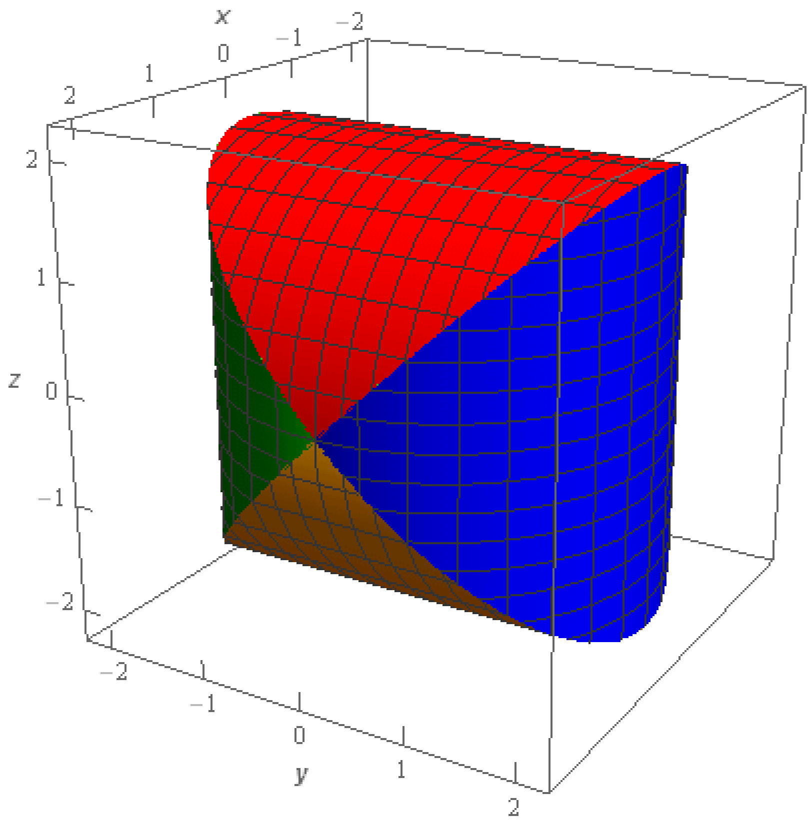

z
x
y


x
y
z

The blue part of this sketch is the portion of $x^2 + y^2 = 4$ that is inside of $x^2 + z^2 = 4$ and the red part of this sketch is the portion of $x^2 + z^2 = 4$ that is inside of $x^2 + y^2 = 4$. Taken together we get the surface/region we are interested in.

Now, when we did the volume of this region the sketches above were sufficient for our purposes. This time let's do one more set of sketches.





In this sketch the blue and green parts of the sketch are the right and left portions of $x^2 + y^2 = 4$ respectively and the red and yellow parts of the sketch are the upper and lower portion of $x^2 + z^2 = 4$ respectively.

Step 2
Now, we need to set up an integral that will give the surface area of this surface/region.

Let's first notice that by symmetry the red and yellow surfaces must have the same surface area since they are simply the upper/lower portion of the same cylinder. Likewise the blue and green surfaces will have the same surface area.

Next, because both of the cylinders that make up this surface have the same radius the upper/lower and right/left portions also must have the same surface area.

So, this means that all we need to do is get the surface area of one of the four surfaces shown above and we can them get the full surface area by multiplying that number by four.

So, let's get the surface area of the red portion of the sketch above. To do this we'll need to solve $x^2 + z^2 = 4$ for *z* to get,

$$z = \pm\sqrt{4 - x^2}$$

The "+" equation will give the red portion of the sketch and the "-" equation will give the yellow portion of the sketch.

So, let's set up the integral that will give the surface area of the red portion. The integral is,

$$S = \iint\limits_D \sqrt{\left[f_x\right]^2 + \left[f_y\right]^2 + 1}\,dA = \iint\limits_D \sqrt{\left[\frac{-x}{\sqrt{4-x^2}}\right]^2 + \left[0\right]^2 + 1}\,dA$$

$$= \iint\limits_D \sqrt{\frac{x^2}{4-x^2} + 1}\,dA = \iint\limits_D \sqrt{\frac{x^2 + \left(4-x^2\right)}{4-x^2}}\,dA = \iint\limits_D \sqrt{\frac{4}{4-x^2}}\,dA$$

$$= \iint\limits_D \frac{2}{\sqrt{4-x^2}}\,dA$$

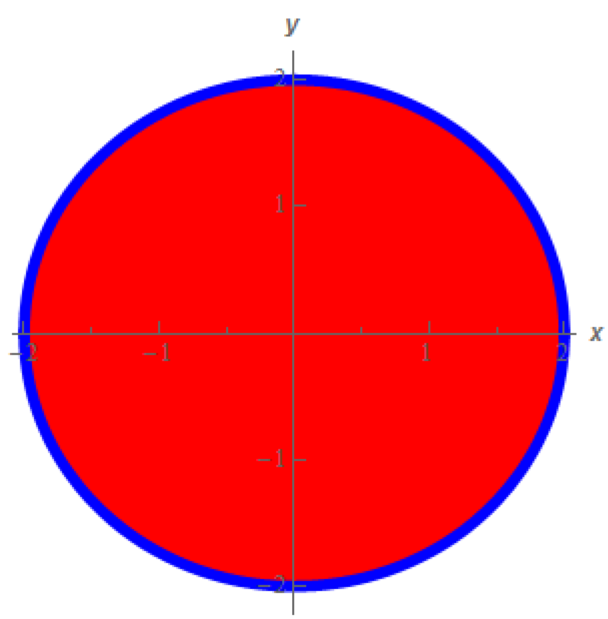
Do not get excited about the fact that one of the derivatives in the formula ended up being zero. This will happen on occasion so we don't want to get excited about it when it happens.

Note as well that we did a fair amount of simplification. Notably, after taking the x derivative and squaring we set up the terms under the radical to have the same dominator. This in turn gave us a single rational expression under the radical that we could easily take the root of both the numerator and denominator.

As we will eventually see all this simplification will make the integral evaluation work a lot simpler.

Step 3

Now, we need to determine the region D for our integral. If we looked at the surface from directly above (*i.e.* down along the z-axis) we see the following figure.



The blue circle is in fact the cylinder $x^2 + y^2 = 4$ and the red area is the upper portion of $x^2 + z^2 = 4$ that is inside $x^2 + y^2 = 4$.

Now, in general, seeing this region would (and probably should) suggest that we use polar coordinates to do the integral. However, while using polar coordinates to do this integral wouldn't be that difficult, (you might want to do it for the practice) it actually turns out to be easier (in this case!) to use Cartesian coordinates.

In terms of Cartesian coordinates let's use the following limits for the red disk.

$$-2 \le x \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

For the *y* limits we are just going from the lower portion of the circle to the upper portion of the circle.

Step 4
Okay, let's now set up the integral.

$$S = \iint\limits_D \frac{2}{\sqrt{4-x^2}}\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dy\,dx$$

Step 5
Next, let's evaluate the integral.

$$\begin{aligned} S &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dy\,dx \\ &= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}}\, y\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx \\ &= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}}\left(\sqrt{4-x^2} - \left(-\sqrt{4-x^2}\right)\right)dx \\ &= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}}\left(2\sqrt{4-x^2}\right)dx \\ & \int_{-2}^{2} 4\,dx \\ &= 4x\Big|_{-2}^{2} = \boxed{16} \end{aligned}$$

We put in a few more steps that absolutely required here to make everything clear but you can see that keeping the integral in terms of Cartesian coordinates here was actually quite simple.

It just goes to show that there really are no hard and fast rules about doing things like converting to polar coordinates versus keeping things in Cartesian coordinates. While we are often taught to see a circle and just immediately convert to polar coordinates this problem has shown that there are the occasional problem that doesn't require that to be done!

Step 6
Okay, finally let's get the surface area. So much of this problem was focused on what we did in the last couple of steps that it is easy to forget about just what we were doing.

Recall that the answer from Step 5 was just the surface area of the red portion of our surface. The full surface area is four times this and so the surface area of the whole surface is 16x4 = **64**.

Section 4-10 : Area and Volume Revisited

The intent of the section was just to “recap” the various area and volume formulas from this chapter and so no problems have been (or likely will be in the near future) written.

Chapter 5 : Line Integrals

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Vector Fields – In this section we introduce the concept of a vector field and give several examples of graphing them. We also revisit the gradient that we first saw a few chapters ago.

Line Integrals – Part I – In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length.

Line Integrals – Part II – In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to x, y, and/or z. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.

Line Integrals of Vector Fields – In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to x, y and z.

Fundamental Theorem for Line Integrals – In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.

Conservative Vector Fields – In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.

Green's Theorem – In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Section 5-1 : Vector Fields

1. Sketch the vector field for $\vec{F}\left(x,y\right) = 2x\,\vec{i} - 2\,\vec{j}$.

Step 1

Recall that the graph of a vector field is simply sketching the vectors at specific points for a whole bunch of points. This makes sketching vector fields both simple and difficult. It is simple to compute the vectors and sketch them, but it is difficult to know just which points to pick and how many points to pick so we get a good sketch.

So, let's start off with just computing some vectors at specific points.

$$\vec{F}\left(0,-1\right) = -2\,\vec{j} \qquad \vec{F}\left(1,1\right) = 2\,\vec{i} - 2\,\vec{j} \qquad \vec{F}\left(-1,0\right) = -2\,\vec{i} - 2\,\vec{j}$$

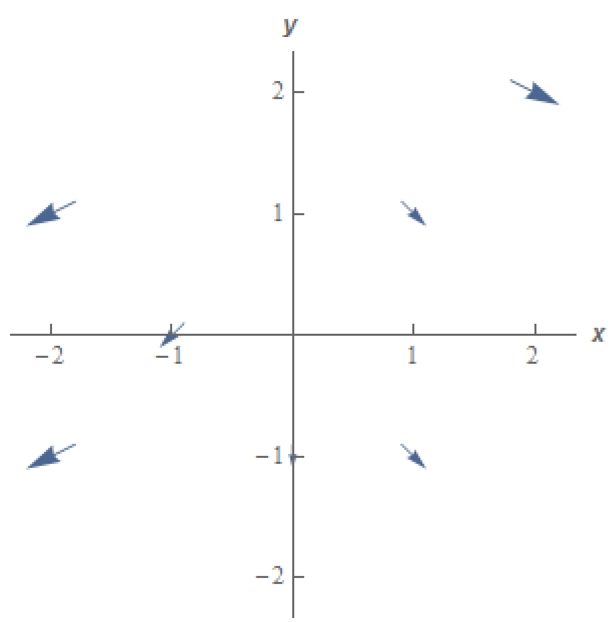
$$\vec{F}\left(-2,-1\right) = -4\,\vec{i} - 2\,\vec{j} \qquad \vec{F}\left(1,-1\right) = 2\,\vec{i} - 2\,\vec{j} \qquad \vec{F}\left(2,2\right) = 4\,\vec{i} - 2\,\vec{j}$$

$$\vec{F}\left(-2,1\right) = -4\,\vec{i} - 2\,\vec{j}$$

Step 2

Now we need to "sketch" each of these vectors at the point that generated them. For example at the point $\left(0,-1\right)$ we'll sketch the vector $-2\,\vec{j}$.

Here is the sketch of these vectors.

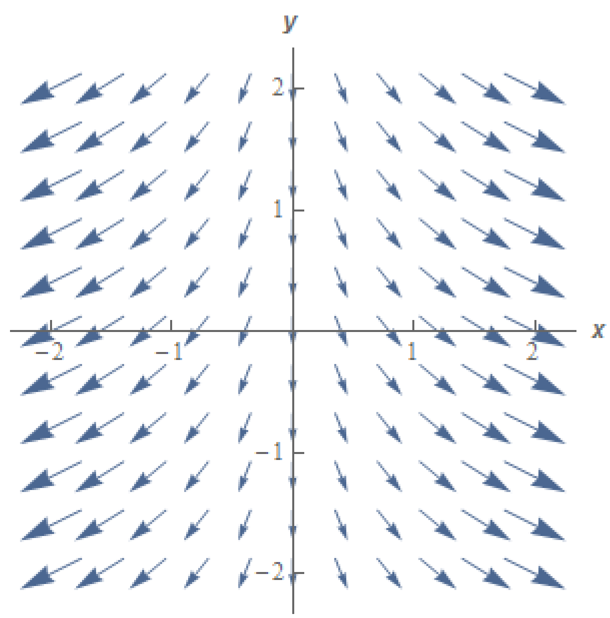


In the sketch above we didn't sketch each of these vectors to scale. In other words we just sketched vectors in the same direction as the indicated vector rather than sketching the vector with "correct" magnitude. The reason for this is to keep the sketch a little easier to see. If we sketched all the vectors to scale we'd just see a mess of overlapping arrows that would be hard to really see what was going on.

Note as well that with the few vectors that we sketched it's difficult to get a real feel for what is going on at any random point. With this sketch we might be able to see some trends but we don't know that those trends will continue into "blank" regions that don't have any vectors sketched in them.

Step 3
Below is a better sketch of the vector field with many more vectors sketched in. We got this sketch by letting a computer just plot quite a few points by itself without actually picking any of them as we did in the previous step.

In general, this is how vector fields are sketched. Computing this number of vectors by hand would so time consuming that it just wouldn't be worth it. Computers however can do all those computations very quickly and so we generally just let them do the sketch.



---

2. Sketch the vector field for $\vec{F}(x,y) = (y-1)\vec{i} + (x+y)\vec{j}$.

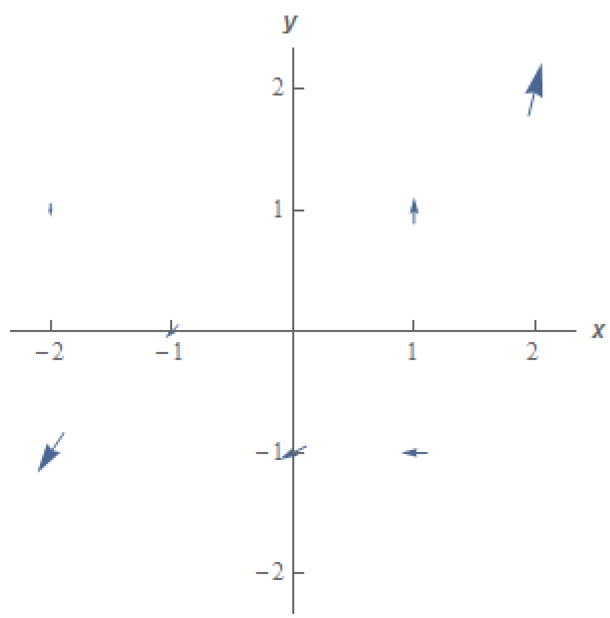
Step 1
Recall that the graph of a vector field is simply sketching the vectors at specific points for a whole bunch of points. This makes sketching vector fields both simple and difficult. It is simple to compute the vectors and sketch them, but it is difficult to know just which points to pick and how many points to pick so we get a good sketch.

So, let's start off with just computing some vectors at specific points.

$$\vec{F}(0,-1) = -2\vec{i} - \vec{j} \qquad \vec{F}(1,1) = 2\vec{j} \qquad \vec{F}(-1,0) = -\vec{i} - \vec{j}$$
$$\vec{F}(-2,-1) = -2\vec{i} - 3\vec{j} \qquad \vec{F}(1,-1) = -2\vec{i} \qquad \vec{F}(2,2) = \vec{i} + 4\vec{j}$$
$$\vec{F}(-2,1) = -\vec{j}$$

Step 2
Now we need to "sketch" each of these vectors at the point that generated them. For example at the point $(0,-1)$ we'll sketch the vector $-2\vec{i} - \vec{j}$.

Here is the sketch of these vectors.

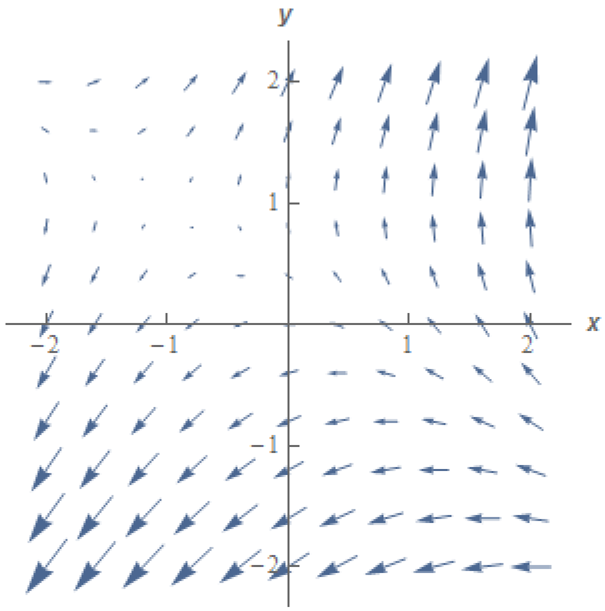


In the sketch above we didn't sketch each of these vectors to scale. In other words we just sketched vectors in the same direction as the indicated vector rather than sketching the vector with "correct" magnitude. The reason for this is to keep the sketch a little easier to see. If we sketched all the vectors to scale we'd just see a mess of overlapping arrows that would be hard to really see what was going on.

Note as well that with the few vectors that we sketched it's difficult to get a real feel for what is going on at any random point let along any trends in the vector field.

Step 3
Below is a better sketch of the vector field with many more vectors sketched in. We got this sketch by letting a computer just plot quite a few points by itself without actually picking any of them as we did in the previous step.

In general, this is how vector fields are sketched. Computing this number of vectors by hand would so time consuming that it just wouldn't be worth it. Computers however can do all those computations very quickly and so we generally just let them do the sketch.



3. Compute the gradient vector field for $f\left(x,y\right)=y^2\cos\left(2x-y\right)$.

Solution

There really isn't a lot to do for this problem. Here is the gradient vector field for this function.

$$\boxed{\nabla f=\left\langle -2y^2\sin\left(2x-y\right),2y\cos\left(2x-y\right)+y^2\sin\left(2x-y\right)\right\rangle}$$

Don't forget to compute partial derivatives for each of these! The first term is the derivative of the function with respect to *x* and the second term is the derivative of the function with respect to *y*.

4. Compute the gradient vector field for $f\left(x,y,z\right)=z^2\mathbf{e}^{x^2+4y}+\ln\left(\frac{xy}{z}\right)$.

Solution

There really isn't a lot to do for this problem. Here is the gradient vector field for this function.

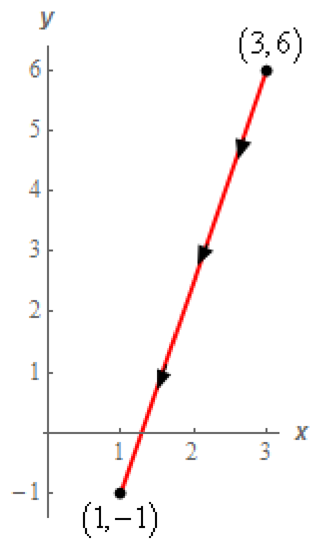
$$\boxed{\nabla f=\left\langle 2xz^2\mathbf{e}^{x^2+4y}+\frac{1}{x},4z^2\mathbf{e}^{x^2+4y}+\frac{1}{y},2z\mathbf{e}^{x^2+4y}-\frac{1}{z}\right\rangle}$$

Don't forget to compute partial derivatives for each of these! The first term is the derivative of the function with respect to x, the second term is the derivative of the function with respect to y and the third term is the derivative of the function with respect to z.

Section 5-2 : Line Integrals - Part I

1. Evaluate $\int\limits_C 3x^2 - 2y\,ds$ where *C* is the line segment from $\left(3,6\right)$ to $\left(1,-1\right)$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Now, with the specified direction we can see that *x* is decreasing as we move along the curve in the specified direction. This means that we can't just determine the equation of the line and use that to work the problem. Using the equation of the line would require us to use increasing *x* since the limits in the integral must go from smaller to larger value.

We could of course use the fact from the notes that relates the line integral with a specified direction and the line integral with the opposite direction to allow us to use the equation of the line. However, for this problem let's just work with problem without the fact to make sure we can do that type of problem

So, we'll need to parameterize this line and we know how to parameterize the equation of a line between two points. Here is the vector form of the parameterization of the line.

$$\vec r\left(t\right) = \left(1-t\right)\left\langle 3,6\right\rangle + t\left\langle 1,-1\right\rangle = \left\langle 3-2t,6-7t\right\rangle \qquad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 3-2t \\ y &= 6-7t \end{aligned} \qquad 0 \le t \le 1$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3
We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}\,'(t) = \langle -2, -7 \rangle \qquad\qquad \left\| \vec{r}\,'(t) \right\| = \sqrt{(-2)^2 + (-7)^2} = \sqrt{53}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the *x*/*y* in the integrand with the *x*/*y* from parameterization. Here is the integrand evaluated at the parameterization.

$$3x^2 - 2y = 3(3-2t)^2 - 2(6-7t) = 3(3-2t)^2 - 12 + 14t$$

Step 4
The line integral is then,

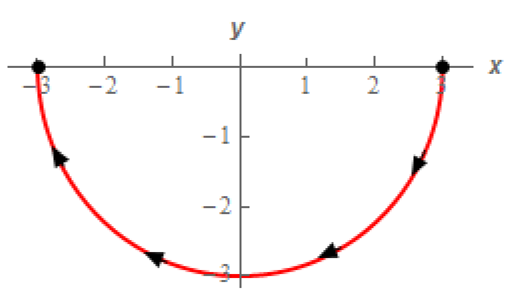
$$\begin{aligned}\int_C 3x^2 - 2y\,ds &= \int_0^1 \left(3(3-2t)^2 - 12 + 14t\right)\sqrt{53}\,dt \\ &= \sqrt{53}\left[-\tfrac{1}{2}(3-2t)^3 - 12t + 7t^2\right]\Big|_0^1 = \boxed{8\sqrt{53}}\end{aligned}$$

Note that we didn't multiply out the first term in the integrand as we could do a quick substitution to do the integral.

---

2. Evaluate $\int\limits_C 2yx^2 - 4x\,ds$ where *C* is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Here is a parameterization for this curve.

$$\vec{r}\left(t\right)=\left\langle 3\cos\left(t\right),-3\sin\left(t\right)\right\rangle \qquad 0\le t\le \pi$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 3\cos\left(t\right) \\ y &= -3\sin\left(t\right) \end{aligned} \qquad 0\le t\le \pi$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3
We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\begin{aligned} \vec{r}\,'\left(t\right) &= \left\langle -3\sin\left(t\right),-3\cos\left(t\right)\right\rangle \\ \left\|\vec{r}\,'\left(t\right)\right\| &= \sqrt{\left(-3\sin\left(t\right)\right)^2+\left(-3\cos\left(t\right)\right)^2} \\ &= \sqrt{9\sin^2\left(t\right)+9\cos^2\left(t\right)}=\sqrt{9\left(\sin^2\left(t\right)+\cos^2\left(t\right)\right)}=\sqrt{9}=3 \end{aligned}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the *x*/*y* in the integrand with the *x*/*y* from parameterization. Here is the integrand evaluated at the parameterization.

$$2yx^2-4x=2\left(-3\sin\left(t\right)\right)\left(3\cos\left(t\right)\right)^2-4\left(3\cos\left(t\right)\right)=-54\sin\left(t\right)\cos^2\left(t\right)-12\cos\left(t\right)$$

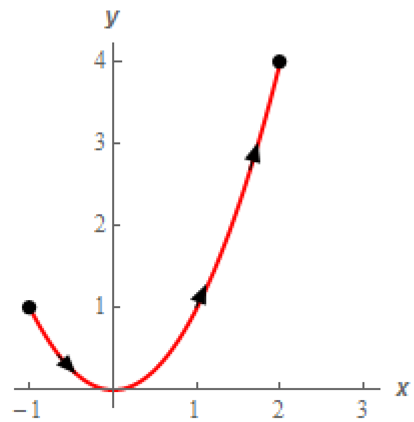
Step 4
The line integral is then,

$$\begin{aligned} \int\limits_C 2yx^2-4x\,ds &= \int_0^\pi \left(-54\sin\left(t\right)\cos^2\left(t\right)-12\cos\left(t\right)\right)3\,dt \\ &= 3\left[18\cos^3\left(t\right)-12\sin\left(t\right)\right]\Big|_0^\pi = \boxed{-108} \end{aligned}$$

---

3. Evaluate $\int\limits_C 6x\,ds$ where *C* is the portion of $y=x^2$ from $x=-1$ to $x=2$. The direction of *C* is in the direction of increasing *x*.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
In this case we can just use the equation of the curve for the parameterization because the specified direction is going in the direction of increasing *x* which will give us integral limits from smaller value to larger value as needed. Here is a parameterization for this curve.

$$\vec r\left(t\right)=\left\langle t,t^2\right\rangle \qquad -1\le t\le 2$$

We could also break this up into parameter form as follows.

$$\begin{array}{l}x=t\\y=t^2\end{array}\qquad -1\le t\le 2$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3
We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec r\,'\left(t\right)=\left\langle 1,2t\right\rangle \qquad\qquad \left\|\vec r\,'\left(t\right)\right\|=\sqrt{\left(1\right)^2+\left(2t\right)^2}=\sqrt{1+4t^2}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the *x*/*y* in the integrand with the *x*/*y* from parameterization. Here is the integrand evaluated at the parameterization.

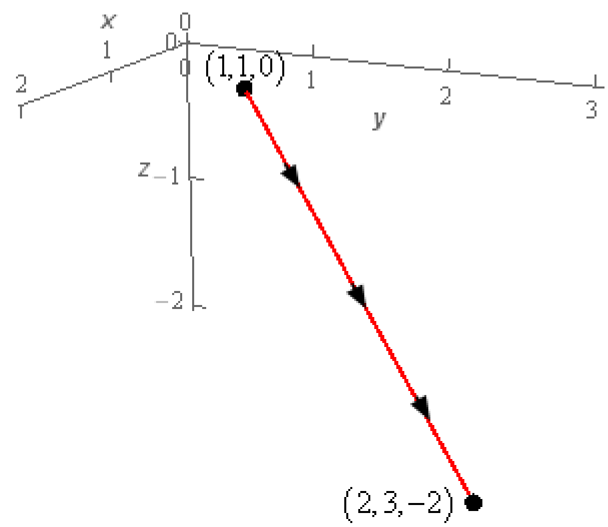
$$6x=6t$$

Step 4
The line integral is then,

$$\int_C 6x\,ds=\int_{-1}^{2}6t\sqrt{1+4t^2}\,dt=\tfrac{1}{2}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_{-1}^{2}=\boxed{\tfrac{1}{2}\left(17^{\frac{3}{2}}-5^{\frac{3}{2}}\right)}$$

4. Evaluate $\int\limits_C xy - 4z\, ds$ where *C* is the line segment from $(1,1,0)$ to $(2,3,-2)$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
We know how to get the parameterization of a line segment so let's just jump straight into the parameterization of the line segment.

$$\vec{r}(t) = (1-t)\langle 1,1,0\rangle + t\langle 2,3,-2\rangle = \langle 1+t, 1+2t, -2t\rangle \qquad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 1+t \\ y &= 1+2t \qquad 0 \le t \le 1 \\ z &= -2t \end{aligned}$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3
We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}\,'(t) = \langle 1,2,-2\rangle \qquad\qquad \left\|\vec{r}\,'(t)\right\| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the *x*/*y* in the integrand with the *x*/*y* from parameterization. Here is the integrand evaluated at the parameterization.

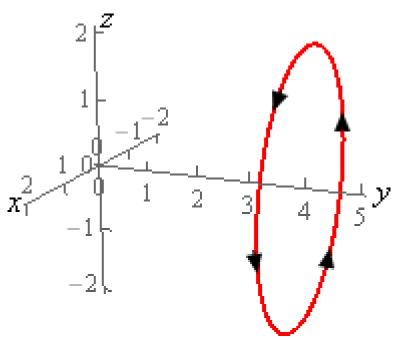
$$xy - 4z = (1+t)(1+2t) - 4(-2t) = 2t^2 + 11t + 1$$

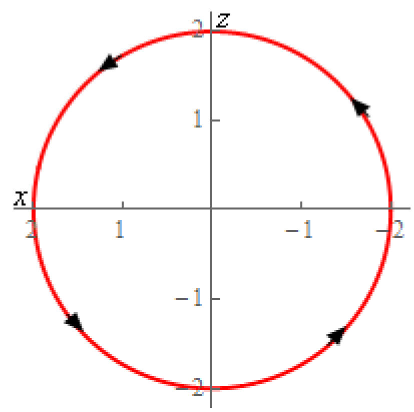
Step 4
The line integral is then,

$$\int_C xy - 4z\,ds = \int_0^1 (2t^2 + 11t + 1)(3)\,dt = 3\left(\tfrac{2}{3}t^3 + \tfrac{11}{2}t^2 + t\right)\Big|_0^1 = \boxed{\tfrac{43}{2}}$$

---

5. Evaluate $\int_C x^2 y^2\, ds$ where *C* is the circle centered at the origin of radius 2 centered on the *y*-axis at $y = 4$. See the sketches below for orientation. Note the "odd" axis orientation on the 2D circle is intentionally that way to match the 3D axis the direction.





Step 1
Before we parameterize the curve note that the "orientation" of the *x*-axis in the 2D sketch above is backwards from what we are used to. In this sketch the positive *x*-axis is to the left and the negative *x*-axis is to the right. This was done to match up with the 3D image.

If we were on the positive *y*-axis (on the 3D image of course) past $y = 4$ and looking towards the origin we would see the 2D sketch. Generating the 2D sketch in this manner will help to make sure that our parameterization has the correct direction.

Speaking of which, here is the parameterization of the curve.

$$\vec{r}(t) = \left\langle 2\cos(t), 4, -2\sin(t) \right\rangle \qquad 0 \le t \le 2\pi$$

If you check the parameterization at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ we can see that we will get the correct (x, z) coordinates for the curve in the 2D sketch and hence we will also get the correct coordinates for the curve in the 3D sketch.

Don't forget that we also need to acknowledge in our parameterization that we are at $y = 4$, *i.e.* the second component of the parameterization. When one of the coordinates on a 3D curve is constant it is often easy to forget to deal with it in the parameterization.

Step 2
We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\begin{aligned} \vec{r}\,'(t) &= \langle -2\sin(t), 0, -2\cos(t) \rangle \\ \left\| \vec{r}\,'(t) \right\| &= \sqrt{\left(-2\sin(t)\right)^2 + (0)^2 + \left(-2\cos(t)\right)^2} \\ &= \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4\left(\sin^2(t) + \cos^2(t)\right)} = \sqrt{4} = 2 \end{aligned}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the *x/y/z* in the integrand with the *x/y/z* from parameterization. Here is the integrand evaluated at the parameterization.

$$x^2 y^2 = \left(2\cos(t)\right)^2 (4)^2 = 64\cos^2(t)$$

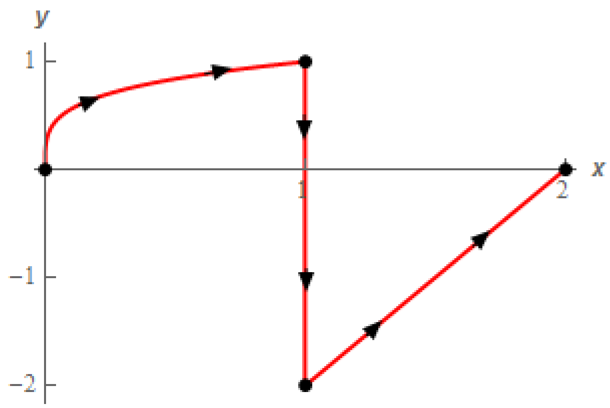
Step 3
The line integral is then,

$$\begin{aligned} \int_C x^2 y^2 \, ds &= \int_0^{2\pi} \left(64\cos^2(t)\right)(2)\,dt = \int_0^{2\pi} 64\left(1 + \cos(2t)\right)dt \\ &= 64\left(t + \tfrac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{128\pi} \end{aligned}$$

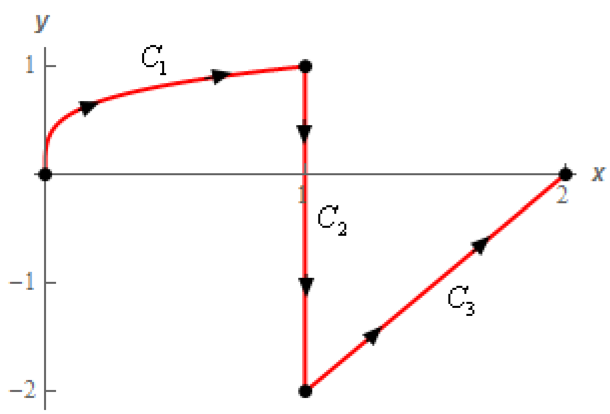
You do recall how to use the double angle formula for cosine to evaluate this integral correct? We'll be seeing a fair number of integrals involving trig functions in this chapter and knowing how to do these kinds of integrals will be important.

---

6. Evaluate $\int\limits_C 16y^5 \, ds$ where *C* is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1,1)$ to $(1,-2)$ which in turn is followed by the line segment from $(1,-2)$ to $(2,0)$. See the sketch below for the direction.



Step 1
To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1 : \vec{r}\left(t\right) = \left\langle t^4, t \right\rangle \qquad 0 \le t \le 1$$

$$C_2 : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 1,1 \right\rangle + t\left\langle 1,-2 \right\rangle = \left\langle 1, 1-3t \right\rangle \qquad 0 \le t \le 1$$

$$C_3 : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 1,-2 \right\rangle + t\left\langle 2,0 \right\rangle = \left\langle 1+t, 2t-2 \right\rangle \qquad 0 \le t \le 1$$

For C_2 we had to use the vector form for the line segment between two points instead of the equation for the line (which is much simpler of course) because the direction was in the decreasing *y* direction and the limits on our integral must be from smaller to larger. We could have used the fact from the notes that tells us how the line integrals for the two directions related to allow us to use the equation of the line if we'd wanted to. We decided to do it this way just for the practice of dealing with the vector form for the line segment and it's not all that difficult to deal with the result and the limits are "nicer".

Note as well that for C_3 we could have solved for the equation of the line and used that because the direction is in the increasing *x* direction. However, the vector form for the line segment between two points is just as easy to use so we used that instead.

Step 2
Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\int_{C_1} 16y^5\,ds = \int_0^1 16(t)^5\sqrt{\left(4t^3\right)^2+(1)^2}\,dt = \int_0^1 16t^5\sqrt{16t^6+1}\,dt$$

$$= \tfrac{1}{9}\left(16t^6+1\right)^{\frac{3}{2}}\Big|_0^1 = \underline{\tfrac{1}{9}\left(17^{\frac{3}{2}}-1\right) = 7.6770}$$

$$\int_{C_2} 16y^5\,ds = \int_0^1 16(1-3t)^5\sqrt{(0)^2+(-3)^2}\,dt = \int_0^1 48(1-3t)^5\,dt$$

$$= -\tfrac{8}{3}(1-3t)^6\Big|_0^1 = \underline{-168}$$

$$\int_{C_3} 16y^5\,ds = \int_0^1 16(2t-2)^5\sqrt{(1)^2+(2)^2}\,dt = \int_0^1 16\sqrt{5}\,(2t-2)^5\,dt$$

$$= \tfrac{4\sqrt{5}}{3}(2t-2)^6\Big|_0^1 = \underline{-\tfrac{256\sqrt{5}}{3} = -190.8111}$$

Step 3
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

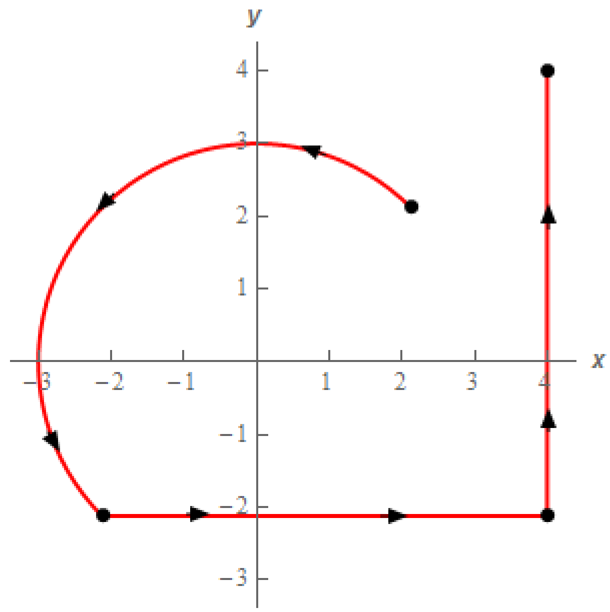
$$\int_C 16y^5\,ds = \left(\tfrac{1}{9}\left(17^{\frac{3}{2}}+1\right)\right)+(-168)+\left(-\tfrac{256\sqrt{5}}{3}\right) = \boxed{-351.1341}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

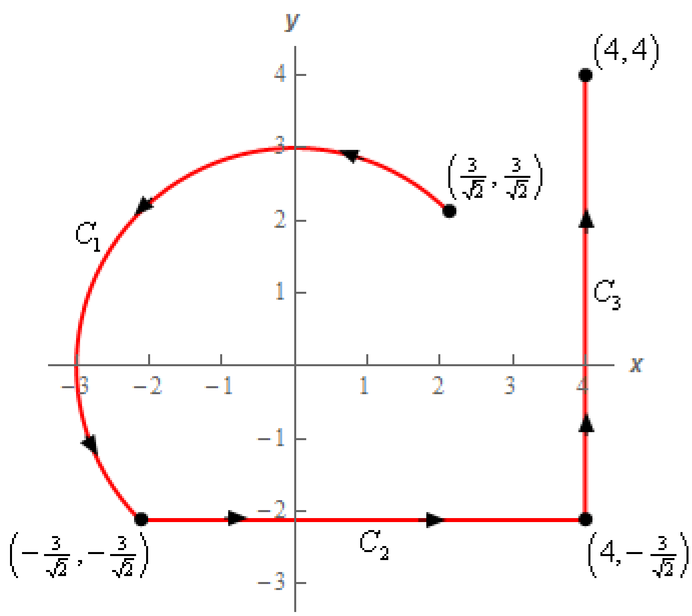
---

7. Evaluate $\int_C 4y - x\,ds$ where *C* is the upper portion of the circle centered at the origin of radius 3 from $\left(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ to $\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ in the counter clockwise rotation followed by the line segment form

$\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ to $\left(4,-\frac{3}{\sqrt{2}}\right)$ which in turn is followed by the line segment from $\left(4,-\frac{3}{\sqrt{2}}\right)$ to $\left(4,4\right)$. See the sketch below for the direction.



Step 1
To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1 : \vec r\left(t\right) = \left\langle 3\cos\left(t\right), 3\sin\left(t\right)\right\rangle \qquad \tfrac{1}{4}\pi \le t \le \tfrac{5}{4}\pi$$

$C_2 : \vec{r}(t) = \left\langle t, -\frac{3}{\sqrt{2}} \right\rangle \qquad -\frac{3}{\sqrt{2}} \le t \le 4$

$C_3 : \vec{r}(t) = \left\langle 4, t \right\rangle \qquad -\frac{3}{\sqrt{2}} \le t \le 4$

The limits for C_1 are actually pretty easy to find. At the starting point, $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, we know that we must have,

$$3\cos(t) = 3\sin(t) = \tfrac{3}{\sqrt{2}} \qquad \to \qquad \cos(t) = \sin(t) = \tfrac{1}{\sqrt{2}} \qquad \to \qquad \tan(t) = 1$$

Since the starting point of C_1 is in the 1st quadrant we know that we must have $t = \frac{1}{4}\pi$.

We can do a similar argument of the final point, $\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$. This point is in the 3rd quadrant and again we'll have $\tan(t) = 1$. Therefore we must have $t = \frac{5}{4}\pi$ at this point.

For C_2 and C_3 we just used the equations of the lines since they were horizontal and vertical respectively and followed increasing *x* and *y* respectively. The limits for them are a little unpleasant but we can't do anything about the fact that there will be messy numbers in these no matter how we do them.

Step 2

Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\begin{aligned}
\int_{C_1} 4y - x\,ds &= \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} \left[4\left(3\sin(t)\right) - 3\cos(t)\right]\sqrt{\left(-3\sin(t)\right)^2 + \left(3\cos(t)\right)^2}\,dt \\
&= \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} \left[12\sin(t) - 3\cos(t)\right]\sqrt{9\sin^2(t) + 9\cos^2(t)}\,dt \\
&= 3\int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} 12\sin(t) - 3\cos(t)\,dt \\
&= 3\left(-12\cos(t) - 3\sin(t)\right)\Big|_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} = \underline{\frac{90}{\sqrt{2}}}
\end{aligned}$$

$$\int_{C_2} 4y - x\,ds = \int_{-\frac{3}{\sqrt{2}}}^{4} \left[4\left(-\tfrac{3}{\sqrt{2}}\right) - t\right]\sqrt{(1)^2 + (0)^2}\,dt$$
$$= \int_{-\frac{3}{\sqrt{2}}}^{4} -\tfrac{12}{\sqrt{2}} - t\,dt$$
$$= \left(-\tfrac{12}{\sqrt{2}}t - \tfrac{1}{2}t^2\,dt\right)\Big|_{-\frac{3}{\sqrt{2}}}^{4} = \underline{-\tfrac{95}{4} - \tfrac{48}{\sqrt{2}}}$$

$$\int_{C_3} 4y - x\,ds = \int_{-\frac{3}{\sqrt{2}}}^{4} \left[4(t) - 4\right]\sqrt{(0)^2 + (1)^2}\,dt$$
$$= \int_{-\frac{3}{\sqrt{2}}}^{4} 4t - 4\,dt$$
$$= \left(2t^2 - 4t\right)\Big|_{-\frac{3}{\sqrt{2}}}^{4} = \boxed{7 - \tfrac{12}{\sqrt{2}}}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C 4y - x\,ds = \left(\tfrac{90}{\sqrt{2}}\right) + \left(-\tfrac{95}{4} - \tfrac{48}{\sqrt{2}}\right) + \left(7 - \tfrac{12}{\sqrt{2}}\right) = \boxed{-\tfrac{67}{4} + \tfrac{30}{\sqrt{2}} = 4.4632}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

8. Evaluate $\int\limits_C y^3 - x^2\,ds$ for each of the following curves.

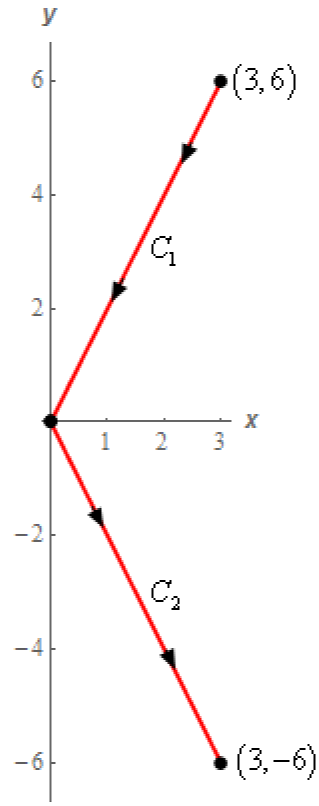
(a) *C* is the line segment from $(3,6)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(3,-6)$.

(b) *C* is the line segment from $(3,6)$ to $(3,-6)$.

(a) *C* is the line segment from $(3,6)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(3,-6)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, this curve comes in two pieces and we'll need to parameterize each of them so let's take care of that next.

$$C_1 : \vec{r}(t) = (1-t)\langle 3,6\rangle + t\langle 0,0\rangle = \langle 3-3t, 6-6t\rangle \qquad 0 \le t \le 1$$

$$C_2 : \vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 3,-6\rangle = \langle 3t, -6t\rangle \qquad 0 \le t \le 1$$

For each of these curves we just used the vector form of the line segment between two points. In the first case we needed to because the direction was in the decreasing *y* direction and recall that the integral limits need to be from smaller to larger value. In the second case the equation would have been just as easy to use but we just decided to use the line segment form for the slightly nicer limits.

Step 2
Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\int_{C_1} y^3 - x^2\, ds = \int_0^1 \left[(6-6t)^3 - (3-3t)^2 \right] \sqrt{(-3)^2 + (-6)^2}\, dt$$
$$= \sqrt{45} \int_0^1 (6-6t)^3 - (3-3t)^2\, dt$$
$$= 3\sqrt{5} \left(-\tfrac{1}{24}(6-6t)^4 + \tfrac{1}{9}(3-3t)^3 \right) \Big|_0^1 = \underline{153\sqrt{5}}$$

$$\int_{C_2} y^3 - x^2\, ds = \int_0^1 \left[(-6t)^3 - (3t)^2 \right] \sqrt{(3)^2 + (-6)^2}\, dt$$
$$= \sqrt{45} \int_0^1 -216t^3 - 9t^2\, dt$$
$$= 3\sqrt{5} \left(-54t^4 - 3t^3 \right) \Big|_0^1 = \underline{-171\sqrt{5}}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

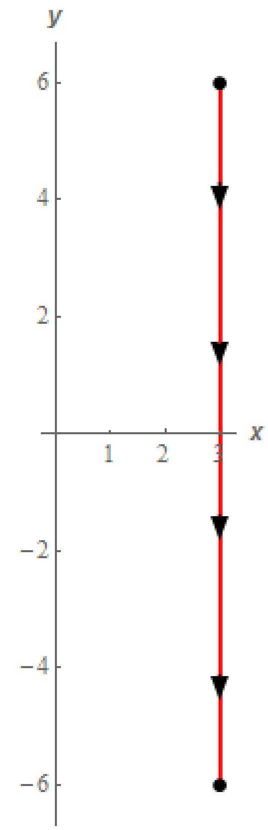
$$\int_C y^3 - x^2\, ds = \left(153\sqrt{5}\right) + \left(-171\sqrt{5}\right) = \boxed{-18\sqrt{5}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

(b) *C* is the line segment from $(3,6)$ to $(3,-6)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(3,6)$ to $(3,-6)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 3,6\right\rangle + t\left\langle 3,-6\right\rangle = \left\langle 3, 6-12t\right\rangle \qquad 0 \le t \le 1$$

Because this curve is in the direction of decreasing *y* and the integral needs its limits to go from smaller to larger values we had to use the vector form of the line segment between two points.

Step 2
Now all we need to do is compute the line integral.

$$\begin{aligned}\int\limits_C y^3 - x^2\,ds &= \int_0^1 \left[\left(6-12t\right)^3 - \left(3\right)^2\right]\sqrt{\left(0\right)^2 + \left(-12\right)^2}\,dt \\ &= 12\int_0^1 \left(6-12t\right)^3 - 9\,dt \\ &= 12\left(-\tfrac{1}{48}\left(6-12t\right)^4 - 9t\right)\Big|_0^1 = \boxed{-108}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

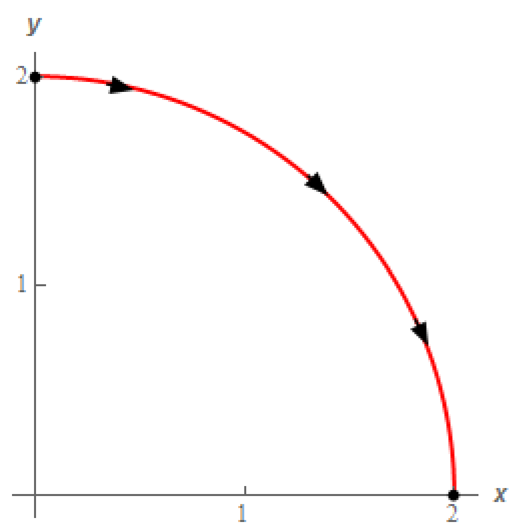
9. Evaluate $\int\limits_C 4x^2\,ds$ for each of the following curves.

(a) *C* is the portion of the circle centered at the origin of radius 2 in the 1st quadrant rotating in the clockwise direction.

(b) *C* is the line segment from $(0,2)$ to $(2,0)$.

(a) *C* is the portion of the circle centered at the origin of radius 2 in the 1st quadrant rotating in the clockwise direction.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for this curve.

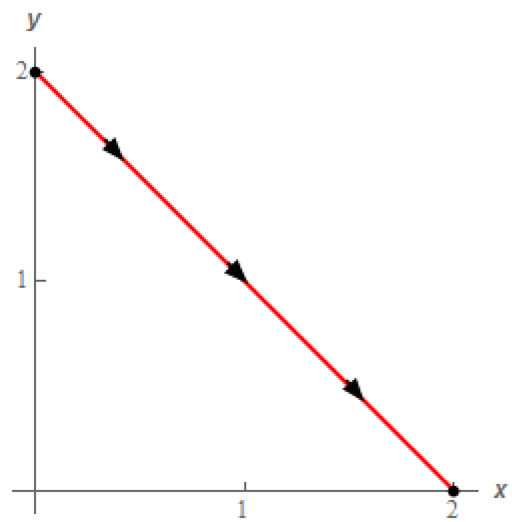
$$C: \vec r\left(t\right) = \left\langle 2\sin\left(t\right), 2\cos\left(t\right)\right\rangle \qquad 0 \le t \le \tfrac{1}{2}\pi$$

Step 2
Here is the line integral for this curve.

$$\begin{aligned}\int_C 4x^2\,ds &= \int_0^{\frac{1}{2}\pi} 4\left(2\sin\left(t\right)\right)^2 \sqrt{\left(2\cos\left(t\right)\right)^2 + \left(-2\sin\left(t\right)\right)^2}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 16\sin^2\left(t\right)\sqrt{4\cos^2\left(t\right)+4\sin^2\left(t\right)}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 16\sin^2\left(t\right)\sqrt{4}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 32\left(\tfrac{1}{2}\right)\left(1-\cos\left(2t\right)\right)dt \\ &= 16\left(t-\tfrac{1}{2}\sin\left(2t\right)\right)\Big|_0^{\frac{1}{2}\pi} = \boxed{8\pi}\end{aligned}$$

(b) *C* is the line segment from $\left(0,2\right)$ to $\left(2,0\right)$.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $\left(0,2\right)$ to $\left(2,0\right)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C:\vec{r}\left(t\right)=\left(1-t\right)\left\langle 0,2\right\rangle + t\left\langle 2,0\right\rangle = \left\langle 2t,2-2t\right\rangle \qquad 0\le t\le 1$$

Note that we could have just found the equation of this curve but it seemed just as easy to just use the vector form of the line segment between two points.

Step 2
Now all we need to do is compute the line integral.

$$\int_C 4x^2\, ds = \int_0^1 4(2t)^2 \sqrt{(2)^2 + (-2)^2}\, dt$$
$$= \sqrt{8}\int_0^1 16t^2\, dt$$
$$= 2\sqrt{2}\left(\tfrac{16}{3}t^3\right)\Big|_0^1 = \boxed{\tfrac{32\sqrt{2}}{3}}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

10. Evaluate $\int\limits_C 2x^3\, ds$ for each of the following curves.

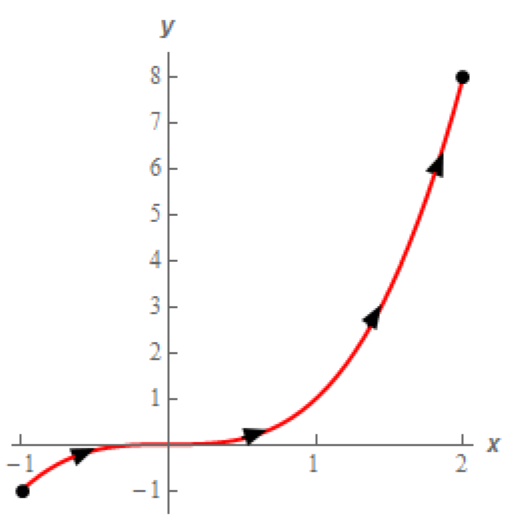
(a) *C* is the portion $y = x^3$ from $x = -1$ to $x = 2$.

(b) *C* is the portion $y = x^3$ from $x = 2$ to $x = -1$.

(a) *C* is the portion $y = x^3$ from $x = -1$ to $x = 2$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of *C*. Here then is the parameterization of C_1.

$$C_1 : \vec{r}(t) = \langle t, t^3 \rangle \qquad -1 \le t \le 2$$

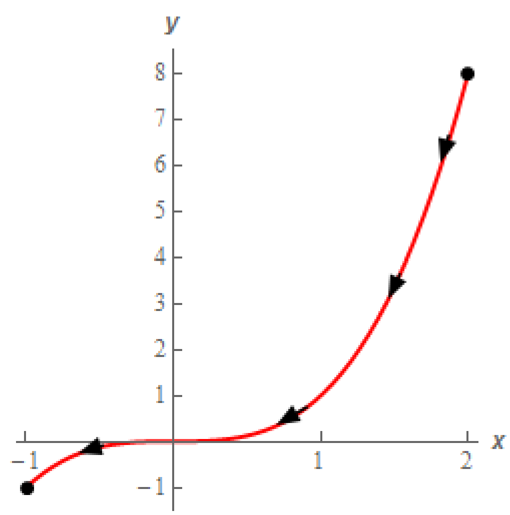
Step 2
Here is the line integral for this curve.

$$\begin{aligned}\int_{C_1} 2x^3\,ds &= \int_{-1}^{2} 2(t)^3\sqrt{(1)^2+(3t^2)^2}\,dt\\ &= \int_{-1}^{2} 2t^3\sqrt{1+9t^4}\,dt\\ &= \tfrac{1}{27}\left(1+9t^4\right)^{\frac{3}{2}}\Big|_{-1}^{2} = \boxed{\tfrac{1}{27}\left(145^{\frac{3}{2}}-10^{\frac{3}{2}}\right)=63.4966}\end{aligned}$$

(b) *C* is the portion $y = x^3$ from $x = 2$ to $x = -1$.

Step 1
Now, as we did in the previous part let's "rename" this curve as C_2 instead of *C*.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2 = -C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2
Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the directions for a curve then the value of this type of line integral doesn't change. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int_{C_2} 2x^3\,ds = \int_{-C_1} 2x^3\,ds = \int_{C_1} 2x^3\,ds = \boxed{\tfrac{1}{27}\left(145^{\frac{3}{2}}-10^{\frac{3}{2}}\right)=63.4966}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the “easy” method for doing this problem. Alternatively we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3
Now, if we are going to parameterize this curve, and follow the indicated direction, we can’t just use the parameterization from the first part and then “flip” the limits on the integral to “go backwards”. Line integrals just don’t work that way. The limits on the line integrals need to go from smaller value to larger value.

We need a new parameterization for this curve that will follow the curve in the indicated direction. Luckily that is actually pretty simple to do in this case. All we need to do is let $x = -t$ as *t* ranges from $t = -2$ to $t = 1$. In this way as *t* increases *x* will go from $x = 2$ to $x = -1$. In other words, at *t* increases *x* will decrease as we need it to in order to follow the direction of the curve.

Now that we have *x* taken care of the *y* is easy because we know the equation of the curve. To get the parametric equation for *y* all we need to do is plug in the parametric equation for *x* into the equation of the curve. Or,

$$y = \left(-t\right)^3 = -t^3$$

Putting all of this together we get the following parameterization of the curve.

$$C_2 : \vec{r}\left(t\right) = \left\langle -t - t^3 \right\rangle \qquad -2 \le t \le 1$$

Step 4
Now all we need to do is compute the line integral.

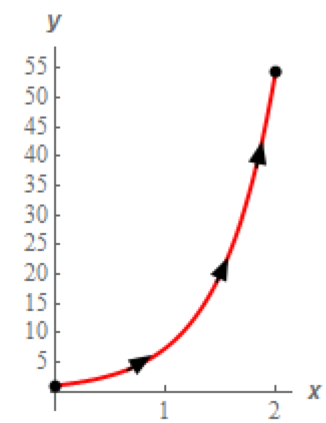
$$\begin{aligned}\int\limits_{C_2} 2x^3\, ds &= \int_{-2}^{1} 2\left(-t\right)^3 \sqrt{\left(-1\right)^2 + \left(-3t^2\right)^2}\, dt \\ &= \int_{-2}^{1} -2t^3\sqrt{1+9t^4}\, dt \\ &= -\tfrac{1}{27}\left(1+9t^4\right)^{\frac{3}{2}}\Big|_{-2}^{1} = \boxed{-\tfrac{1}{27}\left(10^{\frac{3}{2}} - 145^{\frac{3}{2}}\right) = 63.4966}\end{aligned}$$

So, the line integral from this part had exactly the same value as the line integral from the first part as we expected it to.

Section 5-3 : Line Integrals - Part II

1. Evaluate $\int\limits_C \sqrt{1+y}\,dy$ where *C* is the portion of $y = \mathbf{e}^{2x}$ from $x = 0$ to $x = 2$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec{r}(t) = \left\langle t, \mathbf{e}^{2t} \right\rangle \qquad 0 \le t \le 2$$

Step 3
Now we need to evaluate the line integral. Be careful with this type line integral. Note that the differential, in this case, is *dy* and not *ds* as they were in the previous section.

All we need to do is recall that $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the *x* and *y* in the integrand with the *x* and *y* components of the parameterization) and to convert the differential properly.

Here is the line integral.

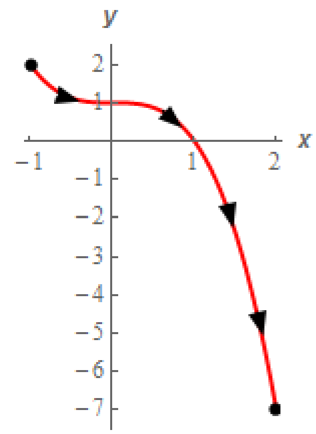
$$\begin{aligned}\int\limits_C \sqrt{1+y}\,dy &= \int_0^2 \sqrt{1+\mathbf{e}^{2t}}\,\left(2\mathbf{e}^{2t}\right)dt \\ &= \left[\tfrac{2}{3}\left(1+\mathbf{e}^{2t}\right)^{\frac{3}{2}}\right]\Bigg|_0^2 = \boxed{\tfrac{2}{3}\left[\left(1+\mathbf{e}^{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = 274.4897}\end{aligned}$$

Note that, in this case, the integral ended up being a simple substitution.

---

2. Evaluate $\int\limits_C 2y\,dx + \left(1-x\right)dy$ where *C* is portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec r\left(t\right) = \left\langle t, 1 - t^3\right\rangle \qquad -1 \le t \le 2$$

Step 3
Now we need to evaluate the line integral. Be careful with this type of line integral. In this case we have both a *dx* and a *dy* in the integrand. Recall that this is just a simplified notation for,

$$\int\limits_C 2y\,dx + \left(1-x\right)dy = \int\limits_C 2y\,dx + \int\limits_C 1 - x\,dy$$

Then all we need to do is recall that $dx = x'\,dt$ and $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the *x* and *y* in the integrand with the *x* and *y* components of the parameterization) and to convert the differentials properly.

Here is the line integral.

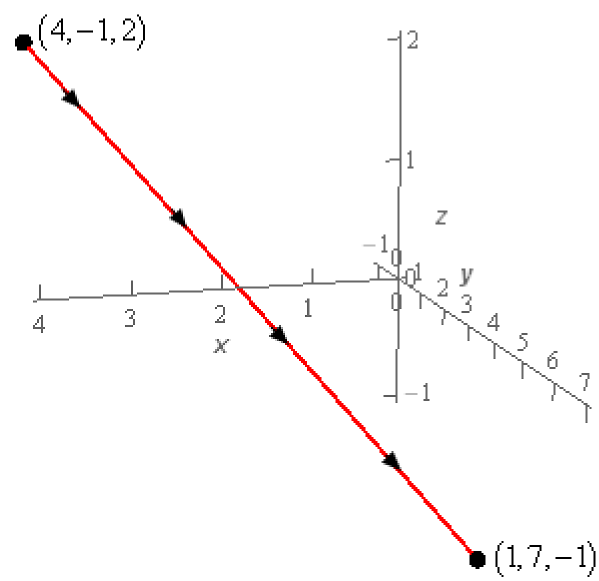
$$\begin{aligned}\int_C 2y\,dx + (1-x)\,dy &= \int_C 2y\,dx + \int_C 1-x\,dy \\ &= \int_{-1}^{2} 2\left(1-t^3\right)(1)\,dt + \int_{-1}^{2}(1-t)\left(-3t^2\right)dt \\ &= \int_{-1}^{2} 2\left(1-t^3\right)dt - 3\int_{-1}^{2} t^2 - t^3\,dt \\ &= \int_{-1}^{2} t^3 - 3t^2 + 2\,dt \\ &= \left[\tfrac{1}{4}t^4 - t^3 + 2t\right]\Big|_{-1}^{2} = \boxed{\tfrac{3}{4}}\end{aligned}$$

Note that, in this case, we combined the two integrals into a single integral prior to actually evaluating the integral. This doesn't need to be done but can, on occasion, simplify the integrand and hence the evaluation of the integral.

---

3. Evaluate $\int\limits_C x^2\,dy - yz\,dz$ where *C* is the line segment from $(4,-1,2)$ to $(1,7,-1)$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec{r}(t) = (1-t)\langle 4,-1,2\rangle + t\langle 1,7,-1\rangle = \langle 4-3t, -1+8t, 2-3t\rangle \qquad 0 \le t \le 1$$

Step 3

Now we need to evaluate the line integral. Be careful with this type of lines integral. In this case we have both a *dy* and a *dz* in the integrand. Recall that this is just a simplified notation for,

$$\int_C x^2\,dy - yz\,dz = \int_C x^2\,dy - \int_C yz\,dz$$

Then all we need to do is recall that $dy = y'\,dt$ and $dz = z'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the *x*, *y* and *z* in the integrand with the *x*, *y* and *z* components of the parameterization) and to convert the differentials properly.

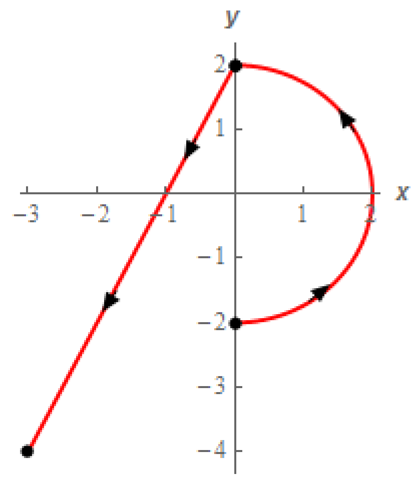
Here is the line integral.

$$\begin{aligned}\int_C x^2\,dy - yz\,dz &= \int_C x^2\,dy - \int_C yz\,dz \\ &= \int_0^1 (4-3t)^2\,(8)\,dt - \int_0^1 (-1+8t)\,(2-3t)(-3)\,dt \\ &= \int_0^1 8(4-3t)^2 - 3\left(24t^2 - 19t + 2\,dt\right)dt \\ &= \left[-\tfrac{8}{9}(4-3t)^3 - 3\left(8t^3 - \tfrac{19}{2}t^2 + 2t\right)\right]\Bigg|_0^1 = \boxed{\tfrac{109}{2}}\end{aligned}$$

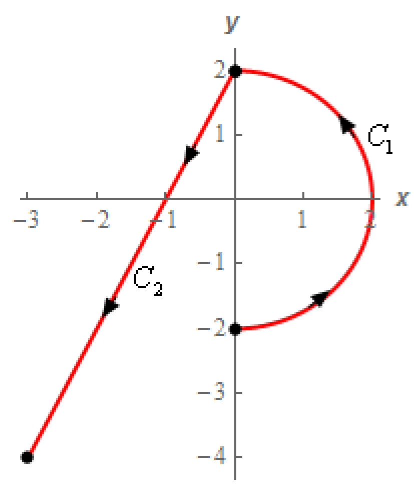
Note that, in this case, we combined the two integrals into a single integral prior to actually evaluating the integral. This doesn't need to be done but can, on occasion, simplify the integrand and hence the evaluation of the integral.

---

4. Evaluate $\int_C 1 + x^3\,dx$ where *C* is the right half of the circle of radius 2 with counter clockwise rotation followed by the line segment from $(0,2)$ to $(-3,-4)$. See the sketch below for the direction.



Step 1
To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1 : \vec{r}\left(t\right) = \left\langle 2\cos\left(t\right), 2\sin\left(t\right)\right\rangle \qquad -\tfrac{1}{2}\pi \le t \le \tfrac{1}{2}\pi$$

$$C_2 : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 0,2\right\rangle + t\left\langle -3,-4\right\rangle = \left\langle -3t, 2-6t\right\rangle \qquad 0 \le t \le 1$$

Step 2
Now we need to compute the line integral for each of the curves.

$$\int_{C_1} 1+x^3\,dx = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left[1+\left(2\cos\left(t\right)\right)^3\right]\left(-2\sin\left(t\right)\right)dt$$
$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} -2\sin\left(t\right)-16\cos^3\left(t\right)\sin\left(t\right)dt$$
$$= \left(2\cos\left(t\right)+4\cos^4\left(t\right)\right)\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \underline{0}$$

$$\int_{C_2} 1+x^3\,dx = \int_0^1 \left[1+\left(-3t\right)^3\right]\left(-3\right)dt$$
$$= \int_0^1 -3+81t^3\,dt$$
$$= \left(-3t+\tfrac{81}{4}t^4\right)\Big|_0^1 = \underline{\tfrac{69}{4}}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

Step 3

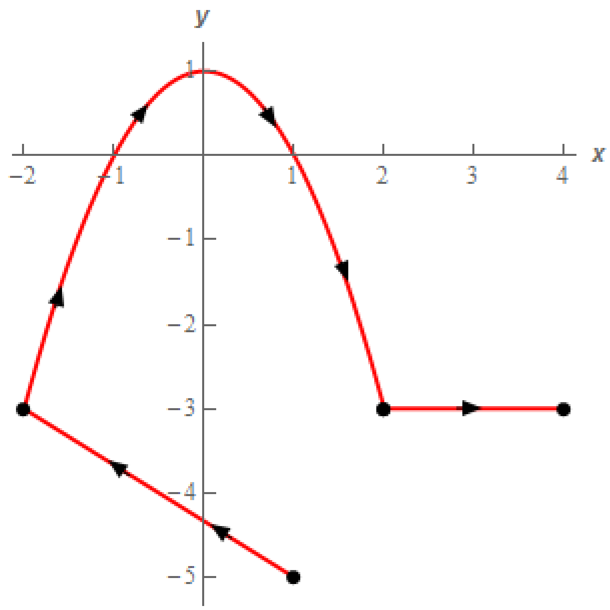
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C 1+x^3\,dx = \left(0\right)+\left(\tfrac{69}{4}\right) = \boxed{\tfrac{69}{4}}$$

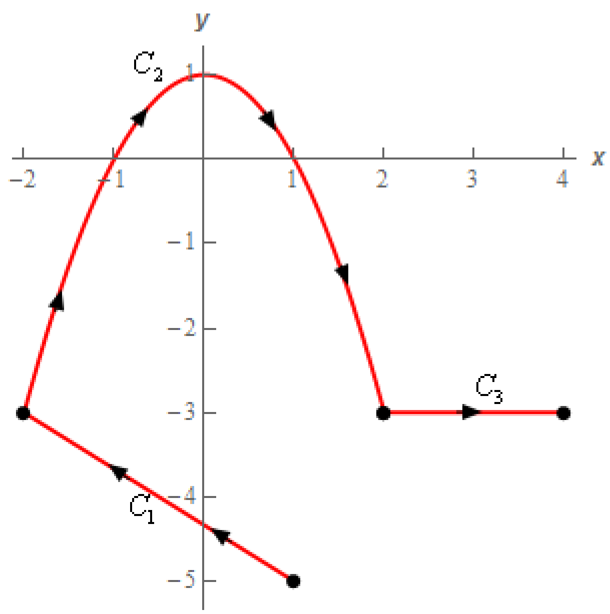
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

5. Evaluate $\int_C 2x^2\,dy - xy\,dx$ where *C* is the line segment from $\left(1,-5\right)$ to $\left(-2,-3\right)$ followed by the portion of $y=1-x^2$ from $x=-2$ to $x=2$ which in turn is followed by the line segment from $\left(2,-3\right)$ to $\left(4,-3\right)$. See the sketch below for the direction.



Step 1
To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1 : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 1,-5\right\rangle + t\left\langle -2,-3\right\rangle = \left\langle 1-3t,-5+2t\right\rangle \quad 0 \le t \le 1$$

$$C_2 : \vec{r}\left(t\right) = \left\langle t,1-t^2\right\rangle \qquad -2 \le t \le 2$$

$$C_3 : \vec{r}\left(t\right) = \left\langle t,-3\right\rangle \qquad 2 \le t \le 4$$

Note that for C_1 we had to use the vector form for the line segment between two points because the specified direction was in the decreasing *x* direction and so the equation of the line wouldn't work since the limits of the line integral need to go from smaller to larger values.

We did just use the equation of the line for C_3 since it was simple enough to do and the limits were also nice enough.

Step 2

Now we need to compute the line integral for each of the curves.

$$\begin{aligned}\int\limits_{C_1} 2x^2\,dy - xy\,dx &= \int\limits_{C_1} 2x^2\,dy - \int\limits_{C_1} xy\,dx \\ &= \int_0^1 2(1-3t)^2(2)\,dt - \int_0^1 (1-3t)(-5+2t)(-3)\,dt \\ &= \int_0^1 4(1-3t)^2 - 3\left(6t^2 - 17t + 5\right)dt \\ &= \left(-\tfrac{4}{9}(1-3t)^3 - 3\left(2t^3 - \tfrac{17}{2}t^2 + 5t\right)\right)\Big|_0^1 = \underline{\tfrac{17}{2}}\end{aligned}$$

$$\begin{aligned}\int\limits_{C_2} 2x^2\,dy - xy\,dx &= \int\limits_{C_2} 2x^2\,dy - \int\limits_{C_2} xy\,dx \\ &= \int_{-2}^2 2(t)^2(-2t)\,dt - \int_{-2}^2 (t)\left(1-t^2\right)(1)\,dt \\ &= \int_{-2}^2 -3t^3 - t\,dt \\ &= \left(-\tfrac{3}{4}t^4 - \tfrac{1}{2}t^2\right)\Big|_{-2}^2 = \underline{0}\end{aligned}$$

$$\begin{aligned}\int\limits_{C_3} 2x^2\,dy - xy\,dx &= \int\limits_{C_3} 2x^2\,dy - \int\limits_{C_2} xy\,dx \\ &= \int_2^4 2(t)^2(0)\,dt - \int_2^4 (t)(-3)(1)\,dt \\ &= \int_2^4 3t\,dt \\ &= \left(\tfrac{3}{2}t^2\right)\Big|_2^4 = \underline{18}\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

Also, don't get excited when one of the differentials "evaluates" to zero as the first one did in the C_3 integral. That will happen on occasion and is not something to get worried about when it does.

Step 3
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C 2x^2\,dy - xy\,dx = \left(\tfrac{17}{2}\right) + \left(0\right) + \left(18\right) = \boxed{\tfrac{53}{2}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

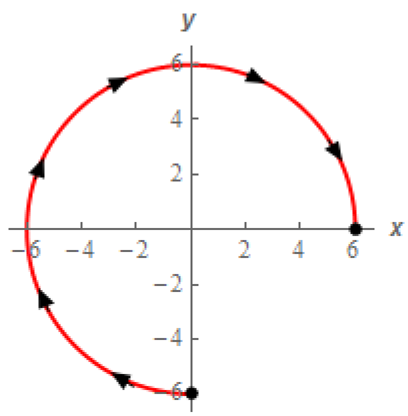
6. Evaluate $\int_C (x-y)\,dx - yx^2\,dy$ for each of the following curves.

   (a) *C* is the portion of the circle of radius 6 in the 1st, 2nd and 3rd quadrant with clockwise rotation.

   (b) *C* is the line segment from $(0,-6)$ to $(6,0)$.

(a) *C* is the portion of the circle of radius 6 in the 1st, 2nd and 3rd quadrant with clockwise rotation.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for this curve.

$$C : \vec{r}\left(t\right) = \left\langle 6\cos\left(t\right), -6\sin\left(t\right)\right\rangle \qquad \tfrac{1}{2}\pi \le t \le 2\pi$$

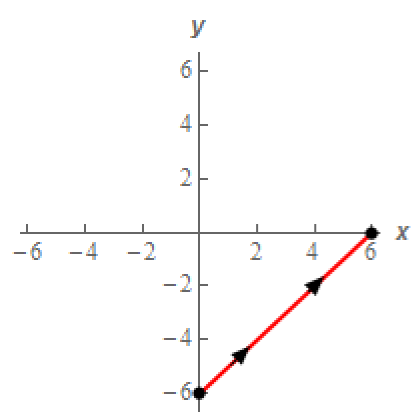
Step 2
Here is the line integral for this curve.

$$\begin{aligned}\int_C (x-y)\,dx - yx^2\,dy &= \int_C (x-y)\,dx - \int_C yx^2\,dy\\ &= \int_{\frac{1}{2}\pi}^{2\pi} \left[6\cos(t) - (-6\sin(t))\right](-6\sin(t))\,dt\\ &\qquad\qquad - \int_{\frac{1}{2}\pi}^{2\pi}\left[(6\sin(t))(6\cos(t))^2\right](-6\cos(t))\,dt\\ &= \int_{\frac{1}{2}\pi}^{2\pi} -36\cos(t)\sin(t) - 36\sin^2(t) - 1296\sin(t)\cos^3(t)\,dt\\ &= \int_{\frac{1}{2}\pi}^{2\pi} -18\sin(2t) - 18(1-\cos(2t)) - 1296\sin(t)\cos^3(t)\,dt\\ &= \left(9\cos(2t) - 18t + 9\sin(2t) + 324\cos^4(t)\right)\Big|_{\frac{1}{2}\pi}^{2\pi}\\ &= \boxed{342 - 27\pi = 257.1770}\end{aligned}$$

Don’t forget to correctly deal with the differentials when converting the line integral into a “standard” integral.

(b) *C* is the line segment from $(0,-6)$ to $(6,0)$.

Step 1
Let’s start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(0,-6)$ to $(6,0)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we’ll need to go through the work and see what we get from the line integral.

We’ll need to parameterize the curve so let’s take care of that.

$$C : \vec r(t) = (1-t)\langle 0,-6\rangle + t\langle 6,0\rangle = \langle 6t, -6+6t\rangle \qquad\qquad 0 \le t \le 1$$

Note that we could have just found the equation of this curve but it seemed just as easy to just use the vector form of the line segment between two points.

Step 2
Now all we need to do is compute the line integral.

$$\begin{aligned}\int\limits_C (x-y)\,dx - yx^2\,dy &= \int\limits_C (x-y)\,dx - \int\limits_C yx^2\,dy \\ &= \int_0^1 \left[6t-(-6+6t)\right](6)\,dt - \int_0^1 \left[(-6+6t)(6t)^2\right](6)\,dt \\ &= \int_0^1 36+1296t^2-1296t^3\,dt \\ &= \left(36t+432t^3-324t^4\right)\Big|_0^1 = \boxed{144}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

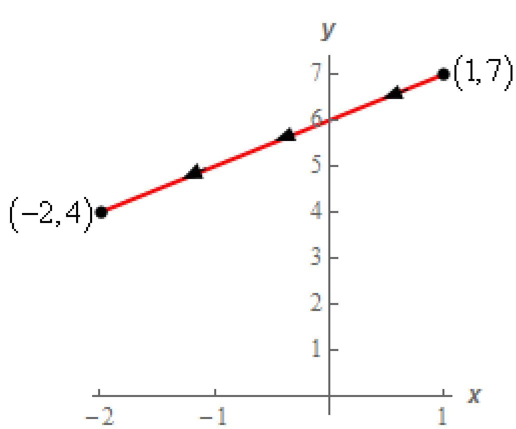
7. Evaluate $\int\limits_C x^3\,dy - (y+1)\,dx$ for each of the following curves.

(a) C is the line segment from $(1,7)$ to $(-2,4)$.

(b) C is the line segment from $(-2,4)$ to $(1,7)$.

(a) C is the line segment from $(1,7)$ to $(-2,4)$.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of *C*. Here then is the parameterization of C_1.

$$C_1 : \vec{r}\left(t\right) = \left(1-t\right)\left\langle 1,7\right\rangle + t\left\langle -2,4\right\rangle = \left\langle 1-3t, 7-3t\right\rangle \qquad 0 \le t \le 1$$

Step 2
Here is the line integral for this curve.

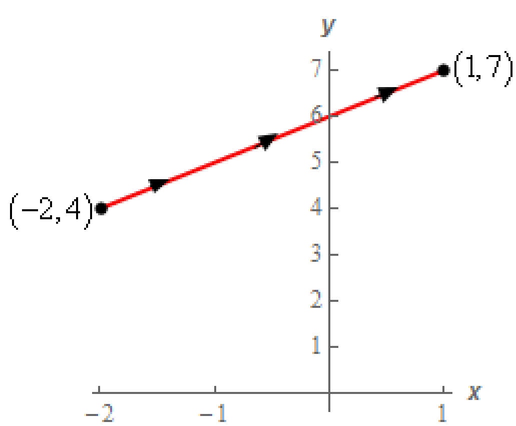
$$\begin{aligned}\int_{C_1} x^3\,dy - \left(y+1\right)dx &= \int_{C_1} x^3\,dy - \int_{C_1} y+1\,dx \\ &= \int_0^1 \left(1-3t\right)^3\left(-3\right)dt - \int_0^1 \left(7-3t+1\right)\left(-3\right)dt \\ &= \int_0^1 -3\left(1-3t\right)^3 + 3\left(8-3t\right)dt \\ &= \left[\tfrac{1}{4}\left(1-3t\right)^4 + 24t - \tfrac{9}{2}t^2\right]\Bigg|_0^1 = \boxed{\tfrac{93}{4}}\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

(b) *C* is the line segment from $\left(-2,4\right)$ to $\left(1,7\right)$.

Step 1
Now, as we did in the previous part let's "rename" this curve as C_2 instead of *C*.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2 = -C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2
Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the direction of a curve then the value of this type of line integral will just change signs. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int_{C_2} x^3\,dy - (y+1)\,dx = \int_{-C_1} x^3\,dy - (y+1)\,dx = -\int_{C_1} x^3\,dy - (y+1)\,dx = \boxed{-\tfrac{93}{4}}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the "easy" method for doing this problem. Alternatively, we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3
Here is the parameterization for this curve.

$$C_2 : \vec{r}(t) = (1-t)\langle -2,4\rangle + t\langle 1,7\rangle = \langle -2+3t, 4+3t\rangle \qquad 0 \le t \le 1$$

Step 4
Now all we need to do is compute the line integral.

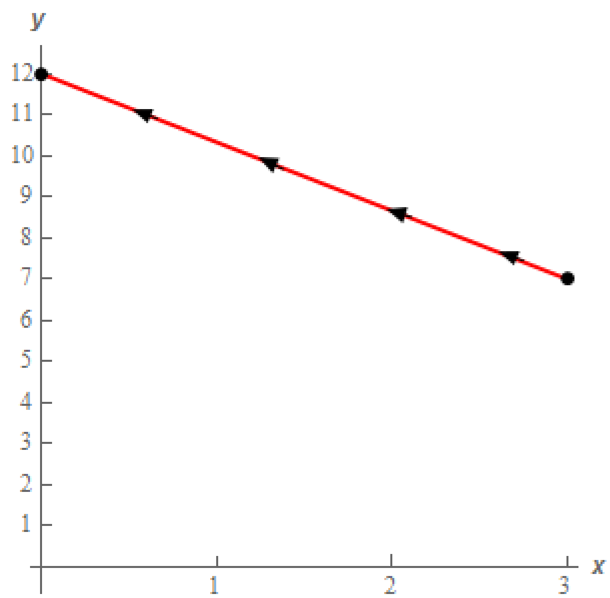
$$\begin{aligned} \int_{C_2} x^3\,dy - (y+1)\,dx &= \int_{C_2} x^3\,dy - \int_{C_2} y+1\,dx \\ &= \int_0^1 (-2+3t)^3(3)\,dt - \int_0^1 (4+3t+1)(3)\,dt \\ &= \int_0^1 3(-2+3t)^3 - 3(5+3t)\,dt \\ &= \left[\tfrac{1}{4}(-2+3t)^4 - 15t - \tfrac{9}{2}t^2\right]\Big|_0^1 = \boxed{-\tfrac{93}{4}} \end{aligned}$$

So, the line integral from this part had the same value, except for the sign, as the line integral from the first part as we expected it to.

Section 5-4 : Line Integrals of Vector Fields

1. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}\left(x,y\right) = y^2\,\vec{i} + \left(3x - 6y\right)\vec{j}$ and *C* is the line segment from $\left(3,7\right)$ to $\left(0,12\right)$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec{r}\left(t\right) = \left(1-t\right)\left\langle 3,7\right\rangle + t\left\langle 0,12\right\rangle = \left\langle 3-3t, 7+5t\right\rangle \qquad 0 \le t \le 1$$

Step 3
In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in *x* and *y* from the parameterization into the vector field).

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \left(7+5t\right)^2\vec{i} + \left(3\left(3-3t\right) - 6\left(7+5t\right)\right)\vec{j} = \left(7+5t\right)^2\vec{i} + \left(-33-39t\right)\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}\,'\left(t\right) = \left\langle -3, 5\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right) = -3\left(7+5t\right)^2 - 5\left(33+39t\right)$$

Step 4
Now all we need to do is evaluate the integral.

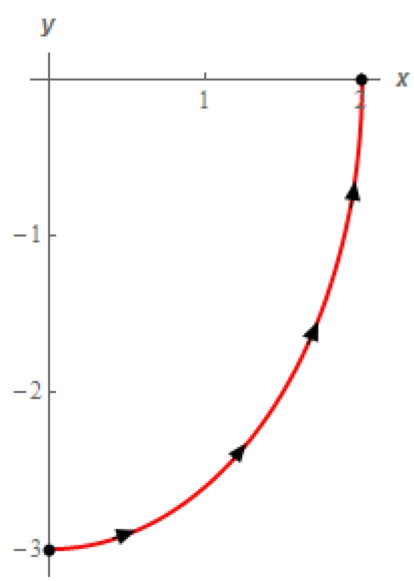
$$\int_C \vec{F}\bullet d\vec{r} = \int_0^1 -3\left(7+5t\right)^2 - 5\left(33+39t\right)dt$$

$$= \left[-\tfrac{1}{5}\left(7+5t\right)^3 - 165t - \tfrac{195}{2}t^2\right]\Big|_0^1 = \boxed{-\tfrac{1079}{2}}$$

---

2. Evaluate $\int\limits_C \vec{F}\bullet d\vec{r}$ where $\vec{F}\left(x,y\right) = \left(x+y\right)\vec{i} + \left(1-x\right)\vec{j}$ and *C* is the portion of $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ that is in the 4th quadrant with the counter clockwise rotation.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec{r}\left(t\right) = \left\langle 2\cos\left(t\right), 3\sin\left(t\right)\right\rangle \qquad \tfrac{3}{2}\pi \le t \le 2\pi$$

Step 3

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in *x* and *y* from the parameterization into the vector field).

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \left(2\cos\left(t\right) + 3\sin\left(t\right)\right)\vec{i} + \left(1 - 2\cos\left(t\right)\right)\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}\,'\left(t\right) = \left\langle -2\sin\left(t\right), 3\cos\left(t\right)\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right) &= \left(2\cos\left(t\right) + 3\sin\left(t\right)\right)\left(-2\sin\left(t\right)\right) + \left(1 - 2\cos\left(t\right)\right)\left(3\cos\left(t\right)\right)\\ &= -4\cos\left(t\right)\sin\left(t\right) - 6\sin^2\left(t\right) + 3\cos\left(t\right) - 6\cos^2\left(t\right)\\ &= -4\cos\left(t\right)\sin\left(t\right) - 6\left[\sin^2\left(t\right) + \cos^2\left(t\right)\right] + 3\cos\left(t\right)\\ &= -2\sin\left(2t\right) + 3\cos\left(t\right) - 6\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral! In this case that meant using the double angle formula for sine to "simplify" the first term for the integral.

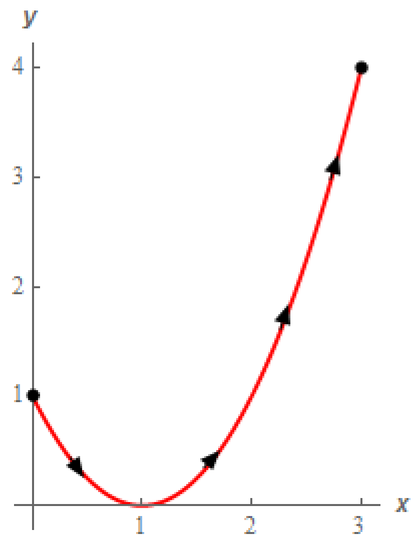
Step 4
Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_C \vec{F}\bullet d\vec{r} &= \int_{\frac{3}{2}\pi}^{2\pi} -2\sin\left(2t\right) + 3\cos\left(t\right) - 6\,dt\\ &= \left[\cos\left(2t\right) + 3\sin\left(t\right) - 6t\right]\Big|_{\frac{3}{2}\pi}^{2\pi} = \boxed{5 - 3\pi}\end{aligned}$$

---

3. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}\left(x,y\right) = y^2\,\vec{i} + \left(x^2 - 4\right)\vec{j}$ and *C* is the portion of $y = \left(x-1\right)^2$ from $x = 0$ to $x = 3$.

Step 1
Here is a quick sketch of *C* with the direction specified in the problem statement shown.



Step 2
Next, we need to parameterize the curve.

$$\vec r\left(t\right) = \left\langle t,\left(t-1\right)^2\right\rangle \qquad 0 \le t \le 3$$

Step 3
In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in *x* and *y* from the parameterization into the vector field).

$$\vec F\left(\vec r\left(t\right)\right) = \left[\left(t-1\right)^2\right]^2\vec i + \left(\left(t\right)^2 - 4\right)\vec j = \left(t-1\right)^4\vec i + \left(t^2-4\right)\vec j$$

The derivative of the parameterization is,

$$\vec r\,'\left(t\right) = \left\langle 1, 2\left(t-1\right)\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec F\left(\vec r\left(t\right)\right)\centerdot\vec r\,'\left(t\right) &= \left(t-1\right)^4\left(1\right) + \left(t^2-4\right)\left(2t-2\right)\\ &= \left(t-1\right)^4 + 2t^3 - 2t^2 - 8t + 8\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral!

Step 4
Now all we need to do is evaluate the integral.

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^3 (t-1)^4 + 2t^3 - 2t^2 - 8t + 8\,dt$$

$$= \left[\tfrac{1}{5}(t-1)^5 + \tfrac{1}{2}t^4 - \tfrac{2}{3}t^3 - 4t^2 + 8t \right]\Big|_0^3 = \boxed{\tfrac{171}{10}}$$

---

4. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y,z) = \mathbf{e}^{2x}\,\vec{i} + z(y+1)\,\vec{j} + z^3\,\vec{k}$ and *C* is given by $\vec{r}(t) = t^3\,\vec{i} + (1-3t)\,\vec{j} + \mathbf{e}^t\,\vec{k}$ for $0 \le t \le 2$.

Step 1
Okay, for this problem we've been given the parameterization of the curve in the problem statement so we don't need to worry about that for this problem and we can jump right into the work needed to evaluate the line integral. This means that we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in *x* and *y* from the parameterization into the vector field).

$$\vec{F}(\vec{r}(t)) = \mathbf{e}^{2t^3}\,\vec{i} + \mathbf{e}^t(1-3t+1)\,\vec{j} + \left(\mathbf{e}^t\right)^3\vec{k} = \mathbf{e}^{2t^3}\,\vec{i} + \mathbf{e}^t(2-3t)\,\vec{j} + \mathbf{e}^{3t}\,\vec{k}$$

The derivative of the parameterization is,

$$\vec{r}\,'(t) = 3t^2\,\vec{i} - 3\,\vec{j} + \mathbf{e}^t\,\vec{k}$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t) = \mathbf{e}^{2t^3}\left(3t^2\right) + \mathbf{e}^t(2-3t)(-3) + \mathbf{e}^{3t}\left(\mathbf{e}^t\right)$$

$$= 3t^2\mathbf{e}^{2t^3} - 3\mathbf{e}^t(2-3t) + \mathbf{e}^{4t}$$

Make sure that you simplify the dot product with an eye towards doing the integral!

Step 2
Now all we need to do is evaluate the integral.

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^2 3t^2\mathbf{e}^{2t^3} - 3\mathbf{e}^t(2-3t) + \mathbf{e}^{4t}\,dt$$

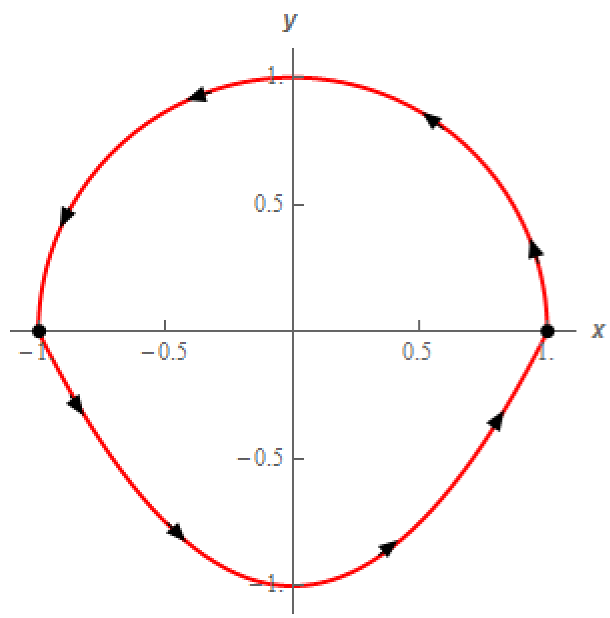
$$= \left[\tfrac{1}{2}\mathbf{e}^{2t^3} - 3\mathbf{e}^t(5-3t) + \tfrac{1}{4}\mathbf{e}^{4t}\,dt \right]\Big|_0^2 = \boxed{\tfrac{57}{4} + 3\mathbf{e}^2 + \tfrac{1}{4}\mathbf{e}^8 + \tfrac{1}{2}\mathbf{e}^{16}}$$

Note that the second term in the integral involved integration by parts (you do recall how to do that right?). We'll leave the integration by parts details to you to verify and note that we did simplify the results a little bit.

Also, do not get excited about the "messy" answer here! You will get these kinds of answers on occasion.

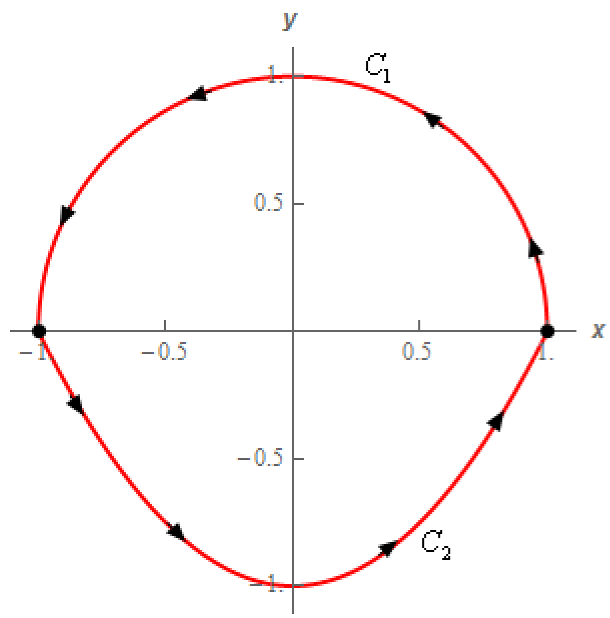
---

5. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = 3y\,\vec{i} + (x^2 - y)\,\vec{j}$ and *C* is the upper half of the circle centered at the origin of radius 1 with counter clockwise rotation and the portion of $y = x^2 - 1$ from $x = -1$ to $x = 1$. See the sketch below.



Step 1

To help with the problem let's label each of the curves as follows,



The parameterization of each curve is,

$$C_1 : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle \qquad 0 \le t \le \pi$$

$$C_2 : \vec{r}(t) = \langle t, t^2 - 1 \rangle \qquad -1 \le t \le 1$$

Step 2

Now we need to compute the line integral for each of the curves. In the first few problems in this section we evaluated the vector function along the curve, took the derivative of the parameterization and computed the dot product separately. For this problem we'll be doing all that work in the integral itself.

Here is the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} \vec{F} \bullet d\vec{r} &= \int_0^\pi \langle 3\sin(t), \cos^2(t) - \sin(t) \rangle \bullet \langle -\sin(t), \cos(t) \rangle \, dt \\ &= \int_0^\pi -3\sin^2(t) + \cos^3(t) - \sin(t)\cos(t) \, dt \\ &= \int_0^\pi -\tfrac{3}{2}\left(1 - \cos(2t)\right) + \cos(t)\left(1 - \sin^2(t)\right) - \tfrac{1}{2}\sin(2t) \, dt \\ &= \left(-\tfrac{3}{2}\left(t - \tfrac{1}{2}\sin(2t)\right) + \sin(t) - \tfrac{1}{3}\sin^3(t) + \tfrac{1}{4}\cos(2t) \, dt\right)\Big|_0^\pi = \underline{-\tfrac{3}{2}\pi}\end{aligned}$$

$$\int\limits_{C_2} \vec{F}\bullet d\vec{r} = \int_{-1}^{1} \left\langle 3\left(t^2-1\right), t^2-\left(t^2-1\right)\right\rangle \bullet \left\langle 1, 2t\right\rangle dt$$
$$= \int_{-1}^{1} 3\left(t^2-1\right)+2t\,dt$$
$$= \left(t^3-3t+t^2\right)\Big|_{-1}^{1} = \underline{-4}$$

You do recall how to deal with all those trig functions that we saw in the first integral don't you? If not you should go back to the Calculus II material and work some practice problems. You'll be seeing a fair number of integrals involving trig functions from this point on.

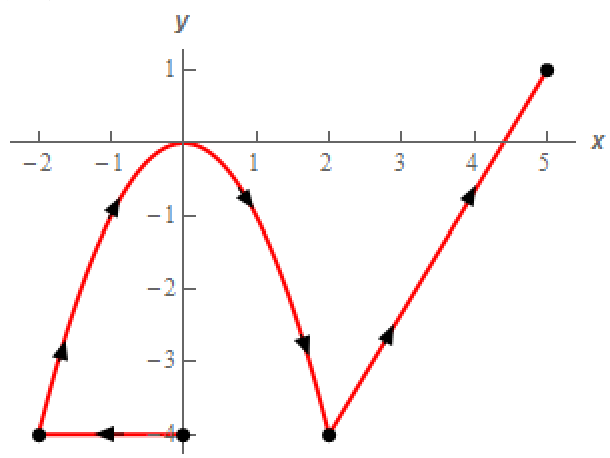
Step 3
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_{C} \vec{F}\bullet d\vec{r} = \left(-\tfrac{3}{2}\pi\right)+\left(-4\right) = \boxed{-4-\tfrac{3}{2}\pi = -8.7124}$$

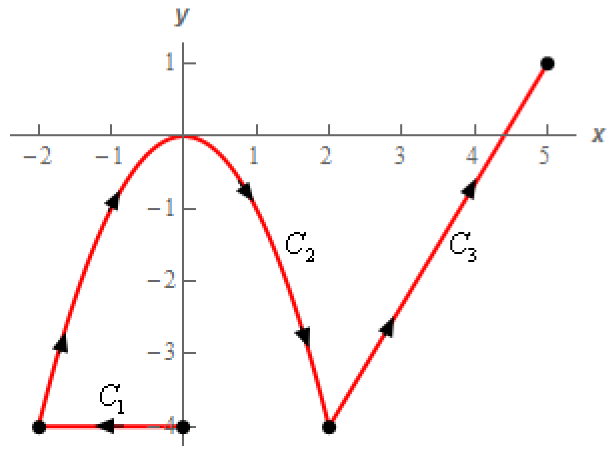
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

6. Evaluate $\int\limits_{C} \vec{F}\bullet d\vec{r}$ where $\vec{F}\left(x,y\right) = xy\,\vec{i} + \left(1+3y\right)\vec{j}$ and *C* is the line segment from $\left(0,-4\right)$ to $\left(-2,-4\right)$ followed by portion of $y=-x^2$ from $x=-2$ to $x=2$ which is in turn followed by the line segment from $\left(2,-4\right)$ to $\left(5,1\right)$. See the sketch below.



Step 1
To help with the problem let's label each of the curves as follows,



The parameterization of each curve is,

$$C_1 : \vec r(t) = (1-t)\langle 0,-4\rangle + t\langle -2,-4\rangle = \langle -2t,-4\rangle \qquad 0 \le t \le 1$$

$$C_2 : \vec r(t) = \langle t,-t^2\rangle \qquad -2 \le t \le 2$$

$$C_3 : \vec r(t) = (1-t)\langle 2,-4\rangle + t\langle 5,1\rangle = \langle 2+3t,-4+5t\rangle \qquad 0 \le t \le 1$$

Step 2

Now we need to compute the line integral for each of the curves. In the first few problems in this section we evaluated the vector function along the curve, took the derivative of the parameterization and computed the dot product separately. For this problem we'll be doing all that work in the integral itself.

Here is the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} \vec F \bullet d\vec r &= \int_0^1 \left\langle (-2t)(-4), 1+3(-4)\right\rangle \bullet \langle -2,0\rangle\, dt \\ &= \int_0^1 -16t\, dt = \left(-8t^2\right)\Big|_0^1 = \underline{-8}\end{aligned}$$

$$\begin{aligned}\int_{C_2} \vec F \bullet d\vec r &= \int_{-2}^2 \left\langle (t)(-t^2), 1+3(-t^2)\right\rangle \bullet \langle 1,-2t\rangle\, dt \\ &= \int_{-2}^2 5t^3 - 2t\, dt = \left(\tfrac{5}{4}t^4 - t^2\right)\Big|_{-2}^2 = \underline{0}\end{aligned}$$

$$\begin{aligned}\int_{C_3} \vec F \bullet d\vec r &= \int_0^1 \left\langle (2+3t)(-4+5t), 1+3(-4+5t)\right\rangle \bullet \langle 3,5\rangle\, dt \\ &= \int_0^1 45t^2 + 69t - 79\, dt = \left(15t^3 + \tfrac{69}{2}t^2 - 79t\right)\Big|_0^1 = \underline{-\tfrac{59}{2}}\end{aligned}$$

Step 3
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C \vec{F} \bullet d\vec{r} = (-8) + (0) + \left(-\tfrac{59}{2}\right) = \boxed{-\tfrac{75}{2}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

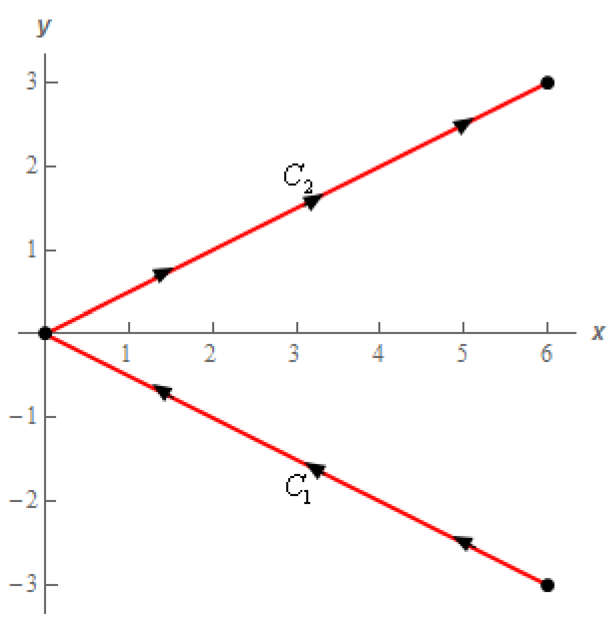
7. Evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = (6x-2y)\vec{i} + x^2\vec{j}$ for each of the following curves.

(a) C is the line segment from $(6,-3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(6,3)$.

(b) C is the line segment from $(6,-3)$ to $(6,3)$.

(a) C is the line segment from $(6,-3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(6,3)$.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for each of these curves.

$$C_1 : \vec{r}(t) = (1-t)\langle 6,-3\rangle + t\langle 0,0\rangle = \langle 6-6t, -3+3t\rangle \qquad 0 \le t \le 1$$

$$C_2 : \vec{r}\left(t\right) = \left\langle t, \tfrac{1}{2}t \right\rangle \qquad\qquad 0 \le t \le 6$$

For C_2 we used the equation of the line to get the parameterization because it gave a slightly nicer form to work with. We couldn't do this with C_1 because the specified direction of the curve was in the decreasing *x* direction and the limits of the integral need to be from smaller value to larger value.

Step 2
Here is the line integral for each of these curves.

$$\begin{aligned}\int\limits_{C_1} \vec{F} \bullet d\vec{r} &= \int_0^1 \left\langle 6\left(6-6t\right) - 2\left(-3+3t\right), \left(6-6t\right)^2 \right\rangle \bullet \left\langle -6, 3 \right\rangle dt \\ &= \int_0^1 252t - 252 + 3\left(6-6t\right)^2 \, dt \\ &= \left(126t^2 - 252t - \tfrac{1}{6}\left(6-6t\right)^3\right)\Big|_0^1 = \underline{-90}\end{aligned}$$

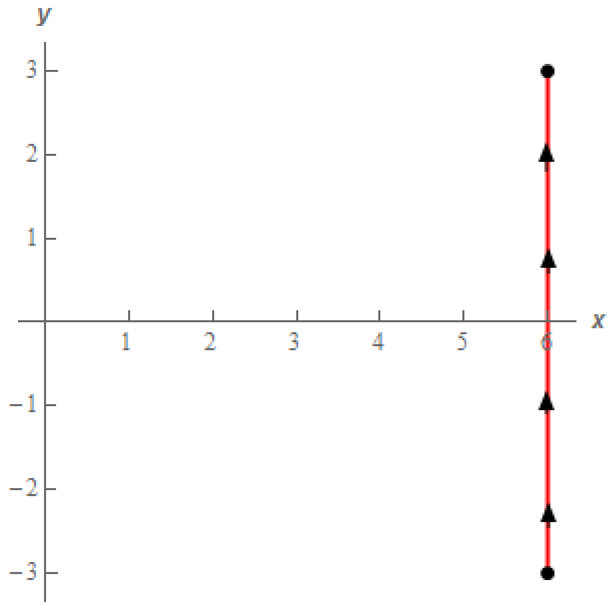
$$\begin{aligned}\int\limits_{C_2} \vec{F} \bullet d\vec{r} &= \int_0^6 \left\langle 6\left(t\right) - 2\left(\tfrac{1}{2}t\right), \left(t\right)^2 \right\rangle \bullet \left\langle 1, \tfrac{1}{2} \right\rangle dt \\ &= \int_0^6 5t + \tfrac{1}{2}t^2 \, dt = \left(\tfrac{5}{2}t^2 + \tfrac{1}{6}t^3\right)\Big|_0^6 = \underline{126}\end{aligned}$$

Step 3
The line integral for this part is then,

$$\int\limits_C \vec{F} \bullet d\vec{r} = \left(-90\right) + \left(126\right) = \boxed{36}$$

(b) *C* is the line segment from $\left(6,-3\right)$ to $\left(6,3\right)$.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(6,-3)$ to $(6,3)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore, we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C : \vec{r}\left(t\right) = \left\langle 6, t \right\rangle \qquad\qquad -3 \le t \le 3$$

Step 2

Now all we need to do is compute the line integral.

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= \int_{-3}^{3} \left\langle 6\left(6\right) - 2\left(t\right), \left(6\right)^2 \right\rangle \bullet \left\langle 0,1 \right\rangle dt \\ &= \int_{-3}^{3} 36\, dt = \left(36t\right)\Big|_{-3}^{3} = \boxed{216}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

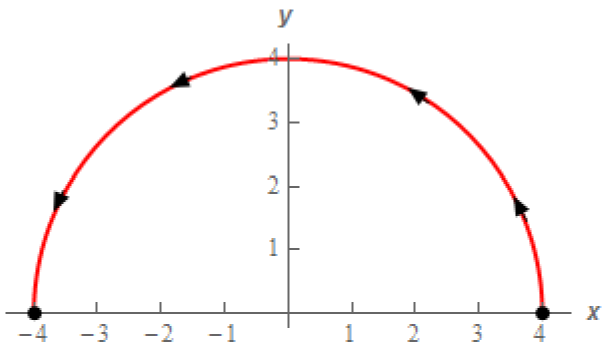
8. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}\left(x,y\right) = 3\vec{i} + \left(xy - 2x\right)\vec{j}$ for each of the following curves.

(a) *C* is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

(b) *C* is the upper half of the circle centered at the origin of radius 4 with clockwise rotation.

(a) *C* is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

Step 1
Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of *C*. Here then is the parameterization of C_1.

$$C_1 : \vec{r}\left(t\right) = \left\langle 4\cos\left(t\right), 4\sin\left(t\right)\right\rangle \qquad 0 \le t \le \pi$$

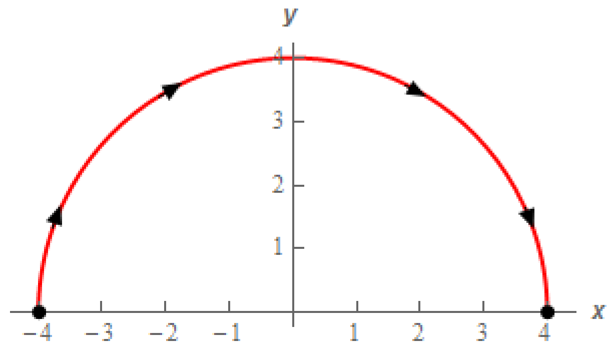
Step 2
Here is the line integral for this curve.

$$\begin{aligned}\int\limits_{C_1} \vec{F}\bullet d\vec{r} &= \int_0^{\pi} \left\langle 3, \left(4\cos\left(t\right)\right)\left(4\sin\left(t\right)\right) - 2\left(4\cos\left(t\right)\right)\right\rangle \bullet \left\langle -4\sin\left(t\right), 4\cos\left(t\right)\right\rangle dt \\ &= \int_0^{\pi} -12\sin\left(t\right) + 64\sin\left(t\right)\cos^2\left(t\right) - 32\cos^2\left(t\right) dt \\ &= \int_0^{\pi} -12\sin\left(t\right) + 64\sin\left(t\right)\cos^2\left(t\right) - 16\left(1 + \cos\left(2t\right)\right) dt \\ &= \left(12\cos\left(t\right) - \tfrac{64}{3}\cos^3\left(t\right) - 16t - 8\sin\left(2t\right)\right)\Big|_0^{\pi} = \boxed{\tfrac{56}{3} - 16\pi = -31.5988}\end{aligned}$$

(b) *C* is the upper half of the circle centered at the origin of radius 4 with clockwise rotation.

Step 1
Now, as we did in the previous part let's "rename" this curve as C_2 instead of *C*.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2 = -C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2
Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the direction of a curve then the value of this type of line integral will just change signs. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int_{C_2} \vec{F} \bullet d\vec{r} = \int_{-C_1} \vec{F} \bullet d\vec{r} = -\int_{C_1} \vec{F} \bullet d\vec{r} = \boxed{16\pi - \tfrac{56}{3} = 31.5988}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the "easy" method for doing this problem. Alternatively, we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3
Here is the parameterization for this curve.

$$C_2 : \vec{r}\left(t\right) = \left\langle -4\cos\left(t\right), 4\sin\left(t\right)\right\rangle \qquad\qquad 0 \le t \le \pi$$

Step 4
Now all we need to do is compute the line integral.

$$\begin{aligned}\int_{C_2} \vec{F} \bullet d\vec{r} &= \int_0^{\pi} \left\langle 3, \left(-4\cos\left(t\right)\right)\left(4\sin\left(t\right)\right) - 2\left(-4\cos\left(t\right)\right)\right\rangle \bullet \left\langle 4\sin\left(t\right), 4\cos\left(t\right)\right\rangle dt \\ &= \int_0^{\pi} 12\sin\left(t\right) - 64\sin\left(t\right)\cos^2\left(t\right) + 32\cos^2\left(t\right) dt \\ &= \int_0^{\pi} 12\sin\left(t\right) - 64\sin\left(t\right)\cos^2\left(t\right) + 16\left(1 + \cos\left(2t\right)\right) dt \\ &= \left(-12\cos\left(t\right) + \tfrac{64}{3}\cos^3\left(t\right) + 16t + 8\sin\left(2t\right)\right)\Big|_0^{\pi} = \boxed{16\pi - \tfrac{56}{3} = 31.5988}\end{aligned}$$

So, the line integral from this part had the same value, except for the sign, as the line integral from the first part as we expected it to.

Section 5-5 : Fundamental Theorem for Line Integrals

1. Evaluate $\int\limits_C \nabla f \bullet d\vec r$ where $f\left(x,y\right) = x^3\left(3 - y^2\right) + 4y$ and *C* is given by $\vec r\left(t\right) = \left\langle 3 - t^2, 5 - t \right\rangle$ with $-2 \le t \le 3$.

Step 1
There really isn't all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we'll need the following two "points".

$$\vec r\left(-2\right) = \left\langle -1, 7 \right\rangle \qquad\qquad \vec r\left(3\right) = \left\langle -6, 2 \right\rangle$$

Remember that we are thinking of these as the position vector representations of the points $\left(-1,7\right)$ and $\left(-6,2\right)$ respectively.

Step 2
Now simply apply the Fundamental Theorem to evaluate the integral.

$$\int\limits_C \nabla f \bullet d\vec r = f\left(\vec r\left(3\right)\right) - f\left(\vec r\left(-2\right)\right) = f\left(-6,2\right) - f\left(-1,7\right) = 224 - 74 = \boxed{150}$$

---

2. Evaluate $\int\limits_C \nabla f \bullet d\vec r$ where $f\left(x,y\right) = y\mathbf{e}^{x^2 - 1} + 4x\sqrt{y}$ and *C* is given by $\vec r\left(t\right) = \left\langle 1 - t, 2t^2 - 2t \right\rangle$ with $0 \le t \le 2$.

Step 1
There really isn't all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we'll need the following two "points".

$$\vec r\left(0\right) = \left\langle 1, 0 \right\rangle \qquad\qquad \vec r\left(2\right) = \left\langle -1, 4 \right\rangle$$

Remember that we are thinking of these as the position vector representations of the points $\left(1,0\right)$ and $\left(-1,4\right)$ respectively.

Step 2
Now simply apply the Fundamental Theorem to evaluate the integral.

$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\left(2\right)\right) - f\left(\vec{r}\left(0\right)\right) = f\left(-1,4\right) - f\left(1,0\right) = -4 - 0 = \boxed{-4}$$

---

3. Given that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path compute $\int_C \vec{F} \cdot d\vec{r}$ where *C* is the ellipse given by $\frac{\left(x-5\right)^2}{4} + \frac{y^2}{9} = 1$ with the counter clockwise rotation.

Solution
At first glance this problem seems to be impossible since the vector field isn't even given for the problem. However, it's actually quite simple and the vector field is not needed to do the problem.

There are two important things to note in the problem statement.

First, and somewhat more importantly, we are told in the problem statement that the integral is independent of path.

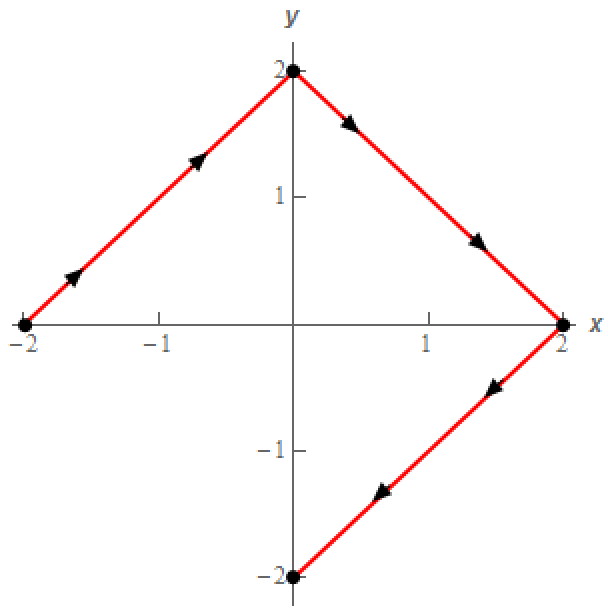
Second, we are told that the curve, C, is the full ellipse. It isn't the fact that *C* is an ellipse that is important. What is important is the fact that *C* is a closed curve.

Now all we need to do is use Fact 4 from the notes. This tells us that the value of a line integral of this type around a closed path will be zero if the integral is independent of path. Therefore,

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$$

---

4. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f\left(x,y\right) = \mathbf{e}^{xy} - x^2 + y^3$ and *C* is the curve shown below.



Solution

This problem is much simpler than it appears at first. We do not need to compute 3 different line integrals (one for each curve in the sketch).

All we need to do is notice that we are doing a line integral for a gradient vector function and so we can use the Fundamental Theorem for Line Integrals to do this problem.

Using the Fundamental Theorem to evaluate the integral gives the following,

$$\begin{aligned}\int_C \nabla f \bullet d\vec{r} &= f\left(\text{end point}\right) - f\left(\text{start point}\right)\\ &= f\left(0,-2\right) - f\left(-2,0\right)\\ &= -7-\left(-3\right) = \boxed{-4}\end{aligned}$$

Remember that all the Fundamental Theorem requires is the starting and ending point of the curve and the function used to generate the gradient vector field.

Section 5-6 : Conservative Vector Fields

1. Determine if the following vector field is conservative.

$$\vec{F} = \left(x^3 - 4xy^2 + 2\right)\vec{i} + \left(6x - 7y + x^3y^3\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify *P* and *Q* then run through the test.

So,

$$P = x^3 - 4xy^2 + 2 \qquad P_y = -8xy$$
$$Q = 6x - 7y + x^3y^3 \qquad Q_x = 6 + 3x^2y^3$$

Okay, we can clearly see that $P_y \neq Q_x$ and so the vector field is **not conservative.**

---

2. Determine if the following vector field is conservative.

$$\vec{F} = \left(2x\sin\left(2y\right) - 3y^2\right)\vec{i} + \left(2 - 6xy + 2x^2\cos\left(2y\right)\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify *P* and *Q* then run through the test.

So,

$$P = 2x\sin\left(2y\right) - 3y^2 \qquad P_y = 4x\cos\left(2y\right) - 6y$$
$$Q = 2 - 6xy + 2x^2\cos\left(2y\right) \qquad Q_x = -6y + 4x\cos\left(2y\right)$$

Okay, we can clearly see that $P_y = Q_x$ and so the vector field is **conservative.**

---

3. Determine if the following vector field is conservative.

$$\vec{F} = \left(6 - 2xy + y^3\right)\vec{i} + \left(x^2 - 8y + 3xy^2\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify *P* and *Q* then run through the test.

So,

$$P = 6 - 2xy + y^3 \qquad P_y = -2x + 3y^2$$
$$Q = x^2 - 8y + 3xy^2 \qquad Q_x = 2x + 3y^2$$

Okay, we can clearly see that $P_y \neq Q_x$ and so the vector field is **not conservative.**

Be careful with these problems. It is easy to get into a hurry and miss a very subtle difference between the two derivatives. In this case, the only difference between the two derivatives is the sign on the first term. That's it. That is also enough for this vector field to not be conservative.

---

4. Find the potential function for the following vector field.

$$\vec{F} = \left(6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}}\right)\vec{i} - \left(2x^2y - 4 - \sqrt{x}\right)\vec{j}$$

Step 1
Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let's check it anyway just to make sure.

So,

$$P = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} \qquad P_y = -4xy + \frac{1}{2\sqrt{x}}$$
$$Q = -\left(2x^2y - 4 - \sqrt{x}\right) \qquad Q_x = -4xy + \frac{1}{2\sqrt{x}}$$

Okay, we can see that $P_y = Q_x$ and so the vector field is conservative as the problem statement suggested it would be.

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the second component of the vector field. There won't always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won't be able to find a potential function for the vector field!

Step 2
Okay, to find the potential function for this vector field we know that we need to first either integrate *P* with respect to *x* or integrate *Q* with respect to *y*. It doesn't matter which one we use chose to use and, in this case, it looks like neither will be any harder than the other.

So, let's go with the following integration for this problem.

$$
\begin{aligned}
f(x,y) &= \int Q\,dy \\
&= \int -2x^2y + 4 + \sqrt{x}\,dy \\
&= -x^2y^2 + 4y + y\sqrt{x} + g(x)
\end{aligned}
$$

Don't forget that, in this case, because we were integrating with respect to *y* the "constant of integration" will be a function of *x*!

Step 3
Next, differentiate the function from the previous step with respect to *x* and set equal to *P* since we know the derivative and *P* should be the same function.

$$f_x = -2xy^2 + \frac{y}{2\sqrt{x}} + g'(x) = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} = P \qquad \Rightarrow \qquad g'(x) = 6x^2$$

Now, recall that because we integrated with respect to *y* in Step 2 $g(x)$, and hence $g'(x)$, should only be a function of *x*'s (as it is in this case). If there had been any *y*'s in $g'(x)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative. Of course, we verified that it was conservative in Step 1 and so this would in fact mean we'd made a mistake somewhere!

Step 4
We can now integrate both sides of the formula for $g'(x)$ above to get,

$$g(x) = 2x^3 + c$$

Don't forget the " + *c*" on this!

Step 5
Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y) = -x^2y^2 + 4y + y\sqrt{x} + 2x^3 + c}$$

---

5. Find the potential function for the following vector field.

$$\vec{F} = y^2\left(1+\cos(x+y)\right)\vec{i} + \left(2xy - 2y + y^2\cos(x+y) + 2y\sin(x+y)\right)\vec{j}$$

Step 1
Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let's check it anyway just to make sure.

So,

$$\begin{aligned} P &= y^2\left(1+\cos\left(x+y\right)\right) = y^2 + y^2\cos\left(x+y\right) & P_y &= 2y - y^2\sin\left(x+y\right) + 2y\cos\left(x+y\right) \\ Q &= 2xy - 2y + y^2\cos\left(x+y\right) + 2y\sin\left(x+y\right) & Q_x &= 2y - y^2\sin\left(x+y\right) + 2y\cos\left(x+y\right) \end{aligned}$$

Okay, we can see that $P_y = Q_x$ and so the vector field is conservative as the problem statement suggested it would be.

Step 2
Okay, to find the potential function for this vector field we know that we need to first either integrate *P* with respect to *x* or integrate *Q* with respect to *y*. It doesn't matter which one we use chose to use in general, but in in this case integrating *Q* with respect to *y* just looks painful (two integration by parts terms!).

So, let's go with the following integration for this problem.

$$\begin{aligned} f\left(x,y\right) &= \int P\,dx \\ &= \int y^2 + y^2\cos\left(x+y\right)dx \\ &= xy^2 + y^2\sin\left(x+y\right) + h\left(y\right) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to *x* the "constant of integration" will be a function of *y*!

Note, that as this problem has shown, sometimes one integration order will be significantly easier than the other so be on the lookout for which term might be easier to integrate.

Step 3
Next, differentiate the function from the previous step with respect to *y* and set equal to *Q* since we know the derivative and *Q* should be the same function.

$$\begin{aligned} f_y &= 2xy + 2y\sin\left(x+y\right) + y^2\cos\left(x+y\right) + h'\left(y\right) \\ &= 2xy - 2y + y^2\cos\left(x+y\right) + 2y\sin\left(x+y\right) = Q \qquad \Rightarrow \qquad h'\left(y\right) = -2y \end{aligned}$$

Now, recall that because we integrated with respect to *x* in Step 2 $h\left(y\right)$, and hence $h'\left(y\right)$, should only be a function of *y*'s (as it is in this case). If there had been any *x*'s in $h'\left(y\right)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative. Of course, we verified that it was conservative in Step 1 and so this would in fact mean we'd made a mistake somewhere!

Step 4
We can now integrate both sides of the formula for $h'\left(y\right)$ above to get,

$$h(y) = -y^2 + c$$

Don't forget the " + *c*" on this!

Step 5
Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y) = xy^2 + y^2 \sin(x+y) - y^2 + c}$$

---

6. Find the potential function for the following vector field.

$$\vec{F} = \left(2z^4 - 2y - y^3\right)\vec{i} + \left(z - 2x - 3xy^2\right)\vec{j} + \left(6 + y + 8xz^3\right)\vec{k}$$

Step 1
Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don't know how to verify this for a 3D vector field we will just need to trust that it is.

Let's start off the problem by labeling each of the components to make the problem easier to deal with as follows.

$$\begin{aligned} P &= 2z^4 - 2y - y^3 \\ Q &= z - 2x - 3xy^2 \\ R &= 6 + y + 8xz^3 \end{aligned}$$

Step 2
To find the potential function for this vector field we know that we need to first either integrate *P* with respect to *x*, integrate *Q* with respect to *y* or *R* with respect *z*. It doesn't matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the others.

So, let's go with the following integration for this problem.

$$\begin{aligned} f(x,y,z) &= \int Q\,dy \\ &= \int z - 2x - 3xy^2\,dy \\ &= zy - 2xy - xy^3 + h(x,z) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to *y* the "constant of integration" will be a function of *x* and/or *z*!

Step 3
Next, we can differentiate the function from the previous step with respect to *x* and set equal to *P* or differentiate the function with respect to *z* and set equal to *R*.

Again, neither looks any more difficult than the other so let's differentiate with respect to *z*.

$$f_z = y + h_z\left(x,z\right) = 6 + y + 8xz^3 = R \qquad \Rightarrow \qquad h_z\left(x,z\right) = 6 + 8xz^3$$

Now, recall that because we integrated with respect to *y* in Step 2 $h\left(x,z\right)$, and hence $h_z\left(x,z\right)$, should only be a function of *x*'s and *z*'s (as it is in this case). If there had been any *y*'s in $h_z\left(x,z\right)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect $h_z\left(x,z\right)$ to have both *x*'s and *z*'s in it. It is completely possible for one (or both) of the variables to not be present!

Step 4
We can now integrate both sides of the formula for $h_z\left(x,z\right)$ with respect to *z* to get,

$$h\left(x,z\right) = 6z + 2xz^4 + g\left(x\right)$$

Now, because $h\left(x,z\right)$ was a function of both *x* and *z* and we integrated with respect to *z* here the "constant of integration" in this case would need to be a function of *x*, $g\left(x\right)$ in this case.

The potential function is now,

$$f\left(x,y,z\right) = zy - 2xy - xy^3 + 6z + 2xz^4 + g\left(x\right)$$

Step 5
Next, we'll need to differentiate the potential function from Step 4 with respect to *x* and set equal to *P*. Doing this gives,

$$f_x = -2y - y^3 + 2z^4 + g'\left(x\right) = 2z^4 - 2y - y^3 = P \qquad \Rightarrow \qquad g'\left(x\right) = 0$$

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to *x* and so $g\left(x\right)$ should only be a function of *x*. Had $g'\left(x\right)$ contained either *y*'s or *z*'s we'd know that either we'd made a mistake or the vector field was not conservative.

Also, as shown in this problem, it is completely possible for there to be no *x*'s at all in $g'\left(x\right)$.

Step 6

Integrating both sides of the formula for $g'(x)$ from Step 5 and we can see that we must have $g(x) = c$.

Step 7
Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y,z) = zy - 2xy - xy^3 + 6z + 2xz^4 + c}$$

---

7. Find the potential function for the following vector field.

$$\vec{F} = \frac{2xy}{z^3}\vec{i} + \left(2y - z^2 + \frac{x^2}{z^3}\right)\vec{j} - \left(4z^3 + 2yz + \frac{3x^2 y}{z^4}\right)\vec{k}$$

Step 1
Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don't know how to verify this for a 3D vector field we will just need to trust that it is.

Let's start off the problem by labeling each of the components to make the problem easier to deal with as follows.

$$P = \frac{2xy}{z^3}$$
$$Q = 2y - z^2 + \frac{x^2}{z^3}$$
$$R = -\left(4z^3 + 2yz + \frac{3x^2 y}{z^4}\right) = -4z^3 - 2yz - \frac{3x^2 y}{z^4}$$

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the third component of the vector field. There won't always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won't be able to find a potential function for the vector field!

Step 2
To find the potential function for this vector field we know that we need to first either integrate *P* with respect to *x*, integrate *Q* with respect to *y* or *R* with respect *z*. It doesn't matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the other.

In this case the *R* has quite a few terms in it so let's integrate that one first simply because it might mean less work when dealing with *P* and *Q* in later steps.

$$\begin{aligned} f(x,y,z) &= \int R\,dz \\ &= \int -4z^3 - 2yz - 3x^2 y\,z^{-4}\,dz \\ &= -z^4 - yz^2 + x^2 y\,z^{-3} + h(x,y) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to *z* the "constant of integration" will be a function of *x* and/or *y*!

Step 3
Next, we can differentiate the function from the previous step with respect to *x* and set equal to *P* or differentiate the function with respect to *y* and set equal to *Q*.

Let's differentiate with respect to *y* in this case.

$$f_y = -z^2 + x^2\,z^{-3} + h_y(x,y) = 2y - z^2 + x^2 z^{-3} = Q \qquad \Rightarrow \qquad h_y(x,y) = 2y$$

Now, recall that because we integrated with respect to *z* in Step 2 $h(x,y)$, and hence $h_y(x,y)$, should only be a function of *x*'s and *y*'s (as it is in this case). If there had been any *z*'s in $h_y(x,y)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect $h_y(x,y)$ to have both *x*'s and *y*'s in it. It is completely possible for one (or both) of the variables to not be present!

Step 4
We can now integrate both sides of the formula for $h_y(x,y)$ with respect to *y* to get,

$$h(x,y) = y^2 + g(x)$$

Now, because $h(x,y)$ was a function of both *x* and *y* and we integrated with respect to *y* here the "constant of integration" in this case would need to be a function of *x*, $g(x)$ in this case.

The potential function is now,

$$f(x,y,z) = -z^4 - yz^2 + x^2 y\,z^{-3} + y^2 + g(x)$$

Step 5
Next, we'll need to differentiate the potential function from Step 4 with respect to *x* and set equal to *P*. Doing this gives,

$$f_x = 2xy\,z^{-3} + g'(x) = 2xyz^{-3} = P \qquad \Rightarrow \qquad g'(x) = 0$$

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to *x* and $g(x)$ should only be a function of *x*. Had $g'(x)$ contained either *y*'s or *z*'s we'd know that either we'd made a mistake or the vector field was not conservative.

Also, as shown in this problem it is completely possible for there to be no *x*'s at all in $g'(x)$.

Step 6
Integrating both sides of the formula for $g'(x)$ from Step 5 and we can see that we must have $g(x) = c$.

Step 7
Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y,z) = -z^4 - yz^2 + x^2 y z^{-3} + y^2 + c}$$

---

8. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where *C* is the portion of the circle centered at the origin with radius 2 in the 1st quadrant with counter clockwise rotation and $\vec{F}(x,y) = \left(2xy - 4 - \frac{1}{2}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right)\right)\vec{i} + \left(x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right)\right)\vec{j}$.

Step 1
There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably very difficult in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let's go the easy way and check to see if the vector field is conservative.

$$P = 2xy - 4 - \tfrac{1}{2}\sin\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}y\right) \qquad P_y = 2x - \tfrac{1}{4}\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}y\right)$$
$$Q = x^2 + \tfrac{1}{2}\cos\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}y\right) \qquad Q_x = 2x - \tfrac{1}{4}\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}y\right)$$

So, we can see that $P_y = Q_x$ and so the vector field is conservative.

Step 2
Now we just need to find the potential function for the vector field. We'll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let's integrate *P* with respect to *x*.

$$\begin{aligned} f(x,y) &= \int P\,dx \\ &= \int 2xy - 4 - \tfrac{1}{2}\sin\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}y\right)dx \\ &= x^2y - 4x + \cos\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}y\right) + h(y) \end{aligned}$$

Now, differentiate with respect to *y* and set equal to *Q*.

$$f_y = x^2 + \tfrac{1}{2}\cos\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}y\right) + h'(y) = x^2 + \tfrac{1}{2}\cos\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}y\right) = Q \qquad \Rightarrow \qquad h'(y) = 0$$

Solving for $h(y)$ gives $h(y) = c$ and so the potential function for this vector field is,

$$f(x,y) = x^2y - 4x + \cos\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}y\right) + c$$

Step 3
Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

$$\int_C \vec{F}\bullet d\vec{r} = f(\text{end point}) - f(\text{start point})$$

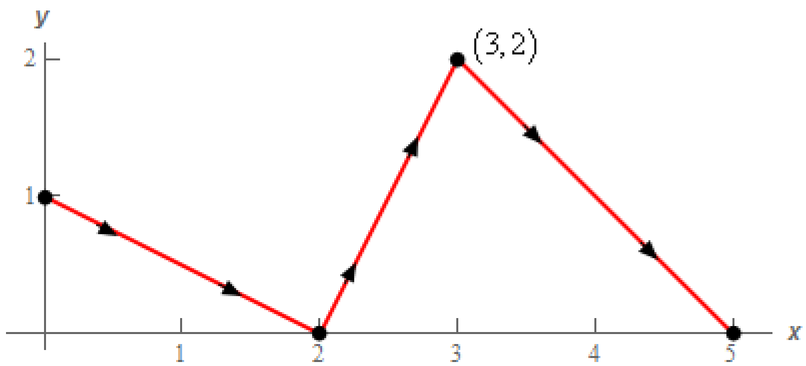
From the problem statement we know that *C* is the portion of the circle of radius 2 in the 1st quadrant with counter clockwise rotation. Therefore, the starting point of *C* is $(2,0)$ and the ending point of *C* is $(0,2)$.

The integral is then,

$$\int_C \vec{F}\bullet d\vec{r} = f(0,2) - f(2,0) = (\sin(1) + c) - (-8 + c) = \boxed{\sin(1) + 8 = 8.8415}$$

---

9. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = \left(2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2}\right)\vec{i} + \left(2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2}\right)\vec{j}$ and *C* is the curve shown below.



Step 1
There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably quite unpleasant in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let's go the easy way and check to see if the vector field is conservative.

$$P = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} \qquad\qquad P_y = 2\mathbf{e}^{xy} + 2xy\mathbf{e}^{xy} - 4xy\mathbf{e}^{x^2-y^2}$$
$$Q = 2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2} \qquad\qquad Q_x = 2\mathbf{e}^{xy} + 2xy\mathbf{e}^{xy} - 4xy\mathbf{e}^{x^2-y^2}$$

So, we can see that $P_y = Q_x$ and so the vector field is conservative.

Step 2
Now we just need to find the potential function for the vector field. We'll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let's integrate *Q* with respect to *y*.

$$\begin{aligned} f(x,y) &= \int Q\,dy \\ &= \int 2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2}\,dy \\ &= 2\mathbf{e}^{xy} + \mathbf{e}^{x^2-y^2} + g(x) \end{aligned}$$

Now, differentiate with respect to *x* and set equal to *P*.

$$f_x = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} + g'(x) = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} = P \qquad \Rightarrow \qquad g'(x) = 0$$

Solving for $g(x)$ gives $g(x) = c$ and so the potential function for this vector field is,

$$f(x,y) = 2\mathbf{e}^{xy} + \mathbf{e}^{x^2-y^2} + c$$

Step 3
Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

$$\int_C \vec{F} \bullet d\vec{r} = f\left(\text{end point}\right) - f\left(\text{start point}\right)$$

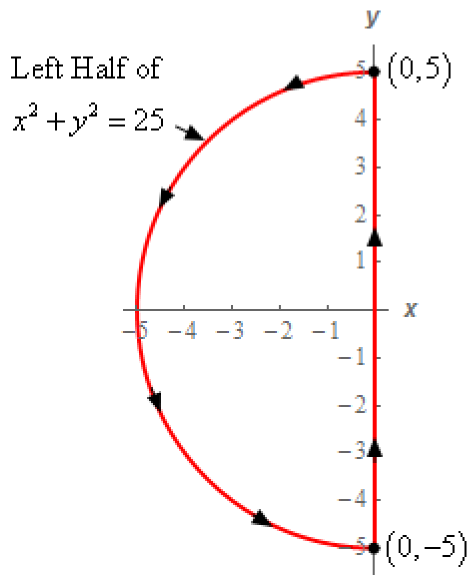
From the graph in the problem statement we can see that the starting point of *C* is $\left(0,1\right)$ and the ending point of *C* is $\left(5,0\right)$.

The integral is then,

$$\int_C \vec{F} \bullet d\vec{r} = f\left(5,0\right) - f\left(0,1\right) = \left(2 + \mathbf{e}^{25} + c\right) - \left(2 + \mathbf{e}^{-1} + c\right) = \boxed{\mathbf{e}^{25} - \mathbf{e}^{-1}}$$

Section 5-7 : Green's Theorem

1. Use Green's Theorem to evaluate $\int\limits_C yx^2\,dx - x^2\,dy$ where *C* is shown below.



Step 1
Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = yx^2 \qquad\qquad Q = -x^2$$

Remember that *P* is multiplied by *dx* and *Q* is multiplied by *dy* and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms.

Step 2
Using Green's Theorem the line integral becomes,

$$\int\limits_C yx^2\,dx - x^2\,dy = \iint\limits_D -2x - x^2\,dA$$

D is the region enclosed by the curve.

Step 3
Since *D* is just a half circle it makes sense to use polar coordinates for this problem. The limits for *D* in polar coordinates are,

$$\tfrac{1}{2}\pi \le t \le \tfrac{3}{2}\pi$$
$$0 \le r \le 5$$

Step 4
Now all we need to do is evaluate the double integral, after first converting to polar coordinates of course.

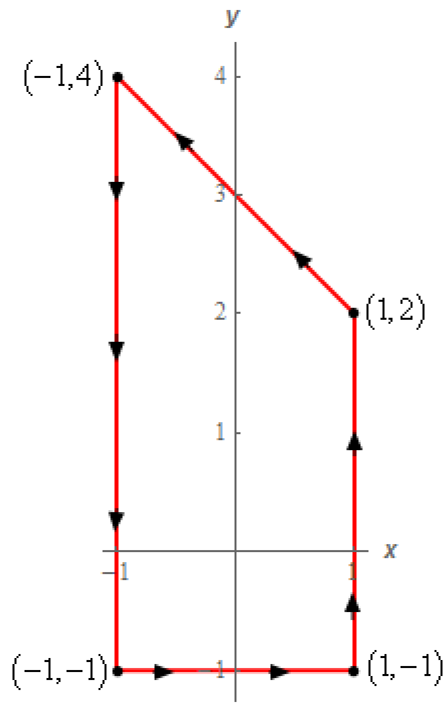
Here is the evaluation work.

$$\begin{aligned}
\int_C yx^2\,dx - x^2\,dy &= \iint_D -2x - x^2\,dA \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_0^5 r\left(-2r\cos\theta - r^2\cos^2\theta\right)dr\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_0^5 -2r^2\cos\theta - \tfrac{1}{2}r^3\left(1+\cos\left(2\theta\right)\right)dr\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\left(-\tfrac{2}{3}r^3\cos\theta - \tfrac{1}{8}r^4\left(1+\cos\left(2\theta\right)\right)\right)\Big|_0^5\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} -\tfrac{250}{3}\cos\theta - \tfrac{625}{8}\left(1+\cos\left(2\theta\right)\right)d\theta \\
&= \left[-\tfrac{250}{3}\sin\theta - \tfrac{625}{8}\left(\theta + \tfrac{1}{2}\sin\left(2\theta\right)\right)\right]\Bigg|_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \\
&= \boxed{\tfrac{500}{3} - \tfrac{625}{8}\pi = -78.7703}
\end{aligned}$$

Don't forget the extra *r* from converting the *dA* to polar coordinates and make sure you recall all the various tirg identities we need to deal with many of the various trig functions that show up in the integrals.

---

2. Use Green's Theorem to evaluate $\int_C \left(6y - 9x\right)dy - \left(yx - x^3\right)dx$ where *C* is shown below.



Step 1
Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = -\left(yx - x^3\right) = x^3 - yx \qquad\qquad Q = 6y - 9x$$

Remember that *P* is multiplied by *dx* and *Q* is multiplied by *dy* and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms. It is also easy to get in a hurry and just assume that *P* is the first term in the integral and *Q* is the second. That is clearly not the case here so be careful!

Step 2
Using Green's Theorem the line integral becomes,

$$\int_C \left(6y - 9x\right)dy - \left(yx - x^3\right)dx = \iint_D -9 - \left(-x\right)dA = \iint_D x - 9\,dA$$

D is the region enclosed by the curve.

Step 3
We'll leave it to you to verify that the equation of the line along the top of the region is given by $y = 3 - x$. Once we have this equation the region is then very easy to get limits for. They are,

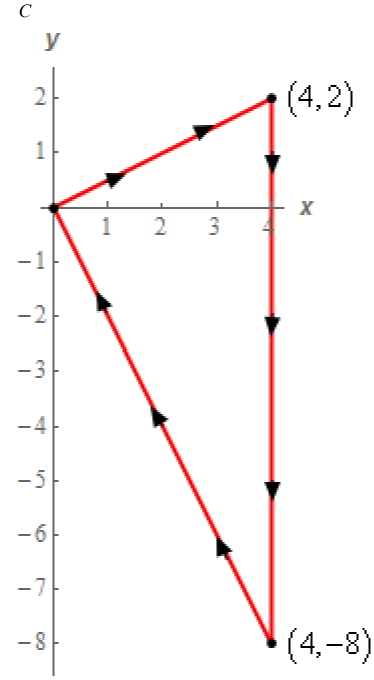
$$-1 \le x \le 1$$
$$-1 \le y \le 3 - x$$

Step 4
Now all we need to do is evaluate the double integral. Here is the evaluation work.

$$\begin{aligned}\int_C (6y - 9x)\,dy - \left(yx - x^3\right)dx &= \iint_D x - 9\,dA \\ &= \int_{-1}^{1}\int_{-1}^{3-x} x - 9\,dy\,dx \\ &= \int_{-1}^{1} (x-9)\,y\Big|_{-1}^{3-x}\,dx \\ &= \int_{-1}^{1} (x-9)(4-x)\,dx \\ &= \int_{-1}^{1} -x^2 + 13x - 36\,dx \\ &= \left[-\tfrac{1}{3}x^3 + \tfrac{13}{2}x^2 - 36x\right]\Big|_{-1}^{1} = \boxed{-\tfrac{218}{3}}\end{aligned}$$

---

3. Use Green's Theorem to evaluate $\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)dy$ where C is shown below.



Step 1
Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will NOT be over the enclosed area and so this path does NOT have the positive orientation. This, in turn, means that we can't actually use Green's Theorem to evaluate the given integral.

However, if *C* has the negative orientation then –*C* will have the positive orientation and we know how to relate the values of the line integrals over these two curves. Specifically, we know that,

$$\int_C x^2 y^2\,dx + \left(yx^3 + y^2\right)dy = -\int_{-C} x^2 y^2\,dx + \left(yx^3 + y^2\right)dy$$

So, instead of using Green's Theorem to compute the value of the integral in the problem statement we'll use Green's Theorem to compute the value of the following integral.

$$\int_{-C} x^2 y^2\,dx + \left(yx^3 + y^2\right)dy$$

From this integral we have,

$$P = x^2 y^2 \qquad\qquad Q = yx^3 + y^2$$

Remember that *P* is multiplied by *dx* and *Q* is multiplied by *dy*.

Step 2

Using Green's Theorem the line integral from over $-C$ becomes,

$$\int_{-C} x^2 y^2\,dx + \left(yx^3 + y^2\right)dy = \iint_D 3yx^2 - 2yx^2\,dA = \iint_D yx^2\,dA$$

D is the region enclosed by the curve.

Step 3

We'll leave it to you to verify that the equation of the line along the top of the region is given by $y = \frac{1}{2}x$ and the equation of the line along the bottom of the region is given by $y = -2x$. Once we have these equations the region is then very easy to get limits for. They are,

$$0 \le x \le 4$$
$$-2x \le y \le \tfrac{1}{2}x$$

Step 4

Now all we need to do is evaluate the double integral to get the value of the line integral from $-C$. Here is the evaluation work.

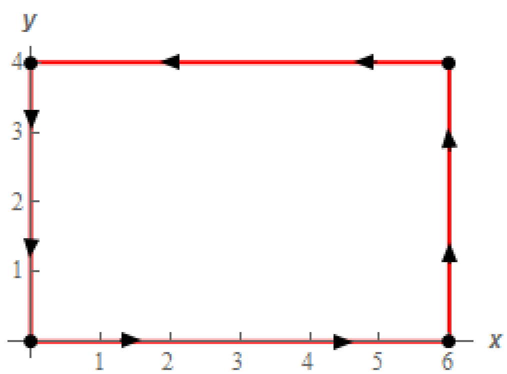
$$\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)dy = \iint_D yx^2\,dA$$
$$= \int_0^4 \int_{-2x}^{\frac{1}{2}x} yx^2\,dy\,dx$$
$$= \int_0^4 \left. \tfrac{1}{2}y^2x^2 \right|_{-2x}^{\frac{1}{2}x} dx$$
$$= \int_0^4 -\tfrac{15}{8}x^4\,dx$$
$$= \left. -\tfrac{3x^5}{8} \right|_0^4 = \boxed{-384}$$

Step 5
Okay, now we can't forget that the integral in Step 4 was not the integral we were asked to find the value of. Using the relationship between the value of the integrals over *C* and –*C* we know that the value of the integral we were asked to compute is,

$$\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)dy = -\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)dy = \boxed{384}$$

---

4. Use Green's Theorem to evaluate $\int_C \left(y^4 - 2y\right)dx - \left(6x - 4xy^3\right)dy$ where *C* is shown below.



Step 1
Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = y^4 - 2y \qquad\qquad Q = -\left(6x - 4xy^3\right) = 4xy^3 - 6x$$

Remember that *P* is multiplied by *dx* and *Q* is multiplied by *dy* and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms.

Step 2
Using Green's Theorem the line integral becomes,

$$\int\limits_C \left(y^4 - 2y\right)dx - \left(4xy^3 - 6x\right)dy = \iint\limits_D 4y^3 - 6 - \left(4y^3 - 2\right)dA = \iint\limits_D -4\,dA = -4\iint\limits_D dA$$

D is the region enclosed by the curve.

Step 3
Okay, we are now pretty much done with this problem. After factoring out the "-4" from the integral we can use the following fact to finish the evaluation of the integral.

$$\begin{aligned}\int\limits_C \left(y^4 - 2y\right)dx - \left(4xy^3 - 6x\right)dy &= -4\iint\limits_D dA \\ &= -4\left(\text{Area of } D\right) = -4\left[(6)(4)\right] = \boxed{-96}\end{aligned}$$

Don't forget the fact that,

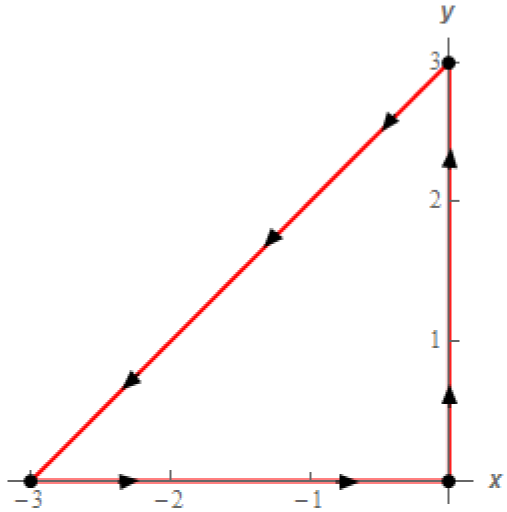
$$\iint\limits_D dA = \text{Area of } D$$

This is a very useful fact. It doesn't come up often but when it does it can reduce the amount of work required to finish out a problem.

Of course, this fact is really only useful if *D* is a region that we can easily determine the area for and in this case *D* was just a rectangle that we could easily get the width, 6, and the height, 4, from the graph in the problem statement.

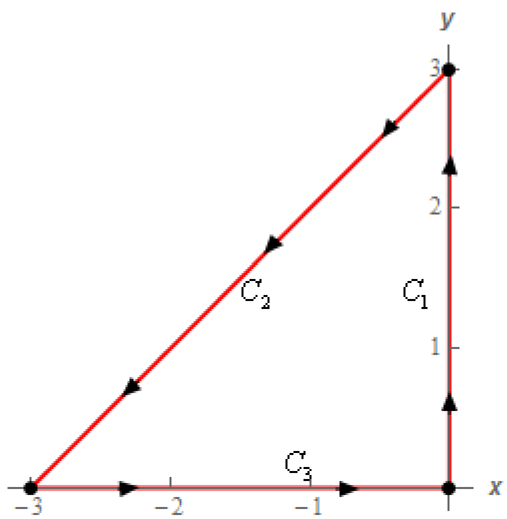
---

5. Verify Green's Theorem for $\oint\limits_C \left(xy^2 + x^2\right)dx + \left(4x - 1\right)dy$ where *C* is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.



(a) computing the line integral directly

Step 1
So, let's start off the problem with labeling the curves as follows,



Following the specified direction for each curve here are the parameterizations for each curve.

$$C_1 : \vec{r}\left(t\right) = \left\langle 0, t \right\rangle \qquad 0 \le t \le 3$$

$$C_2 : \vec{r}\left(t\right) = \left(1 - t\right)\left\langle 0, 3 \right\rangle + t\left\langle -3, 0 \right\rangle = \left\langle -3t, 3 - 3t \right\rangle \qquad 0 \le t \le 1$$

$$C_3 : \vec{r}\left(t\right) = \left\langle t, 0 \right\rangle \qquad -3 \le t \le 0$$

Step 2
Here is the line integral evaluated over each of these curves.

$$\oint_{C_1} \left(xy^2 + x^2\right)dx + \left(4x-1\right)dy = \int_0^3 \left[(0)(t)^2 + (0)^2\right](0)\,dt + \int_0^3 \left[4(0)-1\right](1)\,dt$$
$$= \int_0^3 -1\,dt = -t\Big|_0^3 = \underline{-3}$$

$$\oint_{C_2} \left(xy^2 + x^2\right)dx + \left(4x-1\right)dy = \int_0^1 \left[(-3t)(3-3t)^2 + (-3t)^2\right](-3)\,dt + \int_0^1 \left[4(-3t)-1\right](-3)\,dt$$
$$= \int_0^1 81t^3 - 189t^2 + 81t\,dt + \int_0^1 36t + 3\,dt$$
$$= \int_0^1 81t^3 - 189t^2 + 117t + 3\,dt$$
$$= \left(\tfrac{81}{4}t^4 - 63t^3 + \tfrac{117}{2}t^2 + 3t\right)\Big|_0^1 = \underline{\tfrac{75}{4}}$$

$$\oint_{C_3} \left(xy^2 + x^2\right)dx + \left(4x-1\right)dy = \int_{-3}^0 \left[(t)(0)^2 + (t)^2\right](1)\,dt + \int_{-3}^0 \left[4(t)-1\right](0)\,dt$$
$$= \int_{-3}^0 t^2\,dt = \tfrac{1}{3}t^3\Big|_{-3}^0 = \underline{9}$$

Step 3

Now, all we need to do is add up the results from the previous step to get the value of the line integral over the full curve. This gives,

$$\oint_C \left(xy^2 + x^2\right)dx + \left(4x-1\right)dy = (-3) + \left(\tfrac{75}{4}\right) + (9) = \boxed{\tfrac{99}{4}}$$

(b) using Green's Theorem to compute the line integral

Step 1

Note that as the circle on the integral implies the curve is in the positive direction and so we can use Green's Theorem on this integral.

From the integral we have,

$$P = xy^2 + x^2 \qquad\qquad Q = 4x - 1$$

Remember that *P* is multiplied by *dx* and *Q* is multiplied by *dy*.

Step 2

Using Green's Theorem the line integral becomes,

$$\oint_C \left(xy^2 + x^2\right)dx + \left(4x-1\right)dy = \iint_D 4 - (2xy)\,dA = \iint_D 4 - 2xy\,dA$$

D is the region enclosed by the curve.

Step 3

We'll leave it to you to verify that the equation of the line along the hypotenuse of the region is given by $y = x + 3$. Once we have this equation the region is then very easy to get limits for. They are,

$$\begin{array}{c} -3 \le x \le 0 \\ 0 \le y \le x + 3 \end{array}$$

Step 4

Now all we need to do is evaluate the double integral. Here is the evaluation work.

$$\begin{aligned}
\oint_C \left(xy^2 + x^2 \right) dx + \left(4x - 1 \right) dy &= \iint_D 4 - 2xy \, dA \\
&= \int_{-3}^{0} \int_{0}^{x+3} 4 - 2xy \, dy \, dx \\
&= \int_{-3}^{0} \left. \left(4y - xy^2 \right) \right|_{0}^{x+3} dx \\
&= \int_{-3}^{0} 12 - 5x - 6x^2 - x^3 \, dx \\
&= \left. \left(12x - \tfrac{5}{2} x^2 - 2x^3 - \tfrac{1}{4} x^4 \right) \right|_{-3}^{0} \\
&= \boxed{\tfrac{99}{4}}
\end{aligned}$$

So, we got the same answer after applying Green's Theorem to the line integral as we got by integrating the line integral directly.

Chapter 6 : Surface Integrals

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.

Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals – In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.

Surface Integrals of Vector Fields – In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.

Stokes' Theorem – In this section we will discuss Stokes' Theorem.

Divergence Theorem – In this section we will discuss the Divergence Theorem.

Section 6-1 : Curl and Divergence

1. Compute $\operatorname{div}\vec F$ and $\operatorname{curl}\vec F$ for $\vec F = x^2 y\,\vec i - \left(z^3 - 3x\right)\vec j + 4y^2\vec k$.

Step 1

Let's compute the divergence first and there isn't much to do other than run through the formula.

$$\operatorname{div}\vec F = \nabla \bullet \vec F = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(3x - z^3\right) + \frac{\partial}{\partial z}\left(4y^2\right) = \boxed{2xy}$$

Be careful to watch for minus signs in front of any of the vector components (2nd component in this case!). It is easy to get in a hurry and miss them.

Step 2

The curl is a little more work but still just formula work so here is the curl.

$$\begin{aligned}\operatorname{curl}\vec F = \nabla\times\vec F &= \begin{vmatrix} \vec i & \vec j & \vec k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & 3x - z^3 & 4y^2 \end{vmatrix} \\ &= \frac{\partial}{\partial y}\left(4y^2\right)\vec i + \frac{\partial}{\partial z}\left(x^2 y\right)\vec j + \frac{\partial}{\partial x}\left(3x - z^3\right)\vec k - \frac{\partial}{\partial y}\left(x^2 y\right)\vec k - \frac{\partial}{\partial x}\left(4y^2\right)\vec j - \frac{\partial}{\partial z}\left(3x - z^3\right)\vec i \\ &= 8y\vec i + 3\vec k - x^2\vec k + 3z^2\vec i \\ &= \boxed{\left(8y + 3z^2\right)\vec i + \left(3 - x^2\right)\vec k}\end{aligned}$$

Again, don't forget the minus sign on the 2nd component.

---

2. Compute $\operatorname{div}\vec F$ and $\operatorname{curl}\vec F$ for $\vec F = \left(3x + 2z^2\right)\vec i + \dfrac{x^3 y^2}{z}\vec j - \left(z - 7x\right)\vec k$.

Step 1

Let's compute the divergence first and there isn't much to do other than run through the formula.

$$\operatorname{div}\vec F = \nabla\bullet\vec F = \frac{\partial}{\partial x}\left(3x + 2z^2\right) + \frac{\partial}{\partial y}\left(\frac{x^3 y^2}{z}\right) + \frac{\partial}{\partial z}\left(7x - z\right) = \boxed{2 + \frac{2x^3 y}{z}}$$

Be careful to watch for minus signs in front of any of the vector components (3rd component in this case!). It is easy to get in a hurry and miss them.

Step 2
The curl is a little more work but still just formula work so here is the curl.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x+2z^2 & \dfrac{x^3y^2}{z} & 7x-z \end{vmatrix}$$

$$= \frac{\partial}{\partial y}\left(7x-z\right)\vec{i} + \frac{\partial}{\partial z}\left(3x+2z^2\right)\vec{j} + \frac{\partial}{\partial x}\left(\frac{x^3y^2}{z}\right)\vec{k}$$

$$- \frac{\partial}{\partial y}\left(3x+2z^2\right)\vec{k} - \frac{\partial}{\partial x}\left(7x-z\right)\vec{j} - \frac{\partial}{\partial z}\left(\frac{x^3y^2}{z}\right)\vec{i}$$

$$= 4z\vec{j} + \frac{3x^2y^2}{z}\vec{k} - 7\vec{j} + \frac{x^3y^2}{z^2}\vec{i}$$

$$= \boxed{\frac{x^3y^2}{z^2}\vec{i} + \left(4z-7\right)\vec{j} + \frac{3x^2y^2}{z}\vec{k}}$$

Again, don't forget the minus sign on the 3rd component.

---

3. Determine if the following vector field is conservative.

$$\vec{F} = \left(4y^2 + \frac{3x^2y}{z^2}\right)\vec{i} + \left(8xy + \frac{x^3}{z^2}\right)\vec{j} + \left(11 - \frac{2x^3y}{z^3}\right)\vec{k}$$

Step 1
We know all we need to do here is compute the curl of the vector field.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y^2 + \dfrac{3x^2 y}{z^2} & 8xy + \dfrac{x^3}{z^2} & 11 - \dfrac{2x^3 y}{z^3} \end{vmatrix}$$

$$\begin{aligned} &= \frac{\partial}{\partial y}\left(11 - \frac{2x^3 y}{z^3}\right)\vec{i} + \frac{\partial}{\partial z}\left(4y^2 + \frac{3x^2 y}{z^2}\right)\vec{j} + \frac{\partial}{\partial x}\left(8xy + \frac{x^3}{z^2}\right)\vec{k} \\ &\qquad - \frac{\partial}{\partial y}\left(4y^2 + \frac{3x^2 y}{z^2}\right)\vec{k} - \frac{\partial}{\partial x}\left(11 - \frac{2x^3 y}{z^3}\right)\vec{j} - \frac{\partial}{\partial z}\left(8xy + \frac{x^3}{z^2}\right)\vec{i} \\ &= -\frac{2x^3}{z^3}\vec{i} - \frac{6x^2 y}{z^3}\vec{j} + \left(8y + \frac{3x^2}{z^2}\right)\vec{k} - \left(8y + \frac{3x^2}{z^2}\right)\vec{k} + \frac{6x^2 y}{z^3}\vec{j} + \frac{2x^3}{z^3}\vec{i} \\ &= \underline{\vec{0}} \end{aligned}$$

Step 2

So, we found that $\operatorname{curl}\vec{F} = \vec{0}$ for this vector field and so the vector field **is conservative**.

---

4. Determine if the following vector field is conservative.

$$\vec{F} = 6x\,\vec{i} + \left(2y - y^2\right)\vec{j} + \left(6z - x^3\right)\vec{k}$$

Step 1

We know all we need to do here is compute the curl of the vector field.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6x & 2y - y^2 & 6z - x^3 \end{vmatrix}$$

$$\begin{aligned} &= \frac{\partial}{\partial y}\left(6z - x^3\right)\vec{i} + \frac{\partial}{\partial z}\left(6x\right)\vec{j} + \frac{\partial}{\partial x}\left(2y - y^2\right)\vec{k} \\ &\qquad - \frac{\partial}{\partial y}\left(6x\right)\vec{k} - \frac{\partial}{\partial x}\left(6z - x^3\right)\vec{j} - \frac{\partial}{\partial z}\left(2y - y^2\right)\vec{i} \\ &= \underline{3x^2\vec{j}} \end{aligned}$$

Step 2

So, we found that $\operatorname{curl}\vec{F} = 3x^2\vec{j} \ne \vec{0}$ for this vector field and so the vector field **is NOT conservative**.

Section 6-2 : Parametric Surfaces

1. Write down a set of parametric equations for the plane $7x+3y+4z=15$.

Step 1

There isn't a whole lot to this problem. There are three different acceptable answers here. To get a set of parametric equations for this plane all we need to do is solve for one of the variables and then write down the parametric equations.

For this problem let's solve for *z* to get,

$$z=\tfrac{15}{4}-\tfrac{7}{4}x-\tfrac{3}{4}y$$

Step 2

The parametric equation for the plane is then,

$$\vec{r}\left(x,y\right)=\left\langle x,y,z\right\rangle=\boxed{\left\langle x,y,\tfrac{15}{4}-\tfrac{7}{4}x-\tfrac{3}{4}y\right\rangle}$$

Remember that all we need to do to get the parametric equations is plug in the equation for *z* into the *z* component of the vector $\left\langle x,y,z\right\rangle$.

Also, as noted in Step 1 we could just have easily done either of the following two forms for the parametric equations for this plane.

$$\vec{r}\left(x,z\right)=\left\langle x,g\left(x,z\right),z\right\rangle \qquad\qquad \vec{r}\left(y,z\right)=\left\langle h\left(y,z\right),y,z\right\rangle$$

where you solve the equation of the plane for *y* or *x* respectively. All three set of parametric equations are all perfectly valid forms for the answer to this problem.

---

2. Write down a set of parametric equations for the plane $7x+3y+4z=15$ that lies in the 1st octant.

Step 1

This problem is really just an extension of the previous problem so we'll redo the set of parametric equations for the plane a little quicker this time.

First, we need to solve the equation for any of the three variables. We'll solve for *z* in this case to get,

$$z=\tfrac{15}{4}-\tfrac{7}{4}x-\tfrac{3}{4}y$$

The parametric equation for this plane is then,

$$\vec{r}\left(x,y\right)=\left\langle x,y,z\right\rangle=\left\langle x,y,\tfrac{15}{4}-\tfrac{7}{4}x-\tfrac{3}{4}y\right\rangle$$

Remember that all we need to do to get the parametric equations is plug in the equation for *z* into the *z* component of the vector $\left\langle x,y,z\right\rangle$.

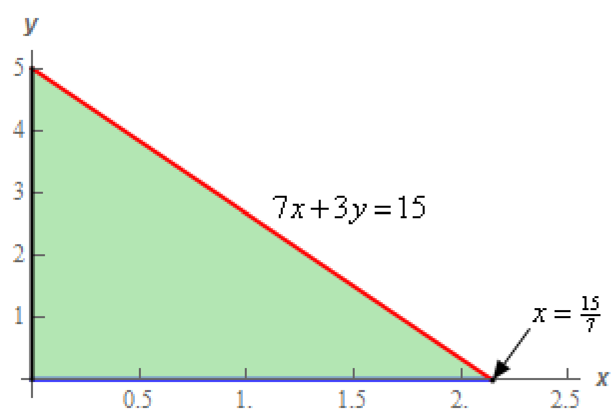
Step 2

Now, the set of parametric equations from above is for the full plane and that isn't what we want in this problem. In this problem we only want the portion of the plane that is in the 1^{st} octant.

So, we'll need to restrict *x* and *y* so that the parametric equation from Step 1 will only give the portion of the plane that is in the 1^{st} octant.

If you recall how to get the region *D* for a triple integral then you know how to do this because it is basically the same idea. In this case we need the region *D* in the *xy*-plane that will give the plane in the 1^{st} octant.

Here is a sketch of this region.



The hypotenuse is just where the plane intersects the *xy*-plane and so we can quickly find the equation of the line by setting $z=0$ in the equation of the plane.

We can either solve this for *x* or *y* to get the ranges for *x* and *y*. It doesn't really matter which we solve for here so let's just solve for *y* to get the following ranges for *x* and *y* to describe this triangle.

$$\begin{gathered}0\le x\le\tfrac{15}{7}\\0\le y\le-\tfrac{7}{3}x+5\end{gathered}$$

Putting this all together we get the following set of parametric equations for the plane that is in the 1^{st} octant.

$$\boxed{\vec{r}\left(x,y\right)=\left\langle x,y,\tfrac{15}{4}-\tfrac{7}{4}x-\tfrac{3}{4}y\right\rangle\qquad 0\le x\le\tfrac{15}{7}\ ,\ 0\le y\le-\tfrac{7}{3}x+5}$$

3. The cylinder $x^2 + y^2 = 5$ for $-1 \le z \le 6$.

Step 1
Because this surface is just a cylinder we just need the cylindrical coordinates conversion formulas with the polar coordinates in the *xy*-plane (since the cylinder is given in terms of *x* and *y*).

The conversion equations are,

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

However, recall that we are actually on the surface of the cylinder and so we know that $r = \sqrt{5}$. The conversion equations are then,

$$x = \sqrt{5}\cos\theta \qquad y = \sqrt{5}\sin\theta \qquad z = z$$

Step 2
We can now write down a set of parametric equations for the cylinder. They are,

$$\vec{r}(z,\theta) = \langle x, y, z\rangle = \left\langle \sqrt{5}\cos\theta, \sqrt{5}\sin\theta, z\right\rangle$$

Remember that all we do is plug the conversion formulas for *x*, *y*, and *z* into the *x*, *y* and *z* components of the vector $\langle x, y, z\rangle$ and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of *z* and θ those will also be the variables for our set of parametric equation.

Step 3
Now, the only issue with the set of parametric equations above is that they are for the full cylinder and we don't want that. We only want the cylinder in the given range of *z* so to finish this problem out all we need to do is add on a set of restrictions or ranges to our variables.

Doing that gives,

$$\boxed{\vec{r}(z,\theta) = \left\langle \sqrt{5}\cos\theta, \sqrt{5}\sin\theta, z\right\rangle \qquad -1 \le z \le 6\ ,\ 0 \le \theta \le 2\pi}$$

Note that the *z* range is just the range given in the problem statement and the θ range is the full zero to 2π range since there was no mention of restricting the portion of the cylinder that we wanted with respect to θ (for example, only the top half of the cylinder).

---

4. The portion of $y = 4 - x^2 - z^2$ that is in front of $y = -6$.

Step 1
Okay, the basic set of parametric equations in this case is pretty easy since we already have the equation in the form of "*y* = ".

The set of parametric equations that will give the full surface is just,

$$\vec{r}\left(x,z\right)=\left\langle x,y,z\right\rangle=\left\langle x,4-x^2-z^2,z\right\rangle$$

Remember that all we need to do to get the parametric equations is plug in the equation for *y* into the *y* component of the vector $\left\langle x,y,z\right\rangle$.

Step 2
Finally, all we need to do is restrict *x* and *z* to get only the portion of the surface we are looking for. That is pretty simple however since we are given that we only want the portion that is in front of $y=-6$.

This is equivalent to requiring that $y\ge-6$ and we do have the equation of the surface so all we need to do is plug that into the inequality and do a little rewrite. Doing this gives,

$$4-x^2-z^2\ge-6 \quad\to\quad x^2+z^2\le 10$$

In other words, we only want the points $\left(x,z\right)$ that are inside the disk of radius $\sqrt{10}$.

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

$$\boxed{\vec{r}\left(x,z\right)=\left\langle x,4-x^2-z^2,z\right\rangle \qquad x^2+z^2\le 10}$$

---

5. The portion of the sphere of radius 6 with $x\ge 0$.

Step 1
Because we have a portion of a sphere we'll start off with the spherical coordinates conversion formulas.

$$x=\rho\sin\varphi\cos\theta \qquad\qquad y=\rho\sin\varphi\sin\theta \qquad\qquad z=\rho\cos\varphi$$

However, we are actually on the surface of the sphere and so we know that $\rho=6$. With this the conversion formulas become,

$$x=6\sin\varphi\cos\theta \qquad\qquad y=6\sin\varphi\sin\theta \qquad\qquad z=6\cos\varphi$$

Step 2

The set of parametric equations that will give the full sphere is then,

$$\vec{r}\left(\theta,\varphi\right)=\left\langle x,y,z\right\rangle=\left\langle 6\sin\varphi\cos\theta,6\sin\varphi\sin\theta,6\cos\varphi\right\rangle$$

Remember that all we do is plug the conversion formulas for *x*, *y*, and *z* into the *x*, *y* and *z* components of the vector $\left\langle x,y,z\right\rangle$ and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of θ and φ those will also be the variables for our set of parametric equation.

Step 3
Finally, we need to deal with the fact that we don't actually want the full sphere here. We only want the portion of the sphere for which $x \ge 0$.

We can restrict *x* to this range if we restrict θ to the range $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

We've not put any restrictions on *z* and so that means that we'll take the full range of possible φ or $0 \le \varphi \le \pi$. Recall that φ is the angle a point in spherical coordinates makes with the positive *z*-axis and so that is the quantity we'd need to restrict if we'd wanted to restrict *z* (for example $z \le 0$).

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

$$\boxed{\vec{r}\left(\theta,\varphi\right)=\left\langle 6\sin\varphi\cos\theta,6\sin\varphi\sin\theta,6\cos\varphi\right\rangle \qquad -\tfrac{1}{2}\pi \le \theta \le \tfrac{1}{2}\pi \quad , \quad 0 \le \varphi \le \pi}$$

---

6. The tangent plane to the surface given by the following parametric equation at the point $\left(8,14,2\right)$.

$$\vec{r}\left(u,v\right)=\left(u^2+2u\right)\vec{i}+\left(3v-2u\right)\vec{j}+\left(6v-10\right)\vec{k}$$

Step 1
In order to write down the equation of a plane we need a point, which we have, $\left(8,14,2\right)$, and a normal vector, which we don't have yet.

However, recall that $\vec{r}_u \times \vec{r}_v$ will be normal to the surface. So, let's compute that.

$$\vec{r}_u=\left(2u+2\right)\vec{i}-2\vec{j} \qquad\qquad \vec{r}_v=3\vec{j}+6\vec{k}$$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2u+2 & -2 & 0\\ 0 & 3 & 6\end{vmatrix}=-12\vec{i}-6\left(2u+2\right)\vec{j}+3\left(2u+2\right)\vec{k}$$

Step 2
Now having $\vec{r}_u \times \vec{r}_v$ is all well and good but it is really only useful if we also know the point, (u,v) for which we are at $(8,14,2)$ so we next need to set the *x*, *y* and *z* coordinates of our point equal to the *x*, *y* and *z* components of our parametric equation to determine the value of *u* and *v* we need.

Here are the equations we get if we do that.

$$\begin{aligned} 8 &= u^2 + 2u \\ 14 &= 3v - 2u \\ 2 &= 6v - 10 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} 0 &= u^2 + 2u - 8 = (u+4)(u-2) \\ 14 &= 3v - 2u \\ 12 &= 6v \end{aligned}$$

Step 3
From the third equation above we can see that we must have $v = 2$ and from the first equation we can see that we must have either $u = -4$ or $u = 2$.

Plugging our only choice for *v* and both choices for *u* into the second equation we can see that we must have $u = -4$.

Step 4
Plugging $u = -4$ and $v = 2$ into the equation for $\vec{r}_u \times \vec{r}_v$ we will arrive at the following normal vector to the surface at $(8,14,2)$.

$$\vec{n} = \left(\vec{r}_u \times \vec{r}_v\right)_{u=-4,v=2} = -12\vec{i} + 36\vec{j} - 18\vec{k}$$

Note that, in this case, the normal vector didn't actually depend on the value of *v*. That won't happen in general, but as we've seen here that kind of thing can happen on occasion so don't get excited about it when it does.

The equation of the tangent plane to the surface at $(8,14,2)$ with normal vector $\vec{n} = -12\vec{i} + 36\vec{j} - 18\vec{k}$ is,

$$-12(x-8) + 36(y-14) - 18(z-2) = 0 \qquad \to \qquad -12x + 36y - 18z = 372$$

Step 5
To get a set of parametric equations for the tangent plane all we need to do is solve the equation for *z* to get,

$$z = -\tfrac{62}{3} - \tfrac{2}{3}x + 2y$$

We can then plug this into the vector $\langle x, y, z \rangle$ to get the following set of parametric equations for the tangent plane.

$$\boxed{\vec{r}\left(x,y\right)=\left\langle x,y,-\tfrac{62}{3}-\tfrac{2}{3}x+2y\right\rangle}$$

Note that there will be no restrictions on *x* and *y* because we wanted the full tangent plane.

---

7. Determine the surface area of the portion of $2x+3y+6z=9$ that is inside the cylinder $x^2+y^2=7$.

Step 1
We first need to parameterize the surface. Because we are wanting the portion that is inside the cylinder centered on the *z*-axis it makes sense to first solve the equation of the plane for *z* to get,

$$z=\tfrac{3}{2}-\tfrac{1}{3}x-\tfrac{1}{2}y$$

The parameterization for the full plane is then,

$$\vec{r}\left(x,y\right)=\left\langle x,y,\tfrac{3}{2}-\tfrac{1}{3}x-\tfrac{1}{2}y\right\rangle$$

We only want the portion that is inside the cylinder given in the problem statement so we'll also need to restrict *x* and *y* to those in the disk $x^2+y^2\le 7$. This will now give only the portion of the plane that is inside the cylinder.

Step 2
Next, we need to compute $\vec{r}_x\times\vec{r}_y$. Here is that work.

$$\vec{r}_x=\left\langle 1,0,-\tfrac{1}{3}\right\rangle \qquad \vec{r}_y=\left\langle 0,1,-\tfrac{1}{2}\right\rangle$$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -\tfrac{1}{3}\\ 0 & 1 & -\tfrac{1}{2}\end{vmatrix}=\tfrac{1}{3}\vec{i}+\tfrac{1}{2}\vec{j}+\vec{k}$$

Now, we what we really need is,

$$\left\|\vec{r}_x\times\vec{r}_y\right\|=\sqrt{\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{2}\right)^2+1}=\sqrt{\tfrac{49}{36}}=\tfrac{7}{6}$$

Step 3
The integral for the surface area is then,

$$A=\iint\limits_D \tfrac{7}{6}\,dA$$

In this case *D* is just the restriction on *x* and *y* that we noted in Step 1. So, *D* is just the disk $x^2 + y^2 \le 7$.

Step 4
Computing the integral in this case is very simple. All we need to do is take advantage of the fact that,

$$\iint_D dA = \text{Area of } D$$

So, the surface area is simply,

$$A = \iint_D \tfrac{7}{6}\, dA = \tfrac{7}{6}\iint_D dA = \tfrac{7}{6}\left[\text{Area of } D\right] = \tfrac{7}{6}\left[\pi\left(\sqrt{7}\right)^2\right] = \boxed{\tfrac{49}{6}\pi}$$

---

8. Determine the surface area of the portion of $x^2 + y^2 + z^2 = 25$ with $z \le 0$.

Step 1
We first need to parameterize the sphere and we've already done a sphere in this problem set so we won't go into great detail with the parameterization here.

The parameterization for the full sphere is,

$$\vec{r}\left(\theta,\varphi\right) = \left\langle 5\sin\varphi\cos\theta, 5\sin\varphi\sin\theta, 5\cos\varphi\right\rangle$$

We don't want the full sphere of course. We only want the lower half of the sphere, *i.e.* the portion with $z \le 0$. This means that we'll need to restrict φ to $\frac{1}{2}\pi \le \varphi \le \pi$. Recall that φ is the angle points make with the positive *z*-axis and because we only want points below the *xy*-plane we'll need the range of $\frac{1}{2}\pi \le \varphi \le \pi$.

We want the full lower half and so we'll use $0 \le \theta \le 2\pi$ for our θ range.

Step 2
Next, we need to compute $\vec{r}_\theta \times \vec{r}_\varphi$. Here is that work.

$$\vec{r}_\theta = \left\langle -5\sin\varphi\sin\theta, 5\sin\varphi\cos\theta, 0\right\rangle \qquad \vec{r}_\varphi = \left\langle 5\cos\varphi\cos\theta, 5\cos\varphi\sin\theta, -5\sin\varphi\right\rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin\varphi\sin\theta & 5\sin\varphi\cos\theta & 0 \\ 5\cos\varphi\cos\theta & 5\cos\varphi\sin\theta & -5\sin\varphi \end{vmatrix}$$

$$= -25\sin^2\varphi\cos\theta\,\vec{i} - 25\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 25\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} - 25\sin^2\varphi\sin\theta\,\vec{j}$$
$$= -25\sin^2\varphi\cos\theta\,\vec{i} - 25\sin^2\varphi\sin\theta\,\vec{j} - 25\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec{k}$$
$$= -25\sin^2\varphi\cos\theta\,\vec{i} - 25\sin^2\varphi\sin\theta\,\vec{j} - 25\sin\varphi\cos\varphi\,\vec{k}$$

Now, we what we really need is,

$$\begin{aligned}\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| &= \sqrt{\left(-25\sin^2\varphi\cos\theta\right)^2 + \left(-25\sin^2\varphi\sin\theta\right)^2 + \left(-25\sin\varphi\cos\varphi\right)^2} \\ &= \sqrt{625\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + 625\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{625\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)} \\ &= 25\left|\sin\varphi\right| \\ &= 25\sin\varphi\end{aligned}$$

Note that we can drop the absolute value bars on the sine because we know that sine will be positive in $\frac{1}{2}\pi \le \varphi \le \pi$.

Step 3
The integral for the surface area is then,

$$A = \iint\limits_D 25\sin\varphi\,dA = \int_0^{2\pi}\int_{\frac{1}{2}\pi}^{\pi} 25\sin\varphi\,d\varphi\,d\theta$$

As noted in the integral above *D* is just the ranges of θ and φ we found in Step 1.

Step 4
Now we just need to evaluate the integral to get the surface area.

$$A = \int_0^{2\pi}\int_{\frac{1}{2}\pi}^{\pi} 25\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi} -25\cos\varphi\Big|_{\frac{1}{2}\pi}^{\pi}\,d\theta = \int_0^{2\pi} 25\,d\theta = \boxed{50\pi}$$

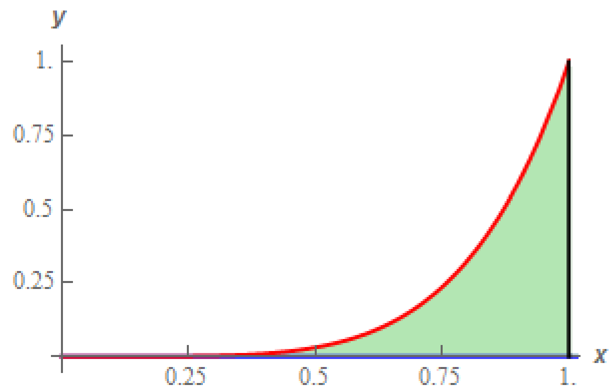
---

9. Determine the surface area of the portion of $z = 3 + 2y + \dfrac{1}{4}x^4$ that is above the region in the *xy*-plane bounded by $y = x^5$, $x = 1$ and the *x*-axis.

Step 1
Parameterizing this surface is pretty simple. We have the equation of the surface in the form $z = f\left(x,y\right)$ and so the parameterization of the surface is,

$$\vec r\left(x,y\right) = \left\langle x, y, 3 + 2y + \tfrac{1}{4}x^4 \right\rangle$$

Now, this is the parameterization of the full surface and we only want the portion that lies over the following region.



So, to get only the portion of the surface we'll need to restrict *x* and *y* to the following ranges,

$$\begin{gathered} 0 \le x \le 1 \\ 0 \le y \le x^5 \end{gathered}$$

On a side note we can see that we are in the 1st quadrant here and so we know that $x \ge 0$ and $y \ge 0$. Therefore, we can see that the surface in the 1st quadrant is always above the *xy*-plane and so will in fact always be above the region above as suggested in the problem statement.

Step 2
Next, we need to compute $\vec r_x \times \vec r_y$. Here is that work.

$$\vec r_x = \left\langle 1, 0, x^3 \right\rangle \qquad\qquad \vec r_y = \left\langle 0, 1, 2 \right\rangle$$

$$\vec r_x \times \vec r_y = \left| \begin{matrix} \vec i & \vec j & \vec k \\ 1 & 0 & x^3 \\ 0 & 1 & 2 \end{matrix} \right| = -x^3\vec i - 2\vec j + \vec k$$

Now, we what we really need is,

$$\left\| \vec r_x \times \vec r_y \right\| = \sqrt{\left(-x^3\right)^2 + \left(-2\right)^2 + \left(1\right)^2} = \sqrt{x^6 + 5}$$

Step 3

The integral for the surface area is then,

$$A = \iint\limits_D \sqrt{x^6 + 5}\, dA = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx$$

As noted in the integral above *D* is just the ranges of *x* and *y* we found in Step 1.

Step 4

Now we just need to evaluate the integral to get the surface area.

$$\begin{aligned} A &= \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx = \int_0^1 y\sqrt{x^6 + 5}\Big|_0^{x^5} dx \\ &= \int_0^1 x^5 \sqrt{x^6 + 5}\, dx = \tfrac{1}{9}\left(x^6 + 5\right)^{\frac{3}{2}}\Big|_0^1 = \boxed{\tfrac{1}{9}\left(6^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = 0.3907} \end{aligned}$$

---

10. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder $u^2 + v^2 = 4$.

$$\vec{r}\left(u, v\right) = \left\langle 2u, vu, 1 - 2v \right\rangle$$

Step 1

We've already been given the parameterization of the surface in the problem statement so we don't need to worry about that for this problem. All we really need to do yet is to acknowledge that we'll need to restrict *u* and *v* to the disk $u^2 + v^2 \le 4$.

Step 2

Next, we need to compute $\vec{r}_u \times \vec{r}_v$. Here is that work.

$$\vec{r}_u = \left\langle 2, v, 0 \right\rangle \qquad \vec{r}_v = \left\langle 0, u, -2 \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & v & 0 \\ 0 & u & -2 \end{vmatrix} = -2v\vec{i} + 4\vec{j} + 2u\vec{k}$$

Now, we what we really need is,

$$\left\| \vec{r}_u \times \vec{r}_v \right\| = \sqrt{\left(-2v\right)^2 + \left(4\right)^2 + \left(2u\right)^2} = \sqrt{4u^2 + 4v^2 + 16} = 2\sqrt{u^2 + v^2 + 4}$$

Step 3

The integral for the surface area is then,

$$A = \iint_D 2\sqrt{u^2 + v^2 + 4}\,dA$$

Where *D* is the disk $u^2 + v^2 \le 4$.

Step 4
Because *D* is a disk the best bet for this integral is to use the following "version" of polar coordinates.

$$u = r\cos\theta \qquad v = r\sin\theta \qquad u^2 + v^2 = r^2 \qquad dA = r\,dr\,d\theta$$

The polar coordinate limits for this *D* is,

$$\begin{aligned} &0 \le \theta \le 2\pi \\ &0 \le r \le 2 \end{aligned}$$

So, the integral to converting to polar coordinates gives,

$$A = \iint_D 2\sqrt{u^2 + v^2 + 4}\,dA = \int_0^{2\pi}\int_0^2 2r\sqrt{r^2 + 4}\,dr\,d\theta$$

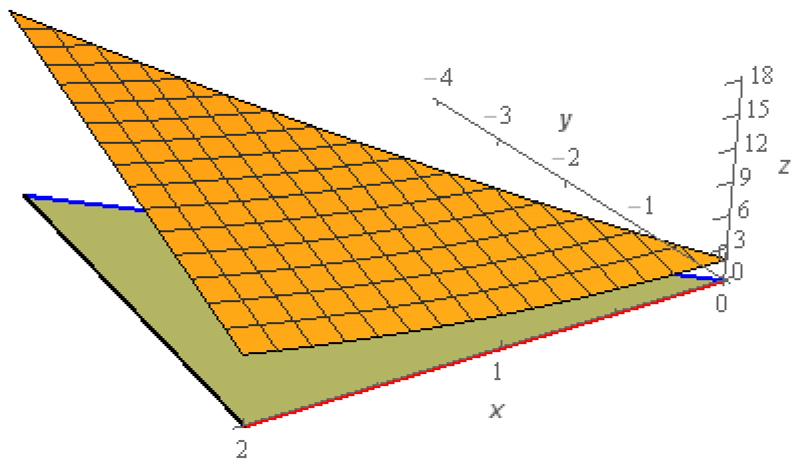
Step 5
Now we just need to evaluate the integral to get the surface area.

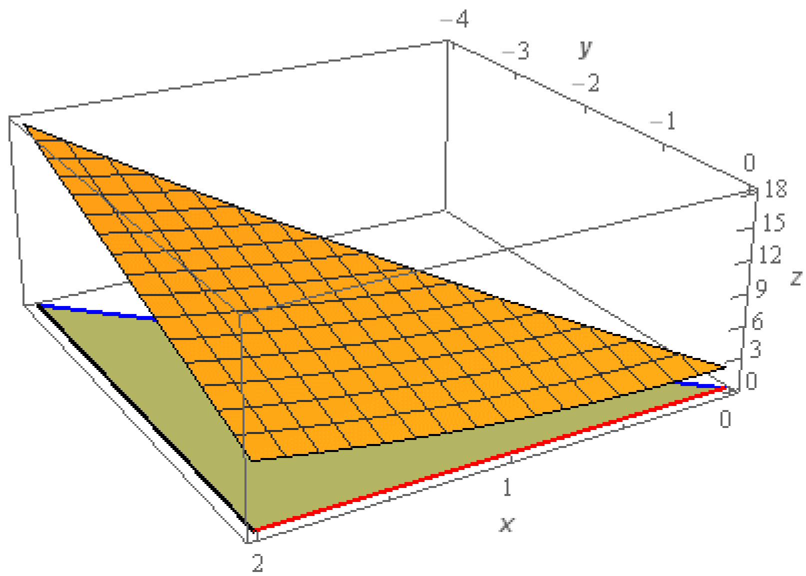
$$\begin{aligned} A &= \int_0^{2\pi}\int_0^2 2r\sqrt{r^2 + 4}\,dr\,d\theta = \int_0^{2\pi} \left.\tfrac{2}{3}\left(r^2 + 4\right)^{\frac{3}{2}}\right|_0^2 d\theta \\ &= \int_0^{2\pi} \tfrac{2}{3}\left(8^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)d\theta = \left.\tfrac{2}{3}\left(8^{\frac{3}{2}} - 8\right)\theta\right|_0^{2\pi} = \boxed{\tfrac{32}{3}\pi\left(\sqrt{8} - 1\right) = 61.2712} \end{aligned}$$

Section 6-3 : Surface Integrals

1. Evaluate $\iint\limits_S z + 3y - x^2 \, dS$ where *S* is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the *xy*-plane with vertices $(0,0)$, $(2,0)$ and $(2,-4)$.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.

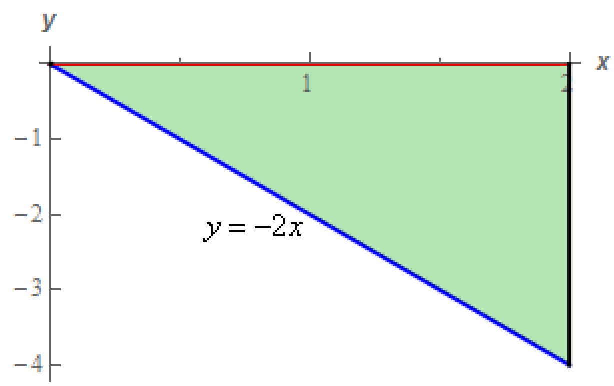




We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

The orange surface is the sketch of $z = 2 - 3y + x^2$ that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region *D* for this problem.

Here is a quick sketch of *D* just to get a better view of it than the mostly obscured view in the sketch above.



We could use either of the following sets of limits to describe *D*.

$$\begin{array}{r|r} 0 \le x \le 2 & -4 \le y \le 0 \\ -2x \le y \le 0 & -\frac{1}{2}y \le x \le 2 \end{array}$$

We'll decide which set to use in the integral once we get that set up.

Step 2

Let's get the integral set up now. In this case the surface is in the form,

$$z = g\left(x,y\right) = 2 - 3y + x^2$$

so we'll use the following formula for the surface integral.

$$\iint\limits_S f\left(x,y,z\right)dS = \iint\limits_D f\left(x,y,g\left(x,y\right)\right)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dA$$

The integral is then,

$$\begin{aligned} \iint\limits_S z + 3y - x^2\,dS &= \iint\limits_D \left[\left(2 - 3y + x^2\right) + 3y - x^2\right]\sqrt{\left(2x\right)^2 + \left(-3\right)^2 + 1}\,dA \\ &= \iint\limits_D 2\sqrt{4x^2 + 10}\,dA \end{aligned}$$

Don't forget to plug the equation of the surface into *z* in the integrand and recall that *D* is the triangle sketched in Step 1.

Step 3
Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

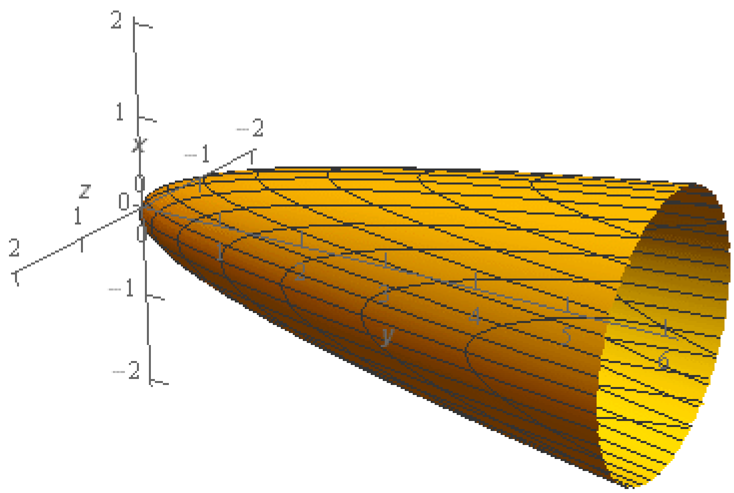
First note that from the integrand it should be pretty clear that we'll want to integrate with respect to *y* first (unless you want to do a trig substitution of course….). So, the integral becomes,

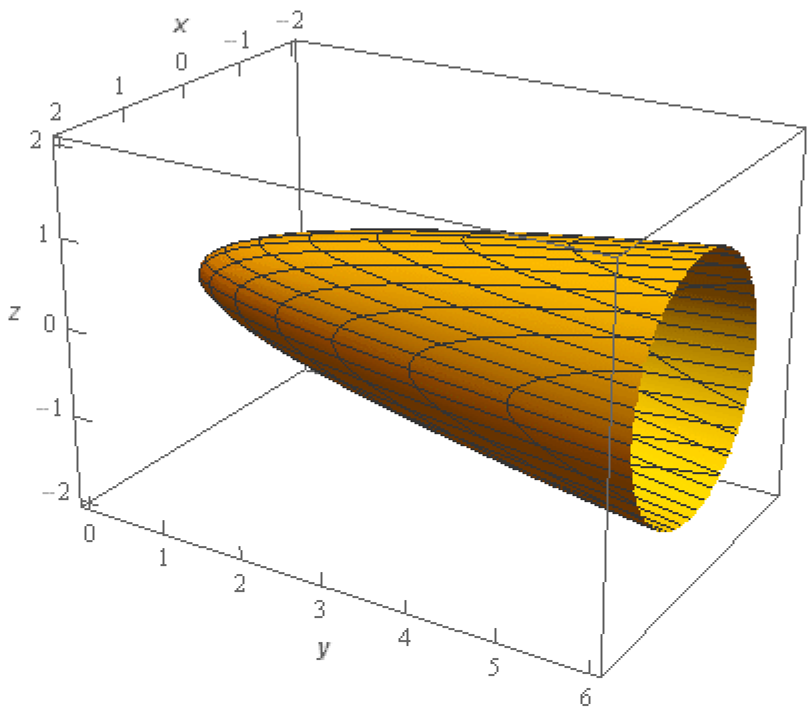
$$\begin{aligned}\iint\limits_S z+3y-x^2\,dS &= \iint\limits_D 2\sqrt{4x^2+10}\,dA\\ &= \int_0^2\int_{-2x}^0 2\sqrt{4x^2+10}\,dy\,dx\\ &= \int_0^2 \left(2y\sqrt{4x^2+10}\right)\Big|_{-2x}^{0}\,dx\\ &= \int_0^2 4x\sqrt{4x^2+10}\,dx\\ &= \tfrac{1}{3}\left(4x^2+10\right)^{\frac{3}{2}}\Big|_0^2 = \boxed{\tfrac{1}{3}\left(26^{\frac{3}{2}}-10^{\frac{3}{2}}\right)=33.6506}\end{aligned}$$

---

2. Evaluate $\iint\limits_S 40y\,dS$ where *S* is the portion of $y=3x^2+3z^2$ that lies behind $y=6$.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the elliptic paraboloid and does not include the “cap” at $y = 6$. We would only include the “cap” if the problem had specified that in some manner to make it clear.

In this case *D* will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \qquad \Rightarrow \qquad x^2 + z^2 = 2$$

So, D will be the disk $x^2+z^2 \le 2$.

Step 2
Let's get the integral set up now. In this case the surface is in the form,

$$y = g(x,z) = 3x^2 + 3z^2$$

so we'll use the following formula for the surface integral.

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f\left(x, g(x,z), z\right)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2}\,dA$$

The integral is then,

$$\begin{aligned}\iint\limits_S 40y\,dS &= \iint\limits_D 40\left(3x^2+3z^2\right)\sqrt{(6x)^2+1+(6z)^2}\,dA \\ &= \iint\limits_D 120\left(x^2+z^2\right)\sqrt{36\left(x^2+z^2\right)+1}\,dA\end{aligned}$$

Don't forget to plug the equation of the surface into y in the integrand and recall that D is the disk we found in Step 1.

Step 3
Now, for this problem it should be pretty clear that we'll want to use polar coordinates to do the integral. We'll use the following set of polar coordinates.

$$x = r\cos\theta \qquad z = r\sin\theta \qquad x^2+z^2 = r^2$$

Also, because D is the disk $x^2+z^2 \le 2$ the limits for the integral will be,

$$\begin{aligned}&0 \le \theta \le 2\pi \\ &0 \le r \le \sqrt{2}\end{aligned}$$

Converting the integral to polar coordinates gives,

$$\begin{aligned}\iint\limits_S 40y\,dS &= \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^2\sqrt{36r^2+1}\,(r)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^3\sqrt{36r^2+1}\,dr\,d\theta\end{aligned}$$

Don't forget to pick up the extra r when converting the dA into polar coordinates.

Step 4
Now all that we need to do is evaluate the double integral and this one can be a little tricky unless you've seen this kind of integral done before.

We'll use the following substitution to do the integral.

$$u = 36r^2 + 1 \qquad \to \qquad du = 72r\,dr \qquad \to \qquad \tfrac{1}{72}du = r\,dr$$

The problem is that this doesn't seem to work at first glance because the differential will only get rid of one of the three *r*'s in front of the root. However, we can also solve the substitution for r^2 to get,

$$r^2 = \tfrac{1}{36}(u-1)$$

and we can now convert the remaining two *r*'s into *u*'s.

So, using the substitution the integral becomes,

$$\begin{aligned}\iint\limits_S 40y\,dS &= \int_0^{2\pi}\int_1^{73} 120\left(\tfrac{1}{72}\right)\left(\tfrac{1}{36}\right)(u-1)u^{\frac{1}{2}}\,du\,d\theta \\ &= \int_0^{2\pi}\int_1^{73} \tfrac{5}{108}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du\,d\theta\end{aligned}$$

Note that we also converted the *r* limits in the original integral into *u* limits simply by plugging the "old" *r* limits into the substitution to get "new" *u* limits.

We can now easily finish evaluating the integral.

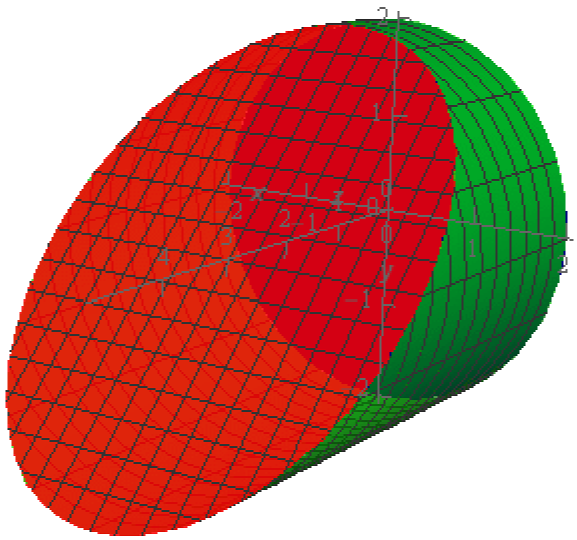
$$\begin{aligned}\iint\limits_S 40y\,dS &= \int_0^{2\pi} \tfrac{5}{108}\left(\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^{73} d\theta \\ &= \int_0^{2\pi} \tfrac{5}{108}\left[\tfrac{2}{5}\left(73^{\frac{5}{2}}\right) - \tfrac{2}{3}\left(73^{\frac{3}{2}}\right) - \left(-\tfrac{4}{15}\right)\right]d\theta \\ &= \boxed{\tfrac{5\pi}{54}\left[\tfrac{2}{5}\left(73^{\frac{5}{2}}\right) - \tfrac{2}{3}\left(73^{\frac{3}{2}}\right) + \tfrac{4}{15}\right] = 5176.8958}\end{aligned}$$

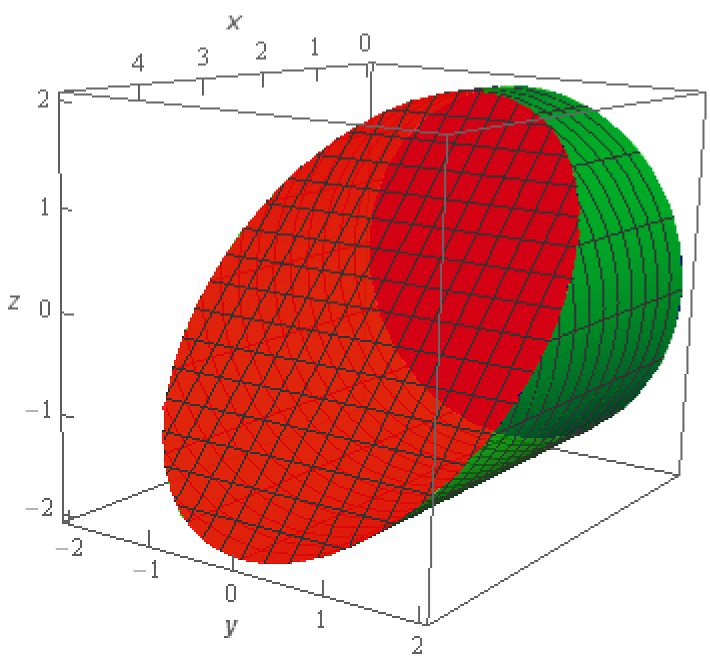
Kind of messy integral with a messy answer but that will happen on occasion so we shouldn't get too excited about it when that does happen.

---

3. Evaluate $\iint\limits_S 2y\,dS$ where *S* is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the cylinder itself. The “caps” of the cylinder are not part of this surface despite the red “cap” in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the “caps” if the problem had specified that in some manner to make it clear.

Step 2

Now because our surface is a cylinder we’ll need to parameterize it and use the following formula for the surface integral.

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\left\|\vec{r}_u \times \vec{r}_v\right\| dA$$

where u and v will be chosen as needed when doing the parameterization.

We saw how to parameterize a cylinder in the previous section so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}(x,\theta) = \left\langle x, 2\sin\theta, 2\cos\theta \right\rangle \qquad 0 \le \theta \le 2\pi, \quad 0 \le x \le 3 - z = 3 - 2\cos\theta$$

We'll use the full range of θ since we are allowing it to rotate all the way around the *x*-axis. The *x* limits come from the two planes that "bound" the cylinder and we'll need to convert the upper limit using the parameterization.

Next, we'll need to compute the cross product.

$$\vec{r}_x = \left\langle 1,0,0 \right\rangle \qquad \vec{r}_\theta = \left\langle 0, 2\cos\theta, -2\sin\theta \right\rangle$$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 2\cos\theta & -2\sin\theta \end{vmatrix} = 2\sin\theta\,\vec{j} + 2\cos\theta\,\vec{k}$$

The magnitude of the cross product is,

$$\left\|\vec{r}_x \times \vec{r}_\theta\right\| = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$$

The integral is then,

$$\iint_S 2y\,dS = \iint_D 2(2\sin\theta)(2)\,dA = \iint_D 8\sin\theta\,dA$$

Don't forget to plug the *y* component of the surface parametrization into the integrand and *D* is just the limits on *x* and θ we noted above in the parameterization.

Step 3
Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

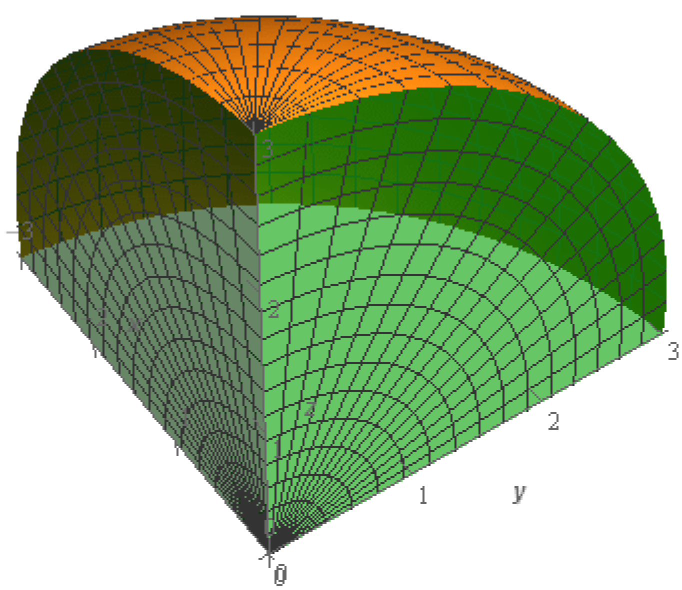
The integral is then,

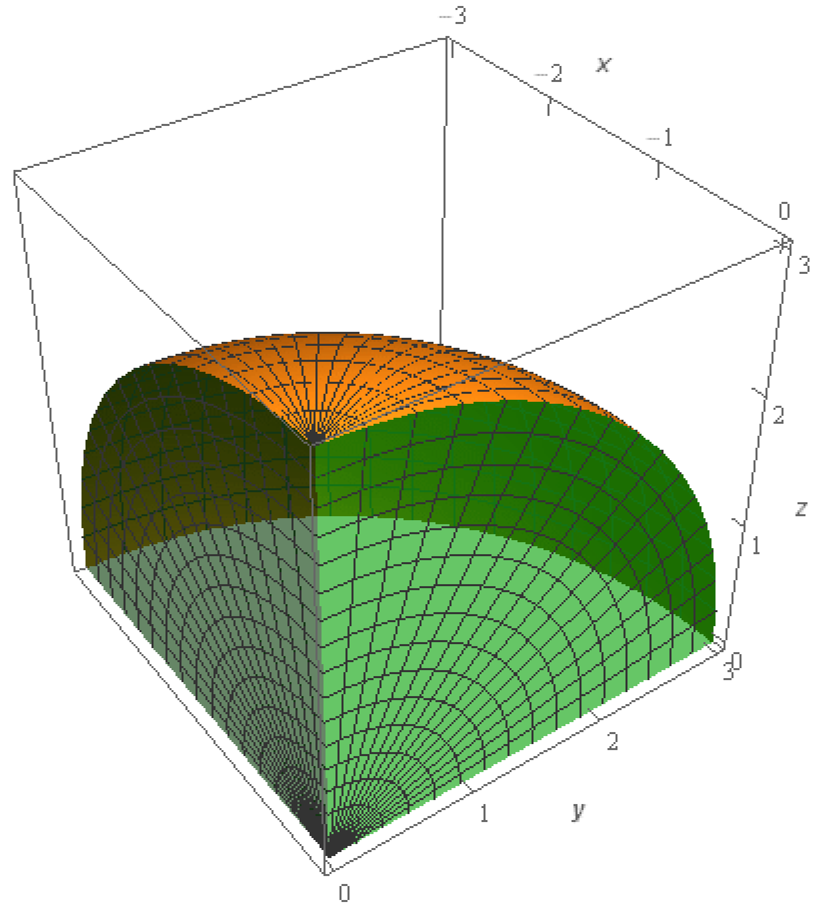
$$\iint_S 2y\,dS = \int_0^{2\pi}\int_0^{3-2\cos\theta} 8\sin\theta\,dx\,d\theta$$
$$= \int_0^{2\pi} \left(8x\sin\theta\right)\Big|_0^{3-2\cos\theta}\,d\theta$$
$$= \int_0^{2\pi} 8\left(3-2\cos\theta\right)\sin\theta\,d\theta$$
$$= \int_0^{2\pi} 24\sin\theta - 16\sin\theta\cos\theta\,d\theta$$
$$= \int_0^{2\pi} 24\sin\theta - 8\sin\left(2\theta\right)d\theta$$
$$= \left(-24\cos\theta + 4\cos\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{0}$$

---

4. Evaluate $\iint\limits_S xz\,dS$ where *S* is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the part of the sphere itself. The "edges" (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the "edges" if the problem had specified that in some manner to make it clear.

Step 2

Now because our surface is a sphere we'll need to parameterize it and use the following formula for the surface integral.

$$\iint\limits_S f\left(x,y,z\right)dS = \iint\limits_D f\left(\vec{r}\left(u,v\right)\right)\left\|\vec{r}_u \times \vec{r}_v\right\| dA$$

where u and v will be chosen as needed when doing the parameterization.

We saw how to parameterize a sphere in the previous section so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}\left(\theta,\varphi\right) = \left\langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi\right\rangle \qquad \tfrac{1}{2}\pi \le \theta \le \pi, \quad 0 \le \varphi \le \tfrac{1}{2}\pi$$

We needed the restriction on φ to make sure that we only get a portion of the upper half of the sphere (*i.e.* $z \ge 0$). Likewise the restriction on θ was needed to get only the portion that was in the 2nd quadrant of the *xy*-plane (*i.e.* $x \le 0$ and $y \ge 0$).

Next, we'll need to compute the cross product.

$$\vec{r}_\theta = \left\langle -3\sin\varphi\sin\theta, 3\sin\varphi\cos\theta, 0\right\rangle \qquad\qquad \vec{r}_\varphi = \left\langle 3\cos\varphi\cos\theta, 3\cos\varphi\sin\theta, -3\sin\varphi\right\rangle$$

$$\begin{aligned}\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \end{vmatrix} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 9\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} - 9\sin^2\varphi\sin\theta\,\vec{j} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec{k} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\,\vec{k}\end{aligned}$$

The magnitude of the cross product is,

$$\begin{aligned}\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| &= \sqrt{\left(-9\sin^2\varphi\cos\theta\right)^2 + \left(-9\sin^2\varphi\sin\theta\right)^2 + \left(-9\sin\varphi\cos\varphi\right)^2} \\ &= \sqrt{81\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + 81\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{81\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)} \\ &= 9\left|\sin\varphi\right| \\ &= 9\sin\varphi\end{aligned}$$

The integral is then,

$$\iint\limits_S xz\,dS = \iint\limits_D \left(3\sin\varphi\cos\theta\right)\left(3\cos\varphi\right)\left(9\sin\varphi\right)dA = \iint\limits_D 81\cos\varphi\sin^2\varphi\cos\theta\,dA$$

Don't forget to plug the *x* and *z* component of the surface parameterization into the integrand and *D* is just the limits on θ and φ we noted above in the parameterization.

Step 3

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

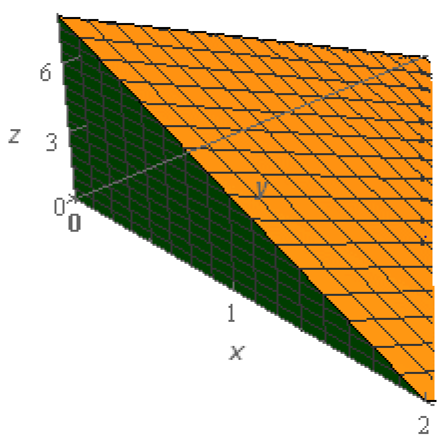
The integral is then,

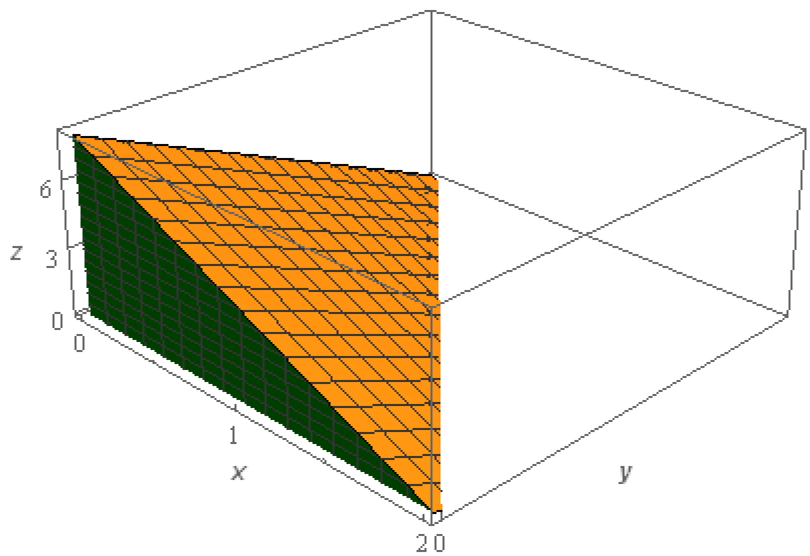
$$\iint\limits_S xz\,dS = \iint\limits_D 81\cos\varphi\sin^2\varphi\cos\theta\,dA$$
$$= \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi} 81\cos\varphi\sin^2\varphi\cos\theta\,d\varphi\,d\theta$$
$$= \int_{\frac{1}{2}\pi}^{\pi}\left(27\sin^3\varphi\cos\theta\,d\varphi\right)\Big|_0^{\frac{1}{2}\pi}\,d\theta$$
$$= \int_{\frac{1}{2}\pi}^{\pi} 27\cos\theta\,d\theta$$
$$= \left(27\sin\theta\right)\Big|_{\frac{1}{2}\pi}^{\pi} = \boxed{-27}$$

---

5. Evaluate $\iint\limits_S yz + 4xy\,dS$ where *S* is the surface of the solid bounded by $4x+2y+z=8$, $z=0$, $y=0$ and $x=0$. Note that all four surfaces of this solid are included in *S*.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of *S* so let's define each of them as follows.

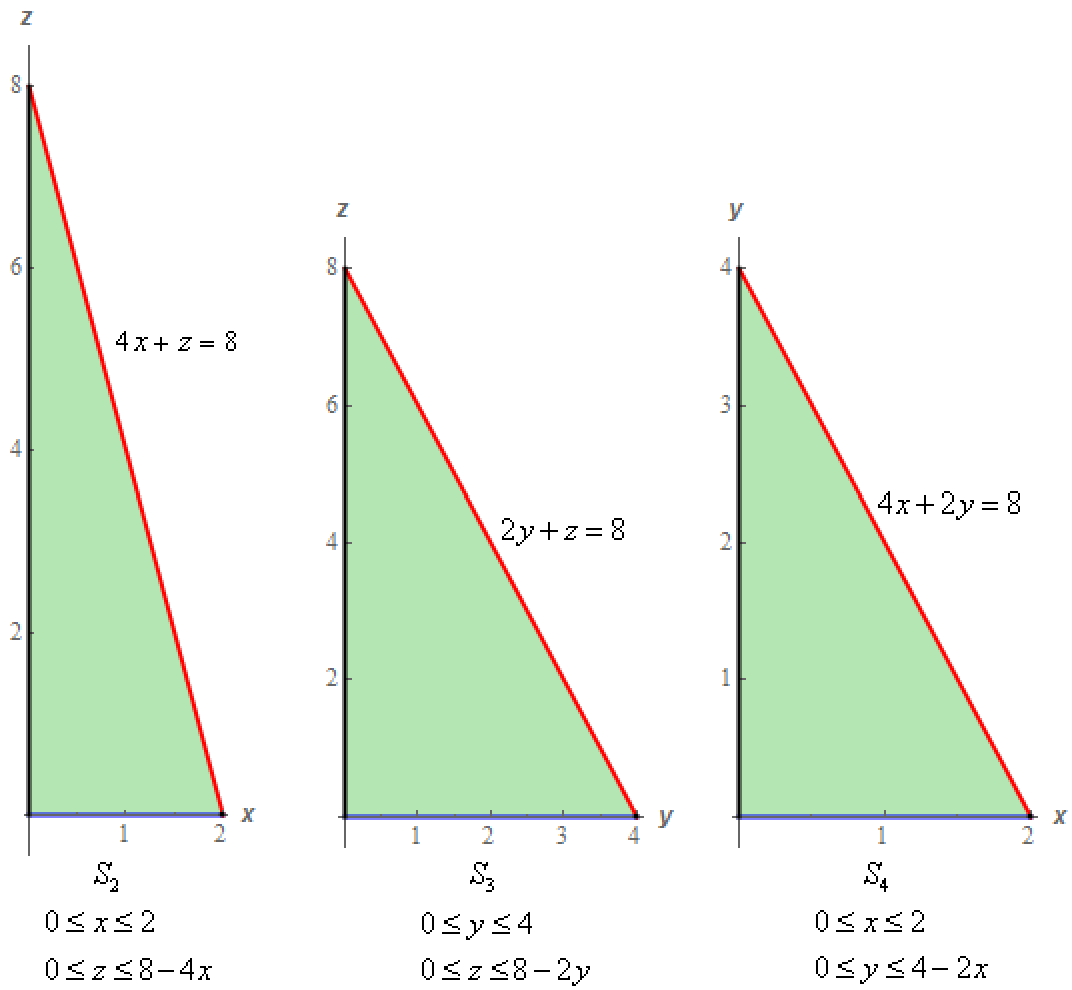
S_1 : Plane given by $4x+2y+z=8$ $\left(i.e \text{ the top of the solid}\right)$

S_2 : Plane given by $y=0$ $\left(i.e \text{ the triangle on right side of the solid}\right)$

S_3 : Plane given by $x=0$ $\left(i.e \text{ the triangle at back of the solid - not shown in sketch}\right)$

S_4 : Plane given by $z=0$ $\left(i.e \text{ the triangle on bottom of the solid - not shown in sketch}\right)$

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.



With each of the sketches we gave limits on the variables for each of them since we'll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. In this case the surface can easily be solved for z to get,

$$z = 8 - 4x - 2y$$

With the equation of the surface written in this manner the region *D* will be in the *xy*-plane and if you think about it you'll see that in fact *D* is nothing more than S_4!

The integral in this case is,

$$\iint\limits_{S_1} yz + 4xy\,dS = \iint\limits_{D} \left[y\left(8-4x-2y\right)+4xy \right]\sqrt{\left(-4\right)^2+\left(-2\right)^2+1}\,dA$$
$$= \sqrt{21}\iint\limits_{D} 8y - 2y^2\,dA$$

Don't forget to plug the equation of the surface into *z* in the integrand and don't forget to use the equation of the surface in the computation of the root!

Now, as noted above *D* for this surface is nothing more than S_4 and so we can use the limits from the sketch of S_4 in Step 1.

Now let's compute the integral for this surface.

$$\iint\limits_{S_1} yz + 4xy\,dS = \sqrt{21}\int_0^2\int_0^{4-2x} 8y - 2y^2\,dy\,dx$$
$$= \sqrt{21}\int_0^2 \left[4y^2 - \tfrac{2}{3}y^3 \right]\Big|_0^{4-2x}\,dx$$
$$= \sqrt{21}\int_0^2 4\left(4-2x\right)^2 - \tfrac{2}{3}\left(4-2x\right)^3\,dx$$
$$= \sqrt{21}\left[-\tfrac{2}{3}\left(4-2x\right)^3 + \tfrac{1}{12}\left(4-2x\right)^4 \right]\Big|_0^2 = \underline{\tfrac{64\sqrt{21}}{3} = 97.7616}$$

Step 3

Next we'll take care of S_2. In this case the equation for the surface is simply $y = 0$ and *D* is given in the sketch of S_2 in Step 1.

The integral in this case is,

$$\iint\limits_{S_2} yz + 4xy\,dS = \iint\limits_{D} 0\sqrt{\left(0\right)^2+1+\left(0\right)^2}\,dA = \iint\limits_{D} 0\,dA = \underline{0}$$

So, in this case we didn't need to actually compute the integral. Sometimes we'll get lucky like this, although it probably won't happen all that often.

Step 4

Now we can take care of S_3. In this case the equation for the surface is simply $x = 0$ and *D* is given in the sketch of S_3 in Step 1.

The integral in this case is,

$$\iint\limits_{S_3} yz + 4xy\,dS = \iint\limits_{D} \left[yz + 4\left(0\right)y \right]\sqrt{1+\left(0\right)^2+\left(0\right)^2}\,dA = \iint\limits_{D} yz\,dA$$

Don't forget to plug the equation of the surface into *x* in the integrand and don't forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for *D* from the sketch in Step 1 we can quickly evaluate the integral for this surface.

$$\begin{aligned}\iint\limits_{S_3} yz + 4xy\,dS &= \int_0^4 \int_0^{8-2y} yz\,dz\,dy \\ &= \int_0^4 \left[\tfrac{1}{2} yz^2\right]\Big|_0^{8-2y} dy \\ &= \int_0^4 32y - 16y^2 + 2y^3\,dy \\ &= \left[16y^2 - \tfrac{16}{3}y^3 + \tfrac{1}{2}y^4\right]\Big|_0^4 = \underline{\tfrac{128}{3} = 42.6667}\end{aligned}$$

Step 5
Finally let's take care of S_4. In this case the equation for the surface is simply $z = 0$ and *D* is given in the sketch of S_4 in Step 1.

The integral in this case is,

$$\iint\limits_{S_4} yz + 4xy\,dS = \iint\limits_{D} \left[y(0) + 4xy\right]\sqrt{(0)^2 + (0)^2 + 1}\,dA = \iint\limits_{D} 4xy\,dA$$

Don't forget to plug the equation of the surface into *z* in the integrand and don't forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for *D* from the sketch in Step 1 we can quickly evaluate the integral for this surface.

$$\begin{aligned}\iint\limits_{S_4} yz + 4xy\,dS &= \int_0^2 \int_0^{4-2x} 4xy\,dy\,dx \\ &= \int_0^2 \left[2xy^2\right]\Big|_0^{4-2x} dx \\ &= \int_0^2 32x - 32x^2 + 8x^3\,dx \\ &= \left[16x^2 - \tfrac{32}{3}x^3 + 2x^4\right]\Big|_0^2 = \underline{\tfrac{32}{3} = 10.6667}\end{aligned}$$

Step 6
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

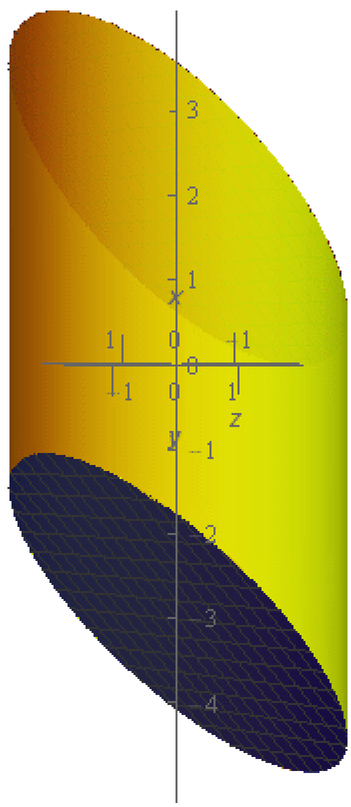
$$\iint\limits_S yz + 4xy\,dS = \left(\tfrac{64\sqrt{21}}{3}\right) + \left(0\right) + \left(\tfrac{128}{3}\right) + \left(\tfrac{32}{3}\right) = \tfrac{64\sqrt{21}}{3} + \tfrac{160}{3} = \boxed{151.0949}$$

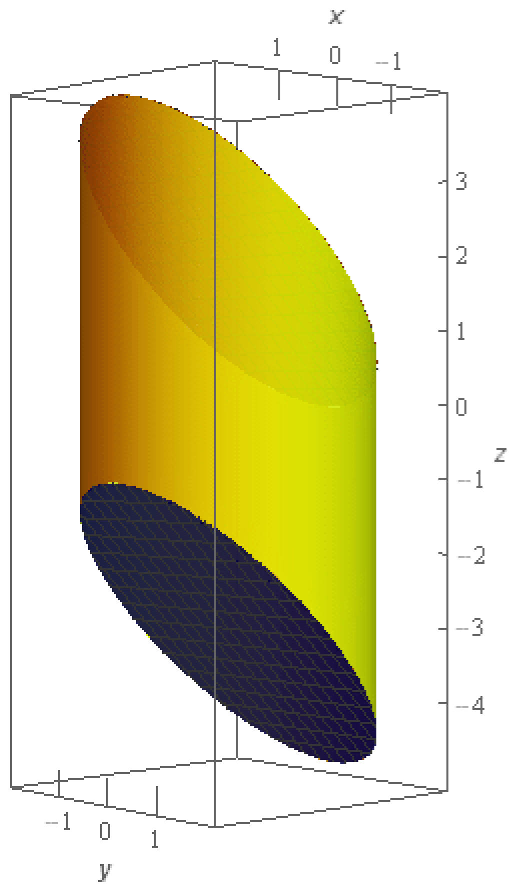
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

---

6. Evaluate $\iint\limits_S x - z\,dS$ where *S* is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$, and $z = x + 2$. Note that all three surfaces of this solid are included in *S*.

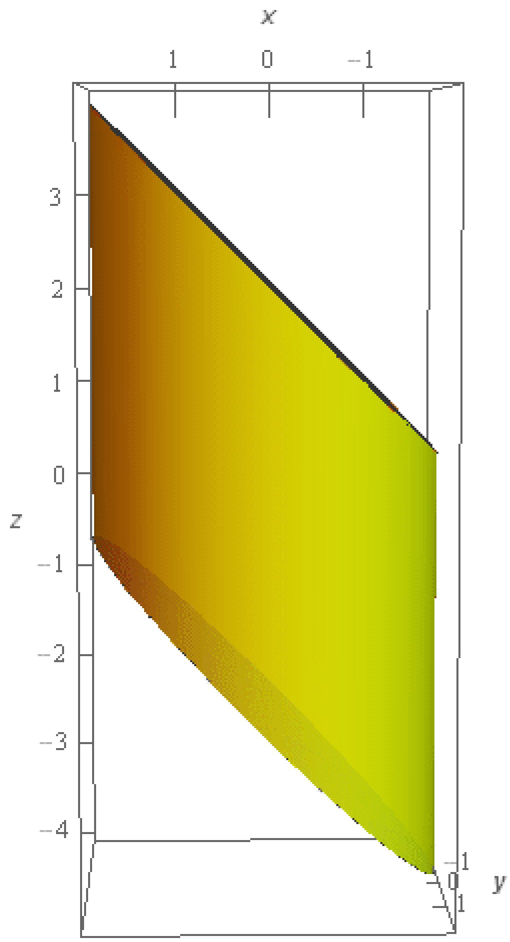
Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





As noted in the problem statement there are three surfaces here. The "top" of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.



From this view we can see that the top and bottom planes that "cap" the cylinder are parallel.

Let's define the three surfaces in the sketch as follows.

S_1 : Cylinder given by $x^2 + y^2 = 4$ $\left(i.e \text{ the walls of the solid}\right)$
S_2 : Plane given by $z = x + 2$ $\left(i.e \text{ the top cap of the cylinder}\right)$
S_3 : Plane given by $z = x - 3$ $\left(i.e \text{ the bottom cap of the cylinder}\right)$

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2
Let's start with S_1. The surface in this case is a cylinder and so we'll need to parameterize it. The parameterization of the surface is,

$$\vec r\left(z,\theta\right) = \left\langle 2\cos\theta, 2\sin\theta, z\right\rangle$$

The limits on *z* and θ are,

$$0 \le \theta \le 2\pi, \quad 2\cos\theta - 3 = x - 3 \le z \le x + 2 = 2\cos\theta + 2$$

With the *z* limits we'll need to make sure that we convert the *x*'s into their parameterized form.

In order to evaluate the integral in this case we'll need the cross product $\vec{r}_z \times \vec{r}_\theta$ so here is that work.

$$\vec{r}_z = \langle 0,0,1 \rangle \qquad \vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$$

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -2\sin\theta & 2\cos\theta & 0 \end{vmatrix} = -2\cos\theta\vec{i} - 2\sin\theta\vec{j}$$

Next, we'll need the magnitude of the cross product so here is that.

$$\left\| \vec{r}_z \times \vec{r}_\theta \right\| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$$

The integral in this case is,

$$\iint\limits_{S_1} x - z\, dS = \iint\limits_{D} \left[2\cos\theta - z\right](2)\, dA = \iint\limits_{D} 4\cos\theta - 2z\, dA$$

Don't forget to plug the parameterization of the surface into the integrand and don't forget to add in the magnitude of the cross product!

Now, *D* for this surface is nothing more than the limits on *z* and θ we gave above.

Now let's compute the integral for this surface.

$$\begin{aligned} \iint\limits_{S_1} x - z\, dS &= \int_0^{2\pi} \int_{2\cos\theta - 3}^{2\cos\theta + 2} 4\cos\theta - 2z\, dz\, d\theta \\ &= \int_0^{2\pi} \left(4z\cos\theta - z^2\right)\Big|_{2\cos\theta - 3}^{2\cos\theta + 2} d\theta \\ &= \int_0^{2\pi} 4\cos\theta\left[2\cos\theta + 2 - \left(2\cos\theta - 3\right)\right] - \left[\left(2\cos\theta + 2\right)^2 - \left(2\cos\theta - 3\right)^2\right] d\theta \\ &= \int_0^{2\pi} 5\, d\theta = \underline{10\pi} \end{aligned}$$

Do not forget to simplify! As we saw with this problem after the *z* integration the integrand looked really messy but after some pretty simple simplification it reduced down to an incredibly simple integrand.

Step 3

Next we'll take care of S_2. In this case the equation for the surface is simply $z = x + 2$ and *D* is the disk $x^2 + y^2 \le 4$.

The integral in this case is,

$$\iint_{S_2} x - z\,dS = \iint_D \left[x - (x+2)\right]\sqrt{(1)^2 + (0)^2 + 1}\,dA = \iint_D -2\sqrt{2}\,dA = -2\sqrt{2}\iint_D dA$$

Okay, in this case we don't need to actually do the evaluation of the integral because we know that,

$$\iint_D dA = \text{Area of } D$$

and in this case *D* is just a disk and we can quickly determine its area without any evaluation.

So, the integral for this surface is then just,

$$\iint_{S_2} x - z\,dS = -2\sqrt{2}\left(\text{Area of } D\right) = -2\sqrt{2}\left[(2)^2\pi\right] = \underline{-8\sqrt{2}\pi}$$

Step 4
Finally, let's integrate over S_3. In this case the equation for the surface is simply $z = x - 3$ and *D* is the disk $x^2 + y^2 \le 4$.

The integral in this case is,

$$\begin{aligned}\iint_{S_3} x - z\,dS &= \iint_D \left[x - (x-3)\right]\sqrt{(1)^2 + (0)^2 + 1}\,dA \\ &= \iint_D 3\sqrt{2}\,dA = 3\sqrt{2}\iint_D dA = 3\sqrt{2}\left(4\pi\right) = \underline{12\sqrt{2}\pi}\end{aligned}$$

So, the integral in this case ended up being every similar to the integral in Step 3 and so we didn't put in any of the explanation here.

Step 5
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,

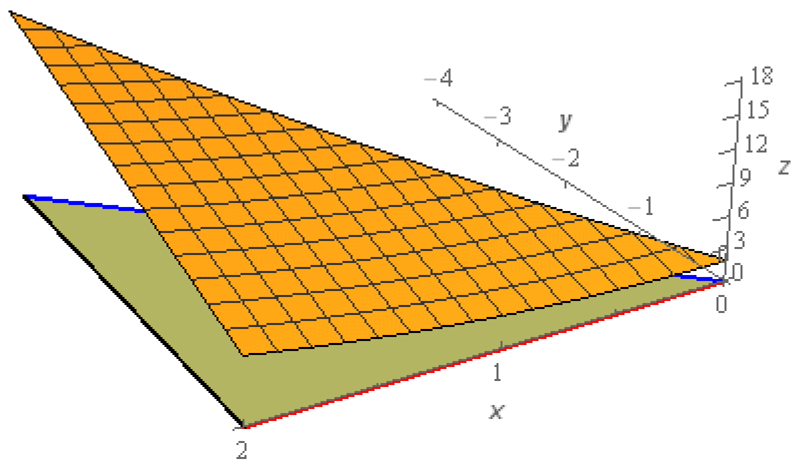
$$\iint_S x - z\,dS = \left(10\pi\right) + \left(-8\sqrt{2}\pi\right) + \left(12\sqrt{2}\pi\right) = \left(10 + 4\sqrt{2}\right)\pi = \boxed{49.1875}$$

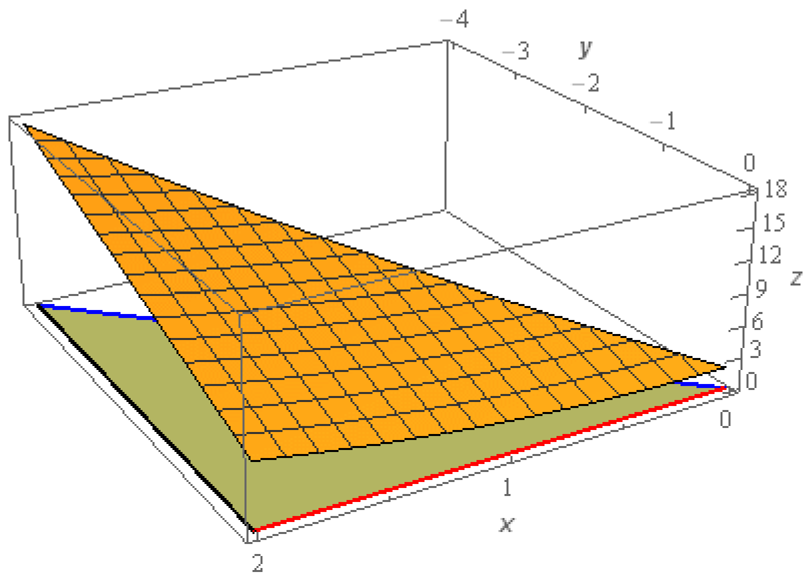
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

Section 6-4 : Surface Integrals of Vector Fields

1. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = 3x\,\vec{i} + 2z\,\vec{j} + \left(1 - y^2\right)\vec{k}$ and S is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the *xy*-plane with vertices $\left(0,0\right)$, $\left(2,0\right)$ and $\left(2,-4\right)$ oriented in the negative *z*-axis direction.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.

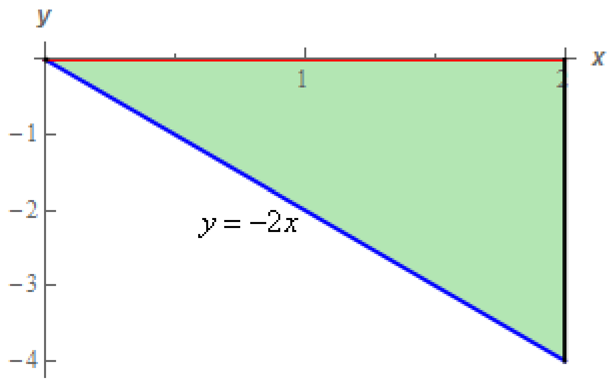




We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

The orange surface is the sketch of $z = 2 - 3y + x^2$ that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region *D* for this problem.

Here is a quick sketch of *D* just to get a better view of it than the mostly obscured view in the sketch above.



We could use either of the following sets of limits to describe *D*.

$$\begin{array}{c|c} 0 \le x \le 2 & -4 \le y \le 0 \\ -2x \le y \le 0 & -\tfrac{1}{2}y \le x \le 2 \end{array}$$

We'll decide which set to use in the integral once we get that set up.

Step 2

Let's get the integral set up now. In this case the we can write the equation of the surface as follows,

$$f(x, y, z) = 2 - 3y + x^2 - z = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 2x, -3, -1 \rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above has the correct orientation. We were told in the problem statement that the orientation was in the negative *z*-axis direction and this means that the normal vector should always have a downwards direction (*i.e.* a negative *z* component) and this one does.

Step 3

Next, we'll need to compute the following dot product.

$$\begin{aligned}\vec{F}\left(x,y,2-3y+x^2\right)\bullet\vec{n} &= \left\langle 3x,2\left(2-3y+x^2\right),1-y^2\right\rangle\bullet\frac{\left\langle 2x,-3,-1\right\rangle}{\left\|\nabla f\right\|}\\ &= \tfrac{1}{\left\|\nabla f\right\|}\left(6x^2-6\left(2-3y+x^2\right)-\left(1-y^2\right)\right)\\ &= \tfrac{1}{\left\|\nabla f\right\|}\left(y^2+18y-13\right)\end{aligned}$$

Remember that we needed to plug in the equation of the surface, $z=2-3y+x^2$, into *z* in the vector field!

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \iint\limits_S \tfrac{1}{\left\|\nabla f\right\|}\left(y^2+18y-13\right)dS\\ &= \iint\limits_D \tfrac{1}{\left\|\nabla f\right\|}\left(y^2+18y-13\right)\left\|\nabla f\right\|dA\\ &= \iint\limits_D y^2+18y-13\,dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that *D* was given in Step 1. We had two sets of limits to use here but it seems like the first set is probably just as easy to use so we'll use that one in the integral.

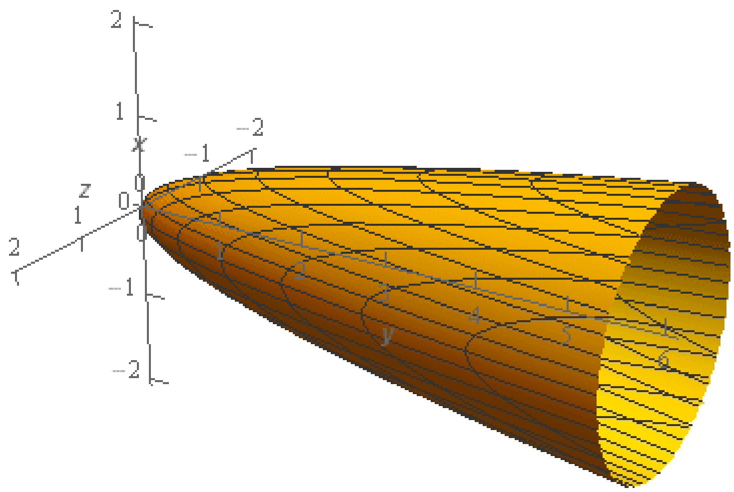
Step 4
Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

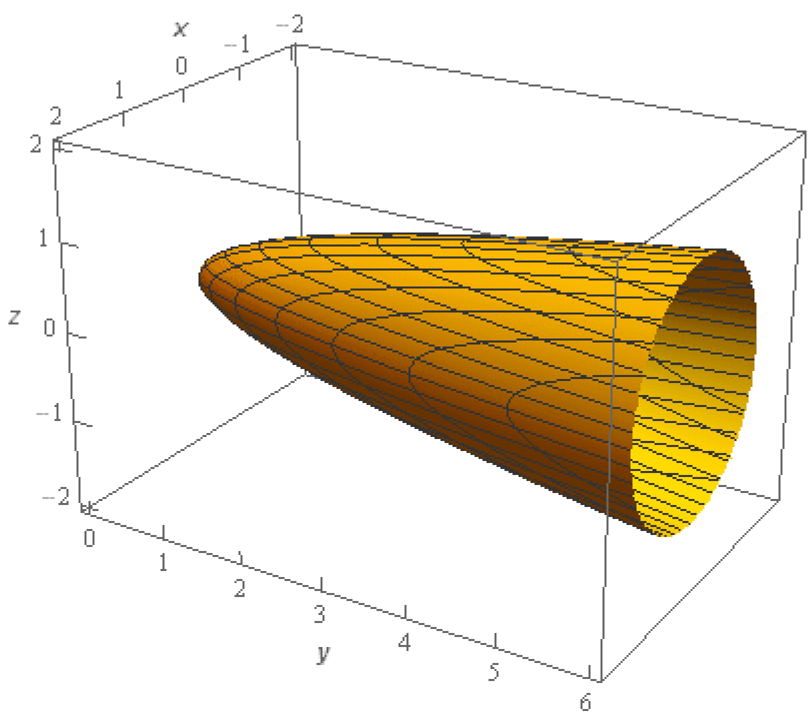
Here is the integral,

$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \iint\limits_D y^2+18y-13\,dA\\ &= \int_0^2\int_{-2x}^0 y^2+18y-13\,dy\,dx\\ &= \int_0^2\left(\tfrac{1}{3}y^3+9y^2-13y\right)\Big|_{-2x}^0\,dx\\ &= \int_0^2 \tfrac{8}{3}x^3-36x^2-26x\,dx\\ &= \left(\tfrac{2}{3}x^4-12x^3-13x^2\right)\Big|_0^2 = \boxed{-\tfrac{412}{3}}\end{aligned}$$

2. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = -x\,\vec{i} + 2y\,\vec{j} - z\,\vec{k}$ and S is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$ oriented in the positive y-axis direction.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the elliptic paraboloid and does not include the "cap" at $y = 6$. We would only include the "cap" if the problem had specified that in some manner to make it clear.

In this case *D* will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \qquad \Rightarrow \qquad x^2 + z^2 = 2$$

So, *D* will be the disk $x^2 + z^2 \le 2$.

Step 2
Let's get the integral set up now. In this case the we can write the equation of the surface as follows,

$$f\left(x, y, z\right) = 3x^2 + 3z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle 6x, -1, 6z \right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above does not have the correct orientation. We were told in the problem statement that the orientation was in the positive *y*-axis direction and this means that the normal vector should always point in the general direction of the positive *y*-axis (*i.e.* a positive *y* component) and this one does not.

That is easy to fix however. All we need to do is multiply the above normal vector by minus one and we'll get what we need. So, here is the normal vector we need for this problem.

$$\vec{n} = -\frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle -6x, 1, -6z \right\rangle}{\left\|\nabla f\right\|}$$

As we can see this normal vector does in fact have a positive *y* component as we need.

Step 3
Next, we'll need to compute the following dot product.

$$\vec{F}\left(x,3x^2+3z^2,z\right)\bullet\vec{n}=\left\langle -x,2\left(3x^2+3z^2\right),-z\right\rangle\bullet\frac{\left\langle -6x,1,-6z\right\rangle}{\left\|\nabla f\right\|}$$
$$=\tfrac{1}{\left\|\nabla f\right\|}\left(6x^2+2\left(3x^2+3z^2\right)+6z^2\right)$$
$$=\tfrac{1}{\left\|\nabla f\right\|}\left[12\left(x^2+z^2\right)\right]$$

Remember that we needed to plug in the equation of the surface, $y=3x^2+3z^2$, into *y* in the vector field!

The integral is then,

$$\iint\limits_S \vec{F}\bullet d\vec{S}=\iint\limits_S \tfrac{1}{\left\|\nabla f\right\|}\left[12\left(x^2+z^2\right)\right]dS$$
$$=\iint\limits_D \tfrac{1}{\left\|\nabla f\right\|}\left[12\left(x^2+z^2\right)\right]\left\|\nabla f\right\|dA$$
$$=\iint\limits_D 12\left(x^2+z^2\right)dA$$

As noted above we didn't need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that *D* was given in Step 1 and is just the disk $x^2+z^2\le 2$

Step 4
Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

Note as well that we'll want to use polar coordinates in the double integral. We'll use the following set of polar coordinates.

$$x=r\cos\theta \qquad z=r\sin\theta \qquad x^2+z^2=r^2$$

The polar limits for *D* are,

$$0\le\theta\le 2\pi$$
$$0\le r\le\sqrt{2}$$

The integral is then,

$$\iint\limits_S \vec{F} \bullet d\vec{S} = \iint\limits_D 12\left(x^2 + z^2\right) dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{2}} 12r^3 \, dr \, d\theta$$
$$= \int_0^{2\pi} 3r^4 \Big|_0^{\sqrt{2}} \, d\theta$$
$$= \int_0^{2\pi} 12 \, d\theta = \boxed{24\pi}$$

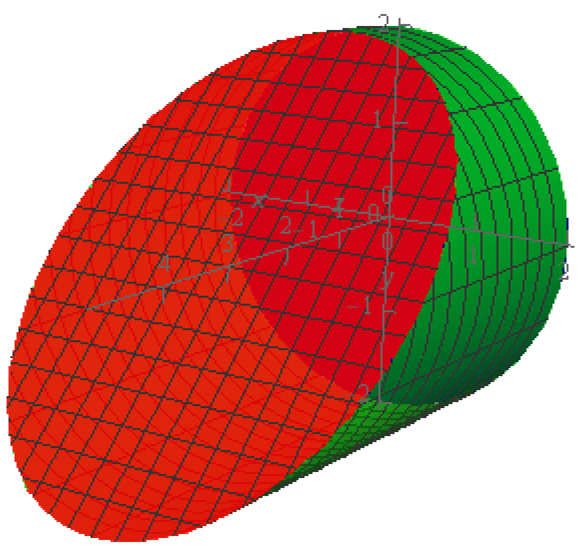
Don't forget that we pick up an extra *r* from the *dA* when converting to polar coordinates.

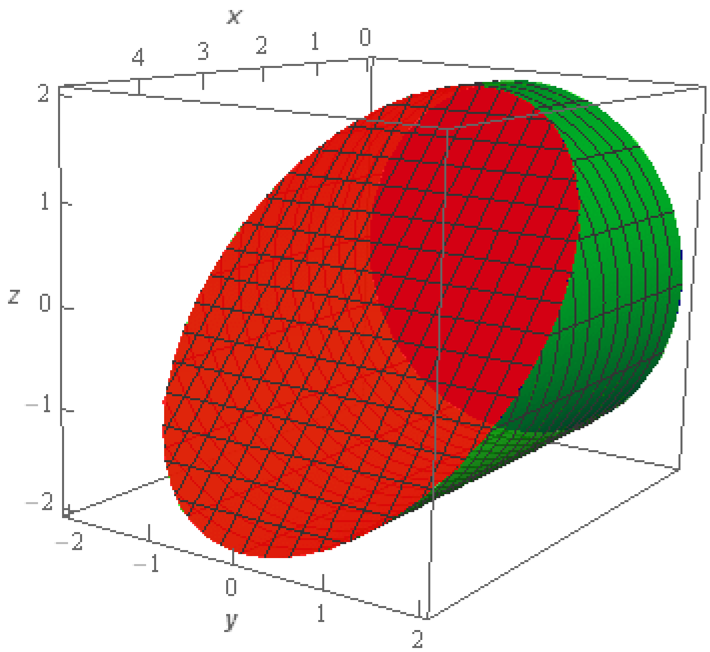
---

3. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = x^2 \,\vec{i} + 2z\,\vec{j} - 3y\,\vec{k}$ and *S* is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$ oriented outwards (*i.e.* away from the *x*-axis).

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the cylinder itself. The "caps" of the cylinder are not part of this surface despite the red "cap" in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the "caps" if the problem had specified that in some manner to make it clear.

Step 2
Let's get the integral set up now. In this case the we are integrating over a cylinder and so we'll need to set up a parameterization for the surface.

We saw how to parameterize a cylinder in the first section of this chapter so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}\left(x,\theta\right)=\left\langle x,2\sin\theta,2\cos\theta\right\rangle \qquad 0\le\theta\le2\pi,\quad 0\le x\le 3-z=3-2\cos\theta$$

We'll use the full range of θ since we are allowing it to rotate all the way around the *x*-axis. The *x* limits come from the two planes that "bound" the cylinder and we'll need to convert the upper limit using the parameterization.

Next, we'll need to compute the cross product.

$$\vec{r}_x=\left\langle 1,0,0\right\rangle \qquad \vec{r}_\theta=\left\langle 0,2\cos\theta,-2\sin\theta\right\rangle$$

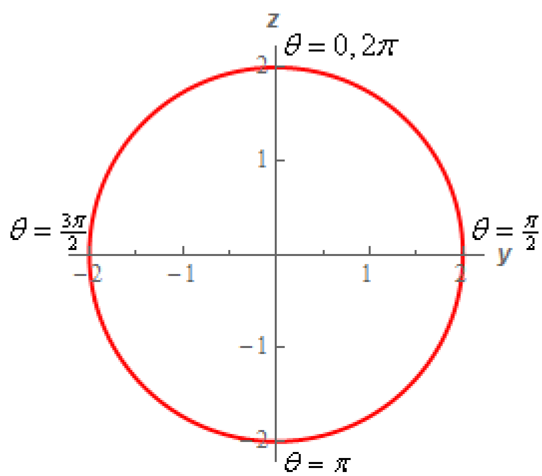
$$\vec{r}_x\times\vec{r}_\theta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & 2\cos\theta & -2\sin\theta\end{vmatrix}=2\sin\theta\,\vec{j}+2\cos\theta\,\vec{k}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_\theta}{\left\| \vec{r}_x \times \vec{r}_\theta \right\|} = \frac{\left\langle 0, 2\sin\theta, 2\cos\theta \right\rangle}{\left\| \vec{r}_x \times \vec{r}_\theta \right\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. First let's look at the cylinder from in front of the cylinder and directly along the *x*-axis. This is what we'd see.



In this sketch the *x* axis will be coming straight out of the sketch at the origin. Plugging in a few value of θ into the parameterization we can see that we'll be at the points listed above.

Now, in the range $0 \le \theta \le \frac{1}{2}\pi$ we know that sine and cosine are both positive and so in the normal vector both the *y* and *z* components will be positive. This means that in the 1st quadrant above the normal vector would need to be pointing out away from the origin. This is exactly what we need to see since the orientation was given as pointing away from the *x*-axis and recall that the *x*-axis is coming straight out of the sketch from the origin.

Next, if we look at $\frac{1}{2}\pi \le \theta \le \pi$ (so we're in the 4th quadrant of the graph above….) we know that in this range sine is still positive but cosine is now negative. From our unit vector above this means that the *y* component is positive (so pointing in positive *y* direction) and the *z* component is negative (so pointing in negative *z* direction). Together this again means that we have to be pointing away from the origin in the 4th quadrant which is again the orientation we want.

We could continue in this fashion looking at the remaining two quadrants but once we've done a couple and gotten the correct orientation we know we'll continue to get the correct orientation for the rest.

Step 3
Next, we'll need to compute the following dot product.

$$\vec{F}\left(\vec{r}\left(x,\theta\right)\right)\bullet\vec{n} = \left\langle x^2, 2\left(2\cos\theta\right), -3\left(2\sin\theta\right)\right\rangle \bullet \frac{\left\langle 0, 2\sin\theta, 2\cos\theta\right\rangle}{\left\|\vec{r}_x \times \vec{r}_\theta\right\|}$$
$$= \tfrac{1}{\left\|\vec{r}_x \times \vec{r}_\theta\right\|}\left(-4\sin\theta\cos\theta\right)$$

Remember that we needed to plug in the parameterization for the surface into the vector field!

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \iint\limits_S \tfrac{1}{\left\|\vec{r}_x \times \vec{r}_\theta\right\|}\left(-4\sin\theta\cos\theta\right) dS \\ &= \iint\limits_D \tfrac{1}{\left\|\vec{r}_x \times \vec{r}_\theta\right\|}\left(-4\sin\theta\cos\theta\right)\left\|\vec{r}_x \times \vec{r}_\theta\right\| dA \\ &= \iint\limits_D -4\sin\theta\cos\theta\, dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that *D* is given by the limits on *x* and θ we found at the start of Step 2.

Step 4

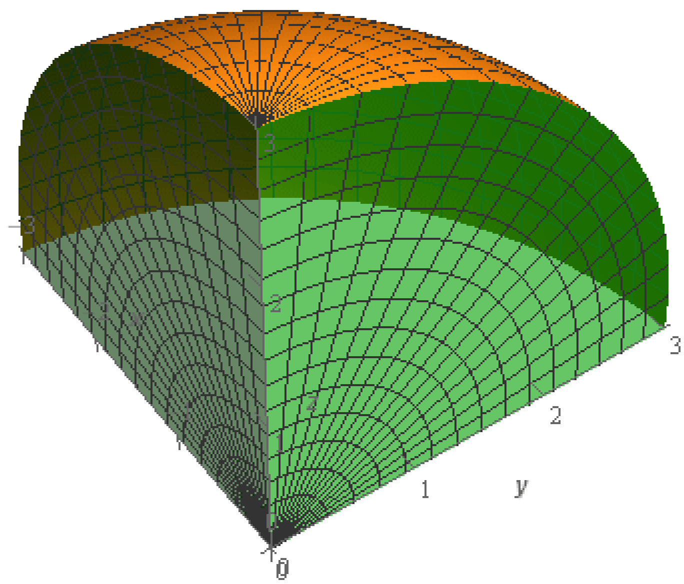
Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

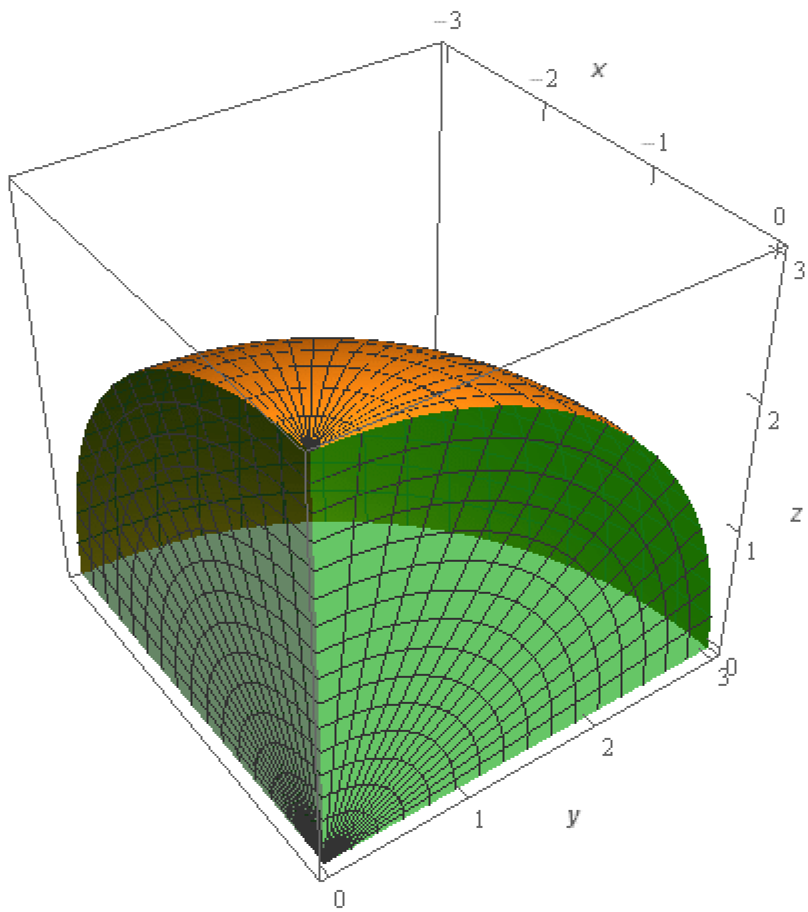
The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \iint\limits_D -4\sin\theta\cos\theta\, dA \\ &= \int_0^{2\pi}\int_0^{3-2\cos\theta} -4\sin\theta\cos\theta\, dx\, d\theta \\ &= \int_0^{2\pi} -4x\sin\theta\cos\theta\Big|_0^{3-2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} -4\left(3-2\cos\theta\right)\sin\theta\cos\theta\, d\theta \\ &= \int_0^{2\pi} -12\sin\theta\cos\theta + 8\sin\theta\cos^2\theta\, d\theta \\ &= \int_0^{2\pi} -6\sin\left(2\theta\right) + 8\sin\theta\cos^2\theta\, d\theta \\ &= \left(3\cos\left(2\theta\right) - \tfrac{8}{3}\cos^3\theta\right)\Big|_0^{2\pi} = \boxed{0}\end{aligned}$$

4. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \vec{i} + z\,\vec{j} + 6x\,\vec{k}$ and *S* is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$ oriented inward (*i.e.* towards the origin).

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





Note that the surface in this problem is only the part of the sphere itself. The "edges" (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the "edges" if the problem had specified that in some manner to make it clear.

Step 2

Let's get the integral set up now. In this case the we are integrating over a sphere and so we'll need to set up a parameterization for the surface.

We saw how to parameterize a sphere in the first section of this chapter so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}\left(\theta,\varphi\right)=\left\langle 3\sin\varphi\cos\theta,3\sin\varphi\sin\theta,3\cos\varphi\right\rangle \qquad \tfrac{1}{2}\pi\le\theta\le\pi,\quad 0\le\varphi\le\tfrac{1}{2}\pi$$

We needed the restriction on φ to make sure that we only get a portion of the upper half of the sphere (*i.e.* $z\ge0$). Likewise the restriction on θ was needed to get only the portion that was in the 2nd quadrant of the *xy*-plane (*i.e.* $x\le0$ and $y\ge0$).

Next, we'll need to compute the cross product.

$$\vec{r}_\theta = \left\langle -3\sin\varphi\sin\theta, 3\sin\varphi\cos\theta, 0\right\rangle \qquad \vec{r}_\varphi = \left\langle 3\cos\varphi\cos\theta, 3\cos\varphi\sin\theta, -3\sin\varphi\right\rangle$$

$$\begin{aligned}\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \end{vmatrix} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 9\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} - 9\sin^2\varphi\sin\theta\,\vec{j} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec{k} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\,\vec{k}\end{aligned}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_\theta \times \vec{r}_\varphi}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|} = \frac{\left\langle -9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi\right\rangle}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. We know that the normal vector needs to point in towards the origin. Let's think about what that would mean for a normal vector on the upper half of a sphere and it won't matter which quadrant in the *xy*-plane we are in.

If we are on the upper half of a sphere and the normal vectors must point towards the origin then we know that they will all need to point downwards. They could point in the positive or negative *x* (or *y*) direction depending on which quadrant from the *xy*-plane we are on but they will have to all point downwards. Or in other words, the *z* component must be negative.

So, the *z* component of the normal vector above is $-9\sin\varphi\cos\varphi$ and we know that we are restricted to $0 \le \varphi \le \frac{1}{2}\pi$ for the portion of the sphere we are working on in this problem. In this range of φ we know that both sine and cosine are positive and so the *z* component must always be negative. This means that the normal vector above has the correct orientation for this problem.

Note that if we were on the lower half of a sphere (not relevant for this problem but useful to think about anyway) and the normal vector would be pointing towards the origin and so they would have to all be pointing upwards.

Also note that if the normal vectors were all pointing out away from the origin then we'd just need to multiply the normal vector above by minus one to get the normal vector we'd need.

Step 3
Next, we'll need to compute the following dot product.

$$\vec{F}\left(\vec{r}\left(\theta,\varphi\right)\right)\bullet\vec{n} = \left\langle 1, 3\cos\varphi, 18\sin\varphi\cos\theta\right\rangle \bullet \frac{\left\langle -9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi\right\rangle}{\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|}$$
$$= \tfrac{1}{\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|}\left(-9\sin^2\varphi\cos\theta - 27\sin^2\varphi\cos\varphi\sin\theta - 162\sin^2\varphi\cos\varphi\cos\theta\right)$$
$$= \tfrac{1}{\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|}\left(-\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta + 162\cos\theta\right)\right)$$

Note that we did a little simplification for the integration process in the last step above.

The integral is then,

$$\iint\limits_S \vec{F}\bullet d\vec{S} = \iint\limits_S \tfrac{1}{\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|}\left(-\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta+162\cos\theta\right)\right)dS$$
$$= \iint\limits_D \tfrac{1}{\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|}\left(-\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta+162\cos\theta\right)\right)\left\|\vec{r}_\theta\times\vec{r}_\varphi\right\|dA$$
$$= \iint\limits_D -\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta+162\cos\theta\right)dA$$

As noted above we didn’t need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a “normal” double integral.

Also, recall that *D* is given by the limits on θ and φ we found at the start of Step 2.

Step 4
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

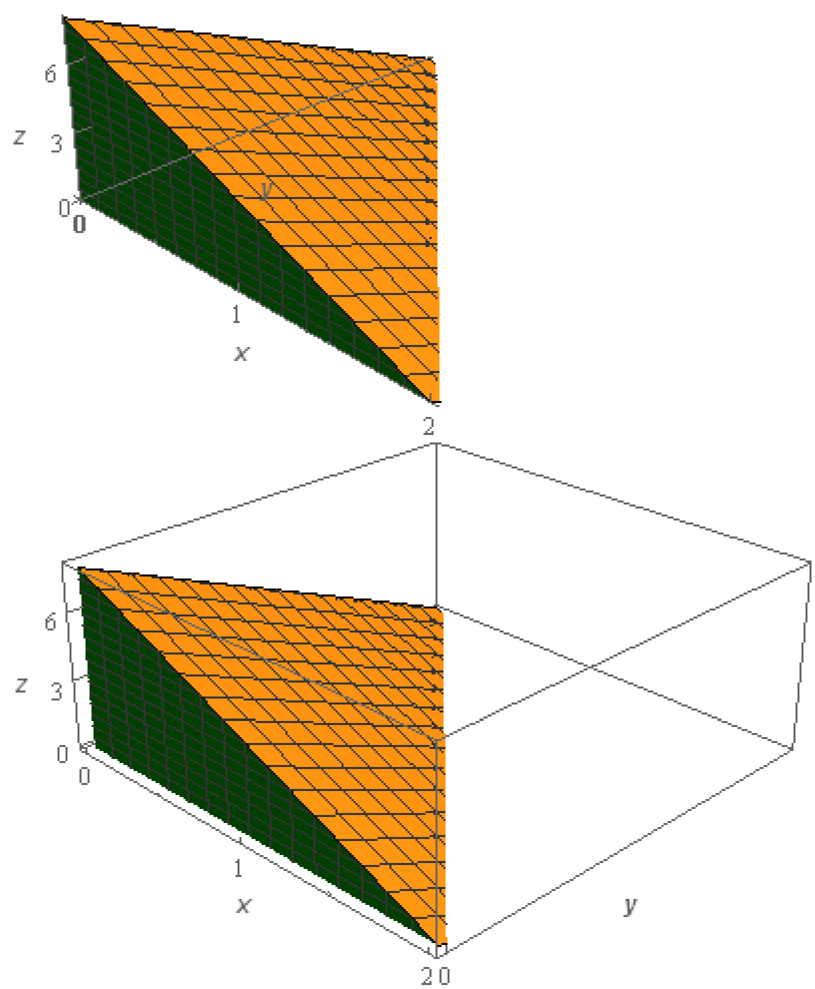
The integral is then,

$$\iint\limits_S \vec{F}\bullet d\vec{S} = \iint\limits_D -\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta+162\cos\theta\right)dA$$
$$= \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi} -\tfrac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos\theta - \sin^2\varphi\cos\varphi\left(27\sin\theta+162\cos\theta\right)d\varphi\,d\theta$$
$$= \int_{\frac{1}{2}\pi}^{\pi} -\tfrac{9}{2}\left(\varphi - \tfrac{1}{2}\sin\left(2\varphi\right)\right)\cos\theta - \tfrac{1}{3}\sin^3\varphi\left(27\sin\theta+162\cos\theta\right)\Big|_0^{\frac{1}{2}\pi}\,d\theta$$
$$= \int_{\frac{1}{2}\pi}^{\pi} -\tfrac{9}{4}\pi\cos\theta - 9\sin\theta - 54\cos\theta\,d\theta$$
$$= \left(-\tfrac{9}{4}\pi\sin\theta + 9\cos\theta - 54\sin\theta\right)\Big|_{\frac{1}{2}\pi}^{\pi} = \boxed{\tfrac{9}{4}\pi + 45}$$

---

5. Evaluate $\iint\limits_S \vec F \bullet d\vec S$ where $\vec F = y\,\vec i + 2x\,\vec j + (z-8)\,\vec k$ and *S* is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$ with the positive orientation. Note that all four surfaces of this solid are included in *S*.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.



Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of *S* so let's define each of them as follows.

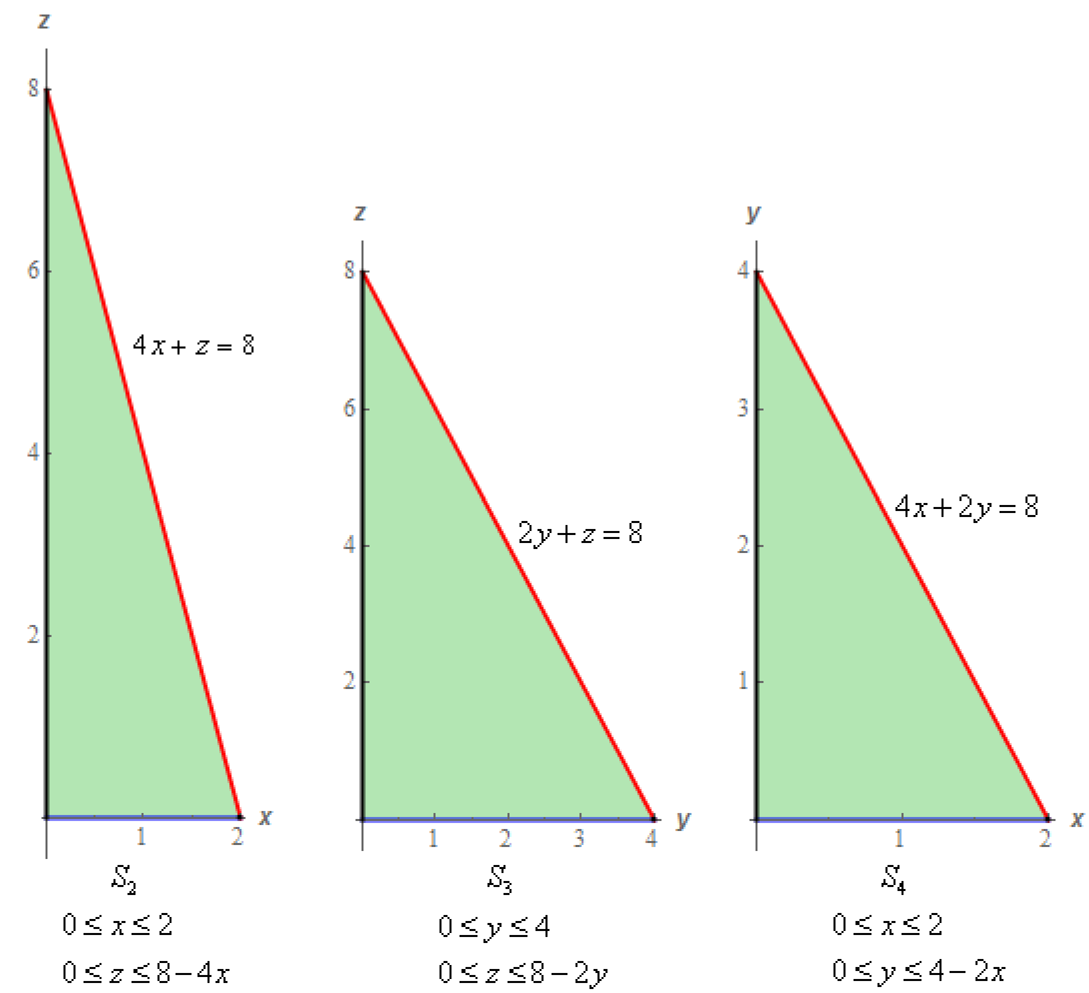
S_1 : Plane given by $4x + 2y + z = 8$ $\left(i.e \text{ the top of the solid} \right)$

S_2 : Plane given by $y = 0$ $\left(i.e \text{ the triangle on right side of the solid} \right)$

S_3 : Plane given by $x = 0$ $\left(i.e \text{ the triangle at back of the solid - not shown in sketch} \right)$

S_4 : Plane given by $z = 0$ $\left(i.e \text{ the triangle on bottom of the solid - not shown in sketch} \right)$

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.



With each of the sketches we gave limits on the variables for each of them since we'll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. In this case we can write the equation of the plane as follows,

$$f\left(x, y, z\right) = 4x + 2y + z - 8 = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 4,2,1\rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The surface has the positive orientation and so must point outwards from the region enclosed by the surface. This means that normal vectors on this plane will need to be pointing generally upwards (*i.e.* have a positive z component) which this normal vector does.

Now we'll need the following dot product and don't forget to plug in the equation of the plane (solved for z of course) into z in the vector field.

$$\begin{aligned}\vec{F}\left(x,y,8-4x-2y\right)\bullet\vec{n} &= \left\langle y,2x,-4x-2y\right\rangle\bullet\frac{\langle 4,2,1\rangle}{\|\nabla f\|}\\ &= \tfrac{1}{\|\nabla f\|}\left(2y\right)\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_1}\vec{F}\bullet d\vec{S} &= \iint\limits_{S_1}\tfrac{1}{\|\nabla f\|}\left(2y\right)dS\\ &= \iint\limits_{D}\tfrac{1}{\|\nabla f\|}\left(2y\right)\|\nabla f\|\,dA\\ &= \iint\limits_{D}2y\,dA\end{aligned}$$

In this case *D* is just S_4 and so we can now finish out the integral.

$$\begin{aligned}\iint\limits_{S_1}\vec{F}\bullet d\vec{S} &= \iint\limits_{D}2y\,dA\\ &= \int_0^2\int_0^{4-2x}2y\,dy\,dx\\ &= \int_0^2 y^2\Big|_0^{4-2x}\,dx\\ &= \int_0^2\left(4-2x\right)^2dx\\ &= -\tfrac{1}{6}\left(4-2x\right)^3\Big|_0^2 = \underline{\tfrac{32}{3}}\end{aligned}$$

Step 3

Next, we'll take care of S_2. In this case the equation for the surface is simply $y = 0$ and *D* is given in the sketch of S_2 in Step 1.

In this case S_2 is simply a portion of the *xz*-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n} = -\vec{j} = \left\langle 0,-1,0 \right\rangle$.

The dot product for this surface is,

$$\vec{F}\left(x,0,z\right)\bullet\vec{n} = \left\langle 0,2x,z-8\right\rangle\bullet\frac{\left\langle 0,-1,0\right\rangle}{1} = -2x$$

Don't forget to plug $y = 0$ into the vector field and note that the magnitude of the gradient is,

$$\left\|\nabla f\right\| = \sqrt{\left(0\right)^2 + \left(1\right)^2 + \left(0\right)^2} = 1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_2} \vec{F}\bullet d\vec{S} &= \iint\limits_{S_2} -2x\,dS \\ &= \iint\limits_{D} -2x\left(1\right)dA \\ &= \int_0^2\int_0^{8-4x} -2x\,dz\,dx \\ &= \int_0^2 -2xz\Big|_0^{8-4x}\,dx \\ &= \int_0^2 8x^2 - 16x\,dx \\ &= \left(\tfrac{8}{3}x^3 - 8x^2\right)\Big|_0^2 = \underline{-\tfrac{32}{3}}\end{aligned}$$

Step 4

Now we can deal with S_3. In this case the equation for the surface is simply $x = 0$ and *D* is given in the sketch of S_3 in Step 1.

In this case S_3 is simply a portion of the *yz*-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n} = -\vec{i} = \left\langle -1,0,0 \right\rangle$.

The dot product for this surface is,

$$\vec{F}\left(0,y,z\right)\bullet\vec{n} = \left\langle y,0,z-8\right\rangle\bullet\frac{\left\langle -1,0,0\right\rangle}{1} = -y$$

Don't forget to plug $x = 0$ into the vector field and note that the magnitude of the gradient is,

$$\left\|\nabla f\right\| = \sqrt{\left(1\right)^2 + \left(0\right)^2 + \left(0\right)^2} = 1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_3} \vec{F}\bullet d\vec{S} &= \iint\limits_{S_3} -y\,dS \\ &= \iint\limits_{D} -y\left(1\right)dA \\ &= \int_0^4\int_0^{8-2y} -y\,dz\,dy \\ &= \int_0^4 -yz\Big|_0^{8-2y}\,dy \\ &= \int_0^4 2y^2 - 8y\,dy \\ &= \left(\tfrac{2}{3}y^3 - 4y^2\right)\Big|_0^4 = \underline{-\tfrac{64}{3}}\end{aligned}$$

Step 5

Finally let's take care of S_4. In this case the equation for the surface is simply $z = 0$ and *D* is given in the sketch of S_4 in Step 1.

In this case S_4 is simply a portion of the *xy*-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n} = -\vec{k} = \left\langle 0,0,-1\right\rangle$.

The dot product for this surface is,

$$\vec{F}\left(x,y,0\right)\bullet\vec{n} = \left\langle y,2x,-8\right\rangle\bullet\frac{\left\langle 0,0,-1\right\rangle}{1} = 8$$

Don't forget to plug $z = 0$ into the vector field and note that the magnitude of the gradient is,

$$\left\|\nabla f\right\| = \sqrt{\left(0\right)^2 + \left(0\right)^2 + \left(1\right)^2} = 1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_4} \vec{F} \bullet d\vec{S} &= \iint\limits_{S_4} 8\, dS \\ &= \iint\limits_{D} 8\, dA \\ &= 8\iint\limits_{D} dA \\ &= 8\left(\text{Area of } D\right) \\ &= 8\left(\tfrac{1}{2}\right)\left(2\right)\left(4\right) = \underline{32}\end{aligned}$$

In this case notice that we didn't have to actually compute the double integral since *D* was just a right triangle and we can easily compute its area.

Step 6

Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

$$\iint\limits_{S} \vec{F} \bullet d\vec{S} = \left(\tfrac{32}{3}\right) + \left(-\tfrac{32}{3}\right) + \left(-\tfrac{64}{3}\right) + \left(32\right) = \boxed{\tfrac{32}{3}}$$

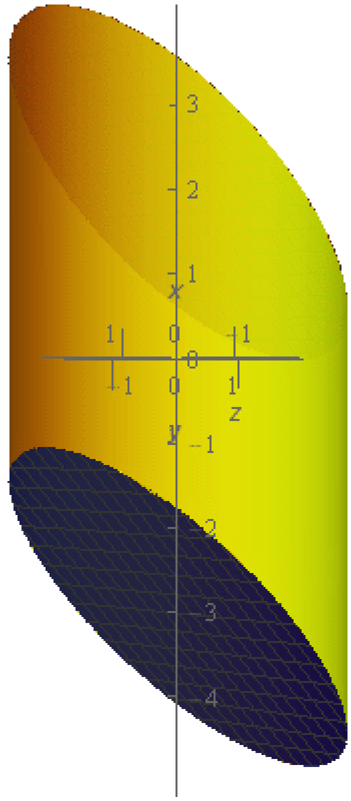
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

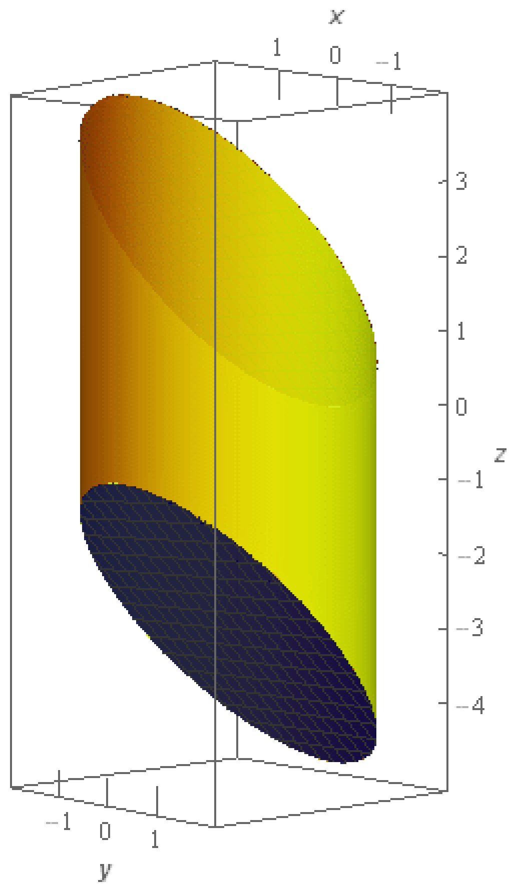
---

6. Evaluate $\iint\limits_{S} \vec{F} \bullet d\vec{S}$ where $\vec{F} = yz\,\vec{i} + x\,\vec{j} + 3y^2\,\vec{k}$ and *S* is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$, and $z = x + 2$ with the negative orientation. Note that all three surfaces of this solid are included in *S*.

Step 1

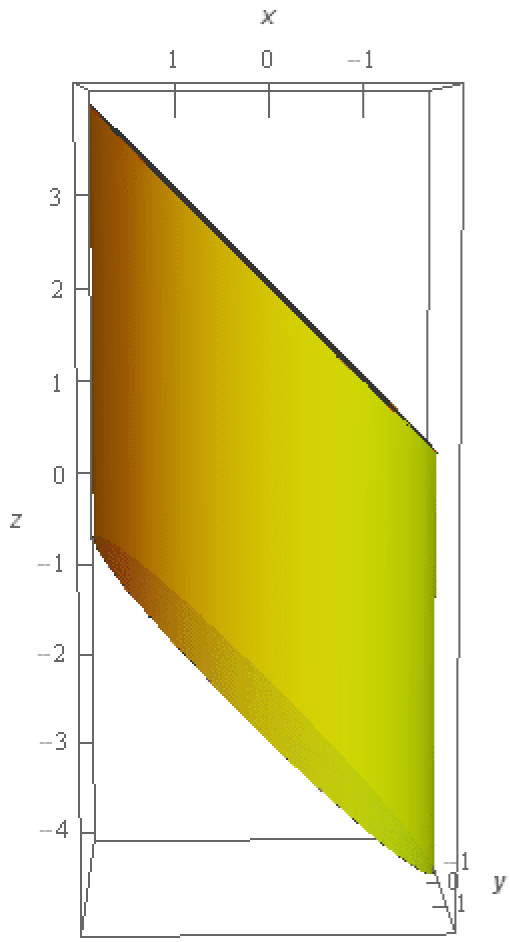
Let's start off with a quick sketch of the surface we are working with in this problem.


x
y
z
3
2
1
0
-1
-2
-3
-4



As noted in the problem statement there are three surfaces here. The “top” of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.



From this view we can see that the top and bottom planes that "cap" the cylinder are parallel.

Let's define the three surfaces in the sketch as follows.

$$S_1: \text{ Cylinder given by } x^2+y^2=4 \;\left(i.e \text{ the walls of the solid}\right)$$
$$S_2: \text{ Plane given by } z=x+2 \;\left(i.e \text{ the top cap of the cylinder}\right)$$
$$S_3: \text{ Plane given by } z=x-3 \;\left(i.e \text{ the bottom cap of the cylinder}\right)$$

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. The surface in this case is a cylinder and so we'll need to parameterize it. The parameterization of the surface is,

$$\vec r\left(z,\theta\right)=\left\langle 2\cos\theta, 2\sin\theta, z\right\rangle$$

The limits on *z* and θ are,

$$0 \le \theta \le 2\pi, \quad 2\cos\theta - 3 = x - 3 \le z \le x + 2 = 2\cos\theta + 2$$

With the *z* limits we'll need to make sure that we convert the *x*'s into their parameterized form.

In order to evaluate the integral in this case we'll need the cross product $\vec{r}_z \times \vec{r}_\theta$ so here is that work.

$$\vec{r}_z = \langle 0, 0, 1 \rangle \qquad\qquad \vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$$

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -2\sin\theta & 2\cos\theta & 0 \end{vmatrix} = -2\cos\theta\,\vec{i} - 2\sin\theta\,\vec{j}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_z \times \vec{r}_\theta}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|} = \frac{\langle -2\cos\theta, -2\sin\theta, 0 \rangle}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

The surface has the negative orientation and so must point in towards the region enclosed by the surface. This means that normal vectors on cylinder will need to point in towards the *z*-axis and this vector does point in that direction.

To see that this vector points in towards the *z*-axis consider the $0 \le \theta \le \frac{\pi}{2}$. In this range both sine and cosine are positive and so the *x* and *y* component of the normal vector will be negative and so will point in towards the *z*-axis.

Now we'll need the following dot product and don't forget to plug in the parameterization of the surface in the vector field.

$$\begin{aligned} \vec{F}\left(\vec{r}\left(z,\theta\right)\right) \bullet \vec{n} &= \left\langle 2z\sin\theta, 2\cos\theta, 12\sin^2\theta \right\rangle \bullet \frac{\langle -2\cos\theta, -2\sin\theta, 0 \rangle}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|} \\ &= \tfrac{1}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|}\left(-4z\sin\theta\cos\theta - 4\sin\theta\cos\theta\right) \\ &= \tfrac{1}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|}\left[-4\sin\theta\cos\theta\left(z+1\right)\right] \end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_1} \vec{F} \bullet d\vec{S} &= \iint\limits_{S_1} \tfrac{1}{\left\|\vec{r}_z \times \vec{r}_\theta\right\|}\left[-2\sin\left(2\theta\right)\left(z+1\right)\right] dS \\ &= \iint\limits_{D} \tfrac{1}{\left\|\vec{r}_z \times \vec{r}_\theta\right\|}\left[-2\sin\left(2\theta\right)\left(z+1\right)\right]\left\|\vec{r}_z \times \vec{r}_\theta\right\| dA \\ &= \iint\limits_{D} -2\sin\left(2\theta\right)\left(z+1\right) dA\end{aligned}$$

In this case *D* is is nothing more than the limits on *z* and θ we gave above and so we can now finish out the integral.

$$\begin{aligned}\iint\limits_{S_1} \vec{F} \bullet d\vec{S} &= \iint\limits_{D} -4\sin\theta\cos\theta\left(z+1\right) dA \\ &= \int_0^{2\pi}\int_{2\cos\theta-3}^{2\cos\theta+2} -4\sin\theta\cos\theta\left(z+1\right) dz\, d\theta \\ &= \int_0^{2\pi} -4\sin\theta\cos\theta\left(\tfrac{1}{2}z^2+z\right)\Big|_{2\cos\theta-3}^{2\cos\theta+2} d\theta \\ &= \int_0^{2\pi} -10\sin\theta\cos\theta - 40\sin\theta\cos^2\theta\, d\theta \\ &= \int_0^{2\pi} -5\sin\left(2\theta\right) - 40\sin\theta\cos^2\theta\, d\theta \\ &= \left(\tfrac{5}{2}\cos\left(2\theta\right) + \tfrac{40}{3}\cos^3\theta\right)^3\Big|_0^{2\pi} = \underline{0}\end{aligned}$$

Step 3

Next we'll take care of S_2. In this case the equation for the surface is can be written as $z - x - 2 = 0$ and *D* is the disk $x^2 + y^2 \le 4$.

A unit normal vector for S_2 is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle -1, 0, 1\right\rangle}{\left\|\nabla f\right\|}$$

The region has the negative orientation and so must point into the enclosed region and so must point downwards (since this is the top "cap" of the cylinder). The normal vector above points upwards (it has a positive *z* component) and so we'll need to multiply this by minus one to get the normal vector we need for this surface.

The correct normal vector is then,

$$\vec{n} = -\frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle 1, 0, -1\right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The dot product we'll need for this surface is,

$$\begin{aligned}\vec{F}\left(x,y,x+2\right)\bullet\vec{n} &= \left\langle y\left(x+2\right),x,3y^2\right\rangle\bullet\frac{\left\langle 1,0,-1\right\rangle}{\left\|\nabla f\right\|}\\ &= \tfrac{1}{\left\|\nabla f\right\|}\left(xy+2y-3y^2\right)\end{aligned}$$

Don't forget to plug the equation of the surface into *z* in the vector field.

The integral is then,

$$\begin{aligned}\iint\limits_{S_2}\vec{F}\bullet d\vec{S} &= \iint\limits_{S_2}\tfrac{1}{\left\|\nabla f\right\|}\left(xy+2y-3y^2\right)dS\\ &= \iint\limits_{D}\tfrac{1}{\left\|\nabla f\right\|}\left(xy+2y-3y^2\right)\left\|\nabla f\right\|dA\\ &= \iint\limits_{D}xy+2y-3y^2\,dA\end{aligned}$$

Note that we'll need to finish this integral with polar coordinates and the polar limits will be,

$$\begin{aligned}&0\le\theta\le2\pi\\ &0\le r\le2\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_2}\vec{F}\bullet d\vec{S} &= \iint\limits_{D}xy+2y-3y^2\,dA\\ &= \int_0^{2\pi}\int_0^2\left(r^2\sin\theta\cos\theta+2r\sin\theta-3r^2\sin^2\theta\right)\left(r\right)dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2\tfrac{1}{2}r^3\sin\left(2\theta\right)+2r^2\sin\theta-\tfrac{3}{2}r^3\left(1-\cos\left(2\theta\right)\right)dr\,d\theta\\ &= \int_0^{2\pi}\tfrac{1}{8}r^4\sin\left(2\theta\right)+\tfrac{2}{3}r^3\sin\theta-\tfrac{3}{8}r^4\left(1-\cos\left(2\theta\right)\right)\Big|_0^2\,d\theta\\ &= \int_0^{2\pi}2\sin\left(2\theta\right)+\tfrac{16}{3}\sin\theta-6\left(1-\cos\left(2\theta\right)\right)d\theta\\ &= \left(-\cos\left(2\theta\right)-\tfrac{16}{3}\cos\theta-6\left(\theta-\tfrac{1}{2}\sin\left(2\theta\right)\right)\right)\Big|_0^{2\pi} = \underline{-12\pi}\end{aligned}$$

Step 4

Finally, let's integrate over S_3. In this case the equation for the surface is can be written as $z-x+3=0$ and *D* is the disk $x^2+y^2\le4$.

A unit normal vector for S_2 is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -1, 0, 1\rangle}{\|\nabla f\|}$$

The region has the negative orientation and so must point into the enclosed region and so must point upwards (since this is the bottom "cap" of the cylinder). The normal vector above does point upwards (it has a positive *z* component) and so is the normal vector we'll need.

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The dot product we'll need for this surface is,

$$\begin{aligned}\vec{F}(x, y, x-3)\bullet\vec{n} &= \left\langle y(x-3), x, 3y^2\right\rangle \bullet \frac{\langle -1, 0, 1\rangle}{\|\nabla f\|} \\ &= \tfrac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right)\end{aligned}$$

Don't forget to plug the equation of the surface into *z* in the vector field.

The integral is then,

$$\begin{aligned}\iint\limits_{S_3} \vec{F}\bullet d\vec{S} &= \iint\limits_{S_3} \tfrac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right) dS \\ &= \iint\limits_{D} \tfrac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right)\|\nabla f\|\, dA \\ &= \iint\limits_{D} -xy + 3y + 3y^2 \, dA\end{aligned}$$

Note that we'll need to finish this integral with polar coordinates and the polar limits will be,

$$\begin{aligned}&0 \le \theta \le 2\pi \\ &0 \le r \le 2\end{aligned}$$

The integral is then,

$$\iint\limits_{S_3} \vec{F} \bullet d\vec{S} = \iint\limits_{D} -xy + 3y + 3y^2 \, dA$$
$$= \int_0^{2\pi} \int_0^2 \left(-r^2 \sin\theta\cos\theta + 3r\sin\theta + 3r^2\sin^2\theta\right)(r)\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 -\tfrac{1}{2} r^3 \sin(2\theta) + 3r^2 \sin\theta + \tfrac{3}{2} r^3 \left(1 - \cos(2\theta)\right) dr\, d\theta$$
$$= \int_0^{2\pi} -\tfrac{1}{8} r^4 \sin(2\theta) + r^3 \sin\theta + \tfrac{3}{8} r^4 \left(1 - \cos(2\theta)\right)\Big|_0^2 \, d\theta$$
$$= \int_0^{2\pi} -2\sin(2\theta) + 16\sin\theta + 6\left(1 - \cos(2\theta)\right) d\theta$$
$$= \left(\cos(2\theta) - 16\cos\theta + 6\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\right)\Big|_0^{2\pi} = \underline{12\pi}$$

Step 5
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,

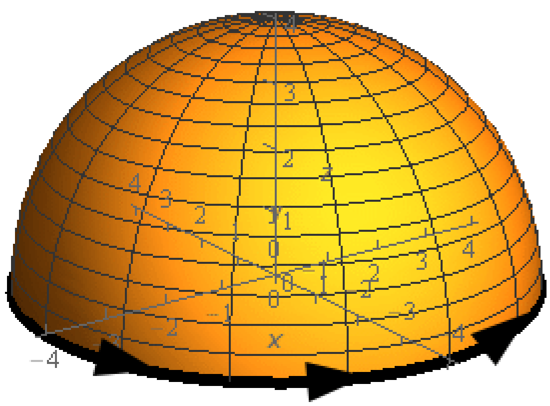
$$\iint\limits_{S} \vec{F} \bullet d\vec{S} = (0) + (-12\pi) + (12\pi) = \boxed{0}$$

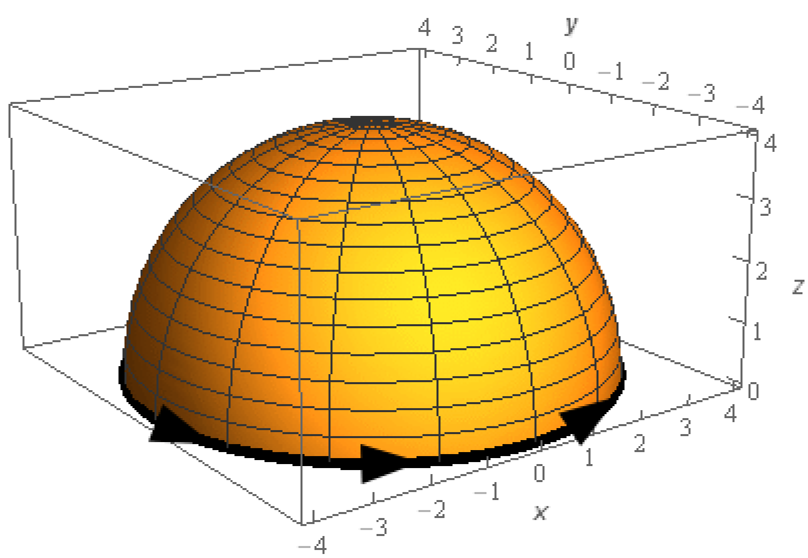
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

Section 6-5 : Stokes' Theorem

1. Use Stokes' Theorem to evaluate $\iint\limits_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = y\,\vec{i} - x\,\vec{j} + yx^3\,\vec{k}$ and *S* is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is upwards then all the normal vectors will be pointing outwards. So, if we walk along the edge of the surface, *i.e.* the curve *C*, in the direction indicated with our head pointed away from the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of *C*.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it upside down on a piece of paper on a table. Sketch a set of axis on the piece of paper that will represent the plane the cup/bowl is sitting on to really help with the visualization. Then cut out a little stick figure and put a face on the “front” side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with it’s head pointing in the direction of the normal vectors (out away from the sphere/cup/bowl in our case) with its left hand over the surface. The direction that the “face” on the stick figure is facing is the direction you’d need to walk along the surface to get the positive orientation for *C*.

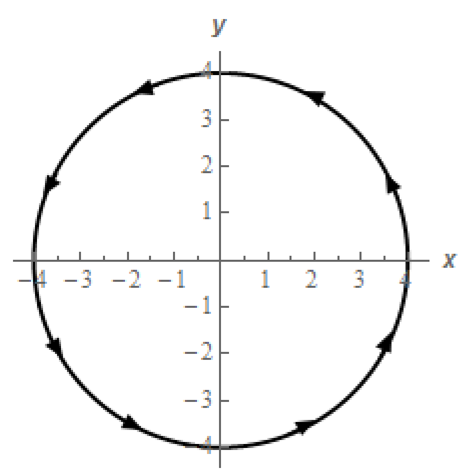
Step 2
We are going to use Stokes’ Theorem in the following direction.

$$\iint\limits_{S} \operatorname{curl} \vec{F} \bullet d\vec{S} = \int\limits_{C} \vec{F} \bullet d\vec{r}$$

We’ve been given the vector field in the problem statement so we don’t need to worry about that. We will need to deal with *C*.

Because *C* is just the curve along the bottom of the upper half of the sphere we can see that *C* in fact will be the intersection of the sphere and the *xy*-plane (*i.e.* $z = 0$). Therefore, *C* is just the circle of radius 4.

If we look at the sphere from above we get the following sketch of *C*.



The parameterization of *C* is given by,

$$\vec{r}\left(t\right) = \left\langle 4\cos t, 4\sin t, 0\right\rangle \qquad\qquad 0 \le t \le 2\pi$$

The *z* component of the parameterization is zero because *C* lies in the *xy*-plane.

Step 3

Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right) &= \left\langle 4\sin t, -4\cos t, 256\sin t\cos^3 t\right\rangle\cdot\left\langle -4\sin t, 4\cos t, 0\right\rangle \\ &= -16\sin^2 t - 16\cos^2 t \\ &= -16\end{aligned}$$

Don't forget to plug the parameterization of *C* into the vector field!

Step 4

Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

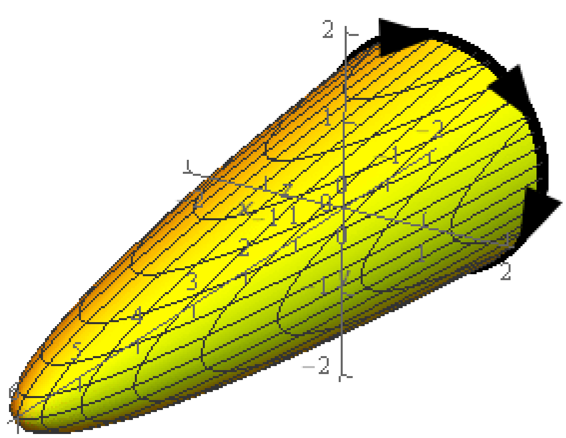
$$\begin{aligned}\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} &= \int\limits_C \vec{F}\cdot d\vec{r} \\ &= \int_0^{2\pi} -16\,dt = \boxed{-32\pi}\end{aligned}$$

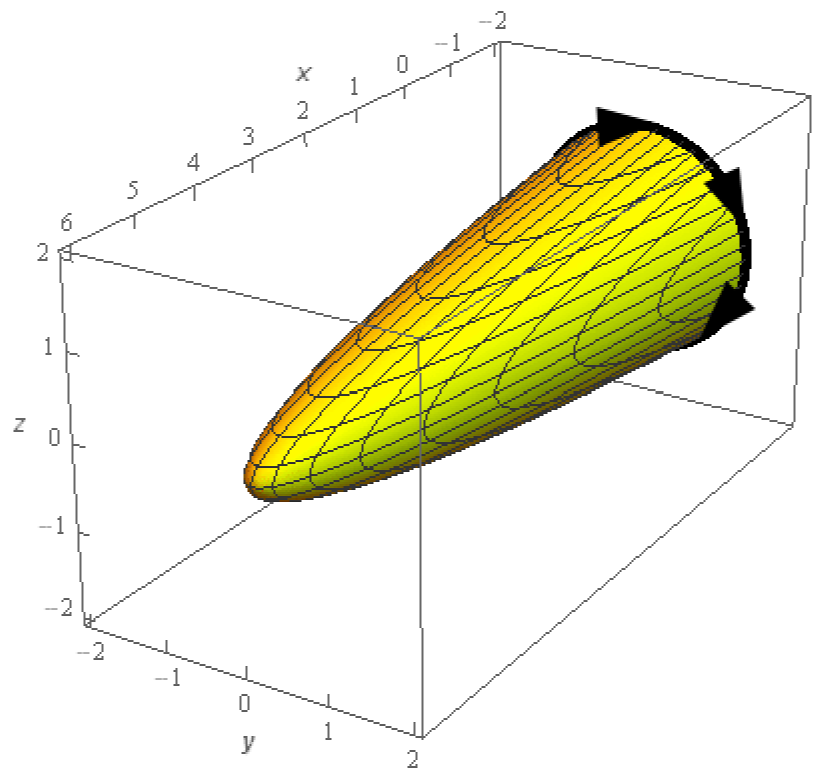
---

2. Use Stokes' Theorem to evaluate $\displaystyle\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ where $\vec{F} = \left(z^2 - 1\right)\vec{i} + \left(z + xy^3\right)\vec{j} + 6\vec{k}$ and *S* is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative *x*-axis direction.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is towards the negative *x*-axis all the normal vectors will be pointing into the region enclosed by the surface. So, if we walk along the edge of the surface, *i.e.* the curve *C*, in the direction indicated with our head pointed into the region enclosed by the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore, the direction indicated in the sketch is the positive orientation of *C*.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it on its side with a piece of paper behind it. Sketch a set of axes on the piece of paper that will represent the plane the cup/bowl is sitting in front of to really help with the visualization. Then cut out a little stick figure and put a face on the "front" side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with its head pointing in the direction of the normal vectors (into the paraboloid/cup/bowl in our case) with its left hand over the surface. The direction that the "face" on the stick figure is facing is the direction you'd need to walk along the surface to get the positive orientation for *C*.

Step 2

We are going to use Stokes' Theorem in the following direction.

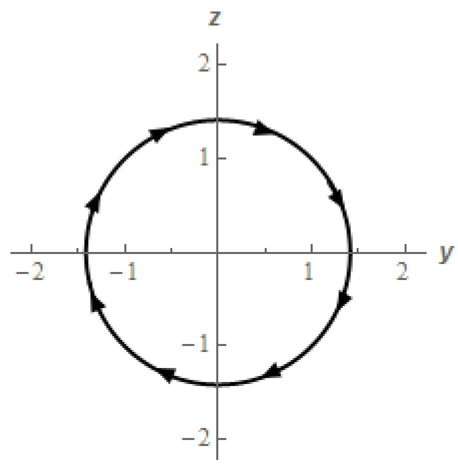
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

We've been given the vector field in the problem statement so we don't need to worry about that. We will need to deal with *C*.

In this case *C* is the curve we get by setting the two equations in the problem statement equal. Doing this gives,

$$-2 = 6 - 4y^2 - 4z^2 \quad \to \quad 4y^2 + 4z^2 = 8 \quad \Rightarrow \quad y^2 + z^2 = 2$$

We will see following sketch of *C* if we are in front of the paraboloid and look directly along the *x*-axis.



One possible parameterization of *C* is given by,

$$\vec r\left(t \right) = \left\langle { - 2,\sqrt 2 \sin t,\sqrt 2 \cos t} \right\rangle \qquad 0 \le t \le 2\pi$$

The *x* component of the parameterization is –2 because *C* lies at $x = -2$.

Step 3
Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}\vec F\left({\vec r\left(t \right)} \right)\centerdot\vec r'\left(t \right) & = \left\langle {2{{\cos }^2}t - 1,\sqrt 2 \cos t - 4\sqrt 2 {{\sin }^3}t,6} \right\rangle \centerdot \left\langle {0,\sqrt 2 \cos t, - \sqrt 2 \sin t} \right\rangle \\ & = \sqrt 2 \cos t\left({\sqrt 2 \cos t - 4\sqrt 2 {{\sin }^3}t} \right) - 6\sqrt 2 \sin t\\ & = 2{\cos ^2}t - 8\cos t{\sin ^3}t - 6\sqrt 2 \sin t\\ & = \left({1 + \cos \left({2t} \right)} \right) - 8\cos t{\sin ^3}t - 6\sqrt 2 \sin t\end{aligned}$$

Don't forget to plug the parameterization of *C* into the vector field!

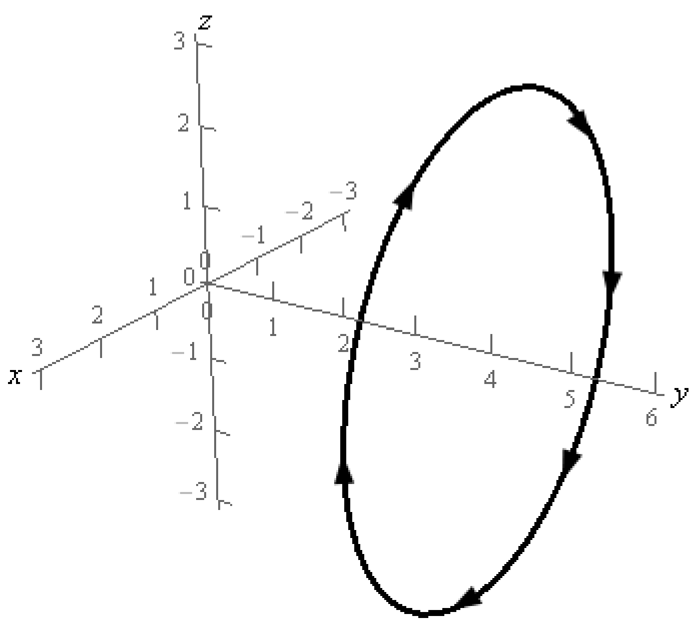
We also did a little simplification on the first term with an eye towards the integration.

Step 4
Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

$$\begin{aligned}\iint\limits_S \operatorname{curl}\vec F \bullet d\vec S &= \int\limits_C \vec F \bullet d\vec r \\ &= \int_0^{2\pi} \left(1+\cos\left(2t\right)\right) - 8\cos t\sin^3 t - 6\sqrt{2}\sin t\,dt \\ &= \left(t + \tfrac{1}{2}\sin\left(2t\right) - 2\sin^4 t + 6\sqrt{2}\cos t\,dt\right)\Big|_0^{2\pi} = \boxed{2\pi}\end{aligned}$$

---

3. Use Stokes' Theorem to evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F = -yz\,\vec i + \left(4y+1\right)\vec j + xy\,\vec k$ and *C* is is the circle of radius 3 at $y = 4$ and perpendicular to the *y*-axis. *C* has a clockwise rotation if you are looking down the *y*-axis from the positive *y*-axis to the negative *y*-axis. See the figure below for a sketch of the curve.



Step 1
Okay, we are going to use Stokes' Theorem in the following direction.

$$\int\limits_C \vec F \bullet d\vec r = \iint\limits_S \operatorname{curl}\vec F \bullet d\vec S$$

So, let's first compute $\operatorname{curl}\vec F$ since that is easy enough to compute and might be useful to have when we go to determine the surface *S* we're going to integrate over.

The curl of the vector field is then,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 4y+1 & xy \end{vmatrix} = x\vec{i} - y\vec{j} + z\vec{k} - y\vec{j} = \underline{x\vec{i} - 2y\vec{j} + z\vec{k}}$$

Step 2
Now we need to find a surface *S* with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation. This can seem to be a daunting task at times but it's not as bad as it might appear to be. First we know that the boundary curve needs to be a circle. This means that we're going to be looking for a surface whose cross section is a circle and we know of several surfaces that meet this requirement. We know that spheres, cones and elliptic paraboloids all have circles as cross sections.

The question becomes which of these surfaces would be best for us in this problem. To make this decision remember that we'll eventually need to plug this surface into the vector field and then take the dot product of this with normal vector (which will also come from the surface of course).

In general, it won't be immediately clear from the curl of the vector field by itself which surface we should use and that is the case here. The curl of the vector field has all three components and none of them are that difficult to deal with but there isn't anything that suggests one surface might be easier than the other.

So, let's consider a sphere first. The issue with spheres is that its parameterization and normal vector are lengthy and many lead to messy integrands. So, because the curl of the vector field does have all three components to it which may well lead to long and/or messy integrands we'll not work with a sphere for this problem.

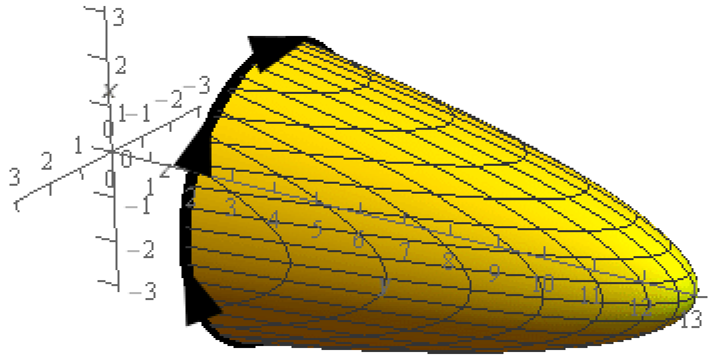
Now let's think about a cone. Equations of cones aren't that bad but they will involve a square root and in this case would need to be in the form $y = \sqrt{ax^2 + bz^2}$ because the boundary curve is centered on the *y*-axis. The normal vector will also contain roots and this will often lead to messy integrands. So, let's not work with a cone either in this problem.

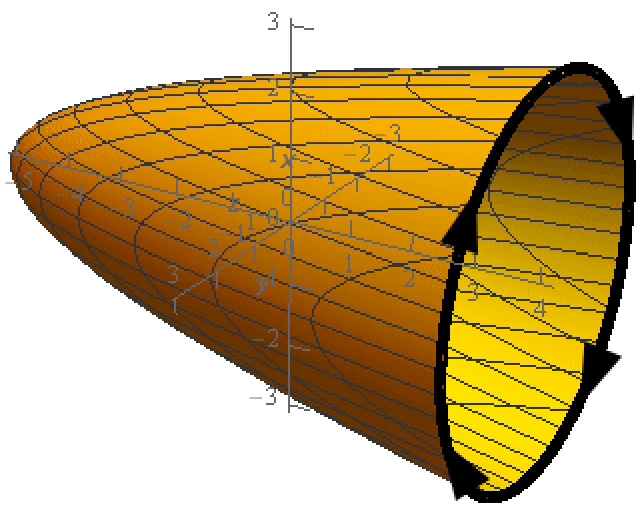
That leaves elliptic paraboloids and we probably should have considered them first. The equations are simple and the normal vectors are even simpler so they seem like a good choice of surface for this problem.

Note that we're not saying that spheres and cones are never good choices for the surface. For some vector fields the curl may end up being very simple with one of these surfaces and so they would be perfectly good choices.

Step 3
We have two possibilities for elliptic paraboloids that we could use here. Both will be centered on the *y*-axis but one will open in the negative *y* direction while the other will open in the positive *y* direction.

Here is a couple of sketches of possible elliptic paraboloids we could use here.





Let's get an equation for each of these. Note that for each of these if we set the equation of the paraboloid and the plane $y = 4$ equal we need to get the circle $x^2 + z^2 = 9$ since this is the boundary curve that should occur at $y = 4$.

Let's get the equation of the first paraboloid (the one that opens in the negative *y* direction. We know that the equation of this paraboloid should be $y = a - x^2 - z^2$ for some value of *a*. As noted if we set this equal to $y = 4$ and do some simplification we know what equation we should get. So, let's set the two equations equal.

$$4 = a - x^2 - z^2 \qquad \to \qquad x^2 + z^2 = a - 4 = 9 \qquad \to \qquad a = 13$$

As shown we know that the $a-4$ should be 9 and so we must have $a=13$. Therefore, the equation of the paraboloid that open in the negative *y* direction is,

$$y = 13 - x^2 - z^2$$

Next, let's get the equation of the paraboloid that opens in the positive *y* direction. The equation of this paraboloid will be in the form $y = x^2 + y^2 + a$ for some *a*. Setting this equal to $y=4$ gives,

$$4 = x^2 + z^2 + a \qquad \to \qquad x^2 + z^2 = 4 - a = 9 \qquad \to \qquad a = -5$$

The equation of the paraboloid that opens in the positive *y* direction is then,

$$y = x^2 + z^2 - 5$$

Either of these surfaces could be used to do this problem.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, *C*, that matches the orientation that was given in the problem statement.

We'll find the normal vectors for each surface despite the fact that we really only need to do it for one of them since we only need one of the surfaces to do the problem as noted in the previous step. Determining the orientation of the surface can be a little tricky for some folks so doing an extra one might help see what's going on here.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the surface itself. We now need to determine if we are walking along the outside of the surface or the inside of the surface.

If we are walking along the outside of the surface then our heads, and hence the normal vectors, will be pointing away from the region enclosed by the surface. On the other hand, if we are walking along the inside of the surface then our heads, and hence the normal vectors, will be pointing into the region enclosed by the surface.

To help visualize this for our two surfaces it might help to get a cup or bowl that we can use to represent the surface. The edge of the cup/bowl will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, put the cup/bowl on its side so it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Do we need to put the stick figure on the inside or outside of the cup/bowl to do this?

Okay, let's do this for the first surface, $y = 13 - x^2 - z^2$. In this case our stick figure would need to be standing on the inside of the cup/bowl/surface. Therefore, the normal vectors on the surface would all need to be point in towards the region enclosed by the surface. This also will mean that all the normal vectors will need to have a negative *y* component. Again, to visualize this take the stick figure and move it into the region and toward the end of cup/bowl/surface and you'll see it start to point more and more in the negative *y* direction (and hence will have a negative *y* component). Note that the *x* and *z* component can be either positive or negative depending on just where we are on the interior of the surface.

Now, let's take a look at the first surface, $y = x^2 + z^2 - 5$. For this surface our stick figure would need to be standing on the outside of the cup/bowl/surface. So, in this case, the normal vectors would point out away from the region enclosed by the surface. These will also have a negative *y* component and you can use the method we discussed in the above paragraph to help visualize this.

Step 5
We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$f\left(x,y,z\right) = 13 - x^2 - z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle -2x, -1, -2z\right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the *y* component is negative.

Next, we'll need to compute the following dot product.

$$\begin{aligned}\operatorname{curl}\vec{F}\bullet\vec{n} &= \left\langle x, -2\left(13 - x^2 - z^2\right), z\right\rangle \bullet \frac{\left\langle -2x, -1, -2z\right\rangle}{\left\|\nabla f\right\|} \\ &= \tfrac{1}{\left\|\nabla f\right\|}\left(-2x^2 + 2\left(13 - x^2 - z^2\right) - 2z^2\right) \\ &= \tfrac{1}{\left\|\nabla f\right\|}\left(26 - 4x^2 - 4z^2\right)\end{aligned}$$

Step 6
Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S}\\ &= \iint_S \tfrac{1}{\|\nabla f\|}\left(26-4x^2-4z^2\right)dS\\ &= \iint_D \tfrac{1}{\|\nabla f\|}\left(26-4x^2-4z^2\right)\|\nabla f\|\,dA\\ &= \iint_D 26-4x^2-4z^2\,dA\end{aligned}$$

Step 7
To finish this integral out then we'll need to convert to polar coordinates using the following polar coordinates.

$$x = r\cos\theta \qquad z = r\sin\theta \qquad x^2+z^2=r^2$$

In this case *D* is just the disk $x^2+z^2 \le 9$ and so the limits for the integral are,

$$\begin{aligned}&0\le\theta\le 2\pi\\ &0\le r\le 3\end{aligned}$$

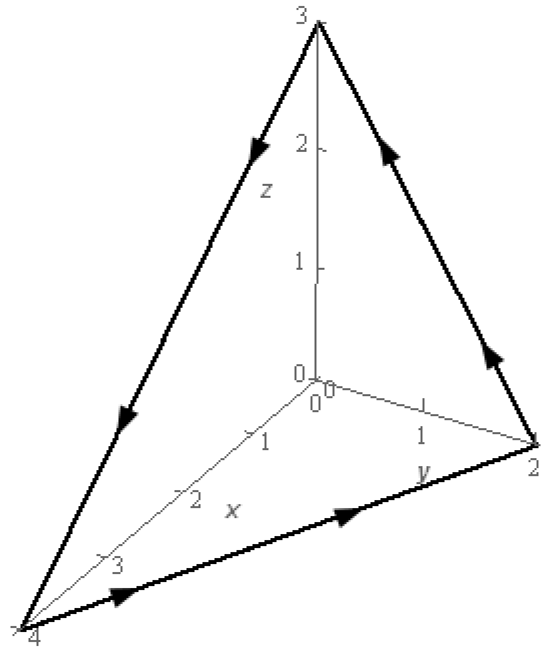
The integral is then,

$$\begin{aligned}\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S}\\ &= \int_0^{2\pi}\int_0^3 \left(26-4r^2\right)r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3 26r-4r^3\,dr\,d\theta\\ &= \int_0^{2\pi}\left.\left(13r^2-r^4\right)\right|_0^3 d\theta\\ &= \int_0^{2\pi} 36\,d\theta\\ &= \boxed{72\pi}\end{aligned}$$

Don't forget to pick up an extra *r* from converting the *dA* to polar coordinates.

---

4. Use Stokes' Theorem to evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F} = \left(3yx^2+z^3\right)\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and *C* is is triangle with vertices $(0,0,3)$, $(0,2,0)$ and $(4,0,0)$. *C* has a counter clockwise rotation if you are above the triangle and looking down towards the *xy*-plane. See the figure below for a sketch of the curve.



Step 1
Okay, we are going to use Stokes' Theorem in the following direction.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

So, let's first compute $\text{curl}\,\vec{F}$ since that is easy enough to compute and might be useful to have when we go to determine the surface *S* we're going to integrate over.

The curl of the vector field is then,

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3yx^2 + z^3 & y^2 & 4yx^2 \end{vmatrix} = 4x^2\vec{i} + 3z^2\vec{j} - 3x^2\vec{k} - 8yx\vec{j} = \underline{4x^2\vec{i} + \left(3z^2 - 8yx\right)\vec{j} - 3x^2\vec{k}}$$

Step 2
Now we need to find a surface *S* with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation.

In this case we can see that the triangle looks like the portion of a plane and so it makes sense that we use the equation of the plane containing the three vertices for the surface here.

The curl of the vector field looks a little messy so using a plane here might be the best bet from this perspective as well. It will (hopefully) not make the curl of the vector field any messier and the normal vector, which we'll get from the equation of the plane, will be simple and so shouldn't make the curl of the vector field any worse.

Step 3
Determining the equation of the plane is pretty simple. We have three points on the plane, the vertices, and so we can quickly determine the equation.

First, let's "label" the points as follows,

$$P = (4,0,0) \qquad\qquad Q = (0,2,0) \qquad\qquad R = (0,0,3)$$

Then two vectors that must lie in the plane are,

$$\overrightarrow{QP} = \langle 4,-2,0\rangle \qquad\qquad \overrightarrow{QR} = \langle 0,-2,3\rangle$$

To write the equation of a plane recall that we need a normal vector to the plane. Now, we know that the cross product of these two vectors will be orthogonal to both of the vectors. Also, since both of the vectors lie in the plane the cross product will also be orthogonal, or normal, to the plane. In other words, we can use the cross product of these two vectors as the normal vector to the plane.

The cross product is,

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = -6\vec{i} - 12\vec{j} - 8\vec{k}$$

Now we can use any of the points in our equation. We'll use *Q* for our point. The equation of the plane is then,

$$-6(x-0) - 12(y-2) - 8(z-0) = 0 \qquad \to \qquad \begin{aligned} -6x - 12y - 8z &= -24 \\ \underline{3x + 6y + 4z = 12}& \end{aligned}$$

Note that we divided the equation by –2 to make the equation a little "nicer" to work with.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, *C*, that matches the orientation that was given in the problem statement.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the plane. We now need to determine if we are walking along the top or bottom of the plane.

If we are walking along the top of the plane then our heads, and hence the normal vectors, will be pointing in a generally upwards direction. On the other hand, if we are walking along the bottom of the plane then our heads, and hence the normal vectors, will be pointing generally downwards.

To help visualize this for our plane it might help to cut out a triangular piece of paper that we can use to represent the plane. The edge of the piece of paper will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, hold the piece of paper so that it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Doing this we'll quickly see that we must be walking along the top of the surface. Therefore, the normal vectors on the surface need to be pointing in a generally upwards direction (and hence will have a positive *z* component).

Step 5
We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$z = 3 - \tfrac{3}{4}x - \tfrac{3}{2}y$$

Recall that if we aren't going to parameterize the surface we need it to be written as $z = g\left(x, y\right)$ so that the magnitude of the normal vector will eventually cancel.

Now, that we have the surface written in the "proper" form let's define,

$$f\left(x, y, z\right) = z - 3 + \tfrac{3}{4}x + \tfrac{3}{2}y = 0$$

The unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle \tfrac{3}{4}, \tfrac{3}{2}, 1\right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the *z* component is positive.

Next, we'll need to compute the following dot product.

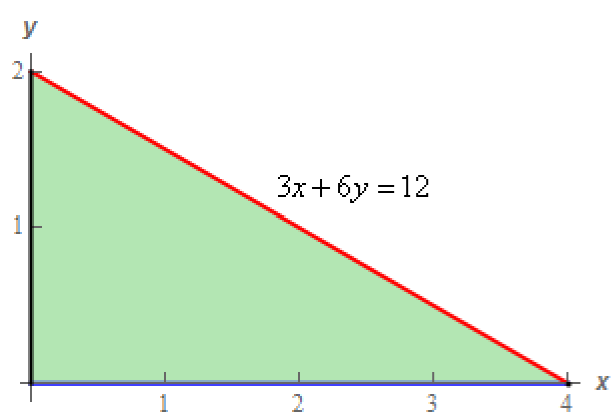
$$\begin{aligned}\operatorname{curl}\vec{F}\cdot\vec{n} &= \left\langle 4x^2, \left(3\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-8yx\right), -3x^2\right\rangle \cdot \frac{\left\langle \tfrac{3}{4},\tfrac{3}{2},1\right\rangle}{\|\nabla f\|} \\ &= \tfrac{1}{\|\nabla f\|}\left(3x^2+\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy-3x^2\right) \\ &= \tfrac{1}{\|\nabla f\|}\left[\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\right]\end{aligned}$$

Don't forget to plug the equation of the surface into the curl of the vector field.

Step 6
Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} \\ &= \iint_S \tfrac{1}{\|\nabla f\|}\left[\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\right]dS \\ &= \iint_D \tfrac{1}{\|\nabla f\|}\left[\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\right]\|\nabla f\|\,dA \\ &= \iint_D \tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\,dA\end{aligned}$$

Step 7
To finish this integral we just need to determine *D*. In this case *D* just the triangle in the *xy*-plane that lies below the plane. Here is a quick sketch of *D*.



The integral doesn't seem to suggest one integration order over the other so let's use the following set of limits for our integral.

$$\begin{aligned}&0 \le x \le 4 \\ &0 \le y \le 2-\tfrac{1}{2}x\end{aligned}$$

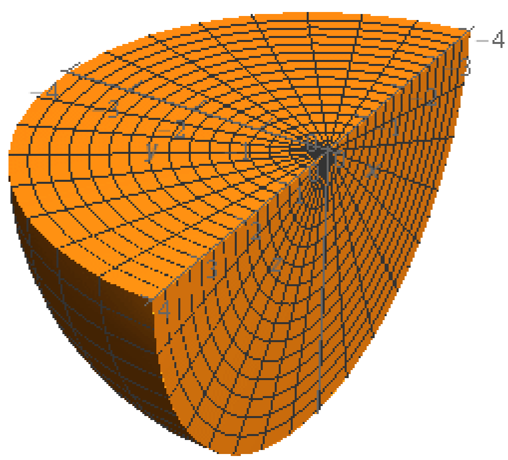
The integral is then,

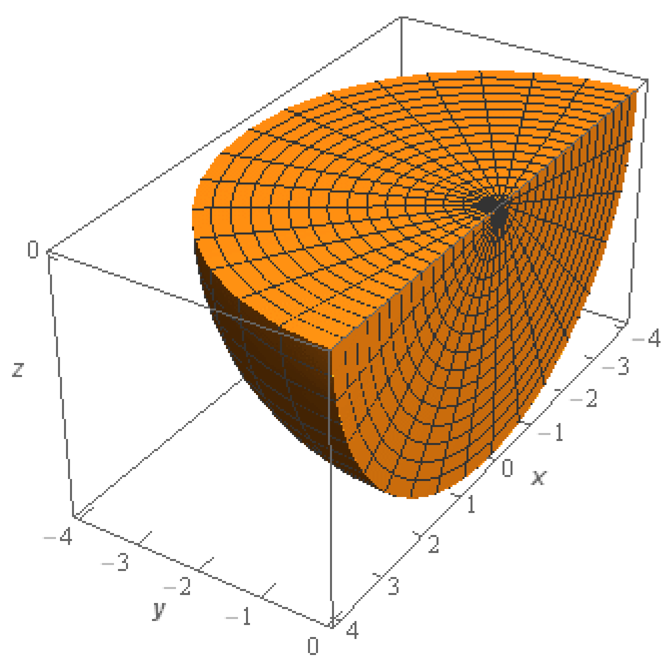
$$\begin{aligned}
\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} \\
&= \int_0^4 \int_0^{2-\frac{1}{2}x} \tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2 - 12xy\,dy\,dx \\
&= \int_0^4 \left(-\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^3 - 6xy^2\right)\Big|_0^{2-\frac{1}{2}x} dx \\
&= \int_0^4 \left(3-\tfrac{3}{4}x\right)^3 - 6x\left(2-\tfrac{1}{2}x\right)^2 dx \\
&= \int_0^4 \left(3-\tfrac{3}{4}x\right)^3 - 24x + 12x^2 - \tfrac{3}{2}x^3\,dx \\
&= \left(-\tfrac{1}{3}\left(3-\tfrac{3}{4}x\right)^4 - 12x^2 + 4x^3 - \tfrac{3}{8}x^4\right)\Big|_0^4 \\
&= \boxed{-5}
\end{aligned}$$

Section 6-6 : Divergence Theorem

1. Use the Divergence Theorem to evaluate $\iint\limits_S \vec F \cdot d\vec S$ where $\vec F = yx^2\,\vec i + \left(xy^2 - 3z^4\right)\vec j + \left(x^3 + y^2\right)\vec k$ and *S* is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in *S*.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Note as well here that because we are including all three surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the portion of the sphere shown above.

Step 2
We are going to use the Divergence Theorem in the following direction.

$$\iint\limits_{S} \vec{F} \bullet d\vec{S} = \iiint\limits_{E} \operatorname{div} \vec{F} \, dV$$

where *E* is just the solid shown in the sketches from Step 1.

Because *E* is a portion of a sphere we'll be wanting to use spherical coordinates for the integration. Here are the spherical limits we'll need to use for this region.

$$\begin{aligned} &\pi \le \theta \le 2\pi \\ &\tfrac{\pi}{2} \le \varphi \le \pi \\ &0 \le \rho \le 4 \end{aligned}$$

One of the restrictions on the region in the problem statement was $y \le 0$. This means that if we look at this from above we'd see the portion of the circle of radius 4 that is below the *x* axis and so we need the given range of θ above to cover this region.

We were also told in the problem statement that $z \le 0$ and so we only want the portion of the sphere that is below the *xy*-plane. We therefore need the given range of φ to make sure we are only below the *xy*-plane.

We'll also need the divergence of the vector field so here is that.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}\left(yx^2\right) + \frac{\partial}{\partial y}\left(xy^2 - 3z^4\right) + \frac{\partial}{\partial z}\left(x^3 + y^2\right) = 4xy$$

Step 3
Now let's apply the Divergence Theorem to the integral and get it converted to spherical coordinates while we're at it.

$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\,dV\\ &= \iiint\limits_E 4xy\,dV\\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\left(\rho\sin\varphi\cos\theta\right)\left(\rho\sin\varphi\sin\theta\right)\left(\rho^2\sin\varphi\right)d\rho\,d\varphi\,d\theta\\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\rho^4\sin^3\varphi\cos\theta\sin\theta\,d\rho\,d\varphi\,d\theta\end{aligned}$$

Don't forget to pick up the $\rho^2\sin\varphi$ when converting the *dV* to spherical coordinates.

Step 4
All we need to do then in evaluate the integral.

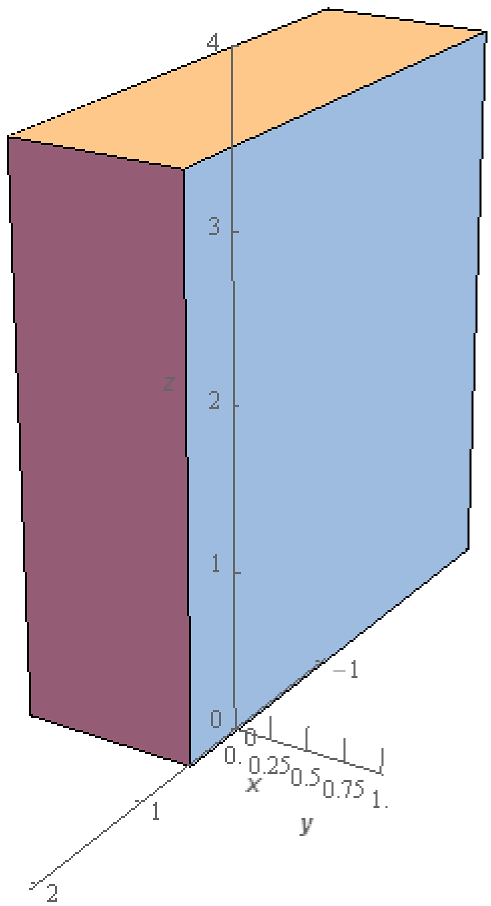
$$\begin{aligned}\iint\limits_S \vec{F}\bullet d\vec{S} &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\rho^4\sin^3\varphi\cos\theta\sin\theta\,d\rho\,d\varphi\,d\theta\\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\left(\tfrac{4}{5}\rho^5\sin^3\varphi\cos\theta\sin\theta\right)\Big|_0^4\,d\varphi\,d\theta\\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\tfrac{4096}{5}\sin\varphi\left(1-\cos^2\varphi\right)\cos\theta\sin\theta\,d\varphi\,d\theta\\ &= \int_{\pi}^{2\pi}\left(-\tfrac{4096}{5}\left(\cos\varphi-\tfrac{1}{3}\cos^3\varphi\right)\cos\theta\sin\theta\right)\Big|_{\frac{1}{2}\pi}^{\pi}\,d\theta\\ &= \int_{\pi}^{2\pi}\tfrac{8192}{15}\cos\theta\sin\theta\,d\theta\\ &= \int_{\pi}^{2\pi}\tfrac{4096}{15}\sin\left(2\theta\right)d\theta\\ &= -\tfrac{2048}{15}\cos\left(2\theta\right)\Big|_{\pi}^{2\pi} = \boxed{0}\end{aligned}$$

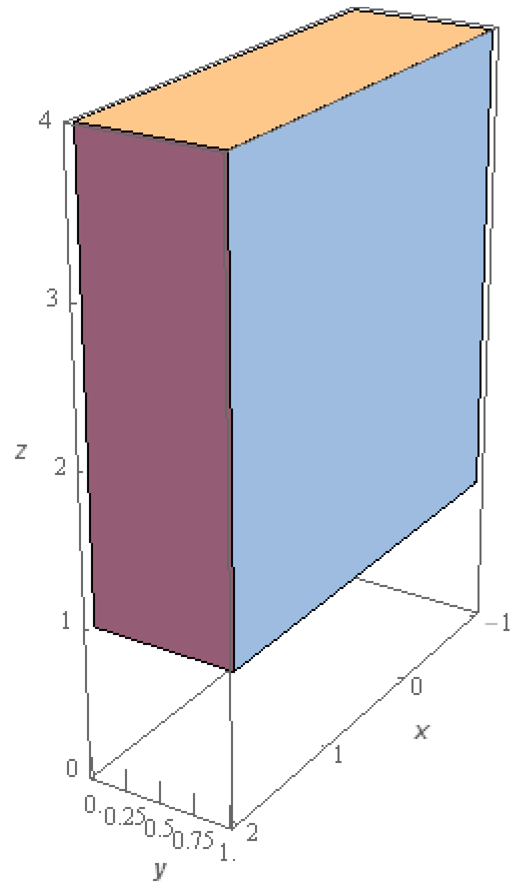
Make sure you can do use your trig formulas as we did here to deal with these kinds of integrals!

---

2. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = \sin\left(\pi x\right)\vec{i} + zy^3\,\vec{j} + \left(z^2+4x\right)\vec{k}$ and *S* is the surface of the box with $-1 \le x \le 2$, $0 \le y \le 1$ and $1 \le z \le 4$. Note that all six sides of the box are included in *S*.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Note as well here that because we are including all six sides of the box shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) for the box.

Step 2

We are going to use the Divergence Theorem in the following direction.

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = \iiint\limits_{E} \operatorname{div} \vec{F} \, dV$$

where E is just the solid shown in the sketches from Step 1.

E is just a box and the limits defining it where given in the problem statement. The limits for our integral will then be,

$$
\begin{aligned}
-1 &\le x \le 2 \\
0 &\le y \le 1 \\
1 &\le z \le 4
\end{aligned}
$$

We'll also need the divergence of the vector field so here is that.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}\left(\sin\left(\pi x\right)\right) + \frac{\partial}{\partial y}\left(zy^3\right) + \frac{\partial}{\partial z}\left(z^2 + 4x\right) = \pi\cos\left(\pi x\right) + 3zy^2 + 2z$$

Step 3
Now let's apply the Divergence Theorem to the integral and get it converted to a triple integral.

$$
\begin{aligned}
\iint\limits_S \vec{F} \bullet d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\, dV \\
&= \iiint\limits_E \pi\cos\left(\pi x\right) + 3zy^2 + 2z\, dV \\
&= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos\left(\pi x\right) + 3zy^2 + 2z\, dz\, dy\, dx
\end{aligned}
$$

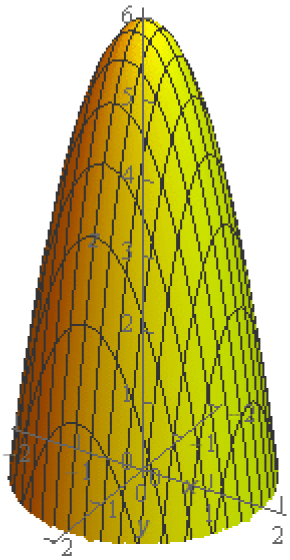
Step 4
All we need to do then in evaluate the integral.

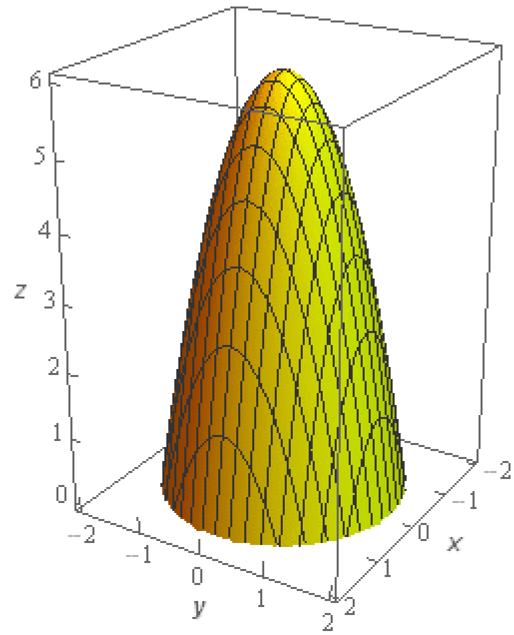
$$
\begin{aligned}
\iint\limits_S \vec{F} \bullet d\vec{S} &= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos\left(\pi x\right) + 3zy^2 + 2z\, dz\, dy\, dx \\
&= \int_{-1}^{2}\int_{0}^{1} \left(\pi z\cos\left(\pi x\right) + \tfrac{3}{2}z^2y^2 + z^2\right)\Big|_1^4\, dy\, dx \\
&= \int_{-1}^{2}\int_{0}^{1} 3\pi\cos\left(\pi x\right) + \tfrac{45}{2}y^2 + 15\, dy\, dx \\
&= \int_{-1}^{2} \left(3y\pi\cos\left(\pi x\right) + \tfrac{15}{2}y^3 + 15y\right)\Big|_0^1\, dx \\
&= \int_{-1}^{2} 3\pi\cos\left(\pi x\right) + \tfrac{45}{2}\, dx \\
&= \left(3\sin\left(\pi x\right) + \tfrac{45}{2}x\right)\Big|_{-1\pi}^{2} = \boxed{\tfrac{135}{2}}
\end{aligned}
$$

---

3. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = 2xz\vec{i} + \left(1 - 4xy^2\right)\vec{j} + \left(2z - z^2\right)\vec{k}$ and *S* is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$. Note that both of the surfaces of this solid included in *S*.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface. The bottom “cap” of the elliptic paraboloid is also included in the surface but isn’t shown.

Note as well here that because we are including both of the surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the region.

Step 2

We are going to use Stokes’ Theorem in the following direction.

$$\iint\limits_{S} \vec{F} \bullet d\vec{S} = \iiint\limits_{E} \operatorname{div} \vec{F}\, dV$$

where *E* is just the solid shown in the sketches from Step 1.

The region *D* for that we’ll need in converting the triple integral into iterated integrals is just the intersection of the two surfaces from the problem statement. This is,

$$0 = 6 - 2x^2 - 2y^2 \qquad \to \qquad x^2 + y^2 = 3$$

So, *D* is a disk and so we’ll eventually be doing cylindrical coordinates for this integral. Here are the cylindrical limits for the region *E*.

$$\begin{aligned} &0 \le \theta \le 2\pi \\ &0 \le r \le \sqrt{3} \\ &0 \le z \le 6 - 2x^2 - 2y^2 = 6 - 2r^2 \end{aligned}$$

Don't forget to convert the *z* limits into cylindrical coordinates as well!

We'll also need the divergence of the vector field so here is that.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(2xz\right) + \frac{\partial}{\partial y}\left(1-4xy^2\right) + \frac{\partial}{\partial z}\left(2z - z^2\right) = 2 - 8xy$$

Step 3

Now let's apply the Divergence Theorem to the integral and get it converted to cylindrical coordinates while we're at it.

$$\begin{aligned}
\iint\limits_S \vec{F}\bullet d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\,dV \\
&= \iiint\limits_E 2 - 8xy\,dV \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} \left(2 - 8r^2\cos\theta\sin\theta\right) r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} 2r - 8r^3\cos\theta\sin\theta\,dz\,dr\,d\theta
\end{aligned}$$

Don't forget to pick up the *r* when converting the *dV* to cylindrical coordinates.

Step 4

All we need to do then in evaluate the integral.

$$\begin{aligned}
\iint\limits_S \vec{F}\bullet d\vec{S} &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} 2r - 8r^3\cos\theta\sin\theta\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - 8r^3\cos\theta\sin\theta\right)z\Big|_0^{6-2r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - 8r^3\cos\theta\sin\theta\right)\left(6 - 2r^2\right)dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 12r - 4r^3 - \left(48r^3 - 16r^5\right)\cos\theta\sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \left(6r^2 - r^4 - \left(12r^4 - \tfrac{8}{3}r^6\right)\cos\theta\sin\theta\right)\Big|_0^{\sqrt{3}}\,d\theta \\
&= \int_0^{2\pi} 9 - 36\cos\theta\sin\theta\,d\theta \\
&= \int_0^{2\pi} 9 - 18\sin\left(2\theta\right)d\theta \\
&= \left(9\theta - 9\cos\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{18\pi}
\end{aligned}$$

