2. CONTINUITY

Continuity

A function f is continuous at $x = a$ if f is defined at $x = a$ and if

$$\lim_{x \to a} f(x) = f(a).$$

Continuity

A function f is continuous at $x = a$ if f is defined at $x = a$ and if

$$\lim_{x \to a} f(x) = f(a).$$

Notice that the previous definition implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f);
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

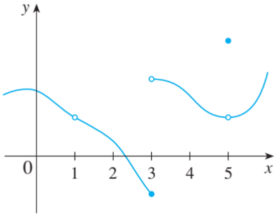
A function f is continuous at $x = a$ if f is defined at $x = a$ and if

$$\lim_{x \to a} f(x) = f(a).$$

Notice that the previous definition implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f);
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

At what numbers f is discontinuous? why?



Examples:
$f(x) = \frac{1}{x}$

$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

Examples:
$f(x) = \frac{1}{x}$

$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Examples:
$f(x) = \frac{1}{x}$

$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$
A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

The following types of functions are continuous at every element in their domains:

polynomials, rational functions, root functions,
trigonometric functions, inverse trigonometric functions,
exponential functions, logarithmic functions.

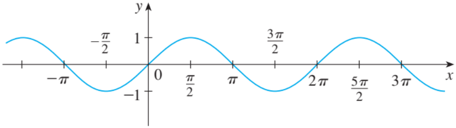
Example: Trigonometric functions

$$f(x) = \sin(x), \qquad f(x) = \cos(x), \qquad f(x) = \tan(x)$$

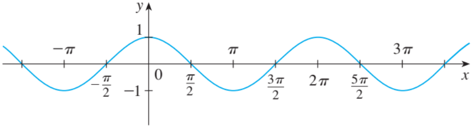
$$f(x) = \cot(x), \qquad f(x) = \sec(x), \qquad f(x) = \csc(x)$$

Example: Trigonometric functions

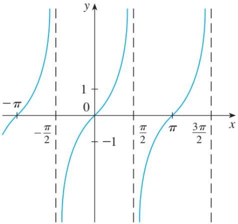
$$f(x) = \sin(x), \qquad f(x) = \cos(x), \qquad f(x) = \tan(x)$$
$$f(x) = \cot(x), \qquad f(x) = \sec(x), \qquad f(x) = \csc(x)$$



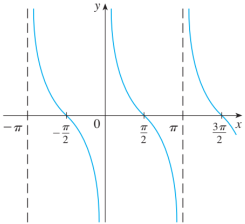
(a) $f(x) = \sin x$



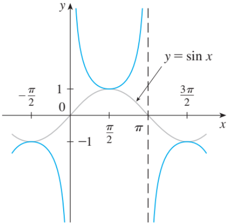
(b) $g(x) = \cos x$



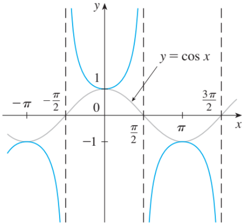
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$



(d) $y = \sec x$

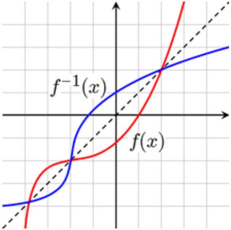
A function has an inverse if and only if its graph intersects any horizontal line at most once. A function is invertible if and only if it is strictly increasing or strictly decreasing on its domain. If the function f is invertible, its inverse f^{-1} is defined as follows:

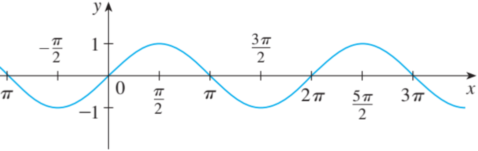
$$f^{-1}(y) = x \text{ means } f(x) = y.$$

A function has an inverse if and only if its graph intersects any horizontal line at most once. A function is invertible if and only if it is strictly increasing or strictly decreasing on its domain. If the function f is invertible, its inverse f^{-1} is defined as follows:

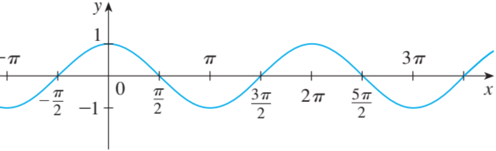
$$f^{-1}(y) = x \text{ means } f(x) = y.$$

The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

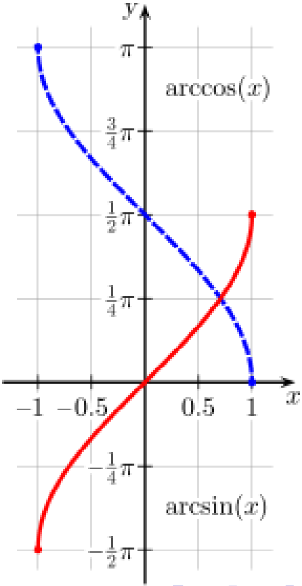




(a) $f(x) = \sin x$



(b) $g(x) = \cos x$



• If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$f + g, \quad f - g, \quad cf, fg,$$

$$f/g \ \text{ provided } \ g(a) \neq 0,$$

$$f^n \ \text{ with } \ n \in \mathbb{N}.$$

• If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$f+g, \quad f-g, \quad cf, fg,$$

$$f/g \;\text{ provided }\; g(a)\neq 0,$$

$$f^n \;\text{ with }\; n\in\mathbb{N}.$$

• If f is continuous at a and g is continuous at $f(a)$, then the composite $g\circ f$ is continuous at a.

Theorem. If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right).$$

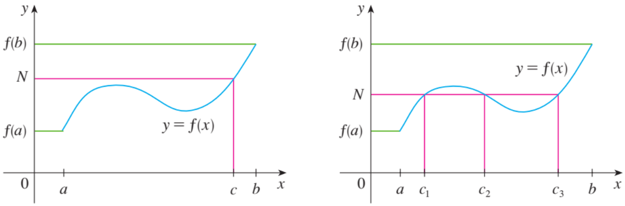
Example: $\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right).$

Theorem. If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right).$$

Example: $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

Theorem (The intermediate value theorem). Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $[a, b]$ such that $f(c) = N$.



A consequence for root finding

Suppose that $f(x)$ is continuous at every point of a closed interval $[a, b]$ and that $f(a)$ and $f(b)$ differ in sign. Then zero lies between $f(a)$ and $f(b)$, so there is at least one number c between a and b where $f(c) = 0$.

1. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

1. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

2. Is there any real number that is 1 less than its cube?

Continuous extensions

If c is a point where f is not defined but where $\lim_{x \to c} f(x)$ exists, we can define $f(c)$ to be the value of the limit.

Continuous extensions

If c is a point where f is not defined but where $\lim_{x \to c} f(x)$ exists, we can define $f(c)$ to be the value of the limit.

Is it possible to define $f(2)$ in a way that extends

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

to be continuous at $x = 2$?

Continuous extensions

If c is a point where f is not defined but where $\lim_{x \to c} f(x)$ exists, we can define $f(c)$ to be the value of the limit.

Is it possible to define $f(2)$ in a way that extends

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

to be continuous at $x = 2$?

Is possible to extend $f(x) = \frac{\sin x}{x}$ to be continuous at $x = 0$?