APPLICATIONS OF DIFFERENTIATION

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leqslant f(x)$ for all x in D.

The maximum and minimum values of f are called extreme values of f.

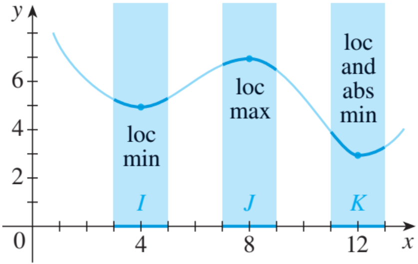
Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leqslant f(x)$ for all x in D.

The maximum and minimum values of f are called extreme values of f.

The number $f(c)$ is a

- local maximum value of f on D if $f(c) \geqslant f(x)$ when x is near c.
- local minimum value of f on D if $f(c) \leqslant f(x)$ when x is near c.



The extreme value theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The extreme value theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

A critical point of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

• If f has a local maximum or minimum at c, then c is a critical point of f.

The extreme value theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

A critical point of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

• If f has a local maximum or minimum at c, then c is a critical point of f.

• To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical points of f in $[a, b]$.
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Rolle′s Theorem. Let f be a function that satisfes the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b);
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Rolle's Theorem. Let f be a function that satisfes the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b);
3. $f(a) = f(b)$.

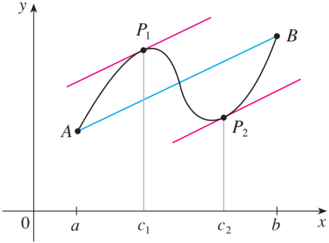
Then there is a number c in (a, b) such that $f'(c) = 0$.

The mean value theorem. Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

Ex. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 256 km on a toll road with speed limit 104 km/hr. The trucker was cited for speeding. Why?

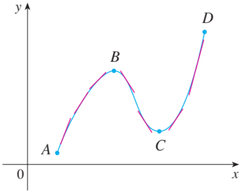
- If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

- If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

- If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

What does f' says about f?

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



Suppose that c is a critical point of a continuous function f.
(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

What does f'' says about f?

• If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

• If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

What does f'' says about f?

- If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

- If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on the graph of the function $f(x)$ is called an inflection point if f is continuous there and the graph changes from concave upward to concave downward or from concave downward to concave upward at P.

- At a inflection point $(c, f(c))$, either $f''(c) = 0$ or f'' does not exists.

What does f'' says about f?

- If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

- If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on the graph of the function $f(x)$ is called an inflection point if f is continuous there and the graph changes from concave upward to concave downward or from concave downward to concave upward at P.

- At a inflection point $(c, f(c))$, either $f''(c) = 0$ or f'' does not exists.

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Strategy for graphing $y = f(x)$

1) Identify the domain of f and any symmetries the curve may have.
2) Find the derivatives y' and y''.
3) Find the critical points of f, f any, and identify the function's behavior at each one.
4) Find where the curve is increasing and where it is decreasing.
5) Find the points of inflection, if any occur, and determine the concavity of the curve.
6) Identify any asymptotes that may exist.
7) Plot key points, such as the intercepts and the points found in steps 3)-5) and sketch the curve together with any asymptotes that exist.

Strategy for graphing $y = f(x)$

1) Identify the domain of f and any symmetries the curve may have.
2) Find the derivatives y' and y''.
3) Find the critical points of f, f any, and identify the function's behavior at each one.
4) Find where the curve is increasing and where it is decreasing.
5) Find the points of inflection, if any occur, and determine the concavity of the curve.
6) Identify any asymptotes that may exist.
7) Plot key points, such as the intercepts and the points found in steps 3)-5) and sketch the curve together with any asymptotes that exist.

Find the graph of the function $f(x) = x^4 - 4x^3$.

Optimization

Example

An open top box is to be made by cutting congruent squares from the corners of a 12 by 12 inch. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).