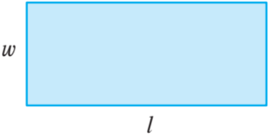
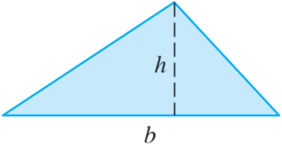
INTEGRALS

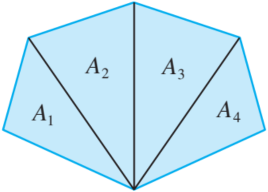
Area problem



$$A = lw$$



$$A = \frac{1}{2}bh$$



$$A = A_1 + A_2 + A_3 + A_4$$

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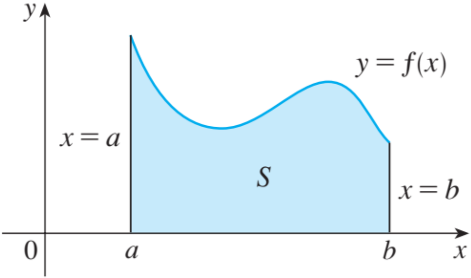
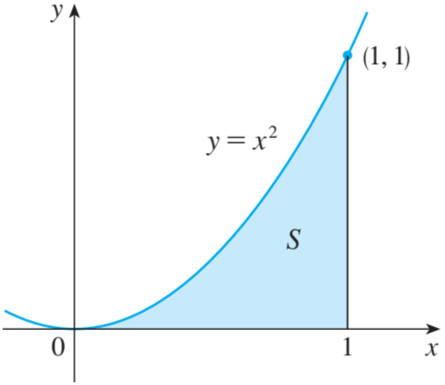
We first approximate the region S by rectangles and then we take the limit of the sum of the areas of these rectangles when we increase the number of rectangles

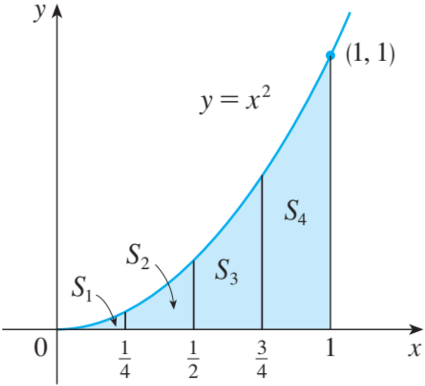


Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



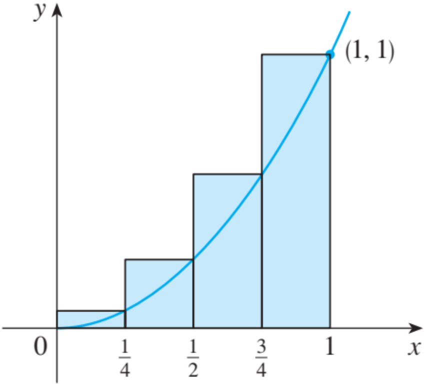
• The area of S must be somewhere between 0 and 1.

• Suppose we divide S into four strips $S1, S2, S3$ and $S4$ by drawing the vertical lines $x = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$.



• We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\frac{1}{4}^2, \frac{1}{2}^2, \frac{3}{4}^2$ and 1^2.



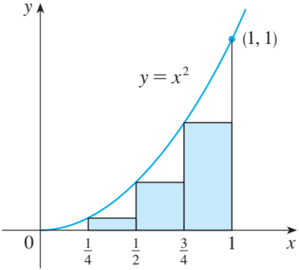
Let R_4 be the sum of the areas of these approximating rectangles.
The area A of S is less than R_4.

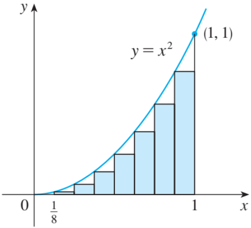
$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = \frac{15}{32} = 0.46875$$

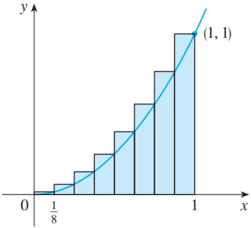
We could use the smaller rectangles in whose heights are the values of f at the left endpoints of the subintervals. (The first rectangle has collapsed because its height is 0). The sum of the areas of these approximating rectangles is

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875,$$

$L_4 < A$.

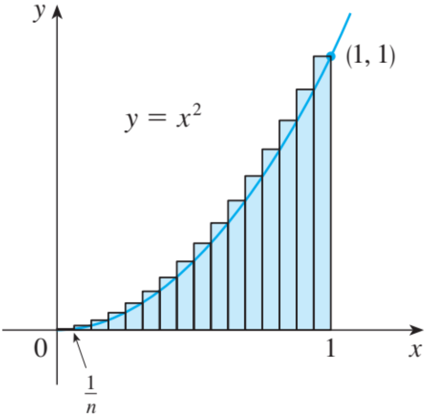






n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$



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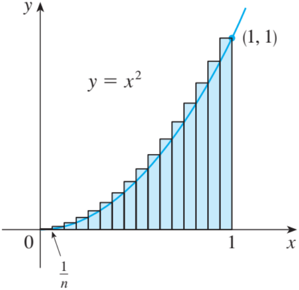

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$$

k is the summation index, in this sum it starts from 0 and ends at n, a_k is the formula for the k-th term in the sum.

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Useful sums:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

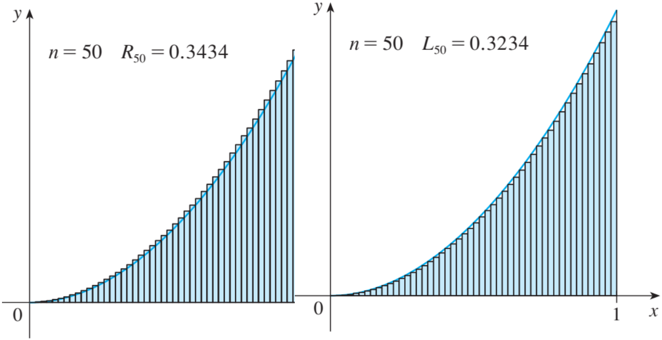
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}
\end{aligned}$$

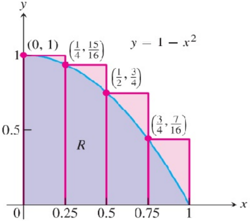
It can be shown also that $\lim_{n\to\infty} L_n = \frac{1}{3}$, it is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}.$$

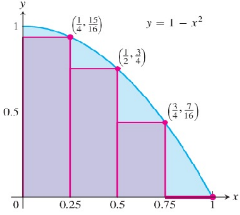


How to find the area below the graph of $y = f(x)$, $x \in [a, b]$?

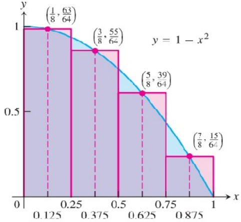
Divide interval $[a, b]$ into n (equal) intervals and form rectangles with heights $f(x)$ for a point x in the base interval of the rectangle. The total area of these rectangles is an approximation of the area under the graph.



(b)



(a)



(b)

Upper sum 0.78125 Lower sum 0.53125 Midpoint rule 0.671875

Exact value: 2/3.

Antiderivatives

A function $F(x)$ is an antiderivative of a function $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I.

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Exercise. Find an antiderivative of:

a) $f(x) = 2x + 4$

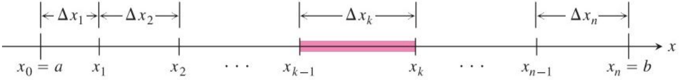
b) $g(x) = \frac{1}{x}$

The definite Integral

Let a function f be defined on $[a, b]$. We divide the interval into subintervals:

$$a = x_0 < x_1 < x_2 < \; < x_{n-1} < x_n = b$$

- Denote the length of k-th subinterval by Δx_k, i.e. $\Delta x_k = x_k - x_{k-1}$.



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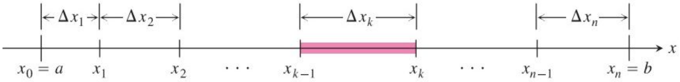
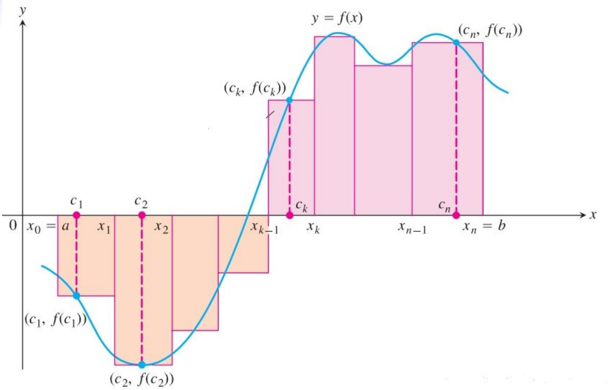
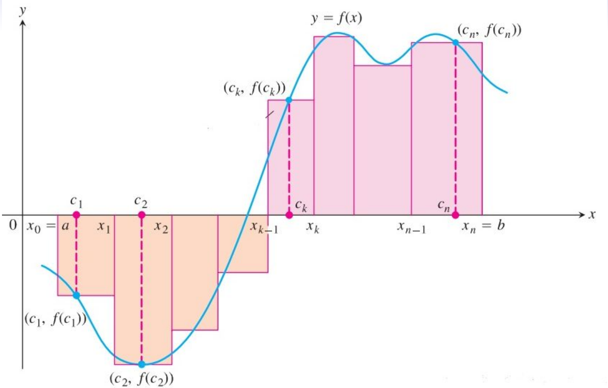


- In each subinterval we select a point $c_k \in [x_{k-1}, x_k]$ and form a rectangle with height $f(c_k)$ and width Δx_k.



y
y = f(x)
(c_n, f(c_n))
(c_k, f(c_k))
c_1
c_2
c_k
c_n
x
0
x_0 = a
x_1
x_2
x_{k−1}
x_k
x_{n−1}
x_n = b
(c_1, f(c_1))
(c_2, f(c_2))



• Add the areas of all those rectangles (if the rectangle is below the x-axis, we take the area with the opposite sign):

$$S = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

This sum is called a Riemann sum for f on the interval $[a, b]$.

- Denote $\Delta = \max_k \Delta x_k$.

If the limit

$$\lim_{\Delta\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists and does not depend on the choice of x_k and c_k, then this sum is called the definite integral (Riemann integral) of f over $[a, b]$ and denoted by $\int_a^b f(x)dx$. The function f is then called integrable over $[a, b]$.

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The symbol $\int$ is an integral sign, a is the lower limit of integration and b the upper limit of integration. The function f is the integrand of the integral, and x is the variable of integration.

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Exercise.Calculate $\int_0^1 xdx$

Theorem. If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

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Let f and g be integrable over $[a, b]$. Then:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx; \qquad \int_a^a f(x)dx = 0$$

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$(b-a)\min f \leq \int_a^b f(x)dx \leq (b-a)\max f$$

If $f(t)$ is an integrable function over interval I, then the integral from a fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)dt.$$

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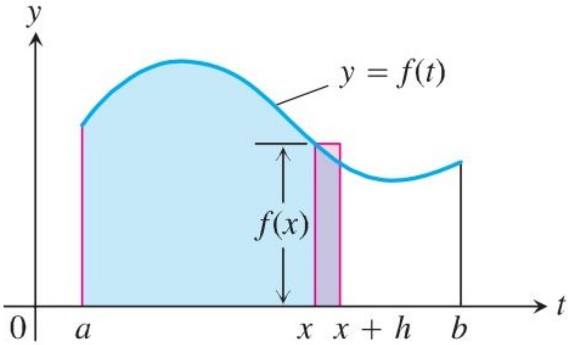

The fundamental theorem of calculus 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

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Exercise. Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}dt$

The fundamental theorem of calculus 2

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This formula is called Newton-Leibniz rule.

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Exercise. Evaluate $\int_0^1 4dx$, $\int_0^1 x^2dx$, $\int_0^1 (4+3x^2)dx$

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. subdivide $[a, b]$ at the zeros of f;
2. integrate f over each subinterval;
3. add the absolute values of the integrals.

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Exercise. Find the area between the graph of $y = \sin x$, $x \in [0, 2\pi]$ and the x-axis.

The Indefinite Integral

The collection of all antiderivatives of f is called the indefinite integral of f with respect to x, and is denoted by $\int f(x)dx$: The indefinite integral of a function f always includes an arbitrary constant C

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Exercise. Find $\int x^2 dx$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative. Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx.$$

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Theorem (Substitution rule). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

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In case of definite integrals the substitution rule takes the form

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$