2. LIMITS

The velocity problem

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If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo′s law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$

$$\begin{aligned}
\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\
&= \frac{s(5.1) - s(5)}{0.1} \\
&= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}
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Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

The instantaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is

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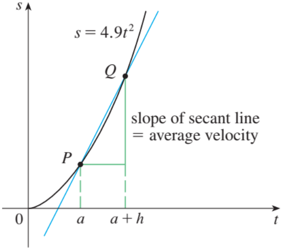

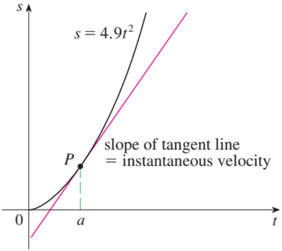

Given an arbitrary function $y = f(x)$, the average rate of change of f over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h},$$

where $h = x_2 - x_1 \neq 0$.
The average rate of change is the slope of a secant to the curve.
The instantaneous rate of change is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero.

Limits of functions

Lets investigate the behavior of the function $f(x) = x^2 - x + 2$ for values of x near 2.

Notation: $\lim_{x \to 2} x^2 - x + 2 = 4$.

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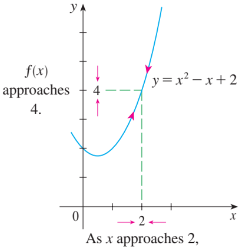
x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001



Informal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L.$$

A Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

One side limits

a) We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a.

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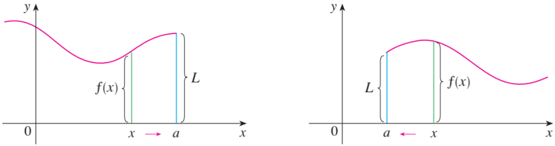
b) Analogously we define the right-hand limit of $f(x)$ as x approaches a is equal to L and we write

$$\lim_{x \to a^+} f(x) = L.$$

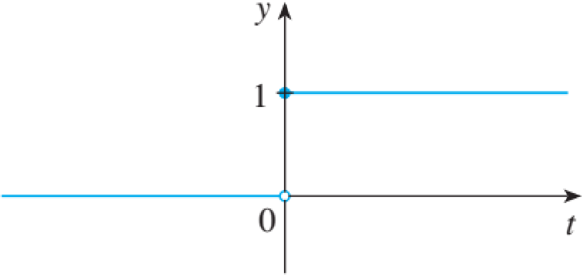


(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

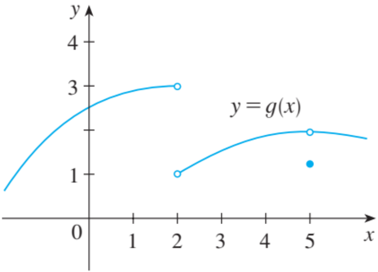
$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Example

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geqslant 0 \end{cases}$$



Example



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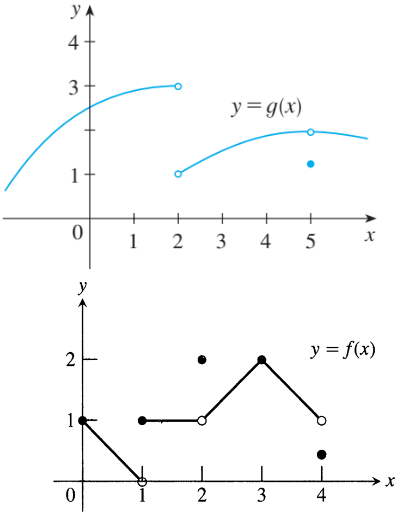


Infinite limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$1) \quad \lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

$$2) \quad \lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty$$

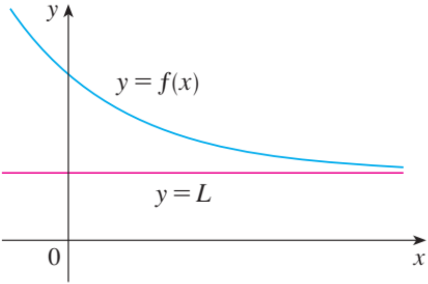
$$\lim_{x \to a} f(x) = -\infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty$$

Limits at infinity

- Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

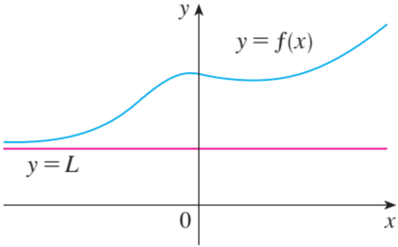
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.



- Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.



- The line $y = L$ is called a horizontal asymptote of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\displaystyle \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\displaystyle \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

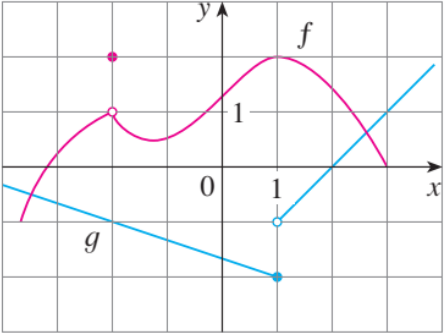
3. $\displaystyle \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\displaystyle \lim_{x \to a} [f(x)\,g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist

$$\lim_{x \to -2} (f(x) + 5g(x)), \quad \lim_{x \to 1} (f(x)g(x)).$$



6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Direct substitution property

If $f(x)$ is any function from the lists of essential functions and c is in the domain of f, then

$$\lim_{x \to c} f(x) = f(c).$$

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Find

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \qquad Sol. -\frac{1}{11}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \qquad Sol.\ 2$$

Theorem. If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

Theorem (The sandwich theorem). If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then,

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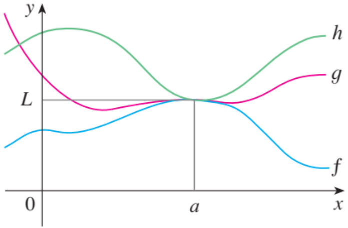

Exercise.

Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

An important limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

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Proof. We will show that $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

Consider $0 < x < \frac{\pi}{2}$. As we can see in the figure, we have that $\sin x < x < \tan x$. Hence,

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x},$$

this is

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

So,

$$\cos x < \frac{\sin x}{x} < 1$$ and by the sandwich theorem que have that

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Since $\sin x$ and x are odd, then

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1.$$