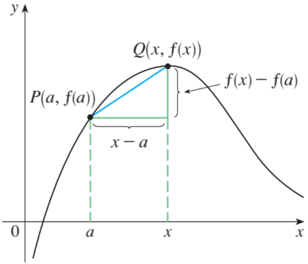
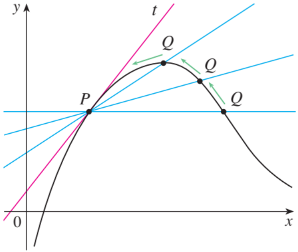
1. DERIVATIVES

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.





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- Find the derivative of the function

$$f(x) = x^2 - 8x + 9$$

at a.

- Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

If we replace a by the variable x, we have

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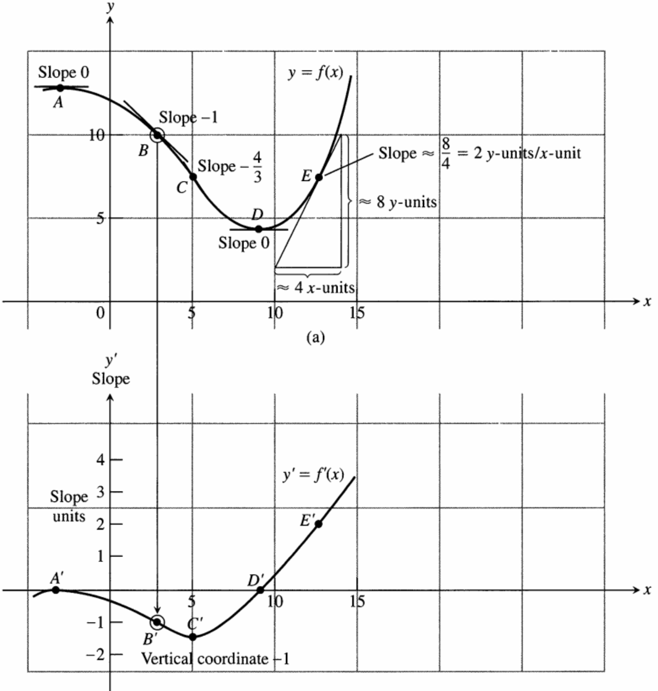
This function is called the derivative of f.



y
Slope 0
A
y = f(x)
Slope −1
10
B
Slope ≈ 8/4 = 2 y-units/x-unit
Slope − 4/3
E
C
≈ 8 y-units
D
5
Slope 0
≈ 4 x-units
x
0
5
10
15
(a)
y′
Slope
4
y′ = f′(x)
Slope units
3
2
E′
1
A′
D′
x
5
10
15
−1
C′
B′
−2
Vertical coordinate −1

Notation

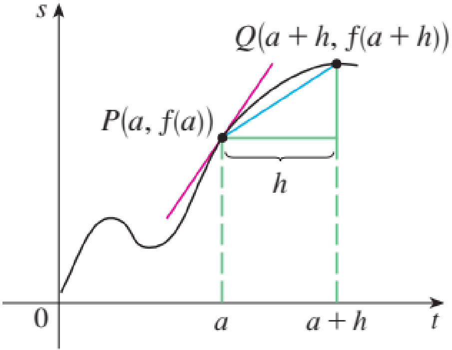
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position (also called displacement) is $f(a + h) - f(a)$.

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant PQ.



Suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the velocity (or instantaneous velocity) $v(a)$ at time $t = a$ to be the limit of these average velocities:

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Speed is the absolute value of velocity.

• A function f is differentiable at a if $f'(a)$ exists.

• It is differentiable on an open interval (a, b) [or (a, ∞),or $(-\infty, a)$, or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

• It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the one-side derivatives $f'(a^+)$ and $f'(b^-)$ exist at the endpoints, here

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{righ-hand derivative at } a.$$

$$f'(b^-) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{left-hand derivative at } b.$$

A function has a derivative at a point if and only if it has left-hand and right-hand derivatives there and these one-sided derivatives are equal.

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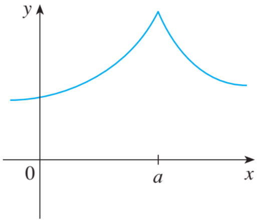
- If f is differentiable at a, then f is continuous at a.

How can a function fail to be differentiable at a point a?

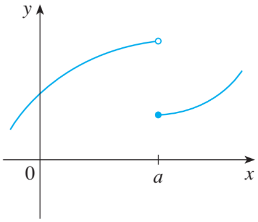
$a)$ The graph of a function f has a corner or kink in it.

$b)$ f is not continuous at a.

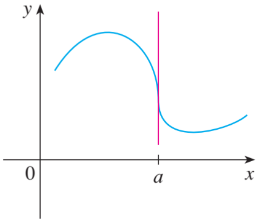
$c)$ The graph of a function f has a vertical tangent line when $x = a$.



(a) A corner



(b) A discontinuity



(c) A vertical tangent

Differentiation Rules

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}[cf(x)] = cf'(x)$

3. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

4. $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

5. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

7. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

8. $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\dfrac{d}{dx}(e^x) = e^x$

10. $\dfrac{d}{dx}(b^x) = b^x \ln b$

11. $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$

12. $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$

Trigonometric Functions

13. $\dfrac{d}{dx}(\sin x) = \cos x$

14. $\dfrac{d}{dx}(\cos x) = -\sin x$

15. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

16. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

17. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

18. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

19. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$

20. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1 - x^2}}$

21. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$

22. $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$

23. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

24. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1 + x^2}$

Higher derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f.

If f'' is differentiable, its derivative f''', is the third derivative and so on. In general, the nth derivative of $y = f(x)$ is written as $f^{(n)}(x)$.

Acceleration is the derivative of velocity with respect to time

$$a(t) = v'(t) = s''(t).$$

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Examples

1) The position of a moving body is given by the equation $160t - 16t^2$, with s in feet and t in seconds. Find the body's velocity and acceleration at time t.

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Examples

1) The position of a moving body is given by the equation $160t - 16t^2$, with s in feet and t in seconds. Find the body's velocity and acceleration at time t.

2) A heavy rock blasted vertically upward with a velocity 160 ft/sec reaches a height of $s = 160t - 16t^2$ ft after t seconds.
a) How high does the rock go?
b) How fast is the rock traveling when it is 256 ft above the ground on the way up? on the way down?

Related Rates Problems

Problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a related rates problem.

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Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time, they are some kind of functions at time t.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Related Rates Problems

Example. Water runs into a conical tank at the rate of $9m^3/min$. The tank stands point down and has a height of 10 m and a base radius of 5 m. How fast is the water level rising when the water is 6 m deep?

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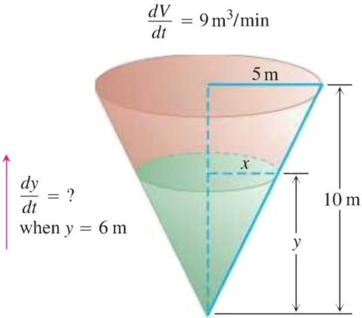

$$V = \frac{1}{3}\pi x^2 y \qquad \frac{x}{y} = \frac{5}{10}$$

$$V = \frac{\pi}{12} y^3$$

$$\frac{dV}{dt} = \frac{\pi}{12} 3y^2 \frac{dy}{dt}$$