LTMS.00.002 Higher Mathematics

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Autumn 2021

Topics

1. Functions
2. Limits and Continuity
3. Derivatives
4. Applications of Differentiation
5. Integrals
6. Applications of Integrals
7. Ordinary Differential Equations
8. Infinite Sequences and Series
9. Complex numbers
10. Vector Calculus and Geometry of Space
11. Linear Algebra
12. Partial Differential Equations

Assessment methods and criteria

• 3 tests (25 points each; 1st test: September 29th (Groups 1,2) September 30th (Delta-group), 2nd test: October 27th (Groups 1,2), October 28th (Delta-group), 3rd test: December 1st (Groups 1,2), December 2nd (Delta-group); possible to retake once, the last score counts)

• 10 quizzes (2 points each)

• 10 Moodle tests (2 points each; unlimited number of trials within a week, best score counts)

• 3 individual homework sets (5 points each)

• Final exam (70 points) (Choose one day from: December 17th, January 13th,).

Re-exam: January 20th.

In total you can get 200 points. You need to collect at least 60 points before you are permitted to take the exam.

Total grade

180-200 – excellent (A)

160-179 – very good (B)

140-159 – good (C)

120-139 – satisfactory (D)

100-119 – poor (E)

0- 99 – fail (F)

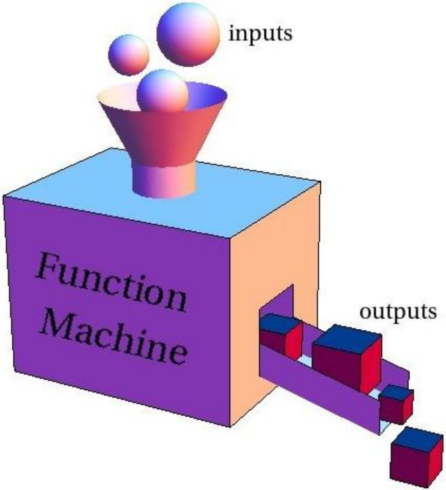
Lecture slides (Bibliography)

• J. Stewart, *Single variable calculus early transcendentals*, 8th edition.

• G. B. Thomas, R. L. Finney, *Calculus and Analytic Geometry* (9th edition), Addison Wesley,1998.

• G. B. Thomas, Jr., M. D. Weir, J. Hass, C. Heil ,*Thomas's Calculus early trascendentals* (13th edition), Pearson.

1. Functions



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The domain and the range could be any set, but we will consider sets of real numbers. If the domain is not stated explicitly or restricted by the context, then we consider the domain as the largest set of real numbers such that when we applied the rule defined by the function gives real values.

1) Function (definition);
2) Independent and dependent variable;
3) Domain and Range of a function.

Their representations

A function can be represented by formulas that describe how to get an output value from certain input value.

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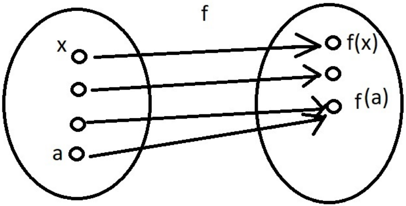
A function can be also represented as an arrow diagram:



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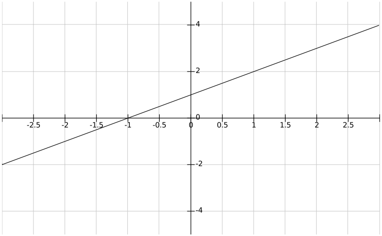

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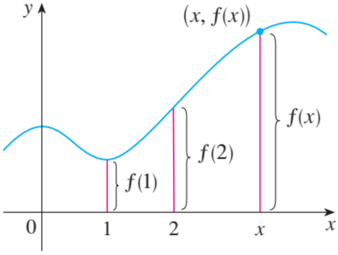
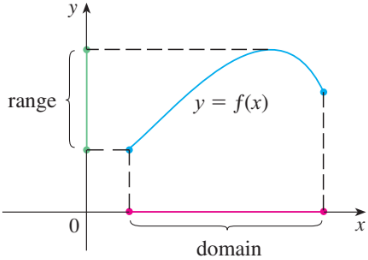
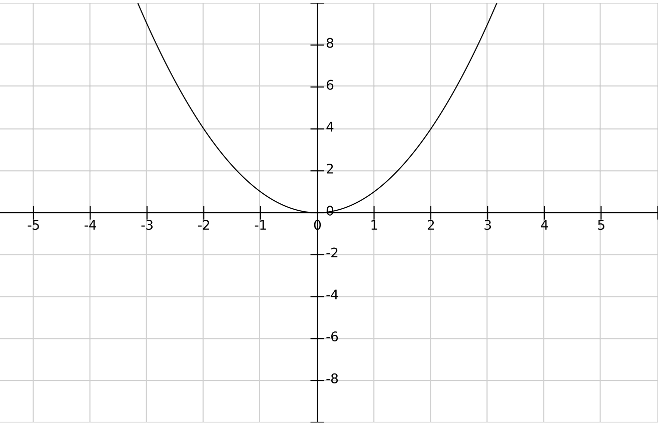




Sometimes also the functions are represented by tables, it enables to represent only discrete data (finite number of data points). But formula enables to represent discrete and continuous data.

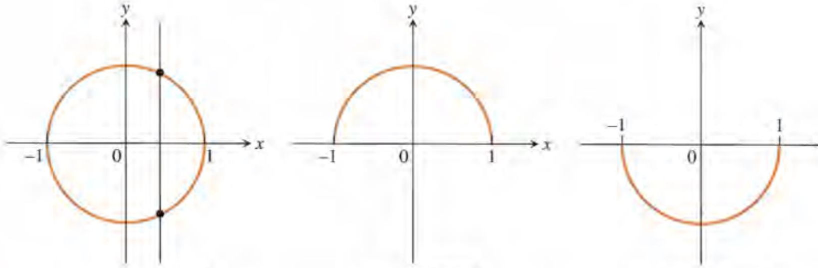


x	$y=x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Representations

1) Formulas;
2) Arrow diagram;
3) Graph;
4) Tables.

Not every curve in the cartesiane plane is a function.
Since for every point in the domain of a function, there exists only one value $f(x)$, then no vertical line can intersect the graph of a function more than once.
So, the circle is not the graph of a function.



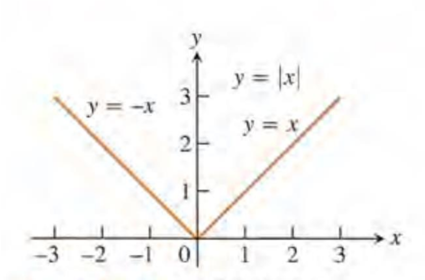
(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Sometimes a function is defined by different formulas on different parts of its domain

Example: absolute value

$$|x| = \begin{cases} x & \text{if} \quad x \geqslant 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

This function has domain $(-\infty, \infty)$ and range $[0, \infty)$.



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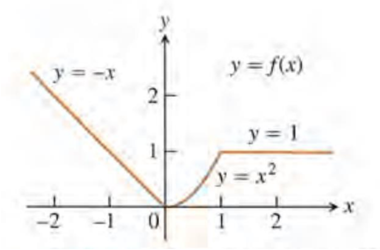

A function is convex (also called concave up) if every line segment joining two points on its graph does not lie below the graph at any point. A function is concave (concave down) if every line segment joining two points on its graph does not lie above the graph at any point. A line is both convex and concave.

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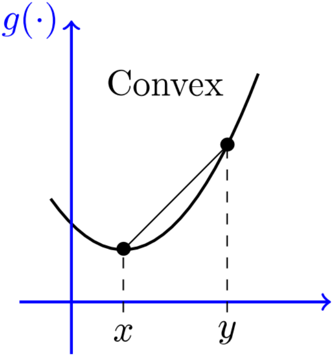

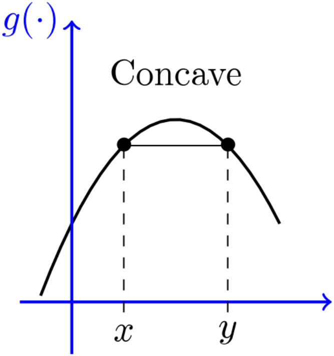

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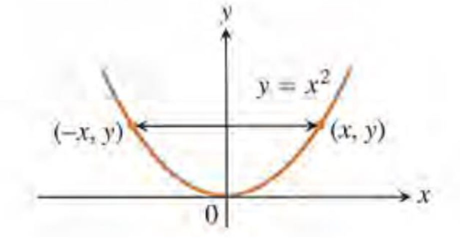

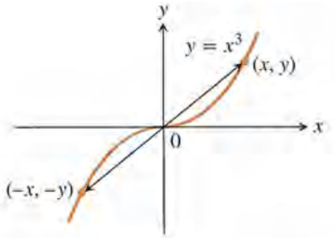

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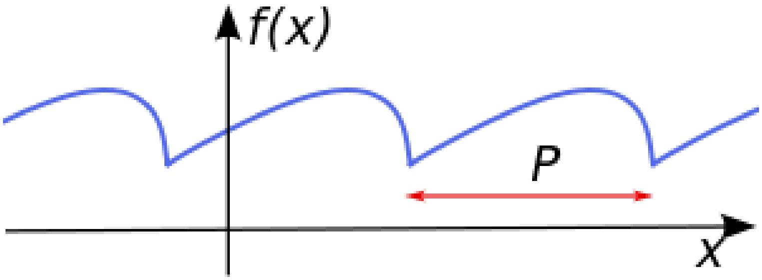

Some properties of the functions

1) Increasing;
2) Decreasing;
3) Convex;
4) Concave;
5) Even;
6) Odd;
7) Periodic.

Essential functions

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1) Linear functions

When we say that $f(x)$ is a linear function, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write the function as

$$f(x) = mx + b,$$

where m is the slope and b is the $y-$intercept. The function $f(x) = x$ (where $m = 1$ and $b = 0$) is called the identity function.

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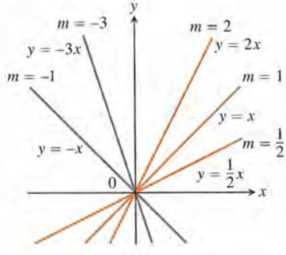
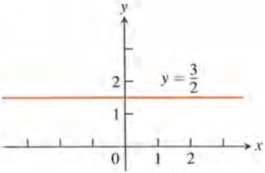


(a)



(b)

The slope of a linear function f, can be calculated from values of the function at two points, x_1, x_2 using the formula

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We will say that y and x are proportional if $y = kx, k \neq 0$ and are inversely proportional if $y = \frac{k}{x}, k \neq 0$.

2) Polynomials

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A function f is called a polynomial if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ are called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R}$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

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Examples:

$f(x) = mx + b$, where $m, b \in \mathbb{R}$

$f(x) = x^2 + x + 1$

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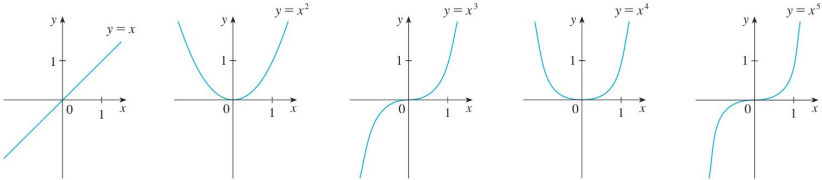
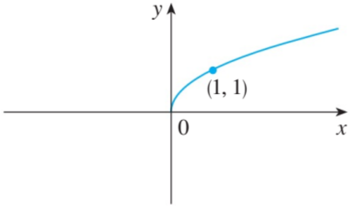
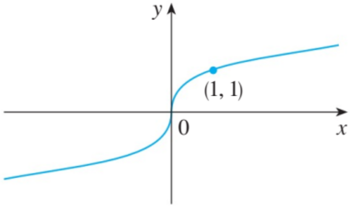


$ii)$ $f(x) = x^a$, where $a = \frac{1}{n}, \;\; n \in \mathbb{N} - \{0\}$;

ii) $f(x) = x^a$, where $a = \frac{1}{n}$, $n \in \mathbb{N} - \{0\}$;
The first graph is the graph of the function $f(x) = \sqrt[2]{x}$ and the second one is the graph of the function $\sqrt[3]{x}$:





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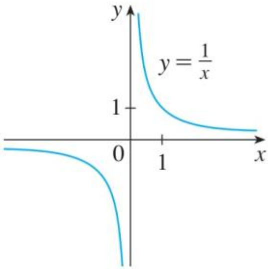

4) Rational functions

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials. The domain consist of all the values of x such that $q(x) \neq 0$.

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$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

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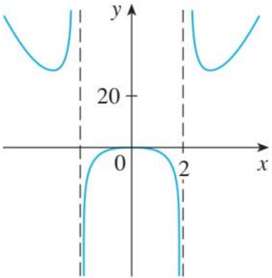

5) Trigonometric functions

$$f(x) = \sin(x), \qquad f(x) = \cos(x), \qquad f(x) = \tan(x)$$
$$f(x) = \cot(x), \qquad f(x) = \sec(x), \qquad f(x) = \csc(x)$$

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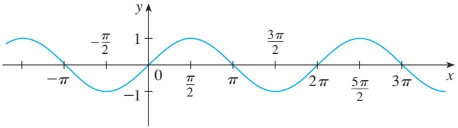
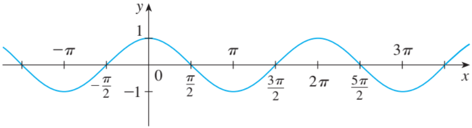
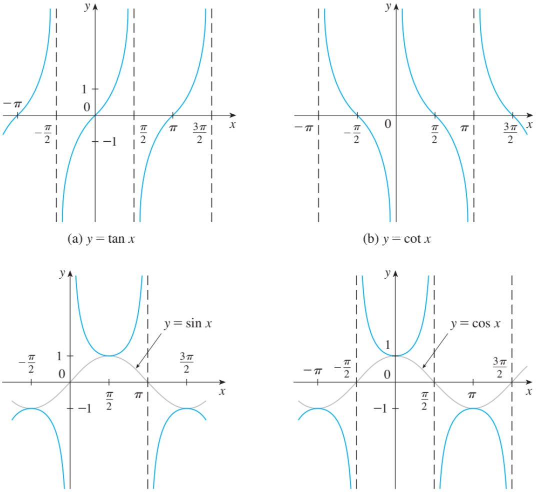


(a) $f(x) = \sin x$



(b) $g(x) = \cos x$



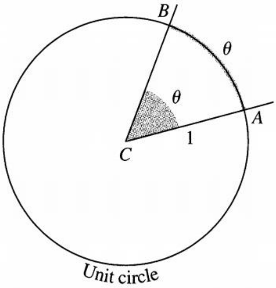
(a) $y = \tan x$

(b) $y = \cot x$

(c) $y = \csc x$

(d) $y = \sec x$

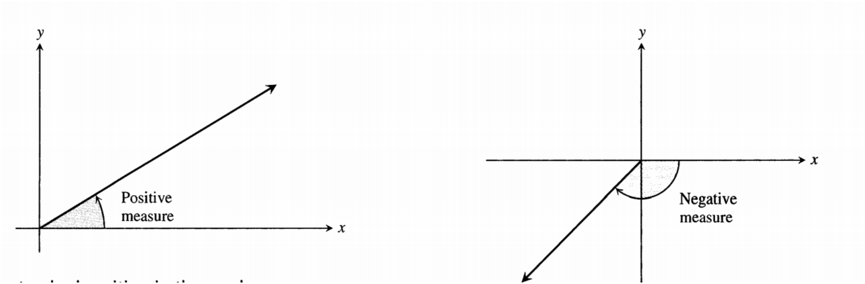
Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

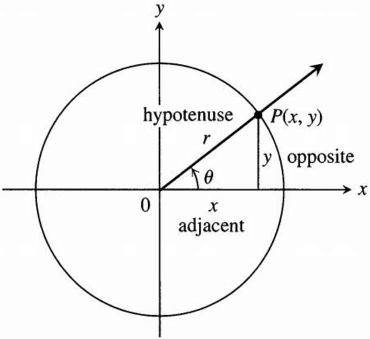
The **radian measure** $\boldsymbol{\theta}$ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by the following equation.

$$\boxed{\pi \text{ radians} = 180°}$$


y
x
Positive
measure
y
x
Negative
measure

We define the trigonometric functions in terms of the coordinates of the point $P(x, y)$, where the angle's terminal ray intersects the circle.

Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

6) Exponential functions

The exponential functions are the functions of the form

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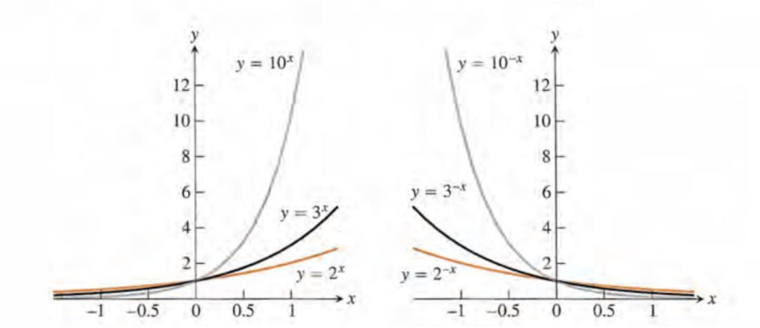

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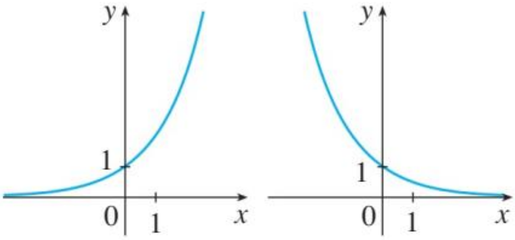

(a) $y = 2^x$ (b) $y = (0.5)^x$

A function has an inverse if and only if its graph intersects any horizontal line at most once. A function is invertible if and only if it is strictly increasing or strictly decreasing on its domain. If the function f is invertible, its inverse f^{-1} is defined as follows:

$$f^{-1}(y) = x \text{ means } f(x) = y.$$

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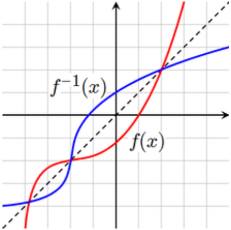
The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.



7) Logarithmic functions

The logarithmic functions

$$f(x) = \log_a x,$$

where $a \in \mathbb{R}^+ - \{1\}$, are the inverse functions of the exponential functions. It means

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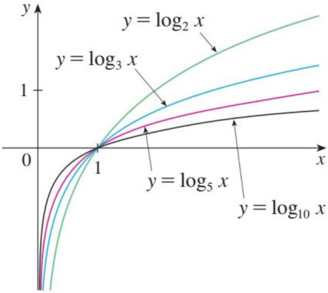

Essential functions

1) Linear functions;
2) Polynomials;
3) Power functions;
4) Rational functions;
5) Trigonometric functions;
6) Exponential functions;
7) Logarithmic functions.

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Some properties of logarithms

$$\log_a 1 = 0, \;\; \log_a(xy) = \log_a x + \log_a y, \;\; \log_a(x^p) = p \log_a(x)$$

$$a^{\log_a x} = x, \;\; \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y, \;\; \log_a x = \frac{\log_b x}{\log_b a}.$$

8) Any function constructed from previous functions using algebraic operations (addition, substraction, multiplication, division and taking roots).

Sums, Differences, Products and Quotients

Sums, Differences, Products and Quotients

If f and g are functions, then for every x that belongs to the domain of both. We define functions $f+g$, $f-g$ and fg as follows:
$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(fg)(x) = f(x)g(x)$.
At any point, x, that belongs to the domain of both functions f and g such that $g(x) \neq 0$, we can also define the function

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

where $g(x) \neq 0$.
If $c \in \mathbb{R}$, then we define cf, for every x in the domain of f as follows

$$(cf)(x) = cf(x).$$

Example

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = \sqrt{1 - x}$	$(-\infty, 1]$
$3g$	$3g(x) = 3\sqrt{1 - x}$	$(-\infty, 1]$
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

If f and g are functions, the composition $f \circ g$ of f with g is defined by

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of x in the domain of g for which $g(x)$ lies in the domain of f.

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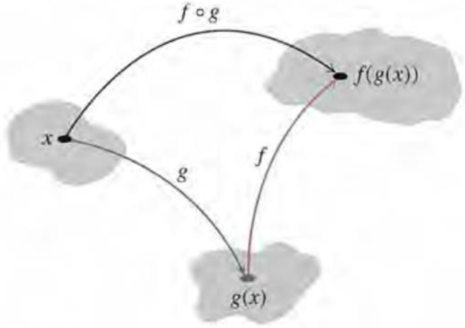

Example

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

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Solution. We have

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Notice that in general $f \circ g \neq g \circ f$.

1) Sums;
2) Differences;
3) Products;
4) Quotients;
5) Compositions.

Transformations of functions

Vertical and Horizontal shifts

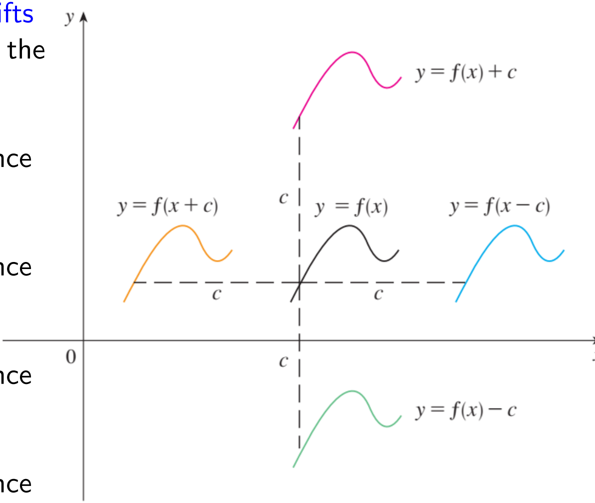
Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward,

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward,

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right,

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left.



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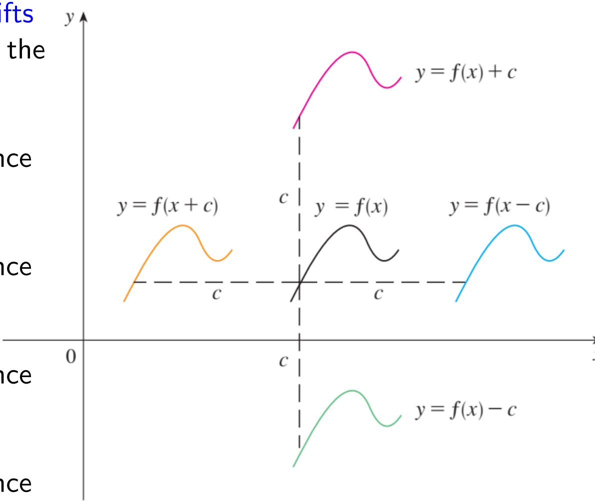
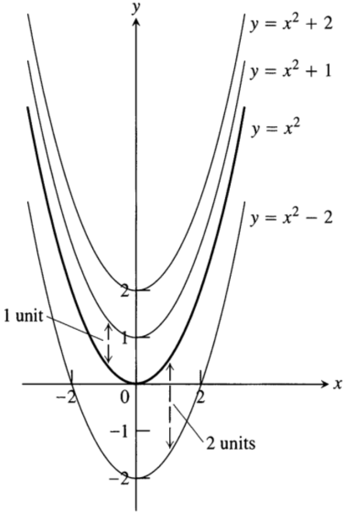
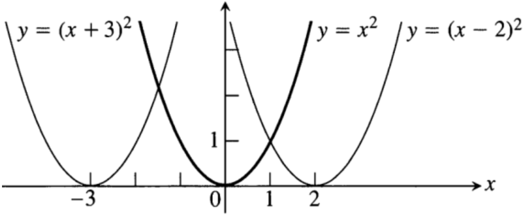




y
$y = x^2 + 2$
$y = x^2 + 1$
$y = x^2$
$y = x^2 - 2$
2
1 unit
1
−2
0
2
x
−1
2 units
−2



$y = (x + 3)^2$
$y = x^2$
$y = (x - 2)^2$
1
−3
0
1
2
x

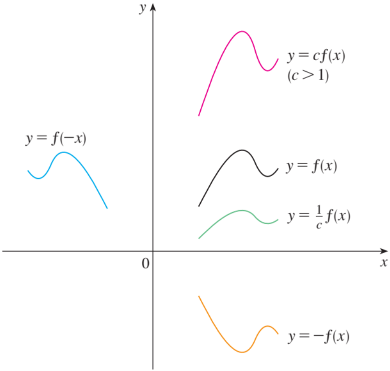
Vertical and horizontal stretching and reflecting

Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretches the graph of $y = f(x)$ vertically by a factor of c,

$y = \frac{1}{c}f(x)$, compresses the graph of $y = f(x)$ vertically by a factor of c ,

$y = f(cx)$, compresses the graph of $y = f(x)$ horizontally by a factor of c,



$y = f\left(\frac{x}{c}\right)$, stretches the graph of $y = f(x)$ horizontally by a factor of c,

$y = -f(x)$, reflects the graph of $y = f(x)$ across/over/about the x-axis,

$y = f(-x)$, reflects the graph of $y = f(x)$ cross/over/about the y-axis.