On the Capacity Region of Broadcast Packet Erasure Relay Networks with Feedback

Quan Geng	Google Research, New York
Hieu T. Do	Ericsson Research, Stockholm
Rui Wu	University of Illinois, Urbana-Champaign
Mindi Yuan	University of Illinois, Urbana Champaign
Yun Li	University of Illinois, Urbana Champaign
Wei Ding	Google Research, New York

IEEE International Conference on Communications
20-24 May 2019 // Shanghai, China
Empowering Intelligent Communications

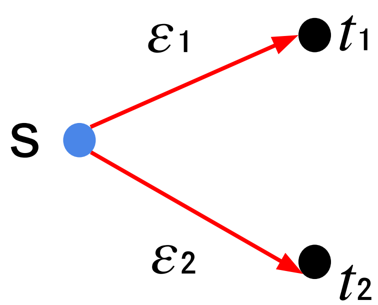
Outline

- **Background**
  - Broadcast Packet Erasure Channel (PEC)
  - Multiple-Input Broadcast Packet Erasure Channel
- **Model**
  - Multiple-Input Broadcast Packet Erasure Channel with Feedback
  - Broadcast Packet Erasure Relay Netwo*rk with Feedback*

- **Our Contributions**
  - New Outer-bounds on
    - Multiple Input Broadcast PEC with Feedback
    - Broadcast Packet Erasure Relay Networks with Feedback
  - Tightness of the outer bounds
    - Outer bounds achieved via network coding across sub-channels
    - Engineering implication: the network coding packets should be sent over correlated broadcast subchannels

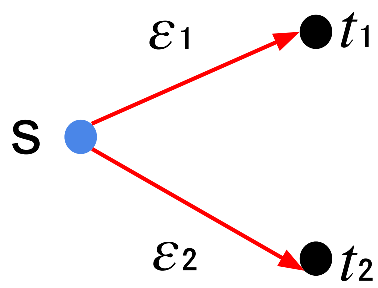
Background

- Broadcast Packet Erasure Channel
  - single transmitter/source
  - multiple receivers/destinations
  - the same packet is sent over the communication channels to destinations
  - each receiver receives
    - either the entire packet
    - or no packet
  - packet erasure events are independent over the time (memoryless)



Background

- Broadcast Packet Erasure Channel
    - **Capacity region achieved by timesharing**

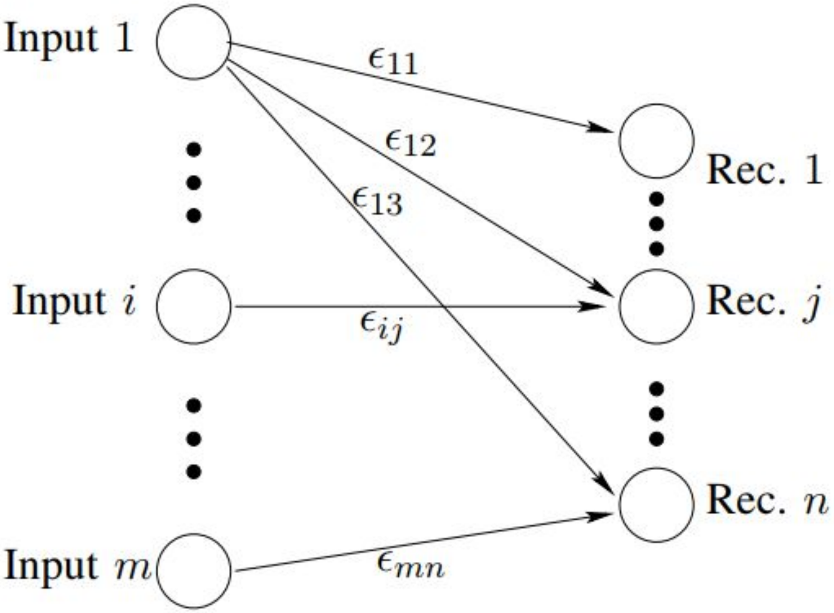


Capacity Region

$$\{(R_1, R_2) | \frac{R_1}{1-\epsilon_1} + \frac{R_2}{1-\epsilon_2} \leq 1\}$$

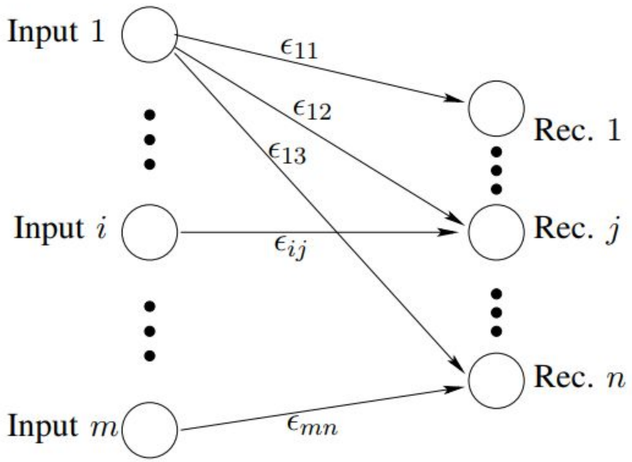
Background

- Multiple-Input Broadcast Packet Erasure Channel
  - one transmitter
  - n receivers
  - m broadcast packet erasure sub-channels
  - no interference at each receiver



Background

- Multiple-Input Broadcast Packet Erasure Channel



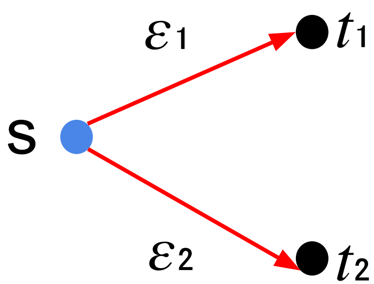
[Dana&Hassibi] Capacity region achieved by timesharing.

$$\boxed{\mathcal{R}_T = \bigcup_{\underline{\alpha}} \{(R_1, \ldots, R_n) | 0 \leq R_j < \sum_{i=1}^{m} \alpha_{ij}(1-\epsilon_{ij})\},}$$

1) $\alpha_{ij} \geq 0$
2) $\sum_{j=1}^{n} \alpha_{ij} = 1$ for all inputs $1 \leq i \leq m$.

Background

- Broadcast Packet Erasure Channel with **Feedback**
  - [Georgiadis&Tassiulas] Feedback can improve the capacity region



Capacity Region without Feedback

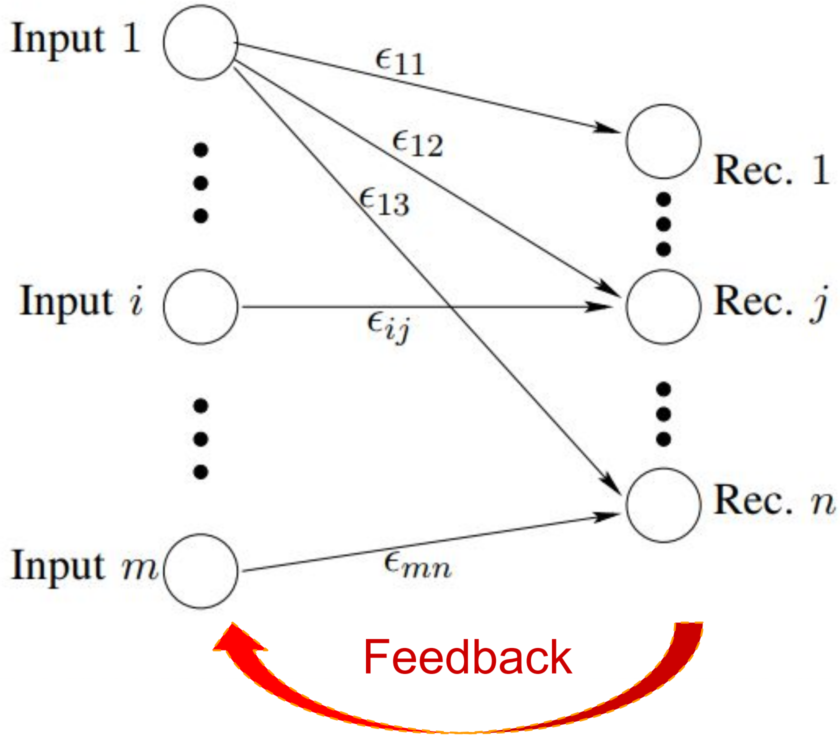
$$\{(R_1, R_2) | \frac{R_1}{1-\epsilon_1} + \frac{R_2}{1-\epsilon_2} \leq 1\}$$

Capacity Region with Feedback

$$\{(R_1, R_2) | \frac{R_1}{1-\epsilon_1\epsilon_2} + \frac{R_2}{1-\epsilon_2} \leq 1, \frac{R_1}{1-\epsilon_1} + \frac{R_2}{1-\epsilon_1\epsilon_2} \leq 1\}$$

Model

- Multiple-Input Broadcast Packet Erasure Channel with **Feedback**
  - one transmitter
  - n receivers
  - m broadcast packet erasure sub-channels
  - no interference at each receiver
  - channel output feedback at the transmitter



Outer bound

- Multiple-Input Broadcast Packet Erasure Channel with **Feedback**

Theorem 1. *For any achievable rate tuple* $\boldsymbol{R} \triangleq (R_1, R_2, \ldots, R_K)$, *it must hold that for any permutation function* $\pi : [K] \to [K]$,

$$\boldsymbol{R} \in \boldsymbol{C}_\pi,$$

where

$$\boldsymbol{C}_\pi \triangleq \Big\{(R_1, R_2, \ldots, R_K) \Big| \sum_{j=1}^{K} \frac{R_{i\pi(j)}}{1 - \epsilon_{i\pi([j])}} \leq 1,$$

$$R_k = \sum_{m=1}^{M} R_{mk}, R_{i\pi(k)} \geq 0, \forall i \in [M], k \in [K].\Big\}$$

Outer bound

- Multiple-Input Broadcast Packet Erasure Channel with **Feedback**

Theorem 1. *For any achievable rate tuple* $\boldsymbol{R} \triangleq (R_1, R_2, \ldots, R_K)$*, it must hold that for any permutation function* $\pi : [K] \to [K]$*,*

$$\boldsymbol{R} \in \boldsymbol{C}_\pi,$$

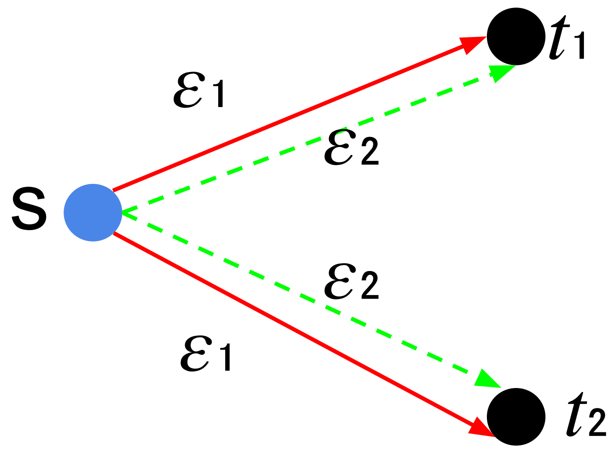
where

$$\boldsymbol{C}_\pi \triangleq \Big\{(R_1, R_2, \ldots, R_K) \Big| \sum_{j=1}^{K} \frac{R_{i\pi(j)}}{1 - \epsilon_{i\pi([j])}} \leq 1,$$
$$R_k = \sum_{m=1}^{M} R_{mk}, R_{i\pi(k)} \geq 0, \forall i \in [M], k \in [K].\Big\}$$

Proof: For any permutation function π, we construct new multiple input broadcast erasure channels with feedback by creating information pipes connecting node $t_{\pi(j)}$ to node $t_{\pi(j+1)}$, so that $t_{\pi(j+1)}$ will get all packets node $t_{\pi(j)}$ receives, for all $j \in [K - 1]$.

This creates a physically degraded channel, for which feedback does not improve the capacity. Then apply the result in [Dana&Hassibi].

Tightness of the Outer bound

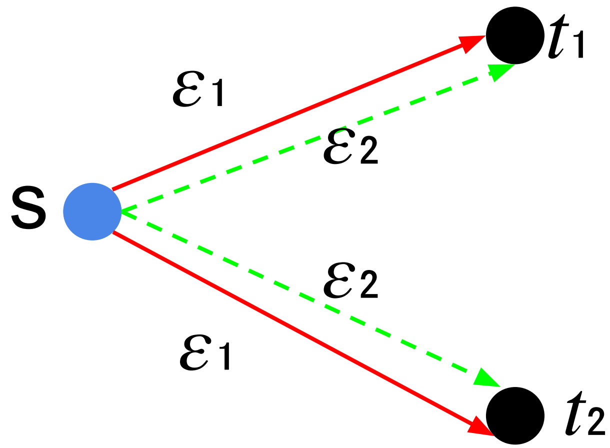


Erasure events on the first sub-channel is **independent**

Erasure events on the second sub-channel is **the same**.

Theorem 3. *If* $\epsilon_2 \geq 1 - \frac{(1-\epsilon_1)\epsilon_1}{2}$ *and* $0 < \epsilon_1, \epsilon_2 < 1$, *then the maximum sum rate of the outer bound in Theorem 1 is tight and can be achieved by coding across subchannels, which thus characterizes the sum capacity of this channel. In addition, the inner bound without coding across subchannels is strictly sub-optimal.*

Inner bound without coding across sub-channels



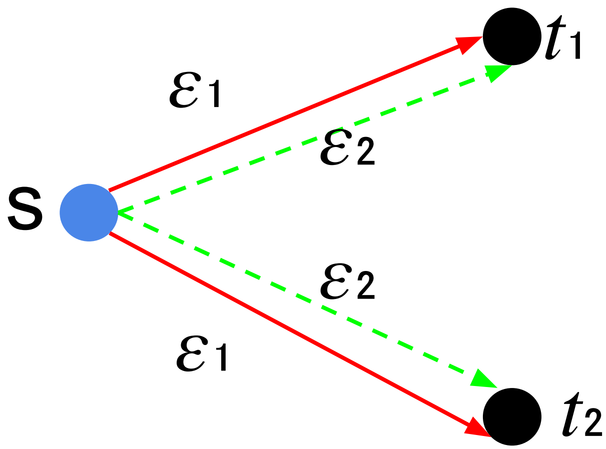
$$\{(R_{11}, R_{12}) \mid \frac{R_{11}}{1-\epsilon_1} + \frac{R_{12}}{1-\epsilon_1^2} \leq 1, R_{11}, R_{12} \geq 0\}$$

and

$$\{(R_{11}, R_{12}) \mid \frac{R_{11}}{1-\epsilon_1^2} + \frac{R_{12}}{1-\epsilon_1} \leq 1, R_{11}, R_{12} \geq 0\}$$

$$R_{sum}^{inner} = \frac{2(1-\epsilon_1^2)}{2+\epsilon_1} + 1 - \epsilon_2$$

Outer bound on Sum Capacity



$$\max_{(R_1,R_2)\in\mathcal{V}} R_1 + R_2.$$

$$\mathcal{V} \triangleq \mathcal{V}_1 \cap \mathcal{V}_2.$$

$$\begin{aligned}\mathcal{V}_1 \triangleq \{(R_1, R_2)| &\frac{R_{11}}{1-\epsilon_1^2} + \frac{R_{12}}{1-\epsilon_1} \le 1,\\ &R_{21} + R_{22} \le 1 - \epsilon_2,\\ &R_1 = R_{11} + R_{21},\\ &R_2 = R_{12} + R_{22},\\ &R_{ij} \ge 0, \forall i, j \in [2]\},\end{aligned}$$

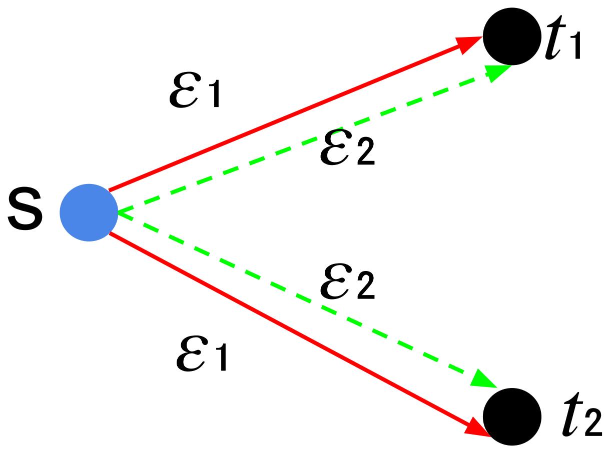
and

$$\begin{aligned}\mathcal{V}_2 \triangleq \{(R_1, R_2)| &\frac{R_{11}}{1-\epsilon_1} + \frac{R_{12}}{1-\epsilon_1^2} \le 1,\\ &R_{21} + R_{22} \le 1 - \epsilon_2,\\ &R_1 = R_{11} + R_{21},\\ &R_2 = R_{12} + R_{22},\\ &R_{ij} \ge 0, \forall i, j \in [2]\}.\end{aligned}$$

$$\Downarrow$$

$$R_{sum}^{outer} = \frac{2(2-\epsilon_1-\epsilon_2)(1+\epsilon_1)}{2+\epsilon_1}.$$

Inner bound < Outer bound

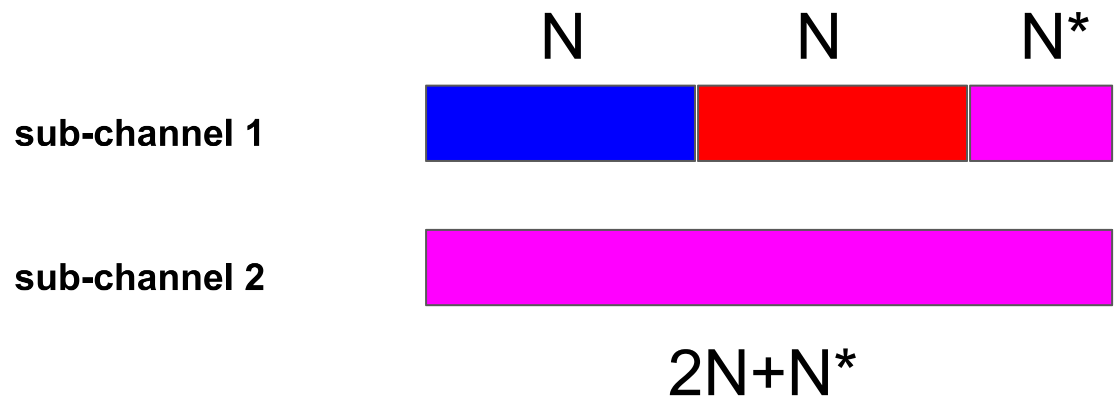


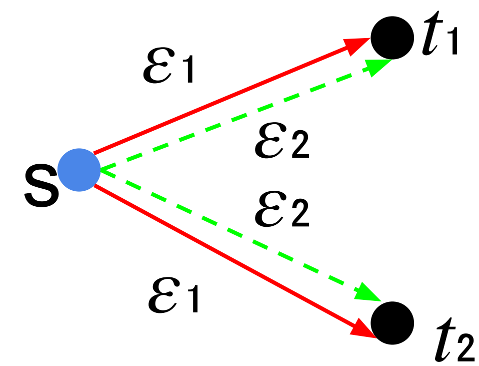
$$R_{sum}^{inner} = \frac{2(1-\epsilon_1^2)}{2+\epsilon_1} + 1 - \epsilon_2$$

$$R_{sum}^{outer} = \frac{2(2-\epsilon_1-\epsilon_2)(1+\epsilon_1)}{2+\epsilon_1}.$$

$$\begin{aligned} & R_{sum}^{outer} - R_{sum}^{inner} \\ = & \frac{2(2-\epsilon_1-\epsilon_2)(1+\epsilon_1)}{2+\epsilon_1} - \left(\frac{2(1-\epsilon_1^2)}{2+\epsilon_1} + 1 - \epsilon_2\right) \\ = & \frac{\epsilon_1(1-\epsilon_2)}{2+\epsilon_1} > 0, \end{aligned}$$

Outer bound achievable by network coding





Packets for t_1 only, resend if neither received

Packets for t_2 only, resend if neither received

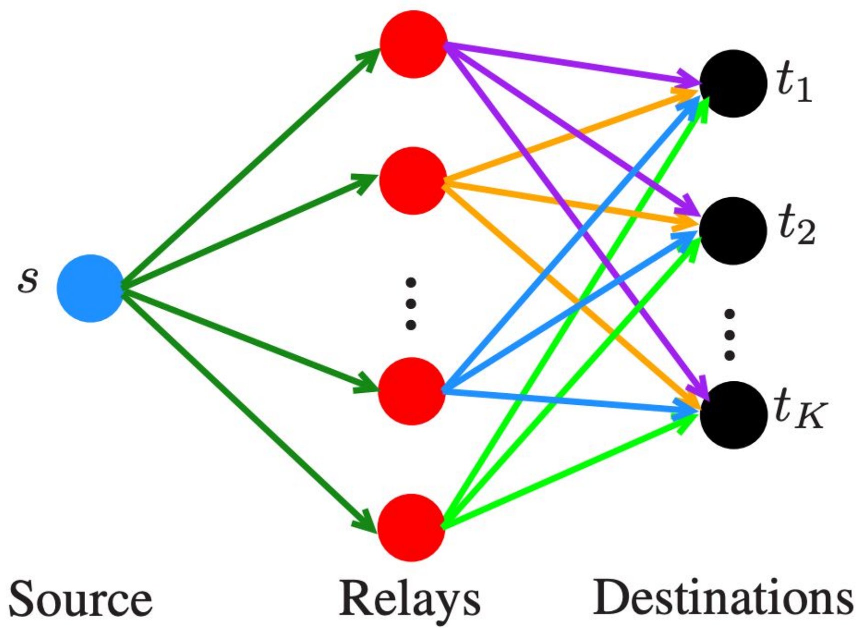
Network coding of packets not received by desired receivers

$$N^* = \frac{(1-\epsilon_1)\epsilon_1 - 2(1-\epsilon_2)}{2-\epsilon_1-\epsilon_2} N$$

which is positive when $\epsilon_2 \geq 1 - \frac{(1-\epsilon_1)\epsilon_1}{2}$.

Broadcast Packet Erasure Relay Networks with Feedback

- one source
- multiple destinations
- broadcast erasure channels connecting nodes
- channel output feedback at each transmitter



A three-layer 1 to K broadcast erasure relay networks with feedback.

Broadcast Packet Erasure Relay Networks with Feedback

- New outer bound based on multiple-input broadcast packet erasure channels with feedback

- Convert the network to multiple-input broadcast PEC with feedback

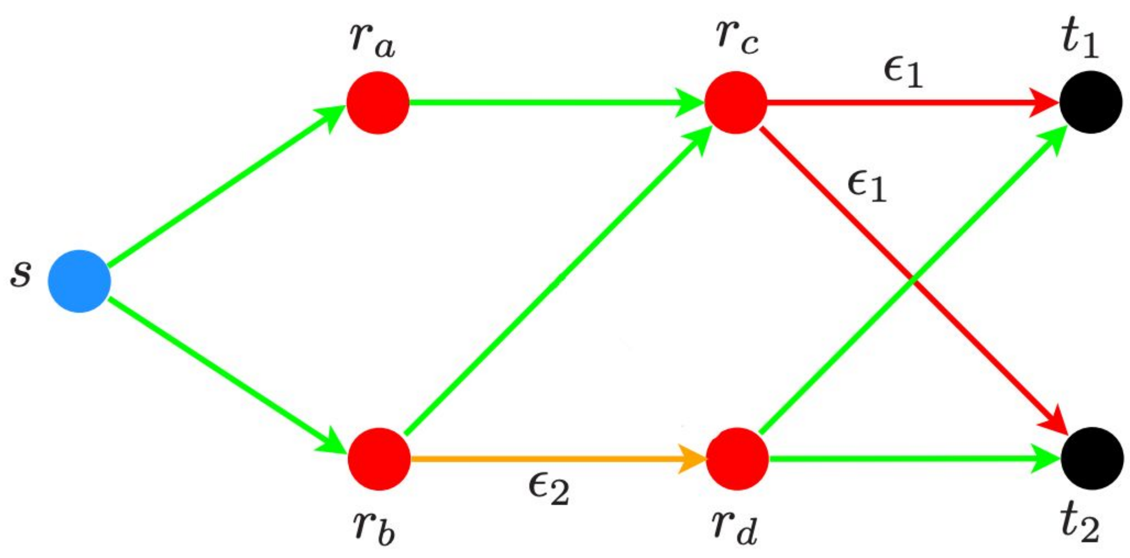
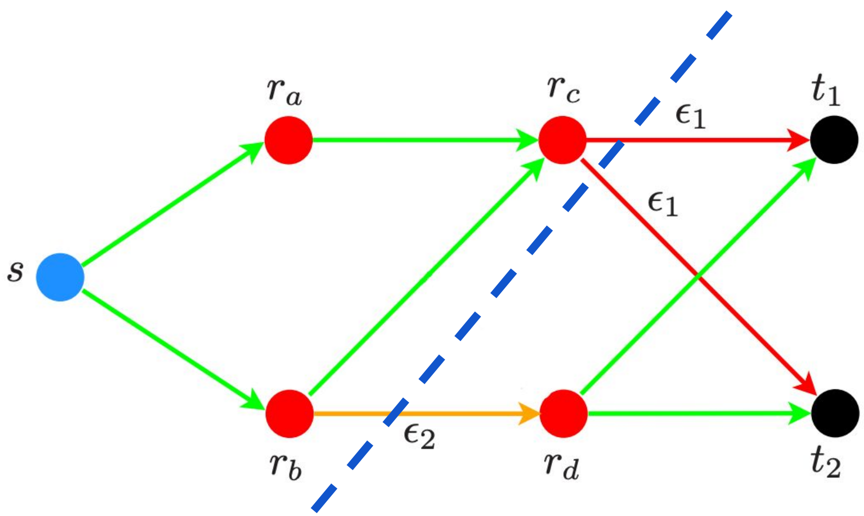
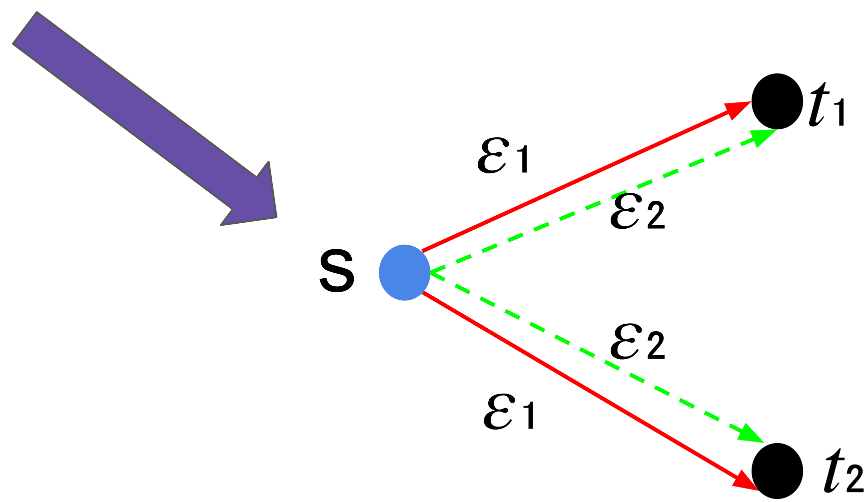


Fig. 3. A 1-to-2 broadcast packet erasure relay network.

Broadcast Packet Erasure Relay Networks with Feedback

- Convert the network to multiple-input broadcast PEC with feedback

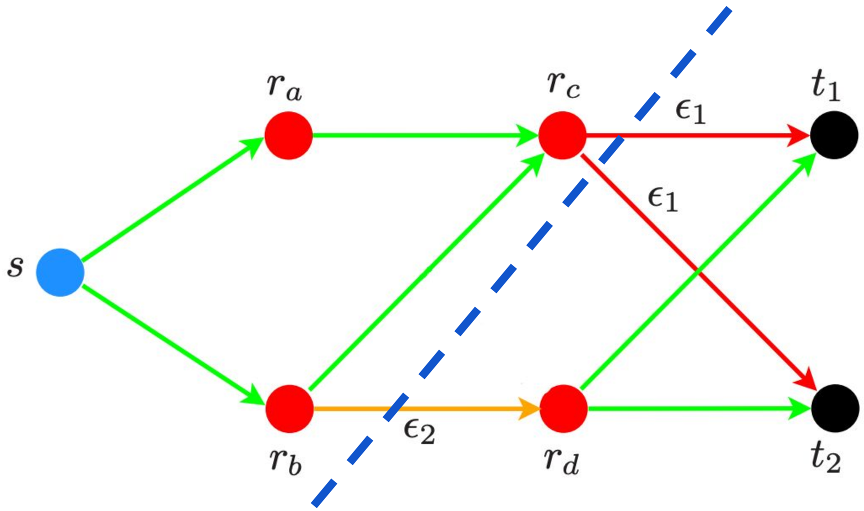




- Merge s, r_a,r_b,r_c into a super node s
- remove r_d, and connect with s to t_1,t_2 directly with the correlated broadcast erasure channel with erasure probability ε_2

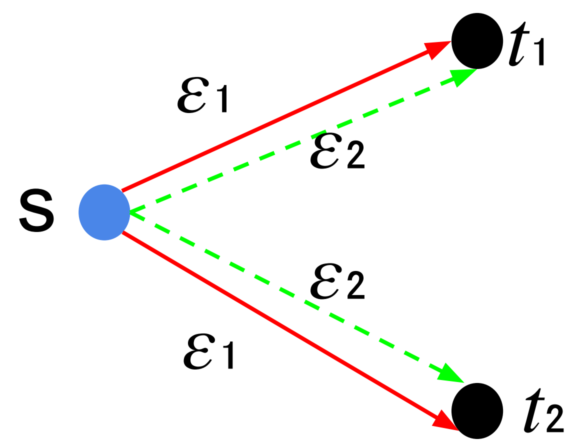
Broadcast Packet Erasure Relay Networks with Feedback

- Convert the network to multiple-input broadcast PEC with feedback



$$R_{sum}^{outer} = \frac{2(2-\epsilon_1-\epsilon_2)(1+\epsilon_1)}{2+\epsilon_1}.$$

- Merge s, r_a,r_b,r_c into a super node s
- remove r_d, and connect with s to t_1,t_2 directly with the correlated broadcast erasure channel with erasure probability ε_2



Summary

- **Background**
  - Broadcast Packet Erasure Channel (PEC)
  - Multiple-Input Broadcast Packet Erasure Channel
- **Model**
  - Multiple-Input Broadcast Packet Erasure Channel with Feedback
  - Broadcast Packet Erasure Relay Netwo*rk with Feedback*

- **Our Contributions**
  - New Outer-bounds on
    - Multiple Input Broadcast PEC with Feedback
    - Broadcast Packet Erasure Relay Networks with Feedback
  - Tightness of the outer bounds
    - Outer bounds achieved via network coding across sub-channels
    - Engineering implication: the network coding packets should be sent over correlated broadcast subchannels

Thank you!