Category Theory Study Group

Second Session

For the second session, read 1.5 - 1.6.

1. Every category $\mathbf{C}$, with sets of arrows between objects is isomorphic to a subcategory of $\mathbf{Set}$ (Theorem 1.6). This theorem places all such categories on equal footings. Furthermore, we can apply reasoning tools for the category $\mathbf{Set}$ to the Cayley representation of a category and the results reflect back into the original category (the Yoneda principle). For now, let us try to understand the Cayley representation itself. Consider the diagram of the category $\mathbf{C}$ below.

$$
\begin{tikzcd}
 & A & \\
B \arrow[ur, "f_1"] & C \arrow[u, "f_2"] & D \arrow[ul, "f_3"'] \\
E \arrow[u, "f_4"] \arrow[ur, "f_6"] & F \arrow[ul, "f_5"] \arrow[ur, "f_8"] & G \arrow[ul, "f_7"] \arrow[u, "f_9"'] \\
 & H \arrow[ul, "f_{10}"] \arrow[u, "f_{11}"] \arrow[ur, "f_{12}"'] &
\end{tikzcd}
$$

   How many objects and arrows does the Cayley representation $\overline{\mathbf{C}}$ of $\mathbf{C}$ have? How does the set $\overline{E}$, $\overline{C}$, and $\overline{A}$ look like? What is the result of $\overline{f}_6(id_E)$, $\overline{f}_6(f_{10})$, $\overline{f}_2(id_C)$ and $\overline{f}_2(f_6 \circ f_{10})$?

   Given a diagram in $\mathbf{C}$ commutes, is there an corresponding commuting diagram in $\overline{\mathbf{C}}$? Show that if $f_1 \circ f_5 = f_3 \circ f_8$, then $\overline{f}_1 \circ \overline{f}_5 = \overline{f}_3 \circ \overline{f}_8$.

   Show that the Cayley representation can be defined as a functor from $\mathbf{C} \to \mathbf{Set}$.

2. A contravariant functor is a functor $F : \mathbf{C} \to \mathbf{D}$ that maps arrows $f : A \to B$ in $\mathbf{C}$ to $F(f) : F(B) \to F(A)$ in $\mathbf{D}$. Show that functors $F : \mathbf{C^{op}} \to \mathbf{D}$ give rise to a contravariant functor $\mathbf{C} \to \mathbf{D}$. Show that there is a dual Cayley representation defined by a functor from $\mathbf{C^{op}} \to \mathbf{Set}$.

3. Consider the category $\mathbf{C}$ of exercise 1 and that the upper and the lower side of the cube commutes. Draw a diagram of the opposite category $\mathbf{C^{op}}$, the arrow category $\mathbf{C}^{\rightarrow}$ and the slice category $\mathbf{C}/A$. Do not draw objects and arrows containing identities.

4. Draw a diagram of the product category of the following two categories.

$$
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_3"] & B \arrow[d, "f_2"] \\
C \arrow[r, "f_4"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{I} \arrow[d, "g_1"] \arrow[dr, "g_2"] & \\
\mathrm{II} \arrow[r, "g_3"] & III
\end{tikzcd}
$$

5. For two functors $F : \mathbf{C} \to \mathbf{D}$, and $G : \mathbf{C} \to \mathbf{E}$, define a functor $H : \mathbf{C} \to \mathbf{D} \times \mathbf{E}$, such that $\pi_1 \circ H = F$ and $\pi_2 \circ H = G$.

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dl, "F"'] \arrow[d, "H"] \arrow[dr, "G"] & \\
\mathbf{D} & \mathbf{D} \times \mathbf{E} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbf{E}
\end{tikzcd}
$$

Prove that H is the only functor that satisfies these conditions by showing that for all other functors $H' : \mathbf{C} \to \mathbf{D} \times \mathbf{E}$ with $\pi_1 \circ H' = F$ and $\pi_2 \circ H' = G$, it follows $H = H'$.

Use the previous lemma to construct a functor called the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ and a functor from the arrow category into the product category that make the following diagrams commute.

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dl, "id_{\mathbf{C}}"'] \arrow[d, "\Delta"] \arrow[dr, "id_{\mathbf{C}}"] & \\
\mathbf{C} & \mathbf{C} \times \mathbf{C} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbf{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{C}^{\rightarrow} \arrow[dl, "Dom"'] \arrow[d] \arrow[dr, "Cod"] & \\
\mathbf{C} & \mathbf{C} \times \mathbf{C} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbf{C}
\end{tikzcd}
$$

6. Prove a few simple facts about the opposite categories:
   - $(\mathbf{C^{op}})^{\mathbf{op}} \cong \mathbf{C}$
   - $(\mathbf{C} \times \mathbf{D})^{\mathbf{op}} \cong \mathbf{C^{op}} \times \mathbf{D^{op}}$
   - $(\mathbf{C}^{\rightarrow})^{\mathbf{op}} \cong (\mathbf{C^{op}})^{\rightarrow}$
   - for all objects X of $\mathbf{C}$, $(\mathbf{C^{op}}/X)^{\mathbf{op}} \cong X/\mathbf{C}$ and $(X/\mathbf{C^{op}})^{\mathbf{op}} \cong \mathbf{C}/X$