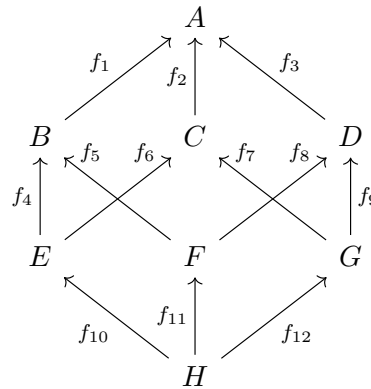


Category Theory Study Group

Second Session

For the second session, read 1.5 - 1.6.

1. Every category \mathbf{C} , with sets of arrows between objects is isomorphic to a subcategory of **Set** (Theorem 1.6). This theorem places all such categories on equal footings. Furthermore, we can apply reasoning tools for the category **Set** to the Cayley representation of a category and the results reflect back into the original category (the Yoneda principle). For now, let us try to understand the Cayley representation itself. Consider the diagram of the category \mathbf{C} below.



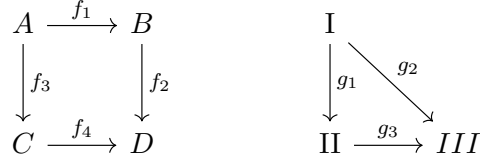
How many objects and arrows does the Cayley representation $\overline{\mathbf{C}}$ of \mathbf{C} have? How does the set \overline{E} , \overline{C} , and \overline{A} look like? What is the result of $\overline{f}_6(id_E)$, $\overline{f}_6(f_{10})$, $\overline{f}_2(id_C)$ and $\overline{f}_2(f_6 \circ f_{10})$?

Given a diagram in \mathbf{C} commutes, is there an corresponding commuting diagram in $\overline{\mathbf{C}}$? Show that if $f_1 \circ f_5 = f_3 \circ f_8$, then $\overline{f}_1 \circ \overline{f}_5 = \overline{f}_3 \circ \overline{f}_8$.

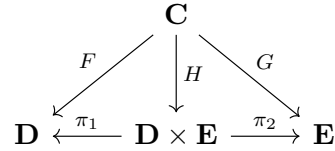
Show that the Cayley representation can be defined as a functor from $\mathbf{C} \rightarrow \mathbf{Set}$.

2. A contravariant functor is a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ that maps arrows $f : A \rightarrow B$ in \mathbf{C} to $F(f) : F(B) \rightarrow F(A)$ in \mathbf{D} . Show that functors $F : \mathbf{C}^{\mathbf{op}} \rightarrow \mathbf{D}$ give rise to a contravariant functor $\mathbf{C} \rightarrow \mathbf{D}$. Show that there is a dual Cayley representation defined by a functor from $\mathbf{C}^{\mathbf{op}} \rightarrow \mathbf{Set}$.

3. Consider the category \mathbf{C} of exercise 1 and that the upper and the lower side of the cube commutes. Draw a diagram of the opposite category \mathbf{C}^{op} , the arrow category \mathbf{C}^{\rightarrow} and the slice category \mathbf{C}/A . Do not draw objects and arrows containing identities.
4. Draw a diagram of the product category of the following two categories.



5. For two functors $F : \mathbf{C} \rightarrow \mathbf{D}$, and $G : \mathbf{C} \rightarrow \mathbf{E}$, define a functor $H : \mathbf{C} \rightarrow \mathbf{D} \times \mathbf{E}$, such that $\pi_1 \circ H = F$ and $\pi_2 \circ H = G$.



Prove that H is the only functor that satisfies these conditions by showing that for all other functors $H' : \mathbf{C} \rightarrow \mathbf{D} \times \mathbf{E}$ with $\pi_1 \circ H' = F$ and $\pi_2 \circ H' = G$, it follows $H = H'$.

Use the previous lemma to construct a functor called the diagonal functor $\Delta : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ and a functor from the arrow category into the product category that make the following diagrams commute.



6. Prove a few simple facts about the opposite categories:

- $(\mathbf{C}^{\text{op}})^{\text{op}} \cong \mathbf{C}$
- $(\mathbf{C} \times \mathbf{D})^{\text{op}} \cong \mathbf{C}^{\text{op}} \times \mathbf{D}^{\text{op}}$
- $(\mathbf{C}^{\rightarrow})^{\text{op}} \cong (\mathbf{C}^{\text{op}})^{\rightarrow}$
- for all objects X of \mathbf{C} , $(\mathbf{C}^{\text{op}}/X)^{\text{op}} \cong X/\mathbf{C}$ and $(X/\mathbf{C}^{\text{op}})^{\text{op}} \cong \mathbf{C}/X$