

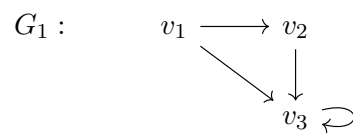
Category Theory Study Group

Fourth Session

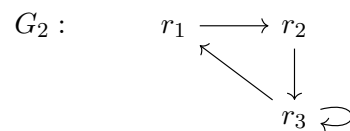
For the fourth session, read 2.3. - 2.8.

1. Remember the category of directed graphs and graph homomorphism: The objects are graphs consisting of a set of vertices V and edges $E \subseteq V \times V$. An arrow $f^* : (V, E) \rightarrow (V', E')$ between two graphs is a mapping between the vertices $f : V \rightarrow V'$, such that if $(v_1, v_2) \in E$ then $(f(v_1), f(v_2)) \in E'$.

What is an initial/terminal object in this category. How many *points* does the following graph have?

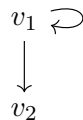


Prove that the previous graph is not isomorphic to the following graph by selecting a suitable “test-object” X and show that the hom-sets have different cardinality, i.e. $|\text{Hom}_{\mathbf{Graphs}}(X, G_1)| \neq |\text{Hom}_{\mathbf{Graphs}}(X, G_2)|$.

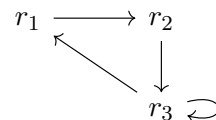


2. What is the product graph of the following two graphs?

$G_1 :$



$G_2 :$



Consider the following graph

$$G_3 : \begin{array}{ccc} s_1 & \longrightarrow & s_2 \\ \downarrow & & \downarrow \\ s_3 & \longrightarrow & s_4 \end{array}$$

and graph homomorphisms $f : G_3 \rightarrow G_1$ and $g : G_3 \rightarrow G_2$ defined by

$$f(x) = \begin{cases} v_1, & x \in \{s_1, s_2, s_3\} \\ v_2, & x = s_4 \end{cases}$$

and

$$g(x) = \begin{cases} r_1, & x = s_1 \\ r_2, & x = s_2 \\ r_3, & x \in \{s_3, s_4\}. \end{cases}$$

Use the UMP of the product graph to construct a unique mapping into $G_1 \times G_2$ as in the following diagram:

$$\begin{array}{ccccc} & & G_3 & & \\ & f \swarrow & \downarrow \scriptstyle (f,g) & \searrow g & \\ G_1 & \xleftarrow{\pi_1} & G_1 \times G_2 & \xrightarrow{\pi_2} & G_2 \end{array}$$

3. Prove the following simple facts about products and terminal objects:

- $(A \times 1) \cong A$
- $(1 \times B) \cong B$
- $(A \times B) \times C \cong A \times (B \times C)$

Where have we seen these laws before? Do these propositions also hold for coproducts and initial objects?

Are products commutative, i.e. $A \times B \cong B \times A$?

4. For any index set I , define the product $\prod_{i \in I} X_i$ of an I -indexed of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).

Show that in **Sets**, for any set X the set X^I of all functions $f : I \rightarrow X$ has this UMP, with respect to the “constant family” where $X_i = X$ for all $i \in I$, and thus

$$X^I \cong \prod_{i \in I} X.$$

5. Given a category \mathbf{C} with object A and B , define the category $\mathbf{C}_{A,B}$ to have objects (X, x_1, x_2) , where $x_1 : X \rightarrow A$, $x_2 : X \rightarrow B$, and with arrows $f : (X, x_1, x_2) \rightarrow (Y, y_1, y_2)$ being arrows $f : X \rightarrow Y$ with $y_1 \circ f = x_1$ and $y_2 \circ f = x_2$.

Show that $\mathbf{C}_{A,B}$ has a terminal object just in case A and B have a product in \mathbf{C} .