Category Theory Study Group

Fourth Session

For the fourth session, read 2.3. - 2.8.

1. Remember the category of directed graphs and graph homomorphism: The objects are graphs consisting of a set of vertices V and edges $E \subseteq V \times V$. An arrow $f^* : (V, E) \to (V', E')$ between two graphs is a mapping between the vertices $f : V \to V'$, such that if $(v_1, v_2) \in E$ then $(f(v_1), f(v_2)) \in E'$.

   What is an initial/terminal object in this category. How many *points* does the following graph have?

$$
\begin{tikzcd}
G_1 : & v_1 \arrow[r] \arrow[dr] & v_2 \arrow[d] \\
& & v_3 \arrow[loop right]
\end{tikzcd}
$$

   Prove that the previous graph is not isomorphic to the following graph by selecting a suitable "test-object" X and show that the hom-sets have different cardinality, i.e. $|\mathrm{Hom}_{\mathbf{Graphs}}(X, G_1)| \neq |\mathrm{Hom}_{\mathbf{Graphs}}(X, G_2)|$.

$$
\begin{tikzcd}
G_2 : & r_1 \arrow[r] & r_2 \arrow[d] \\
& & r_3 \arrow[ul] \arrow[loop right]
\end{tikzcd}
$$

2. What is the product graph of the following two graphs?

$$
\begin{tikzcd}
G_1 : & & G_2 : & \\
v_1 \arrow[loop right] \arrow[d] & & r_1 \arrow[r] & r_2 \arrow[d] \\
v_2 & & & r_3 \arrow[ul] \arrow[loop right]
\end{tikzcd}
$$

   Consider the following graph

$$G_3: \quad \begin{tikzcd} s_1 \arrow[r] \arrow[d] & s_2 \arrow[d] \\ s_3 \arrow[r] & s_4 \end{tikzcd}$$

and graph homomorphisms $f : G_3 \to G_1$ and $g : G_3 \to G_2$ defined by

$$f(x) = \begin{cases} v_1, & x \in \{s_1, s_2, s_3\} \\ v_2, & x = s_4 \end{cases}$$

and

$$g(x) = \begin{cases} r_1, & x = s_1 \\ r_2, & x = s_2 \\ r_3, & x \in \{s_3, s_4\}. \end{cases}$$

Use the UMP of the product graph to construct a unique mapping into $G_1 \times G_2$ as in the following diagram:

$$\begin{tikzcd} & G_3 \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ G_1 & G_1 \times G_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G_2 \end{tikzcd}$$

3. Prove the following simple facts about products and terminal objects:
   - $(A \times 1) \cong A$
   - $(1 \times B) \cong B$
   - $(A \times B) \times C \cong A \times (B \times C)$

   Where have we seen these laws before? Do these propositions also hold for coproducts and initial objects?

   Are products commutative, i.e. $A \times B \cong B \times A$?

4. For any index set I, define the product $\prod_{i \in I} X_i$ of an I-indexed of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).

   Show that in **Sets**, for any set X the set X^I of all functions $f : I \to X$ has this UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus
   $$X^I \cong \prod_{i \in I} X.$$

5. Given a category $\mathbf{C}$ with object A and B, define the category $\mathbf{C}_{A,B}$ to have objects (X, x_1, x_2), where $x_1 : X \to A$, $x_2 : X \to B$, and with arrows $f : (X, x_1, x_2) \to (Y, y_1, y_2)$ being arrows $f : X \to Y$ with $y_1 \circ f = x_1$ and $y_2 \circ f = x_2$.

   Show that $\mathbf{C}_{A,B}$ has a terminal object just in case A and B have a product in $\mathbf{C}$.