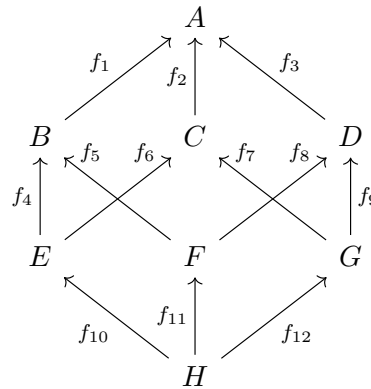


# Category Theory Study Group

## Second Session

For the second session, read 1.5 - 1.6.

1. Every category  $\mathbf{C}$ , with sets of arrows between objects is isomorphic to a subcategory of  $\mathbf{Set}$  (Theorem 1.6). This theorem places all such categories on equal footings. Furthermore, we can apply reasoning tools for the category  $\mathbf{Set}$  to the Cayley representation of a category and the results reflect back into the original category (the Yoneda principle). For now, let us try to understand the Cayley representation itself. Consider the diagram of the category  $\mathbf{C}$  below.



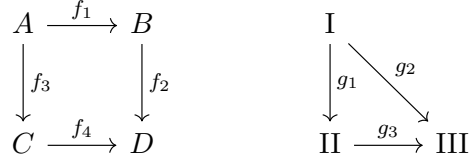
How many objects and arrows does the Cayley representation  $\overline{\mathbf{C}}$  of  $\mathbf{C}$  have? How does the set  $\overline{E}$ ,  $\overline{C}$ , and  $\overline{A}$  look like? What is the result of  $\overline{f}_6(id_E)$ ,  $\overline{f}_6(f_{10})$ ,  $\overline{f}_2(id_C)$  and  $\overline{f}_2(f_6 \circ f_{10})$ ?

Given a diagram in  $\mathbf{C}$  commutes, is there an corresponding commuting diagram in  $\overline{\mathbf{C}}$ ? Show that if  $f_1 \circ f_5 = f_3 \circ f_8$ , then  $\overline{f}_1 \circ \overline{f}_5 = \overline{f}_3 \circ \overline{f}_8$ .

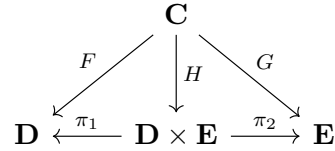
Show that the Cayley representation can be defined as a functor from  $\mathbf{C} \rightarrow \mathbf{Set}$ .

2. A contravariant functor is a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  that maps arrows  $f : A \rightarrow B$  in  $\mathbf{C}$  to  $F(f) : F(B) \rightarrow F(A)$  in  $\mathbf{D}$ . Show that functors  $F : \mathbf{C}^{\mathbf{op}} \rightarrow \mathbf{D}$  give rise to a contravariant functor  $\mathbf{C} \rightarrow \mathbf{D}$ . Show that there is a dual Cayley representation defined by a functor from  $\mathbf{C}^{\mathbf{op}} \rightarrow \mathbf{Set}$ .

3. Consider the category  $\mathbf{C}$  of exercise 1 and that the upper and the lower side of the cube commutes. Draw a diagram of the opposite category  $\mathbf{C}^{\text{op}}$ , the arrow category  $\mathbf{C}^{\rightarrow}$  and the slice category  $\mathbf{C}/A$ . Do not draw objects and arrows containing identities.
4. Draw a diagram of the product category of the following two categories.



5. For two functors  $F : \mathbf{C} \rightarrow \mathbf{D}$ , and  $G : \mathbf{C} \rightarrow \mathbf{E}$ , define a functor  $H : \mathbf{C} \rightarrow \mathbf{D} \times \mathbf{E}$ , such that  $\pi_1 \circ H = F$  and  $\pi_2 \circ H = G$ .



Prove that  $H$  is the only functor that satisfies these conditions by showing that for all other functors  $H' : \mathbf{C} \rightarrow \mathbf{D} \times \mathbf{E}$  with  $\pi_1 \circ H' = F$  and  $\pi_2 \circ H' = G$ , it follows  $H = H'$ .

Use the previous lemma to construct a functor called the diagonal functor  $\Delta : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$  and a functor from the arrow category into the product category that make the following diagrams commute.



6. Prove a few simple facts about the opposite categories:

- $(\mathbf{C}^{\text{op}})^{\text{op}} \cong \mathbf{C}$
- $(\mathbf{C} \times \mathbf{D})^{\text{op}} \cong \mathbf{C}^{\text{op}} \times \mathbf{D}^{\text{op}}$
- $(\mathbf{C}^{\rightarrow})^{\text{op}} \cong (\mathbf{C}^{\text{op}})^{\rightarrow}$
- for all objects  $X$  of  $\mathbf{C}$ ,  $(\mathbf{C}^{\text{op}}/X)^{\text{op}} \cong X/\mathbf{C}$  and  $(X/\mathbf{C}^{\text{op}})^{\text{op}} \cong \mathbf{C}/X$