The Snake Eyes Paradox

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Problem Statement

You are offered a gamble. A pair of six-sided dice are rolled and unless they come up snake eyes you get a bajillion dollars. If they do come up snake eyes, you're devoured by snakes.

So far it sounds like you have a 1/36 chance of dying, right?

Now the twist. First, I gather up an unlimited number of people willing to play the game, including you. I take 1 person from that pool and let them play. Then I take 2 people and have them play together, where they share a dice roll and either get the bajillion dollars each or both get devoured. Then I do the same with 4 people, and then 8, 16, and so on.

Eventually one of those groups will be devoured by snakes—hopefully not the group you're in—and then I stop. Is the probability that you'll die, given that you're chosen to play, still 1/36?

Argument for NO: Due to the doubling, the final group of people that die is slightly bigger than all the surviving groups put together. So if you're chosen to play you have about a 50% chance of dying! 😬 🐍

Argument for YES: The dice rolls are independent and whenever you're chosen, what happened in earlier rounds is irrelevant. Your chances of death are the chances of snake eyes on your round: 1/36. 😅

So which is it? What's your probability of dying, conditional on being chosen to play?

Some clarifications:

1. The game is not adversarial and the dice rolls are independent and truly random.
2. Choosing each group also happens uniformly randomly and without replacement.
3. This is technically undefined with an infinite pool of people but we can cap it and say that if no one has died after N rounds then the game ends and no one dies. We just need to then find the limit as N goes to infinity.
4. Importantly, in the finite version it's possible for no one to die. Just that the probability of that approaches zero as the size of the pool approaches infinity.
5. Again, we want the conditional probability that you die given that you are chosen to play. In other words, of the people chosen, what fraction, in expectation, die? In the unbounded case there's a 0% chance of infinitely many people being chosen and none dying; in the bounded case it's a tiny chance of a huge number chosen and none dying.

Solution

We want the probability that you die given that you are chosen to play, Pr(death | chosen). It seems like we can ignore the 0% chance of rolling not-snake-eyes forever and say that eventually about half the people who are chosen die, but let's Bayes it out carefully:

$$\begin{aligned}\Pr(\text{death} \mid \text{chosen}) &= \frac{\Pr(\text{chosen} \mid \text{death})\Pr(\text{death})}{\Pr(\text{chosen})} \\ &= \frac{1 \cdot \Pr(\text{death})}{\Pr(\text{chosen})}.\end{aligned}$$

In the uncapped case, that conditional probability is undefined. You're part of an infinite pool so you have a 0% chance of being chosen and a 0% chance of dying. The probability we want is 0/0. **robot-with-smoke-coming-out-of-its-ears-emoji**

Since we can't directly calculate the probability in the infinite case, we have to take a limit.

To get a feel for where we're going, suppose you're 1 person in a huge but finite pool. Now suppose you are actually chosen. There are two ways that can happen:

1. The pool runs out and everyone survives.
2. The pool doesn't run out and you have about a 50% chance of dying.

But knowing that you are chosen is Bayesian evidence that we had many, many rounds of survival. If an early group died then most of the pool wasn't chosen, so probably you weren't chosen.

Thinking like a Bayesian means shifting your probability in light of evidence by seeing how surprised you'd be in various universes by that evidence. If an early group died then most people aren't chosen and in that universe you're surprised to be chosen. If *no* group died then everyone was chosen and in that universe you're fully unsurprised that you were chosen. That's the sense in which being chosen is Bayesian evidence that more people survived. In particular it's at least weak evidence that everyone survived.

So even with an absurdly huge pool of people, where there's *essentially* a 0% chance of everyone surviving, if you know you were chosen (which itself has near zero probability, but, you know, *if*) then that means you're more likely to be in that essentially-0%-probability universe where everyone survives.

Enough hand-waving and appeals to intuition. Let's Bayes it out to see what Pr(death | chosen) is exactly, in the version where we stop after N rounds. Once we have that, we can take the limit as N goes to infinity.

First, let M be the size of the pool:

$$M = \sum_{i=1}^{N} 2^{i-1} = 2^N - 1.$$

And let p be the probability of snake eyes, 1/36. We can now compute the probability of being chosen by summing up (1) the probability you're chosen for the first round, $1/M$, plus (2) the probability that the first group survives, $1-p$, and that you're chosen for the 2nd round, $2/M$, plus (3) the probability that the first two groups survive and you're chosen for the 3rd round, etc. Writing that out as an equation gives this:

$$\begin{aligned}\Pr(\text{chosen}) &= \tfrac{1}{M} + (1-p)\tfrac{2}{M} \\ &\quad + (1-p)^2\tfrac{4}{M} \\ &\quad + (1-p)^3\tfrac{8}{M} \\ &\quad + \ldots \\ &\quad + (1-p)^{N-1}\frac{2^{N-1}}{M} \\ &= \sum_{i=1}^{N} \tfrac{1}{M} 2^{i-1}(1-p)^{i-1}.\end{aligned}$$

For Pr(death) the calculation is very similar but every term is multiplied by p. To die, you have to be chosen and then roll snake eyes. This can happen on any round, all of which are mutually exclusive. We can then factor that p out and we have

$$\Pr(\text{death}) = p \cdot \Pr(\text{chosen}).$$

Working out that expression for Pr(chosen) wasn't even necessary! We compute Pr(death | chosen) like so:

$$\begin{aligned}\Pr(\text{death} \mid \text{chosen}) &= \frac{\Pr(\text{death})}{\Pr(\text{chosen})}\\ &= \frac{p \cdot \Pr(\text{chosen})}{\Pr(\text{chosen})} = p.\end{aligned}$$

It doesn't depend on N at all! The limit as N goes to infinity is just... p or 1/36, the probability of rolling snake eyes. □

Discussion and Dead-Horse Beating

What about the argument that, with unlimited people, there will necessarily be a finite round n at which snake eyes is rolled? And for every possible such n, at least half of the chosen players die. After all, the probability of rolling not-snake-eyes forever is zero. (More precisely, in the limit as n goes to infinity, the probability of rolling not-snake-eyes n times in a row goes to zero.)

That's all true but let's work out the probability of rolling not-snake-eyes forever *conditional on you being chosen*. Starting with Pr(snake eyes) as the probability that a game rolls snake eyes—unambiguously 1—we have, by the definition of conditional probability:

$$\Pr(\text{snake eyes} \mid \text{chosen}) = \frac{\Pr(\text{chosen} \wedge \text{snake eyes})}{\Pr(\text{chosen})}.$$

In the infinite setting that's $\frac{0}{0}$ because you have a 0% chance of being chosen from an infinite pool. So let's work it out in the limit with a cap of N rounds and finite pool M as before:

$$\frac{\sum_{n=1}^{N}(1-p)^{n-1}p \cdot \frac{2^n-1}{M}}{\sum_{i=1}^{N}\frac{1}{M}2^{i-1}(1-p)^{i-1}}.$$

In the numerator we're summing over every possible round n at which we could roll snake eyes, saying that we need to roll not-snake-eyes $n-1$ times followed by one snake eyes *and* that we are chosen in any round from 1 through n. The denominator, Pr(chosen), is the same as in the previous section.

Now algebra ensues! We multiply the numerator and denominator by M to get rid of the $1/M$ factor:

$$\frac{\sum_{n=1}^{N}(1-p)^{n-1}p \cdot (2^n-1)}{\sum_{i=1}^{N}2^{i-1}(1-p)^{i-1}}.$$

Then distribute the $(1-p)^{n-1}p$ over the 2^n-1 and split it into two summations like so:

$$\frac{\left(\sum_{n=1}^{N}2^n(1-p)^{n-1}p\right) - \left(\sum_{n=1}^{N}(1-p)^{n-1}p\right)}{\sum_{i=1}^{N}2^{i-1}(1-p)^{i-1}}.$$

These are finite sums so that's kosher. As N goes to ∞, the right side of the numerator—which is again the probability of rolling snake eyes by round N—goes to one:

$$\frac{\left(\sum_{n=1}^{N}2^n(1-p)^{n-1}p\right) - 1}{\sum_{i=1}^{N}2^{i-1}(1-p)^{i-1}}.$$

Almost there! Pull a $2p$ out of the sum in the numerator to get this:

$$\begin{aligned}&\frac{2p\left(\sum_{n=1}^{N}2^{n-1}(1-p)^{n-1}\right) - 1}{\sum_{i=1}^{N}2^{i-1}(1-p)^{i-1}}\\ =&2p - \frac{1}{\sum_{i=1}^{N}2^{i-1}(1-p)^{i-1}}\\ =&2p - \frac{1}{\sum_{i=1}^{N}\left(2(1-p)\right)^{i-1}}.\end{aligned}$$

The remaining summation is a finite geometric series with common ratio $2(1-p)$. As long as $2(1-p) \geq 1$ (i.e., $p \leq 1/2$ which for us is the case, namely $p = 1/36$), the summation in the denominator diverges and the above approaches $2p$ in the limit as N goes to ∞.

In conclusion, the probability of eventually rolling snake eyes, conditional on you being chosen to play, approaches $2p = 1/18$ in the limit. Which is to say that the conditional probability of rolling not-snake-eyes literally forever is $17/18$. 🤯

This vindicates our initial intuitive argument that being chosen is Bayesian evidence of never rolling snake eyes. And it contradicts the intuition that any event which occurs with an absolute probability of 1, such as rolling snake eyes eventually, must be independent of any other event. If being chosen and rolling snake eyes were independent then

$$
\begin{aligned}
1/18 &= \frac{\Pr(\text{chosen} \wedge \text{snake eyes})}{\Pr(\text{chosen})} \\
&= \frac{\Pr(\text{chosen}) \Pr(\text{snake eyes})}{\Pr(\text{chosen})} \\
&= \Pr(\text{snake eyes})
\end{aligned}
$$

which contradicts $\Pr(\text{snake eyes}) = 1$. The temptation to treat $\Pr(X)$ as $\Pr(X \mid \text{snake eyes})$ since $\Pr(\text{snake eyes}) = 1$ leads us astray!