# The Snake Eyes Paradox


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## 1 Problem Statement

You are offered a gamble. A pair of six-sided dice are rolled and unless they come up snake eyes you get a bajillion dollars. If they do come up snake eyes, you're devoured by snakes.

So far it sounds like you have a 1/36 chance of dying, right?

Now the twist. First, I gather up an unlimited number of people willing to play the game, including you. I take 1 person from that pool and let them play. Then I take 2 people and have them play together, where they share a dice roll and either get the bajillion dollars each or both get devoured. Then I do the same with 4 people, and then 8, 16, and so on.

Eventually one of those groups will be devoured by snakes—hopefully not the group you're in—and then I stop. Is the probability that you'll die, given that you're chosen to play, still 1/36?

**Argument for NO:** Due to the doubling, the final group of people that die is slightly bigger than all the surviving groups put together. So if you're chosen to play you have about a 50% chance of dying! 😬 🐍

**Argument for YES:** The dice rolls are independent and whenever you're chosen, what happened in earlier rounds is irrelevant. Your chances of death are the chances of snake eyes on your round: 1/36. 😅

So which is it? What's your probability of dying, conditional on being chosen to play? If you learn that your friend was chosen to play in this game and the game is now over, how worried are you? If you wanted to play the one-shot version, do you still want to play the doubling groups version?

*Fine print: The game is not adversarial and the dice rolls are independent and truly random. Groups are chosen uniformly and without replacement. Void where prohibited.*

## 2 Solution (With Limits)

We want the probability that you die given that you are chosen to play, Pr(death | chosen). It seems like we can ignore the 0% chance of rolling not-snake-eyes forever and say that eventually about half the people who are chosen die, but let's Bayes it out carefully:

$$\begin{aligned}\Pr(\text{death} \mid \text{chosen}) &= \frac{\Pr(\text{chosen} \mid \text{death}) \Pr(\text{death})}{\Pr(\text{chosen})} \\ &= \frac{1 \cdot \Pr(\text{death})}{\Pr(\text{chosen})}.\end{aligned}$$

But if you're part of an infinite pool, you have a 0% chance of being chosen and a 0% chance of dying. The probability we want is 0/0. **robot-with-smoke-coming-out-of-its-ears-emoji**

Since we can't directly calculate the probability in the infinite case, a natural thing to do is to take a limit.

To get a feel for where we're going, suppose you're one person in a huge but finite pool. Now suppose you are actually chosen. There are two ways that can happen:

1. The pool runs out and everyone survives.

2. The pool doesn't run out and you have about a 50% chance of dying.

But knowing that you are chosen is Bayesian evidence that we had many, many rounds of survival. If an early group died then most of the pool wasn't chosen, so probably you weren't chosen.

Thinking like a Bayesian means shifting your probability in light of evidence by seeing how surprised you'd be in various universes by that evidence. If an early group died then most people aren't chosen and in that universe you're surprised to be chosen. If *no* group died then everyone was chosen and in that universe you're fully unsurprised that you were chosen. That's the sense in which being chosen is Bayesian evidence that more people survived. In particular it's at least weak evidence that everyone survived.

So even with an absurdly huge pool of people, where there's *essentially* a 0% chance of everyone surviving, if you know you were chosen (which itself has near zero probability, but, you know, *if*) then that means you're more likely to be in that essentially-0%-probability universe where everyone survives.

Enough hand-waving and appeals to intuition. Let's Bayes it out to see what Pr(death | chosen) is exactly, in a finite version where we stop after $N$ rounds. Once we have that, we can take the limit as $N$ goes to infinity.

First, let $M$ be the size of the pool:

$$M = \sum_{i=1}^{N} 2^{i-1} = 2^N - 1.$$

And let $p$ be the probability of snake eyes, 1/36. We can now compute the probability of being chosen by summing up (1) the probability you're chosen for the first round, $1/M$, plus (2) the probability that the first group survives, $1-p$, and that you're chosen for the 2nd round, $2/M$, plus (3) the probability that the first two groups survive and you're chosen for the 3rd round, etc. Writing that out as an equation gives this:

$$\begin{aligned}\Pr(\text{chosen}) = \tfrac{1}{M} &+ (1-p)\tfrac{2}{M} \\ &+ (1-p)^2\tfrac{4}{M} \\ &+ (1-p)^3\tfrac{8}{M} \\ &+ \dots \\ &+ (1-p)^{N-1}\frac{2^{N-1}}{M} \\ = \sum_{i=1}^{N} &\tfrac{1}{M} 2^{i-1}(1-p)^{i-1}.\end{aligned}$$

For Pr(death) the calculation is very similar but every term is multiplied by $p$. To die, you have to be chosen and then roll snake eyes. This can happen on any round, all of which are mutually exclusive. We can then factor that $p$ out and we have

$$\Pr(\text{death}) = p \cdot \Pr(\text{chosen}).$$

Working out that expression for Pr(chosen) wasn't even necessary! We compute Pr(death | chosen) like so:

$$\begin{aligned}\Pr(\text{death} \mid \text{chosen}) &= \frac{\Pr(\text{death})}{\Pr(\text{chosen})} \\ &= \frac{p \cdot \Pr(\text{chosen})}{\Pr(\text{chosen})} = p.\end{aligned}$$

It doesn't depend on $N$ at all! The limit as $N$ goes to infinity is just... $p$ or $1/36$, the probability of rolling snake eyes. □

## 3 Can We Roll Not-Snake-Eyes Forever?

What about the argument that, with unlimited people, there will necessarily be a finite round $n$ at which snake eyes is rolled? And for every possible such $n$, at least half of the chosen players die. After all, the probability of rolling not-snake-eyes forever is zero. (More precisely, in the limit as $n$ goes to infinity, the probability of rolling not-snake-eyes $n$ times in a row goes to zero.)

That's all true but let's work out the probability of rolling not-snake-eyes forever *conditional on you being chosen*. Starting with Pr(snake eyes) as the probability that a game rolls snake eyes—unambiguously 1—we have, by the definition of conditional probability:

$$\Pr(\text{snake eyes} \mid \text{chosen}) = \frac{\Pr(\text{chosen} \wedge \text{snake eyes})}{\Pr(\text{chosen})}.$$

In the infinite setting that's $\frac{0}{0}$ because you have a 0% chance of being chosen from an infinite pool. So let's work it out in the limit with a cap of $N$ rounds and finite pool $M$ as before:

$$\frac{\sum_{i=1}^{N}(1-p)^{i-1}p \cdot \dfrac{2^i-1}{M}}{\sum_{i=1}^{N} \frac{1}{M} 2^{i-1}(1-p)^{i-1}}.$$

In the numerator we're summing over every possible round $i$ at which we could roll snake eyes, saying that we need to roll not-snake-eyes $i-1$ times followed by one snake eyes *and* that we are chosen in any round from 1 through $i$. The denominator, Pr(chosen), is the same as in the previous section.

Now algebra ensues. We multiply the numerator and denominator by $M$ to get rid of the $1/M$ factor, then distribute the $(1-p)^{i-1}p$ over the $2^i - 1$ and split it into two summations:

$$\frac{\left(\sum_{i=1}^{N} 2^i(1-p)^{i-1}p\right) - \left(\sum_{i=1}^{N}(1-p)^{i-1}p\right)}{\sum_{i=1}^{N} 2^{i-1}(1-p)^{i-1}}.$$

These are finite sums so that's kosher. The right side of the numerator is the probability of rolling snake eyes by round $N$, which is Pr(snake eyes) in the limit as $N$ goes to infinity, so we replace that sum by one:

$$\frac{\left(\sum_{i=1}^{N} 2^i(1-p)^{i-1}p\right) - 1}{\sum_{i=1}^{N} 2^{i-1}(1-p)^{i-1}}.$$

Almost there! Pull a $2p$ out of the sum in the numerator to get this:

$$\frac{2p\left(\sum_{i=1}^{N} 2^{i-1}(1-p)^{i-1}\right) - 1}{\sum_{i=1}^{N} 2^{i-1}(1-p)^{i-1}}.$$

Notice that the sums in the numerator and denominator are now identical. We distribute the denominator,

$$2p - \frac{1}{\sum_{i=1}^{N} 2^{i-1}(1-p)^{i-1}},$$

and combine the terms in the sum,

$$2p - \frac{1}{\sum_{i=1}^{N} \left(2(1-p)\right)^{i-1}},$$

to see that the denominator is a finite geometric series with common ratio $2(1-p)$. As long as the common ratio is greater than or equal to 1, the denominator diverges and the above approaches $2p$ in the limit as $N$ goes to infinity. How do we know $2(1-p) \geq 1$? Because we can rearrange it as $p \leq 1/2$ and that's true for us, namely $p = 1/36$.[1]

In conclusion, the probability of eventually rolling snake eyes, conditional on you being chosen to play, approaches $2p = 1/18$ in the limit. Which is to say that the conditional probability of rolling not-snake-eyes literally forever is $17/18$. 🤯

(Or to say it less sensationally: For any finite $N$, the conditional probability of taking more than $N$ rolls to hit snake eyes is greater than $17/18$.)

This vindicates our initial intuitive argument that being chosen is Bayesian evidence—strong Bayesian evidence, it turns out!—of never rolling snake eyes. And it invalidates the intuition that we can safely condition on snake eyes being rolled just because it definitely will be rolled (unconditionally). Another version of that intuition is that any event with probability 1, such as rolling snake eyes eventually, must be independent of any other event. But if being chosen and rolling snake eyes were independent then, by definition of independence, $\Pr(\text{chosen} \wedge \text{snake eyes}) = \Pr(\text{chosen}) \cdot \Pr(\text{snake eyes})$. And if that were true, we'd conclude from the above derivation of $\Pr(\text{snake eyes} \mid \text{chosen})$ that

$$\begin{aligned} &\Pr(\text{snake eyes}) \\ =&\frac{\Pr(\text{chosen}) \Pr(\text{snake eyes})}{\Pr(\text{chosen})} \\ =&\frac{\Pr(\text{chosen} \wedge \text{snake eyes})}{\Pr(\text{chosen})} \\ =&\Pr(\text{snake eyes} \mid \text{chosen}) \\ =&1/18. \end{aligned}$$

Which contradicts $\Pr(\text{snake eyes}) = 1$. The temptation to treat $\Pr(X)$ as $\Pr(X \mid \text{snake eyes})$ since $\Pr(\text{snake eyes}) = 1$ leads us astray!

## 4 To Infinity And Beyond (With A Nonuniform Prior)

What if we reject the whole idea of defining a finite version of Snake Eyes to take a limit of? Can we math out an answer for the infinite game directly? Yes! The only monkey wrench is that we can't have a uniform prior over an infinite set.[2] So let's just say we don't *quite* have a uniform prior. Maybe you think you're equally likely to be any of the first trillion people chosen to play and that it gradually becomes less likely after that. We can make that "trillion" as high as we like.

As long as the probability of being chosen isn't exactly zero, there's no division-by-zero problem like before.

Is that fair though, to reject the stipulation in the problem statement that you're chosen uniformly? Well, it's arguably less of a leap than we made before in defining a finite version of the game where it's possible for no one to die. We're just saying you're not quite chosen uniformly because you *can't* be and have any probability of being chosen at all. But we can get arbitrarily close to uniform! We can even consider the limit as the distribution approaches uniform. Great, let's get to it!

Let $\text{CH}_c$ be the event that you're chosen to play in round $c$ and let $\text{SE}_s$ be the event that snake eyes is rolled in round $s$. Define $p_{cs} = \Pr(\text{CH}_c \wedge \text{SE}_s)$ as the probability of a game where you're chosen in round $c$ and snake eyes is rolled in round $s$. In this formulation, $\text{CH}_c$ and $\text{SE}_s$ are independent for all $c$ and $s$. So $c > s$ is possible, just that it means a game where you're not chosen because snake eyes was

[1] What would happen if we had $p > 1/2$? In that case, by the preceding derivation, $\Pr(\text{snake eyes} \mid \text{chosen}) = 1$ so no chance of everyone surviving. That makes sense because the whole paradox is ruined if $p > 1/2$. The probability of dying in the one-shot version is already greater than the fraction of people who die when the game ends in snake eyes.

[2] Not in standard analysis anyway. If infinitely many things are all equally likely then they all have zero probability. Or to be slightly more formal, there's an elegant proof by contradiction: First, the sum of the probabilities of each element of the set must be 1. That's part of what it means to have a prior over a set of possibilities. Now suppose every element in your infinite set has equal probability $\epsilon$. That's what we mean by a uniform prior. Further suppose that $\epsilon = 0$. Then the sum of the probabilities is 0. So that's no good; we must have $\epsilon > 0$. But the sum of an infinite number of positive $\epsilon$'s is infinity. So that's no good either. $\rightarrow\leftarrow$

rolled before we got to you. Summing $p_{cs}$ over every possible $c$ and $s$—every possible game—necessarily gives us 1:

$$\sum_{s=1}^{\infty}\sum_{c=1}^{\infty} p_{cs} = 1.$$

The independence of $\mathrm{CH}_c$ and $\mathrm{SE}_s$ gives us the following:

$$\begin{aligned} p_{cs} &= \Pr(\mathrm{CH}_c \wedge \mathrm{SE}_s) \\ &= \Pr(\mathrm{CH}_c) \cdot \Pr(\mathrm{SE}_s) \\ &= \Pr(\mathrm{CH}_c) \cdot (1-p)^{s-1} \cdot p. \end{aligned} \tag{1}$$

That final line is because the only way to get snake eyes on round $s$ is by rolling not-snake-eyes $s-1$ times in a row followed by one snake eyes.

We can write the unconditional probability of death like this:

$$\Pr(\text{death}) = \sum_{i=1}^{\infty} p_{ii}. \tag{2}$$

That's just summing up all the infinite ways you can be chosen on the same round that snake eyes is rolled.

For the unconditional probability of being chosen to play, we can get it two ways:

$$\Pr(\text{chosen}) = \sum_{s=1}^{\infty}\sum_{c=1}^{s} p_{cs} = \sum_{c=1}^{\infty}\sum_{s=c}^{\infty} p_{cs}. \tag{3}$$

In the first double sum, the outer sum iterates over every round $s$ on which we might roll snake eyes and the inner sum covers all the cases where you're chosen on or before $s$. In the second double sum, the outer sum iterates over every round $c$ in which you can be chosen and the inner sum covers all the cases where snake eyes is rolled on or after $c$.

Eventually we want to find the probability of death given that you're chosen. As we saw in the original derivation, Bayes' Law tells us that this is $\Pr(\text{death})/\Pr(\text{chosen})$. But first let's compute $\Pr(\text{death} \mid \mathrm{CH}_c)$, your probability of death given that you're chosen on a particular round $c$. We expect that to be $p = 1/36$ because it amounts to the one-shot scenario: a specific round $c$ when you're chosen means there's exactly one way to die, namely, rolling snake eyes on that specific round. To be totally sure, and to sanity-check our $p_{cs}$ definition, let's now compute it rigorously. We start with the definition of conditional probability:

$$\Pr(\text{death} \mid \mathrm{CH}_c) = \frac{\Pr(\text{death} \wedge \mathrm{CH}_c)}{\Pr(\mathrm{CH}_c)}.$$

The numerator can also be written $\Pr(\mathrm{SE}_c \wedge \mathrm{CH}_c)$ or $p_{cc}$, the probability that you're both chosen in round $c$ and that snake eyes is rolled on round $c$. And we can write the denominator in terms of $p_{cs}$ by summing over all the ways you can be chosen in round $c$:

$$\frac{p_{cc}}{\sum\limits_{s=c}^{\infty} p_{cs}}. \tag{4}$$

Now we use (1) to expand that to

$$\frac{\Pr(\mathrm{CH}_c) \cdot (1-p)^{c-1} \cdot p}{\sum\limits_{s=c}^{\infty} \Pr(\mathrm{CH}_c)(1-p)^{s-1}p}$$

and cancel common factors (notice we're summing over $s$, not $c$) to get this:

$$\frac{(1-p)^{c-1}}{\sum\limits_{s=c}^{\infty}(1-p)^{s-1}}.$$

Because the denominator is a geometric series starting at $(1-p)^{c-1}$ and with common ratio $1-p$ we can replace it with its closed form and simplify the above to this:

$$\frac{(1-p)^{c-1}}{\frac{(1-p)^{c-1}}{p}}.$$

And that simplifies to $p$. Phew!

Knowing that (4) equals $p$ implies that

$$\sum_{s=c}^{\infty} p_{cs} = \frac{p_{cc}}{p}. \tag{5}$$

Finally we have everything we need to work out your chances of dying if you're chosen to play. Recall that

$$\begin{aligned}\Pr(\text{death} \mid \text{chosen}) &= \frac{\Pr(\text{chosen} \mid \text{death}) \Pr(\text{death})}{\Pr(\text{chosen})} \\ &= \frac{\Pr(\text{death})}{\Pr(\text{chosen})}.\end{aligned}$$

By (2) and (3), that becomes

$$\frac{\sum_{i=1}^{\infty} p_{ii}}{\sum_{c=1}^{\infty} \sum_{s=c}^{\infty} p_{cs}}.$$

Coup de grâce coming up. The inner sum in the denominator is the left-hand side of (5) so we can substitute that in like so:

$$\frac{\sum_{i=1}^{\infty} p_{ii}}{\sum_{c=1}^{\infty} \frac{p_{cc}}{p}}.$$

And we're home free. Factor out the $1/p$ and the sums are the same sum:

$$\frac{\sum_{i=1}^{\infty} p_{ii}}{\frac{1}{p} \sum_{c=1}^{\infty} p_{cc}}.$$

They cancel and the $1/p$ flips to the top as $p$ and we're done! □

Amazingly, we didn't need to define a finite version of the game. We just need a valid prior on when you're chosen. And even more amazingly, the answer is completely independent of what that prior is. For example, say it's uniform for the first $N$ possible values of where you are in the queue of people in the pool. Now compute Pr(death | chosen) in terms of $N$. The answer, as we just saw, is $p$. No $N$ in sight. So in the limit as our prior approaches uniform? Still $p$.

Or maybe you don't like that the above prior has a finite cutoff. No problem. Here's a prior that's both arbitrarily close to uniform and puts positive probability on all infinitely many future rounds in which you could be picked:

- Your probability of being chosen first is 1 in a million
- Your probability of being $n$th in the queue is 99.9999% as much as your probability of being $n-1$st.

In the limit as that "million" goes to infinity (and the 99.9999% correspondingly goes to 1) we again have a uniform prior. Paradox: resolved and double-resolved.