Hyper-Real and Infinitesimal Analysis

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Abstract

Something about non-standard analysis is bothersome. Namely the deletion of epsilons when they were such a valuable tool. In this work we build on nonstandard analysis and generalize the dual numbers.

Keywords 1. *hyper-complex numbers, nonstandard analysis, rings*

1 Basic Properties of Infinitesimals

Definition 1 (Hyper-real ring). Let $\mathbb{K}$ be a field (e.g., $\mathbb{R}$ or $\mathbb{C}$). The *hyper-real ring* $\mathbb{HR}[\mathbb{K}]$ is the commutative ring of formal Laurent polynomials in an indeterminate ϵ with coefficients in $\mathbb{K}$:

$$\mathbb{HR}[\mathbb{K}] = \left\{ \sum_{n=k}^{m} a_n \epsilon^n \;\middle|\; a_n \in \mathbb{K},\ k, m \in \mathbb{Z},\ k \leq m \right\}$$

where ϵ satisfies $\epsilon^n \epsilon^m = \epsilon^{n+m}$ for all $n, m \in \mathbb{Z}$ [1]. The operations are: and where ϵ satisfies $\lim_{x \to 0} x$ and $\epsilon \neq 0$.

$$\text{Addition: } \sum a_n \epsilon^n + \sum b_n \epsilon^n = \sum (a_n + b_n) \epsilon^n$$

$$\text{Multiplication: } \left(\sum a_i \epsilon^i\right) \left(\sum b_j \epsilon^j\right) = \sum_n \left(\sum_{i+j=n} a_i b_j \right) \epsilon^n$$

Proof of Ring Axioms for $\mathbb{HR}[\mathbb{K}]$

Proof. We verify all ring axioms for $\mathbb{HR}[\mathbb{K}]$:

Additive Closure: For $v = \sum_{n=k_v}^{m_v} a_n \epsilon^n$, $w = \sum_{n=k_w}^{m_w} b_n \epsilon^n$, let $k = \min(k_v, k_w)$, $m = \max(m_v, m_w)$. Then:

$$v + w = \sum_{n=k}^{m} (a'_n + b'_n) \epsilon^n \in \mathbb{HR}[\mathbb{K}]$$

where $a'_n = a_n$ if $k_v \leq n \leq m_v$ (0 otherwise), and similarly for b'_n.

Additive Associativity: For $u, v, w \in \mathbb{HR}[\mathbb{K}]$:

$$(u + v) + w = \sum [(a_n + b_n) + c_n] \epsilon^n = \sum [a_n + (b_n + c_n)] \epsilon^n = u + (v + w)$$

by associativity in $\mathbb{K}$.

Additive Identity: $0 = \sum 0 \cdot \epsilon^n$ satisfies $v + 0 = v$ for all v.

Additive Inverses: For $v = \sum a_n \epsilon^n$, define $-v = \sum (-a_n) \epsilon^n$. Then $v + (-v) = 0$.

Commutative Addition: $v + w = \sum (a_n + b_n) \epsilon^n = \sum (b_n + a_n) \epsilon^n = w + v$.

Multiplicative Closure: For $v = \sum_{i=k_v}^{m_v} a_i \epsilon^i$, $w = \sum_{j=k_w}^{m_w} b_j \epsilon^j$:

$$v \cdot w = \sum_{n=k_v+k_w}^{m_v+m_w} \left(\sum_{\substack{i+j=n \\ k_v \leq i \leq m_v \\ k_w \leq j \leq m_w}} a_i b_j \right) \epsilon^n \in \mathbb{HR}[\mathbb{K}]$$

The inner sum has finitely many terms since i, j are bounded.

Multiplicative Associativity: Follows from associativity in $\mathbb{K}$ and $\epsilon^i(\epsilon^j\epsilon^k) = \epsilon^{i+j+k} = (\epsilon^i\epsilon^j)\epsilon^k$.

Distributivity: For $u, v, w \in \mathbb{HR}[\mathbb{K}]$:

$$\begin{aligned} u(v+w) &= \sum_i a_i\epsilon^i \sum_j (b_j + c_j)\epsilon^j \\ &= \sum_n \sum_{i+j=n} a_i(b_j + c_j)\epsilon^n \\ &= \sum_n \left(\sum_{i+j=n} a_i b_j + \sum_{i+j=n} a_i c_j \right) \epsilon^n \\ &= uv + uw \end{aligned}$$

Multiplicative Identity: $1 = 1\epsilon^0$ satisfies $1 \cdot v = v$ for all v.

Thus $\mathbb{HR}[\mathbb{K}]$ is a commutative ring. □

Derivative Properties

Derivatives in this space are different than in $\mathbb{C}$ or in $\mathbb{R}$. To show this we shall use the limit definition of the derivative on a simple function: $f(x) = x^2$.

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x + \epsilon \end{aligned}$$

□

There is no non-trivial function with its own derivative in this space.

Proof. We take:

$$\begin{aligned} f(x) &= f'(x) \\ f(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f(x)(\epsilon + 1) &= f(x + \epsilon) \\ f(x) &= 0 \end{aligned}$$

□

Integral Properties

Integration in this space is different than in $\mathbb{C}$ or $\mathbb{R}$ as well. To show this we shall compute the antiderivative of a trivial function. Namely $f(x) = x$.

Proof.

$$\begin{aligned} f'(x) &= x \\ f(x) &= \frac{x^2}{2} - \frac{\epsilon x}{2} \end{aligned}$$

□

There is no non-trivial function that is its own antiderivative in this space besides the zero function.

Proof. See the proof there is no $f(x) = f'(x)$. $\square$

Recovering Hypercomplex Numbers

If we truncate a power of ϵ such that $\epsilon^k = \{-1, 0, +1\}$ for some $k \in \mathbb{N}$ we recover the hypercomplex numbers.

Proof. A hypercomplex number is an element of a unital algebra generated by a basis in $\{1, i_1, i_2, ..., i_n\}$ with coefficients in a field [2] such that $i_k^2 = \{-1, 0, +1\}$. We can map directly powers of ϵ^n to these i_n components and then take $n^2 = k$ for $n, k \in \mathbb{N}$ and we are done. $\square$

References

[1] Abraham Robinson. *Nonstandard Analysis.* North-Holland, 1966.

[2] Henry Taber. On hypercomplex number systems. *Transactions of the American Mathematical Society*, 5(4):509–548, October 1904. Early foundational work on hypercomplex algebras.