COLORADO SCHOOL OF MINES

MATH437/537 **Fall, 2022**

Homework 2. Due September 22

1. Suppose a symmetric 4×4 matrix $\boldsymbol{S}$ has eigenvector decomposition $\boldsymbol{S} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\top$, where $\boldsymbol{D} = \text{diag}\{4, 2, -1, -5\}$.

   (a) Determine all the singular values of $\boldsymbol{S}$

   (b) Determine all the singular values of the orthogonal matrix $\boldsymbol{U}$

   (c) Provide an example of an orthogonal 3×3 matrix different from the identity

2. Consider the linear model
$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$
where $\boldsymbol{A}$ is a fixed $n \times p$ matrix of rank p and $\boldsymbol{\varepsilon}$ is an $n \times 1$ random vector with mean zero and nonsingular covariance matrix $\boldsymbol{\Sigma}$. The weighted least-squares estimate, $\widehat{\boldsymbol{\beta}}$, of $\boldsymbol{\beta}$ is obtained by minimizing
$$F(\boldsymbol{x}) = (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})$$
over $\boldsymbol{x} \in \mathbb{R}^p$.

   (a) Differentiate F with respect to $\boldsymbol{x}$ to find $\widehat{\boldsymbol{\beta}}$

   (b) Find the mean and covariance matrix of $\widehat{\boldsymbol{\beta}}$ using the properties of $\mathbb{E}$, $\mathbb{V}\text{ar}$ and $\mathbb{C}\text{ov}$

   (c) Find the mean and covariance matrix of the residual vector $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{A}\widehat{\boldsymbol{\beta}}$

3. Let
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$
Determine each of the following:

   (a) $\boldsymbol{a} \otimes \boldsymbol{A}$

   (b) $\boldsymbol{A} \otimes \boldsymbol{b}$

   (c) $\boldsymbol{I}_2 \otimes \boldsymbol{A}$

   (d) $\boldsymbol{A} \otimes \boldsymbol{I}_2$

   (e) $\boldsymbol{A} \otimes \boldsymbol{A}$

   (f) $(\boldsymbol{A} \otimes \boldsymbol{A})(\boldsymbol{I}_2 \otimes \boldsymbol{A})$

4. Suppose Z_1, Z_2, Z_3, Z_4 are iid $N(0, 1)$. Set $\boldsymbol{Y} = (Z_1 + Z_2, Z_1 - Z_4, Z_3, Z_3^2)^\top$.

   (a) Find $\mathbb{E}\,\boldsymbol{Y}$.

(b) Find $\mathbb{V}\text{ar}(\boldsymbol{Y})$.

(c) Let $\boldsymbol{U}$ be the random matrix

$$\boldsymbol{U} = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.$$

Find $\mathbb{E}\,\boldsymbol{U}$ and $\mathbb{V}\text{ar}(\boldsymbol{U})$.

5. Let $\boldsymbol{X} = (X_1, X_2)^\top$ be multivariate Gaussian with $\boldsymbol{\mu} = \mathbb{E}\,\boldsymbol{X} = (1,1)^\top$ and

$$\mathbb{V}\text{ar}(\boldsymbol{X}) = \boldsymbol{\Sigma} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}.$$

(a) Determine the distribution of X_1 and that of X_2

(b) Find the joint distribution of X_1 and $X_1 - X_2$

(c) Find the joint distribution of $X_1 + X_2$ and $X_1 - X_2$

(d) Determine the mean and covariance matrix of $\boldsymbol{AX}$ where $\boldsymbol{A}$ is defined in question 2

(e) Determine the mean and covariance matrix of $\boldsymbol{AXb}$, where $\boldsymbol{b}$ is the vector defined in question 3 (use the transformation formulas from class)

(f) Find a linear combination $aX_1 + bX_2$ that is independent of $X_1 + X_2$

6. Show that the covariance matrix $\boldsymbol{\Sigma}$ of any $p \times 1$ random vector has to be non-negative definite. That is, $\boldsymbol{a}^\top \boldsymbol{\Sigma} \boldsymbol{a} \geq 0$ for any $p \times 1$ vector $\boldsymbol{a}$.