1. Let N be a normal subgroup of a group G. Use properties of group homomorphisms to show that every subgroup of G/N has the form H/N, where H is a subgroup of G. Let $\phi : G \to G/N$ be the canonical group homomorphism that sends an element $g \in G$ to its equivelence class $[g]$. Let K be a subgroup of G/H. Since the inverse image of a subgroup under a group homomorphism is a subgroup we have $\phi^{-1}(K)$ is a subgroup of G containing $ker(\phi) = N$. Lets denote $\phi^{-1}(K)$ by H. Then H is a subgroup of G and $\phi(H) = K$. Since $\phi|_H : H \to \phi(H)$ is surjective by first isomorphism theorem $\phi(H) \cong H/ker(\phi|_H)$. Since $N \subset H$ we have $ker(\phi|_H) = ker(\phi) \cap H = N \cap H = N$. Hence $K = \phi(H) \cong H/N$. Since K was arbitrary this completes the proof.

2. The Second Isomorphism Theorem states:

   If K is a subgroup of G and N is a normal subgroup of G, then $K/(K \cap N) \cong KN/N$.

   (a) Draw a diagram (similar to a subgroup lattice) that shows the relationship between the

$$\begin{tikzcd}[arrows=dash] & G \arrow[d] & \\ & KN \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & N \arrow[dl] \\ & K \cap N \arrow[d] & \\ & \{e\} & \end{tikzcd}$$

   groups G, K, N, KN, $K \cap N$, and $\{e\}$.

   (b) Implied in the conclusion of the theorm is that $KN \leq G$, $K \cap N \triangleleft K$ and $N \triangleleft KN$. Justify why this is true given the hypotheses. Since K and N are subgroups and N is normal for each $k \in K, kNk^{-1} = N$. Therefore $kN = Nk, \forall k \in K$. Hence $KN = NK$ and therefore $KN \leq G$.

   (c) Prove the theorem. We have $K \to KN \to KN/N$, where the first map is inclusion and the second map is the canonical surjection. We denote the first map by i and the second map by j. Since the first map is injective the kernal of the composite map must be the kerneal of the second map instersected with K ie $K \cap N$, We will be done by first isomorphism theorjem if we can show that the composite map is surjective. For any $[a] \in KN/N \exists b \in KN$ such that $j(b) = [a]$. But all elements of KN are of the form kn for $k \in K$ and $n \in N$. Therefore $b = k_1 n_1$ for some $k_1 \in K$ and some $n_1 \in N$. We now see that $j(i(k_1)) = j(k_1) = j(k_1 n_1) = [a]$, as $n_1 \in N, j(k_1) = j(n_1 k_1)$. Therefore the composite map $K \to KN/N$ is surjective with kernal $K \cap N$ and hence by first isomorphism theorem $K/(K \cap N) \cong KN/N$.

3. The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is group under multiplication modulo 45.

   (a) Write G as an internal direct product of cyclic subgroups of prime-power order. (No justification needed here.) $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ Since G has two elements of order 2 it cannot be isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$.

   (b) Determine the isomorphism class of G by writing G as an external direct product of cyclic groups of prime-power order. (No justification needed here.) $\langle 19 \rangle = \{1, 19\}$ and

$\langle 26\rangle = \{1, 26\}$. $\langle 19\rangle \times \langle 26\rangle = \{1, 19, 26, 44\}$. Since $\langle 16\rangle = \{1, 16, 31\}$ we can conclude $G = \langle 19\rangle \times \langle 16\rangle \times \langle 26\rangle$

(c) Justify your answers to parts (a) and (b).

4. Let G be an abelian group of order p^n for some prime p and positive integer n. Prove that G is cyclic if and only if G has exactly $p-1$ elements of order p. Since $p|p^n$ for every divsor (p) of a finite abelian group there is $\phi(p) = p-1$ elemenets of order p. For all finite abelian groups if they have unique cyclic subgroups for every divisor of the order of the group then the group is cyclic. To prove the converse part let there be $p-1$ elements of order p. Now for finite abelian groups the number of cyclic subgroups of order p is equal to the number of elements of order p divided by $\phi(p)$. $\frac{p-1}{\phi(p)} = \frac{p-1}{p-1} = 1$. Given that there are $p-1$ elements of order p. Thus the proof is complete.