Homework 7

Unless otherwise indicated, you must justify all answers/steps. See the Canvas assignment for more information about the homework requirements.

1. Let N be a normal subgroup of a group G. Use properties of group homomorphisms to show that every subgroup of G/N has the form H/N, where H is a subgroup of G.

2. The Second Isomorphism Theorem states:

   If K is a subgroup of G and N is a normal subgroup of G, then $K/(K \cap N) \cong KN/N$.

   (a) Draw a diagram (similar to a subgroup lattice) that shows the relationship between the groups G, K, N, KN, $K \cap N$, and $\{e\}$.

   (b) Implied in the conclusion of the theorm is that $KN \leq G$, $K \cap N \triangleleft K$ and $N \triangleleft KN$. Justify why this is true given the hypotheses.

   (c) Prove the theorem.

3. The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is group under multiplication modulo 45.

   (a) Write G as an internal direct product of cyclic subgroups of prime-power order. (No justification needed here.)

   (b) Determine the isomorphism class of G by writing G as an external direct product of cyclic groups of prime-power order. (No justification needed here.)

   (c) Justify your answers to parts (a) and (b).

4. Let G be an abelian group of order p^n for some prime p and positive integer n. Prove that G is cyclic if and only if G has exactly $p - 1$ elements of order p.