Homework 8

Unless otherwise indicated, you must justify all answers/steps. See the Canvas assignment for more information about the homework requirements.

1. Prove proposition 6.6 from the Module 6 notes.

   Proposition 6.6 (Properties of Multiplication in Rings). Let a, b, c belong to a ring R. Then

   (a) $a0 = 0a = 0$

   (b) $a(-b) = (-a)(b) = -(ab)$

   (c) $(-a)(-b) = (ab)$

   (d) $a(b-c) = ab - ac$ and $(b-c)a = ba - ca$

   If R has a unity element 1, then

   (e) $(-1)a = -a$

   (f) $(-1)(-1) = 1$

2. Suppose that a and b belong to a commutative ring R with unity. If a is a unit and $b^2 = 0$, show that $a + b$ is a unit.

3. The set $\mathbb{R}[x]$ of all polynomials in the variable x with real coefficients under ordinary addition and multiplication is a commutative ring.

   (a) What is unity in $\mathbb{R}[x]$? What are the units of $\mathbb{R}[x]$? Explain.

   (b) Show that $\mathbb{Z}[x]$ forms a subring of R, where $\mathbb{Z}[x]$ is the subset of $\mathbb{R}[x]$ with integer coefficients.

4. An element a in a ring R with unity is called *nilpotent* if there exists a positive integer n such that $a^n = 0$.

   (a) Give an example of a nontrivial ring R and a nonzero nilpotent element a.

   (b) Show that for an arbitrary ring R with unity, if a is a nilpotent element of R, then $1-a$ is a unit. (Hint: Consider $(1-a)(1+a+a^2+\cdots+a^{n-1})$.)

   (c) Show that for a *commutative* ring R with unity, the set of nilpotent elements forms a subring.

5. Let R and S be commutative rings. Prove that (a, b) is a zero-divisor in $R \oplus S$ if and only if a or b is a zero-divisor or exactly one of a or b is 0.