1. A ring element a is called *idempotent* if $a^2 = a$. The following are problems regarding idempotent elements. Each problem is otherwise unrelated.
   (a) Prove that is a is an idempotent ring element, then $a^n = a$ for all positive integers n. a is indempotent so $a^2 = a$.Now for $m = 1$. $a^m = a^1 = a$. Now for $m = k + 1$. $a^{k+1} = a^k a = aa = a$ since $a^k = a$ and $a^2 = a$ by the induction hypothesis. So we can say it is true for all natural numbers.
   (b) Show that any idempotent element in a commutative ring with unity other than 0 or 1 is a zero-divisor. $a \neq 0, 1$ Now $a(1-a) = a(1+(-a)) = a + a(-a) = a - a^2 = a - a$ since the ring is commutive with unity. Now we have $a - a = 0$ So with $a \neq 0, 1$ we have an element $(1-a)$ such that $a(1-a) = 0$. So a is a zero divisor.
2. Find all units, zero-divisors, idempotents, and nilpotent elements in ring $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. (Recall: an element a is nilpotent if $a^n = 0$ for some positive integer n.) Zero element of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ is (0,0). Identity element of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ is (1,1). Unit let $(a,b) and (c,d) \in \mathbb{Z}_3 \oplus \mathbb{Z}_6$ such that $(a,b)\dot{(}c,d) = (1,1)$. Then we have $(ac, bd) = (1,1), ac = 1, bd = 1$. So a is a unit and b is a unit. Now we know a is a unit in $\mathbb{Z}_n$ for some integer n if $gcd(n,a) = 1$ So 1,2 are units in $\mathbb{Z}_3$ and 1,5 are units in $\mathbb{Z}_6$. Hence the units of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ are (1,1),(1,5),(2,1),and(2,5). Zero divisors: let (a,b) bye a zero divisor of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. Then there exists non zero element (c,d) in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ such that (ac,bd)=(0,0). So ac=0 and bd=0. 0 is the only zero divisor of $\mathbb{Z}_3$ as it is a field. the zero divisors of $\mathbb{Z}_6$ are 0,2,3,4. So our zero divisors of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ are (0,0),(0,2),(0,3),(0,4). Indepotent: Let (a,b) be indempotent of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. Then $(a,b)^2 = (a,b)$ so $a^2 = a$ and $b^2 = b$. So a and b are indempotent in $\mathbb{Z}_3$ and $\mathbb{Z}_6$ respectfully. 0,1 are indempotent in $\mathbb{Z}_3$ 0,1,3,4 are indempotent in $\mathbb{Z}_6$. So the indempotents of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ are (0,0),(0,1),(0,3),(0,4),(1,0),(1,1),(1,3), and (1,4). Nilpotent. Similarly if we can show (a,b) is nilpotent in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ then a,b are nilpotent in $\mathbb{Z}_3$ and $\mathbb{Z}_6$. 0 is the only nilpotent element in $\mathbb{Z}_3$. 0 is the only nilpotent element in $\mathbb{Z}_6$. So (0,0) is the only nilpotent element of $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.
3. Suppose that a and b belong to an integral domain R. If $a^m = b^m$ and $a^n = b^n$, where m and n positive integers that are relatively prime, prove that $a = b$. Note that the elements a, b are not necessarily units, so we cannot assume a^{-1} or b^{-1} (or powers of a^{-1} or b^{-1}) exist. Because gcd(m,n)=1 $\exists x, y \in \mathbb{Z}$ such that mx+ny=1. $a^1 = a^{mx+ny} = a^{mx}a^{ny} = (a^m)^x(a^n)^y = (b^m)^x(b^n)^y = b^{mx+ny} = b$ so $a = b$.
4. The following are problems regarding the characteristic of a ring.
   (a) Let R be a ring with m elements. Show that the characteristic of R divides m. Let r be the characteristic of R. Since the characteristic is r there $\exists x \in R$ such that the group generated by x under addition of the ring has size r. By lagrange theroem $r | \|R\|$ hence $r|m$.
   (b) Show that any finite field has order p^n, where p is a prime. (Hint: Use facts about finite abelian groups.) Let F be a finit efield. Le tthe smallest mutltipe of 1 that gives be p. p is the characteristic of the field. We have $p.1 = 0$ Let it be possible p is not prime. then p=rs for some integers r and s less than p. p.1=(r.1)(s.1). Since p is the smallest possible integer such that p.1 =0 we have $(r.1) \neq 0$ and $(s.1) \neq 0$ but $(r.1)(s.1) = 0$ this contradicts that F is a field. Hence p is prime. Finally if q is prime other than p such that $q | \|F\|$ the since (R,+) is an abelian group $\exists x \in F$ such that $x \neq 0, x + x ...(qtimes) + x = 0$ hence x(q.1)=0. Since p is the characyeristic of the field and p doesn't divide q we have $q.1 \neq 0$ hence a contracdiction. Hence p is the only prime divisor of $\|F\|$.