Homework 9

Unless otherwise indicated, you must justify all answers/steps. See the Canvas assignment for more information about the homework requirements.

1. A ring element a is called *idempotent* if $a^2 = a$. The following are problems regarding idempotent elements. Each problem is otherwise unrelated.

   (a) Prove that is a is an idempotent ring element, then $a^n = a$ for all positive integers n.

   (b) Show that any idempotent element in a commutative ring with unity other than 0 or 1 is a zero-divisor.

2. Find all units, zero-divisors, idempotents, and nilpotent elements in ring $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. (Recall: an element a is nilpotent if $a^n = 0$ for some positive integer n.)

3. Suppose that a and b belong to an integral domain R. If $a^m = b^m$ and $a^n = b^n$, where m and n positive integers that are relatively prime, prove that $a = b$. Note that the elements a, b are not necessarily units, so we cannot assume a^{-1} or b^{-1} (or powers of a^{-1} or b^{-1}) exist.

4. The following are problems regarding the characteristic of a ring.

   (a) Let R be a ring with m elements. Show that the characteristic of R divides m.

   (b) Show that any finite field has order p^n, where p is a prime. (Hint: Use facts about finite abelian groups.)