

1.

$$\begin{aligned}
 f(y) &= (\theta + 1)y^\theta \\
 \mu &= \int_{\mathcal{D}} f(y)dy = \int_0^1 (\theta + 1)y^\theta dy \\
 &= \int_0^1 (\theta + 1)y^{\theta+1} dy \\
 &= \frac{(\theta + 1)}{(\theta + 2)} y^{\theta+2} \Big|_0^1 \\
 &= \frac{(\theta + 1)}{(\theta + 2)} \\
 \bar{Y} &= \frac{(\theta + 1)}{(\theta + 2)} \\
 \bar{Y}(\theta + 2) &= \theta + 1 \\
 \bar{Y}\theta + 2\bar{Y} &= \theta + 1 \\
 \bar{Y}\theta - \theta &= 1 - 2\bar{Y} \\
 \theta(\bar{Y} - 1) &= 1 - 2\bar{Y} \\
 \hat{\theta} &= \frac{1 - 2\bar{Y}}{\bar{Y} - 1}
 \end{aligned}$$

2. (a)

$$l(\theta; X) = \binom{100}{33} \theta^{33} (1 - \theta)^{100-33}$$

(b)

$$\begin{aligned}
 \mu &= n\theta \\
 \frac{\bar{X}}{100} &= \hat{\theta}
 \end{aligned}$$

3. (a)

$$\begin{aligned}
 \mu &= \frac{1}{\lambda} \\
 \bar{X} &= \frac{1}{\bar{\lambda}} \\
 \frac{1}{\bar{X}} &= \hat{\lambda}
 \end{aligned}$$

(b)

$$\begin{aligned}
l(\lambda; X) &= \prod_{n=1}^N \lambda e^{-\lambda X_n} \\
\ln(l(\lambda; X)) &= \ln\left(\prod_{n=1}^N \lambda e^{-\lambda X_n}\right) \\
\ln(l(\lambda; X)) &= N \ln(\lambda) - \lambda \sum_{n=1}^N X_n \\
\frac{d}{d\lambda} \ln(l(\lambda; X)) &= \frac{d}{d\lambda} [N \ln(\lambda) - \lambda \sum_{n=1}^N X_n] \\
\frac{d}{d\lambda} \ln(l(\lambda; X)) &= \frac{N}{\lambda} - \sum_{n=1}^N X_n \\
\frac{d}{d\lambda} \ln(l(\lambda; X)) &= 0 \\
0 &= \frac{N}{\lambda} - \sum_{n=1}^N X_n \\
\sum_{n=1}^N X_n &= \frac{N}{\lambda} \\
\frac{\sum_{n=1}^N X_n}{N} &= \frac{1}{\lambda} \\
\frac{1}{\bar{X}} &= \hat{\lambda}
\end{aligned}$$

(c)

$$\begin{aligned}
\sqrt{n}(g(\hat{\lambda}) - g(\mu)) &\rightarrow N(0, g'(\mu)^2 \sigma^2) \\
g'(\mu) &= -\mu^{-2} \\
g'\left(\frac{1}{\lambda}\right) &= -\lambda^2 \\
\sqrt{n}\left(\frac{1}{\bar{X}} - \lambda\right) &\rightarrow N(0, \lambda^2)
\end{aligned}$$

4. (a)

$$P(X = 1) = \lambda e^{-\lambda}$$

(b)

$$MLE = \bar{X} e^{-\bar{X}}$$

(c)

$$\sqrt{n}(\bar{X} e^{-\bar{X}} - \lambda e^{-\lambda}) \rightarrow N(0, \lambda(-\lambda e^{-\lambda} + e^{-\lambda})^2)$$

(d) $\lambda \neq 1$

(e)

$$\hat{MLE} = \bar{X}^2$$

(f)

$$\sqrt{n}(\bar{X}^2 - \lambda^2) \rightarrow N(0, 4\lambda^3)$$

(g) $\lambda > 0$