1. The function:

$$f(t,u) = u^2 e^{-t^2} sin(t)$$

is Lipschitz continous in $u$ on:

$$\mathcal{D} = \{(t,u) : t \in \mathbb{R}, u \in [0,2]\}$$

if

$$\begin{aligned}
\exists L \in \mathbb{R}_+ : |f(t,u) - f(t,u^*)| &\leq L\,|u-u^*| \\
\left|u^2 e^{-t^2} sin(t) - u^{*2} e^{-t^2} sin(t)\right| &\leq L\,|u-u^*| \\
\left|u^2 - u^{*2}\right| \left|e^{-t^2} sin(t)\right| &\leq L\,|u-u^*| \\
|u-u^*|\,|u+u^*| \left|e^{-t^2} sin(t)\right| &\leq L\,|u-u^*|
\end{aligned}$$

Since $f(t,u)$ is differentiable with respect to $u$ and bounded in $\mathcal{D}$ then we can set the Lipschitz constant:

$$\begin{aligned}
L &= \max_{(t,u)\in\mathcal{D}} |f_u(t,u)| \\
&= \max_{(t,u)\in\mathcal{D}} \left|2ue^{-t^2} sin(t)\right| \\
&\approx 4 * .397 \\
&\approx 1.588
\end{aligned}$$

2. (a) Since $f(u)$ is differentible at all points $u \in \mathbb{R}$, $f(u)$ is Lipshitz at every point $u \in \mathbb{R}$. With L $\leq$ 2.

   (b) Take Dirchlet's 'jagged' discontinuous function which is continuous nowhere and is differentiable nowhere but is nonetheless bounded by 1.

3.

$$f = \begin{bmatrix} u_2 \\ -\frac{u_1}{(u_1^2+u_3^2)^{3/2}} \\ u_4 \\ -\frac{u_3}{(u_1^2+u_3^2)^{3/2}} \end{bmatrix}$$

$$J(f) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-2u_1^2+u_3^2}{(u_1^2+u_3^2)^{5/2}} & 0 & \frac{-3u_1u_3}{(u_1^2+u_3^2)^{5/2}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-3u_1u_3}{(u_1^2+u_3^2)^{5/2}} & 0 & \frac{-2u_3^2+u_1^2}{(u_1^2+u_3^2)^{5/2}} & 0 \end{bmatrix}$$

$$||A||_F = \sqrt{\sum_i \sum_j |A_{ij}|^2}$$

$$||A||_F = \sqrt{1^2 + (\frac{-2u_1^2 + u_3^2}{(u_1^2+u_3^2)^{5/2}})^2 + (\frac{3u_1u_3}{(u_1^2+u_3^2)^{5/2}})^2 + 1^2 + (\frac{3u_1u_3}{(u_1^2+u_3^2)^{5/2}})^2 + (\frac{-2u_3^2 + u_1^2}{(u_1^2+u_3^2)^{5/2}})^2}$$

$$||A||_F = \sqrt{2 + \frac{4u_1^4 + u_3^4 - 4u_1^2u_3^2}{(u_1^2+u_3^2)^5} + \frac{18u_1^2u_3^2}{(u_1^2+u_3^2)^5} + \frac{4u_3^4 + u_1^4 - 4u_3^2u_1^2}{(u_1^2+u_3^2)^5}}$$

$$||A||_F = \sqrt{2 + \frac{5u_3^4 + 5u_1^4 + 10u_3^2u_1^2}{(u_1^2+u_3^2)^5}}$$

$$||A||_F = \sqrt{2 + \frac{5}{(u_1^2+u_3^2)^3}}$$

$$(||A||_F)^2 = 2 + \frac{5}{(u_1^2+u_3^2)^3}$$

$$(||A||_F)^2 - 2 = \frac{5}{(u_1^2+u_3^2)^3}$$

$$(u_1^2+u_3^2)^3 = \frac{5}{(||A||_F)^2 - 2}$$

$$\mathcal{D} := \{u : 1/4 \leq u_1^2 + u_3^2\}$$

Apply this domain and we achieve an upper bound on $(||A||_F)^2$

$$(\frac{1}{4})^3 \leq \frac{5}{(||A||_F)^2 - 2}$$
$$(||A||_F)^2 \leq 322$$

And is therefore Lipschitz.

4.