

1. The function:

$$f(t, u) = u^2 e^{-t^2} \sin(t)$$

is Lipschitz continuous in u on:

$$\mathcal{D} = \{(t, u) : t \in \mathbb{R}, u \in [0, 2]\}$$

if

$$\begin{aligned} \exists L \in \mathbb{R}_+ : |f(t, u) - f(t, u^*)| &\leq L |u - u^*| \\ \left| u^2 e^{-t^2} \sin(t) - u^{*2} e^{-t^2} \sin(t) \right| &\leq L |u - u^*| \\ |u^2 - u^{*2}| \left| e^{-t^2} \sin(t) \right| &\leq L |u - u^*| \\ |u - u^*| |u + u^*| \left| e^{-t^2} \sin(t) \right| &\leq L |u - u^*| \end{aligned}$$

Since $f(t, u)$ is differentiable with respect to u and bounded in \mathcal{D} then we can set the Lipschitz constant:

$$\begin{aligned} L &= \max_{(t, u) \in \mathcal{D}} |f_u(t, u)| \\ &= \max_{(t, u) \in \mathcal{D}} \left| 2ue^{-t^2} \sin(t) \right| \\ &\approx 4 * .397 \\ &\approx 1.588 \end{aligned}$$

2. (a) Since $f(u)$ is differentiable at all points $u \in \mathbb{R}$, $f(u)$ is Lipschitz at every point $u \in \mathbb{R}$. With $L \leq 2$.
- (b) Take Dirichlet's 'jagged' discontinuous function which is continuous nowhere and is differentiable nowhere but is nonetheless bounded by 1.
- 3.

$$\begin{aligned} f &= \begin{bmatrix} u_2 \\ -\frac{u_1}{(u_1^2 + u_3^2)^{3/2}} \\ u_4 \\ -\frac{u_3}{(u_1^2 + u_3^2)^{3/2}} \end{bmatrix} \\ J(f) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-2u_1^2 + u_3^2}{(u_1^2 + u_3^2)^{5/2}} & 0 & \frac{-3u_1 u_3}{(u_1^2 + u_3^2)^{5/2}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-3u_1 u_3}{(u_1^2 + u_3^2)^{5/2}} & 0 & \frac{-2u_3^2 + u_1^2}{(u_1^2 + u_3^2)^{5/2}} & 0 \end{bmatrix} \end{aligned}$$

$$\|A\|_F = \sqrt{\sum_i \sum_j |A_{ij}|^2}$$

$$\|A\|_F = \sqrt{1^2 + \left(\frac{-2u_1^2 + u_3^2}{(u_1^2 + u_3^2)^{5/2}}\right)^2 + \left(\frac{3u_1u_3}{(u_1^2 + u_3^2)^{5/2}}\right)^2 + 1^2 + \left(\frac{3u_1u_3}{(u_1^2 + u_3^2)^{5/2}}\right)^2 + \left(\frac{-2u_3^2 + u_1^2}{(u_1^2 + u_3^2)^{5/2}}\right)^2}$$

$$\|A\|_F = \sqrt{2 + \frac{4u_1^4 + u_3^4 - 4u_1^2u_3^2}{(u_1^2 + u_3^2)^5} + \frac{18u_1^2u_3^2}{(u_1^2 + u_3^2)^5} + \frac{4u_3^4 + u_1^4 - 4u_3^2u_1^2}{(u_1^2 + u_3^2)^5}}$$

$$\|A\|_F = \sqrt{2 + \frac{5u_3^4 + 5u_1^4 + 10u_3^2u_1^2}{(u_1^2 + u_3^2)^5}}$$

$$\|A\|_F = \sqrt{2 + \frac{5}{(u_1^2 + u_3^2)^3}}$$

$$(\|A\|_F)^2 = 2 + \frac{5}{(u_1^2 + u_3^2)^3}$$

$$(\|A\|_F)^2 - 2 = \frac{5}{(u_1^2 + u_3^2)^3}$$

$$(u_1^2 + u_3^2)^3 = \frac{5}{(\|A\|_F)^2 - 2}$$

$$\mathcal{D} := \{u : 1/4 \leq u_1^2 + u_3^2\}$$

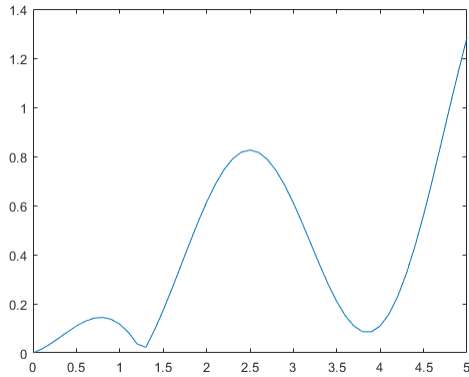
Apply this domain and we achieve an upper bound on $(\|A\|_F)^2$

$$\left(\frac{1}{4}\right)^3 \leq \frac{5}{(\|A\|_F)^2 - 2}$$

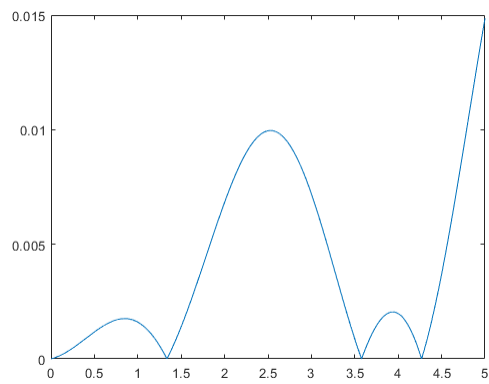
$$(\|A\|_F)^2 \leq 322$$

And is therefore Lipschitz.

4. (a)
(b) 1.2830

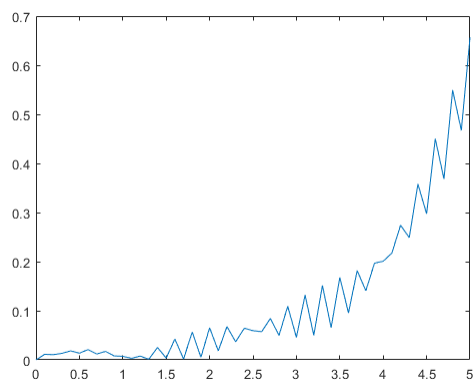


(c) 0.0149

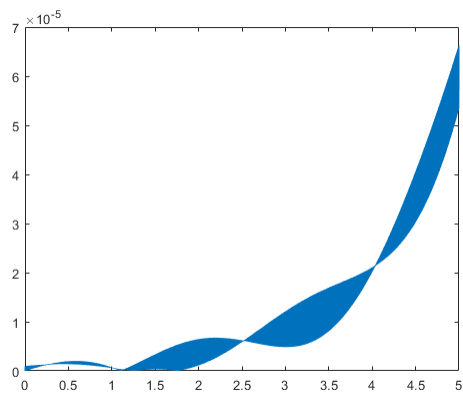


5. (a)

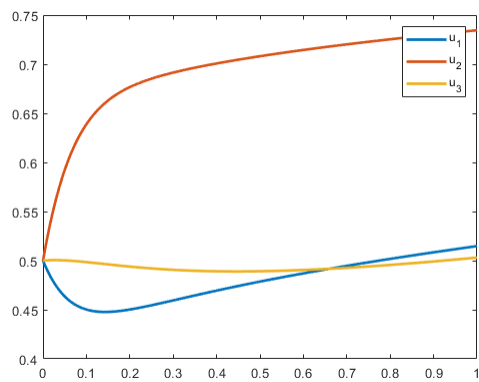
(b) .6563



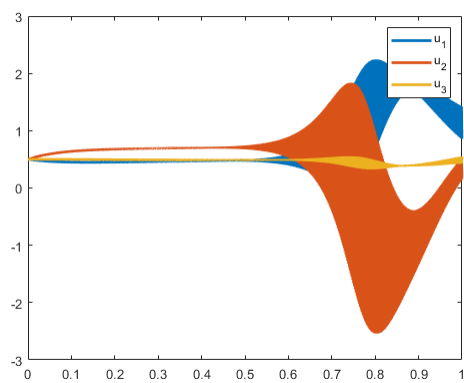
(c) 6.6262E-05



6. (a) Backwards Euler



(b) Leapfrog



(c) Leapfrog with smaller k

